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A LOGIC FOR USING INFORMATION

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Abstract

Logics of information have abounded in the last two decades. A major point of conceptual and formal difference among various such logics has been represented by the interaction they model between information and truth. Floridi's logic of "being informed" defines semantic information as the truthful basis for knowledge; Allo's revisitation of it restraints this relation from the agent's perspective; Primiero's logic for "becoming informed" focuses on information as denoting assertibility conditions, whereas truth is only granted by their verification. The present article extends the debate to include aspects originating in semiotics and pragmatics to the study of information, in particular with respect to computational systems. We stress the role of the different informational users in exchanging and determining validity conditions for information and formalize a weighted multi-agent modal logic for "using information". We present both a semantics and axiomatic systems, prove standard meta-theoretical results and show which fragments correspond to different interpretations of use of the computational systems under interpretation.

Keywords: Logic of Information, Logic of Computation, Pragmatic Theory of Information, Function Theory, Modal logic

1. Introduction

In the last hundred years, the concept of information has become an important subject of research in both the humanities and the sciences. Two main interpretations of the notion of information are available in the literature:

- *Quantitative theories of information*, "associated with different narratives (counting, receiving messages, gathering information, computing) rooted in the same basic mathematical framework" (Adriaans 2020), and
- *Qualitative theories of information*, further divided in two branches, regarding:
  - *Semantic Information*: first formulated in Bar-Hillel and Carnap (1953), and reinvigorated since the turn of the century,
  - *Information as a state of an agent*: originated in Hintikka (1962), further developed by Dretske (1981).

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In recent decades, with the rise of the internet, social media, and other forms of digital communication, "being informed", "being informative", and "becoming informed", have all reached new levels of significance. Floridi (2006)'s explanation of the logic of "being informed", distinguishing it from the original interpretation by Bar-Hillel and Carnap, takes information not only as well-formed and meaningful, but also as truthful and non-introspective, hence distinct from both belief and knowledge. Allo (2011) builds on Floridi's work arguing that being informed is not a one-size-fits-all concept, but rather depends on the actual informational state in which the information is being consumed. Therefore information is also distinct from true belief, given that "information privileges the actual world by never excluding it" (18), and consequently, not only can it not be false, but also it cannot be derived from false data. Primiero (2009) proposes a proof-theoretical account, in which the truth of information is rejected, in favour of its interpretation as assertibility conditions, i.e. pre-conditions for knowledge which is the only epistemic status which ascribes truth to contents. The resulting *Epistemic Constructive Definition of Information* (ECDI) is the theoretical basis to develop the logic for "becoming informed".

Such approaches, despite relevant in the investigation of various central aspects in the definition of information, have all neglected the pragmatic component of information, as complementary to both semantic and epistemic ones. Pragmatics is the study of how language and communication are used in real-world contexts. In the context of information, pragmatics refers to the ways in which information is used to achieve specific goals or objectives. For example, a news article may be used to inform readers about current events, but it may also be used to influence public opinion or promote a particular agenda. The pragmatic aspect of information is closely tied to the concept of information literacy, which is the ability to find, evaluate, and use information effectively. Information literacy is not just about acquiring knowledge, but also about applying that knowledge in practical situations. This includes being able to identify relevant information, evaluate its credibility, and use it to make informed decisions. In short, the pragmatic aspect of information is essential for making effective use of the vast amounts of data available to us today. By evaluating information in terms of its practical usefulness and usability, we can make informed decisions that help us achieve our goals and succeed in our endeavors. The present paper aims at offering a logic for "using information", providing an applied semantics for the use, relevance and evolution of information. Such a pragmatic component is notoriously of general relevance for any user-based information exchange, and recently has been reconsidered for the ontology and epistemology of computational artefacts in Buda and Primiero (2023). Computational artefacts are especially prone to being considered a natural environment for the conceptual and formal interpretation of information contents and informational users. Although some relevant aspects have already been considered from this perspective (non-introspection by Floridi, agent's rootedness by Allo, context-dependency by Primiero), many other aspects of the role of informational users within computational artefacts have been neglected: their being structured around levels; the abstraction-implementation relation among those levels; and the difference between standard and non-standard uses of functionalities, from specifications to deviant uses. The Logic for Pragmatic Information (LPI), introduced in the present paper, aims at addressing these aspects. We believe this basic step is essential also for a generalization to the simulation of computational systems where information results also from non-deterministic processes, as in the case of interaction with AIs.

The remainder of this paper is structured as follows. In Section 2 we offer an overview of relevant logics of information, as well as of ontological and epistemological principles from the literature crucial in the definition of a pragmatic notion of information. In Section 3 we introduce the logic LPI in the form of a relational semantics with graded modalities, aimed at expressing partial or total inclusion of functionalities in a specification, in terms of the access agents have to them at a level of abstraction. Further, this modal logic adds two new modalities to express relations across levels of abstractions. In Section 4, we first consider the axiomatic characterization of LPI for what we call "passive use", or the situation of well-functioning computational artefacts where all functionalities are fully accessed at all levels of abstraction; for this axiomatic characterization, we offer standard metatheoretical results of soundness and completeness extending known results for modal logics to the additional modalities of our logic. In Section 5 we explain how the different uses of computational artifacts can be modelled by the logic LPI: idiosyncratic use, where one agent discovers a new functionality; innovative use, where such novelty is shared with multiple agents within a level; combinations of both; expert redesign, where an idiosyncratic use is shared across levels and induces correction of the newly acquired functionalities; product design, where these become again fully available to all users at all levels. In Section 6 we investigate which reformulations of the ontological and epistemological principles from the literature are necessary for the notion of pragmatic information modelled by our logic. In Section 7 we sum up the results of the present contribution and sketch future lines of investigation.

2. Related works

2.1. *Logics of Information*

Floridi (2006)'s information logic (IL), designed to be independent of both epistemic logic (EL) and doxastic logic (DL) and capable of formalizing the relation "a is informed that p" ($I_a p$) as a different cognitive state from either $K_a p$ or $B_a p$, gave rise to a flourishing philosophical debate. Allo (2011) refers to this distinction between information, knowledge and belief as the Basic Independence Thesis (BIT). The General Definition of Information (GDI) (Floridi 2006, 2011) as well-formed meaningful data is standardly based on the following principles of neutrality (Primiero 2007):

- Typological Neutrality: information cannot be dataless, and everything can be a datum;
- Taxonomical Neutrality: a datum is a relational entity, so is information;
- Ontological Neutrality: data implementing information are physical;
- Genetic Neutrality: data (and therefore information) can have a semantics independently of any informer;
- Alethic Neutrality: meaningful and well-formed data qualify as information, no matter whether they represent or convey a truth or a falsehood or have no alethic value at all.

These principles can be considered, to different degrees, controversial. In particular, the Alethic Neutrality Principle is the one that has generated the most criticisms. In a second formulation of the GDI, Floridi introduces an additional condition, "suggesting that information should 'encapsulate' truthfulness" (392). This definition is equivalent to the *Veridicality Thesis* (VT), or in other words, truthfulness is a necessary feature for a non-empty set of meaningful and well formed data to be considered as information. IL, as developed by Floridi, is meant as a logical interpretation of the VT, "arguing that the axiom schemata of the normal modal logic (NML) KTB (also known as B or Br or Brouwer's system) are well suited to formalise the relation of 'being informed"' (Floridi 2006, 1). This set of axioms ensures the validity of the VT, by means of axiom T, whereas it does not validate *introspection*, since both Axiom S4 and Axiom S5 are not valid. Axiom B has to be intended as a way to build up the set of information an agent holds. According to Floridi, Axiom B does not contradict the anti-reflective principle that made us exclude Axiom S4 and Axiom S5.

The main objections raised against the logic of information as defined by Floridi concern the VT and its inability to satisfy the *strong independence* *thesis* (SIT). Moreover, "its implementation of being informed as a non-mentalistic state is questionable as well" (Allo 2011, 2). Allo (2011) offers a KTB analysis disagreeing on its approach and philosophical framework. The main objection is based on the fact that, due to its purely syntactic formulation, the KTB-IL analysis "can only scratch at [...] the syntactical surface of the axiomatic presentation" (5) without giving a positive characterization of the notion of *being informed*. In addition to that, the reverse-engineering method used by Floridi, consisting in taking an existing approach, *i.e.*, a normal modal logic for knowledge and belief, and applying it to a new domain, *i.e.*, the logic of being informed, is able only to identify an axiomatic difference, without telling where such a difference comes from. Allo starts from the unsatisfactory distinction settled by the BIT according to which information is neither knowledge, because it has very few modal properties, nor belief, because it has to be true. He adds the SIT, by further specifying that information is different from true belief, since information can be derived only from other information (true data), whereas true belief can be obtained from false ones. Moreover, "information privileges the actual world by never excluding it" (18). The latter consideration and the above mentioned poor modal properties of informational accessibility lead to a first formulation of a non-standard semantics, suitable to model holding information only when the actual world is involved. Unfortunately, it validates transitivity when non-actual worlds are taken into account as base worlds. To solve this problem, a further "level of abstraction" (Floridi 2008) is needed to dig beyond the *de dicto*, sentential representation of data, and consider also the *de re* reading, which sees data as lack of uniformity in the real world. This allows to distinguish between data and their representation, specifying which data an agent has access to and hence giving an explanation to the failure of persistence of knowledge implied by the transitivity of the accessibility relation. The *de re* reading still endorses the most controversial part of the logic of being informed, *i.e.*, the *Veridicality Thesis*.

Primiero (2007) provides an *Epistemic Constructive Definition of Information* (ECDI), "based on the conceptual difference between the notions of knowledge and information, the latter being formulated as the meaningful content presupposed and required by an assertion of the former" (402). The simple intuition underlying this formulation is that information is dynamic, and when moving from one informational state to another the truth of a given informational content may no longer be guaranteed. This leads to a rejection of the VT and, in accordance with an intuitionistic-constructive approach, to the formulation of the *Verificationist Principle of Truth* (VPT) (396), according to which knowledge can be defined as verified information. A knowledge state, with respect to a certain propositional content, is expressed by a judgement, and is produced in presence of a justification or proof. This approach focuses on a deep investigation of the relation between truth, *justification (of knowledge)*, and *conditions (for knowledge)*. Besides knowledge states, the new epistemic notion of *Informational State* is defined as including meaningfulness, accessibility, syntactic and semantic extendibility (Primiero 2007, 402). The epistemic role of such an informational state is to formulate the conditions on which it is possible for the agent to acquire proper knowledge. The formal framework for ECI is given by an epistemic informational logic based on the notion of dependent justification, then admitting proof-conditions rather than truth conditions for propositional contents, resulting in a logic for "becoming informed". Becoming informed that ϕ is an epistemic state which does not satisfy the VT, and then it cannot be confused with being informed that ϕ or knowing that ϕ. Taking into account the basic principle according to which a proof is a justification for the truth of a propositional content, it can be concluded that knowledge, *i.e.*, the proper object of predication for truth and falsity, is actually verified information.

All Neutrality principles valid for GDI but the ontological one are revised in order to fit this constructive-epistemological framework. Data included in informational contents can be produced by different sources, and these contents can induce different epistemic states in different receivers, producing different levels of informativity, depending on the ability of the receiver to produce the related judgements. Hence, information is user-dependent and "not every content is (equally) 'informative' for an agent: rather, everything is a judgeable content (for any agent) provided its basic conditions of assertability (meaningfulness) and provability (constructions) are met." (405). Informational sources are not all equally reliable, and one may consider some sources more valuable, or even more accessible, than others. One may also consider unreliable (or not accessible) a source which, in contrast, is considered reliable (or easily accessible) by someone else. On these lines, from an epistemic perspective, any content which is always truthfully provided by all possible sources and is accepted by any possible agent, is no longer considered as information, it rather qualifies as proper knowable content. From a first-person perspective, information is source-dependent, whereas knowledge "is actually *independent from any possible source* producing it" (408). Also the independence of data from the receiver is rejected. Data qualify as information, and information qualifies as knowledge, with respect to what an agent can do with them, *i.e.*, in terms of what operations the agent can perform on them. A certain propositional content may be considered as information by someone unable to provide a justification for it, whereas it can be marked as proper knowledge by other agents, able to construct a proof for that content: “there is no ‘pure information’ flowing independently from the agent predisposed to receive and to interpret it” (408). Finally, the constructive version of AN aims at expressing “the relation between the ontological nature of informational contents (as expressed according to the restricted version of TyN.) and their truthfulness” (409). One can attribute truth values to the content of a knowledge act only on condition that the related informational content is meaningful. Obviously, also assumptions, *i.e.*, contents for which truth is not justified but only assumed, qualify as information, given their necessary presupposed meaningfulness. This leads to an epistemic constructive definition of information as condition for knowledge independent of alethic constraints.

Such variety of approaches seems to suggest that fully supporting or rejecting the Veridicality Thesis might be ill suited to describe one of the characteristic properties of information, which the agent-based approach has timidly let us guess: the stratification of information on different levels of abstraction and interpretation. The same kind of critique can be aimed at accounts which are fully *de re* or *de dicto*. In the next section, we will show how all the above issues combine through the lens of technology, inquiring what information *is made of*.

2.2. *Information (and) Technology*

Information and technology have always been hand in glove, and not just for what the acronym *IT* can suggest. As already underlined in the constructivist approach to the principles of neutrality, information is such only when it is transferred from (at least) an informer agent, to (at least) an informed agent. This alone could raise some doubts about total support (or rejection) of both VT and an entirely *de re* or entirely *de dicto* thesis, since it is only on the basis of the *informational capabilities* of the source(s) and of the receiver(s) that the information is true or false, verifiable or refutable, meaningful or not, correct or incorrect, well formed or not, physical or symbolic.

One way to model “being informed” and “becoming informed” from this agent-based perspective is represented by *Depth-bounded Boolean Logic* (DBBL) (D’Agostino 2013, 2015; D’Agostino et al. 2024), its extension to *multi-agent* systems (MA-DBBL) (Cignarale and Primiero 2021), and its specific formulation for “becoming informed” (DBBL-BI) (Larotonda and

Primiero 2023). In all of these accounts information is acquired or transferred by iterated applications of some inferential rules (originally the *Rule of Bivalence* (RB) ($\theta \vee \neg\theta$) in D'Agostino (2013), a modal formula for information transmission in the multi-agent settings) whenever uncertainty is faced, therefore they presuppose agents with at least some sort of informational (computational) power. Nonetheless, the original formulation of the notion of "being informed" was intended to include entities with very limited or zero computational power, and no ability at all to discriminate truth from falsity, such as a notebook or a silicon cluster inside an hard-disk. This is the notion of agent we refer to.

Every flow of information, from the most primordial one expressed through gestures and guttural monosyllables, to the most technologically advanced contemporary analogues such as social media posts, shows a stratification neglected by the theories that have been analyzed so far. This layered abstraction may be difficult to identify in the most primitive forms of information, however it is quite evident if we analyze the concept of information in the philosophy of computer science (Primiero 2016). By means of the methodology of the Level of Abstraction (LoA) (Floridi 2008), first successful especially in mathematical and philosophical research, Primiero (2016) shows that computational systems are constituted by several LoAs. Each LoA embeds a pair consisting of an epistemological construct and an ontological domain appropriately related by an instantiation relation of implementation across different LoAs. At the lowest level of *structured data* realized in the physical world there is no symbolic representation, hence no *de dicto* component of information; as well as no truth involved, given that truth and falsity are properties of sentences, not of physical matter; hence, at this level there are not even the preconditions for VT to hold. Moving to the next level, where the *operational information* embedded in machine code is qualified as *well-formed performative data*, we are in a mixed *de re/de dicto* situation, since we have a lack of uniformity at the level of symbolic representation, directly reflected in a lack of uniformity in the world. As for the previous level, VT doesn't hold, since this type of purely quantitative information possesses neither semantics nor alethic value. At programming language level, *instructional information*, described in terms of *well-formed and meaningful data*, totally abstracts from the physical layer, therefore, from now on, any kind of *de re* consideration loses its meaning. This is exactly what programming languages were invented for: to write programs without having to consider the physical machine. Like in the previous case, also at this level there is still no alethic assessment, hence no VT. Instead, the content of an algorithm is defined as *abstract, correctness- and truth-determining information* for all previous levels, and at the final level *intentional information* is characterized by being abstract, semantically loaded, and truth-determining for the algorithm. At these two levels the Veridicality thesis properly holds. These considerations lead to a philosophical formulation of (digital) computing as the systematic study of the ontologies and epistemology of information structures (122), and to a definition of information as *well-formed* data, characterized by meaningfulness, correctness and truth at different levels of abstraction. This analysis of the stratification of information allows to resolve the debates over the Veridicality Thesis and the *de re/de dicto* characterization of information.

However, as extensively argued in Buda and Primiero (2023), the rigid informational structure given by LoAs, forces us to accept a continuous shift of syntax and semantics. This is particularly hard to justify when taking into account the indefinite, and possibly infinite, growth of LoAs within computational systems, due to the improvement of technology and *usability*. Human-friendly interfaces and simplified dialects are increasingly being created to make software development easier and affordable for more and more users, underlining the importance of the role played by the users, in the evolution of Information Technologies. To this aim, Buda and Primiero (2023) introduce a pragmatic-inspired version of the LoA structure, namely the *User Level* (UL) structure, developed to overcome the rigidity and lack of resilience to evolutionary changes of LoAs. The aim of the next section is to introduce a corresponding logic for "using information".

3. A logic for "*using* information"

Within computational artefacts, the semantics of information cannot be confined to a single LoA. Instead, it seems to propagate along all levels. For the programmers of ENIAC (Electronic Numerical Integrator and Computer) instructions and operations coincide in a single level of abstraction; conversely, the very same algorithm, *e.g.*, Euclid's algorithm for computing the greatest common divisor, can be written in machine language, in assembly language, and in any high-level programming language. The *User Levels* (UL) model (Buda and Primiero 2023) is developed to include any use (including deviant ones) by which an agent (not necessarily human) with their ability (information-base, computational power, skills, luck) makes information meaningful in a particular context. ULs offer a user-centered characterization of the following definition of Pragmatic Information:

Definition 1. (Pragmatic Information - PI) Information is transferred data characterized by being:

- Abstract and truth-bearing,
- Instructional and meaningful,
- Operational, correct, and well-formed

with respect to the UL considered.

On this basis, we develop a logic for "using information" which we call *LPI* – Logic of Pragmatic Information. It includes the *de re* and the *de signo/de dicto* readings of contents, and it is capable of accounting for many types of use, from passive use to product designing (Vermaas and Houkes 2006, 12).

LPI is able to model the transfer of information between different ULs: different sets of users, with different informational capabilities, *i.e.*, different expressive abilities as sources and different elaboration powers as receivers. As in the LoAs account, we will refer to information flow in computational systems, however the approach can be easily extended to information at large. Computational artifacts are just paradigmatic cases that make the stratification of information explicit. The term *user* is preferred to *agent*, for two reasons: on the one hand, we want to underline the anti-realistic approach based on the meaning-as-use paradigm, largely inspired by Wittgenstein's pragmatics and Saussure's semiotics; on the other hand, as already briefly mentioned, we want to model a type of informational entity not necessarily able to *act on*, but still able to *use* (and *be used* as a source of) information. The language we will define must be thought of as a *meta-language* for any given UL, where the latter is characterized by its own vocabulary suitable to convey instructional information across informational states in *intra-level* communication, but incompatible with those of other ULs, therefore not apt for *inter-level* communication. The transfer of information across levels is assured by appropriate relations of *abstraction*, by means of abstract information, and *implementation*, *via* operational information. Hence, inter-level communication do not correspond to literal translation, nor to *one-to-one* correspondence between informational states of different ULs. We assume that lower ULs, namely those closer to the physical world, are characterized by having a greater number of simple informational states, *i.e.*, informational states with low informational content, whereas higher levels have less, more complex states. After all, we need millions of bits to have one single low-quality image. Lastly, evolution of information through idiosyncratic use happens not *across* levels, but *within* the same UL, by means of instructional information, starting in the context of use of one singular user, and spreading within the UL as innovative using. To pass across ULs, another channel of communication is needed, one in which users, belonging to different ULs in the original channel, are at the same level, *e.g.*, chatting on a forum or in a bar. Once this happens, we have the preconditions for expert redesigning (and further iterations of product designing). To model the evolution of specification for computational systems, we use a weighted modal logic inspired by Legastelois, Lesot, and d'Allonnes (2017).

We now introduce the language for LPI:

DEFINITION 2 (Language).

$$\phi := \ p \mid \neg\phi \mid \phi \wedge \phi$$

$$\psi := \phi \mid \ \Box_\alpha\psi \mid \Diamond_\alpha\psi \mid \neg\psi \mid \psi \wedge \psi \ with\ \alpha \in [0, 1]$$

$$\xi := \ \phi \mid A^{i \mapsto j}\xi \mid I^{j \mapsto i}\xi \mid \neg\xi \ with\ i < j$$

$$\chi := \psi \mid \xi \mid \neg\chi \mid \chi \wedge \chi$$

In the following, our interest is in expressing user-level specifications, which by the nature of the application domain are to be seen as always finite. For this reason, naturally we constrain the language to finite sets of atoms p to express atomic specifications, denoted with $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$. Since Σ is finite, not all valid formulae in UL are included in Σ. Where appropriate we will refer to the set of atoms as `atom`, to the set of literals (i.e. atoms or their negations) as `literal` and to propositional formulae as `PROP`. Given the application under consideration, it seems more useful to be able to speak directly about combinations of specifications and invalid specifications. Accordingly, our language contains connectives for negation and conjunction. Less useful seems to be the case of disjunctions and implications between specifications. Naturally, these can be recovered from the connectives in the syntax, and in the following we provide examples where disjunction and implication are used as well. Beyond standard Boolean formulae, we extend the language with four new modalities:

- $\Box_\alpha\phi$ is interpreted as "ϕ is included in the specification with degree α"; a full gloss of this modal formula would be "ϕ is a correct use up to degree α of the computational system in view of the specification, with respect to the user level/context in consideration", with $\alpha \in [0, 1]$;
- $\Diamond_\alpha\phi$ is interpreted as "ϕ is not excluded from the specification with degree α"; a full gloss of this modal formula would be "ϕ is a possible use up to degree α of the computational system in view of the specification, with respect to the user level/context in consideration", with $\alpha \in [0, 1]$;

- $A^{i \mapsto j}\phi$ is interpreted as "ϕ is abstracted from user level i to user level j"; a full gloss of this modal formula would be "ϕ is a correct use of the computational system in view of the specification at user level i and it is abstracted at user level j";
- $I^{j \mapsto i}\phi = \neg A^{i \mapsto j} \neg \phi$ is interpreted as "ϕ is implemented from user level j in user level i"; a full gloss of this modal formula would be "ϕ is a correct use of the computational system in view of the specification at user level j and it is implemented at user level i".

Note that $\Box/\Diamond$ and A/I modalities range over propositions and iterate. The reason to split them is to avoid mixed iteration. While the modality for implementation is technically a dual of abstraction (as shown below in Theorem 1), we consider its semantic interpretation crucial for the domain of application and therefore we explicitly add it to the language.

3.1. *Semantics*

We start the semantic characterization of *LPI* by an appropriate interpretation of a standard frame structure. We will model three kinds of information (abstract, instructional, and operational) through distinct accessibility relations across informational states and ULs:

- R_{Abs}: *Abstraction accessibility relation* from a lower level to one above,
- R_{Ins}: *Instruction accessibility relation* within any given level,
- R_{Imp}: *Implementation accessibility relation* from a higher level to one below.

Definition 3 (User Context). A *user context*, or *context of use* $C_u = \langle W^u, R^u_{Ins} \rangle$ is a pair composed by

- a finite set W^u of accessible states for user u and
- an instructional accessibility relation $R^u_{Ins} \subseteq W^u \times W^u$.

We denote with $R^u_{Ins_w} = \{w' \in W^u \mid wR^u_{Ins}w'\}$.

Definition 4 (User Level). Given a set of users $U = \{u_1, u_2, ..., u_n\}$, a *user level* is a pair $UL = \langle W^U, R^U_{Ins} \rangle$ composed by

- a non-empty set $W^U = \bigsqcup_{i=1}^{n} W^{u_i}$ *i.e.*, the disjoint union of all informational states for all users in U, and
- an instructional accessibility relation $R^U_{Ins} = \bigsqcup_{i=1}^{n} R^{u_i}_{Ins}$, *i.e.*, the disjoint union of all accessibility relations of all users in U.

We denote with $R^U_{Ins_w} = \{w' \in W^U \mid wR^U_{Ins}w'\}$.

Note that the ordered pairs of worlds in R^U_{Ins}, as well as the worlds in $R^U_{Ins_w}$, are indexed with respect of the user they refer to, such that $\{\{w\},\{w,w'\}\}_{u_i} \neq \{\{w\},\{w,w'\}\}_{u_j}$, and $w'_{u_i} \neq w'_{u_j}$ $\forall u_i, u_j \in U$.

Definition 5 (Informational Frame). An *informational frame* is a triple $F = \langle \mathcal{L}, R_{Abs}, R_{Imp} \rangle$ composed by

- a non-empty finite set $\mathcal{L} = \{UL_0, UL_1, ..., UL_n\}$ of ULs,
- a partial order $<$ on the indexes of ULs such that, given UL_i and UL_j, $i < j$ if and only if UL_i is closer than UL_j to the physical level, identified as UL_0,
- an abstraction accessibility relation $R_{Abs} \subseteq W^{U_i} \times W^{U_j}$,
- and an implementation accessibility relation $R_{Imp} \subseteq W^{U_j} \times W^{U_i}$,

such that W^{U_i} denotes $W^U \in UL_i$ for any $i < j$, *i.e.*, UL_i is closer to the physical level than UL_j.

Given two ULs UL_i and UL_j, with $i < j$, R_{Abs_w} identifies the set of all states w' in W^{U_j} *abstracting* all functionalities that hold at $w \in W^{U_i}$:

$$R_{Abs_w} = \{w' \in W^{U_j} \mid wR_{Abs}w'\} \; with \; w \in W^{U_i}$$

Conversely, $R_{Imp_{w'}}$ identifies the set of all states in W^{U_i} *implementing* all functionalities that hold at $w' \in W^{U_j}$:

$$R_{Imp_{w'}} = \{w \in W^{U_i} \mid w'R_{Imp}w\} \; with \; w' \in W^{U_j}$$

We now extend the frame structure to appropriate models for each context of use, for each UL and for each set of ULs. We associate to each an appropriate set of valuation clauses.

Definition 6 (Context Model). A user-context model $M_{C_j} = \langle C_{u_j}, v_j \rangle$ is a pair composed by

- a user-context C_{u_j} and
- a valuation $v_j := W^{u_j} \mapsto 2^{\mathtt{atom}}$ and if $j \neq 0$ then it is such that $\forall w \in W^{u_j}.\forall p \in \mathtt{atom}$ if $p \in v(w)$ then $\exists w' \in W^{u_i}$ and $v' \in M_{C_{j-1}}$ such that $p \in v(w')$ and if $p \notin v(w)$ then $\exists w' \in W^{u_i}$ and $v' \in M_{C_{j-1}}$ such that $p \notin v(w')$.

Starting from the formulae valid *within* one user context, the standard definitions for propositional logic operators and classical modal operators $\Box\phi$ and $\Diamond\phi = \neg\Box\neg\phi$ for instructional accessibility relation are given in the following. Below, we denote with $R^u_{Ins_w}(\phi)$ the set of worlds accessible from the current one w according to R_{Ins_w} by agent u, in which ϕ holds. As we are confined to a single user context, and there is no risk of confusion between accessibility relations, R_{Ins} is abbreviated as R for readability reasons. Note, moreover, that here and in the following semantic clauses we use ϕ as a generic formula respecting the appropriate formation clauses as given in Definition 2.

Definition 7 (Semantic valuation in a User Context).

$$M_{C_u}, w \models p \text{ iff } p \in v(w)$$

$$M_{C_u}, w \models \neg\phi \text{ iff } M_{C_u}, w \not\models \phi$$

$$M_{C_u}, w \models \phi_1 \wedge \phi_2 \text{ iff } M_{C_u}, w \models \phi_1 \textit{ and } M_{C_u}, w \models \phi_2$$

$$M_{C_u}, w \models \Box_\alpha \phi \text{ iff } \frac{|R^u_w(\phi)|}{|R^u_w|} \geq \alpha \text{ if } R^u_w \neq \emptyset$$

$$M_{C_u}, w \models \Diamond_\alpha \phi \text{ iff } \frac{|R^u_w(\phi)|}{|R^u_w|} > 1 - \alpha \text{ if } R^u_w \neq \emptyset$$

The proposed semantics is well-defined on finite models. In particular, the intended domain of application as mentioned above is meant to refer to specifications for computational systems, which are always finite. In this sense, the ratios on the number of accessibility relations occurring in the definition of $\Box_\alpha$ and $\Diamond_\alpha$ always return standard values in $[0, 1]$. As mentioned in Legastelois, Lesot, and d'Allonnes (2017), one can replace this counting approach by a probabilistic framework to account for domains of infinite size. The loose inequality for $\Box$ becomes a strict one for $\Diamond$ to preserve the duality constraint $\Diamond_\alpha\phi \rightarrow \neg\Box_\alpha\neg\phi$. It follows directly that $\alpha = 1$ corresponds to classical modalities, and $\alpha = 0$ corresponds to a tautology for $\Box$ and a contradiction for $\Diamond$ (345). In particular, $\forall\alpha \in [0, 1]$ it holds that $\models \Diamond_\alpha\phi \rightarrow \Box_{1-\alpha}\phi$ and $\not\models \Box_\alpha\phi \rightarrow \Diamond_{1-\alpha}\phi$. The condition on finite models will be further strengthened below in the metatheory, where we restrict our analysis to the $\Box_1$ modality. The above clauses can be glossed as follows. A specification p is valid in a state within a context of use if and only if that instance of use makes p hold; it validates its negation otherwise. Clauses for conjunction, disjunction and implication respect classical criteria. The satisfiability clause for $\Box_\alpha\phi$ says that a formula ϕ is included in the specification with degree α within a context of use if and only if the size of the set of states that make ϕ hold over all meaningful states accessible by that context of use is equal or greater than α. As in normal modal logics, if $R^u_w = \emptyset$, then $M_{C_u}, w \models \Box_1 \phi$, *i.e.*, the classical modality $\Box\phi$. The satisfiability clause for $\Diamond_\alpha \phi$ says that formula ϕ is not excluded by the specification with degree α within a context of use if and only if the size of the set of states that make ϕ hold over all meaningful states accessible by that context of use is greater than $1 - \alpha$. If $R^u_w = \emptyset$, then $M_{C_u}, w \not\models \Diamond_\alpha \phi$.

The following relation is a direct consequence of Definition (7):

Proposition 1. Given a user context, for any user-context model $M_{C_u} = \langle\langle W^u, R^u \rangle, v\rangle$ and any $w \in W^u$,

$$M_{C_u}, w \models \Diamond_1 \phi \text{ iff } R^u_w \neq \emptyset \text{ and } M_{C_u}, w \models \Box_{\frac{1}{|R^u_w|}} \phi$$

Definition 8 (Truth in a User-Context Model). A formula ϕ is true in a user-context model M_{C_u} written $M_{C_u} \models \phi$ if and only if $M_{C_u}, w \models \phi$ for every $w \in W$.

Example 1. Consider the user context in Figure 1, composed by two informational states, w_1, w_2, and an instructional accessibility relation for user u_1. This can represent the context of a final user of a spreadsheet (u_1), who can fill cells with content (r) and change size and background color of the cells in the sheet, by using the *change size tool* (p in w_1) and the *change background tool* (q in w_3). In the example, only w_2 validates q, therefore, according to Definition 7, $M_{C_{u_1}}, w_1 \models \Box_{1/2} q$, given that only one in two worlds accessible to u_1 from w_1 validates q, and similarly $M_{C_{u_1}}, w_2 \models \Box_{1/2} q$. Note that $M_{C_{u_1}}, w_1 \models \Box_1 r$, as well as $M_{C_{u_1}}, w_2 \models \Box_1 r$, hence both $\Box_1 r$ and r are true in the user-context model $M_{C_{u_1}}$. Accordingly, $M_{C_{u_1}}, w_1 \models \Box_1 \Box_{1/2} p$, and also $M_{C_{u_1}}, w_2 \models \Box_1 \Box_{1/2} p$ (same for q).

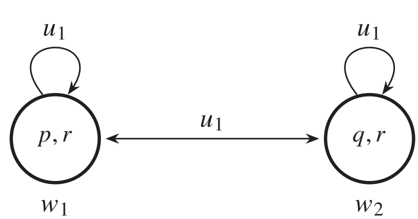


Figure 1: A serial and reflexive user context

For formulae valid *within* a UL, i.e. for all agents that have similar contexts of use, we define a UL model and corresponding semantic evaluation clauses as follows (again, as we are limiting our analysis to a single UL, R_{Ins} is abbreviated as R for readability reasons):

DEFINITION 9 (UL Model). A user-level model $M_{UL} = \langle UL, v\rangle$ is a pair composed by

- a user level UL with user contexts $C_1, \ldots, C_n$, and
- a valuation $v := \bigcup_{i=1}^{n} v_i$,

where each pair $\langle C_i, v_i\rangle$ is a user-context model M_{C_i}.

When referring to a specific UL, we will use the notation $M_{UL_i} = \langle UL_i, v_i\rangle$.

DEFINITION 10 (Semantic valuation in a User Level).

$$M_{UL}, w \models p \text{ iff } p \in v(w)$$

$$M_{UL}, w \models \neg\phi \text{ iff } M_{UL}, w \not\models \phi$$

$$M_{UL}, w \models \phi_1 \wedge \phi_2 \text{ iff } M_{UL}, w \models \phi_1 \textit{ and } M_{UL}, w \models \psi$$

$$M_{UL}, w \models \Box_\alpha\phi \text{ iff } \frac{\sum_{i=1}^{n} |R_w^{u_i}(\phi)|}{\sum_{i=1}^{n} |R_w^{u_i}|} \geq \alpha \text{ if } R_w \neq \emptyset$$

$$M_{UL}, w \models \Diamond_\alpha\phi \text{ iff } \frac{\sum_{i=1}^{n} |R_w^{u_i}(\phi)|}{\sum_{i=1}^{n} |R_w^{u_i}|} > 1 - \alpha \text{ if } R_w \neq \emptyset$$

These semantic clauses remain unchanged for atoms, and Boolean operators, extending their scope from a user context to a UL. The satisfiability clause for $\Box_\alpha\phi$ says that a formula ϕ is included in the specification with degree α within a UL if and only if the sum of the sizes of the sets of states that make ϕ hold for all agents over all meaningful states accessible by all agents in that UL is equal or greater than α. The satisfiability clause for $\Diamond_\alpha\phi$ says that formula ϕ is not excluded by the specification with degree α within a UL if and only if the sum of the sizes of the sets of states that make ϕ hold for all agents over all meaningful states accessible by all agents in that UL is greater than $1 - \alpha$. As for user contexts, if $R_w^U = \emptyset$, then $M_{UL}, w \models \Box_1\phi$, and $M_{UL}, w \not\models \Diamond_\alpha\phi$. Note that

$$\sum_{i=1}^{n} |R_w^{u_i}| = |\bigcup_{i=1}^{n} R_w^{u_i}| = |R_w^U|.$$

Proposition 1 holds also within a UL, and the following relation is direct consequence of Definition 10:

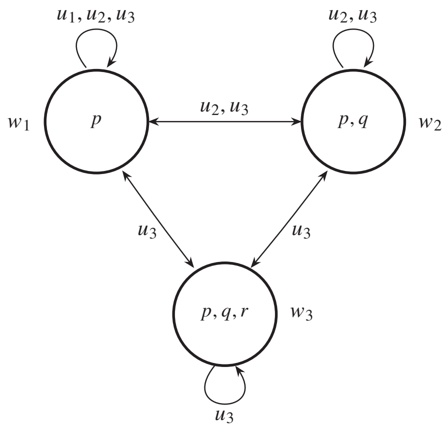


Figure 2: A serial UL consisting of three user contexts $C_{u_1} = \langle\{w_1\}, R^{u_1}_{Ins}\rangle$, $C_{u_2} = \langle\{w_1, w_2\}, R^{u_2}_{Ins}\rangle$, $C_{u_3} = \langle\{w_1, w_2, w_3,\}, R^{u_3}_{Ins}\rangle$.

Proposition 2. Given a UL and its set of users $U = \{u_1, u_2, ..., u_n\}$, for any user-level model $M_{UL} = \langle\langle W^U, R\rangle, v\rangle$ and any $w \in W^U$,

$$M_{UL}, w \models \Diamond_1\phi \text{ iff } R_w \neq \emptyset \text{ and } M_{UL}, w \models \Box_{\frac{1}{\sum_{i=1}^{n} |R_w^{u_i}|}}\phi$$

Definition 11 (Truth in a User-Level Model). A formula ϕ is true in a user-level model M_{UL} written $M_{UL} \models \phi$ if and only if $M_{UL}, w \models \phi$ for every $w \in W^U$ and every $C_u \in UL$.

Example 2. Consider the UL in Figure 2, composed by three user contexts, $C_{u_1} = \langle\{w_1\}, R^{u_1}_{Ins}\rangle$, $C_{u_2} = \langle\{w_1, w_2\}, R^{u_2}_{Ins}\rangle$, $C_{u_3} = \langle\{w_1, w_2, w_3\}, R^{u_3}_{Ins}\rangle$. This can represent the front-end developer UL, *e.g.*, *Javascript* developer UL: u_1 can be responsible of the graphic interface as intended by the interface designer; u_2 will be in charge of the interaction of the computational artefact with the final user, as intended by the interaction designer, on the basis of the graphics realized by the interface designer; u_3 is responsible for the communication with the data structures realized by the database designer. Data received from the interface designer (p in w_1) can be read by u_1, u_2 and u_3, and both u_2 and u_3 can read data received by the interaction designer (q in w_2), but only u_3 has access to data as established by the database designer (r in w_3). According to Definition 10, $M_{UL}, w_1 \models \Box_{1/6}r$: indeed the sum of the cardinalities of the sets of worlds accessible from w_1 by the accessibility relations of all users in the UL is 6, as follows: w_1 is accessible by $R^{u_1}_{Ins}$

(cardinality 1), w_1, w_2 are accessible by $R^{u_2}_{Ins}$ (cardinality 2), w_1, w_2, w_3 are accessible by $R^{u_3}_{Ins}$ (cardinality 3); the cardinality of the set of worlds among these which satisfy r is instead equal to 1: w_3 by means of $R^{u_3}_{Ins}$. Note that, for every w in every user context, both p and $\Box_1 p$ are valid formulae, hence p and $\Box_1 p$ are true in the user-level model M_{UL}.

We consider a class of UL models in the standard sense as composed by all UL models with the same properties. Then we define the notion of validity in a UL.

Definition 12 (Validity in a UL). A formula ϕ is valid in a class of UL models, denoted $C_{UL} \vDash \phi$, if $M_{UL}, w \vDash \phi$ for every world w in every UL model in C_{UL}.

Proposition 3. If $C_{UL} \vDash \phi$ then $C_{UL} \vDash \Box_1 \phi$.

Proof. Assume $C_{UL} \vDash \phi$ and let $M_{UL} \in C_{UL}$ and $w \in M_{UL}$. If $wR^{u_i}w'$, for some $u_i \in UL$ since $C_{UL} \vDash \phi$ then $M_{UL}, w' \vDash \phi$ and hence $M_{UL}, w' \vDash \Box_1 \phi$. Since both $M_{UL} \in C_{UL}$ and $w \in M_{UL}$ were arbitrary, $C_{UL} \vDash \Box_1 \phi$. □

Distinct ULs are connected by means of the abstraction accessibility relation R_{Abs} and the implementation accessibility relation R_{Imp}, which actually constitute a single symmetric *abstraction/implementation* relation R_{AI}. You gotta keep them separated to understand the direction of the relation, and consequently from which perspective, *i.e.*, UL, we are considering it. The symmetry of this relation means that, ideally, each Σ for each UL should correctly and coherently implement the Σ of the level immediately above, and should be correctly and coherently implemented in the Σ of the level immediately below, or the other way around. The highest UL should be reachable via abstraction by any level below, and the lowest UL should be reachable via implementation by any level above; with respect to a single UL, a correct specification includes only those atoms which are valid at lower ULs, although not necessarily all of them. In fact, any specification at a UL will be the abstraction of only a coherent subset of implemented formulae. This formally reflects the notion of implementation for ULs given in Buda and Primiero (2023).

Definition 13 (Informational Model). An informational model $M_F = \langle F, v \rangle$ is a pair composed by

- an informational frame F,with user levels $UL_0, \dots, UL_n$, and
- a valuation $v := \bigcup_{i=1}^{n} v_i \in M_{UL_i}$

where each pair $\langle UL_i, v_i \rangle$ is a user-level model M_{UL_i}.

As briefly mentioned at the beginning of the section, we assume that lower ULs, *i.e.*, levels closer to the physical world, are composed by a greater number of simpler informational states than higher levels.

Definition 14 (Semantic valuation in an Informational Frame). The semantics of Abstraction and Implementation operators for a formula ϕ with respect to some existing set of minimal cardinality of literals $X_\phi = \{l^1, \ldots, l^k\}$ such that $X_\phi \vDash \phi$ is defined by the following two clauses:

- $M_F, w_j \vDash A^{i \mapsto j}\phi$ iff
    1. $M_{UL_j}, w_j \vDash \phi$
    2. $\forall l^i \in X_\phi . \exists w_i$ such that $w_i R_{Abs} w_j$ and $M_{UL_i}, w_i \vDash l^i$
- $M_F, w_j \vDash I^{j \mapsto i}\phi$ iff
    1. $M_{UL_j}, w_j \vDash \phi$
    2. $\exists w_i$ such that $w_j R_{Imp} w_i . \exists l^i \in X_\phi$ and $M_{UL_i}, w_i \vDash l^i$

A formula ϕ is abstracted from w_i to w_j if and only if it is true at w_j and for all elements of some of the minimal sets X_ϕ of literals such that ϕ is derivable from X_ϕ, there is a world in UL_i connected with w_j by means of the abstraction accessibility relation that satisfies at least one such literal. A formula ϕ is implemented from w_j to w_i if and only if it is true at w_j and there is an element of X_ϕ derivable at a world w_i, connected to w_j by the implementation accessibility relation.

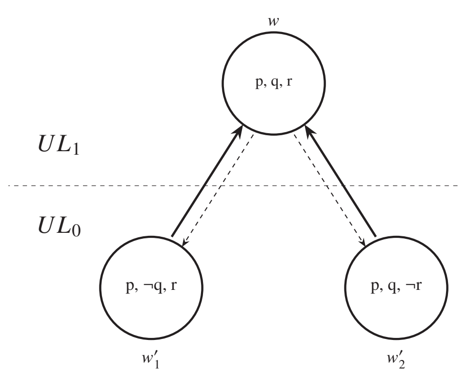


Figure 3: The informational frame described in Example 3. In this and the following examples, the abstraction accessibility relation R_{Abs} is is represented with thick arrows and the implementation accessibility relation R_{Imp} with dashed arrows.

Example 3. Consider the frame in Fig. 3. In the world w the following (non exhaustive list of) formulae hold:

- non-modal formulae: $M_{UL_j}, w \vDash p$, $M_{UL_j}, w \vDash q$, $M_{UL_j}, w \vDash r$, $M_{UL_j}, w \vDash p \wedge q$, $M_{UL_j}, w \vDash p \wedge r$, $M_{UL_j}, w \vDash q \wedge r$, $M_{UL_j}, w \vDash p \wedge q \wedge r$;
- abstraction formulae: $M_{UL_j}, w \vDash A^{i \mapsto j} p$, $M_{UL_j}, w \vDash A^{i \mapsto j} q$, $M_{UL_j}, w \vDash A^{i \mapsto j} r$, $M_{UL_j}, w \vDash A^{i \mapsto j}(p \wedge q)$, $M_{UL_j}, w \vDash A^{i \mapsto j}(p \wedge r)$, $M_{UL_j}, w \vDash A^{i \mapsto j}(q \wedge r)$, $M_{UL_j}, w \vDash A^{i \mapsto j}(p \wedge q \wedge r)$;
- implementation formulae: $M_{UL_j}, w \vDash I^{j \mapsto i} p$, $M_{UL_j}, w \vDash I^{j \mapsto i} q$, $M_{UL_j}, w \vDash I^{j \mapsto i} r$, $M_{UL_j}, w \vDash I^{j \mapsto i} p \wedge q$, $M_{UL_j}, w \vDash I^{j \mapsto i} p \wedge r$, $M_{UL_j}, w \vDash I^{j \mapsto i} q \wedge r$, $M_{UL_j}, w \vDash I^{j \mapsto i} p \wedge q \wedge r$;

EXAMPLE 4. Consider the frame in Fig. 4. The following formulae are valid:
- $M_{UL_i}, w'_1 \vDash \neg p$, $M_{UL_i}, w'_2 \vDash \neg p$ and $M_{UL_j}, w \vDash A^{i \mapsto j} \neg p$.
- $M_{UL_j}, w \vDash \neg p$, $M_{UL_i}, w'_1 \vDash I^{j \mapsto i} \neg p$, and $M_{UL_i}, w'_2 \vDash I^{j \mapsto i} \neg p$.

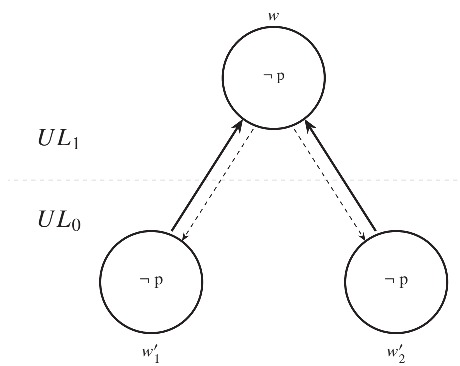


Figure 4: The informational frame described in Example 4.

The Abstraction operator allows for a consistent selection of possibly inconsistent formulae to be abstracted from a lower to a higher level, i.e. a formula $\Box_\alpha \phi$ with $\alpha < 1$ valid at UL_i to UL_j. In this case, whether ϕ or $\neg\phi$ is abstracted may depend on different design choices; in general abstracted formula are those as by specification, and an entrenching relation may be additionally defined on formulae to say which between a formula and its negation should be abstracted; alternatively, also some quantitative principle may be defined, e.g. a formula $\Box_\alpha \phi$ is abstracted iff $\alpha > 0.5$; similarly for the Implementation operator. In both cases, we leave such criteria to design choices.

EXAMPLE 5. Consider the frame in Fig. 5. The following formulae are valid:
- $M_{UL_i}, w'_1 \vDash p$, $M_{UL_i}, w'_2 \vDash p, q$ and $M_{UL_j}, w \vDash A^{i \mapsto j} p \wedge q$.
- $M_{UL_j}, w \vDash p, q$, $M_{UL_i}, w'_1 \vDash I^{j \mapsto i} p$, $M_{UL_i}, w'_2 \vDash I^{j \mapsto i} p, I^{j \mapsto i} q$.

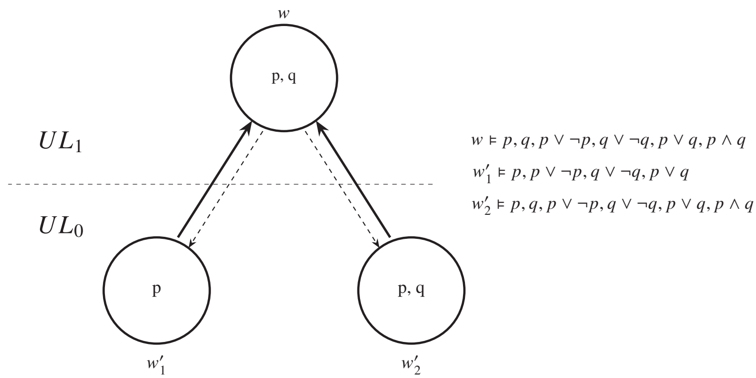


Figure 5: The informational frame described in Examples 5, 6.

Abstraction and Implementation operators allow to abstract and implement all the consequences and tautologies derivable from abstracted and implemented formulae:

EXAMPLE 6. Consider again the frame in Fig. 5. The following formulae are valid:

- $M_{UL_i}, w'_1 \vDash p$,
  $M_{UL_i}, w'_2 \vDash p, q$ and
  $M_{UL_j}, w \vDash A^{i\mapsto j}(p \vee \neg p), A^{i\mapsto j}(q \vee \neg q), A^{i\mapsto j}(p \wedge q)$.
- $M_{UL_j}, w \vDash p, q$,
  $M_{UL_i}, w'_1 \vDash I^{j\mapsto i}p, I^{j\mapsto i}(p \vee \neg p), I^{j\mapsto i}(q \vee \neg q), I^{j\mapsto i}(p \vee q)$,
  $M_{UL_i}, w'_2 \vDash p, q, I^{j\mapsto i}p, I^{j\mapsto i}q, I^{j\mapsto i}(p \vee \neg p), I^{j\mapsto i}(q \vee \neg q), I^{j\mapsto i}(p \wedge q)$

EXAMPLE 7. Consider the informational frame in Figure(6), composed by two ULs:

- $UL_0 = \langle W^{UL_0}, R_{Ins_{UL_0}} \rangle$, with $W^{UL_0} = \{w_3, w_4, w_5, w_6, w_7,\}$ and $R_{Ins_{UL_0}}$, and
- $UL_1 = \langle W^{UL_1}, R_{Ins_{UL_1}} \rangle$, with $W^{UL_1} = \{w_1, w_2\}$ and $R_{Ins_{UL_1}}$.

The abstraction accessibility relation $R_{Abs} \subseteq W^{UL_0} \times W^{UL_1}$ is marked with thick arrows and the implementation accessibility relation $R_{Imp} \subseteq W^{UL_1} \times W^{UL_0}$ with dashed arrows. UL_0 and UL_1 can respectively represent the front-end developer UL and the final-user UL, where the high-level functionality "*click the button to send the message*" (a, b, c, d in w_1) is implemented in the low-level functions "*onClick*" (a in w_3) and "*getURL*" (b in w_4), combined with "*getMessage*" and "*getSender*" (c and d in w_5). Formally, in W^{UL_1}, the formula $\neg(a \wedge e)$ holds in both w_1 and w_2, as it follows from the formulae in both worlds; similarly in W^{UL_0}; therefore, the formula $\neg(a \wedge e)$ is correctly implemented from and abstracted to both w_1 and w_2.

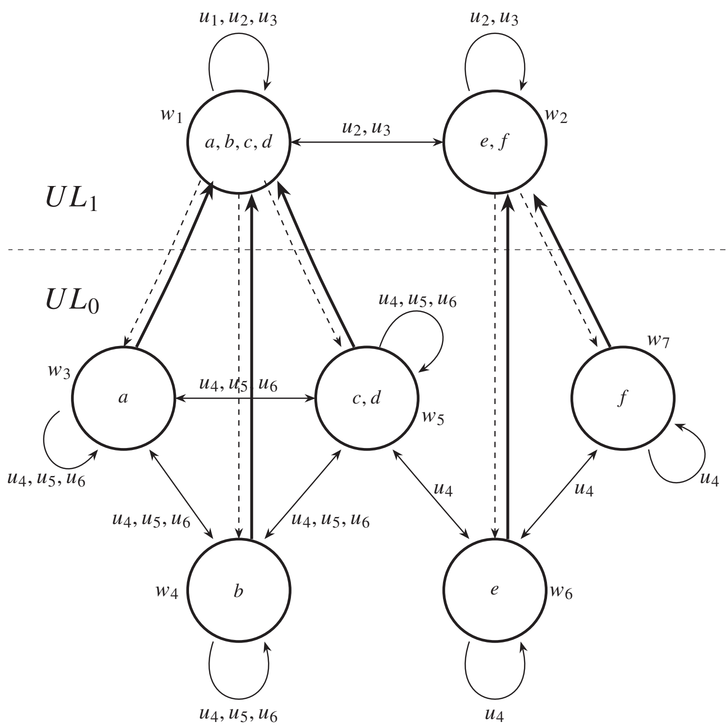


Figure 6: An informational frame consisting of two ULs $UL_0 = \langle W^{UL_0}, R_{Ins_{UL_0}} \rangle$ and $UL_1 = \langle W^{UL_1}, R_{Ins_{UL_1}} \rangle$.

Theorem 1. We show that the following equivalence holds:

$$M_F, w_j \vDash A^{i \mapsto j}\phi \leftrightarrow M_F, w_j \vDash \neg I^{j \mapsto i} \neg \phi$$

Proof. We prove the implication in both directions:

- LTR: $M_F, w_j \vDash A^{i \mapsto j}\phi \rightarrow M_F, w_j \vDash \neg I^{j \mapsto i}\neg\phi$. According to Definition 14, if $M_F, w_j \vDash A^{i \mapsto j}\phi$, then $M_F, w_j \vDash \phi$. By contradiction, assume now $M_F, w_j \vDash \neg\neg I^{j \mapsto i}\neg\phi$, *i.e.*, $M_F, w_j \vDash I^{j \mapsto i}\neg\phi$, then $M_F, w_j \vDash \neg\phi$, against the assumption.
- RTL: $M_F, w_j \vDash \neg I^{j \mapsto i}\neg\phi \rightarrow M_F, w_j \vDash A^{i \mapsto j}\phi$. According to Definition 14, if $M_F, w_j \vDash \neg I^{j \mapsto i}\neg\phi$ then two cases apply:
  1. if $M_F, w_j \nvDash \neg\phi$ then $M_F, w_j \vDash \phi$; against the first condition for $M_F, w_j \vDash \neg A^{i \mapsto j}\phi$ as by Definition 14;

2. if $\forall w_i : w_i R_{Abs} w_j \neg\exists\chi^i \in X_\phi$ for $X \vDash \phi$ such that $M_{UL_i}, w_i \vDash \chi^i$, against the construction of $v := W^{u_j} \mapsto 2^{\mathtt{atom}}$ as by Definition 6. □

DEFINITION 15 (Truth in an Informational Model). A formula ϕ is true in an informational model M_F written $M_F \models \phi$ if and only if $M_F, w \models \phi$ for every $w \in W^U$, $C_u \in UL$ and $UL \in \mathcal{L}$.

With these basic semantic definitions in place, we present rules and axioms valid for the $\Box$ operator and for the pair $A^{i \mapsto j}$ and $I^{j \mapsto i}$.

4. Axiomatization and Metatheory for Full Specification

On the basis of the applied semantic interpretation we gave in Section 3, the logic of "using information" is equivalent to "being informed" when considering $\Box_1$ modalities, corresponding to the axioms of KTB. The possibility to axiomatize graded modalities has been studied in Legastelois, Lesot, and d'Allonnes (2017), where it is specified for which axioms of a normal modal logic it is possible to provide graded versions and for which ones not. In this case, only axioms that can be graded become valid, the corresponding logic being a version of KD.

On the other hand, the relation of abstraction/implementation is symmetric by definition and transitivity is a desirable property for both R_{Abs} and R_{Imp} when three distinct levels are involved. Therefore, "using information" expresses in these enriched frames a cooperative condition in which we have an $S5$ system across ULs. This means that when considering the combined action of different ULs, "using information" is a broader epistemic condition, with a stronger logic constrained by order in the case of the abstraction and implementation modalities, as well as by ULs. In the following, we are interested in the restricted task of considering the impact of the new modalities A and I on this logic. For this reason, we consider only the case in which all the propositional variables in consideration are included in the specification Σ for each UL. Hence, for our meta-theoretical results, showing that the A and I modalities are conservative extensions, we rely on the full specification case expressed with only grade 1 modalities.

In the following we will use $\vdash_u$ for reasoning within a single user context, $\vdash_{UL_i}$ for reasoning within a UL, and use a different index when interacting with a distinct $\vdash_{UL_j}$. We denote with $\vdash_{LPI}$ the inference relation for the entire logic of pragmatic information.

- All PL tautologies.

- *Modus Ponens*:
  $\vdash_{UL_i} \phi, \vdash_{UL_i} \phi \rightarrow \psi \;\Rightarrow \vdash_{UL_i} \psi$.
- *Necessitation Rule*:
  $\vdash_{UL_i} \phi \Rightarrow \vdash_{UL_i} \Box_1\phi$,
  $\vdash_{UL_i} \phi \Rightarrow \vdash_{UL_j} A^{i\mapsto j}\phi$,
  $\vdash_{UL_i} \phi \Rightarrow \vdash_{UL_j} I^{j\mapsto i}\phi$.
- *Axiom K*:
  $\Box_1(\phi \rightarrow \psi) \rightarrow (\Box_1\phi \rightarrow \Box_1\psi)$,
  $A^{i\mapsto j}(\phi \rightarrow \psi) \rightarrow (A^{i\mapsto j}\phi \rightarrow A^{i\mapsto j}\psi)$ with $i < j$,
  $I^{j\mapsto i}(\phi \rightarrow \psi) \rightarrow (I^{j\mapsto i}\phi \rightarrow I^{j\mapsto i}\psi)$ with $i < j$.
- *Axiom D*
  $\Box_1\phi \rightarrow \Diamond_1\phi$,
  $A^{i\mapsto j}\phi \rightarrow I^{j\mapsto i}\phi$.
- *Axiom T*:
  $\Box_1\phi \rightarrow \phi$,
  $A^{i\mapsto j}\phi \rightarrow \phi$ with $i < j$,
  $I^{j\mapsto i}\phi \rightarrow \phi$ with $i < j$.
- *Axiom B*:
  $\phi \rightarrow \Box_1\Diamond_1\phi$,
  $\phi \rightarrow A^{i\mapsto j}I^{j\mapsto i}\phi$ with $i < j$,
  $\phi \rightarrow I^{j\mapsto i}A^{i\mapsto j}\phi$ with $i < j$.
- *Axiom 4*:
  $A^{i\mapsto k}\phi \rightarrow A^{i\mapsto j}A^{j\mapsto k}\phi$ with $i < j < k$,
  $I^{k\mapsto i}\phi \rightarrow I^{k\mapsto j}I^{j\mapsto i}\phi$ with $i < j < k$.

The following metatheoretical results of soundness and completeness of LPI are formulated for the standard case in which all the propositional variables in consideration are included in the specification for each UL, *i.e.*, $\alpha = 1$. This means that we prove such properties only with respect to a normal modal logic whose frames are characterized by equivalence relations and euclideaness, and not with respect to its graded counterpart. This allows us to repurpose known results of the modal logic $S5$: our main aim is to show that such results are preserved under the addition of accessibility relations of abstraction and implementation across ULs. To this aim, it is particularly important to notice that finiteness of the canonical model is granted by results on local tabularity – i.e. with finitely many atoms we can express only finitely many meanings – of the modal logic $S5$ based on its finite height. For these results on local tabularity see Shapirovsky and Shehtman (2016) and Shapirovsky (2020).

THEOREM 2 (Soundness). If $\vdash_{LPI} \phi$ then $\models \phi$, where LPI is the Logic of Pragmatic Information and $\models$ denotes the class of informational frames satisfying an equivalence relation.

Proof. By cases:

- Modus Ponens is sound for each user at each UL;
- Necessitation is sound:
- we have to show that $M_{UL_i}, w_i \models \phi \rightarrow \Box_1\phi$ for any world w_i (as the rule is for UL_i); consider any arbitrary world $w_i \in UL_i$ such that $M_{UL_i}, w_i \models \phi$, then by Proposition 3 $M_{UL_i}, w_i \models \Box_1\phi$;
- we have to show that if $M_F, w_i \models \phi$ for all worlds in UL_i, then $M_F, w_j \models A^{i\mapsto j}\phi$, for all worlds in UL_j. Assume that there is an arbitrary world $w_j \in UL_j$ such that $w_i R_{Abs} w_j$, and $w_j \models \neg(A^{i\mapsto j}\phi)$. Then, for Definition 14,
  - either $w_j \models \neg\phi$ against the construction of $v := W^{u_j} \mapsto 2^{\texttt{atom}}$ as by Definition 6;
  - or $\exists w_i : w_i R_{Abs} w_j . w_i \nvDash \chi$ such that $\chi \in X$, with $X = \{\chi^1, \ldots, \chi^k\}$, set of minimal cardinality of literals such that $X \models \phi$, *i.e.*, $w_i \models \neg\chi$, hence $w_i \models \neg\phi$, against the assumption.
- we have to show that if $M_F, w_i \models \phi$ for all worlds in UL_i, then $M_F, w_j \models I^{j\mapsto i}\phi$, for all worlds in UL_j. Assume that there is an arbitrary world $w_j \in UL_j$ such that $w_i R_{Abs} w_j$ and $w_j \models \neg(I^{j\mapsto i}\phi)$. Then, for Definition 14,
  - either $w_j \models \neg\phi$ against the construction of $v := W^{u_j} \mapsto 2^{\texttt{atom}}$ as by Definition 6;
  - or $\forall w_i : w_i R_{Abs} w_j . w_i \nvDash \chi$ such that $\chi \in X$, with $X = \{\chi^1, \ldots, \chi^k\}$, set of minimal cardinality of literals such that $X \models \phi$, *i.e.*, $w_i \models \neg\chi$,hence $w_i \models \neg\phi$ against the assumption.
- K is sound:
- The proof for the weighted version of K is given in Legastelois, Lesot, and d'Allonnes (2017, 351); the case for $\alpha = 1$ is the limit case.
- we have to show that if $M_F, w_j \models A^{i\mapsto j}(\phi \rightarrow \psi)$ and $M_F, w_j \models A^{i\mapsto j}\phi$, then $M_F, w_j \models A^{i\mapsto j}\psi$; by assumptions and by Definition 14, we know that $M_{UL_j}, w_j \models \phi \rightarrow \psi$, $M_{UL_j}, w_j \models \phi$ hence, by Modus Ponens, $M_{UL_j}, w_j \models \psi$; now, assume $M_F, w_j \models \neg A^{i\mapsto j}\psi$. Then:
  - either $M_F, w_j \models \neg\psi$, against the assumptions;
  - or $\exists w_i : w_i R_{Abs} w_j . w_i \nvDash \chi$ such that $\chi \in X$, with $X = \{\chi^1, \ldots, \chi^k\}$, set of minimal cardinality of literals such that $X \models \psi$, *i.e.*, $w_i \models \neg\chi$, hence $w_i \models \neg\psi$, against the construction of $v := W^{u_j} \mapsto 2^{\texttt{atom}}$ as by Definition 6.

- we have to show that if $M_F, w_j \models I^{j\mapsto i}(\phi \rightarrow \psi)$ and $M_F, w_j \models I^{j\mapsto i}\phi$, then $M_F, w_j \models I^{j\mapsto i}\psi$; by assumptions and by Definition 14, we know that $M_{UL_j}, w_j \models \phi \rightarrow \psi$, $M_{UL_j}, w_j \models \phi$ hence, by Modus Ponens, $M_{UL_j}, w_j \models \psi$; now, assume $M_F, w_j \vDash \neg A^{i\mapsto j}\psi$. Then:
  - either $M_F, w_j \models \neg\psi$, against the assumptions;
  - or $\forall w_i : w_i R_{Abs} w_j . w_i \nvDash \chi$ such that $\chi \in X$, with $X = \{\chi^1, \dots, \chi^k\}$, set of minimal cardinality of literals such that $X \vDash \psi$, *i.e.*, $w_i \vDash \neg\chi$, hence $w_i \vDash \neg\psi$, against the construction of $v := W^{u_j} \mapsto 2^{\mathtt{atom}}$ as by Definition 6.
- D is sound:
- The proof for the weighted version of D is given in Legastelois, Lesot, and d'Allonnes (2017, 352); the case for $\alpha = 1$ is the limit case;
- we have to show that if $M_F, w_j \models A^{i\mapsto j}\phi$, then $M_F, w_j \models I^{j\mapsto i}\phi$; assume $M_F, w_j \vDash A^{i\mapsto j}\phi$, then by Definition 14 it follows directly that $M_F, w_j \vDash I^{j\mapsto i}\phi$.
- T is sound:
- we have to show that $M_{UL_i}, w_i \models \Box_1\phi \rightarrow \phi$; consider $M_{UL_i}, w_i \models \Box_1\phi$, then for any $w_i \in UL_i$ by Modus Ponens $M_{UL_i}, w_i \models \phi$;
- we have to show that $M_F, w_j \models A^{i\mapsto j}\phi \rightarrow \phi$; by Definition 14, $A^{i\mapsto j}\phi$ holds at some world w if and only if also ϕ holds at w;
- similarly for $I^{j\mapsto i}\phi$.
- B is sound:
- we have to show that $M_{UL_i}, w_i \models \phi \rightarrow \Box_1\Diamond_1\phi$; consider a world $M_{UL_i}, w_i \models \phi$, then $M_{UL_i}, w_i \models \Diamond_1\phi$ by Definition 10 as there is at least one world from where $\Box_1\phi$, namely w_i; hence by Necessitation $M_{UL_i}, w_i \models \Box_1\Diamond_1\phi$;
- we have to show that $M_F, w_j \models \phi \rightarrow A^{i\mapsto j}I^{j\mapsto i}\phi$, which amounts to prove $M_F, w_j \models I^{j\mapsto i}A^{i\mapsto j}\phi \rightarrow \phi$. By Definition 14, for a formula such as $I^{j\mapsto i}A^{i\mapsto j}\phi$ to be valid in a world w_j, it requires that w_j also validates $A^{i\mapsto j}\phi$, which in turn also requires the formula ϕ to be valid.
- similarly for $M_F, w_j \models \phi \rightarrow I^{j\mapsto i}A^{i\mapsto j}\phi$.
- 4 is sound:
- the validity of $A^{i\mapsto k}\phi \rightarrow A^{j\mapsto k}A^{i\mapsto j}\phi$ with $i < j < k$ reflects the fact that abstraction holds across any three distinct levels UL_i, UL_j, UL_k, with $i < j < k$ in the informational frame, *i.e.*, if $w_i R_{Abs} w_k$, $w_i R_{Abs} w_j$, $w_j R_{Abs} w_k$, and a valid formula ϕ is abstracted from w_i to w_k, then if it's abstracted from w_i to w_j, it's also abstracted from w_j to w_k. If $M_F, w_k \vDash A^{i\mapsto k}\phi$ then, by Definition 14 $M_F, w_k \vDash \phi$ and there exists a set of subformulae of ϕ such that, for all subformulae, w_i

validates at least one of them. If also $M_F, w_j \vDash A^{i\mapsto j}\phi$, then $M_F, w_j \vDash \phi$. Given that $w_j R_{Abs} w_k$, and w_j validates at least a subformula of ϕ, *i.e.*, ϕ, then $M_F, w_k \vDash A^{j\mapsto k}\phi$.
- Similarly for $I^{k\mapsto i}\phi \rightarrow I^{k\mapsto j}I^{j\mapsto i}\phi$ with $i < j < k$.

□

We now consider this axiomatization for a completeness result.

Definition 16. Let $\Gamma \nvdash_{LPI} \bot$, then Γ is maximal LPI-consistent if $\Gamma' \nvdash_{LPI} \bot$ for any $\Gamma' \supseteq \Gamma$.

This means that for the purposes of the current meta-theoretical results we assume that all functionalities are coherent and uses thereof are always made in accordance with the intended specification. Note that in the next section we will consider some cases where this assumption does not hold.

Proposition 4. Let Γ be a maximal LPI-consistent set of modal formulae of LPI. Then

1. Γ is closed under modus ponens and the axioms of LPI.
2. $LPI \subseteq \Gamma$.
3. for all formulae ϕ, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.
4. if $\phi, \psi \in \Gamma$, then $\phi \wedge \psi \in \Gamma$.
5. if $\phi \in \Gamma$, then $\phi \vee \psi \in \Gamma$.
6. if $\neg\phi$ or $\psi \in \Gamma$, then $\phi \rightarrow \psi \in \Gamma$.

Lemma 1 (Lindenbaum's Lemma). Let Γ be a LPI-consistent set of formulae. Then there exists a maximal LPI-consistent set Γ' such that $\Gamma \subseteq \Gamma'$.

Proof. Standard for Lindenbaum Lemma. □

Definition 17 (Canonical Model). The canonical model for the logic LPI is a tuple

$$M := \langle M_F, R_{Ins} \rangle$$

such that

- $F \in M_F$ includes in $\mathcal{L}$ all ULs for which formulas are being considered and W^U is the set of all maximal LPI-consistent sets of worlds;
- the accessibility relation R_{Ins} holds for $w, w' \in W^{U_i}$ if for all $\Box_1\phi \in w$, then $\phi \in w'$; then w, w' are related and we say they are both at level UL_i;
- the accessibility relation R_{Abs} holds for $w \in W^{U_i}, w' \in W^{U_j}$ iff for all ϕ such that for all $\mathtt{l} \in X_\phi$ such that X_ϕ is a minimal set of literals such that $X_\phi \vDash \phi$ and $\mathtt{l} \in w$ then $A^{i\mapsto j}\phi \in w'$;

- the accessibility relation R_{Imp} holds for $w' \in W^{U_j}$, $w \in W^{U_i}$ iff for all $\phi \in w'$ such that there exists $\mathtt{l} \in X$ such that X a minimal set of literals such that $X \models \phi$ and $\mathtt{l} \in w$, then $I^{j \mapsto i}\phi \in w'$;
- $v \in M_F$ is the union of all valuation functions for all user contexts at all ULs.

We constrain the Existence Lemma to the interesting formulae, considering that the lemma for $\Box_1\phi$ is trivially valid by contruction of R_{Ins}:

Lemma 2 (Existence Lemma). Given a world $w \in M$ and ϕ an arbitrary formula:

- if $\neg\Box_1\phi \in w$, then there exists a world w' such that $wR_{Ins}w'$ and $\neg\phi \in w'$, with $w, w' \in W^{U_i}$.
- if $A^{i \mapsto j}\varphi \in w'$, then there exists for every $l \in X_\varphi$, a world w on user level i such that $wR_{Abs}w'$ and $\mathtt{l} \in w$ with $w \in W^{U_i}$ and $w' \in W^{U_j}$.
- if $I^{j \mapsto i}\phi \in w'$, then there exists a $\mathtt{l} \in X_\phi$ for any of the minimal sets X_ϕ such that $X_\phi \models \phi$, and a world w such that $w'R_{Imp}w$ and $\mathtt{l} \in w$, with $w' \in W^{U_j}$ and $w \in W^{U_i}$.

Proof. For the three cases:

- Take $\neg\Box_1\phi \in w$, for some $w \in W^{U_i}$. We show that there exists w' which verifies the statement. Consider $X = \{\neg\phi\} \cup \{\psi \mid \Box_1\psi \in w\}$. We show that X is consistent. Suppose it is not. This means that, for $\psi_1, \ldots, \psi_n \in \{\psi \mid \Box_1\psi \in w\}$, $\psi_1 \wedge \cdots \wedge \psi_n \rightarrow \phi$. If this is the case though, by necessitation and axiom K, we also have that $\Box_1\psi_1 \wedge \cdots \wedge \Box_1\psi_n \rightarrow \Box_1\phi$ is a theorem of our logic. But then, by Proposition 4 on w, we also have that $\Box_1\phi \in w$. But this is impossible since we assumed that $\neg\Box_1\phi \in w$. Hence, X is consistent. Consider now, by Lemma 1, a w' which is maximally consistent such that $w' \supset X$. By Definition 17, $wR_{Ins}w'$. Moreover, $\neg\phi \in w'$.
- Let l be an arbitrary element of $X_\varphi \subseteq w'$. Since l is a literal in the maximally consistent set w, it must be that either $l = p$ or $l = \neg p$ for some propositional variable p. Without loss of generality, consider the case $l = p$. Since $A^{i \mapsto j}p \vee A^{i \mapsto j}\neg p$ is a theorem of the logic, it is an element of w'. Suppose that $A^{i \mapsto j}\neg p \in w'$. Since, $A^{i \mapsto j}\neg p$ implies $\neg p$ and $p \in w'$, we obtain a contradiction. Indeed, p and a formula implying $\neg p$ cannot inhabit the same maximally consistent set w'. We thus obtain that $A^{i \mapsto j}p \in w'$. But this means precisely that there is a world w on a lower user level i such that $p = l \in w$, by definition of the abstraction relation of the canonical model. Since l was chosen

arbitrarily, this concludes the proof.[1]

- Take $I^{j\mapsto i}\psi \in w'$, for some $w' \in W^{U_j}$. Let now X_ψ be any minimal set of literals that makes any such ψ true, and $X_\psi \subset w'$. Let moreover l be any element of X_ψ. We need to show that
  - there exists a world w such that $l \in w$ and
  - $w' R_{Imp} w$ and
  - $w \in W^{U_i}$

  First we construct $\bigcup_{\psi \in \{\psi | A^{i\mapsto j}\psi \in w'\}} X_\psi$. Note that such union is consistent because it is a subset of w', which is maximally consistent. Now consider the following two facts:
  1. for every atom p, the formula $I^{j\mapsto i}p \vee I^{j\mapsto i}\neg p$ is a theorem of the logic and hence it holds in w';
  2. because $I^{j\mapsto i}\psi \in w'$ implies that $\psi \in w'$, then by Proposition 4 for every p it holds that either $p \in \bigcup_{\psi \in \{\psi | A^{i\mapsto j}\psi \in w'\}} X_\psi$ or $\neg p$ is.

  Then, by Lemma 1, there exists a maximally consistent set of literals w such that $w \supset \bigcup_{\psi \in \{\psi | I^{j\mapsto i}\psi \in w'\}} X_\psi$. Since $l \in \bigcup_{\psi \in \{\psi | I^{j\mapsto i}\psi \in w'\}} X_\psi$, then also $l \in w$.
  Now we want to show that w' is in an implementation relation with this w and that the latter is in W^{UL_i}. Due to maximal consistency, for no literal $\mathtt{l} \in w$ it can be the case that $\mathtt{l} \in Y_\chi$ for some minimal set $Y_\chi \models \chi$ for any $I^{j\mapsto i}\chi \notin w'$. From this it follows directly that by the accessibility relation in Definition 17, $w' R_{Imp} w$ and $w \in W^{U_i}$.

□

LEMMA 3 (Truth Lemma). For any well-formed formula ϕ, and world $w \in M$

$$M_F, w \vDash \phi \text{ iff } \phi \in w$$

Proof. We proceed by induction on ϕ:

- $M_{UL}, w \vDash p$ holds by Definition iff $p \in v(w)$;
- for $M_{UL}, w \vDash \neg\phi$: for the left-to-right direction, by semantic definition if $M_{UL}, w \vDash \neg\phi$ then if $M_{UL}, w \nvDash \phi$; by induction hypothesis, the latter implies $\phi \notin w$; by maximal consistency $\neg\phi \in w$; for the opposite direction: by maximal consistency $\neg\phi \in w$ implies $\phi \notin w$; by induction hypothesis then $M_{UL}, w \nvDash \phi$; by semantic definition $M_{UL}, w \vDash \neg\phi$.
- for $M_{UL}, w \vDash \phi \wedge \psi$, by Definition $M_{UL}, w \vDash \phi$ iff $\phi \in w$ and $M_{UL}, w \vDash$

1. We would like to reserve a special acknowledgment to the anonymous reviewer for suggesting us this elegant and concise formulation of the proof.

ψ iff $\psi \in w$. Suppose first that $M_{UL}, w \vDash \phi \wedge \psi$. Then $M_{UL}, w \vDash \phi$ and $M_{UL}, w \vDash \psi$, and it follows $\phi, \psi \in w$. Hence by $\phi \wedge \psi \in w$. Suppose instead that $\phi \wedge \psi \in w$. Then, $\phi \in w$ and $\psi \in w$. By assumption, therefore, $M_{UL}, w \vDash \phi$ and $M_{UL}, w \vDash \psi$, and it follows that $M_{UL}, w \vDash \phi \wedge \psi$.

- similar for $\vee$ and $\rightarrow$
- LTR: Assume $M_{UL}, w \vDash \Box_1 \phi$, then, for any w', if $wR_{Ins}w'$ by the semantics of $\Box_1$ it holds $M_{UL}, w' \vDash \phi$. By inductive hypothesis $\phi \in w'$. Suppose $\Box_1 \phi \notin w$ then $\neg\Box_1 \phi \in w$. By Lemma 2 there is $wR_{Ins}w'$ such that $\neg\phi \in w'$, against what shown since w' is consistent. RTL: Assume $\Box_1 \phi \in w$, then if $wR_{Ins}w'$ by Definition of R_{Ins} it holds $\phi \in w'$. By induction hypothesis, $M, w' \vDash \phi$, as w' is an arbitrary successor of w, then $M, w \vDash \Box_1 \phi$.
- LTR: Assume $M_F, w \vDash A^{i \mapsto j}\phi$, then by Definition 14 for all $\mathtt{l} \in X_\phi$ for some minimal set of literals $X_\phi \vDash \phi$ there is a world w' such that $w' R_{Abs} w$ which satisfies $\mathtt{l}$. Now suppose $A^{i \mapsto j}\phi \notin w$. Then by Lemma 2 there cannot be any world $w' R_{Abs} w$ which satisfies $\mathtt{l} \in X_\phi$ for all elements of X_ϕ, against the hypothesis. RTL: Assume $A^{i \mapsto j}\phi \in w$, with $w \in W^{U_j}$; then, by Lemma 2, there exists a $w' \in W^{U_i}$ such that $w' R_{Abs} w$ and such that $X_\phi \subset w'$ for a minimal set for which $X_\phi \vDash \phi$. By the Inductive hypothesis $M_F, w' \vDash l$ for any $l \in X_\phi$. Thus, both conditions are satisfied for $M_F, w \vDash A^{i \mapsto j}\phi$.
- LTR: Assume $M_F, w \vDash I^{j \mapsto i}\phi$, then by Definition 14 for some $\mathtt{l} \in X_\phi$ of some minimal set X_ϕ such that $X_\phi \vDash \phi$, there is a world w' such that $w' R_{Imp} w$ and $w' \vDash \mathtt{l}$. Now suppose $I^{j \mapsto i}\phi \notin w$. Then by Lemma 2 there cannot be any world $w' R_{Imp} w$ which satisfies an $\mathtt{l} \in X_\phi$, against the hypothesis. RTL: Assume $I^{j \mapsto i}\phi \in w$, with $w \in W^{U_j}$; then, by Lemma 2, there exists a $w' \in W^{U_i}$ such that $w' R_{Imp} w$ and such that $l \in w'$ for some $l \in X_\phi$ and $X_\phi \vDash \phi$. By the Inductive hypothesis $M_F, w' \vDash l$. Thus, both conditions are satisfied for $M_F, w \vDash I^{j \mapsto i}\phi$.

□

THEOREM 3 (Completeness). For any consistent set Γ of LPI-formulae, if $\Gamma \vDash \phi$ then $\Gamma \vdash_{LPI} \phi$.

Proof. Assume $\Gamma \vDash \phi$. By Lindenbaum's Lemma there is a LPI-maximally consistent set $\Gamma' \supset \Gamma$, and thus a world $w \in M$. By the Truth Lemma $M, \Gamma' \vDash \Gamma$. Assume now $\Gamma_{LPI} \nvdash \phi$. Then Γ is consistent (else it proves everything) and $\Gamma \cup \neg\phi$ must also be LPI-consistent. But if it exists, $M, w \vDash \Gamma \cup \neg\phi$ and, by maximal consistency, both $M, w \vDash \phi$ and $M, w \vDash \neg\phi$. That is $\Gamma \nvDash \phi$. □

5. Types of Use

In this section we illustrate how to interpret (some fragments of) our logic to model specific types of use of computational systems by users. The interpretations range from standard (passive) use to product design. For each type of use we provide an appropriate weakening of LPI or some variant semantic clauses for its modalities.

5.1. *Passive Using*

The full logic LPI with non-weighted modalities can be understood to model the case of passive using, which is the most trivial or common one: it concerns a computational artefact like a spreadsheet, used according to what has been designed for. This means that every user performs only actions which are valid in the context of the specification of the artefact. And each implementation is made correctly with respect to the corresponding functionality; and each program actions can be correctly abstracted to a given expected functionality. Moreover, in LPI we assume that all functionalities are transparent for each user at each level. Formally, this corresponds to use classic modalities, *i.e.*, $\alpha = 1$ for both $\Box_\alpha$ and $\Diamond_\alpha$:

Definition 18 (Passive Using). Given an informational frame M_F and the corresponding set of ULs $\mathcal{L} = \{UL_0, UL_1, ..., UL_n\}$, a formula ϕ is passively used iff

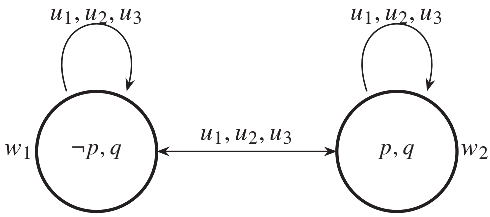
$$M_{UL_i} \models \Box_1\phi$$
$$M_F \models A^{i\mapsto j}\phi$$
$$M_F \models I^{j\mapsto i}\phi$$

for all $UL_i, UL_j \in \mathcal{L}$

Example 8. Consider the model in Fig. 7 representing a case of passive use of a spreadsheet. All user in the UL can resize all cells in the same square shape (p) and can change their background color (q). The specification for this system is $\Sigma = (p \vee \neg p), q$, meaning that $\Box_1(p \vee \neg p)$, and $\Box_1 q$ are valid formulae.

5.2. *Idiosyncratic Using*

Deviant using differs from passive using in that it can be the result of the ability, experience or even luck of a single user in finding unintended applications of some functionality. To analyze the case of idiosyncratic use, we need



Passive Using
$M_{UL} \models \Box_1(p \vee \neg p)$
$M_{UL} \models \Box_1 q$

Figure 7: Passive using.

to focus on the information flow as it occurs not across ULs, but within a UL. We want to model the case where a given functionality becomes available to one user within a UL, but remains hidden to other users (and to other ULs). Therefore, our main object of analysis is the instruction accessibility relation R_{Ins} and its modal operator $\Box_\alpha$. In particular, idiosyncratic uses are such that not all users within a level have access to some functionality ϕ, hence the weight on the modality for ϕ will be weakened to < 1. Moreover, for the functionality under observation we will assume neither abstraction nor implementation relations, given that ϕ is not available in other ULs.

Definition 19 (Idiosyncratic Using). Given a specification $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$, a user level UL, the corresponding set of users $U = \{u_1, u_2, ..., u_n\}$, and their user contexts $C_{u_1}, C_{u_2}, ..., C_{u_n}$, a formula ϕ is idiosyncratically used iff:

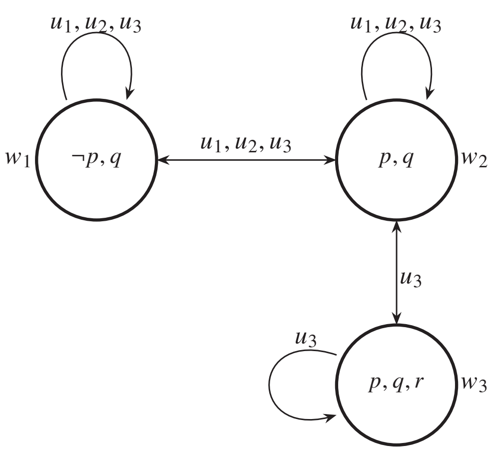
- $\phi, \neg\phi \notin \Sigma$;
- $\exists u_j \in U$ with the related user context C_{u_j} such that:
  - $\exists w \in W^{u_j}$ such that $M_{C_{u_j}}, w \models \phi, \Box_\alpha \phi$, with $\alpha < 1$,
  - $M_{C_{u_j}}, w' \models \Box_\beta \phi$, with $\beta < \alpha < 1, \forall w' \in R^{u_j}_w$,
  - $M_{C_{u_j}}, w'' \models \Box_0 \phi, \forall w'' \notin R^{u_j}_w$;
- and for any other user $u_i \in U$, $M_{C_{u_i}} \models \Box_0 \phi$.

Accordingly, the following hold:

- $M_{UL}, w \models \phi, \Box_\alpha \phi$, with $\alpha < 1$,
- $M_{UL}, w' \models \Box_\gamma \phi$, with $\gamma < \beta < \alpha < 1$,
- $M_{UL}, w'' \models \Box_0 \phi$.

The first condition for a formula to be used in an idiosyncratic way is that neither the formula nor its negation should be already included in the specification: if a formula is included in the specification then it is passively used; if its negation is included in the specification, the formula cannot be used without provoking bugs or malfunctioning. Secondly, there must be a user with access to a new informational state which validates both the formula and its boxed version with a degree $\alpha < 1$; the validity of the boxed formula decreases to β when considering the single user context, and to γ when the entire UL is under analysis (with $\gamma < \beta < \alpha$), if evaluated in a world with access to the new informational state. It drops to zero in all the informational states which are not related to the new one. Lastly, the degree of validity of the boxed formula is zero also in every other user context.

Example 9. Consider Fig. 8 representing a case of idiosyncratic use of a spreadsheet to create pixel-art. A final-user u_3 may have the intuition that $(p \wedge q) \rightarrow r$, *e.g.*, that resizing all the cells in the same square shape, and changing color to their background in a certain way, allows her to create amazing pixel-artworks. A new informational state (w_3), only accessible to her and in which r holds, is added to the frame: neither r nor $\neg r$ are included in the specification Σ and other users in the same UL may not have the computational or informational abilities to realize the implication between $p \wedge q$ and r; other ULs may not even have the right vocabulary to formulate such implication. In w_3 the value for α in $\Box_\alpha r$ is equal to $1/2 < 1$. For every other world with access to w_3, namely w_2, $\Box_\beta r$ holds with $\beta = 1/3$ in the user context C_{u_3}, and $\Box_\gamma r$ holds with $\gamma = 1/7$ when considering the UL. In every other user context the only value for which the boxed formula holds is zero, given that no accessible worlds in such contexts validate r.



Idiosyncratic Using

$M_{UL} \models \Box_1(p \vee \neg p)$
$M_{UL} \models \Box_1 q$

$M_{C_{u_1}} \models \Box_0 r$
$M_{C_{u_2}} \models \Box_0 r$

$M_{C_{u_3}}, w_1 \models \Box_0 r$
$M_{C_{u_3}}, w_2 \models \Box_{1/3} r$
$M_{C_{u_3}}, w_3 \models \Box_{1/2} r$

$M_{UL}, w_1 \models \Box_0 r$
$M_{UL}, w_2 \models \Box_{1/7} r$
$M_{UL}, w_3 \models \Box_{1/2} r$

Figure 8: A serial UL representing a case of idiosyncratic use.

We now look at what axioms hold for M_{UL} with the evaluation clauses enunciated above.

- *Axiom* K_α: $\Box_\alpha(\phi \rightarrow \psi) \rightarrow (\Box_\beta\phi \rightarrow \Box_\gamma\psi)$ where
  $\alpha \leq \frac{|R_w(\phi \rightarrow \psi)|}{|R_w|}$,
  $\beta \leq \frac{|R_w(\phi)|}{|R_w|}$ and
  $\gamma \geq \frac{|R_w(\phi \rightarrow \psi) \bigcap R_w(\phi)|}{|R_w|}$

For K_α, the schema is valid taking into account only the worlds in which the implication and the antecedent actually hold. In our scenario, p, q may hold for users u_1, u_2, u_3, but the implication to r only be available to u_3, hence a new accessibility relation to a world where r holds only be valid within $R_w^{u_3}(r)$.

- *Axiom* D_α: $\Box_\alpha\phi \rightarrow \Diamond_{1-\alpha+\epsilon}\phi, \forall\alpha \in (0, 1]$ and $\forall\epsilon \in (0, \alpha]$

The proof for D_α can be found in Legastelois, Lesot, and d'Allonnes (2017, 352). In our interpretation, the validity of this schema means that any idiosyncratic use must preserve the seriality of the frame, *i.e.*, the world in which the idiosyncratic use is present cannot be an isolated world nor a *dead end*, in order to preserve the original functionalities of the system (together with Axiom B, see next). In other words, when the seriality of the system is lost we have a complete change of intended use in which we no longer have the possibility of accessing the original functions.

- *Axiom B*: $\phi \rightarrow \Box_1\Diamond_1\phi$

According to Legastelois, Lesot, and d'Allonnes (2017), Axiom B is part of the set of unweightable axioms, hence valid only in its normal form (349). This means that the instruction accessibility relation can be symmetrical only with respect to the original functionalities of the system when considering the entire UL, or with respect to the idiosyncratic use when considering the deviant user context (C_{u_3} in the example of Figure 8). As for seriality, any idiosyncratic use must preserve the symmetry of the relation, to preserve the access to the original functionalities of the system and prevent revolutionary changes in intended use.

Also Axiom T is unweightable, *i.e.*, valid only in its normal form with $\alpha = 1$, hence is not a valid schema in the idiosyncratic case given that no world in no context validates $\Box_1\phi \rightarrow \phi$: this represents the fact that $\phi \notin \Sigma$.

5.3. *Innovative using*

The case of innovative using is a direct evolution of the idiosyncratic one. User u_3 may share her discovery by means of several media, with the result that the expanded context C_{u_3} becomes accessible to other members of the UL, increasing the value of both $|R_w|$ and $|R_w(r)|$.

Definition 20 (Innovative use). Given a specification $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$, a user level UL, the corresponding set of users $U = \{u_1, u_2, ..., u_n\}$, and their user contexts $C_{u_1}, C_{u_2}, ..., C_{u_n}$, a formula ϕ is innovatively used iff:

- $\phi, \neg\phi \notin \Sigma$;
- $\exists U' \subset U$ with the related user context $C_{U'}$ such that:
  - $\exists w \in W^{U'}$ such that $M_{C_{U'}}, w \models \Box_\alpha \phi$, with $\alpha < 1$,
  - $M_{C_{U'}}, w' \models \Box_\beta \phi$, with $\beta < \alpha < 1, \forall w' \in R_w^{U'}$,
  - $M_{C_{U'}}, w'' \models \Box_0 \phi, \forall w'' \notin R_w^{U'}$;
- and for any other user $u_i \in U, M_{C_{u_i}} \models \Box_0 \phi$;

Consequently,

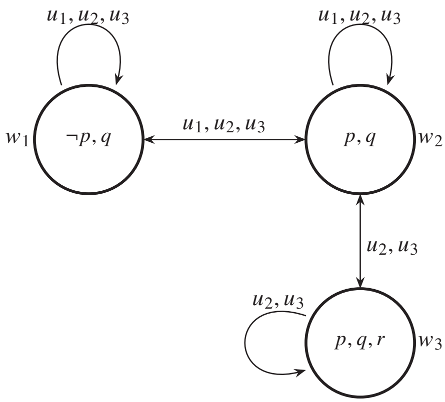
- $M_{UL}, w \models \Box_\alpha \phi$, with $\alpha < 1$,
- $M_{UL}, w' \models \Box_\delta \phi$, with $\gamma < \delta < \beta < \alpha < 1$ with γ representing the degree of validity for ϕ in the idiosyncratic case,
- $M_{UL}, w'' \models \Box_0 \phi$.

The conditions for idiosyncratic using valid for one user are extended to a set of users $U' \subset U$, and the related user contexts. The main difference is the increasing degree of validity of the boxed formula in worlds connected to the informative state where the innovative formula holds ($\delta > \gamma$).

Example 10. Consider the model in Fig. 9 representing a case of innovative use of a spreadsheet to create pixel-art. In this case, more than one user in the UL has access to the new functionality, increasing the degree of validity of the boxed formula when considering worlds, as part of the UL, in relation with the new informational state where the innovative formula holds ($M_{UL}, w_2 \models \Box_{1/4} r$ with $\delta = 1/4 > \gamma = 1/7$).

The dual for $\Diamond_\alpha \phi$ is easily derivable. The clause from idiosyncratic use is here generalised, taking as element of the proportion returning value α the number of accessibility relations of any user in the UL to a state satisfying ϕ over the number of all accessibility relations of all users for any formula.

The weighted axiom schemata valid for idiosyncratic using are also valid for the innovative one, with an appropriate value of α. Innovative using identifies the necessary precondition for the next step in the array of using: expert redesigning. Idiosyncratic and innovative using can be combined to



Innovative Using

$M_{UL} \models \Box_1(p \vee \neg p)$
$M_{UL} \models \Box_1 q$

$M_{C_{u_1}} \models \Box_0 r$

$M_{C_{u_2}}, w_1 \models \Box_0 r \quad M_{C_{u_3}}, w_1 \models \Box_0 r$
$M_{C_{u_2}}, w_2 \models \Box_{1/3} r \quad M_{C_{u_3}}, w_2 \models \Box_{1/3} r$
$M_{C_{u_2}}, w_3 \models \Box_{1/2} r \quad M_{C_{u_3}}, w_3 \models \Box_{1/2} r$

$M_{UL}, w_1 \models \Box_0 r$
$M_{UL}, w_2 \models \Box_{1/4} r$
$M_{UL}, w_3 \models \Box_{1/2} r$

Figure 9: A serial UL representing a case of innovative use.

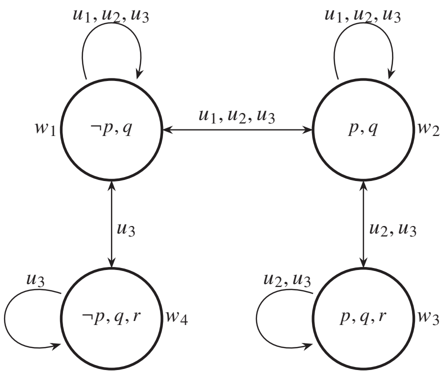
model the evolution in the use of a system before expert redesigning, as shown in the following examples.

Example 11. Consider the model in Figure 10 representing the combination of idiosyncratic and innovative using of the same formula: the new functionality "create art using a spreadsheet" (r), can be achieved by means of an innovative combination of the original functionality "resize all cells as squares" (p) and "change background color to cells" (q), or by other means, *e.g.*, resizing cells in different rectangular shapes ($\neg p$). In worlds w_1, w_4 is represented an idiosyncratic use, in w_2, w_3 an innovative one.

Example 12. Consider the model in Figure 11 representing a system in which a new idiosyncratic use of original functionalities, *e.g*, "use the border of cells to create an ink effect on ASCII art" (s), is developed on top of the combination of idiosyncratic and innovative using of the same formula as shown in the previous example. Note the increasing value of the boxed formula for r, due to the fact that the new idiosyncratic functionality s is built on top of r.

5.4. *Expert redesigning*

As already mentioned, given the lack of scientific and theoretical knowledge, both idiosyncratic and innovative using are not considered as cases of proper designing. Expertise redesign is a modification requiring professional



Idiosyncratic and innovative Using

$M_{UL} \models \Box_1(p \vee \neg p)$
$M_{UL} \models \Box_1 q$

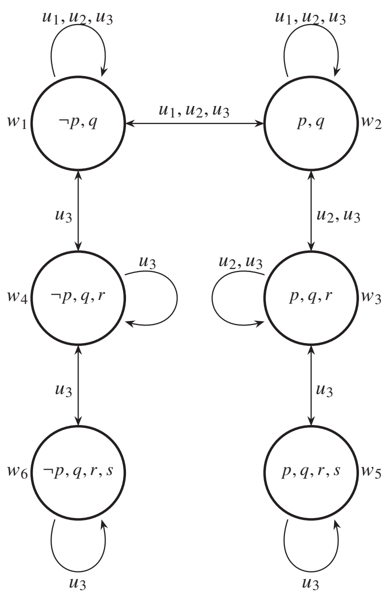
$M_{C_{u_1}} \models \Box_0 r$

$M_{C_{u_2}}, w_1 \models \Box_0 r$
$M_{C_{u_2}}, w_2 \models \Box_{1/3} r$
$M_{C_{u_2}}, w_3 \models \Box_{1/2} r$

$M_{C_{u_3}}, w_1 \models \Box_{1/3} r$ $M_{C_{u_3}}, w_2 \models \Box_{1/3} r$
$M_{C_{u_3}}, w_4 \models \Box_{1/2} r$ $M_{C_{u_3}}, w_3 \models \Box_{1/2} r$

$M_{UL}, w_1 \models \Box_{1/7} r$
$M_{UL}, w_2 \models \Box_{1/4} r$
$M_{UL}, w_3 \models \Box_{1/2} r$
$M_{UL}, w_4 \models \Box_{1/2} r$

Figure 10: Idiosyncratic and innovative using.



Extended Idiosyncratic and Innovative Using

$M_{UL} \models \Box_1(p \vee \neg p)$
$M_{UL} \models \Box_1 q$

$M_{C_{u_1}} \models \Box_0 r, \Box_0 s$

$M_{C_{u_2}}, w_1 \models \Box_0 r, \Box_0 s$
$M_{C_{u_2}}, w_2 \models \Box_{1/3} r, \Box_0 s$
$M_{C_{u_2}}, w_3 \models \Box_{1/2} r, \Box_0 s$

$M_{C_{u_3}}, w_1 \models \Box_{1/3} r, \Box_0 s$ $M_{C_{u_3}}, w_2 \models \Box_{1/3} r, \Box_0 s$
$M_{C_{u_3}}, w_4 \models \Box_{2/3} r, \Box_{1/3} s$ $M_{C_{u_3}}, w_3 \models \Box_{2/3} r, \Box_{1/3} s$
$M_{C_{u_3}}, w_6 \models \Box_1 r, \Box_{1/2} s$ $M_{C_{u_3}}, w_5 \models \Box_1 r, \Box_{1/2} s$

$M_{UL}, w_1 \models \Box_{1/7} r, \Box_0 s$
$M_{UL}, w_2 \models \Box_{1/4} r, \Box_0 s$
$M_{UL}, w_3 \models \Box_{3/5} r, \Box_{1/5} s$
$M_{UL}, w_4 \models \Box_{2/3} r, \Box_{1/3} s$
$M_{UL}, w_5 \models \Box_1 r, \Box_{1/2} s$
$M_{UL}, w_6 \models \Box_1 r, \Box_{1/2} s$

Figure 11: Extended idiosyncratic and innovative using.

practical skills and scientific theoretical knowledge to be applied, *i.e.*, requiring access to lower ULs. In Figure 12, user u_4 is a spreadsheet user with expertise in design and development, who can create new implementations of the discovery made by u_3. User u_4 can write *macros* useful for pixel-art

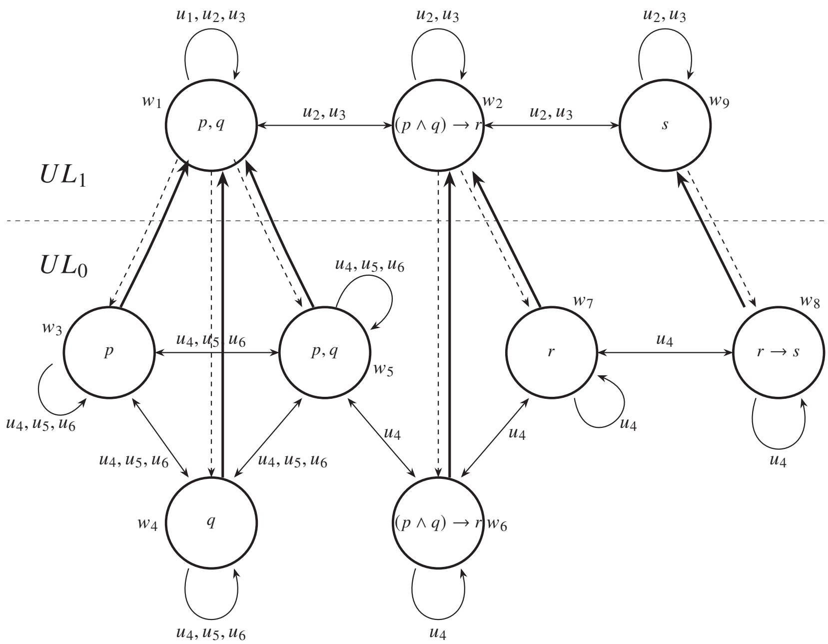


Figure 12: Two ULs (UL_1 with $U_1 = \{u_4, u_5, u_6\}$ and $W^{U_1} = \{w_3, w_4, w_5, w_6, w_7.w_8\}$, and UL_2 with $U_2 = \{u_1, u_2, u_3\}$ and $W^{U_2} = \{w_1, w_2, w_9\}$) involved in Expert redesign. Abstraction relation is marked with thick arrows and Implementation with dashed arrows.

design ($s \in w_8$) and share them to final users (abstracting s from w_8 to w_9). The formula s must be derivable from Σ. If $\Sigma \models \neg s$ then s may generate bugs or malfunctioning.

Definition 21 (Expert redesign). Given a set of ULs $\mathcal{L} = \{UL_0, UL_1, ..., UL_n\}$, a new functionality ϕ, not included in the original specification Σ, represents a case of expert redesign iff there are at least two distinct ULs $UL_i, UL_j \in \mathcal{L}$ with $i < j$, with their respective sets of users $U_i = \{u_1, u_2, ..., u_n\}$ and $U_j = \{u'_1, u'_2, ..., u'_l\}$, and a formula θ, iff:

1. $M_{UL_j}, w_j \models I^{j \mapsto i}\theta$,
2. $M_{UL_i}, w_i \models \phi$, where ϕ represents an idiosyncratic use derived from θ;
3. $M_{UL_j}, w'_j \models A^{i \mapsto j}\phi$.

Example 13. Consider a case of expert redesigning as represented in Figure 12, a new user context (C_{u_4}) is then involved in representing the new behaviours. The innovative functionality ($(p \wedge q) \to r$) is implemented from

w_2 to w_7 ($M_F, w_2 \models I^{j \mapsto i}(p \wedge q) \to r$), it is then idiosyncratically evolved as a new functionality ($M_{C_{u_4}}, w_7 \models \Box_\alpha r \to s; M_{C_{u_4}}, w_8 \models s$), and finally abstracted as expertly redesigned function ($M_F, w_9 \models A^{i \mapsto j} s$).

The new functionality start spreading across different levels, however, as already mentioned, to pass across ULs, another channel of communication is needed, one in which users, belonging to different ULs in the original channel, are at the same level. At this stage, the new features do not involve all ULs, but only part of them, *e.g*, final user UL, graphic designer UL, and developer UL. The axiom schemata valid for idiosyncratic and innovative using become valid in all the ULs interested by expert redesign.

Given the meta-linguistic nature of LPI, it must be remarked that the linguistic expressions for p, q, r in one UL are not the same as for p, q, r in another UL, rather they are the result of mutual abstraction/implementation relations, differing from UL to UL.

5.5. *Product designing*

Once a certain diffusion has been reached, the new functionality ϕ must be evaluated by the original designers, to decide whether to include it in the specification, in the case of non-harmful and potentially valuable functionalities, or whether to make changes to it so as not to make deviant behavior possible, in the case of bugs or vulnerabilities. The inclusion of ϕ in the specification corresponds to $\Sigma \models \phi$, hence for any $w \in W^U$, for all $UL \in \mathcal{L}$, $M_F \models A^{i \mapsto j}\phi, M_F \models I^{j \mapsto i}\phi$, and $M_{UL}, w \models \Box_\alpha \phi$ with $\alpha = 1$, *i.e.*, the classical modality $\Box\phi$, that in our interpretation means that ϕ is included in the specification Σ ($\phi \in \Sigma$). From the final user point of view, a new version of the computational artefact, enriched by the new feature ϕ, has been released by the original designers, hence we are back to the first element in the array of using: passive using phase.

These operations can be formalised semantically in the form of model updates or by means of the *knowledge update operator* $\diamond$ (Baral and Zhang 2005), or syntactically, by means of expansion ($K + A$), update ($K \diamond A$), and contraction ($K \circ A$) on contexts (Primiero 2012).

6. Philosophical Considerations

The notion of pragmatic information underlying LPI offers a reading of the notion of information in which the neutrality principles argued for in Floridi (2006) can be reconsidered. Such reformulation will deviate from the one given for ECDI in Primiero (2007), with the same epistemological perspective in mind.

We start by considering the only principle not reformulated in Primiero (2007): Ontological Neutrality. The original version of this principle states that data implementing information are physical. However, in the formal model designed above and in particular in the case of idiosyncratic uses, we refer to physical data which may not be directly accessible (and usable) by all users, or at all users levels in the case of innovative uses. On the contrary, it is possible that only a limited number of users may have access to the original source of data. Moreover, in the case of expert redesign, we illustrate the case in which innovative uses operations of implementation and abstraction are in place, such that some new functionality is only abstracted from the original physical data. This leads to the following reformulation:

Definition 22 (Pragmatic Ontological Neutrality). Data implementing information can be physical, symbolic, and/or abstract, with respect to the user's ability to access them. All non-physical data are derived from, and grounded in, physical data.

For Typological Neutrality, the standard version affirms that information cannot be dataless and everything can be a datum. According to a pragmatic interpretation, the second conjunct of the principle is quite imprecise, the first one is wrong. As already stated in Primiero (2007), information is user-dependent, hence everything can be considered as information, provided that there is at least one user for which it is well-formed, correct, and meaningful. Consider the case of a baby's cry: for an outsider it will certainly not be well formed, correct or meaningful, therefore it cannot be a source of information. On the contrary, for the child's mother, the cry will not only be well formed and correct, but will also be meaningful of a particular discomfort (sleep, hunger, need to be cleaned, etc.) based on the expressed form, so it can be properly considered as information. On the other hand, information can also be dataless. Consider again the case of a mother and her child. The absence of data, *e.g.*, the silence of a sleepy night, can be surely considered by the mother as the comforting information of the fact that the baby sleeps quietly.

Definition 23 (Pragmatic Typological Neutrality). Everything, including nothing, can be a source of information, provided that there is at least one user for which it is well-formed, correct, and meaningful.

Concerning Taxonomical Neutrality, the relational nature of information as stated by the original version seems granted. However, its formulation does not explain in what this relation consists. A first step in doing so resulted in the constructive version of this principle in Primiero (2007), stating that it consists not only in the relation between a source of data and an interpreting receiver, but also between data sent and data the receiver considers as knowledge (as in the example of the new functionalities for the spreadsheet). Second, by means of the LoA account, it has been shown how several pairs of ontological and epistemological constructs are *related* when considering information in computational artefacts. Thirdly, this has been generalized in the UL theory, in which information is characterized by a tripartite relation of Abstraction/Instruction/Operation crawling across levels, from abstract intentions to physical entities (and *vice versa*). This results in the following reformulation:

Definition 24 (Pragmatic Taxonomical Neutrality). The relational nature of information is realised

- between source (with its expressive capabilities) and receiver (with its elaboration power),
- as an abstraction/implementation relation across ULs,
- as an instructional relation across informational states, within each UL.

The rejection of the Alethic Neutrality principle to avoid the Bar-Hillel Carnap Paradox (BCP), is probably the most controversial move in the logic of "being informed", both in its original form in Floridi (2006) and in its revised version in Allo (2011). In general, this approach seems rather counterintuitive, since history of science and several epistemological accounts have shown how (the greatest part of) our knowledge is doomed to be provisional, and subject to changes and revisions. In Floridi's account in particular, this sounds rather problematic, given that, as briefly mentioned at the end of Section 2, information is necessarily and always true, but necessarily true sentences, such as tautologies, qualify as uninformative. This approach is motivated by the basic and strong versions of the independence thesis, i.e. the preliminary intuition that "being informed" is a cognitive state distinct from both knowing and believing, situated somewhere in between the two. In Floridi's account, information is more than belief, because it is true; according to Allo, it's also more than true belief, because it can be derived only from true contents. On the contrary, in both the constructive approach in Primiero (2007), and in our pragmatic one, information provides the building blocks for both knowledge and belief, hence it is defined by meaningfulness, well-formedness, and correctness with respect to the user's informational states and capabilities, independently of its truth value.

Definition 25 (Pragmatic Alethic Neutrality Principle). Independently of their alethic value, data qualify as information, provided that there is at least one user for which it is coherent, well-formed, correct, and meaningful, i.e. *usable*.

Finally, Genetic Neutrality: having underlined the importance of information usability, inevitably leads to a substantial re-consideration of the genetic principle. According to the constructive version, any informational content has to be evaluated by the receiver with respect to the source informativeness. In the present framework, this connection is bidirectional: on the one hand, the receiver abstracts and elaborates information on the basis of the source's ability to implement it; on the other, the source must implement usable data with respect to the receiver ability to abstract and elaborate information. Let us take again the example of a mother and her crying baby. Consider the case where there is no reason to cry other than boredom. If the mother shouted at the baby "stop crying!", the effect would probably be the opposite. Even if the mother addressed the child gently, whispering "please stop crying", in many cases the message transmitted would not be received by the child correctly. On the other hand, the use of gibberish sounds, which in any other occasion would certainly not be considered well-formed and meaningful data, would probably have the desired effect, making the right information usable for the intended receiver. As a consequence, the Genetic Neutrality Principle does not hold in our structuralist account, in accordance with the "no pure information" principle.

7. Conclusions and Future Investigations

Information has become an integral part of our lives, permeating every aspect of our existence. In a world where data is constantly being generated, processed, and disseminated, the question of how we make sense of it all becomes increasingly important. To address this challenge, scholars from a range of disciplines have explored the logic of information, seeking to understand the principles that govern our ability to acquire, interpret, and use data.

In Floridi (2006), a pure semantics for information is provided, characterized by embedding meaning and truth. This has been expanded in Allo (2011), by providing also an applied semantics, based on the *de re* reading of data. The role of the agent in defining information based on proofs and assertible judgments is emphasized in Primiero (2007). Our attempt to underline the strengths and eliminate the weaknesses of all three approaches has led to a weighted multi-agent logic for "using information" characterized by well-formedness, correctness, and meaningfulness, with respect to the user perspective, and coherence with respect to other users informational states and expressions. This type of information is particularly apt to model the contemporary tendency of being both sources and receivers of an ever increasing amount of information, in one word, users. From an ethical perspective, this points to the shared responsibility of users in providing and transfer data *as true as possible*, with respect to their informational states and their informational capabilities as both sources and receivers.

Some of the next steps in this research are: the formulation of a proof theoretical counterpart to the model-theoretical approach above proposed; an extension to include interactions with mechanical users based on machine learning techniques, e.g. automatic code generators and classifiers, requiring to consider uncertainty under probabilistic computations and associated trust evaluation in the spirit of the calculus introduced in D'Asaro, Genco, and Primiero (2025) and the associated world-based semantics in Kubyshkina and Primiero (2024); the exploration of possible semiotic algebras for the relations of implementation and abstraction across distinct ULs as applied to software development in Pitsiladis and Stefaneas (2023) inspired by Goguen (1999). These formal systems will ground a "semiotic foundation of computing", complementary to the mathematical, the engineering, and the experimental ones as described in Primiero (2020).

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