



Some geometric properties of generalized φ -vacuum static spaces

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Abstract

In this paper we study the geometry of generalized φ -vacuum static spaces, proving estimates for the φ -scalar curvature and for the first eigenvalue of the Jacobi operator, and also rigidity under various geometric assumptions; in particular, we prove a result related to the famous Cosmic no-hair conjecture of Boucher, Gibbons and Horowitz.

1 Introduction and statement of the results

The aim of the present paper is to study the geometric properties of what we call a *generalized φ -vacuum static space*, that is, an m -dimensional Riemannian manifold (M^m, g) , $m \geq 3$, with possibly non-empty boundary ∂M , for which we have a smooth solution u of the problem

$$\begin{cases} u\text{Ric}^\varphi - \text{Hess}(u) = \Lambda g, \\ u\tau(\varphi) + d\varphi(\nabla u) = 0, \end{cases} \quad (1.1)$$

with $u > 0$, $u \equiv 0$ on ∂M in case the latter is non-empty and where Λ is a smooth function on M , φ is a smooth map from (M^m, g) to a second Riemannian manifold (N^n, h) of dimension n , Ric^φ is the φ -Ricci tensor (see below for the definition) and $\tau(\varphi)$ is the *tension field* of the map φ , that is the trace, with respect to g , of the generalized second fundamental tensor $\nabla d\varphi$ of the map φ , which extends the notion of mean curvature vector field for isometric immersions (see [12]). We recall the definition of the φ -Ricci tensor:

$$\text{Ric}^\varphi = \text{Ric} - \alpha\varphi^*h, \quad (1.2)$$

where α is a non-null real coupling constant; this $(0, 2)$ -tensor has been first introduced by B. List, [25], in his study of the Ricci-harmonic maps flow. We shall introduce other modified curvature tensors of a similar nature: they merge the Riemannian geometry of M with that

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of the map φ . Clearly, in case φ is constant, Ric^φ coincides with its classical counterpart Ric (and this holds also for all the remaining φ -curvatures that we will define later); the φ -scalar curvature is obtained from (1.2) by contraction in the metric g , thus

$$S^\varphi = S - \alpha|d\varphi|^2; \tag{1.3}$$

note that $\frac{1}{2}|d\varphi|^2$ is the *energy density* of the map φ . For suitable choices of the function Λ , system (1.1) describes several important structures, both from the mathematical and physical point of view: for instance, in case φ is constant and

$$\Lambda = \frac{S}{m-1}u, \tag{1.4}$$

system (1.1) recovers the equation of *vacuum static spaces*; for

$$\Lambda = -\frac{S}{m-1}u - \frac{\sigma}{m-1}, \tag{1.5}$$

where σ is a real constant, we obtain the *V-static equation* (see Miao and Tam, [28]). Observe that, in the paper of Miao and Tam, the constant σ is equal to -1 . However, it is easy to check that their result can be extended to the case of $\sigma \in \mathbb{R}$; moreover, in this case S is constant (see Theorem 7 of [9, 28]). For

$$\Lambda = (\mu - p)\frac{1}{m-1}u, \tag{1.6}$$

with μ and p smooth functions on M to be interpreted as *energy density* and *pressure* of a perfect fluid, we recover the *static perfect fluid equation* (see e.g., [10, 20] and the references therein). In the aforementioned examples, φ is constant; otherwise, φ is to be considered as a non-linear field, possibly interacting with a potential $U : N \rightarrow \mathbb{R}$. However, we shall not consider this latter case here; for details, we refer to [6].

Our first result gives a lower bound on the φ -scalar curvature S^φ : we recall that, for $f \in C^\infty(M)$, the *f-Laplacian* operator is defined as $\Delta_f = \Delta - g(\nabla f, \cdot)$; moreover, we define

$$S_*^\varphi = \inf_M S^\varphi.$$

Theorem 1.1 *Let (M^m, g) be a complete manifold of dimension $m \geq 2$, satisfying (1.1). Assume $\alpha > 0$,*

$$\Delta_{-2 \log u} \left(\frac{\Lambda}{u} \right) \leq 0 \tag{1.7}$$

and

$$\lambda = \inf_M \left(\frac{\Lambda}{u} \right) > -\infty. \tag{1.8}$$

Then:

- i) if $\lambda > 0$, then M is compact and $(m - 1)\lambda \leq S_*^\varphi \leq m\lambda$;
- ii) if $\lambda = 0$, then $S_*^\varphi = 0$;
- iii) if $\lambda < 0$, then $S_*^\varphi \geq m\lambda$.

Remark 1.2 Note that:

- i) conditions (1.7) and (1.8) are automatically satisfied for a vacuum static space, that is for Λ given in (1.4), since in this case S is constant.

ii) For a static perfect fluid, that is, for the choice of Λ in (1.6), (1.7) amounts to the request

$$\Delta(\mu - p) + 2g(\nabla \log u, \nabla(\mu - p)) \leq 0, \tag{1.9}$$

while (1.8) becomes

$$\inf(\mu - p) > -\infty. \tag{1.10}$$

However, in this case, by the dynamics of the fluid we know that

$$\nabla p = -(\mu + p) \frac{\nabla u}{u}, \tag{1.11}$$

so that (1.9) becomes

$$\Delta(\mu + p) \leq \frac{2}{(\mu + p)} g(\nabla p, \nabla(\mu - p)) \tag{1.12}$$

whenever $\mu + p > 0$, which, in turn, can be written as

$$\Delta \log(\mu + p) \leq -\frac{(|\nabla p|^2 - |\nabla \mu|^2 + (|\nabla p| - |\nabla \mu|)^2)}{(\mu + p)}$$

and (1.8) is satisfied. What is interesting, in the latter case, is that neither (1.7) nor (1.8) depend on u . Similar considerations hold for φ non-constant.

iii) For the V -static equation, since S is constant, (1.7) becomes

$$-\frac{\sigma}{m-1} \Delta_{-2 \log u}(u^{-1}) \leq 0,$$

which is equivalent to

$$\sigma \Delta u \leq 0,$$

while (1.8) becomes

$$\inf_M \left(-\frac{\sigma}{u} \right) > -\infty,$$

which is automatically satisfied for $\sigma < 0$ (which is the case of Miao and Tam, see the discussion above).

Remark 1.3 Observe also that Theorem 1.1 provides a generalization of Theorem 7.29 of [3], where the authors studied Riemannian manifolds carrying a smooth solution of the system

$$\begin{cases} \text{Ric}^\varphi + \text{Hess}(f) - \mu df \otimes df = \lambda g, \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{1.13}$$

for $\mu, \lambda \in \mathbb{R}$. Indeed, the main difference is that in our case Λ is a smooth function and we have to deal with the quantity $\frac{\Delta}{u}$. Note that, in case the latter is constant our result recovers that of [3] for $\mu = 1$.

Our second result is related to the famous *Cosmic no-hair conjecture* of Boucher, Gibbons and Horowitz (see [4]), that can be rephrased in the following form:

“an m -dimensional, compact, static manifold (M, g) with positive scalar curvature and connected boundary $\partial M \neq \emptyset$ is isometric to a round hemisphere $\mathbb{S}_+^m(c)$ with an appropriate radius c ”. This conjecture has been confirmed under different further hypotheses, but disproved for $\dim(M) \geq 4$ (for more information, see e.g. [10, 14, 15]). We prove the following

Theorem 1.4 *Let (M, g) be a compact, connected manifold of dimension $m \geq 2$ with boundary $\partial M \neq \emptyset$. Let $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map, $\alpha > 0$, $\Lambda \in C^\infty(M)$ and suppose that u is a C^2 -solution of*

$$\begin{cases} u\text{Ric}^\varphi - \text{Hess}(u) = \Lambda g, \\ u > 0 \text{ on } \text{int}(M), \\ \partial M = u^{-1}(\{0\}), \end{cases} \tag{1.14}$$

with $\frac{\Lambda}{u} \in C^0(M)$. Assume the following conditions:

$$uS^\varphi \geq (m - 1)\Lambda, \tag{1.15}$$

$$\Delta_{(2m-3)\log u} \left(S^\varphi - m\frac{\Lambda}{u} \right) \leq 0 \text{ on } \text{int}(M), \tag{1.16}$$

$$m(m - 1)|\nabla u|^2|_{\partial M} \leq \max_M \{m\Lambda u - S^\varphi u^2\}. \tag{1.17}$$

Then

$$m\frac{\Lambda}{u} - S^\varphi = c^2, \tag{1.18}$$

with c a positive constant, and (M, g) is isometric to $\mathbb{S}_+^m(c^2)$.

Some comments on the assumptions are in order. Suppose φ is constant and $\Lambda = \frac{S}{m-1}u$, that is, we are dealing with a vacuum static space: then (1.15) is automatically satisfied. The same is true for (1.16), since $S - m\frac{\Lambda}{u} = -\frac{S}{m-1}$, which is a constant. Note that (1.17) becomes

$$|\nabla u|^2|_{\partial M} \leq \frac{S}{m(m-1)} \max_M \{u^2\}, \tag{1.19}$$

which is the usual gravitational constraint imposed on the boundary ∂M (see e.g. [5]): it is meaningful since, in this case, $S > 0$ and, obviously, $\max_M \{u^2\} > 0$. This latter fact is not at all obvious for the general condition expressed in (1.17); however, to show that (1.17) is meaningful we may reason as follows. Set

$$\tilde{A} = S^\varphi - m\frac{\Lambda}{u} \in C^0(M), \tag{1.20}$$

so that, tracing the first equation in (1.14), we deduce

$$\Delta u = \tilde{A}u. \tag{1.21}$$

We claim that there exist $q \in \text{int}(M)$ such that

$$\tilde{A}(q) < 0. \tag{1.22}$$

Indeed, suppose the contrary and let $\tilde{A} \geq 0$ on $\text{int}(M)$ and thus, by continuity, on M . Since u attains its maximum on $\text{int}(M)$, by (1.21) and the maximum principle u is constant, and therefore null, since $u \equiv 0$ on ∂M , and this is a contradiction. Hence (1.22) holds: this is enough to guarantee that the right-hand side of (1.17) is strictly positive.

Our third result is a rigidity theorem: first we recall that a manifold (M, g) is *harmonic-Einstein* (with respect to $\varphi : (M, g) \rightarrow (N, h)$ and $\alpha \in \mathbb{R} \setminus \{0\}$) if

$$\begin{cases} \text{Ric}^\varphi = \frac{S^\varphi}{m}g, \\ \tau(\varphi) = 0, \end{cases} \tag{1.23}$$

and we require S^φ constant in case $m = 2$ (in case $m \geq 3$, S^φ is automatically constant, see [3]).

Theorem 1.5 *Let (M, g) be a complete manifold of dimension $m \geq 2$ satisfying, for some $\Lambda \in C^\infty(M)$,*

$$\begin{cases} u\text{Ric}^\varphi - \text{Hess}(u) = \Lambda g, \\ u\tau(\varphi) + d\varphi(\nabla u) = 0, \end{cases} \tag{1.24}$$

with

$$u > 0 \quad \text{on } M \tag{1.25}$$

Assume S^φ is constant and that $|\mathring{\text{Ric}}^\varphi| \in L^\infty(M)$, $|\nabla u| \in L^1(M)$; then (M, g) is harmonic-Einstein. Furthermore, if u is non-constant, then one of the following alternatives hold

- i) (M, g) is φ -Ricci flat
- ii) φ is constant and (M, g) is isometric to a Euclidean sphere $\mathbb{S}^m(k)$ of constant sectional curvature

$$k = \frac{S^\varphi}{m(m-1)} > 0;$$

- iii) φ is constant and (M, g) is isometric to a Hyperbolic space $\mathbb{H}^m(k)$ of constant sectional curvature

$$k = \frac{S^\varphi}{m(m-1)} < 0;$$

- iv) (M, g) is the Riemannian product $(\overline{M}, \overline{g}) \times (\mathbb{R}, g_{\mathbb{R}})$ of a complete Riemannian manifold $(\overline{M}, \overline{g})$ and the real line $(\mathbb{R}, g_{\mathbb{R}})$.

Remark 1.6 Observe that in Theorem 1.5 we assume $|\mathring{\text{Ric}}^\varphi| \in L^\infty(M)$, $|\nabla u| \in L^1(M)$; however, it is sufficient to require a weaker hypothesis, that is

$$\liminf_{R \rightarrow \infty} \int_{B_{2R} \setminus B_R} \mathring{\text{Ric}}^\varphi(\nabla u, \cdot)^\sharp dV_g = 0,$$

where B_R denotes the geodesic ball of radius R centered at some $p \in M$. Indeed this assumption is needed to apply an extended version of Stokes' theorem for non-compact manifolds by Karp ([23]).

Remark 1.7 We point out that when (M, g) is compact Theorem 1.5 recovers an analogous result, proved in [3], for compact Riemannian manifolds carrying a solution of system (1.13) when $\lambda \in C^\infty(M)$ and $\mu = 1$.

To introduce our last result, we need the following

Definition 1.1 (Jacobi Operator) Let (M, g) be a smooth Riemannian manifold with boundary ∂M and let $\phi \in C^\infty(\partial M)$, then the *Jacobi operator*, denoted by J_g , is defined as

$$J_g \phi = \Delta_g \phi + \left(\text{Ric}_g(v, v) + |\text{II}|_g^2 \right) \phi, \tag{1.26}$$

where Δ_g denotes the Laplace operator with respect to the metric g on the boundary, v is the outward unit normal and II denotes the second fundamental form of ∂M . The *first eigenvalue* of J_g is defined as

$$(\lambda_1)_g := \inf_{\phi \neq 0} \frac{-\int_{\partial M} J_g \phi dV_{g|_{\partial M}}}{\int_{\partial M} \phi^2 dV_{g|_{\partial M}}}. \tag{1.27}$$

Moreover, given $\varphi : (M, g) \rightarrow (N, h)$, where (X, g) and (N, h) are smooth Riemannian manifolds and φ is a smooth function, we define the φ -Yamabe invariant as

$$Y(M, [g])^\varphi = \inf_{\tilde{g} \in [g]} \text{Vol}(M)^{-\frac{n-2}{n}} \int_M \tilde{S}^\varphi dV_{\tilde{g}}, \tag{1.28}$$

where $[g]$ denotes the conformal class of g . We say that $\tilde{g} \in [g]$ is a φ -Yamabe minimizer if it achieves the infimum in (1.28); we also denote with $\tilde{\varphi}$ the map $\varphi : (M, \tilde{g}) \rightarrow (N, h)$.

Theorem 1.8 *Let (M, g) be a compact Riemannian manifold of dimension $m \geq 3$ with non-empty connected boundary, and let $\varphi : (M, g) \rightarrow (N, g_N)$ be a smooth map, where (N, g_N) is a second Riemannian manifold of dimension n . Let us assume the validity of the equation*

$$u\text{Ric}^\varphi - \text{Hess}(u) = \Lambda g, \tag{1.29}$$

where $u \in C^\infty(M)$, $u > 0$ on M and $\partial M = u^{-1}(\{0\})$. Let $\tilde{g} \in [g]$ be such that $\tilde{g}|_{\partial M}$ is a φ -Yamabe minimizer on ∂M , $\Lambda \in C^\infty(M)$, $\frac{\Lambda}{u} \in C^0(M)$. Then

$$(\lambda_1)_{\tilde{g}} \leq \frac{1}{2} \left((m-1)(m-2)\omega_{m-1}^{\frac{2}{m-1}} \text{Vol}(\partial M)^{-\frac{2}{m-1}} - \tilde{S}_{min}^{\tilde{\varphi}} \right),$$

where $\tilde{S}^{\tilde{\varphi}}$ is the $\tilde{\varphi}$ -scalar curvature of the metric \tilde{g} and ω_{m-1} is the volume of the unit $(m-1)$ -sphere. Moreover, if equality holds, then $\tilde{S}^{\tilde{\varphi}}$ is constant on ∂M , $\tilde{\varphi}_m^a \equiv 0$, ∂M is totally geodesic in (M, \tilde{g}) and $(\partial M, \tilde{g}|_{\partial M})$ is conformally equivalent to the standard sphere.

Remark 1.9 The technique used in the proof of Theorem 1.8 is similar to the one used by Andrade in [2], where she considers Riemannian manifold (M, g) with boundary satisfying the equation

$$u\text{Ric} - \text{Hess}(u) = \Lambda g.$$

In particular, she provides an estimate for the first Jacobi eigenvalue (with respect to the metric g) in terms of $\text{Vol}(\partial M)$ and the Yamabe invariant of the $(m-1)$ -sphere. However, note that the assumptions in [2] are stronger than ours, indeed the boundary is assumed to be Einstein with positive scalar curvature.

2 Examples and physical motivation

In this section we provide a mathematical and physical motivation and a connection with established physical models, justifying both the relevance and naturalness of (1.1).

From a mathematical point of view, we point out that the map $\varphi : (M, g) \rightarrow (N, h)$ arise naturally in the context of the *harmonic-Ricci flow*, first introduced by List in [25] and later studied by Buzano in [29]. This flow combines the classical Ricci flow equation (introduced by Hamilton in [19]), with the presence of a harmonic map $\varphi : (M, g) \rightarrow (N, h)$, where (N, h) is a target manifold. In particular, the coupled system is obtained by evolving simultaneously a Riemannian metric $g(t)$ with a map $\varphi(t)$:

$$\begin{cases} \partial_t g(t) = -2\text{Ric}_{g(t)} + 2\alpha\varphi(t)^*h; \\ \partial_t \varphi(t) = \tau(\varphi(t)). \end{cases}$$

A central geometric object in this theory is the modified curvature tensor Ric^φ defined in (1.2), that is

$$\text{Ric}^\varphi = \text{Ric} - \alpha\varphi^*h,$$

which plays a role analogous to the Ricci tensor in the classical Ricci flow. Self-similar solutions of the Ricci-harmonic flow are known as *Ricci-harmonic solitons* and are strictly related to harmonic-Einstein manifolds.

It is important to remark that the presence of φ is not relevant only from a mathematical perspective, but it also has a physical meaning. Indeed, system (1.1) is strictly related to solutions of the Einstein equations in the Lorentzian setting. Let (\hat{M}, \hat{g}) *Lorentzian manifold* of dimension $m + 1$, i.e. a smooth $(m + 1)$ -dimensional manifold with a non-degenerate $(0, 2)$ -symmetric tensor \hat{g} with signature $(-, +, \dots, +)$. Then, we say that it satisfies the Einstein equations, if

$$\hat{\text{Ric}} - \frac{\hat{S}}{2}\hat{g} = \hat{T}, \tag{2.1}$$

where, $\hat{\text{Ric}}, \hat{S}$ are the Ricci tensor and the scalar curvature relative to \hat{g} , respectively, and \hat{T} denotes the *stress-energy tensor*. When \hat{T} is the stress-energy tensor of a *scalar field* $\hat{\varphi}$, it satisfies following expression

$$T_{\hat{\varphi}} = \alpha \hat{\varphi}^* h - \alpha \frac{|d\hat{\varphi}|}{2} \hat{g} + U(\hat{\varphi}) \hat{g}.$$

This tensor can be obtained *via* a Lagrangian formulation, computing the Euler-Lagange equations of a suitable functional (see Chapter 3 of [20] for more details). We point out that here $\hat{\varphi} : (\hat{M}, \hat{g}) \rightarrow (N, h)$, is a smooth map, (N, h) is a target Riemannian manifold and $U : (N, h) \rightarrow \mathbb{R}$ is a smooth function.

If (\hat{M}, \hat{g}) can be written as a warped product of an m -dimensional Riemannian factor, (M, g) , and the real line, $\hat{M} = \mathbb{R}_u \times M$ with metric

$$\hat{g} = -u^2 dt \otimes dt + g, \tag{2.2}$$

$u \in C^\infty(M)$, $u > 0$ on M , we have that the components of the curvature tensors of \hat{M} can be rewritten in terms of those of M and, in particular, given $\varphi : (M, g) \rightarrow (N, h)$ as above, we can consider the scalar field $\hat{\varphi} := \varphi \circ \pi_M$, where π_M is the projection on M (note that in the above definition of \hat{g} we have used a somewhat loose style: we should have written $\hat{g} = \pi_{M*}(-u^2)\pi_{\mathbb{R}*}(dt \otimes dt) + \pi_{M*}g$, where $\pi_{\mathbb{R}}$ and π_M are the canonical projection of \hat{M} onto its factors).

As a consequence, using the relations between $\hat{\text{Ric}}, \hat{S}$ and Ric, S , (see e.g. Chapter 2 of [6] for more details and an explicit computation) we have that (\hat{M}, \hat{g}) satisfies (2.1) if and only if u is a solution of the following system

$$\begin{cases} u\text{Ric} - \text{Hess}(u) - \left(\frac{S}{2} - \Delta u\right)g = u\left(U(\varphi)g + \alpha\varphi^*h - \alpha\frac{|d\varphi|^2}{2}g\right), \\ \frac{S}{2} = -U(\varphi) + \alpha\frac{|d\varphi|^2}{2}, \end{cases}$$

which gives us

$$\begin{cases} u\text{Ric}^\varphi - \text{Hess}(u) = \Lambda(x)g, \\ \frac{S^\varphi}{2} = -U(\varphi), \end{cases}$$

for $\Lambda(x) = \Lambda_{\varphi,u}(x) = \frac{S^\varphi}{m-1}u$.

In our setting, we consider a generic smooth function Λ gaining in generality and recovering at the same time a large number of examples.

Another important physical motivation is that system (1.1) can be obtained variationally computing the Euler-Lagrange equations of the functional

$$S[g, \hat{\varphi}] := \int_{\Omega} \left(\hat{S} - \alpha |d\hat{\varphi}|^2 - U(\hat{\varphi}) \right) dV_{\hat{g}}, \tag{2.3}$$

for $\Omega \subset\subset \hat{M}$. The corresponding field equations are

$$\hat{\text{Ric}} = \alpha \hat{\varphi}^* h + U(\hat{\varphi}), \quad \tau(\hat{\varphi}) + U(\hat{\varphi}) = 0$$

(for more details we refer the reader to [11]); in particular, when (\hat{M}, \hat{g}) is a warped product as above, we have the validity of

$$u\text{Ric}^{\varphi} - \text{Hess}(u) = \Lambda g,$$

where $u = e^{-f}$ and $\Lambda = \frac{u}{m-1} U(\varphi)$. The second equation of system (1.1) is strictly related to the *conservation of energy* (see e.g. [6, 26] for more details): in particular, whenever $\nabla U(\varphi) = 0$, the equations of motion reduce to the second equation of (1.1) and they are satisfied by several physical examples, such as the Schwarzschild space: see Example 2.1 for more details. As a matter of fact, well-known examples of non-trivial potential $U(\varphi)$ are given by $U(\varphi) = \eta^2 \varphi^2$ (see [16, 20]), whose corresponding equations of motion lead to the so-called *Klein-Gordon equation*, and the *Higgs potential* (see [21, 27]), $U(\varphi) = \eta^2 \varphi^2 + \gamma \varphi^4$, which reduces to the previous case when $\gamma = 0$. In both settings (N, h) is $(\mathbb{R}, dx \otimes dx)$.

Let us now give a further explicit example of a family of manifolds satisfying (1.1), for a non-trivial choice of $\varphi : (M, g) \rightarrow (N, h)$:

Example 2.1 A scalar field solution to (1.1) is the *Fisher solution* (see [13]), that we describe hereafter following [7] (see, in particular, equations (56-58) in this reference). These solutions are 3-dimensional Riemannian manifolds depending on two parameters $k, \eta \in \mathbb{R}$ such that $k \neq 0, |\eta| \leq |k|$ and such that

$$M := (2k_+, +\infty) \times \mathbb{S}^2; \quad g := P(w)^{-\frac{\eta}{k}} dw \otimes dw + P(w)^{1-\frac{\eta}{k}} w^2 g_{\mathbb{S}^2},$$

where $k_+ := \max(k, 0)$, w is the coordinate function on $(2k_+, +\infty)$ and

$$P(w) := 1 - \frac{2k}{w}.$$

These spaces satisfy

$$\text{Ric} - df \otimes df - d\phi \otimes d\phi + \text{Hess}(f) = 0,$$

where

$$f := -\frac{\eta}{2k} \ln P(w), \quad \phi := -\sqrt{\frac{1}{2} \left(1 - \frac{\eta^2}{k^2} \right)} \ln P(w).$$

Hence, taking $(N, h) = (\mathbb{R}, dx \otimes dx)$ let $\alpha = 1, \varphi = \phi, u = e^{-f}$ and $\Lambda = 0$ we have that the space (M, g, u, φ) satisfies (1.1).

We point out that the Lorentzian metric associated to these solution is provided by

$$\hat{g} = P(w)^{-\frac{\eta}{k}} dw \otimes dw + P(w)^{1-\frac{m}{k}} w^2 g_{\mathbb{S}^2} - P(w)^{\frac{m}{k}} dt \otimes dt,$$

where, for the sake of simplicity we have, once again, omitted the projections on the factors. We remark that, for $\eta = k$, we obtain $\phi = 0$ and (M, g) coincides with the Schwarzschild space, which satisfies

$$u\text{Ric} - \text{Hess}(u) = 0.$$

3 Proof of Theorem 1.1

In the rest of the paper we shall freely use the moving frame notation: indices will run in the ranges

$$1 \leq i, j, \dots, \leq m, \quad 1 \leq a, b, \dots, \leq n,$$

and $\{\theta^i\}, \{\omega^a\}$ will be, respectively, local orthonormal coframes on open sets $U \subseteq M, V \subseteq N$ such that $\varphi(U) \subseteq V$. To prove Theorem 1.1 we shall need the following

Proposition 3.1 *Let (M, g) be a Riemannian manifold of dimension $m \geq 2$ satisfying system (1.1). Then, on the set $\{x \in M : u(x) > 0\}$,*

$$\frac{1}{2} \Delta_{-3 \log u} S^\varphi = - \left(S^\varphi - m \frac{\Lambda}{u} \right) \left(S^\varphi - (m-1) \frac{\Lambda}{u} \right) + (m-1) \Delta_{-2 \log u} \left(\frac{\Lambda}{u} \right). \tag{3.1}$$

Proof We take the covariant derivative of the first equation in (1.1), obtaining

$$u_k R_{ij}^\varphi + u R_{ij,k}^\varphi - u_{ijk} = \Lambda_k \delta_{ij}. \tag{3.2}$$

Reversing the roles of j and k in (3.2) and subtracting from (3.2) we deduce

$$u_k R_{ij}^\varphi - u_j R_{ik}^\varphi + u \left(R_{ij,k}^\varphi - R_{ik,j}^\varphi \right) = u_{ijk} - u_{ikj} + \Lambda_k \delta_{ij} - \Lambda_j \delta_{ik}. \tag{3.3}$$

From the standard Ricci commutation relations we know that

$$u_{ikj} = u_{ijk} + u_t R_{tikj};$$

thus, from (3.3) we get

$$u \left(R_{ij,k}^\varphi - R_{ik,j}^\varphi \right) = -u_t R_{tikj} + \Lambda_k \delta_{ij} - \Lambda_j \delta_{ik} - u_k R_{ij}^\varphi + u_j R_{ik}^\varphi. \tag{3.4}$$

Contracting with respect to i and j , using the definition of Ric^φ and the φ -Schur's identity, that is,

$$R_{ij,j}^\varphi = \frac{1}{2} S_i^\varphi - \alpha \varphi_{it}^a \varphi_i^a \tag{3.5}$$

(see for instance [3] for a proof), we obtain

$$\frac{1}{2} u S_k^\varphi = -\alpha \varphi_k^a (u_t \varphi_t^a + u \varphi_{tt}^a) + (m-1) \Lambda_k - u_k S^\varphi, \tag{3.6}$$

and therefore, using the second equation in (1.1), we obtain

$$\frac{1}{2} u S_k^\varphi = (m-1) \Lambda_k - u_k S^\varphi. \tag{3.7}$$

Contracting (3.7) with $\frac{u_k}{u^2}$ yields

$$\frac{1}{2} S_k^\varphi \frac{u_k}{u} = (m-1) \frac{\Lambda_k u_k}{u^2} - S^\varphi \frac{|\nabla u|^2}{u^2}. \tag{3.8}$$

Now we compute the divergence of ∇S^φ , using the relation

$$\frac{1}{2} S_k^\varphi = (m-1) \frac{\Lambda_k}{u} - S^\varphi \frac{u_k}{u},$$

which is obtained from (3.7) dividing by u . After some algebraic manipulations we get

$$\begin{aligned} \frac{1}{2} \Delta S^\varphi &= \frac{m-1}{u} \left(\Delta \Lambda - \frac{\Lambda}{u} \Delta u \right) + \left(\frac{m-1}{u} \Lambda - S^\varphi \right) \frac{\Delta u}{u} + S^\varphi \frac{|\nabla u|^2}{u^2} \\ &\quad - \frac{m-1}{u^2} g(\nabla \Lambda, \nabla u) - \frac{1}{u} g(\nabla S^\varphi, \nabla u). \end{aligned} \tag{3.9}$$

We observe that, tracing the first equation in (1.1), we have

$$\frac{\Delta u}{u} = S^\varphi - m \frac{\Lambda}{u}; \tag{3.10}$$

moreover, an easy computation shows that

$$\Delta_{-2 \log u} \left(\frac{\Lambda}{u} \right) = \frac{1}{u} \left(\Delta \Lambda - \frac{\Lambda}{u} \Delta u \right), \tag{3.11}$$

so that, inserting (3.8), (3.10) and (3.11) into (3.9) we get

$$\frac{1}{2} \left(\Delta S^\varphi + 3g \left(\nabla S^\varphi, \frac{\nabla u}{u} \right) \right) = (m-1) \Delta_{\log u^{-2}} \left(\frac{\Lambda}{u} \right) - \left(S^\varphi - m \frac{\Lambda}{u} \right) \left(S^\varphi - (m-1) \frac{\Lambda}{u} \right),$$

that is, equation (3.1). □

We are now ready for the

Proof of Theorem 1.1 We perform the change of variable

$$u = e^{-f} \tag{3.12}$$

so that system (1.1) becomes

$$\begin{cases} \text{Ric}^\varphi + \text{Hess}(f) - df \otimes df = \Lambda e^f g, \\ \tau(\varphi) = d\varphi(\nabla f); \end{cases} \tag{3.13}$$

having set $v = -S^\varphi$, equation (3.1) becomes

$$\frac{1}{2} \Delta_{3f} v = \left(v + m \Lambda e^f \right) \left(v + (m-1) \Lambda e^f \right) - (m-1) \Delta_{2f} (\Lambda e^f),$$

so that, by assumption (1.7),

$$\frac{1}{2} \Delta_{3f} v \geq \left(v + m \lambda e^f \right) \left(v + (m-1) \Lambda e^f \right). \tag{3.14}$$

From the first equation in (3.13), $\alpha > 0$ and (1.8) we have

$$\text{Ric} + \text{Hess } f - df \otimes df \geq \lambda g. \tag{3.15}$$

We now consider the three possible cases.

1) $\lambda > 0$. Completeness of (M, g) implies, by Theorem 5 of Z. Qian [30], that M is compact. It follows that there exists $x_0 \in M$ such that $-v(x_0) = S_*^\varphi$; we now show that

$$S_*^\varphi \geq (m-1)\lambda. \tag{3.16}$$

By contradiction, suppose this is not the case, that is,

$$S_*^\varphi < (m-1)\lambda$$

or, equivalently,

$$\Omega = \{x \in M : v(x) > -(m - 1)\lambda\} \neq \emptyset. \tag{3.17}$$

Since $(m - 1)\lambda < m\lambda$ we also have

$$v(x) > -m\lambda \quad \text{on } \Omega.$$

Using (1.8) it follows that, on Ω ,

$$v + (m - 1)\Delta e^f \geq v + (m - 1)\lambda > 0$$

and

$$v + m\Delta e^f \geq v + m\lambda > 0.$$

Then (3.14) gives

$$\frac{1}{2}\Delta_{3f}v \geq (v + m\lambda)(v + (m - 1)\lambda) \quad \text{on } \Omega, \tag{3.18}$$

but $x_0 \in \Omega$ and is a maximum of v : it follows that

$$0 \geq (-S_*^\varphi + m\lambda)(-S_*^\varphi + (m - 1)\lambda)$$

from which we immediately deduce the validity of the inequality in the conclusion i).

2) $\lambda = 0$. We show that $S_*^\varphi \geq 0$. By contradiction suppose that this is not the case, so that

$$\Omega = \{x \in M : v(x) > 0\} \neq \emptyset.$$

Using (1.8) it follows that, on Ω ,

$$v + (m - 1)\Delta e^f \geq v > 0 \quad \text{and} \quad v + m\Delta e^f \geq v > 0.$$

Thus (3.14) gives

$$\frac{1}{2}\Delta_{3f}v \geq v^2 \quad \text{on } \Omega; \tag{3.19}$$

from (3.15) we obtain

$$\text{Ric} + \text{Hess}(f) \geq 0. \tag{3.20}$$

By Proposition 8.11 of [1] for $\lambda \leq 0$, from (3.15) we deduce

$$\liminf_{r \rightarrow +\infty} \frac{1}{r^2} \log \text{vol}_f(B_r) < +\infty, \tag{3.21}$$

where $\text{vol}_f(B_r) = \int_{B_r(o)} e^{-f}$ for some fixed origin $o \in M$. Now we first apply Theorem 4.2 of [1] to deduce from (3.19) that $v^* < +\infty$, and then Theorem 4.1, again from [1], to infer $v^* \leq 0$, so that $S_*^\varphi \geq 0$.

3) $\lambda < 0$. We claim that $S_*^\varphi \geq \lambda$. Suppose the contrary: then

$$\Omega = \{x \in M : v(x) > -m\lambda\} \neq \emptyset.$$

Since $(m - 1)\lambda > m\lambda$ we also have

$$v(x) > -(m - 1)\lambda \quad \text{on } \Omega;$$

then (3.14) gives

$$\frac{1}{2}\Delta_{3f}v \geq (v + m\lambda)(v + (m - 1)\lambda) \quad \text{on } \Omega. \tag{3.22}$$

The validity of (3.21) and the completeness of (M, g) enable us to apply again Theorems 4.2 and 4.1 of [1] to conclude. □

4 Proof of Theorem 1.4

We begin with the following observation, whose proof is, by now, quite standard; the aim is to emphasize the role of the assumption $\frac{\Lambda}{u} \in C^0(M)$. Let $M = \text{int}(M) \cup \partial M$.

Lemma 4.1 *Let (M, g) be a compact manifold with boundary $\partial M \neq \emptyset$ and let $u \in C^2(M)$ be a solution of*

$$u\text{Ric}^\varphi - \text{Hess}(u) = \Lambda g \tag{4.1}$$

on M for some $\Lambda \in C^\infty(M)$ and $\varphi : (M, g) \rightarrow (N, h)$ a smooth map. Suppose $u > 0$ on $\text{int}(M)$ and $\partial M = u^{-1}(\{0\})$; then $|\nabla u|$ is a positive constant on each connected component of ∂M and the inclusion $\iota : \partial M \hookrightarrow M$ is totally geodesic, provided

$$\frac{\Lambda}{u} \in C^0(M). \tag{4.2}$$

Proof For the sake of completeness, we include the proof here. Using (4.1) we obtain

$$\nabla|\nabla u|^2 = 2u \left[\text{Ric}^\varphi(\nabla u, \cdot)^\sharp - \frac{\Lambda}{u} \nabla u \right].$$

Hence $u \equiv 0$ on ∂M and (4.2) show that $\nabla|\nabla u|^2 \equiv 0$ on ∂M . We now show that $|\nabla u|(p) \neq 0$ for $p \in \partial M$; towards this aim, let v be the outward unit normal to ∂M at p , $\varepsilon > 0$ sufficiently small and $\gamma : [0, \varepsilon) \rightarrow M$ a unit speed geodesic such that $\gamma(0) = p$ and $\dot{\gamma}(0) = -v$. Define

$$v(t) = (u \circ \gamma)(t) \quad \text{on } [0, \varepsilon).$$

Using (4.1) and the fact that γ is a geodesic we have

$$\begin{cases} v'' = \text{Ric}^\varphi(\dot{\gamma}, \dot{\gamma})v - \Lambda(\gamma), \\ v'(0) = g(\nabla u(p), \dot{\gamma}(0)), \\ v(0) = u(p) = 0. \end{cases}$$

Therefore, if $\nabla u(p) = 0, v'(0) = 0$: in this case we have $v \equiv 0$ on $[0, \varepsilon')$ for some $0 < \varepsilon' \leq \varepsilon$. This is a contradiction, since $\gamma((0, \varepsilon')) \subseteq \text{int}(M)$ and $u > 0$ on $\text{int}(M)$. It follows that $|\nabla u|$ is a positive constant on each connected component of ∂M ; in particular,

$$v = -\frac{\nabla u}{|\nabla u|}$$

is the outward unit normal on ∂M and the second fundamental form in the direction of v is given by

$$\text{II} = \frac{1}{|\nabla u|} \text{Hess}(u)|_{T\partial M \times T\partial M} = \frac{u}{|\nabla u|} \left(\text{Ric}^\varphi - \frac{\Lambda}{u} g \right),$$

so that, using assumption (4.2), we deduce that $\iota : \partial M \hookrightarrow M$ is totally geodesic. □

Lemma 4.2 *Let (M, g) be a manifold of dimension $m \geq 2$ and let $u \in C^2(M), u > 0$ on $\text{int}(M)$, be a solution of*

$$u\text{Ric}^\varphi - \text{Hess}(u) = \Lambda g \tag{4.3}$$

for some $\Lambda \in C^\infty(M)$ and $\varphi : (M, g) \rightarrow (N, h)$ a smooth map. Let Z be the vector field defined on $\text{int}(M)$ by

$$Z = \frac{1}{u} \nabla \left[|\nabla u|^2 - \left(\frac{S^\varphi}{m} - \frac{\Lambda}{u} \right) u^2 \right]. \tag{4.4}$$

Then the divergence of Z is given by

$$\begin{aligned} \operatorname{div}(Z) = & \frac{2}{u} \left[|\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right] + \frac{2}{u} \left(S^\varphi - (m-1) \frac{\Lambda}{u} \right) |\nabla u|^2 + \frac{2\alpha}{u} |d\varphi(\nabla u)|^2 \\ & - \frac{u}{m} \left[\Delta \left(S^\varphi - m \frac{\Lambda}{u} \right) - \frac{2m-3}{u} g \left(\nabla \left(S^\varphi - m \frac{\Lambda}{u} \right), \nabla u \right) \right]. \end{aligned} \tag{4.5}$$

Proof Tracing equation (4.3) we get

$$\Delta u = \frac{A}{m-1} u, \tag{4.6}$$

where we have set

$$A = (m-1) \left(S^\varphi - m \frac{\Lambda}{u} \right) \tag{4.7}$$

to simplify the writing. Using (4.6) we infer

$$\frac{2}{u} g(\nabla \Delta u, \nabla u) = \frac{2}{u^2} \Delta u |\nabla u|^2 + 2g \left(\nabla \left(\frac{A}{m-1} \right), \nabla u \right). \tag{4.8}$$

We rewrite the components of Z in (4.4) in the form

$$Z_k = \frac{2}{u} u_{ik} u_i - \frac{2}{m(m-1)} A u_k - \frac{u}{m(m-1)} A_k.$$

Computing the divergence and recalling the definition of $\operatorname{Ric}^\varphi$ we obtain

$$\begin{aligned} \operatorname{div}(Z) = & \frac{2}{u} \left[|\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right] - \frac{u}{m} \left[\frac{3}{u} g \left(\nabla \left(\frac{A}{m-1} \right), \nabla u \right) + \Delta \left(\frac{A}{m-1} \right) \right] \\ & + \frac{2}{u} \left\{ -\frac{1}{u} \operatorname{Hess}(u)(\nabla u, \nabla u) + \operatorname{Ric}^\varphi(\nabla u, \nabla u) + g(\nabla \Delta u, \nabla u) + \alpha |d\varphi(\nabla u)|^2 \right\}. \end{aligned} \tag{4.9}$$

Using (4.3), (4.6) and (4.8) into (4.9) we deduce

$$\begin{aligned} \operatorname{div}(Z) = & \frac{2}{u} \left[|\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right] + \frac{2}{u} \left(S^\varphi - (m-1) \frac{\Lambda}{u} \right) |\nabla u|^2 + \frac{2}{u} \alpha |d\varphi(\nabla u)|^2 \\ & - \frac{u}{m} \left[\Delta \left(\frac{A}{m-1} \right) - (2m-3) g \left(\nabla \left(\frac{A}{m-1} \right), \frac{\nabla u}{u} \right) \right]. \end{aligned} \tag{4.10}$$

Inserting (4.7) we obtain(4.5). □

We are now ready for the

Proof of Theorem 1.4 We set

$$v = |\nabla u|^2 - \frac{1}{m(m-1)} A u^2, \tag{4.11}$$

where A has been defined in (4.7). We now show that v is constant on M : towards this aim, since M is compact, by Lemma 4.1 we have that there exists a constant $c > 0$ such that

$$|\nabla u|^2 \geq c^2$$

on each connected component of ∂M . We can thus fix $\delta > 0$ sufficiently small such that, for each $0 < \varepsilon \leq \delta$ and

$$M_\varepsilon = \{x \in M : u(x) > \varepsilon\}$$

$\frac{1}{u}$ is positive and bounded on M_ε and ∂M_ε is a C^1 -hypersurface. Under assumptions (1.15) and (1.16) of the theorem, together with $\alpha > 0$, equation (4.5) of Lemma 4.2 gives the validity of

$$\Delta v - \frac{1}{u}g(\nabla u, \nabla v) \geq 0 \quad \text{on } M_\varepsilon. \tag{4.12}$$

Hence, by the maximum principle,

$$\max_{M_\varepsilon} v = \max_{\partial M_\varepsilon} v \quad \text{for } 0 < \varepsilon \leq \delta. \tag{4.13}$$

Since, for $0 < \varepsilon \leq \delta$, $M_\delta \subseteq M_\varepsilon$, from (4.13) we deduce

$$\max_{\partial M_\delta} v \leq \max_{\partial M_\varepsilon} v. \tag{4.14}$$

Now

$$\lim_{\varepsilon \downarrow 0^+} \max_{\partial M_\varepsilon} v \leq \max_{\partial M} |\nabla u|^2, \tag{4.15}$$

and therefore

$$\max_{\partial M_\delta} v \leq \max_{\partial M} |\nabla u|^2. \tag{4.16}$$

Using the previous relation and assumption (1.17) we infer

$$\max_{\partial M_\delta} v \leq \max_M \left\{ -\frac{1}{m(m-1)} Au^2 \right\}. \tag{4.17}$$

We let

$$K = \left\{ p \in M : -\frac{1}{m(m-1)} Au^2(p) = \max_M \left\{ -\frac{1}{m(m-1)} Au^2 \right\} \right\}.$$

K is closed, and therefore compact (since M is compact), and for $p \in K$ we have $u(p) > 0$. It follows that

$$\min_K u = 2\eta \tag{4.18}$$

for some $\eta > 0$. Then, choosing $\delta \leq \eta$,

$$K \subseteq M_\delta \setminus \partial M_\delta,$$

and using (4.17), for each $p \in K$ we have

$$\max_K \left\{ -\frac{1}{m(m-1)} Au^2 \right\} = -\frac{1}{m(m-1)} Au^2(p) \leq v(p) \leq \max_{M_\delta} v \leq \max_M \left\{ -\frac{1}{m(m-1)} Au^2 \right\}.$$

Hence, for each $p \in K$,

$$v(p) = \max_{M_\delta} \left\{ -\frac{1}{m(m-1)} Au^2 \right\},$$

that is, v assumes its absolute maximum at the interior point p of M_δ . Using (4.12) and the maximum principle we deduce that

$$v \equiv \max_M \left\{ -\frac{1}{m(m-1)} Au^2 \right\} \tag{4.19}$$

on the connected components C_δ of M_δ such that $C_\delta \cap K \neq \emptyset$. Letting $\delta \downarrow 0^+$ we infer that (4.19) holds on M and v is constant; indeed, let $p, q \in \text{int}(M)$: then, by connectedness,

there exists a path $\gamma : [0, 1] \rightarrow \text{int}(M)$ such that $\gamma(0) = p, \gamma(1) = q$. Since $\gamma([0, 1])$ is compact, then $\inf_{\gamma([0,1])} u > 0$; it follows that $\gamma([0, 1]) \subset M_\delta$ for $0 < \delta \ll 1$. Hence q and p are in the same connected component of p . Now choose $p \in K$; since v is constant, the vector field Z defined in (4.4) is identically null and (4.10) gives

$$\text{Hess}(u) = \frac{\Delta u}{m} g \tag{4.20}$$

and

$$\left(S^\varphi - (m - 1) \frac{\Lambda}{u} \right) |\nabla u|^2 = 0 \tag{4.21}$$

on $\text{int}(M)$ and, by continuity, on M itself. From (4.6) and (4.7) we have

$$\Delta u = \left(S^\varphi - m \frac{\Lambda}{u} \right) u = u S^\varphi - m \Lambda,$$

so that inserting into (4.20) yields

$$\text{Hess}(u) = \left(\frac{S^\varphi}{m} u - \Lambda \right) g \quad \text{on } M. \tag{4.22}$$

Taking the gradient of the constant function v , from (4.11) we obtain

$$0 = 2 \text{Hess}(u)(\nabla u, \cdot)^\sharp - \frac{u^2}{m(m-1)} \nabla A - \frac{2}{m(m-1)} Au \nabla u. \tag{4.23}$$

Thus, using (4.22) into (4.23) and (4.7) we infer

$$\nabla \left(m \frac{\Lambda}{u} - S^\varphi \right) \equiv 0 \quad \text{on } M.$$

Connectedness of M and (1.22) imply

$$m \frac{\Lambda}{u} - S^\varphi = C^2$$

for some constant $C \neq 0$. We can thus rewrite (4.22) as

$$\text{Hess}(u) = -\frac{C^2}{m} u g,$$

with u non-constant. Furthermore, since $\frac{\Lambda}{u} \in C^0(M), \iota : \partial M \hookrightarrow M$ is totally geodesic, and therefore we can apply Lemma 3 of Reilly ([31]) to deduce that M is isometric to $S_+^m(C^2)$. \square

Remark 4.3 Note that assumption (1.15) and conclusion (1.18) of Theorem 1.4 imply

$$(0 <) C^2 \leq \frac{\Lambda}{u} \quad \text{on } M.$$

5 Proof of Theorem 1.5

For the proof of Theorem 1.5 we need a series of formulas.

Proposition 5.1 *Let (M, g) be a manifold of dimension $m \geq 3$, X a conformal vector field on $M, \alpha \in \mathbb{R}, \alpha \neq 0$ and $\varphi : (M, g) \rightarrow (N, h)$ a smooth map. Set*

$$w = \text{div}(X). \tag{5.1}$$

Then

$$\begin{aligned} \text{Hess}(w) &= \frac{S^\varphi}{(m-1)(m-2)}wg + \frac{m}{2(m-1)(m-2)}g(\nabla S^\varphi, X)g - \frac{m}{m-2}\mathcal{L}_X\text{Ric}^\varphi \\ &\quad - \frac{m}{m-2}\alpha\left(\mathcal{L}_X(\varphi^*h) - \frac{1}{2(m-1)}\text{tr}(\mathcal{L}_X(\varphi^*h))g\right), \end{aligned} \tag{5.2}$$

where \mathcal{L}_X denotes the Lie derivative in the direction of X and tr is the trace with respect to the metric g . In particular,

$$\Delta w = -\frac{S^\varphi}{m-1}w - \frac{m}{2(m-1)}g(\nabla S^\varphi, X) - \frac{m}{2(m-1)}\alpha \text{tr}(\mathcal{L}_X(\varphi^*h)). \tag{5.3}$$

Remark 5.2 Note that, when φ is constant, (5.2) and (5.3) reduce, respectively, to

$$\text{Hess}(w) = \frac{S}{(m-1)(m-2)}wg + \frac{m}{2(m-1)(m-2)}g(\nabla S, X)g - \frac{m}{m-2}\mathcal{L}_X\text{Ric}, \tag{5.4}$$

$$\Delta w = -\frac{S}{m-1}w - \frac{m}{2(m-1)}g(\nabla S, X). \tag{5.5}$$

These two formulas are known (see for instance the book [32]) and we shall use them to prove (5.2). However, for the sake of completeness we shall provide a proof here.

Proof (of Proposition 5.1) To show the validity of (5.2) we shall use (5.4); thus, we begin to prove the latter, and towards this aim we first establish (5.5). Since X is conformal,

$$X_{ij} + X_{ji} = \frac{2}{m}w\delta_{ij}; \tag{5.6}$$

contracting (5.6) with the Ricci tensor we infer

$$R_{ij}X_{ij} = \frac{w}{m}S. \tag{5.7}$$

By the definition of w , and using the commutation relation

$$X_{ijk} - X_{ikj} = X_iR_{tijk}, \tag{5.8}$$

we compute

$$\begin{aligned} \Delta w &= (X_{ii})_{jj} = (X_{iij})_j = (X_{iji} + X_kR_{kii})_j = X_{ijij} - (X_iR_{ij})_j \\ &= (X_{ij})_{ij} - R_{ij,j}X_i - R_{ij}X_{ij}. \end{aligned}$$

With the aid of (5.6), Schur’s identity and (5.7), the last equality can be written in the form

$$\begin{aligned} \Delta w &= \left(-X_{ji} + \frac{2}{m}w\delta_{ij}\right)_{ij} - \frac{1}{2}S_iX_i - \frac{S}{m}w \\ &= -X_{jii} + \frac{2}{m}\Delta w - \frac{1}{2}S_iX_i - \frac{S}{m}w. \end{aligned} \tag{5.9}$$

Using the commutation relation (see e.g. [8])

$$X_{ijkl} - X_{ijlk} = X_{tj}R_{tikl} + X_{it}R_{tjkl}$$

we immediately obtain

$$X_{jii} = X_{jji}, \tag{5.10}$$

and inserting into (5.9) we infer

$$\Delta w = -(X_{jij})_i + \frac{2}{m} \Delta w - \frac{1}{2} S_i X_i - \frac{S}{m} w.$$

Using once again (5.8) and (5.6), Schur’s identity and (5.7) we get

$$\begin{aligned} \Delta w &= -(X_{jji} + X_k R_{kji})_i + \frac{2}{m} \Delta w - \frac{1}{2} S_i X_i - \frac{S}{m} w \\ &= -X_{jji} - (R_{ki} X_k)_i + \frac{2}{m} \Delta w - \frac{1}{2} S_i X_i - \frac{S}{m} w \\ &= -\Delta w - R_{ki,i} X_k - R_{ki} X_{ki} + \frac{2}{m} \Delta w - \frac{1}{2} S_i X_i - \frac{S}{m} w \\ &= -\Delta w - S_i X_i - \frac{2}{m} S w + \frac{2}{m} \Delta w, \end{aligned}$$

that is, (5.5). We now compute Hess(w).

$$\begin{aligned} w_{ij} &= X_{l ij} = X_{l il} + X_{t j} R_{t l i} + X_t R_{t l i, j} \\ &= X_{l i j} + X_{t i} R_{t l j} + X_{l t} R_{t i j} - X_{t j} R_{t i} - X_t R_{t i, j} \\ &= \left(-X_{i l} + \frac{2}{m} X_{t i} \delta_{i l} \right)_{j l} - X_{t i} R_{t j} - X_{t j} R_{t i} + X_{l t} R_{t i j} - X_t R_{t i, j} \\ &= -X_{i l j} + \frac{2}{m} X_{t t j} - X_{t i} R_{t j} - X_{t j} R_{t i} + X_{l t} R_{t i j} - X_t R_{t i, j} \\ &= -X_{i j l} - X_{t l} R_{t i j} - X_t R_{t i j, l} + \frac{2}{m} X_{t t j} - X_t R_{t i, j} - X_{t i} R_{t j} - X_{t j} R_{t i} + X_{l t} R_{t i j} \\ &= -X_{i j l} + \frac{2}{m} X_{t t j} - X_{t l} R_{t i j} + X_{l t} R_{t i j} - X_t R_{i j, t} + X_t R_{t j, i} - X_t R_{t i, j} \\ &\quad - X_{t i} R_{t j} - X_{t j} R_{t i} \\ &= -X_{i j l} + \frac{2}{m} X_{t t j} - X_{l t} R_{t i j} + X_t (R_{t j, i} - R_{t i, j}) - (X_t R_{i j, t} + X_{t i} R_{t j} + X_{t j} R_{t i}) \\ &= -X_{i j l} + \frac{2}{m} X_{t t j} - X_{l t} R_{t i j} + X_t (R_{t j, i} - R_{t i, j}) - (\mathcal{L}_X \text{Ric})_{i j}. \end{aligned}$$

Since $w_{ij} = w_{ji}$, symmetrizing with respect to the indexes i and j we get

$$\begin{aligned} X_{l ij} &= -\frac{1}{2} (X_{ij} + X_{ji})_{l l} + \frac{2}{m} X_{l ij} - (\mathcal{L}_X \text{Ric})_{i j} \\ &= -\frac{1}{m} X_{k k l} \delta_{i j} + \frac{2}{m} X_{l ij} - (\mathcal{L}_X \text{Ric})_{i j}. \end{aligned}$$

Hence, rearranging the terms and using (5.5), we obtain (5.4).

Starting from (5.4), we now prove (5.2). Towards this aim, we observe that from $S = S^\varphi + \alpha |d\varphi|^2$, differentiating and contracting with X we have

$$S_t X_t = S_t^\varphi X_t + 2\alpha \varphi_{k t}^a \varphi_k^a X_t. \tag{5.11}$$

From the symmetry of $\nabla d\varphi$ we get

$$\varphi_{k t}^a X_t = \varphi_{t k}^a X_t = (\varphi_t^a X_t)_k - \varphi_t^a X_{t k}, \tag{5.12}$$

and therefore

$$2\alpha \varphi_{k t}^a \varphi_k^a X_t = 2\alpha (\varphi_t^a X_t)_k \varphi_k^a - 2\alpha \varphi_t^a \varphi_k^a X_{t k}.$$

Thus, using (5.6),

$$2\alpha\varphi_{kt}^a\varphi_k^aX_t = 2\alpha(\varphi_t^aX_t)_k\varphi_k^a - \frac{2\alpha}{m}w|d\varphi|^2. \tag{5.13}$$

Inserting (5.13) into (5.11) we infer

$$S_tX_t = S_t^\varphi X_t + 2\alpha(\varphi_t^aX_t)_k\varphi_k^a - \frac{2\alpha}{m}w|d\varphi|^2. \tag{5.14}$$

Now we analyze $\mathcal{L}_X\text{Ric}$: since

$$R_{ij} = R_{ij}^\varphi + \alpha\varphi_i^a\varphi_j^a,$$

we have

$$R_{ij,t} = R_{ij,t}^\varphi + \alpha(\varphi_{it}^a\varphi_j^a + \varphi_i^a\varphi_{jt}^a).$$

Hence, using (5.12),

$$\begin{aligned} X_tR_{ij,t} + X_{ti}R_{tj} + X_{tj}R_{it} &= X_tR_{ij,t}^\varphi + \alpha X_t(\varphi_{it}^a\varphi_j^a + \varphi_i^a\varphi_{jt}^a) + X_{ti}R_{tj}^\varphi \\ &\quad + \alpha X_{ti}\varphi_t^a\varphi_j^a + X_{tj}R_{it}^\varphi + \alpha X_{ij}\varphi_i^a\varphi_t^a \\ &= (\mathcal{L}_X\text{Ric}^\varphi)_{ij} + \alpha(\varphi_t^aX_t)_i\varphi_j^a + \alpha(\varphi_t^aX_t)_j\varphi_i^a, \end{aligned}$$

that is,

$$(\mathcal{L}_X\text{Ric})_{ij} = (\mathcal{L}_X\text{Ric}^\varphi)_{ij} + \alpha(\mathcal{L}_X(\varphi^*h))_{ij}. \tag{5.15}$$

Inserting (5.11) and (5.15) in the local form of (5.4), we find (5.2). □

As a consequence of Proposition 5.1 we have the following

Corollary 5.3 *In the assumptions of Proposition (5.1), suppose*

$$\mathcal{L}_X(\varphi^*h) = 0. \tag{5.16}$$

Then

$$\text{Hess}(w) = \frac{S^\varphi}{(m-1)(m-2)}wg + \frac{m}{2(m-1)(m-2)}g(\nabla S^\varphi, X) - \frac{m}{m-2}\mathcal{L}_X\text{Ric}^\varphi, \tag{5.17}$$

$$\Delta w = -\frac{S^\varphi}{m-1}w - \frac{m}{2(m-1)}g(\nabla S^\varphi, X). \tag{5.18}$$

Remark 5.4 Equation (5.16) geometrically means that φ^*h is invariant under the flow of X .

Note that, for a conformal vector field X ,

$$\mathcal{L}_X\text{Ric}^\varphi = \mathcal{L}_X\overset{\circ}{\text{Ric}}^\varphi + \frac{1}{m}\left(g(\nabla S^\varphi, X) + \frac{2}{m}S^\varphi w\right)g, \tag{5.19}$$

where $\overset{\circ}{\text{Ric}}^\varphi$ is the traceless φ -Ricci tensor, so that (5.17) can be written as

$$\text{Hess}(w) - w\text{Ric}^\varphi - \Delta wg = \frac{1}{2}g(\nabla S^\varphi, X)g - \frac{m}{m-2}\mathcal{L}_X\overset{\circ}{\text{Ric}}^\varphi - w\overset{\circ}{\text{Ric}}^\varphi. \tag{5.20}$$

Furthermore, suppose that X is conformal and it satisfies

$$\nabla(d\varphi(X)) = 0; \tag{5.21}$$

then it also satisfies (5.16). Indeed, we have

$$(\mathcal{L}_X(\varphi^*h))_{ij} = (\varphi_t^a X_t)_i \varphi_j^a + (\varphi_t^a X_t)_j \varphi_i^a. \tag{5.22}$$

From the identity

$$((\varphi_t^a X_t)_s)_s = 0,$$

conformality of X , the commutation relations

$$\varphi_{ijk}^a = \varphi_{ikj}^a + R_{tijk} \varphi_t^a + {}^N R_{abcd} \varphi_t^b \varphi_j^c \varphi_k^d$$

and

$$X_{tst} = X_{tts} + X_i R_{is}$$

we deduce

$$\begin{aligned} 0 &= ((\varphi_t^a X_t)_s)_s \\ &= \varphi_{tss}^a X_t + \varphi_{ts}^a X_{ts} + \varphi_{st}^a X_{st} + \varphi_t^a X_{tss} \\ &= \varphi_{sts}^a X_t + \frac{2}{m} w \varphi_{tt}^a + \varphi_t^a X_{tts} \\ &= X_t \left(\varphi_{sst}^a + \varphi_k^a R_{kt} + {}^N R_{abcd} \varphi_s^b \varphi_t^c \varphi_s^d \right) + \frac{2}{m} w \varphi_{tt}^a + \varphi_t^a \left(-X_{sts} + \frac{2}{m} w_t \right) \\ &= X_t \varphi_{sst}^a + X_t \varphi_k^a R_{kt} + {}^N R_{abcd} \varphi_s^b \varphi_t^c \varphi_s^d X_t + \frac{2}{m} w \varphi_{tt}^a + \frac{2}{m} \varphi_t^a w_t - \varphi_t^a (X_{sst} + X_k R_{kt}) \\ &= X_t \varphi_{sst}^a + {}^N R_{abcd} \varphi_s^b \varphi_t^c \varphi_s^d X_t + \frac{2}{m} w \varphi_{tt}^a + \frac{2}{m} \varphi_t^a w_t - \varphi_t^a w_t \\ &= X_t \varphi_{sst}^a + {}^N R_{abcd} \varphi_s^b \varphi_t^c \varphi_s^d X_t + \frac{2}{m} w \varphi_{tt}^a - \frac{m-2}{m} \varphi_t^a w_t. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{m-2}{m} (w \varphi_{tt}^a + w_t \varphi_t^a) &= X_t \varphi_{sst}^a + {}^N R_{abcd} \varphi_s^b \varphi_t^c \varphi_s^d X_t + \frac{2}{m} w \varphi_{tt}^a - \frac{m-2}{m} \varphi_t^a w_t \\ &\quad + \frac{m-2}{m} (w \varphi_{tt}^a + w_t \varphi_t^a) \\ &= w \varphi_{tt}^a + \varphi_{sst}^a X_t + {}^N R_{abcd} \varphi_s^b \varphi_t^c \varphi_s^d X_t \end{aligned}$$

that gives the validity of the formula

$$w\tau(\varphi) + d\varphi(\nabla w) = \frac{m}{m-2} [w\tau(\varphi) + g(\nabla\tau(\varphi), X) + {}^N R_{abcd} \varphi_s^b \varphi_t^c \varphi_s^d X_t E_a], \tag{5.23}$$

where $\{E_a\}$ is the frame dual to the local orthonormal coframe $\{\omega^a\}$ considered on N .

Now, the condition

$$X \in \text{Ker}(d\varphi) \tag{5.24}$$

implies (5.21), and in this case (5.23) simplifies to

$$w\tau(\varphi) + d\varphi(\nabla w) = \frac{m}{m-2} [w\tau(\varphi) + g(\nabla\tau(\varphi), X)]. \tag{5.25}$$

We now state the following consequence of equations (5.20) and (5.25).

Proposition 5.5 *Let (M, g) be a harmonic-Einstein manifold of dimension $m \geq 3$, with respect to $\varphi : (M, g) \rightarrow (N, h)$ and $\alpha \in \mathbb{R}, \alpha \neq 0$. Let X be a conformal vector field on M such that $X \in \text{Ker}(d\varphi)$. Then $w = \text{div } X$ solves the system*

$$\begin{cases} \text{Hess}(w) - \Delta w g - w \text{Ric}^\varphi = 0, \\ w \tau(\varphi) + d\varphi(\nabla w) = 0. \end{cases} \tag{5.26}$$

Remark 5.6 Of course, $w \neq 0$ if and only if X is not a Killing field.

Proof Simply observe that, since (M, g) is harmonic-Einstein, then $\overset{\circ}{\text{Ric}}^\varphi \equiv 0, \tau(\varphi) \equiv 0$ and S^φ is constant. Then the conclusion follows from equations (5.20) and (5.25). \square

Remark 5.7 Note that, in the present case, the second equation in (5.26) becomes $d\varphi(\nabla w) = 0$, that is, $\nabla w \in \text{Ker}(d\varphi)$.

Here is another consequence of Proposition 5.1, extending a well-known result of Yano and Nagano [33]. The role of the Einstein condition in [33] is here replaced by the harmonic-Einstein request.

Proposition 5.8 *Let (M, g) be a complete manifold of dimension $m \geq 2$, which is harmonic-Einstein with respect to $\varphi : (M, g) \rightarrow (N, h)$ and $\alpha \in \mathbb{R}, \alpha \neq 0$. If there exists a conformal, non-Killing vector field X on M satisfying*

$$\nabla d\varphi(X) = 0, \tag{5.27}$$

then one the following alternatives occurs:

- i) (M, g) is φ -Ricci-flat;
- ii) (M, g) is isometric to a Euclidean sphere $\mathbb{S}^m(k)$ of constant sectional curvature $k = \frac{S^\varphi}{m(m-1)} > 0$;
- iii) (M, g) is isometric to a Hyperbolic space $\mathbb{H}^m(k)$ of constant sectional curvature $k = \frac{S^\varphi}{m(m-1)} < 0$.

Proof From (5.27) we know that $\mathcal{L}_X(\varphi^*h) = 0, S^\varphi$ is constant and $\overset{\circ}{\text{Ric}}^\varphi \equiv 0$, thus from (5.20) we deduce

$$\text{Hess}(w) - \Delta w g - \frac{S^\varphi}{m} w g = 0, \tag{5.28}$$

with $w = \text{div } X$. Tracing (5.28) gives

$$\Delta w = -\frac{S^\varphi}{m-1} w; \tag{5.29}$$

inserting into (5.28) yields

$$\text{Hess}(w) + \frac{S^\varphi}{m(m-1)} w g = 0. \tag{5.30}$$

Suppose now that (M, g) is not φ -Ricci flat: then w cannot be constant, because otherwise $w = a \neq 0$, since X is not Killing, and from (5.29) $S^\varphi \equiv 0$, contradicting the fact that (M, g) is not φ -Ricci flat. Thus w is a non-constant solution of (5.30) and either $S^\varphi > 0$ or $S^\varphi < 0$, corresponding, by a theorem of Kanai [22], to cases ii) and iii), respectively. \square

Remark 5.9 In particular, in cases ii) and iii), the constant sectional curvatures are given by $\frac{S^\varphi}{m(m-1)}$ and

$$\text{Ric} = \frac{S^\varphi}{m} g.$$

Thus, using the harmonic-Einstein condition, we deduce

$$\frac{S^\varphi}{m} g = \text{Ric}^\varphi = \text{Ric} - \alpha\varphi^*h = \frac{S^\varphi}{m} g - \alpha\varphi^*h$$

and since $\alpha \neq 0$

$$\varphi^*h \equiv 0,$$

so that φ is constant.

Remark 5.10 In case M is compact, only alternative ii) can occur. Indeed, integrating (5.29) on M , we obtain

$$\int_M |\nabla w|^2 = \frac{S^\varphi}{m-1} \int_M w^2,$$

showing, since $w \not\equiv 0$, that $S^\varphi \geq 0$. However, $S^\varphi = 0$ implies, by (5.29), $\Delta w = 0$ on M , so that w is constant. Thus

$$w \text{Vol}(M) = \int_M \text{div } X = 0,$$

and then $w = 0$, which is a contradiction. It follows that $S^\varphi > 0$.

We need now a further preliminary result.

Lemma 5.11 *Let (M, g) be a manifold of dimension $m \geq 2$, $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map and X a vector field on M . Then*

$$\text{div} \left(\overset{\circ}{\text{Ric}}^\varphi(X, \cdot)^\sharp \right) = \frac{(m-2)}{2m} g(X, \nabla S^\varphi) + \frac{1}{2} \text{tr} \left(\overset{\circ}{\text{Ric}}^\varphi \circ \mathcal{L}_X g \right) - \alpha h(\tau(\varphi), d\varphi(X)). \tag{5.31}$$

Proof We recall the φ -Schur's identity (see equation (3.5))

$$R_{ij,j}^\varphi = \frac{1}{2} S_i^\varphi - \alpha \varphi_{it}^a \varphi_i^a$$

and we compute

$$\begin{aligned} \left(\overset{\circ}{R}_{ij}^\varphi X_i \right)_j &= \left(R_{ij,j}^\varphi - \frac{S_i^\varphi}{m} \right) X_i + R_{ij}^\varphi X_{ij} - \frac{S^\varphi}{m} \delta_{ij} X_{ij} \\ &= X_i \left(\frac{1}{2} S_i^\varphi - \alpha \varphi_{it}^a \varphi_i^a \right) - \frac{1}{m} S_i^\varphi X_i + \frac{1}{2} R_{ij}^\varphi (X_{ij} + X_{ji}) - \frac{1}{2} \frac{S^\varphi}{m} \delta_{ij} (X_{ij} + X_{ji}) \\ &= \frac{m-2}{2m} g(X, \nabla S^\varphi) - \alpha h(\tau(\varphi), d\varphi(X)) + \frac{1}{2} \text{tr} \left(\overset{\circ}{\text{Ric}}^\varphi \circ \mathcal{L}_X g \right), \end{aligned}$$

that is, (5.31). □

We are now ready for the

Proof (of Theorem 1.5) We let u be the solution of (1.24) and we set $X = \nabla u$ in (5.31) to obtain

$$\operatorname{div} \left(\operatorname{Ric}^\circ(X, \cdot)^\sharp \right) = \frac{(m-2)}{2m} g(X, \nabla S^\varphi) + \left(\left| \operatorname{Ric}^\circ \right|^2 + \alpha |\tau(\varphi)|^2 \right) u. \tag{5.32}$$

Since $\left| \operatorname{Ric}^\circ \right| \in L^\infty(M)$ and $|\nabla u| \in L^1(M)$, we have that

$$\liminf_{R \rightarrow \infty} \int_{B_{2R} \setminus B_R} \operatorname{Ric}^\circ(X, \cdot)^\sharp dV_g = 0,$$

where B_R denotes the geodesic ball of radius R centered at some $p \in M$. As a consequence, we can apply an extension of Stokes’ theorem for non-compact manifolds due to Karp ([23]), which implies

$$\int_M \operatorname{div} \left(\operatorname{Ric}^\circ(X, \cdot)^\sharp \right) = 0.$$

Thus, integrating (5.32) on M and using the constancy of S^φ we infer

$$\int_M \left(\left| \operatorname{Ric}^\circ \right|^2 + \alpha |\tau(\varphi)|^2 \right) u = 0.$$

Since $\alpha > 0$ and $u > 0$ on M , from the above we obtain

$$\begin{cases} \operatorname{Ric}^\circ = \frac{S^\varphi}{m} g, \\ \tau(\varphi) = 0, \end{cases}$$

that is, (M, g) is harmonic-Einstein. Suppose now that u is non-constant; from the first equation in (1.24) and the (just proved) validity of (1.23) we have

$$\operatorname{Hess}(u) = \left(\frac{S^\varphi}{m} - \frac{\Lambda}{u} \right) u g,$$

so that ∇u is a conformal vector field on M . Furthermore, when M is compact

$$d\varphi(\nabla u) = -\tau(\varphi)u \equiv 0,$$

and ∇u is non-Killing because otherwise, from (1.23), $\operatorname{Hess}(u) \equiv 0$ and, by compactness of M , u would be constant, a contradiction. From Proposition 5.8 and Remark 5.7 applied to $X = \nabla u$, we have the validity of conclusion ii) of Proposition 5.8, and hence the theorem. When M is non-compact, we conclude by Proposition 5.8 if ∇u is non-Killing. On the other hand, if it is a Killing vector field $\Delta u = 0$, that is

$$0 = \Delta u = S^\varphi u - \Lambda m.$$

As a consequence, we have

$$\operatorname{Hess}(u) = 0$$

and by Theorem B of [22] (see also [33] for a proof) we conclude that (M, g) is the the Riemannian product of a complete Riemannian manifold and the real line with its standard metric. □

6 Proof of Theorem 1.8

This section is dedicated to the proof of Theorem 1.8, which provides an estimate for the first Jacobi eigenvalue of the boundary, with respect to a suitable metric, of a Riemannian manifold satisfying the first equation of (1.1). The proof is based on the existence of a solution of the φ -Yamabe problem.

Let (M, g) be a Riemannian manifold of dimension $m \geq 3$; we will denote by $[g]$ the conformal class of the metric g , i.e.,

$$[g] := \left\{ \tilde{g} = e^{2f} g, f \in C^\infty(M) \right\}.$$

Yamabe, Trudinger, Aubin, and Schoen (see [24] and the references therein for more details) proved that on a closed Riemannian manifold (M, g) it is always possible to find a metric $\tilde{g} \in [g]$ such that the scalar curvature of \tilde{g} is constant. The problem of finding such a metric is known as the *Yamabe problem*: its solution involves variational methods applied to the normalized total scalar curvature functional.

Under the conformal change $\tilde{g} = u^{\frac{4}{m-2}} g$, for $m \geq 3$, $u \in C^\infty(M)$ positive, the scalar curvature transforms according to the well-known formula

$$S_{\tilde{g}} = u^{-\frac{m+2}{m-2}} \left(\frac{-4(m-1)}{m-2} \Delta_g u + S_g u \right) =: u^{-\frac{m+2}{m-2}} L_g u, \tag{6.1}$$

where the operator L_g is called the *conformal Laplacian*. Solving the Yamabe problem, for $m \geq 3$, is equivalent to finding a positive solution of the equation

$$L_g u = \lambda u^{\frac{m+2}{m-2}}, \text{ with } \lambda \in \mathbb{R}.$$

Note that the above equation is the Euler-Lagrange equation of the normalized Einstein-Hilbert functional

$$\mathfrak{S}(\tilde{g}) = \text{Vol}_{\tilde{g}}(M)^{-\frac{m-2}{m}} \int_M \tilde{S} dV_{\tilde{g}},$$

where $\tilde{g} \in [g]$ and \tilde{S} denotes the scalar curvature with respect to the metric \tilde{g} .

To prove Theorem 1.8 we introduce a suitable modification of the Yamabe invariant, which we will need to obtain a metric on ∂M with constant φ -scalar curvature. We now fix some notation. Let (M, g) be a compact Riemannian manifold of dimension $m \geq 3$; the Yamabe invariant of (M, g) is defined as

$$Y(M, [g]) = \inf_{\tilde{g} \in [g]} \text{Vol}(M)^{-\frac{m-2}{m}} \int_M \tilde{S} dV_{\tilde{g}}.$$

Let $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map, where (N, h) is a second Riemannian manifold; in analogy with the usual Yamabe invariant, define the φ -Yamabe invariant by setting

$$Y^\varphi(M, [g]) = \inf_{\tilde{g} \in [g]} \text{Vol}(M)^{-\frac{m-2}{m}} \int_M \tilde{S} \tilde{\varphi} dV_{\tilde{g}}, \tag{6.2}$$

where $\tilde{\varphi}$ is the map $\varphi : (M, \tilde{g}) \rightarrow (N, h)$. By definition,

$$\begin{aligned} Y^\varphi(M, [g]) &= \inf_{\tilde{g} \in [g]} \text{Vol}(M)^{-\frac{m-2}{m}} \int_M \tilde{S} \tilde{\varphi} dV_{\tilde{g}} \\ &= \inf_{\tilde{g} \in [g]} \text{Vol}(M)^{-\frac{m-2}{m}} \left(\int_M \tilde{S} dV_{\tilde{g}} - \int_M \alpha |\tilde{\varphi}|_{\tilde{g}}^2 dV_{\tilde{g}} \right). \end{aligned}$$

Observe that, when $\alpha > 0$, we have

$$Y^\varphi(M, [g]) \leq Y(M, [g]) \leq Y(\mathbb{S}^m, g_{\mathbb{S}^m}),$$

so that

$$Y^\varphi(M, [g]) \leq m(m - 1)\omega_m^{\frac{2}{m}}.$$

Remark 6.1 We observe that, in analogy with the Yamabe problem, Gursky and LeBrun in [17, 18] introduced the following modification of the Yamabe invariant

$$\widehat{Y}(M, [g]) := \inf_{u \in W^{1,2}(M)} \frac{\int_M u \mathcal{L}_g^t u \, dV_g}{\left(\int_M u^{2m/(m-2)} \, dV_g\right)^{(m-2)/2}},$$

where

$$\mathcal{L}_g^t = -\frac{4(m - 1)}{m - 2} \Delta_g + S_g + t|W_g|_g$$

is a modification of the usual conformal Laplacian. Note that $\widehat{Y}(M, [g])$ is conformally invariant; moreover, making use of a regularization argument, they proved that $\widehat{Y}(M, [g]) \leq 0$ and then the problem of finding a conformal metric $\tilde{g} \in [g]$ which attains $\widehat{Y}(M, [g])$ can be solved adapting the techniques in [24].

For $m \geq 3$, let $\tilde{g} = u^{\frac{4}{m-2}}g$, $u \in C^\infty(M)$ positive; setting $\tilde{\varphi}$ for the map φ , considered ad a map from (M, \tilde{g}) to (N, g_N) , using (6.1) we have

$$\tilde{S}^{\tilde{\varphi}} = u^{-\frac{m+2}{m-2}} \left(-\frac{4(m - 1)}{m - 2} \Delta_g u + S^\varphi u \right).$$

We can rewrite the φ -Yamabe invariant $Y^\varphi(M, [g])$ in the form

$$\begin{aligned} Y^\varphi(M, [g]) &= Y^\varphi(u) = \inf_{u \neq 0, u \in W^{1,2}} \frac{\int_M \left(-\frac{4(m-1)}{m-2} u \Delta_g u + S^\varphi u^2 \right) dV_g}{\|u\|_{L^{\frac{2m}{m-2}}}^2} \\ &= \inf_{u \neq 0, u \in W^{1,2}} \frac{\int_M u \mathcal{L}_g^\varphi u \, dV_g}{\|u\|_{L^{\frac{2m}{m-2}}}^2}, \end{aligned} \tag{6.3}$$

where \mathcal{L}_g^φ is the second order elliptic operator

$$\mathcal{L}_g^\varphi := -\frac{4(m - 1)}{m - 2} \Delta_g + S^\varphi; \tag{6.4}$$

As a direct consequence of the transformation law of the φ -scalar curvature we have the following

Lemma 6.2 *Let $m \geq 3$ and $\tilde{g} = u^{\frac{4}{m-2}}g$; then $\mathcal{L}_g^{\tilde{\varphi}}\psi = u^{-\frac{m+2}{m-2}}\mathcal{L}_g^\varphi(\psi u)$.*

Adapting an argument due to Gursky ([17]), we obtain the next

Lemma 6.3 *Each conformal class of a compact manifold (M^m, g) , $m \geq 3$, admits a C^∞ metric $\tilde{g} = u^{\frac{4}{m-2}}g$ with either $\tilde{S}^{\tilde{\varphi}} > 0$, $\tilde{S}^{\tilde{\varphi}} < 0$ or $\tilde{S}^{\tilde{\varphi}} \equiv 0$; moreover, this three possibilities are mutually exclusive.*

Proof The proof is based on that of Proposition 3.2 of [17]. Let $\mu(g)$ be the principal eigenvalue of \mathcal{L}_g^φ , namely

$$\mu(g) := \inf_{u \neq 0, u \in W^{1,2}} \frac{\int_M u \mathcal{L}_g^\varphi u \, dV_g}{\|u\|_2^2}.$$

Since \mathcal{L}_g^φ is conformally covariant, the sign of $\mu(g)$ is invariant under conformal changes of g . Let ψ denote the first eigenfunction of the eigenvalue $\mu(g)$: by the minimum principle ψ can be assumed positive. Then from the equation

$$\mathcal{L}_g^\varphi \psi = \mu(g) \psi,$$

we have

$$\Delta \psi = -\frac{(m-2)}{4(m-1)} (S^\varphi \psi + \mu(g) \psi)$$

and since S^φ and $\mu(g)$ are smooth, by a bootstrap argument we conclude that ψ is a smooth function. Let $\tilde{g} = \psi^{\frac{4}{m-2}} g$: then

$$\begin{aligned} \tilde{S}^{\tilde{\varphi}} &= \psi^{-\frac{m+2}{m-2}} \mathcal{L}_g^\varphi \psi \\ &= \mu(g) \|\psi\|_2^2 \psi^{-\frac{4}{m-2}}. \end{aligned}$$

Therefore, $\tilde{S}^{\tilde{\varphi}}$ has the same sign of $\mu(g)$: positive, negative or identically zero, and since the sign $\mu(g)$ is invariant under conformal changes of the metric, these three alternatives are mutually exclusive. □

Given (M, g) a closed Riemannian manifold of dimension $m \geq 3$, it is always possible to find a metric $\tilde{g} \in [g]$ such that $\tilde{S}^{\tilde{\varphi}}$ is constant. To prove this, it is sufficient to guarantee

$$Y^\varphi(M, [g]) < Y(S^m, [g_{S^m}])$$

and to apply the arguments in [24] and conclude that in every conformal class there exists a metric with constant φ -scalar curvature. However, for the sake of completeness we provide a proof, which is obtained adapting an argument of Lee and Parker ([24]) to our case. On a compact manifold the Yamabe problem can be solved using a variational approach, provided $Y(M, [g]) < Y(S^m, g_{S^m})$. We focus on Section 4 of [24]. Note that, by construction, for $\alpha > 0$

$$Y^\varphi(M, [g]) \leq Y(M, [g]).$$

Hence, if

$$Y(M, [g]) < Y(S^m, [g_{S^m}])$$

we have

$$Y(M, [g])^\varphi < Y(S^m, [g_{S^m}]).$$

First let us fix some notation, let

$$Q_{\tilde{\varphi}}(\tilde{g}) = \frac{\int_M \tilde{S}^{\tilde{\varphi}} dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{\frac{2}{p}}},$$

where $p = \frac{2m}{m-2}$. As for the classical Yamabe problem Q_φ can be written as

$$Q_{\tilde{\varphi}}(\tilde{g}) = Q_\varphi(u) = \frac{E(u)}{\|u\|_p^2},$$

where

$$E(u) = \int_M \left(-\frac{4(m-1)}{m-2} |\nabla u|^2 + S^\varphi u^2 \right) dV_g.$$

Therefore, u is a critical point for the φ -Yamabe invariant if it satisfies

$$\mathcal{L}_g^\varphi u = \lambda u^{p-1}, \tag{6.5}$$

with $\lambda = \frac{E(u)}{\|u\|_p^p}$.

Note that, as in the classical Yamabe problem, the inclusion

$$W^{1,2} \subset L^p \tag{6.6}$$

is not compact. Let $\{u_i\}$ be a sequence of smooth functions such that $Q_\varphi(u_i) \rightarrow Y^\varphi(M, [g])$ and suppose (by homogeneity) that $\|u_i\|_p = 1$, for any i . Then, by Hölder’s inequality, $\{u_i\}$ is bounded in $W^{1,2}(M)$ and there exists a subsequence converging to a function $u \in W^{1,2}(M)$. However, by (6.6), we cannot guarantee that the constraint $\|u_i\|_p = 1$ is preserved by u ; moreover u may be identically zero.

Following Lee and Parker ([24]) we consider an associated subcritical equation. We define

$$Q_\varphi^s(u) := \frac{E(u)}{\|u\|_s^2}$$

for $2 \leq s \leq p$ and let

$$Y_s^\varphi = \inf \{ Q_\varphi^s(u) : u \in C^\infty(X) \}.$$

Note that if u is a minimizing function with $\|u\|_s = 1$, then

$$\mathcal{L}_g^\varphi u = Y_s^\varphi u^{s-1}. \tag{6.7}$$

We now state the analogous of Proposition 4.2 of [24] in our setting. We omit the proof since it works exactly as that of [24].

Proposition 6.4 *For $s \in [2, p]$ there exists a smooth positive solution u_s to the subcritical equation (6.7), for which $Q_\varphi^s(u) = Y_s^\varphi$ and $\|u_s\|_s = 1$.*

Lemma 6.5 *If $\text{Vol}(M) = 1$, then $|Y_s^\varphi|$ is non-increasing as function of $s \in [2, p]$; moreover, if $Y^\varphi(M, [g]) \geq 0$, then $|Y_s^\varphi|$ is left-continuous.*

Proof Let $u \in C^\infty(M)$, $u \neq 0$. By Hölder inequality if $s \leq s'$, then $\|u\|_s \leq \|u\|_{s'}$. Now

$$Q_\varphi^{s'}(u) = \frac{\|u\|_s^2}{\|u\|_{s'}^2} Q_\varphi^s(u)$$

and then if $s \leq s'$ we have $Q_\varphi^{s'} \leq Q_\varphi^s$. It follows that

$$|Y_{s'}^\varphi| \leq |Y_s^\varphi|. \tag{6.8}$$

Assume that $Y_p^\varphi(M, [g]) = Y_p^\varphi \geq 0$, by (6.8) we have that

$$Y_s^\varphi \geq 0$$

for every $s \in [2, p]$. Take $s \in [2, p]$. Given $\varepsilon > 0$, there exists $u \in C^\infty(M)$ such that

$$Q_\varphi^s(u) < Y_s^\varphi + \varepsilon.$$

By continuity of $\|u\|_s$, for $s' \leq s$, sufficiently close to s we have

$$Y_{s'}^\varphi \leq Q(u)_{\varphi}^{s'} \leq Y_s^\varphi + 2\varepsilon$$

and since Y_s^φ is non-increasing, we deduce that Y_s^φ is continuous from the left. □

Following the proof in [24], we can now prove that if $Y^\varphi(M, [g]) < Y(\mathbb{S}^m, [g_{\mathbb{S}^m}])$ holds, then the φ -Yamabe problem can be solved.

Proposition 6.6 *Let $\{u_s\}$ be the collection of functions given by Proposition 6.4. There are constants $s_0 < p, r > p$ and $C > 0$ such that $\|u_s\|_r \leq C$ for every $s \geq s_0$.*

Proof Let $\delta > 0$. We multiply (6.5) times $u_s^{1+2\delta}$ and we integrate

$$\begin{aligned} Y_s^\varphi \int_M u_s^{s+2\delta} dV_g &= \int_M \left(-\frac{4(m-1)}{m-2} u_s^{1+2\delta} \Delta u_s + S^\varphi u_s^{2+2\delta} \right) dV_g \\ &= \int_M \left(\frac{4(m-1)}{m-2} u_s^{2\delta} (1+2\delta) |\nabla u_s|^2 + S^\varphi u_s^{2+2\delta} \right) dV_g. \end{aligned}$$

Let $w = u_s^{1+\delta}$, then we have

$$\frac{(1+2\delta)}{(1+\delta)^2} \int_X \frac{4(m-1)}{m-2} |\nabla w|^2 dV_g = \int_M (Y_s^\varphi w^2 u_s^{s-2} - S^\varphi w^2) dV_g.$$

To conclude the proof we recall the following theorems (see [24] for more details).

Theorem 6.7 *Let X be a compact manifold. Let σ_M be the optimal Sobolev constant. For every $\varepsilon > 0$ there exists C_ε such that for every $v \in C^\infty(M)$*

$$\|v\|_p^2 \leq (1+\varepsilon)\sigma_M \int_M |\nabla v|^2 dV_g + C_\varepsilon \int_M v^2 dV_g. \tag{6.9}$$

Theorem 6.8 *The m -dimensional Sobolev constant is $\sigma_M = \frac{4(m-1)}{m-2} / \Lambda$, where $\Lambda = Y(\mathbb{S}^m, [g_{\mathbb{S}^m}])$. Thus, the sharp Sobolev inequality on \mathbb{R}^m is*

$$\|v\|_p^2 \leq \frac{4(m-1)}{\Lambda(m-2)} \int_{\mathbb{R}^m} |\nabla v|^2 dV_g. \tag{6.10}$$

Combining (6.9), (6.10) we deduce

$$\begin{aligned} \|w\|_p^2 &\leq (1+\varepsilon) \frac{4(m-1)}{\Lambda(m-2)} \int_M |\nabla w|^2 dV_g + C_\varepsilon \int_M w^2 dV_g \\ &= (1+\varepsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{1}{\Lambda} \int_M Y_s^\varphi w^2 u_s^{s-2} dV_g - (1+\varepsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{4(m-1)}{\Lambda(m-2)} \int_X S^\varphi w^2 dV_g \\ &\quad + C_\varepsilon \int_M w^2 dV_g \\ &\leq (1+\varepsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{1}{\Lambda} \int_M Y_s^\varphi w^2 u_s^{s-2} dV_g + C'_\varepsilon \int_M w^2 dV_g \end{aligned}$$

and using Hölder inequality

$$\|w\|_p^2 \leq (1+\varepsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{Y_s^\varphi}{\Lambda} \|w\|_p^2 \|u_s\|_{\frac{(s-2)m}{2}}^{s-2} + C'_\varepsilon \|w\|_2^2.$$

If $Y^\varphi(M, [g]) < 0$, then

$$\|w\|_p^2 \leq C\|w\|_2^2.$$

If $0 \leq Y^\varphi(M, [g]) < Y(\mathbb{S}^m, [g_{\mathbb{S}^m}])$, since Y_s^φ is non-increasing and left-continuous, we infer the existence of $s_0 < p$, sufficiently close to p such that

$$\frac{Y_s^\varphi}{\Lambda} \leq \frac{Y_{s_0}^\varphi}{\Lambda} < 1,$$

for $s \geq s_0$. Moreover, since $\frac{(s-2)m}{2} < s$, by Hölder’s inequality we have

$$\|u_s\|_{\frac{(s-2)m}{2}} \leq \|u_s\|_s = 1,$$

therefore

$$\|w\|_p^2 \leq C\|w\|_2^2.$$

To conclude the proof observe that

$$\|w\|_2 = \|u_s\|_{2(1+\delta)}^{1+\delta} \leq \|u_s\|_s^{1+\delta}.$$

Therefore, $\|w\|_p$ is bounded independently of s . □

Proposition 6.9 *Let $\{u_s\}$ be the functions given in Proposition 6.4. As $s \rightarrow p$, u_s converges (up to a subsequence) to a positive function $u \in C^\infty(X)$, which satisfies*

$$Q(u)_\varphi^p = Y^\varphi(M, [g]), \quad \mathcal{L}^\varphi u = Y^\varphi(M, [g])u^{p-1},$$

hence the metric $\tilde{g} = u^{p-2}g$ has constant φ -scalar curvature.

Proof By Proposition 6.6, the sequence $\{u_s\}$ is uniformly bounded in $L^r(M)$ and by Theorem 4.1 of [24], they are uniformly bounded in $C^{2,\alpha}$ and by Arzela-Ascoli Theorem a subsequence converges in C^2 norm to a function $u \in C^2(X)$ such that

$$\mathcal{L}_g^\varphi u = \lambda u^{p-1} \quad Q(u) = \lambda,$$

where $\lambda = \lim_{s \rightarrow p} Y_s^\varphi$. If $Y(M, [g]) \geq 0$, by Lemma 6.5 Y_s^φ is left-continuous, so that $\lambda = Y(M, [g])$. If $Y(M, [g]) < 0$, since $|Y_s^\varphi|$ is non-increasing, we have that Y_s^φ is increasing. Thus, $\lambda \leq Y(X, [g])$ and since $Y(M, [g])$ is the infimum of Q_φ^p , equality holds. □

Note that when $Y^\varphi(M, [g]) = Y(\mathbb{S}^m, [g_{\mathbb{S}^m}])$, by definition of the φ -Yamabe invariant, we have that

$$Y^\varphi(M, [g]) = Y^\varphi(M, [g]) = Y(\mathbb{S}^m, [g_{\mathbb{S}^m}])$$

hence, by the classical results of Aubin and Schoen we have that $(M, [g])$ is conformally equivalent to the standard sphere. However, note that when equality holds it is not guaranteed that $\tilde{g} \in [g]$, such that \tilde{S}^φ is constant coincide with the Yamabe metric.

Remark 6.10 Note that the sign of the φ -Yamabe invariant is conformally invariant and it determines the sign of the φ -scalar curvature relative to the metric $\tilde{g} \in [g]$ that attains the minimum.

Before proving Theorem 1.8, we recall that from Gauss equation, we have

$$\begin{aligned} R_{ijkl}^{\partial M} &= R_{ijkl} - \Pi_{il}\Pi_{jk} + \Pi_{ik}\Pi_{jl}; \\ R_{ik}^{\partial M} &= R_{ik}^{\partial M} - R_{imkm} - \Pi_{il}\Pi_{lk} + \Pi_{ik}H(m-1); \\ S^{\partial M} &= S - 2R_{mm} + (m-1)(m-2)H^2, \end{aligned}$$

which imply

$$S^{\partial M, \varphi} = S^\varphi + \alpha(\varphi_m^a)^2 - 2R_{mm} + (m - 1)(m - 2)H^2. \tag{6.11}$$

Moreover, we prove the following

Lemma 6.11 *Let (M, g) be a compact Riemannian manifold with boundary ∂M and let $f \in C^\infty(\partial M)$. Then, there exists $\bar{f} \in C^\infty(M)$ such that $\bar{f}|_{\partial M} = f$.*

Proof Let U_ε be a tubular neighborhood of the boundary ∂M , namely, we consider

$$U_\varepsilon \cong \partial M \times [0, \varepsilon)$$

for $\varepsilon > 0$ sufficiently small.

In particular we can identify a point $x \in U$ as (y, t) , where $y \in \partial M$ and $t \in [0, \varepsilon)$. We define $F : U_\varepsilon \rightarrow \mathbb{R}$, such that for every $x \in U_\varepsilon$, $F(x) = F((y, t)) = f(y)$, note that $F \in C^\infty(U_\varepsilon)$ and $F|_{\partial M} \equiv f$.

Let $\mathcal{U} = \{U_i\}_i$ be an open covering of the boundary such that $U_i \subset U_\varepsilon$ for every i . Since ∂M is compact, we can assume it is finite. We consider a partition of unit $\{\psi_i\}_i$ subordinate to \mathcal{U} ; by construction $\text{supp}(\psi_i) \subseteq U_i$. Then, we define

$$\bar{f}(x) = \begin{cases} \sum_i \psi_i(x)F(x) & \text{if } x \in U_\varepsilon; \\ 0 & \text{if } x \in M \setminus U_\varepsilon. \end{cases}$$

By construction when $y \in \partial M$ $\bar{f}(y) = \sum_i \psi_i(y)f(y) = f(y)$ and $\bar{f} \in C^\infty(M)$. □

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8 By Proposition 6.9 we can always consider a metric $\tilde{g}_b \in [g|_{\partial M}]$ such that the φ -scalar curvature of the boundary associated to \tilde{g}_b is constant. Since $\tilde{g}_b \in [g|_{\partial M}]$, there exists a function f on ∂M such that $\tilde{g}_b = e^{2f}g$. By Lemma 6.11, we can extend f on M ; in particular there exists $\bar{f} \in C^\infty(M)$ such that $\bar{f}|_{\partial M} = f$. Let us consider the metric $\tilde{g} := e^{2\bar{f}}g$. By construction we have $\tilde{g}|_{\partial M} = \tilde{g}_b$ on the boundary and $\tilde{S}^{\tilde{\varphi}, \partial M}$ is constant. Since (M, g) is totally geodesic due to the validity of (1.29), the second fundamental form \tilde{I} of (M, \tilde{g}) writes as

$$\tilde{\Pi}_{ab} = e^{-\bar{f}} \left(\Pi_{ab} + \frac{\partial \bar{f}}{\partial \nu} g_{ab} \right);$$

and the mean curvature \tilde{H} is given by

$$\tilde{H} = \frac{\text{tr}(\tilde{\Pi})}{m - 1} = e^{-\bar{f}} \frac{\partial \bar{f}}{\partial \nu}.$$

Therefore, by definition of $J_{\tilde{g}}$, we deduce

$$J_{\tilde{g}}\phi = \Delta_{\tilde{g}}\phi + \left(\tilde{\text{Ric}}(\nu, \nu) + \frac{\tilde{H}^2}{(m - 1)^2} \right) \phi,$$

where $\tilde{\text{Ric}}$ denotes the Ricci tensor with respect to \tilde{g} . Moreover,

$$\begin{aligned} (\lambda_1)_{\tilde{g}} \int_{\partial M} \phi^2 dV_{\tilde{g}_b} &\leq - \int_{\partial M} \phi J_{\tilde{g}}\phi dV_{\tilde{g}_b} \\ &= \int_{\partial M} \left(|\nabla_{\tilde{g}}\phi|_{\tilde{g}}^2 - \left(\tilde{\text{Ric}}(\nu, \nu) + \frac{\tilde{H}^2}{(m - 1)^2} \right) \phi^2 \right) dV_{\tilde{g}_b}. \end{aligned}$$

Taking $\phi = 1$ in the above equation and using (6.11), we get that when $\alpha > 0$ the following inequalities hold

$$\begin{aligned}
 (\lambda_1)_{\tilde{g}} \text{Vol}_{\tilde{g}_b}(\partial M) &\leq \frac{1}{2} \int_{\partial M} \tilde{S}^{\tilde{\varphi}, \partial M} dV_{\tilde{g}_b} - \frac{1}{2} \int_{\partial M} \tilde{S}^{\tilde{\varphi}} dV_{\tilde{g}_b} - \int_{\partial M} \alpha (\tilde{\varphi}_m^a)^2 dV_{\tilde{g}_b} \\
 &\quad - \int_{\partial M} \frac{\tilde{H}^2}{(m-1)^2} dV_{\tilde{g}_b} \\
 &\leq \frac{1}{2} \int_{\partial M} \tilde{S}^{\tilde{\varphi}, \partial M} dV_{\tilde{g}_b} - \frac{1}{2} \int_{\partial M} \tilde{S}^{\tilde{\varphi}} dV_{\tilde{g}_b} \\
 &\leq \frac{1}{2} \int_{\partial M} \tilde{S}^{\tilde{\varphi}, \partial M} dV_{\tilde{g}_b} - \frac{\tilde{S}_{min}^{\tilde{\varphi}}}{2} \text{Vol}_{\tilde{g}_b}(\partial M) \\
 &\leq \frac{1}{2} Y^\varphi(\partial M, [g|_{\partial M}]) \text{Vol}_{\tilde{g}_b}(\partial M)^{\frac{m-3}{m-1}} - \frac{\tilde{S}_{min}^{\tilde{\varphi}}}{2} \text{Vol}_{\tilde{g}_b}(\partial M),
 \end{aligned}$$

which implies

$$(\lambda_1)_{\tilde{g}} \leq \frac{1}{2} \left((m-1)(m-2) \omega_{m-1}^{\frac{2}{m-1}} \text{Vol}_{\tilde{g}_b}(\partial M)^{-\frac{2}{m-1}} - \tilde{S}_{min}^{\tilde{\varphi}} \right). \tag{6.12}$$

If equality holds,

$$Y^\varphi(\partial M, [g|_{\partial M}]) = Y(\partial M, [g|_{\partial M}]) = Y(\mathbb{S}^{m-1}, [g_{\mathbb{S}^n}])$$

and it follows directly by the classical results of Aubin and Schoen (see [24] and the references therein) that ∂M is conformally equivalent to a sphere. Furthermore, let $\hat{g} \in [g|_{\partial M}]$ be the metric which achieves the infimum of $Y(\partial M, [g|_{\partial M}])$: then

$$\tilde{S}^{\tilde{\varphi}, \partial M} = \hat{S}^{\partial M} \frac{\text{Vol}_{\tilde{g}_b}(M)^{\frac{m-3}{m-1}}}{\text{Vol}_{\hat{g}}(M)^{\frac{m-3}{m-1}}},$$

where

$$\frac{\text{Vol}_{\tilde{g}_b}(M)^{\frac{m-3}{m-1}}}{\text{Vol}_{\hat{g}}(M)^{\frac{m-3}{m-1}}}$$

is constant. Moreover, if equality holds

$$\int_{\partial M} \tilde{H}^2 = 0.$$

which implies that the boundary is totally geodesic. Furthermore, we have that $\tilde{S}^{\tilde{\varphi}} = \tilde{S}_{min}^{\tilde{\varphi}}$ on ∂M , that is $\tilde{S}^{\tilde{\varphi}}$ is constant on ∂M . Moreover, $\tilde{\varphi}_m^a \equiv 0$ on ∂M . □

Remark 6.12 Observe that the proof of Theorem (1.8) is completely independent from the choice of Λ . For instance, when φ is constant and Λ assumes the values discussed in the introduction, we have the validity of the theorem for some well-known models (see the discussion in the introduction).

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Declarations

Conflicts of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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