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Threshold resummation of semi-inclusive deep-inelastic scattering

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Abstract We derive threshold resummation of semi-inclusive deep-inelastic scattering (SIDIS), by building upon previous results by some of us for the resummation of the Drell–Yan process at fixed rapidity, which is related to SIDIS by crossing. We consider both a double-soft limit, in which both the Bjorken and the fragmentation scaling variables tend to their threshold value, and single soft limits in which either of them does. We show that in the former limit only soft radiation contributes, and in the latter limit only collinear radiation, and we derive resummed expressions for the coefficient functions in all cases. We determine explicitly resummation coefficients in the nonsinglet channel up to next-to-next-to-leading log by comparing to recent fixed next-to-next-to-leading order results. Expanding out the single-soft resummation we reproduce recent next-to-leading power results.

Contents

1 Introduction

Interest in Semi-Inclusive Deep Inelastic Scattering (SIDIS) has been growing recently, in view of physics at the forthcoming Electron-Ion Collider (EIC) [1], and indeed two independent groups have recently computed coefficient functions for this process up to next-to-next-to-leading order (NNLO) [2–5]. It was pointed out long ago [6] that, because SIDIS is related by crossing to Drell–Yan production, threshold resummation can be achieved for the former by using well-known results for the latter process.

Of course, SIDIS coefficient functions depend on a pair of scaling variables: the Bjorken variable x, physically related to the fraction of the incoming hadron momentum carried by the incoming parton, and the fragmentation variable z, related to the fraction of the final state parton that fragments into the final-state hadron. The crossing relation therefore relates it to the rapidity distribution of the Drell–Yan process, that can also be parametrized by two scaling variables, x_1 and x_2, related to the momentum fractions of the incoming partons.

The threshold resummation obtained by crossing in Ref. [6] corresponds to the threshold resummation that was derived long ago in Ref. [7] for the Drell–Yan rapidity distribution in the limit in which both scaling variables tend to their threshold value. In the case of Drell–Yan this is the limit in which the available center-of-mass energy tends to the threshold value needed to produce the desired final state, namely the final-state gauge boson mass. This resummation for Drell–Yan has been extended recently up next-to-next-to-leading log (NNLL) matched to NNLO in Ref. [8], and

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next-to-leading power in Ref. [9]. Exploiting crossing, these results have been used to derive the corresponding resummation for SIDIS at increasingly high logarithmic accuracy and then also to next-to-leading power in Refs. [10–13].

The resummation of the Drell–Yan rapidity distribution has been extended to the case in which only one of the scaling variables tends to its threshold value using SCET techniques in Refs. [14,15], and more recently by us [16] using the renormalization-group based approach to soft resummation in direct QCD of Ref. [17]. Here we show that using techniques very similar to those of Ref. [16] it is possible to similarly perform resummation of SIDIS in the limit in which the Bjorken variable tends to its soft limit, but for fixed values of the fragmentation variable, or conversely.

In the case of Drell–Yan these single soft limits correspond to the case in which the final-state gauge boson is boosted by a fixed amount along the direction of one of the two incoming partons in the partonic center of mass frame, and the energy tends to the minimum value which is necessary in order to produce this configuration. This corresponds to the situation in which only the collinear radiation which is necessary in order to produce the required boost survives. In the case of SIDIS, as we shall show, the two soft limits similarly correspond to the case in which only radiation collinear to either the incoming or the fragmenting parton is allowed, and this is why the same resummation formalism applies.

We first derive a general resummation formalism by adapting to this case the formalism of Ref. [16]. This essentially amounts to deriving the kinematic configurations that correspond to the various soft limit and, as in our previous work [16–18], determine the relevant soft scales through a phase-space analysis. The resummation of the dependence on the soft scale then follows using the argument of Ref. [16], and the resummation coefficients can be determined by comparison to the fixed order. We will determine them explicitly in the nonsinglet channel up to NNLL accuracy by matching to the fixed order results of Ref. [3,5], thereby also obtaining a nontrivial consistency check of the resummation.

2 Kinematics and the soft limit

We consider resummation of SIDIS coefficient functions. Because we will match our resummed results to the fixed order results of Refs. [3,5] we follow notation and conventions of these references. The measured cross-section for the process

$$H(P_1) + G(q) \to h(P_2) + X, \tag{1}$$

where G is an off-shell gauge boson, is parametrized by structure functions $\mathcal{F}_F^h(\xi, \zeta, Q^2)$, with $F = L, T, 3$ and

$$Q^2 = -q^2, \tag{2}$$

$$\xi = \frac{Q^2}{2P_1 \cdot q} \tag{3}$$

$$\zeta = \frac{P_1 \cdot P_2}{P_1 \cdot q}. \tag{4}$$

The structure functions can be factorized in terms of parton distributions f_i and fragmentation functions D_j^h as

$$\mathcal{F}_F^h(\xi, \zeta, Q^2) = \sum_{ij} \int_\xi^1 \frac{dx}{x} \int_\zeta^1 \frac{dz}{z} f_i\left(\frac{\xi}{x}, \mu_1^2\right) \times C_{ij}^F\left(x, z, \alpha_s(\mu_R^2), \frac{\mu_R^2}{Q^2}, \frac{\mu_1^2}{Q^2}, \frac{\mu_2^2}{Q^2}\right) D_j^h\left(\frac{\zeta}{z}, \mu_2^2\right). \tag{5}$$

The kinematics of the partonic subprocess

$$\mathcal{Q}_1(p_1) + G(q) \to \mathcal{Q}_2(p_2) + X(k), \tag{6}$$

is parametrized by the scaling variables

$$x = \frac{Q^2}{2p_1 \cdot q}, \tag{7}$$

$$z = \frac{p_1 \cdot p_2}{p_1 \cdot q}. \tag{8}$$

We will henceforth focus on the partonic cross-section, and study the partonic process Eq. (6), which can be thought of as a $2 \to 2$ process, parametrized by the masses of the incoming gauge boson, $q^2 = -Q^2$; the invariant mass of the system X that recoils against the final-state parton, k^2; and the two Mandelstam invariants

$$s = (p_1 + q)^2 = Q^2 \frac{1-x}{x} \tag{9}$$

$$t = (p_1 - p_2)^2 = -Q^2 \frac{z}{x}. \tag{10}$$

Resummation is performed in Mellin space, where the double convolution of Eq. (5) reduces to an ordinary product:

$$\mathcal{F}_F^h(N, M, Q^2) = \sum_{ij} f_i(N, \mu_1^2) C_{ij}^F \times \left(N, M, \alpha_s(\mu_R^2), \frac{\mu_R^2}{Q^2}, \frac{\mu_1^2}{Q^2}, \frac{\mu_2^2}{Q^2}\right) D_j^h(M, \mu_2^2), \tag{11}$$

where

$$\mathcal{F}_F^h(N, M, Q^2) = \int_0^1 d\xi\, \xi^{N-1} \int_0^1 d\zeta\, \zeta^{M-1} \mathcal{F}_F^h\left(\xi, \zeta, Q^2\right), \tag{12}$$

and

$$f_i(N,\mu_F^2) = \int_0^1 dx\, x^{N-1} f_i(x,\mu_F^2), \tag{13}$$

$$D_j^h(M,\mu_F^2) = \int_0^1 dz\, z^{M-1} D_j^h(z,\mu_F^2), \tag{14}$$

$$C_{ij}^F\left(N, M, \alpha_s(\mu_R^2), \frac{\mu_R^2}{Q^2}, \frac{\mu_1^2}{Q^2}, \frac{\mu_2^2}{Q^2}\right) = \int_0^1 dx\, x^{N-1} \int_0^1 dz\, z^{M-1} \times C_{ij}^F\left(x, z, \alpha_s(\mu_R^2), \frac{\mu_R^2}{Q^2}, \frac{\mu_1^2}{Q^2}, \frac{\mu_2^2}{Q^2}\right). \tag{15}$$

By abuse of notation we denote the Mellin transform of cross sections, parton distribution functions and fragmentation functions with the same symbol as the untransformed quantities. This should not cause any confusion, as long as the arguments of the various functions are explicitly indicated.

2.1 Soft limits

Inclusive DIS is a $2 \to 1$ process with one massive object, and thus it is parametrized by two invariants, that can be chosen as Q^2 and the center-of-mass energy s Eq. (9), or equivalently the scaling variable x Eq. (7). The soft limit is defined as that in which $s \to 0$, so $x \to 1$. Because SIDIS is the $2 \to 2$ process of Eq. (1), with two massive objects G and X, we have two more invariants, that we may choose as k^2, the invariant mass of the system that recoils against the final-state parton, and t Eq. (10). We may then define several distinct soft limits, corresponding to different physical limits of both the partonic and the hadronic process. In order to understand them, we first study the kinematic bounds on the invariants of the process.

The kinematic bounds for the Mandelstam invariant t can be easily worked out in the center-of-mass frame, where

$$p_1 = (E, 0, 0, E) \tag{16}$$

$$q = (\sqrt{E^2 - Q^2}, 0, 0, -E) \tag{17}$$

$$p_2 = (E', 0, E'\sin\theta, E'\cos\theta) \tag{18}$$

$$k = (\sqrt{E'^2 + k^2}, 0, -E'\sin\theta, -E'\cos\theta) \tag{19}$$

with $k^2 \geq 0$. In this frame

$$\sqrt{s} = E + \sqrt{E^2 - Q^2} = E' + \sqrt{E'^2 + k^2}, \tag{20}$$

which give E and E' as functions of invariants:

$$E = \frac{s + Q^2}{2\sqrt{s}}; \quad E' = \frac{s - k^2}{2\sqrt{s}}. \tag{21}$$

As a consequence

$$t = (p_1 - p_2)^2 = -2EE'(1 - \cos\theta) = -\frac{(s + Q^2)(s - k^2)}{2s}(1 - \cos\theta) \tag{22}$$

and therefore

$$-\frac{(s + Q^2)(s - k^2)}{s} \leq t \leq 0 \tag{23}$$

when $0 \leq \theta \leq \pi$. The minimum value of t is attained when $k^2 = 0$, hence for fixed s

$$t^{\min} \leq t \leq 0, \tag{24}$$

$$t^{\min} = -\left(s + Q^2\right) = -\frac{Q^2}{x}. \tag{25}$$

The kinematic limit for k^2 can be determined substituting Eqs. (9, 10) in Eq. (23). We find

$$Q^2 \frac{z}{x} \leq \frac{Q^2}{x} - \frac{k^2}{1 - x}. \tag{26}$$

whence

$$k^2 \leq Q^2 \frac{(1 - x)(1 - z)}{x}. \tag{27}$$

The soft limit is in general that in which all radiation is suppressed, so the process tends to the leading-order process. At leading order $p_1 + q$ is the momentum of an on-shell parton, hence Eq. (9) implies that $x = 1$, while $p_1 - p_2 = -q$, hence Eq. (10) implies that $z = x$. We can consequently define: a double soft limit, as the limit in which both $x \to 1$ and $z \to 1$; and two single soft limits, in which $x \to 1$ at fixed z or $z \to 1$ at fixed x. Clearly, in terms of Mandelstam invariants the limit $x \to 1$ at fixed z corresponds to the case in which s tends to its minimum, $s \to 0$ but t takes some fixed value in the limit, while the limit $z \to 1$ at fixed x corresponds to the case in which t tends to its minimum Eq. (25), $t \to t^{\min}$ but s is fixed.

The physical meaning of these limits can be understood in terms of the allowed radiation in each case. In the limit $x \to 1$ from Eq. (7) we have $(p_1 + q)^2 \to 0$. Using momentum conservation,

$$(p_1 + q)^2 = (p_2 + k)^2 = k^2 + 2p_2 \cdot k \to 0, \tag{28}$$

which is achieved when k is either soft, $k \to 0$, or massless and collinear to p_2, $k \to ap_2$. In the limit $z \to 1$ instead from Eq. (8) we have $p_1 \cdot p_2 \to p_1 \cdot q$, so using again momentum conservation

$$p_1 \cdot (p_1 - k) = -p_1 \cdot k \to 0, \tag{29}$$

which is achieved when k is either soft, $k \to 0$, or massless and collinear to p_1, $k \to bp_1$.

It follows that the double soft limit corresponds to the case in which k is soft, while each of the two single soft limits corresponds to k being either collinear to p_1 or to p_2. In particular the two limits for the non-soft scaling variable are respectively given by

$$\lim_{k\to ap_2} z = \lim_{k\to ap_2} \frac{p_1\cdot p_2}{p_1\cdot p_2 + p_1\cdot k} = \frac{1}{1+a}, \tag{30}$$

$$\lim_{k\to bp_1} x = \lim_{k\to bp_1} \frac{-(p_2+k-p_1)^2}{2p_1\cdot(p_2+k)} = 1-b. \tag{31}$$

As pointed out in Ref. [6] and as mentioned in the introduction, SIDIS is the crossed process to Drell–Yan, and indeed these soft limits are in one-to-one correspondence with the soft limits of the (partonic) Drell–Yan rapidity distribution, discussed in Ref. [16]. Specifically, one can view the Drell–Yan partonic process at nonzero rapidity as the $2 \to 2$ process

$$\mathcal{Q}_1(\bar p_1) + \mathcal{Q}_2(\bar p_2) \to G(\bar q) + X(\bar k), \tag{32}$$

in which two incoming massless partons $\mathcal{Q}_i$ produce a gauge boson with mass $q^2 = Q^2$ and a partonic system with invariant mass k^2 that recoils against the gauge boson in order for it to have nonzero rapidity.

The crossing relations between the two sets of momenta are

$$p_1 \leftrightarrow \bar p_1 \tag{33}$$

$$p_2 \leftrightarrow -\bar p_2 \tag{34}$$

$$k \leftrightarrow \bar k \tag{35}$$

$$q \leftrightarrow -\bar q \tag{36}$$

and the relation between Mandelstam invariants is

$$s \leftrightarrow t. \tag{37}$$

The scaling variables for the Drell–Yan process can be defined by choosing the partonic center-of-mass frame, in which

$$\bar p_1 = \frac{\sqrt{s}}{2}(1,0,0,1) \tag{38}$$

$$\bar p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1), \tag{39}$$

and then letting

$$x_1 x_2 = \frac{Q^2}{s} \tag{40}$$

$$\frac{x_1}{x_2} = e^{2y}, \tag{41}$$

where y is the rapidity of the gauge boson.

The double soft limit $x_1 \to 1, x_2 \to 1$ then corresponds to the case in which $s \to Q^2$, so all radiation is suppressed and k is soft. The two single soft limits $x_1 \to 1$ with x_2 fixed and conversely correspond to the case in which the rapidity of the gauge boson is fixed in the limit, and s reaches the minimum value compatible with this requirement. This means that the transverse momentum of the gauge boson then tends to zero, and consequently in the center-of-mass frame k is massless and collinear to the incoming partons, specifically collinear to p_1 or to p_2 according to whether x_1 is fixed and $x_2 \to 1$ or conversely. The physical interpretation of the two limits is thus the same for SIDIS and Drell–Yan.

2.2 The phase space measure in soft limits

In this Section we work out an expression for the phase space measure for the SIDIS process with the emission of n massless partons. Following the approach of Refs. [16–18], the goal is to show that in the soft limit all the dependence on the scaling variable goes through a dimensionful combination that corresponds to the soft scale of the process. The phase space measure can be written as

$$d\phi_{n+1}(p_1+q;\ p_2,\ k_1,\ldots,k_2) = \int_0^{k^2_{\max}} \frac{dk^2}{2\pi}\, d\phi_2(p_1+q;\ p_2,\ k)\, d\phi_n(k;\ k_1,\ldots,k_n) \tag{42}$$

where

$$k^2_{\max} = Q^2\frac{(1-x)(1-z)}{x}, \tag{43}$$

as shown in Sect. 2.1. We first compute the two-body phase space in $d = 4-2\epsilon$ dimensions

$$\begin{aligned} d\phi_2(p_1+q;\, p_2, k) &= \frac{d^{d-1}k}{(2\pi)^{d-1}2k^0}\frac{d^{d-1}p_2}{(2\pi)^{d-1}2p_2^0}(2\pi)^d\,\delta^{(d)}\,(p_1+q-p_2-k) \\ &= \frac{1}{4(2\pi)^{d-2}}\frac{d^{d-1}p_2}{p_2^0k^0}\,\delta\left(p_1^0+q^0-p_2^0-k^0\right). \end{aligned} \tag{44}$$

We adopt the center-of-mass frame defined in the previous subsection, and we switch to $(d-1)$-dimensional polar coordinates for $\vec p_2$. After performing an irrelevant azimuthal integration, we get

$$\begin{aligned} d\phi_2(p_1+q;\, p_2, k) = \frac{1}{4(2\pi)^{d-2}}\frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} &\times \sin^{d-4}\theta d\cos\theta\frac{|\vec p_2|^{d-3}d|\vec p_2|}{\sqrt{|\vec p_2|^2+k^2}} \\ &\times \delta\left(\sqrt{s}-|\vec p_2|-\sqrt{|\vec p_2|^2+k^2}\right). \end{aligned} \tag{45}$$

The energy delta function

$$\delta\left(\sqrt{s}-|\vec{p}_2|-\sqrt{|\vec{p}_2|^2+k^2}\right) = \frac{\sqrt{|\vec{p}_2|^2+k^2}}{\sqrt{s}}\delta\left(|\vec{p}_2|-\frac{s-k^2}{2\sqrt{s}}\right) \tag{46}$$

can be used to perform the $|\vec{p}_2|$ integration. The only remaining independent variable, the scattering angle θ, can be traded for z:

$$z=\frac{p_1\cdot p_2}{p_1\cdot q}=\frac{s-k^2}{2s}(1-\cos\theta) \tag{47}$$

$$\sin\theta=\frac{2\sqrt{s}}{s-k^2}\sqrt{z[s(1-z)-k^2]}. \tag{48}$$

Hence, setting as usual $d=4-2\epsilon$,

$$d\phi_2(p_1+q;\,p_2,k)=\frac{1}{8\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{s-k^2}{2\sqrt{s}}\sin\theta\right)^{-2\epsilon}dz \tag{49}$$

and the phase space in Eq. (42) becomes

$$d\phi_{n+1}(p_1+q;\,p_2,k_1,\ldots,k_n)=dz\,\frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \times\int_0^{k^2_{\max}}\frac{dk^2}{2\pi}\left(\frac{s-k^2}{2\sqrt{s}}\sin\theta\right)^{-2\epsilon}d\phi_n(k;\,k_1,\ldots,k_n). \tag{50}$$

We can now collect all the dimensionful dependence by setting

$$k^2=vk^2_{\max};\qquad 0\le v\le 1, \tag{51}$$

so that

$$dk^2=dv\;Q^2\frac{(1-x)(1-z)}{x}. \tag{52}$$

The factor

$$\left(\frac{s-k^2}{2\sqrt{s}}\sin\theta\right)^2=z[s(1-z)-k^2]=k^2_{\max}z(1-v) \tag{53}$$

is simply the squared transverse momentum of the radiated system, as one can read off Eqs. (19) and (21). We see from Eq. (43) that it vanishes both in the double soft and in the single soft limits. The phase space $d\phi_n(k;k_1,\ldots,k_n)$ has the same structure as in deep-inelastic scattering, with incoming momentum k. As in Ref. [17], it is written in terms of a dimensionless integration measure, with the dimensionful dependence contained in a power of k^2. Thus, following Ref. [17], we obtain

$$d\phi_n(k;\,k_1,\ldots,k_n)=2\pi\left[\frac{N(\epsilon}{2\pi}\right]^{n-1} (k^2)^{n-2-(n-1)\epsilon}d\Omega^{n-1}(\epsilon), \tag{54}$$

where $N(\epsilon)=\frac{1}{2(4\pi)^{2-2\epsilon}}$ and

$$d\Omega^{n-1}(\epsilon)=d\Omega_1\cdots d\Omega_{n-1} \times\int_0^1 dz_{n-1}z_{n-1}^{(n-3)-(n-2)\epsilon}(1-z_{n-1})^{1-2\epsilon}\cdots \int_0^1 dz_2 z_2^{-\epsilon}(1-z_2)^{1-2\epsilon}. \tag{55}$$

The definition of the dimensionless z_i variables is irrelevant here.

We are now in a position to discuss the scale dependence of the phase space in the soft limits. As shown in Sect. 2.1, $k^2\to 0$ as either x or z approach 1. In then follows from Eqs. (50, 54) that the dimensionful dependence of the full phase space is entirely contained in powers $(\Lambda^2)^{n\epsilon}$ of a soft scale

$$\Lambda^2=k^2_{\max}=Q^2(1-x)(1-z)[1+\mathcal{O}(1-x)]. \tag{56}$$

The logarithmic dependence of the cross section then comes from the interference with poles in ϵ that arise when integrating over the squared amplitude with the remaining dimensionless measure [17]. In the double soft limit the soft scale is thus given by Eq. (56), while in each of the two single soft limits the variables $1-z$ or $1-x$ are not soft, and thus the soft scale can be equivalently taken to be equal to $Q^2(1-x)$ or $Q^2(1-z)$ respectively, up to a finite rescaling factor.

3 Resummation up to NNLL accuracy

The main result of Sect. 2 is that the SIDIS structure functions depend on a single soft scale in both the double soft and the two single soft limits. The situation is identical to that encountered when deriving the corresponding limits of the Drell-Yan process, discussed in Ref. [16], and the underlying physical reason is the same, namely, the collinear nature of these limits. Resummation can then be performed following the same argument as in Ref. [16]. We first summarize the resummed results, then determine the resummation coefficients by comparison to the fixed order calculation.

3.1 The resummed coefficient function

Resummation is performed in Mellin space, where the soft scale Eq. (56) becomes [16]

$$\bar{\Lambda}^2 = \frac{Q^2}{MN}. \tag{57}$$

The double soft limit corresponds then to $M \to \infty, N \to \infty$, while the two single soft limits corresponds to either $M \to \infty$ with finite N or conversely. In the two single soft limits the soft scale thus becomes $\frac{Q^2}{M}$ or $\frac{Q^2}{N}$ up to a finite rescaling by the non-soft Mellin variable. Resummation is performed by only retaining terms in Mellin space that do not vanish in the respective limits.

Resummed results are found by a rerun of the argument of Ref. [16]. We note that only the transverse structure function is enhanced in the soft limit. For this reason, to simplify notations, we omit henceforth the F label that distinguishes different structure functions. In the double soft limit, choosing $\mu_R^2 = \mu_1^2 = \mu_2^2 = Q^2$ the resummed coefficient function is given by

$$C_{qq}^{\rm ds}\left(N, M, \alpha_s(Q^2)\right) = C_{c,qq}^{\rm ds}\left(\alpha_s(Q^2)\right) \exp \times \int_{\frac{Q^2}{NM}}^{Q^2} \frac{dk^2}{k^2}\left[A_q(\alpha_s(k^2)) \ln \frac{NMk^2}{Q^2} - D_{qq}^{\rm ds}(\alpha_s^2(k^2))\right], \tag{58}$$

where both parton indices are taken to be q because in the soft limit only the quark-quark channel is enhanced.

In Eq. (58) the functions A_q, $D_{qq}^{\rm ds}$ and $C_{c,qq}^{\rm ds}$ are all power series in α_s

$$A_q(\alpha_s) = \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n A_q^{(n)} \tag{59}$$

$$D_{qq}^{\rm ds}(\alpha_s) = \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n D_{qq}^{F,{\rm ds}(n)} \tag{60}$$

$$C_{c,qq}^{\rm ds}(\alpha_s) = 1 + \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n C_{c,qq}^{{\rm ds}(n)}; \tag{61}$$

A_q is the quark cusp anomalous dimension. Assuming knowledge of the cusp anomalous dimension up to $\mathcal{O}(\alpha_s^{k+1})$, the fixed N^{k}LO result fully determines N^{k}LL resummation; below we will determine $D_{qq}^{\rm ds}$ and $C_{c,qq}^{\rm ds}$ up to $\mathcal{O}(\alpha_s^2)$ by comparison with the fixed order calculation [2–5].

In the single soft limit, up to leading power accuracy, only soft diagonal radiation is allowed from the soft parton, which thus for the DIS process must be a quark, while all collinear radiation is allowed from the non-soft parton. The resummed coefficient function is consequently labeled by a quark index and a parton index j, that runs over all parton flavors, for the non-soft parton. In order to ensure consistency of the resummed result with standard perturbative evolution, which dictates the dependence on $\mu_F^2 = Q^2$ of the coefficient function, the parton index j must run over evolution eigenstates [19]. Considering for definiteness the $x \to 1$, $N \to \infty$ single soft limit, the resummed result is then

$$C_{qj}^{\rm ss}\left(N, M, \alpha_s(Q^2)\right) = C_{c,qj}^{\rm ss}\left(\alpha_s(Q^2), M\right) \exp \times \int_{\frac{Q^2}{N}}^{Q^2} \frac{{\rm d}k^2}{k^2}\left[A_q(\alpha_s(k^2)) \ln \frac{Nk^2}{Q^2} - D_{qj}^{\rm ss}(\alpha_s^2(k^2), M)\right]. \tag{62}$$

The result in the $z \to 1$, $M \to \infty$ limit has the same form, but with $N \leftrightarrow M$. The functions $D_{qj}^{\rm ss}$ and $C_{c,qj}^{\rm ss}$ now depend parametrically on the non-soft variable. In Eq. (62) the soft scale is

$$\bar{\Lambda}_{\rm ss}^2 = \frac{Q^2}{N}. \tag{63}$$

The result can of course be rewritten by choosing any scale $\bar{\Lambda}_{\rm ss}'^2 = k\bar{\Lambda}_{\rm ss}^2$ that differs by a finite factor k, such as specifically the double soft scale $\bar{\Lambda}^2$ Eq. (57).

3.2 Expanded resummed results

In order to determine the resummation coefficients, we match the resummed expression to the fixed order calculation up to $\mathcal{O}(\alpha_s^2)$, specifically using the result of Refs. [3,5]. As mentioned, whereas in the double soft limit only the quark-quark channel is enhanced, in the single soft limit the non-soft parton can correspond to any parton flavor. Here we will only consider the nonsinglet contribution, hence we only match and resum the $\tilde{C}_{qq,\,\rm NS}$ coefficient function. We will drop the NS subscript and implicitly denote with qq the nonsinglet quark channel.

The fixed-order Mellin-space expressions that we used are listed in Appendix A.3. They have been obtained by us starting from the NNLO (x, z)-space expressions of Ref. [3, 5], and performing the Mellin transform of the distributional contributions using the `Mathematica` packages `MT` [20] `HarmonicSums` [21].

The matching procedure requires the expansion of the resummed results in the double soft Eq. (58) and single soft Eq. (62) cases. We note that the double soft and single soft expressions have the same form when expressed in terms of a variable

$$\lambda = \beta_0 \alpha_s(Q^2)\mathcal{L} \tag{64}$$

where $\mathcal{L}$ is a large log, though with different coefficients $A_q^{(n)}$, $D_{ij}^{(n)}$ and $C_{ij}^{(n)}$ Eqs. (59–61), and in general a different choice of the large log $\mathcal{L}$. It is furthermore convenient to write results in terms of modified Mellin variables [7], defined as

$$\bar{N} = Ne^{\gamma_E}; \quad \bar{M} = Me^{\gamma_E}, \tag{65}$$

where γ_E is the Euler–Mascheroni constant, as this leads to a simplification of the resummation coefficients. We then take as large logs in the double soft limit

$$\mathcal{L} = \ln(\bar{N}\bar{M}), \tag{66}$$

while in the single soft limit we give results in terms of the large logs $\ln\bar{N}$ or $\ln\bar{M}$ respectively, but also in terms of $\ln\bar{M}\bar{N}$, which of course when either M or N are fixed and finite differs from them by a finite rescaling of the argument of the log.

Performing the integral over scale, the resummed results have the form

$$C_{qq}\left(N, M, \alpha_s(Q^2)\right) = g_0(\alpha_s)\exp\left(\frac{1}{\alpha_s}g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \cdots\right), \tag{67}$$

where

$$g_0(\alpha_s) = 1 + g_0^{(1)}\frac{\alpha_s}{\pi} + g_0^{(2)}\left(\frac{\alpha_s}{\pi}\right)^2 + \cdots \tag{68}$$

and

$$g_1(\lambda) = \frac{A_q^{(1)}}{\pi\beta_0^2}(\lambda + (1-\lambda)\ln(1-\lambda)), \tag{69}$$

$$g_2(\lambda) = \frac{A_q^{(1)}b_1}{2\pi\beta_0^2}\left(2\lambda + \ln^2(1-\lambda) + 2\ln(1-\lambda)\right) - \frac{A_q^{(2)}}{\pi^2\beta_0^2}(\lambda + \log(1-\lambda)) + \frac{D_{qq}^{(1)}\ln(1-\lambda)}{\pi\beta_0}, \tag{70}$$

$$\begin{aligned} g_3(\lambda) = {} & \frac{A_q^{(1)}b_1^2}{\pi\beta_0^2(1-\lambda)}\left(\frac{\lambda^2}{2} + \frac{1}{2}\ln^2(1-\lambda) + \lambda\ln(1-\lambda)\right) \\ & + \frac{A_q^{(1)}b_2}{\pi\beta_0^2(1-\lambda)}\left(\log(1-\lambda) - \lambda\log(1-\lambda) - \frac{\lambda^2}{2} + \lambda\right) \\ & - \frac{A_q^{(2)}b_1}{\pi^2\beta_0^2(1-\lambda)}\left(\frac{\lambda^2}{2} + \lambda + \log(1-\lambda)\right) \\ & + \frac{A_q^{(3)}\lambda^2}{2\pi^3\beta_0^2(1-\lambda)} - \frac{D_{qq}^{(2)}\lambda}{\pi^2\beta_0(1-\lambda)} \\ & + \frac{D_{qq}^{(1)}b_1}{\pi\beta_0(1-\lambda)}(\lambda + \ln(1-\lambda)), \end{aligned} \tag{71}$$

where $b_i = \frac{\beta_i}{\beta_0}$ for $i \geq 1$. The coefficients β_0 and β_i are explicitly given in Appendix A.1.

Expanding now Eq. (67) up to order α_s^2 we get

$$\begin{aligned} C_{qq}\left(N, M, \alpha_s(Q^2)\right) = {} & 1 + \frac{\alpha_s}{\pi}\left(\frac{A_q^{(1)}}{2}\mathcal{L}^2 - D_{qq}^{(1)}\mathcal{L} + g_0^{(1)}\right) \\ & + \left(\frac{\alpha_s}{\pi}\right)^2\Bigg[\frac{A_q^{(1)2}}{8}\mathcal{L}^4 + \mathcal{L}^3\left(\frac{1}{6}A_q^{(1)}\pi\beta_0 - \frac{1}{2}A_q^{(1)}D_{qq}^{(1)}\right) \\ & + \mathcal{L}^2\left(\frac{1}{2}A_q^{(1)}g_0^{(1)} + \frac{1}{2}A_q^{(2)} - \frac{1}{2}\pi\beta_0 D_{qq}^{(1)} + \frac{1}{2}D_{qq}^{(1)2}\right) \\ & + \mathcal{L}\left(-D_{qq}^{(1)}g_0^{(1)} - D_{qq}^{(2)}\right) + g_0^{(2)}\Bigg] + \mathcal{O}(\alpha_s^3). \end{aligned} \tag{72}$$

Resummed results will be given in both the double soft and single soft limit, in the latter case with two different choices of soft scale, by determining in each case the values of the coefficients $A_q^{(i)}$, $D_{qq}^{(i)}$ and $g_0^{(i)}$ to be used in the expressions Eqs. (67–71). Note in particular that the function g_0 does not in general coincide with the functions C_c^{ds} and C_c^{ss} Eqs. (61–62), because different choices of soft scale can reshuffle constant terms between the exponential and the unexponentiated prefactor.

3.3 Double soft

As mentioned in the introduction, resummation of SIDIS in the double soft limit was already presented in Refs. [10] (NLL), [11] (NNLL) and even N^{3}LL [12], also including next-to-leading power corrections. In these references, resummed results were obtained using crossing symmetry [6], exploiting the close connection of differential to inclusive resummation [7], and using recent higher-order results for the inclusive resummation functions Δ and J (see e.g. [22]). We present here resummation up to NNLL with coefficients determined by matching to the fixed order result, which sets the stage for our new results on resummation in the single soft limit, and also provides an independent cross-check of the consistency of the fixed order and resummed results.

The coefficient function in the double soft limit is at NLO

$$C_{qq}^{(1)} = C_F\left(\frac{\mathcal{L}^2}{2} + \frac{\pi^2}{6} - 4\right) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(\frac{1}{M}\right), \tag{73}$$

and at NNLO

$$\begin{aligned} C_{qq}^{(2)}(N, M) = {} & \frac{\mathcal{L}^4}{8}C_F^2 + C_F\mathcal{L}^3\left(\frac{\pi}{6}\beta_0\right) \\ & + C_F\frac{\mathcal{L}^2}{4}\left[C_F\left(-8 + \frac{\pi^2}{3}\right) + \left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right] \\ & + C_F\frac{\mathcal{L}}{2}\left[\left(\frac{101}{27} - \frac{7}{2}\zeta_3\right)C_A - \frac{14}{27}N_f\right] \\ & + C_F^2\left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4}\zeta_3\right] \end{aligned}$$

$$+ C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36}\zeta_3 \right] + C_F N_f \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta_3}{18} \right] + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(\frac{1}{M}\right). \quad (74)$$

Comparing to Eq. (73) we immediately get the NLL coefficients

$$A_q^{(1)} = C_F \quad (75)$$

$$D_{qq}^{\mathrm{ds}(1)} = 0 \quad (76)$$

$$g_{0,qq}^{\mathrm{ds}(1)} = C_F \left(\frac{\pi^2}{6} - 4 \right) = C_F (\zeta_2 - 4), \quad (77)$$

where of course the value of $A_q^{(1)}$ provides a consistency check, as it is already fixed by LL resummation, and the full NLL resummation requires knowledge of the NLO cusp anomalous dimension, which is given by

$$A_q^{(2)} = \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right]. \quad (78)$$

At order α_s^2, comparing to Eq. (74) we reproduce Eq. (78), and we find

$$D_{qq}^{\mathrm{ds},(2)} = \frac{1}{2} C_F \left[\left(-\frac{101}{27} + \frac{7}{2}\zeta_3 \right) C_A + \frac{14}{27} N_f \right], \quad (79)$$

$$g_{0,qq}^{\mathrm{ds},(2)} = C_F^2 \left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4}\zeta_3 \right] + C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36}\zeta_3 \right] + C_F N_f \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta_3}{18} \right], \quad (80)$$

which fully determine the NNLL resummed result, together with the NNLO cusp anomalous dimension given in Refs. [22, 23]). This agrees with the results of Ref. [11] (note that the definition of the coefficient D_2 in this reference differs by a factor 2 from our own).

3.4 Single soft

We now turn to the new result of this paper, namely the coefficients that determine the single soft resummation in the nonsinglet channel. In order to perform the matching efficiently, it is useful to expand the fixed-order results in powers of either of the large logs corresponding to the two single soft limits, namely

$$C_{qq}(N,M) = 1 + \frac{\alpha_s}{\pi}\left(f_2^{(1)}(M)\ln^2\bar{N} + f_1^{(1)}(M)\ln\bar{N} + f_0^{(1)}(M) \right) + \left(\frac{\alpha_s}{\pi}\right)^2 \left(f_4^{(2)}(M)\ln^4\bar{N} + f_3^{(2)}(M)\ln^3\bar{N} + f_2^{(2)}(M)\ln^2\bar{N} + f_1^{(2)}(M)\ln\bar{N} + f_0^{(2)}(M) \right) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}(\alpha_s^3) \quad (81)$$

$$C_{qq}(N,M) = 1 + \frac{\alpha_s}{\pi}\left(h_2^{(1)}(N)\ln^2\bar{M} + h_1^{(1)}(N)\ln\bar{M} + h_0^{(1)}(N) \right) + \left(\frac{\alpha_s}{\pi}\right)^2 \left(h_4^{(2)}(N)\ln^4\bar{M} + h_3^{(2)}(N)\ln^3\bar{M} + h_2^{(2)}(N)\ln^2\bar{M} + h_1^{(2)}(N)\ln\bar{M} + h_0^{(2)}(N) \right) + \mathcal{O}\left(\frac{1}{M}\right) + \mathcal{O}(\alpha_s^3). \quad (82)$$

The explicit expressions of the functions $f_i^{(j)}(M)$ and $h_i^{(j)}(N)$ are given in Appendix A.3.

3.4.1 *x-single soft*

Taking as expanded resummed expression Eq. (67), but now with

$$\mathcal{L} = \ln\bar{N}, \quad (83)$$

and comparing with the expressions of the functions $f_i^{(j)}(M)$ given in Appendix A.3.1 we get

$$A_q^{(1)} = C_F, \quad (84)$$

$$A_q^{(2)} = \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right], \quad (85)$$

$$D_{qq}^{\mathrm{xss},(1)}(M) = \tilde{\gamma}_{qq}^{(0)}(M), \quad (86)$$

$$D_{qq}^{\mathrm{xss},(2)}(M) = D_{qq}^{\mathrm{ds},(2)} - \pi\beta_0 F(M) + \tilde{\gamma}_{qq,\mathrm{NS}}^{(1),t}(M), \quad (87)$$

$$g_{0,qq}^{\mathrm{xss},(1)}(M) = g_{0,qq}^{\mathrm{ds},(1)} + F(M) = f_0^{(1)}(M), \quad (88)$$

$$g_{0,qq}^{\mathrm{xss},(2)}(M) = f_0^{(2)}(M). \quad (89)$$

Here $\tilde{\gamma}_{qq}^{(0)}(M)$, $\tilde{\gamma}_{qq,\mathrm{NS}}^{(1),t}(M)$ are subtracted N^{i}LO timelike nonsinglet anomalous dimensions, obtained as the Mellin transforms of the N^{i}LO timelike nonsinglet splitting functions, but with contributions proportional to $\delta(1-z)$ not included, as explained in Appendix A.2. The suffix t (for timelike) is only present in the $\mathcal{O}(\alpha_s^2)$ coefficient because at leading order the timelike and spacelike splitting functions coincide. Also,

$$F(M) = \frac{1}{2} C_F \left[S_1^2(M) + \frac{2M^2 - M - 1}{M^2(M+1)^2} + 3\left(S_2(M) - \zeta_2\right) - \frac{S_1(M)}{M(M+1)} \right]. \quad (90)$$

Substituting these coefficients in the resummed expression Eq. (62) gives NNLL resummation in the x-single soft limit.

As mentioned, it might be convenient to rewrite this result by choosing instead as a soft scale the same scale $\bar{\Lambda}^2$ as in the double soft limit, so that now

$$\mathcal{L} = \ln(\bar{N}\bar{M}). \tag{91}$$

In this case, the resummed expression has the same form as in the double soft case, Eq. (58), except that the functions g_0 and D are now M dependent; we will denote them by $\bar{g}^{\text{xss}}_{0,qq}(M)$ and $\bar{D}^{\text{xss}}_{qq}(M)$. We get

$$A^{(1)}_q = C_F, \tag{92}$$

$$A^{(2)}_q = \frac{1}{2}C_F\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right], \tag{93}$$

$$\bar{D}^{\text{xss},(1)}_{qq}(M) = \bar{\gamma}^{(0)}_{qq}(M), \tag{94}$$

$$\bar{D}^{\text{xss},(2)}_{qq}(M) = D^{\text{ds},(2)}_{qq} - \pi\beta_0\bar{F}(M) + \bar{\gamma}^{(1),t}_{qq,\text{NS}}(M), \tag{95}$$

$$\bar{g}^{\text{xss},(1)}_{0,qq}(M) = g^{\text{ds},(1)}_{0,qq} + \bar{F}(M) \tag{96}$$

$$= f^{(1)}_2(M)\ln^2\bar{M} - f^{(1)}_1(M)\ln\bar{M} + f^{(1)}_0(M), \tag{97}$$

$$\bar{g}^{\text{xss},(2)}_{0,qq}(M) = f^{(2)}_4(M)\ln^4\bar{M} - f^{(2)}_3(M)\ln^3\bar{M} + f^{(2)}_2(M) \ln^2\bar{M} - f^{(2)}_1(M)\ln\bar{M} + f^{(2)}_0(M), \tag{98}$$

where the expressions Eqs. (97–98) of $\bar{g}^{\text{xss},(i)}_{0,qq}(M)$ can be obtained by noting that with the choice of scale Λ^2_{ss} Eq. (63), the coefficient of $\ln^k N$ coincides with the corresponding coefficient $f^{(j)}_i(M)$ Eq. (81) of the fixed-order result, and the resummed expression with scale $\bar{\Lambda}^2$ Eq. (57) is obtained by that with scale Λ^2_{ss} Eq. (63) by simply letting everywhere $\ln^k\bar{N} \to (\ln\bar{N} + \ln\bar{M})^k$, expanding the result in powers of $\ln\bar{N}$, and then equating order by order the coefficient of $\ln^k\bar{N}$ to $f^{(j)}_i(M)$. In Eqs. (94–96) we have defined

$$\bar{F}(M) = F(M) + C_F\left[\ln\bar{M}\left(\ln\bar{M} + \frac{1}{2M(M+1)} - S_1(M)\right) - \frac{1}{2}\ln^2\bar{M}\right], \tag{99}$$

and $\bar{\gamma}^{(i),t}_{qq}(M)$ are the anomalous dimensions but now with also all logarithmic contributions subtracted, see Appendix A.2, so that

$$\lim_{M\to\infty}\bar{\gamma}^{(i),t}_{qq}(M) = 0. \tag{100}$$

Furthermore,

$$\lim_{M\to\infty}\bar{F}(M) = 0, \tag{101}$$

because the extra terms in Eq. (99) remove all logarithmic and constant terms from $F(M)$.

Using this form of the resummed expression in the soft limit it is easier to check that when $M \to \infty$ the double soft result is reproduced. Indeed, with this choice of scale the resummed expressions in the double and single soft limits have the same form, hence the single soft limit reduces to the double soft limit provide only all single soft coefficients become identical to their double soft form when both variables tend to to infinity. It is easy to check that this is the case: the coefficients $A^{(i)}$ Eqs. (92–93) are the usual coefficients of the cusp anomalous dimension, and because of Eqs. (100–101) the functions $D^{(i)}(M)$ and $g^{(1)}(M)$ Eqs. (94–115) reduce to the M-independent form that they have in the double soft limit. We have finally checked explicitly that the constant also reduces to its double soft expression Eq. (80) in the limit:

$$\lim_{M\to\infty}\bar{g}^{\text{xss},(2)}_{0,qq}(M) = g^{\text{ds},(2)}_{0,qq}. \tag{102}$$

3.4.2 *z-single soft*

We turn to the case in which $M \to \infty$, so the large log is now

$$\mathcal{L} = \ln\bar{M}. \tag{103}$$

Results can be derived proceeding as in the x-single soft case. We now get

$$A^{(1)}_q = C_F, \tag{104}$$

$$A^{(2)}_q = \frac{1}{2}C_F\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right], \tag{105}$$

$$D^{\text{zss},(1)}_{qq}(N) = \tilde{\gamma}^{(0)}_{qq}(N), \tag{106}$$

$$D^{\text{zss},(2)}_{qq}(N) = D^{\text{ds},(2)}_{qq} - \pi\beta_0 H(N) + \tilde{\gamma}^{(1)}_{qq}(N), \tag{107}$$

$$g^{\text{zss},(1)}_{0,qq}(N) = g^{\text{ds},(1)}_{0,qq} + H(N) = h^{(1)}_0(N), \tag{108}$$

$$g^{\text{zss},(2)}_{0,qq}(N) = h^{(2)}_0(N), \tag{109}$$

where the function $H(N)$ is given by

$$H(N) = \frac{1}{2}C_F\left[S^2_1(N) - \frac{S_1(N)}{N(N+1)} + \frac{2N+1}{N^2(N+1)} + \zeta_2 - S_2(N)\right]. \tag{110}$$

Using instead the double-soft scale Eq. (91) we get

$$A^{(1)}_q = C_F, \tag{111}$$

$$A^{(2)}_q = \frac{1}{2}C_F\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right], \tag{112}$$

$$\bar{D}^{\text{zss},(1)}_{qq}(N) = \bar{\gamma}^{(0)}_{qq}(N), \tag{113}$$

$$\bar{D}^{\text{zss},(2)}_{qq}(N) = D^{\text{ds},(2)}_{qq} - \pi\beta_0\bar{H}(N) + \bar{\gamma}^{(1)}_{qq,\text{NS}}(N), \tag{114}$$

$$\bar{g}_{0,qq}^{\mathrm{zss},(1)}(N) = g_{0,qq}^{\mathrm{ds},(1)} + \bar{H}(N) = h_2^{(1)}(N)\ln^2\bar{N} - h_1^{(1)}(N)\ln\bar{N} + h_0^{(1)}(N), \quad (115)$$

$$g_{0,qq}^{\mathrm{zss},(2)}(M) = h_4^{(2)}(N)\ln^4\bar{N} - h_3^{(2)}(N)\ln^3\bar{N} + h_2^{(2)}(N) \ln^2\bar{N} - h_1^{(2)}(N)\ln\bar{N} + h_0^{(2)}(N), \quad (116)$$

where now

$$\bar{H}(N) = H(N) + C_F\left[\ln\bar{N}\left(\ln\bar{N} + \frac{1}{2N(N+1)} - S_1(N)\right) - \frac{1}{2}\ln^2\bar{N}\right], \quad (117)$$

$\gamma^{(i)}$ are the usual (spacelike) anomalous dimensions, $\tilde{\gamma}^{(i)}$ denotes the anomalous dimensions splitting function with constants subtracted and $\bar{\gamma}^{(i)}$ is the splitting function with both constants and logs subtracted (see Appendix A.2), which satisfies

$$\lim_{M\to\infty}\bar{\gamma}_{qq}^{(i)}(M) = 0. \quad (118)$$

Furthermore,

$$\lim_{M\to\infty}\bar{H}(M) = 0. \quad (119)$$

3.4.3 Discussion

The interpretation of the resummed expressions in the double soft and in the two single soft limits is transparent, and can be summarized as follows:

- In the double soft limit, the resummed result is the same as in the inclusive case, and consequently the origin and meaning of the various contributions to the resummed expression is the same. In particular, as discussed in Sect. 2.1, in this limit all radiation is soft. The A term then contains contributions from radiation that is both soft and collinear, and indeed it is proportional to the cusp anomalous dimension, which is by definition the soft limit of the standard collinear anomalous dimension. The D contribution only starts at NNLL, and it collects further soft but non-collinear contributions.
- Because the double soft limit is a special case of the single soft, if expressed using the same scale Eq. (91), the single soft result reduces to the double soft by simply evaluating all coefficients in the limit in which the non-soft variable goes to infinity.
- The nontrivial single-soft generalization of the double-soft result is embodied in the coefficients $D_{qq}^{\mathrm{xss},(i)}(M)$ or $\bar{D}_{qq}^{\mathrm{xss},(i)}(M)$ and $D_{qq}^{\mathrm{zss},(i)}(N)$ or $\bar{D}_{qq}^{\mathrm{zss},(i)}(N)$. Their meaning is readily understood recalling that, as discussed in Sect. 2.1, the purely single soft radiation is collinear. Focussing on the case in which the same scale Eq. (91) as in the double soft case is adopted, so $\bar{D}_{qq}^{\mathrm{xss},(i)}(M)$ and $\bar{D}_{qq}^{\mathrm{zss},(i)}(N)$ only contain terms that are power suppresses when their respective argument goes to infinity, one notes that the NLL coefficient $\bar{D}_{qq}^{\mathrm{ss},(1)}$ is equal to the subtracted leading-order anomalous dimensions $\bar{\gamma}_{qq}^{(1)}$. In other words, it contains all radiation that is collinear to the non-soft variable, up to the soft scale given by the soft variable, in agreement with the results of Sect. 2.1 (recall Eqs. (28–29), on top of the soft-collinear already contained in the double-soft terms. At NNLL, the coefficients $\bar{D}_{qq}^{\mathrm{xss},(2)}(M)$ and $\bar{D}_{qq}^{\mathrm{zss},(2)}(N)$, which are already nonzero in the double-soft limit, on top of a contribution equal to the NLO subtracted anomalous dimensions, $\bar{\gamma}_{qq}^{(3)}(N)$ or $\bar{\gamma}_{qq}^{(2),t}(M)$, also contain a contribution proportional to the functions $\tilde{F}(M)$ or $\tilde{H}(N)$. These (see Eqs. (96, 115) correspond to the contribution to the NLO constant $\bar{g}_{0,qq}^{\mathrm{ss},(1)}(M)$ or respectively $\bar{h}_{0,qq}^{\mathrm{ss},(1)}(N)$, namely the $O(\alpha_s)$ contribution to the coefficient function which is respectively proportional to $\delta(1-x)$ or $\delta(1-z)$, on top of what is already present in the double soft limit. Their contribution to the $\bar{D}_{qq}^{\mathrm{xss},(2)}(M)$ and $\bar{D}_{qq}^{\mathrm{zss},(2)}(N)$ coefficients amounts to running the argument of α_s from the hard scale Q^2 to the soft scale Q^2/N or respectively Q^2/M.
- The x-single soft and z-single soft resummation exponents coincide up to NLL, because the timelike and spacelike splitting functions coincide at leading order
$$D_{qq}^{\mathrm{xss},(1)}(N) = D_{qq}^{\mathrm{zss},(1)}(N). \quad (120)$$
However, this is no longer the case at NLO, and both the NLO and NNLO constants and the NNLO resummation coefficient found in the two single soft limits differ: $g_{0,qq}^{\mathrm{xss},(i)}(N) \neq g_{0,qq}^{\mathrm{zss},(i)}(N)$ and $D_{qq}^{\mathrm{xss},(1)}(N) \neq D_{qq}^{\mathrm{zss},(1)}(N)$.
- As mentioned in the introduction, NLP corrections to the double soft resummation were determined in Ref. [12], using arguments from Refs. [24,25]. These were found by adding to the double soft resummation coefficient $D_{qq}^{\mathrm{ds},(1)}$ Eq. (76) a contribution equal to $Q^{(1)}\left(\frac{1}{N}+\frac{1}{M}\right)$, with the coefficient $Q^{(1)}$ determined by expanding the LO anomalous dimension $\gamma_{qq}^{(0)}(N)$ as
$$\gamma_{qq}^{(0)}(N) = -A^{(1)}\ln N + \gamma_{q,\delta}^{(0)} + \frac{Q^{(1)}}{N} + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (121)$$
Because of Eqs. (118, 100), this manifestly corresponds to including the first order contribution in the expansion of the single soft coefficients $\bar{D}_{qq}^{\mathrm{xss},(1)}(M)$ Eq. (94) and $\bar{D}_{qq}^{\mathrm{zss},(1)}(N)$ Eq. (113) in powers of of their respective argument about zero. Therefore, our results confirm the

NLP results of Ref. [12], and it extends them to the full single soft limit.

4 Conclusions

We have presented an extension to SIDIS of the asymmetric threshold resummation formalism, previously developed for the Drell–Yan rapidity distribution, using SCET [14,15], and by some of us in direct QCD [16]. This asymmetric resummation of processes with two scaling variables resums large logs arising when only one of the two variables is close to threshold. This situation appears to be especially relevant in the case of SIDIS, where the two scaling variables – the Bjorken variable and the fragmentation variable – play different physical roles. Our results confirm in this somewhat more general and perhaps nontrivial setting the crossing relation between threshold resummation of SIDIS and of the Drell–Yan rapidity distribution first suggested in Ref. [6]. Also, by verifying that the fixed NNLO SIDIS results agree with the prediction from NLL resummation, they provide a further nontrivial check of the correctness of the asymmetric resummation of Refs. [14–16].

These results are one step further in the direction of enabling precision physics studies at the future EIC, by bringing the theory of deep-inelastic scattering and related topics up to the modern standards of precision QCD. They can be used both for precision EIC phenomenology in kinematic regions in which threshold resummation is needed in order to achieve a good accuracy, and also in order to construct an approximate form of higher fixed-order corrections, as already done in Ref. [12] exploiting information from the double soft limit. Both results can be arrived at by combining information from the double soft and the two single soft limits.

This work provides a first proof of concept towards these goals. In order to achieve them it will be necessary to extend our results, that so far are restricted to the nonsinglet quark channel, to include all partonic channels. For instance, at large z and not too high values of Q^2, small-x initial-state partons provide a dominant contribution (see e.g. [5]) , so it is crucial to include the singlet contribution to the non-soft leg in the resummed expression Eq. (62). This entails some technical complications, because the resummation in Eq. (62), as mentioned, must be performed in terms of evolution eigenstates, which, as well-known, are not the same at different perturbative orders. While the way to handle this situation is also well-known [26], its implementation in a resummed framework requires some care. Work in this direction is currently ongoing, towards the construction of fully matched resummed and fixed order results and the study of their phenomenological implications, in particular at the EIC [27].

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Appendix A

We collect for completeness and in order to fix notation and conventions the expressions for the running coupling up to NNLO and of the timelike and spacelike splitting functions up to NLO and of their Mellin transforms (see e.g. Ref. [28]). We then list the coefficients of the expansion of the nonsinglet contribution to the Mellin-space coefficient function $C^T_{qq}(N, M)$ in powers of either of the large logs corresponding to the two single soft limits, following the notation of Eq. (81) up to NNLO, obtained using the results of Refs. [3,5].

A.1 The QCD running coupling

The running coupling of QCD in the $\overline{\text{MS}}$ scheme to NNLL accuracy is given by

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_R)}{\ell}\left[1 - \frac{\alpha_s(\mu_R)}{\ell} b_1 \ln \ell + \left(\frac{\alpha_s(\mu_R)}{\ell}\right)^2 \left(b_1^2\left(\ln^2 \ell - \ln \ell + l - 1\right) - b_2(\ell - 1)\right)\right]$$

$$+\mathcal{O}\left(\alpha_s^3\left(\alpha_s \ln\frac{\mu^2}{\mu_R^2}\right)\right) \tag{122}$$

where

$$\ell = 1+\beta_0\alpha_s(\mu_R)\ln\frac{\mu^2}{\mu_R^2} \tag{123}$$

$$\beta_0 = \frac{1}{12\pi}\left(11C_A-2N_f\right) \tag{124}$$

$$\beta_1 = \frac{1}{24\pi^2}\left(17C_A^2-5C_AN_f-3C_FN_f\right) \tag{125}$$

$$\beta_2 = \frac{1}{64\pi^3}\left(\frac{2857}{54}C_A^3 -\frac{1415}{54}C_A^2N_f-\frac{205}{18}C_AC_FN_f+C_F^2N_f +\frac{79}{54}C_AN_f^2+\frac{11}{9}C_FN_f^2\right), \tag{126}$$

and $b_1=\frac{\beta_1}{\beta_0}$, $b_2=\frac{\beta_2}{\beta_0}$. N_f is the number of light flavors and

$$C_F = \frac{N_c^2-1}{2N_c}=\frac{4}{3};\quad C_A = N_c=3. \tag{127}$$

A.2 NLO splitting functions and anomalous dimensions

We provide expressions for the nonsinglet quark-quark splitting function; all expressions given here (specifically at NLO) refer to the pure nonsinglet contribution. We define

$$P_{qq}=\frac{\alpha_s}{\pi}P_{qq}^{(0)}+\left(\frac{\alpha_s}{\pi}\right)^2P_{qq}^{(1)}+\mathcal{O}(\alpha_s^3). \tag{128}$$

The LO time-like and space-like expressions coincide and are given by

$$P_{qq}^{(0)}(x)=\tilde{P}_{qq}^{(0)}(x)+\frac{C_F}{2}\frac{3}{2}\delta(1-x), \tag{129}$$

where

$$\tilde{P}_{qq}^{(0)}(x)=\frac{C_F}{2}\frac{1+x^2}{(1-x)_+}. \tag{130}$$

Their Mellin transforms are respectively given by

$$\gamma_{qq}^{(0)}(N)=\tilde{\gamma}_{qq}^{(0)}(N)+\frac{3}{2}\frac{C_F}{2}, \tag{131}$$

and

$$\tilde{\gamma}_{qq}^{(0)}(N)=\frac{C_F}{2}\left[\frac{1}{N(N+1)}-2S_1(N)\right]. \tag{132}$$

The subtracted anomalous dimension which enters the NLO single soft coefficients Eqs. (94, 113) is finally given by

$$\bar{\gamma}_{qq}^{(0)}(N)=\frac{C_F}{2}\left[2\ln\bar{N}+\frac{1}{N(N+1)}-2S_1(N)\right]. \tag{133}$$

The NLO coefficient of the space-like splitting function is

$$\begin{aligned}P_{qq,\mathrm{NS}}^{(1)}(x)=\frac{1}{4}\Bigg\{&C_F^2\Bigg[\left(-2\ln x\ln(1-x)-\frac{3\ln x}{2}\right)\\&\times\left(2\left[\frac{1}{1-x}\right]_+-x-1\right)\\&-\frac{1}{2}(x+1)\ln^2x-\left(\frac{7x}{2}+\frac{3}{2}\right)\ln x-5(1-x)\Bigg]\\&+C_FC_A\Bigg[\left(\frac{1}{2}\ln^2x+\frac{11\ln x}{6}-\frac{\pi^2}{6}+\frac{67}{18}\right)\\&\times\left(2\left[\frac{1}{1-x}\right]_+-x-1\right)\\&+(x+1)\ln x+\frac{20(1-x)}{3}\Bigg]\\&+\frac{1}{2}C_FN_f\Bigg[\left(-\frac{2}{3}\ln x-\frac{10}{9}\right)\\&\times\left(2\left[\frac{1}{1-x}\right]_+-x-1\right)-\frac{4(1-x)}{3}\Bigg]\\&+\frac{1}{4}\delta(1-x)\Bigg[C_F^2\left(6\zeta_3+\frac{3}{8}-\frac{\pi^2}{2}\right)\\&-C_FC_A\left(-3\zeta_3+\frac{17}{24}+\frac{11\pi^2}{18}\right)\\&-\frac{1}{2}C_FN_f\left(\frac{1}{6}+\frac{2\pi^2}{9}\right)\Bigg]\Bigg\}.\end{aligned} \tag{134}$$

Its Mellin transform is given by

$$\begin{aligned}\gamma_{qq,\mathrm{NS}}^{(1)}(N)=\tilde{\gamma}_{qq,\mathrm{NS}}^{(1)}(N)+\frac{1}{4}\Bigg(&C_F^2\left(6\zeta_3+\frac{3}{8}-\frac{\pi^2}{2}\right)\\&-C_FC_A\left(-3\zeta_3+\frac{17}{24}+\frac{11\pi^2}{18}\right)-\frac{1}{2}\left(\frac{1}{6}+\frac{2\pi^2}{9}\right)C_FN_F\Bigg),\end{aligned} \tag{135}$$

where

$$\begin{aligned}&\tilde{\gamma}_{qq,\mathrm{NS}}^{(1)}(N)\\&=C_F^2\Bigg(-\frac{(2N+1)S_1(N)}{2N^2(N+1)^2}-\frac{\left(3N^2+3N+2\right)S_2(N)}{4N(N+1)}\\&+\frac{\left(3N^2+3N+2\right)\zeta_2}{4N(N+1)}+\frac{3N^3+N^2-1}{4N^3(N+1)^3}\\&-\zeta_2S_1(N)+S_1(N)S_2(N)+S_3(N)-\zeta_3\Bigg)\\&+C_FN_F\Bigg(-\frac{11N^2+5N-3}{36N^2(N+1)^2}+\frac{5S_1(N)}{18}-\frac{S_2(N)}{6}\end{aligned}$$

$$+\frac{\zeta_2}{6}\Big) + C_F C_A \Bigg(-\frac{\left(11N^2+11N+3\right)\zeta_2}{12N(N+1)} + \frac{151N^4+236N^3+88N^2+3N+18}{72N^3(N+1)^3} + \frac{1}{2}\zeta_2 S_1(N) - \frac{67 S_1(N)}{36} + \frac{11 S_2(N)}{12} - \frac{S_3(N)}{2} + \frac{\zeta_3}{2}\Bigg), \tag{136}$$

and $S_i(N)$ are harmonic sums [29].

The NLO timelike splitting function is given by

$$P^{(1),t}_{qq,\mathrm{NS}}(x) = P^{(1)}_{qq,\mathrm{NS}}(x) + \Delta^{(1)}_{qq,\mathrm{NS}}(x), \tag{137}$$

where

$$\begin{aligned}\Delta^{(1)}_{qq,\mathrm{NS}}(x) &= \frac{1}{16} C_F^2 \Bigg(\Bigg(-32\left[\frac{1}{1-x}\right]_+ + 16x + 16\Bigg) H_{1,0}(x) \\ &+ \Bigg(-16\left[\frac{1}{1-x}\right]_+ + 12x + 12\Bigg) \ln^2 x \\ &+ \Bigg(-32\left[\frac{1}{1-x}\right]_+ + 16x + 16\Bigg) H_2(x) \\ &+ \Bigg(24\left[\frac{1}{1-x}\right]_+ - 4x - 20\Bigg) \ln x\Bigg).\end{aligned} \tag{138}$$

Here H_n and $H_{n,m}$ are harmonic polylogarithms [30]. The Mellin transform of the difference is

$$\begin{aligned}\Delta^{(1)}_{qq,\mathrm{NS}}(N) &= C_F^2 \Bigg[\frac{(2N+1)S_1(N)}{N^2(N+1)^2} + \frac{\left(3N^2+3N+2\right) S_2(N)}{2N(N+1)} \\ &- \frac{\left(3N^2+3N+2\right)\zeta_2}{2N(N+1)} - \frac{6N^3+9N^2+7N+2}{4N^3(N+1)^3} \\ &+ 2\zeta_2 S_1(N) - 2 S_1(N) S_2(N)\Bigg].\end{aligned} \tag{139}$$

The explicit expressions of the spacelike and timelike Mellin-space subtracted anomalous dimensions that enter the NNLO single soft coefficients Eqs. (95, 114) are

$$\begin{aligned}\bar{\gamma}^{(1)}_{qq,\mathrm{NS}}(N) &= C_F^2 \Bigg(-\frac{(2N+1)S_1(N)}{2N^2(N+1)^2} - \frac{\left(3N^2+3N+2\right) S_2(N)}{4N(N+1)} \\ &+ \frac{\left(3N^2+3N+2\right)\zeta_2}{4N(N+1)} + \frac{3N^3+N^2-1}{4N^3(N+1)^3} \\ &- \zeta_2 S_1(N) + S_1(N) S_2(N) + S_3(N) - \zeta_3\Big) \\ &+ C_F N_F \Bigg(-\frac{5\ln\bar{N}}{18} - \frac{11N^2+5N-3}{36N^2(N+1)^2} + \frac{5 S_1(N)}{18} - \frac{S_2(N)}{6} + \frac{\zeta_2}{6}\Bigg) \\ &+ C_F C_A \Bigg(-\frac{1}{2}\zeta_2 \ln\bar{N} + \frac{67\ln\bar{N}}{36} - \frac{\left(11N^2+11N+3\right)\zeta_2}{12N(N+1)} \\ &+ \frac{151N^4+236N^3+88N^2+3N+18}{72N^3(N+1)^3} \\ &+ \frac{1}{2}\zeta_2 S_1(N) - \frac{67 S_1(N)}{36} + \frac{11 S_2(N)}{12} - \frac{S_3(N)}{2} + \frac{\zeta_3}{2}\Bigg),\end{aligned} \tag{140}$$

and

$$\begin{aligned}\bar{\gamma}^{(1),t}_{qq,\mathrm{NS}}(N) &= C_F^2 \Bigg(\frac{(2N+1)S_1(N)}{2N^2(N+1)^2} + \frac{\left(3N^2+3N+2\right) S_2(N)}{4N(N+1)} \\ &- \frac{\left(3N^2+3N+2\right)\zeta_2}{4N(N+1)} - \frac{3N^3+8N^2+7N+3}{4N^3(N+1)^3} \\ &+ \zeta_2 S_1(N) - S_1(N) S_2(N) + S_3(N) - \zeta_3\Big) \\ &+ C_F N_F \Bigg(-\frac{5\ln\bar{N}}{18} - \frac{11N^2+5N-3}{36N^2(N+1)^2} + \frac{5 S_1(N)}{18} - \frac{S_2(N)}{6} + \frac{\zeta_2}{6}\Bigg) \\ &+ C_F C_A \Bigg(-\frac{1}{2}\zeta_2 \ln\bar{N} + \frac{67\ln\bar{N}}{36} - \frac{\left(11N^2+11N+3\right)\zeta_2}{12N(N+1)} \\ &+ \frac{151N^4+236N^3+88N^2+3N+18}{72N^3(N+1)^3} \\ &+ \frac{1}{2}\zeta_2 S_1(N) - \frac{67 S_1(N)}{36} + \frac{11 S_2(N)}{12} - \frac{S_3(N)}{2} + \frac{\zeta_3}{2}\Bigg).\end{aligned} \tag{141}$$

A.3 Coefficient functions in soft limits

We list the coefficients that enter Eq. (81).

A.3.1 x single soft

NLO coefficients

$$f_2^{(1)}(M) = \frac{C_\mathrm{F}}{2} \tag{142}$$

$$f_1^{(1)}(M) = C_\mathrm{F}\left(S_1(M) - \frac{1}{2M^2+2M}\right) \tag{143}$$

$$\begin{aligned} f_0^{(1)}(M) = C_\mathrm{F}\Bigg(&\frac{2M^2-M-1}{2M^2(M+1)^2} + \frac{1}{2}S_1(M)^2 \\ &- \frac{S_1(M)}{2M(M+1)} + \frac{3}{2}S_2(M) - \frac{\zeta_2}{2} - 4\Bigg) \end{aligned} \tag{144}$$

NNLO coefficients

$$f_4^{(2)}(M) = \frac{1}{8}C_\mathrm{F}^2, \tag{145}$$

$$\begin{aligned} f_3^{(2)}(M) &= C_\mathrm{F}^2\left(\frac{S_1(M)}{2} - \frac{1}{4M(M+1)}\right) \\ &\quad + \frac{11C_\mathrm{F}C_\mathrm{A}}{72} - \frac{C_\mathrm{F}n_f}{36}, \end{aligned} \tag{146}$$

$$\begin{aligned} &f_2^{(2)}(M) \\ &= C_\mathrm{F}^2\Bigg(\frac{4M^2-2M-1}{8M^2(M+1)^2} + \frac{3}{4}S_1(M)^2 - \frac{3S_1(M)}{4M(M+1)} \\ &\quad + \frac{3S_2(M)}{4} - \frac{\zeta_2}{4} - 2\Bigg) \\ &\quad + C_\mathrm{F}C_\mathrm{A}\left(\frac{11S_1(M)}{24} - \frac{11}{48M(M+1)} - \frac{\zeta_2}{4} + \frac{67}{72}\right) \\ &\quad + C_\mathrm{F}n_f\left(-\frac{1}{12}S_1(M) + \frac{1}{24M(M+1)} - \frac{5}{36}\right), \end{aligned} \tag{147}$$

$$\begin{aligned} &f_1^{(2)}(M) \\ &= C_\mathrm{F}^2\Bigg(-\frac{\left(3M^2+3M+5\right)S_2(M)}{4M(M+1)} + \frac{3\left(M^2+M+1\right)\zeta_2}{4M(M+1)} \\ &\quad - \frac{\left(16M^4+32M^3+12M^2+6M+3\right)S_1(M)}{4M^2(M+1)^2} \\ &\quad + \frac{8M^4+19M^3+14M^2+8M+4}{4M^3(M+1)^3} \\ &\quad - \frac{3}{2}\zeta_2S_1(M) + \frac{1}{2}S_1(M)^3 - \frac{3S_1(M)^2}{4M(M+1)} \\ &\quad + \frac{5}{2}S_2(M)S_1(M) - S_3(M) + \zeta_3\Bigg) \\ &\quad + C_\mathrm{F}N_f\Bigg(-\frac{\left(10M^2+10M-3\right)S_1(M)}{36M(M+1)} \\ &\quad - \frac{1}{12}S_1(M)^2 - \frac{S_2(M)}{12} \\ &\quad + \frac{5M+8}{36M(M+1)^2} + \frac{\zeta_2}{12} - \frac{7}{27}\Bigg) \\ &\quad + C_\mathrm{F}C_\mathrm{A}\Bigg(\frac{\left(134M^2+134M-33\right)S_1(M)}{72M(M+1)} \\ &\quad - \frac{85M^4+203M^3+154M^2+36M+18}{72M^3(M+1)^3} \\ &\quad - \frac{1}{2}\zeta_2S_1(M) + \frac{11}{24}S_1(M)^2 + \frac{11S_2(M)}{24} \\ &\quad + \frac{S_3(M)}{2} + \frac{\zeta_2}{4M(M+1)} - \frac{11\zeta_2}{24} - \frac{9\zeta_3}{4} + \frac{101}{54}\Bigg), \end{aligned} \tag{148}$$

$$\begin{aligned} f_0^{(2)}(M) = C_\mathrm{F}^2\Bigg(&\frac{1}{8}S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{4}\zeta_2S_1(M)^2 \\ &+ \frac{\left(-M^5+13M^4+41M^3+15M^2+2\right)S_1(M)}{8M^3(M+1)^3} \\ &+ \frac{5\zeta_2S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta_3S_1(M) \\ &+ \frac{S_1(M)^2\left(7M^2(M+1)^2S_2(M) - 2\left(4M^4+8M^3+3M^2+2M+1\right)\right)}{4M^2(M+1)^2} \\ &+ \frac{(-7M(M+1)S_2(M))\,S_1(M)}{4M^2(M+1)^2} \\ &+ \frac{1}{2}\left(5S_3(M) - 2S_{2,1}(M)\right)S_1(M) - \frac{31\zeta_2^2}{40} \\ &+ \frac{-33M^6-50M^5+29M^4+81M^3+65M^2+48M+15}{8M^4(M+1)^4} \\ &+ \frac{\left(4M^2-2M-1\right)\zeta_2}{8M^2(M+1)^2} + \frac{\left(19M^4+38M^3+12M^2+M+1\right)\zeta_2}{4M^2(M+1)^2} \\ &+ \frac{13\zeta_2}{8} - \frac{\zeta_3}{2M(M+1)} - \frac{15\zeta_3}{4} \\ &+ \frac{3\left(7\zeta_3M^2+7\zeta_3M+2\zeta_3\right)}{8M(M+1)} \\ &- \frac{\left(27M^4+54M^3+18M^2+7M+5\right)S_2(M)}{4M^2(M+1)^2} \\ &- \frac{5}{4}\zeta_2S_2(M) - \frac{\left(9M^2+9M+10\right)S_3(M)}{8M(M+1)} - \frac{33S_4(M)}{8} \\ &- \frac{\left(3M^2+3M-2\right)S_{2,1}(M)}{4M(M+1)} \\ &+ \frac{13}{4}S_{2,2}(M) - 2S_{3,1}(M) + \frac{3}{2}S_{2,1,1}(M) + \frac{511}{64}\Bigg) \\ + \frac{1}{36}&(11C_\mathrm{A} - 2N_f)\zeta_3C_\mathrm{F} \\ + N_fC_\mathrm{F}\Bigg(&-\frac{1}{36}S_1(M)^3 - \frac{\left(10M^2+10M-3\right)S_1(M)^2}{72M(M+1)} \\ &- \frac{\left(28M^3+56M^2+13M-24\right)S_1(M)}{108M(M+1)^2} + \frac{1}{12}\zeta_2S_1(M) \\ &- \frac{1}{12}S_2(M)S_1(M) \\ &- \frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta_2}{36M(M+1)} \\ &+ \frac{\zeta_2}{24M(M+1)} + \frac{7\zeta_2}{18} + \frac{\zeta_3}{12} \\ &- \frac{\left(20M^2+20M-3\right)S_2(M)}{72M(M+1)} + \frac{S_3(M)}{36} + \frac{127}{96}\Bigg) \\ + C_\mathrm{A}C_\mathrm{F}\Bigg(&\frac{11}{72}S_1(M)^3 \\ &+ \frac{\left(404M^5+1239M^4+1038M^3-70M^2-273M+54\right)S_1(M)}{216M^2(M+1)^3} \\ &- \frac{11}{24}\zeta_2S_1(M) - \frac{13}{4}\zeta_3S_1(M) \end{aligned}$$

$$
\begin{aligned}
&-\frac{S_1(M)^2\left(-134M^2+36(M+1)S_2(M)M-134M+33\right)}{144M(M+1)}\\
&\frac{-6S_1(M)\left(11M^2+11M+6\right)S_2(M)}{144M(M+1)}\\
&-\frac{1}{2}\left(S_3(M)-2S_{2,1}(M)\right)S_1(M)+\frac{21\zeta_2^2}{20}\\
&+\frac{124M^6-1044M^5-2679M^4-2396M^3-1002M^2-765M-270}{432M^4(M+1)^4}\\
&-\frac{11\zeta_2}{48M(M+1)}-\frac{101\zeta_2}{36}\\
&-\frac{29\zeta_2M^4+58\zeta_2M^3+17\zeta_2M^2-30\zeta_2M-9\zeta_2}{36M^2(M+1)^2}\\
&+\frac{\left(11M^2+11M+39\right)\zeta_3}{24M(M+1)}+\frac{43\zeta_3}{12}\\
&+\frac{\left(250M^3+250M^2-69M-36\right)S_2(M)}{144M^2(M+1)}-\frac{1}{2}\zeta_2S_2(M)\\
&-\frac{\left(11M^2+11M-18\right)S_3(M)}{72M(M+1)}+S_4(M)-\frac{S_{2,1}(M)}{2M(M+1)}\\
&+S_{2,2}(M)+S_{3,1}(M)-\frac{3}{2}S_{2,1,1}(M)-\frac{1535}{192}\Bigg)
\end{aligned}
\tag{149}
$$

A.3.2 *z single soft*

NLO coefficients

$$h_2^{(1)}(N)=\frac{C_F}{2} \tag{150}$$

$$h_1^{(1)}(N)=C_F\left(S_1(N)-\frac{1}{2N^2+2N}\right) \tag{151}$$

$$
\begin{aligned}
h_0^{(1)}(N)=C_F\Bigg(&\frac{2N+1}{2N^2(N+1)}+\frac{1}{2}S_1(N)^2-\frac{S_1(N)}{2N(N+1)}\\
&-\frac{S_2(N)}{2}+\frac{3\zeta_2}{2}-4\Bigg)
\end{aligned}
\tag{152}
$$

NNLO coefficients

$$h_4^{(2)}(N)=\frac{1}{8}C_F^2 \tag{153}$$

$$
\begin{aligned}
h_3^{(2)}(N)=C_F^2&\left(\frac{S_1(N)}{2}-\frac{1}{4N(N+1)}\right)\\
&+\frac{11}{72}C_FC_A-\frac{C_FN_f}{36}
\end{aligned}
\tag{154}
$$

$$
\begin{aligned}
&h_2^{(2)}(N)\\
&=C_F^2\Bigg(\frac{4N^2+6N+3}{8N^2(N+1)^2}+\frac{3}{4}S_1(N)^2-\frac{3S_1(N)}{4N(N+1)}\\
&\quad-\frac{S_2(N)}{4}+\frac{3\zeta_2}{4}-2\Bigg)\\
&\quad+C_FC_A\left(\frac{11S_1(N)}{24}-\frac{11}{48N(N+1)}-\frac{\zeta_2}{4}+\frac{67}{72}\right)\\
&\quad+C_FN_f\left(-\frac{1}{12}S_1(N)+\frac{1}{24N(N+1)}-\frac{5}{36}\right)
\end{aligned}
\tag{155}
$$

$$
\begin{aligned}
&h_1^{(2)}(N)\\
&=C_F^2\Bigg(\frac{3\left(N^2+N+1\right)S_2(N)}{4N(N+1)}-\frac{\left(3N^2+3N+5\right)\zeta_2}{4N(N+1)}\\
&\quad+\frac{8N^3+13N^2+5N-3}{4N^2(N+1)^3}\\
&\quad-\frac{\left(16N^4+32N^3+12N^2-10N-5\right)S_1(N)}{4N^2(N+1)^2}\\
&\quad+\frac{5}{2}\zeta_2S_1(N)+\frac{1}{2}S_1(N)^3\\
&\quad-\frac{3S_1(N)^2}{4N(N+1)}-\frac{3}{2}S_2(N)S_1(N)-S_3(N)+\zeta_3\Bigg)\\
&\quad+C_FN_f\Bigg(-\frac{\left(10N^2+10N-3\right)S_1(N)}{36N(N+1)}\\
&\qquad+\frac{5N^2-4N-6}{36N^2(N+1)^2}\\
&\quad-\frac{1}{12}S_1(N)^2+\frac{S_2(N)}{4}-\frac{\zeta_2}{4}-\frac{7}{27}\Bigg)\\
&\quad+C_FC_A\Bigg(\frac{\left(134N^2+134N-33\right)S_1(N)}{72N(N+1)}\\
&\quad-\frac{85N^4+71N^3-44N^2-30N+18}{72N^3(N+1)^3}\\
&\quad-\frac{1}{2}\zeta_2S_1(N)+\frac{11}{24}S_1(N)^2-\frac{11S_2(N)}{8}+\frac{S_3(N)}{2}\\
&\quad+\frac{11N\zeta_2}{6(N+1)}+\frac{\zeta_2}{4N(N+1)}\\
&\quad+\frac{11\zeta_2}{6(N+1)}-\frac{11\zeta_2}{24}-\frac{9\zeta_3}{4}+\frac{101}{54}\Bigg)
\end{aligned}
\tag{156}
$$

$$
\begin{aligned}
h_0^{(2)}(N)=C_F^2&\left(\frac{1}{8}S_1(N)^4-\frac{S_1(N)^3}{4N(N+1)}+\frac{7}{4}\zeta_2S_1(N)^2\right.\\
&+\frac{\left(N^4+17N^3+4N^2-24N-8\right)S_1(N)}{8N^3(N+1)^2}\\
&-\frac{3\zeta_2S_1(N)}{4N(N+1)}-\frac{\zeta_2S_1(N)}{N^2+N}-\frac{7}{2}\zeta_3S_1(N)\\
&-\frac{\left(8N^4+16N^3+5(N+1)^2S_2(N)N^2+6N^2-6N-3\right)S_1(N)^2}{4N^2(N+1)^2}\\
&+\frac{5S_2(N)S_1(N)}{4N(N+1)}+\frac{3}{2}S_3(N)S_1(N)+S_{2,1}(N)S_1(N)\\
&-\frac{11\zeta_2^2}{40}+\frac{2N^7-25N^6-106N^5-142N^4-83N^3-21N^2-13N-5}{8N^4(N+1)^4}\\
&+\frac{\left(4N^2+6N+3\right)\zeta_2}{8N^2(N+1)^2}-\frac{\left(20N^3+40N^2+13N-3\right)\zeta_2}{4N(N+1)^2}\\
&+\frac{13\zeta_2}{8}-\frac{\zeta_3}{2N(N+1)}-\frac{15\zeta_3}{4}-\frac{3\left(\zeta_3N^2+\zeta_3N-6\zeta_3\right)}{8N(N+1)}\\
&+\frac{\left(12N^3+24N^2+7N-1\right)S_2(N)}{4N(N+1)^2}+\frac{3}{4}\zeta_2S_2(N)\\
&-\frac{3\left(3N^2+3N+2\right)S_3(N)}{8N(N+1)}+\frac{23S_4(N)}{8}+\frac{\left(3N^2+3N-2\right)S_{2,1}(N)}{4N(N+1)}\\
&\left.-\frac{1}{4}S_{2,2}(N)-\frac{1}{2}S_{3,1}(N)-\frac{3}{2}S_{2,1,1}(N)+\frac{511}{64}\right)\\
&+\frac{1}{36}(11C_A-2N_f)\zeta_3C_F
\end{aligned}
$$

$$
\begin{aligned}
&+ C_F \left(\frac{S_1(N)^2 \left(N^2+N+2\right)^2}{16N^2(N+1)^2\left(N^2+N-2\right)} \right.\\
&+ \frac{\zeta_2 \left(N^2+N+2\right)^2}{16N^2(N+1)^2\left(N^2+N-2\right)}\\
&- \frac{S_2(N) \left(N^2+N+2\right)^2}{16N^2(N+1)^2\left(N^2+N-2\right)}\\
&+ \frac{N^{10}+8N^9+33N^8+127N^7+459N^6+1111N^5}{16(N-1)N^4(N+1)^4(N+2)^3}\\
&+ \frac{1725N^4+1776N^3+1192N^2+464N+80}{16(N-1)N^4(N+1)^4(N+2)^3}\\
&\left. - \frac{\left(N^6+12N^5+53N^4+86N^3+80N^2+56N+16\right) S_1(N)}{8(N-1)N^3(N+1)^3(N+2)^2} \right)\\
&+ N_f C_F \left(-\frac{1}{36} S_1(N)^3 - \frac{\left(10N^2+10N-3\right) S_1(N)^2}{72N(N+1)} \right.\\
&- \frac{1}{12}\zeta_2 S_1(N) + \frac{1}{12} S_2(N) S_1(N)\\
&- \frac{7N^2 S_1(N)}{27(N+1)^2} - \frac{14N S_1(N)}{27(N+1)^2} - \frac{S_1(N)}{9N(N+1)^2} - \frac{S_1(N)}{6N^2(N+1)^2}\\
&- \frac{13 S_1(N)}{108(N+1)^2}\\
&- \frac{11N^4+46N^3-19N^2-90N-45}{216N^3(N+1)^3} + \frac{\zeta_2}{24N(N+1)} - \frac{\zeta_2}{36} + \frac{\zeta_3}{12}\\
&\left. + \frac{\left(20N^2+20N-3\right) S_2(N)}{72N(N+1)} - \frac{11 S_3(N)}{36} + \frac{1}{6} S_{2,1}(N) + \frac{127}{96} \right)\\
&+ C_F C_A \left(\frac{11N S_1(N)^3}{72(N+1)} + \frac{11 S_1(N)^3}{72(N+1)} - \frac{3}{4}\zeta_2 S_1(N)^2 \right.\\
&+ \frac{\left(404N^5+781N^4+122N^3-12N^2+144N-54\right) S_1(N)}{216N^3(N+1)^2}\\
&+ \frac{11N\zeta_2 S_1(N)}{24(N+1)} + \frac{3\zeta_2 S_1(N)}{4N(N+1)} + \frac{11\zeta_2 S_1(N)}{24(N+1)} + \frac{5}{4}\zeta_3 S_1(N)\\
&+ \frac{\left(S_1(N)\left(134N^2+72(N+1)S_2(N)N+134N-33\right)\right) S_1(N)}{144N(N+1)}\\
&- \frac{\left(6\left(11N^2+11N+12\right) S_2(N)\right) S_1(N)}{144N(N+1)}\\
&- \frac{108N^7+308N^6-348N^5-999N^4-298N^3+324N^2-153N-162}{432N^4(N+1)^4}\\
&- \frac{3}{2} S_{2,1}(N) S_1(N) - \frac{19\zeta_2^2}{20}\\
&+ \frac{\left(70N^4+140N^3+49N^2-15N-6\right)\zeta_2}{24N^2(N+1)^2}\\
&- \frac{11\zeta_2}{48N(N+1)} - \frac{101\zeta_2}{36} + \frac{\left(11N^2+11N-15\right)\zeta_3}{24N(N+1)} + \frac{43\zeta_3}{12}\\
&- \frac{\left(286N^3+572N^2+181N-33\right) S_2(N)}{144N(N+1)^2} - \frac{1}{4}\zeta_2 S_2(N)\\
&+ \frac{121 S_3(N)}{72} - \frac{5 S_4(N)}{4} - \frac{\left(11N^2+11N-9\right) S_{2,1}(N)}{12N(N+1)}\\
&\left. - \frac{1}{2} S_{2,2}(N) + 2 S_{2,1,1}(N) - \frac{1535}{192} \right) \qquad (157)
\end{aligned}
$$

References

1. R. Abdul Khalek et al., Snowmass 2021 White Paper: Electron Ion Collider for High Energy Physics. arXiv:2203.13199
2. S. Goyal, S.-O. Moch, V. Pathak, N. Rana, V. Ravindran, Next-to-next-to-leading order QCD corrections to semi-inclusive deep-inelastic scattering. Phys. Rev. Lett. **132**(25), 251902 (2024). arXiv:2312.17711
3. L. Bonino, T. Gehrmann, G. Stagnitto, Semi-inclusive deep-inelastic scattering at next-to-next-to-leading order in QCD. Phys. Rev. Lett. **132**(25), 251901 (2024). arXiv:2401.16281
4. S. Goyal, R.N. Lee, S.-O. Moch, V. Pathak, N. Rana, V. Ravindran, NNLO QCD corrections to unpolarized and polarized SIDIS. Phys. Rev. D **111**(9), 094007 (2025). arXiv:2412.19309
5. L. Bonino, T. Gehrmann, M. Löchner, K. Schönwald, G. Stagnitto, Neutral and charged current semi-inclusive deep-inelastic scattering at NNLO QCD. JHEP **10**, 016 (2025). arXiv:2506.19926
6. G.F. Sterman, W. Vogelsang, Crossed threshold resummation. Phys. Rev. D **74**, 114002 (2006). arXiv: hep-ph/0606211
7. S. Catani, L. Trentadue, Resummation of the QCD perturbative series for hard processes. Nucl. Phys. B **327**, 323–352 (1989)
8. P. Banerjee, G. Das, P.K. Dhani, V. Ravindran, Threshold resummation of the rapidity distribution for Drell–Yan production at NNLO+NNLL. Phys. Rev. D **98**(5), 054018 (2018)
9. V. Ravindran, A. Sankar, S. Tiwari, Resummed next-to-soft corrections to rapidity distribution of Higgs boson to NNLO+NNLL¯. Phys. Rev. D **108**(1), 014012 (2023). arXiv:2205.11560
10. D.P. Anderle, F. Ringer, W. Vogelsang, QCD resummation for semi-inclusive hadron production processes. Phys. Rev. D **87**(3), 034014 (2013). arXiv:1212.2099
11. M. Abele, D. de Florian, W. Vogelsang, Approximate NNLO QCD corrections to semi-inclusive DIS. Phys. Rev. D **104**(9), 094046 (2021). arXiv:2109.00847
12. M. Abele, D. de Florian, W. Vogelsang, Threshold resummation at NLL3 accuracy and approximate N3LO corrections to semi-inclusive DIS. Phys. Rev. D **106**(1), 014015 (2022). arXiv:2203.07928
13. S. Goyal, S.-O. Moch, V. Pathak, N. Rana, V. Ravindran, Soft and virtual corrections to semi-inclusive DIS up to four loops in QCD. arXiv:2506.24078
14. G. Lustermans, J.K.L. Michel, F.J. Tackmann, Generalized Threshold Factorization with Full Collinear Dynamics. arXiv:1908.00985
15. B. Mistlberger, G. Vita, Collinear Approximations for LHC Cross Sections: Factorization and Resummation. arXiv:2502.17548
16. L. De Ros, S. Forte, G. Ridolfi, D.M. Tagliabue, Threshold resummation of rapidity distributions at fixed partonic rapidity. arXiv:2601.04309
17. S. Forte, G. Ridolfi, Renormalization group approach to soft gluon resummation. Nucl. Phys. B **650**, 229–270 (2003). arXiv:hep-ph/0209154
18. S. Forte, G. Ridolfi, S. Rota, Threshold resummation of transverse momentum distributions beyond next-to-leading log. JHEP **08**, 110 (2021). arXiv:2106.11321
19. L. De Ros, S. Forte, G. Ridolfi, D.M. Tagliabue, Threshold resummation of rapidity distributions at fixed partonic rapidity. arXiv:2601.04309
20. M. Höschele, J. Hoff, A. Pak, M. Steinhauser, T. Ueda, MT: a Mathematica package to compute convolutions. Comput. Phys. Commun. **185**, 528–539 (2014). arXiv:1307.6925
21. J. Ablinger, Computer Algebra Algorithms for Special Functions in Particle Physics. PhD thesis, Linz U., p. 4 (2012). arXiv:1305.0687
22. S. Moch, J.A.M. Vermaseren, A. Vogt, Higher-order corrections in threshold resummation. Nucl. Phys. B **726**, 317–335 (2005). arXiv:hep-ph/0506288
23. S. Moch, J.A.M. Vermaseren, A. Vogt, The three loop splitting functions in QCD: the nonsinglet case. Nucl. Phys. B **688**, 101–134 (2004). arXiv:hep-ph/0403192
24. M. Kramer, E. Laenen, M. Spira, Soft gluon radiation in Higgs boson production at the LHC. Nucl. Phys. B **511**, 523–549 (1998). arXiv:hep-ph/9611272
25. S. Catani, D. de Florian, M. Grazzini, Higgs production in hadron collisions: soft and virtual QCD corrections at NNLO. JHEP **05**, 025 (2001). arXiv:hep-ph/0102227

26. A. Vogt, Efficient evolution of unpolarized and polarized parton distributions with QCD-PEGASUS. Comput. Phys. Commun. **170**, 65–92 (2005). arXiv:hep-ph/0408244
27. S. Forte, E. Groenendijk, G. Ridolfi, G. Stagnitto, in preparation (2026)
28. R.K. Ellis, W.J. Stirling, B.R. Webber, *QCD and Collider Physics*, vol. 8 (Cambridge University Press, Cambridge, 2011), p.2
29. J.A.M. Vermaseren, Harmonic sums, Mellin transforms and integrals. Int. J. Mod. Phys. A **14**, 2037–2076 (1999). arXiv:hep-ph/9806280
30. E. Remiddi, J.A.M. Vermaseren, Harmonic polylogarithms. Int. J. Mod. Phys. A **15**, 725–754 (2000). arXiv:hep-ph/9905237