On splitting complete manifolds via infinity harmonic functions

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Abstract

In this paper, we prove some splitting results for manifolds supporting a non-constant infinity harmonic function which has at most linear growth on one side. Manifolds with non-negative Ricci or sectional curvature are considered. In dimension 2, we extend Savin's theorem on Lipschitz infinity harmonic functions in the plane to every surface with non-negative sectional curvature.

1 Introduction

The present paper regards the interplay between the geometry of a Riemannian manifold and the qualitative properties of ∞-harmonic functions, i.e., solutions to

$$\Delta_\infty u \doteq \nabla^2 u(\nabla u, \nabla u) = 0 \quad \text{on } M$$

in the viscosity sense. The ∞-Laplace operator and its normalized counterpart

$$\Delta_\infty^N u \doteq \nabla^2 u \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right)$$

gained increasing importance in the field of fully-nonlinear PDEs over the past 60 years, see [6, 15] for a thorough account of the theory, with historical insights and a detailed set of references. The investigation herein is a natural continuation of [3, 34, 35], where the geodesic completeness of a boundaryless Riemannian (or Finsler) manifold was characterized in terms of suitable Liouville properties of viscosity solutions to

$$\Delta_\infty^N u \geq g(u).$$

It is known that u solves $\Delta_\infty u \geq 0$ $(= 0, \leq 0)$ if and only if it solves $\Delta_\infty^N u \geq 0$ $(= 0, \leq 0)$. Therefore, for the purpose of the present paper we will only consider Δ_∞. Among the various equivalent conditions, by [3, Theorem 1.1] (cf. also [34, Theorem 8.1]) a connected Riemannian manifold M without boundary is shown to be complete *if and only if* all solutions to $\Delta_\infty u \leq 0$ whose negative part u_- satisfies

$$u_-(x) = o\big(r(x)\big) \qquad \text{as } x \to \infty \tag{1}$$

are constant[1]. Here, $r(x)$ denotes the distance to a fixed origin. The result extends the known Liouville theorem for positive ∞-superharmonic functions on $\mathbb{R}^m$ proved by Lindqvist and Manfredi in [31, 32], see also [15, p.113], and we stress that the *if* part is its main novelty. The lack of curvature or volume growth requirements on M in order for the aforementioned Liouville property to hold makes the theory of slowly growing ∞-harmonic functions considerably different from that developed for other operators $\mathcal{F}$ like the Laplacian [12, 43], the p-Laplacian [42] and in recent years the minimal hypersurface operator [14, 17, 39]. In these latter cases, $\mathrm{Ric} \geq 0$ is the weakest known condition to guarantee that positive solutions to $\mathcal{F}[u] = 0$ are constant. For solutions satisfying the more general (1), in the minimal hypersurface case further technical conditions on M are needed as of yet, see [13, 18].

Hereafter, M will always denote a complete, connected Riemannian manifold without boundary. A natural problem is then to see what happens to ∞-harmonic functions that grow at most linearly on one side, namely, that satisfy

$$\limsup_{r(q)\to\infty} \frac{u(q)}{r(q)} < \infty. \tag{2}$$

Especially, we shall look for geometric conditions to force a rigidity of M or u, in the sense that M splits as a (possibly warped) product and u only depends on split-off variables. The next example shows that a constraint on the geometry of M is necessary in this case.

Example 1.1. On a Cartan-Hadamard manifold, that is, a simply connected manifold with non-positive sectional curvature Sec, given a ray $\gamma : [0, \infty) \to M$ one can consider the Busemann function

$$b_\gamma : M \to \mathbb{R}, \qquad b_\gamma(x) = \lim_{t\to\infty} \Big(\mathrm{d}(x, \gamma(t)) - t \Big).$$

It is known by [26] that $b_\gamma \in C^2(M)$ and $|\nabla b_\gamma| = 1$ on M, so differentiating we get that b_γ is a globally Lipschitz solution to $\Delta_\infty b_\gamma = 0$ on M. However, in general M does not split off any line.

Remark 1.2. It is known, see Lemma 2.2 below, that for solutions to $\Delta_\infty u \geq 0$ the following identity holds (possibly with infinite values):

$$\limsup_{r(q)\to\infty} \frac{u(q)}{r(q)} = \mathrm{Lip}(u, M),$$

with $\mathrm{Lip}(u, M)$ the Lipschitz constant of u on M. Therefore, non-constant globally Lipschitz solutions to $\Delta_\infty u = 0$ are precisely those for which the limsup in (2) is a positive real number. By scaling u, in our main results we shall assume this number to be one.

Based on the theory of harmonic functions with linear growth developed in [10, 28, 33] and the corresponding results for minimal graphs which appeared in recent years [13, 18, 19], the assumptions

$$\mathrm{Ric} \geq 0 \qquad \text{or} \qquad \mathrm{Sec} \geq 0$$

[1] The implication is (1) $\Leftrightarrow$ (2) in [3, Theorem 1.1], once we observe that $v = -u$ solves $\Delta_\infty v \geq 0$ (hence $\Delta_\infty^N v \geq 0$) with $v_+ = o(r)$.

seem to be appropriate. For Euclidean space, Aronsson in [5, Section 7] proved that any solution of class C^2 on $\mathbb{R}^2$ is affine, see also [20] for the case of dimension $m \geq 3$. Examples therein show that this fails for viscosity solutions which are not C^2, unless one assumes a priori growth of u. On the other hand, Savin's remarkable theorem [40] states that

$$\Delta_\infty u = 0 \quad \text{on } \mathbb{R}^2, \quad u \text{ Lipschitz} \quad \Longrightarrow \quad u \text{ is affine.}$$

As of today, its extension to $\mathbb{R}^m$ for $m \geq 3$ has not been established. In higher dimensions, we are only aware of the next half-space theorem showed by Crandall, Evans and Gariepy [16]:

$$\Delta_\infty u \leq 0, \quad u(x) \geq a + \langle p, x \rangle \text{ on } \mathbb{R}^m \quad \Longrightarrow \quad u = u(0) + \langle p, x \rangle,$$

and the recent work of Hong and Zhao [25], who proved that u is affine by assuming (2) and

$$\lim_{r(p)\to\infty} |Du(p)| = \mathrm{Lip}(u, \mathbb{R}^m).$$

The methods herein are much inspired by those in [16, 25]. As a matter of fact, we show that elaborating on their arguments in a manifold setting, and employing some basic facts of metric geometry, we are able to obtain results with nontrivial geometric content. Let M be complete, connected and without boundary, and assume that u is a nonconstant ∞-harmonic function satisfying (2), so by Remark 1.2 we can assume

$$\limsup_{r(q)\to\infty} \frac{u(q)}{r(q)} = 1.$$

We prove:

- (i) Theorem 3.1. If $\mathrm{Ric} \geq 0$, then any blowdown M_∞ of M splits as $\mathbb{R} \times N_\infty$. Moreover, the blowdown of u only depends on the arclength t of the $\mathbb{R}$ factor, and it is affine in t.
- (ii) Proposition 3.4. In the assumptions of (i), M itself may not split off lines: there exists a manifold M with $\mathrm{Ric} > 0$ carrying a linearly growing ∞-harmonic function. However, by the tangency principle in Proposition 3.5, if the graph of u touches that of a (possibly translated and dilated) Busemann function from above or below, then M splits and u is an affine function of the split direction only.
- (iii) If $\mathrm{Sec} \geq 0$, general theory gives a way to split M itself as $\mathbb{R} \times N$. We prove in Theorem 4.3. that the blowdown of u is unique and that, writing $(x, y) \in \mathbb{R} \times N$ and orienting $\mathbb{R}$ appropriately, it holds

$$\lim_{x\to+\infty} \frac{u(x, y) - u(-x, y)}{2x} = 1 \qquad \text{for each fixed } y \in N.$$

In the assumptions of (iii), whether the function u only depends on x is an open problem even in $\mathbb{R}^m$, whose solution would allow to extend Savin's result to higher dimensions. As pointed out in [15, 16], a positive answer is likely to give new insights on the $C^{1,\alpha}$ regularity property of ∞-harmonic functions. In dimension $m \geq 3$, we have the following sufficient condition:

- (iv) Assume $\mathrm{Sec} \geq 0$, and that there exist a ray γ and a constant C for which either

$$u\big(\gamma(t)\big) \geq t - C \qquad \text{or} \qquad u\big(\gamma(t)\big) \leq -t + C$$

holds for all $t \in \mathbb{R}^+$. Then, referring to the splitting in (iii), we have $u(x, y) = x + C_2$ for some constant C_2.

On the other hand, (iii) strengthens in dimension 2 and gives rise to a full extension of Savin's theorem to any complete surface with non-negative sectional curvature. We get

Theorem 1.3. *Let M be a complete connected surface with* $\mathrm{Sec} \geq 0$*, and let $u \in C(M)$ be a non-constant ∞-harmonic function such that*

$$\limsup_{r(q)\to\infty} \frac{u(q)}{r(q)} < \infty, \tag{3}$$

where r is the distance from a fixed origin. Then, $M = \mathbb{R}^2$ or $M = \mathbb{R} \times \mathbb{S}^1$. Furthermore, u only depends on the arclength x of a split $\mathbb{R}$ factor, and it is affine in x.

Most of the arguments in the present paper extend, almost directly, to RCD spaces, for which we refer to the survey [1] and the references therein. An exception might be the approximation procedure we carried over to prove Theorem 4.3, see Section 5. As a side remark, to approximate we have chosen to use the p-Laplacian instead of the inhomogeneous operator proposed in [22]. In another direction, Finsler manifolds proved to be a quite natural setting for the techniques developed to investigate the ∞-Laplacian, see [3]. However, in such a generality the topological/geometric conclusions that can be achieved from splitting theorems are weaker, apart from the subclass of Berwald metrics, [37]. For these reasons, we decided to stick to the smooth, Riemannian setting to avoid technicalities.

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2 Preliminaries

Busemann functions and convergence

We here collect some basic facts on metric geometry, mostly to fix notation. We refer to [38] for more details. Hereafter, a segment $\gamma : [0, T] \to M$ will be a unit speed geodesic which is minimizing between its endpoints. A unit speed geodesic γ will be called:

- a ray if γ is defined on $[0, \infty)$ and is a segment between any pair of its points;
- a line if γ is defined on $\mathbb{R}$ and is a segment between any pair of its points.

Therefore, a line is characterized by the identity

$$\mathrm{d}(\gamma(t), \gamma(s)) = |t - s| \qquad \forall\, s, t \in \mathbb{R}.$$

Given a ray γ, the Busemann function $b_\gamma : M \to \mathbb{R}$ is defined as the limit

$$b_\gamma(x) = \lim_{t\to\infty} \Big(\mathrm{d}(x, \gamma(t)) - t \Big).$$

Such a limit exists since the family of functions $b_{\gamma,t}(x) = \mathrm{d}(x, \gamma(t)) - t$ is monotone decreasing and bounded as $t \uparrow \infty$, see [38, Sec. 7.3.2]. Given a point $p \in M$, an asymptote of γ issuing from x is a sequential limit $\tilde{\gamma}$ of a sequence of segments $\tilde{\gamma}_j$ joining x to $\gamma(t_j)$ for some $t_j \to \infty$. Notice that $\tilde{\gamma}$ is a ray from x. By [38, Prop. 7.3.8], it holds

$$b_\gamma(x) \le b_\gamma(p) + b_{\tilde{\gamma}}(x)$$

with equality at p, namely, $b_\gamma(p) + b_{\tilde{\gamma}}$ is a support function from above for b_γ at p.

Next, we denote with $\lambda = \{\lambda_j\}$ a sequence with $\lambda_j \to \infty$. For each j, we let M_j^λ be the manifold M with metric $g_j = \lambda_j^{-2} g$, distance $\mathrm{d}_j = \lambda_j^{-1}\mathrm{d}$ and induced volume form $\mathrm{d}V_j = \lambda_j^{-m}\mathrm{d}V$. Also, let B_R^j be geodesic balls in M_j^λ centered at a fixed origin o. A pointed measured Gromov-Hausdorff limit

$$(M, \mathrm{d}_j, \mathrm{d}V_j, o) \longrightarrow (M_\infty^\lambda, \mathrm{d}_\infty, \mathfrak{m}_\infty, o_\infty), \tag{4}$$

see [1, Section 6] for its definition, will be written as $M_j^\lambda \to M_\infty^\lambda$ and named a tangent cone at infinity (or a blowdown) of M at o. Henceforth, given $u : M \to \mathbb{R}$, $\mathrm{Lip}(u, U)$ will denote the Lipschitz constant of u on a subset $U \subset M$. Assume that u is globally Lipschitz. Defining

$$u_j^\lambda : M_j^\lambda \to R, \qquad u_j^\lambda(x) = \frac{u(x) - u(o)}{\lambda_j},$$

we have $\mathrm{Lip}(u_j^\lambda, M_j^\lambda) = \mathrm{Lip}(u, M)$ for each j and therefore, up to a subsequence, u_j^λ converges pointwise in the Gromov-Hausdorff sense (see [38, Lem. 11.1.9]) to a function $v^\lambda : M_\infty^\lambda \to \mathbb{R}$, meaning that, for each $x_j \in M_j^\lambda$, $x_\infty \in M_\infty^\lambda$,

$$q_j \to q_\infty \qquad \Longrightarrow \qquad u_j^\lambda(q_j) \to v^\lambda(q_\infty).$$

Comparison with cones and its consequences

We recall some well-known properties of ∞-subharmonic functions, which can be found in the surveys [6, 15, 41]. Despite the references being set in Euclidean space, the proof of the lemmata below carry over verbatim to any complete Riemannian manifold. For more general metric spaces, we refer to [9].

Let Ω be an open domain of M and let $u \in C(\Omega)$. It is known that $\Delta_\infty u \ge 0$ is equivalent to u enjoying comparison with cones

$$C_x(y) = a + b\,\mathrm{d}(x, y), \qquad a, b \in \mathbb{R}$$

from above, meaning that if $u \le C_x$ in $\partial\Omega \cup \{x\}$, then $u \le C_x$ in Ω, see [16, Section 3] and [9, 15]. As a consequence, if $\Delta_\infty u \ge 0$ then

$$u(y) \le u(x) + \left(\max_{\partial B_R(x)} \frac{u(y) - u(x)}{R}\right) \mathrm{d}(x, y) \quad \forall\, x \in \Omega,\ y \in B_R(x) \Subset \Omega.$$

Even more, by [16, Lem. 2.4] the function

$$R \mapsto S^+_{u,R}(x) = \max_{z \in \partial B_R(x)} \frac{u(z) - u(x)}{R} \tag{5}$$

is non-decreasing for $R < \mathrm{d}(x, \partial\Omega)$, and therefore the limits

$$S^+_u(x) = \lim_{R \to 0} S^+_{u,R}(x) \qquad \text{and} \qquad S^+_{u,\infty}(x) = \lim_{R \to \infty} S^+_{u,R}(x)$$

(the latter, if $\Omega = M$) are well defined. As a direct consequence, $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, see [16, Lem. 2.5]. The following proposition collects some of the properties in [15, Lemm. 4.2 and 4.3].

Proposition 2.1. *Let $\Omega \subset M$ be an open subset and $u \in C(\Omega)$ satisfy $\Delta_\infty u \geq 0$. Then, for each $x \in \Omega$*

$$S_u^+(x) = \lim_{r\to 0} \mathrm{Lip}(u, B_r(x)) = \lim_{r\to 0} \|\nabla u\|_{L^\infty(B_r(x))}.$$

Moreover, if u is differentiable at x, the three quantities equal $|\nabla u(x)|$.

We next state a simple yet very useful consequence of comparison with cones, essentially contained in [15, Prop. 7.1] [25, Prop. 1.1]. We include a proof for the sake of completeness.

Lemma 2.2. *If $u \in C(M)$ satisfies $\Delta_\infty u \geq 0$, and let r be the distance from a fixed origin o. Then,*

$$\mathrm{Lip}(u, M) = S^+_{u,\infty}(x) = \limsup_{r(q)\to\infty} \frac{u(q)}{r(q)},$$

for any $x \in M$, possibly with infinite values.

Proof. We prove the first equality. From the monotonicity of $S^+_{u,R}(x)$ we deduce that $S^+_{u,R}(x) = \max_{z\in \bar{B}_R(x)} \frac{u(z)-u(x)}{R}$. Therefore, for each $w \in M$ we get

$$\begin{aligned} S^+_{u,R}(x) &\leq \max_{\bar{B}_{R+\mathrm{d}(w,x)}(w)} \frac{u(z)-u(x)}{R} \\ &= \max_{\partial B_{R+\mathrm{d}(w,x)}(w)} \left(\frac{u(z)-u(w)}{R+\mathrm{d}(w,x)}\right)\frac{R+\mathrm{d}(w,x)}{R} + \frac{u(w)-u(x)}{R} \\ &\leq \frac{R+\mathrm{d}(w,x)}{R} S^+_{u,R}(w) + \frac{u(w)-u(x)}{R}. \end{aligned}$$

Letting $R \to \infty$ we may conclude $S^+_{u,\infty}(x) \leq S^+_{u,\infty}(w)$. Since x and w are arbitrary, equality holds and the limit $\ell = S^+_{u,\infty}(x)$ (possibly infinite) does not depend on the point x. We now show that $\ell = \mathrm{Lip}(u, M)$. It is clear that $S^+_{u,R}(x) \leq \mathrm{Lip}(u, M)$, thus $\ell \leq \mathrm{Lip}(u, M)$. Assume by contradiction that there exists $C \in (\ell, \mathrm{Lip}(u, M))$ and pick $z, w \in M$ such that $u(z) \geq u(w) + C\mathrm{d}(w, z)$. Then,

$$\ell = S^+_{u,\infty}(w) \geq \frac{u(z)-u(w)}{\mathrm{d}(w,z)} \geq C,$$

contradiction. The second equality follows from $S^+_{u,\infty}(x) = S^+_{u,\infty}(o)$ and the definition of $S^+_{u,\infty}(o)$. ☐

As we shall see, Lemma 2.2 guarantees the non-constancy of any blowdown of u. Thus, it plays the same important role as that of the relation

$$\lim_{R\to\infty} \fint_{B_R} |\nabla u|^2 = \sup_M |\nabla u|^2 \tag{6}$$

in the theory of harmonic functions [10, 33] and minimal graphs [13]. However, we emphasize that the proof of (6) in the above references is considerably subtler than that of Lemma 2.2.

Tightness and the anti-peeling Lemma

We next present two key lemmata which will be often used in the arguments below. The first one adapts [16, Lem. 4.2].

Lemma 2.3. *Let* (N, d_N) *be a metric space and let* v *be a* 1*-Lipschitz function on the product space* $\mathbb{R} \times N$ *such that, for some* $y_0 \in N$,

$$v(x, y_0) = x \qquad \forall\, x \in \mathbb{R}.$$

Then,

$$v(x, y) = x \qquad \forall\, (x, y) \in \mathbb{R} \times N.$$

Proof. Let us fix $\lambda \in \mathbb{R}$. Since v is 1-Lipschitz

$$|v(x, y) - \lambda|^2 = |v(x, y) - v(\lambda, y_0)|^2 \le |x - \lambda|^2 + \mathrm{d}_N(y, y_0)^2. \tag{7}$$

Expanding the squares on both sides and simplifying we get

$$v(x, y)^2 - 2\lambda v(x, y) \le x^2 - 2\lambda x + \mathrm{d}_N(y, y_0)^2. \tag{8}$$

Dividing by $\lambda > 0$ and letting $\lambda \to +\infty$ we obtain $v(x, y) \ge x$. Likewise, dividing by $\lambda < 0$ and letting $\lambda \to -\infty$ we conclude that $v(x, y) \le x$, whence $v(x, y) = x$. □

The second Lemma follows from [15, Prop. 6.2]. We borrowed the name "anti-peeling Lemma" because of its analogy with [7, Thm. 3.2], which is a key result in the theory of the prescribed Lorentzian mean curvature equation.

Lemma 2.4. [Anti-peeling Lemma] *Let* M *be a complete Riemannian manifold,* $\Omega \subset M$ *an open subset, and let* $u : \Omega \to \mathbb{R}$ *satisfy, for some* $x \in \Omega$,

$$\Delta_\infty u \ge 0 \quad \text{on } \Omega, \qquad S_u^+(x) = \|\nabla u\|_{L^\infty(\Omega)} = 1.$$

Then, there exists a segment $\gamma : [0, b) \to \Omega$ *issuing from* x *such that*

$$u(\gamma(t)) - u(\gamma(s)) = t - s \tag{9}$$

for each $0 < s < t < b$. *Moreover,* u *is differentiable at each point of* $\gamma((0, b))$ *with gradient* $\nabla u(\gamma(t)) = \gamma'(t)$, *and if* $b < \infty$ *it holds*

$$\lim_{t \to b} \gamma(t) \in \partial\Omega.$$

In particular, the existence of such γ *occurs if* $\|\nabla u\|_{L^\infty(\Omega)} = 1$ *and there exists a geodesic* $\bar{\gamma} : [0, b') \to M$ *issuing from* x *where* (9) *holds for* $0 < s < t < b'$, *and in this case* γ *extends* $\bar{\gamma}$.

Proof. By Proposition 2.1, $S_u^+(x)$ coincides with $\lim_{r \to 0} \mathrm{Lip}(u, B_r(x))$. It was proved in [15, Prop. 6.2] that there exists a Lipschitz curve $\gamma : [0, b) \to \Omega$ of velocity $|\gamma'| \le 1$ issuing from x and satisfying, among other properties,

$$u(\gamma(t)) \ge u(x) + tS_u^+(x) = u(x) + t, \qquad \lim_{t \to b} \gamma(t) \in \partial\Omega \quad \text{if } b \text{ is finite.}$$

Since u is 1-Lipschitz, $u(\gamma(t)) = u(x) + t$ on $[0, b)$ and therefore

$$|t - s| = |u(\bar{\gamma}(t)) - u(\bar{\gamma}(s))| \le \mathrm{d}(\gamma(t), \gamma(s)) \le |t - s|,$$

whence γ is a segment. By [41, Lem. 3.5], if the domain of a 1-Lipschitz function u contains a segment γ where u has slope 1, then u is differentiable at any interior point of γ. Moreover, its gradient is $\pm\gamma'(t)$ according to whether u grows or decreases along γ. This concludes the first part of the proof. Next, let $\bar{\gamma} : [0, b') \to \Omega$ be a geodesic from x satisfying (9). Using again [41, Lem. 3.5] and Proposition 2.1 we get $S_u^+(y) = 1$ and $\nabla u(y) = \bar{\gamma}'(t)$ at every interior point $y = \bar{\gamma}(t)$ of $\bar{\gamma}$. Applying the first part of the proof, there exists a curve γ issuing from y where u has slope 1. Since u is differentiable at y, $1 = (u \circ \gamma)'(0) = \langle \nabla u(y), \gamma'(0) \rangle \le 1$, whence $\bar{\gamma}' = \gamma'$ at y and γ extends $\bar{\gamma}$. □

3 Manifolds with $\mathrm{Ric} \geq 0$

We begin by investigating manifolds with $\mathrm{Ric} \geq 0$. First, we analyse their blowdowns by adapting an argument in [15, Prop. 7.1], see also Lemma 7.1 therein and [25, Prop. 1].

Theorem 3.1. *Let M^m be a complete manifold with $\mathrm{Ric} \geq 0$, and let $u \in C(M)$ be an ∞-harmonic function such that*

$$\limsup_{r(q)\to\infty} \frac{u(q)}{r(q)} = 1, \tag{10}$$

where r is the distance from a fixed origin. Then, every tangent cone at infinity of M splits as $\mathbb{R} \times N_\infty$ for some $N_\infty \in \mathsf{RCD}(0, m-1)$. Furthermore, the blowdown of u only depends on the arclength τ of the $\mathbb{R}$-factor, and it is affine in τ.

Proof. By Lemma 2.2, $\mathrm{Lip}(u, M) = 1$. Let $M_j^\lambda \to M_\infty^\lambda$ be a tangent cone at infinity centered at $o \in M$, and let $u_j^\lambda \to v^\lambda$ be the associated blowdown of u. We hereafter omit the superscript λ. Fix $R > 0$, and for each j consider a point $z_j^+ \in \partial B_{\lambda_j R}(o) \subset M$ which realizes $S^+_{\lambda_j R}(o)$. By Lemma 2.2,

$$\frac{u_j(z_j^+)}{R} = \frac{u(z_j^+) - u(o)}{\lambda_j R} \to 1 \qquad \text{as } j \to \infty. \tag{11}$$

Likewise, we can consider $z_j^- \in \partial B_{\lambda_j R}(o) \subset M$ which realizes $S^-_{\lambda_j R}(o)$ and obtain

$$\frac{u_j(z_j^-)}{R} = \frac{u(z_j^-) - u(o)}{\lambda_j R} \to -1 \qquad \text{as } j \to \infty.$$

From $z_j^\pm \in \partial B_R^j(o)$ passing to limits as $j \to \infty$ and using the local uniform convergence of u_j, up to subsequences

$$z_j^\pm \to z_R^\pm \in \partial B_R^\infty(o_\infty), \qquad v(z_R^+) = R = -v(z_R^-). \tag{12}$$

Having set $\gamma_R^+ : [0, R] \to M_\infty$ (respectively $\gamma_R^- : [0, R] \to M_\infty$) a segment from o to z_R^+ (resp. from o to z_R^-), we can define $\gamma_R : [-R, R] \to M_\infty$ as

$$\gamma_R(t) = \begin{cases} \gamma_R^-(-t) & \text{for } t \in [-R, 0], \\ \gamma_R^+(t) & \text{for } t \in [0, R]. \end{cases} \tag{13}$$

From (12) we deduce $\mathrm{d}_\infty(z_R^+, z_R^-) \geq u(z_R^+) - u(z_R^-) = 2R$, so by the triangle inequality $\mathrm{d}_\infty(z_R^+, z_R^-) = 2R$. It follows that γ_R is a segment from z_R^- to z_R^+, and by (12) and $\mathrm{Lip}(v, M_\infty) \leq 1$ we deduce

$$v(\gamma_R(t)) = t \qquad \text{for each } t \in [-R, R]. \tag{14}$$

Letting $R \to \infty$, γ_R converges to a line γ_∞ in M_∞. Cheeger-Colding's splitting Theorem in [11, Thm. 6.64] guarantees that M_∞ splits as $\mathbb{R} \times N_\infty$. Moreover, as shown by Gigli's nonsmooth splitting Theorem [23], $(N_\infty, \mathrm{d}') \in \mathsf{RCD}(0, m-1)$. Let $(\tau, y) \in \mathbb{R} \times N_\infty$, with $o = (0, o')$. Since $v(\tau, o') = \tau$ for each $\tau \in \mathbb{R}$, the conclusion $v(\tau, y) = \tau$ on $\mathbb{R} \times N_\infty$ then follows from Lemma 2.3. ☐

Remark 3.2. Notice that the identity $v(x, y) = x$ for $(x, y) \in \mathbb{R} \times N_\infty$, together with (14), imply that each γ_R is the curve (t, o') for $t \in [-R, R]$. Hence, γ_∞ is indeed the extension of each γ_R to the entire real line.

As mentioned above, Theorem 3.1 is not enough to guarantee that M itself splits off a line. The following counterexample describes a manifold with $\mathrm{Ric} > 0$ (hence, not splitting off lines) and carrying a linearly growing ∞-harmonic function. Even more, the example points out that assumption $\mathrm{Sec} \geq 0$ cannot be weakened to the non-negativity of any of the following partial Ricci curvature functions $\mathrm{Ric}^{(\ell)}$ for $\ell \geq 2$:

Definition 3.3. *Let M be a manifold of dimension $m \geq 2$. For $\ell \in \{1, \ldots, m-1\}$, the ℓ-th (normalized) Ricci curvature is the function*

$$v \in T_x M \quad \longmapsto \quad \mathrm{Ric}^{(\ell)}(v) \doteq \inf_{\substack{\mathcal{W} \leq v^\perp \\ \dim \mathcal{W} = \ell}} \left(\frac{1}{\ell} \sum_{j=1}^{\ell} \mathrm{Sec}(v \wedge e_j) \right),$$

where $\{e_j\}$ is an orthonormal basis of $\mathcal{W}$.

We recall that $\mathrm{Ric}^{(\ell)}$ interpolates between the sectional and Ricci curvatures, obtained respectively for $\ell = 1$ and (up to a normalization constant) for $\ell = m-1$. In particular, with our chosen normalization the following implications are immediate:

$$\mathrm{Sec} \geq \kappa \implies \mathrm{Ric}^{(\ell-1)} \geq \kappa \implies \mathrm{Ric}^{(\ell)} \geq \kappa \implies \mathrm{Ric} \geq (m-1)\kappa.$$

Proposition 3.4. *For $m \geq 4$, there exists a complete manifold M with*

$$\mathrm{Ric}^{(2)} \geq 0, \qquad \mathrm{Ric} > 0, \qquad \text{and} \qquad |\mathrm{Sec}| \leq \bar{\kappa}^2$$

for some constant $\bar{\kappa} > 0$, which carries a non-constant linearly growing ∞-harmonic function.

Proof. We consider the example in [29, p. 913], along with the observations made in [13]. Fix $\alpha, \beta \in (0, 1)$ such that $m - 1 - \beta > 2 + \alpha$, and let $0 < \zeta_1, \zeta_2 \in C^\infty(\mathbb{R}^+)$ satisfy

$$\zeta_1(t) = \begin{cases} t & \text{if } t \in (0, 1] \\ t^{-1-\alpha} & \text{if } t \in [2, \infty), \end{cases} \qquad \zeta_2(t) = \int_t^\infty \zeta_1(s) \mathrm{d}s.$$

Then, for $b, c \in \mathbb{R}^+$, define

$$\eta(r) = \frac{1}{2} r + \frac{1}{2\zeta_2(0)} \int_0^r \zeta_2(s) \mathrm{d}s, \qquad f(r) = (b + r^2)^{\frac{\beta+3-m}{2}} + c.$$

We consider $M \doteq \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{m-2}$ with coordinates (t, r, θ) and metric

$$g = f(r)^2 \mathrm{d}t^2 + \mathrm{d}r^2 + \eta(r)^2 \mathrm{d}\theta^2,$$

where $\mathrm{d}\theta^2$ is the round metric on $\mathbb{S}^{m-2}$. Notice that the choice of η implies that g extends smoothly at $r = 0$ and that M is complete. It was shown in [13, Section 9] that $|\mathrm{Sec}|$ is bounded on M, and that $\mathrm{Ric}^{(2)} \geq 0$, $\mathrm{Ric} > 0$ on M if b, c are chosen large enough. Given a function $u : M \to \mathbb{R}$ of the coordinate t alone, it holds

$$|\nabla u| = \frac{\partial_t u}{f}, \qquad \Delta_\infty u = \frac{(\partial_t^2 u)(\partial_t u)^2}{f^4}.$$

Hence, any affine function $u(t) = at + k$ gives rise to an ∞-harmonic function. Also, $|\nabla u| = a/f$ is bounded on M since f is bounded below by a positive constant, thus u has at most linear growth. □

Despite, in general, $\mathrm{Ric} \geq 0$ does not guarantee the splitting of M when the latter supports a non-constant, linearly growing ∞-harmonic function, this happens in some special cases. The next result is a tangency principle between ∞-harmonic functions and (rescaled, translated) Busemann functions.

Proposition 3.5. [tangency principle] *Let M be a complete manifold with* $\mathrm{Ric} \geq 0$*, and let $u \in C(M)$ satisfy $\Delta_\infty u \geq 0$ on M. Let $c > 0$ and assume that there exists a ray γ such that $u - cb_\gamma$ has a global maximum point. Then,*

(i) M splits as $\mathbb{R} \times N$, for some complete manifold N with $\mathrm{Ric}_N \geq 0$*;*

(ii) choosing a suitable arclength parameter x of the $\mathbb{R}$-factor, it holds $u(x, y) = cx$ for each $(x, y) \in \mathbb{R} \times N$;

(iii) γ is a half-line of the type $(-\infty, a] \times \{y_0\}$. In particular, $u - cb_\gamma$ is constant on M.

Proof. First, observe that for any fixed $o \in M$

$$u(x) \leq cb_\gamma(x) + \max_M(u - cb_\gamma) \leq c(b_\gamma(x_0) + \mathrm{d}(x, o)) + \max_M(u - cb_\gamma),$$

whence by Lemma 2.2 u is c-Lipschitz. Let p be a global maximum point of $u - cb_\gamma$. Defining

$$\bar{u} = \frac{1}{c}\big(u - u(p) + cb_\gamma(p)\big)$$

then $\bar{u}$ is 1-Lipschitz and $\bar{u} \leq b_\gamma$ on M, with equality in p. We consider an asymptote $\tilde{\gamma}$ for γ at p. Then, $b_\gamma(p) + b_{\tilde{\gamma}}$ is a support function for b_γ from above at p, which gives

$$\bar{u}(\tilde{\gamma}(t)) \leq b_\gamma(p) + b_{\tilde{\gamma}}(\tilde{\gamma}(t)) = b_\gamma(p) - t$$

for $t \geq 0$, with equality at $t = 0$. Since $\bar{u}$ is 1-Lipschitz, necessarily

$$\bar{u}(\tilde{\gamma}(t)) = b_\gamma(p) - t$$

whence $\bar{u}$ is linear with slope 1 on $\tilde{\gamma}$. Since $\Delta \bar{u} \geq 0$, by the anti-peeling Lemma 2.4 $\tilde{\gamma}$ can be continued to a line $\tilde{\gamma} : \mathbb{R} \to M$ where $\bar{u}$ has slope 1. The splitting theorem guarantees that M splits as $\mathbb{R} \times N$ with the product metric $\mathrm{d}\tau^2 + g_N$, $p = (0, y_0) \in \mathbb{R} \times N$ and $\mathbb{R} \times \{y_0\}$ is the line $\tilde{\gamma}$. This shows (i). Then,

$$\bar{u}(\tau, y_0) = b_\gamma(p) + \tau,$$

thus by Lemma 2.3 we infer $\bar{u}(\tau, y) = b_\gamma(p) + \tau$ on M. Up to choosing $x = \tau + h$ for suitable constant h and recalling the definition of $\bar{u}$, we get $u(x, y) = cx$ on M, which proves (ii). To conclude, observe that since $\tilde{\gamma}$ is an asymptote of γ, then necessarily γ is of the type $(-\infty, a] \times \{y_1\}$ for some y_1. Direct computation of b_γ shows that $b_\gamma(\tau, y) = \tau - a$ and thus $\bar{u} - b_\gamma$, hence $u - cb_\gamma$, is constant. □

Remark 3.6. Note that the completeness assumption on M is crucial. Indeed, on $\mathbb{R}^m \setminus \{0\}$ the function $-|x|$ is ∞-harmonic and $-|x| + x_1$ attains infinitely many maximum points (see also [15, Exercise 2.9]).

4 Manifolds with Sec ≥ 0

Let $u \in C(M)$ satisfy $\Delta_\infty u = 0$ and

$$\limsup_{r(q)\to\infty} \frac{u(q)}{r(q)} = 1. \tag{15}$$

By general theory, if Sec ≥ 0 then blowdowns at any fixed point o are unique (see [24, Lemma 3.4]), so we denote by M_∞ the blowdown and write $M_j^\lambda \to M_\infty$. By Theorem 3.1, λ induces a splitting $M_\infty = \mathbb{R}\times N_\infty$ along a line γ_∞^λ for which the blowdown v^λ of u writes as $v^\lambda(x, y) = x$, $o_\infty = (0, o'_\infty)$ and $\gamma_\infty^\lambda(t) = (t, o'_\infty)$. It is well-known that a splitting of M_∞ induces, if Sec ≥ 0, a splitting of M itself (see [2, Thm. 4.6] for a proof). For our purposes, it is convenient to include the proof in the following lemma, which regards the behaviour of u along the split off line of M.

Lemma 4.1. *If* Sec ≥ 0*, the line* $\gamma_\infty^\lambda(t) = (t, o'_\infty) \in M_\infty = \mathbb{R}\times N_\infty$ *induces a unique line* $\gamma^\lambda : \mathbb{R} \to M$ *passing through an origin* $o \in M$ *whose blowdown is* γ_∞^λ*. Moreover,*

$$\frac{u(\gamma^\lambda(\lambda_j R)) - u(\gamma^\lambda(-\lambda_j R))}{2\lambda_j R} \to 1 \qquad \text{as} \quad j \to \infty. \tag{16}$$

Proof. As we work for fixed λ, we omit its writing. Fix $R > 0$. We refer to the proof of Theorem 3.1 for the construction of γ_∞ and for notation, so let $z_j^\pm \in M_j$ realize $S^\pm_{\lambda_j R}(o)$, let $z_R^\pm \in M_\infty$ be their limits on M_∞ and let $\gamma_R : [-R, R] \to M_\infty$ be the segment built therein to join z_R^- to z_R^+. By Remark 3.2, $z_R^\pm = \gamma_R(\pm R) = \gamma_\infty(\pm R)$. It follows that

$$M_j \ni z_j^\pm \longrightarrow \gamma_\infty(\pm R) \qquad \text{as } j \to \infty, \tag{17}$$

We select segments $\gamma_j^\pm : [0, \lambda_j R] \to M$ joining o to $z_j^\pm$. Up to subsequence, $\gamma_j^\pm \to \gamma^\pm$ for some rays $\gamma^\pm : [0, \infty) \to M$. The concatenation

$$\gamma_j := -\gamma_j^- * \gamma_j^+ = \begin{cases} \gamma_j^-(-t) & \text{for } t \in [-\lambda_j R, 0), \\ \gamma_j^+(t) & \text{for } t \in [0, \lambda_j R], \end{cases}$$

locally uniformly converges to $\gamma = -\gamma^- * \gamma^+ : \mathbb{R} \to M$. We prove that γ is a line, so fix $S > 0$ and $s \leq S$. Then, by Toponogov's Theorem,

$$\mathrm{d}(\gamma_j(-s), \gamma_j(s)) \geq \mathrm{d}(z_j^-, z_j^+)\frac{s}{\lambda_j R}.$$

However, $\mathrm{d}(z_j^-, z_j^+) = \lambda_j \mathrm{d}_j(z_j^-, z_j^+) = 2\lambda_j R(1 + o_j(1))$, whence

$$\mathrm{d}(\gamma_j(-s), \gamma_j(s)) \geq 2s(1 + o_j(1)).$$

Therefore, the excess

$$0 \leq \mathrm{d}(\gamma_j(-s), o) + \mathrm{d}(\gamma_j(s), o) - \mathrm{d}(\gamma_j(-s), \gamma_j(s)) \leq 2s o_j(1) \leq 2S o_j(1)$$

converges to zero uniformly for $s \in [0, S]$, which proves that γ is a line. We next point out that the blowdown of γ is exactly γ_∞. Applying the cosine law to the hinge (o, γ, γ_j) and using that the angle

$$\sphericalangle(\dot\gamma_j^\pm(0), \dot\gamma^\pm(0)) \to 0 \qquad \text{as } j \to \infty,$$

we deduce

$$\mathrm{d}(\gamma(\pm\lambda_j R), z_j^\pm)^2 \leq 2(\lambda_j R)^2 - 2(\lambda_j R)^2 \cos \sphericalangle(\dot{\gamma}_j^\pm(0), \dot{\gamma}^\pm(0)) = o_j(\lambda_j^2 R^2). \tag{18}$$

Rescaling, we get

$$\mathrm{d}_j(\gamma(\pm\lambda_j R), z_j^\pm) \to 0 \qquad \text{as } j \to \infty,$$

and by the triangle inequality and (17) we deduce

$$\gamma(\pm\lambda_j R) \subset M_j^\lambda \to \gamma_\infty(\pm R).$$

Therefore, the blowdown of γ (which is clearly a line in M_∞^λ) restricted to $[-R, R]$ is a segment joining $\gamma_\infty(-R)$ to $\gamma_\infty(R)$. Since γ_∞ is the only such segment, we conclude from the arbitrariness of R that γ blows down to γ_∞. If there were a line $\sigma \neq \gamma$ with $\sigma(0) = o$ whose blowdown is γ_∞, writing

$$\sigma(s) = (b_1 s, \bar{\sigma}(s)) \in \mathbb{R} \times N \qquad \text{with } |b_1| < 1,$$

the curve $\bar{\sigma} : \mathbb{R} \to N$ would be a line in N. Since N has non-negative sectional curvature, the splitting theorem would guarantee that

$$M = \mathbb{R} \times \mathbb{R} \times N', \qquad \text{with} \quad \begin{cases} \gamma(t) = (t, 0, o''), \\ \sigma(s) = (b_1 s, b_2 s, \hat{\sigma}(s)) \end{cases}$$

for some $o'' \in N'$, $b_2 \in (0, 1)$ and line $\hat{\sigma}$ in N', which is incompatible with the assumption that σ blows down to γ_∞.

To prove (16), observe that by definition of $z_j^\pm$,

$$\frac{u_j(z_j^\pm)}{R} = \frac{u(z_j^\pm) - u(o)}{\lambda_j R} \to \pm 1 \qquad \text{as } j \to +\infty. \tag{19}$$

We consider

$$0 \leq 1 - \frac{u(\gamma(\lambda_j R)) - u(\gamma(-\lambda_j R))}{2\lambda_j R} = 1 - \frac{u(z_j^+) - u(z_j^-)}{2\lambda_j R} + A_+ - A_- \tag{20}$$

where

$$A_\pm = \frac{u(\gamma(\pm\lambda_j R)) - u(z_j^\pm)}{2\lambda_j R}.$$

Using $\mathrm{Lip}(u, M) = 1$ and (18) we get

$$|A_\pm| \leq \frac{\mathrm{d}(\gamma(\pm\lambda_j R), z_j^\pm)}{2\lambda_j R} \to 0 \qquad \text{as } j \to \infty,$$

whence letting $j \to \infty$ in (20) and using (19) we conclude (16). □

We first study the 2-dimensional case, where Theorem 3.1 and Savin's result [40] are enough to give a full classification.

Proof of Theorem 1.3. By Theorem 3.1 and Lemma 4.1, $M = \mathbb{R} \times N$ for some 1-dimensional complete manifold N, which is therefore either $\mathbb{R}$ or $\mathbb{S}^1$. If $M = \mathbb{R}^2$, since $u \in \mathrm{Lip}(M)$ we can apply Savin's result to deduce that u is affine. If $M = \mathbb{R} \times \mathbb{S}^1$, we consider the universal covering $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{S}^1$ and the preimage $\bar{u} = u \circ \pi$, which is ∞-harmonic. Savin's result guarantees that $\bar{u}$ is affine on $\mathbb{R}^2$, so $\bar{u}(x, y) = ax + by + c$. By construction, $\bar{u}$ is bounded in the y-coordinate for any fixed x, thus $b = 0$ and $\bar{u}(x, y) = ax + c$ only depends on the first factor. □

In dimension $m \geq 3$, we are going to show uniqueness of the blowdown of u. This depends on refined one-sided gradient estimates for solutions to approximate problems which were obtained, in Euclidean setting, by Evans and Smart [22]. We closely follow the approach therein but with a different approximation, as we use solutions u_p to the p-Laplace equation

$$\begin{cases} \Delta_p u_p = 0 & \text{on } \Omega \Subset M, \\ u_p = u & \text{on } \partial\Omega \end{cases}$$

in the limit $p \to \infty$. The main properties of u_p are put off to the next section as not to interrupt the flow of the discourse. We begin with the following

Lemma 4.2. *Let $M = \mathbb{R} \times \mathbb{R} \times N'$ be a complete manifold and let $x^1, x^2, \pi : M \to \mathbb{R}$ be the natural projections onto the three factors. Consider a smooth function $u : M \to \mathbb{R}$ satisfying*

$$\max_{B_T} |u - b_1 x^1 - b_2 x^2| < \eta T,$$

for some $\eta, T > 0$, where B_T is a geodesic ball centered at $o = (0, 0, \tilde{o})$. Then, there exists an interior point $q_0 \in B_T$ such that

$$|\partial_{x^i} u(q_0) - b_i| \leq 4\eta, \quad \textit{for } i = 1, 2, \tag{21}$$

$$|\nabla^{N'} u(q_0)| \leq 4\eta. \tag{22}$$

Proof. Consider the auxiliary function

$$w = b_1 x^1 + b_2 x^2 - 2\frac{\eta}{T}\rho^2, \qquad \rho(q) \doteq \mathrm{d}_M(q, o).$$

We observe that $(u - w)(o) < \eta T$ and $(u - w)(q) \geq \eta T$ for any $q \in \partial B_T$. Therefore there exists an interior minimum point $q_0 \in B_T$. If ρ is smooth around q_0, the desired conclusion follows from $\nabla(u - w)(q_0) = 0$. Otherwise, we use Calabi's trick by considering a unit speed minimizing geodesic γ from o to q_0 and the function ρ_ε with $\rho_\varepsilon(q) = \varepsilon + \mathrm{d}_M(\gamma(\varepsilon), q)$. From $\rho_\varepsilon \geq \rho$ on M with equality at q_0, and since ρ_ε is smooth near q_0 as shown in [38, end of Lemma 7.1.9], the conclusion follows as above by replacing ρ with ρ_ε. ☐

With the above preparation, we are ready to prove the main result of this section, which generalizes [25, Proposition 2].

Theorem 4.3. *Let M be a complete manifold with* Sec ≥ 0*, and let $u \in C(M)$ be an ∞-harmonic function such that*

$$\limsup_{r(q)\to\infty} \frac{u(q)}{r(q)} = 1, \tag{23}$$

where r is the distance from a fixed origin. Then, for each $o \in M$ the blowdown v_∞ of u at o does not depend on the chosen sequence, and there exists a splitting $M = (\mathbb{R} \times N, \mathrm{d}x^2 + g_N)$ for some complete manifold (N, g_N) such that

$$\lim_{x\to+\infty} \frac{u(x, y) - u(-x, y)}{2x} \to 1 \qquad \forall\, y \in N. \tag{24}$$

Moreover, in the splitting $M_\infty = \mathbb{R} \times N_\infty$ induced by $M = \mathbb{R} \times N$, it holds $v_\infty(x, y) = x$.

Proof. We already know by Lemma 2.2 that $\mathrm{Lip}(u, M) = 1$. We introduce some notation. Let $M_j^\lambda \to M_\infty$ be a tangent cone with associated blowdown $u_j^\lambda \to v^\lambda$, and let $\mathbb{R} \times N$ be the splitting of M induced by the line γ^λ. Write $o = (0, o') \in \mathbb{R} \times N$. Notice that, by Lemma 4.1, N_∞ is the tangent cone of N at o'. Also, (16) guarantees that (24) holds with $y = o'$ provided that the blowdown v^λ does not depend on λ. Once (24) is shown for $y = o'$, its validity for any fixed y immediately follows from the triangle inequality, since

$$\Big| u(x, y) - u(-x, y) - \big(u(x, o') - u(-x, o')\big) \Big| \le 2\mathrm{d}_N(y, o').$$

To conclude the proof we assume, by contradiction, the existence of a sequence $\mu = \{\mu_k\} \to \infty$ such that the blowdown $u_k^\mu \to v^\mu$ associated to the tangent cone $M_k^\mu \to M_\infty$ satisfies $v^\mu \neq v^\lambda$. Consider the line γ_∞^μ in M_∞ induced by μ. From $v^\mu(\gamma_\infty^\mu(s)) = s$ and $v^\mu \neq v^\lambda$, we deduce $\gamma_\infty^\mu \neq \gamma_\infty^\lambda$ and therefore, in the splitting $\mathbb{R} \times N_\infty$ induced by γ_∞^λ, we can write

$$\gamma_\infty^\mu(s) = (b_1 s, \sigma_\infty^\mu(s)), \qquad b_1^2 + |\dot\sigma_\infty^\mu|^2 = 1.$$

Since σ_∞^μ is a line as well, it induces a splitting $N_\infty = \mathbb{R} \times \tilde N_\infty$ and, by Lemma 4.1, a corresponding splitting $N = \mathbb{R} \times \tilde N$. Summarizing, we can write

$$M_\infty = \mathbb{R} \times \mathbb{R} \times \tilde N_\infty, \qquad o_\infty = (0, 0, \tilde o_\infty)$$

with induced projections $(x_\infty^1, x_\infty^2, \pi_\infty) : M_\infty \to \mathbb{R} \times \mathbb{R} \times \tilde N_\infty$, and in these coordinates

$$\gamma_\infty^\lambda(t) = (t, 0, \tilde o_\infty), \qquad \gamma_\infty^\mu(s) = (b_1 s, b_2 s, \tilde o_\infty), \ \text{ with } \ b_1^2 + b_2^2 = 1, \ \ b_2 \neq 0.$$

Accordingly,

$$M = \mathbb{R} \times \mathbb{R} \times \tilde N, \qquad o = (0, 0, \tilde o)$$

with projections (x^1, x^2, π), and

$$\gamma^\lambda(t) = (t, 0, \tilde o), \qquad \gamma^\mu(s) = (b_1 s, b_2 s, \tilde o), \ \text{ with } \ b_1^2 + b_2^2 = 1, \ \ b_2 \neq 0.$$

In these coordinates

$$v^\lambda = x_\infty^1, \qquad v^\mu = b_1 x_\infty^1 + b_2 x_\infty^2,$$

whence

$$u_j^\lambda \to x_\infty^1, \qquad u_k^\mu \to b_1 x_\infty^1 + b_2 x_\infty^2$$

pointwise in the Gromov-Hausdorff sense. Also, associated to the tangent cone $M_j^\lambda \to M_\infty$ we define coordinates

$$x_{\lambda,j}^1 = \frac{x^1}{\lambda_j}, \qquad x_{\lambda,j}^2 = \frac{x^2}{\lambda_j},$$

which are 1-Lipschitz on M_j^λ. By construction, $x_{\lambda,j}^i \to x_\infty^i$ pointwise in the Gromov-Hausdorff sense for each $i \in \{1, 2\}$. Likewise, the coordinates

$$x_{\mu,k}^1 = \frac{x^1}{\mu_k} \qquad \text{and} \qquad x_{\mu,k}^2 = \frac{x^2}{\mu_k} \qquad \text{on } M_k^\mu$$

satisfy $x_{\mu,k}^i \to x_\infty^i$. It easily follows that, for each $\ell > 0$, there exists $j_0(\ell)$ such that

$$\max_{q \in B_{\lambda_j}} |u(q) - x^1(q)| < \ell \lambda_j \qquad \text{for } j \ge j_0(\ell). \tag{25}$$

Indeed, otherwise, there exist $\ell > 0$ and points $q_j \in B_{\lambda_j} \subset M$ such that

$$\ell \le |u_j^\lambda(q_j) - x^1_{\lambda,j}(q_j)|.$$

Up to a subsequence, $q_j \in B_1^j \subset M_j^\lambda$ converges to $q_\infty \in B_1^\infty$ and therefore

$$\begin{aligned}\ell &\le |u_j^\lambda(q_j) - x^1_{\lambda,j}(q_j)| \le |u_j^\lambda(q_j) - x^1_\infty(q_\infty)| + |x^1_\infty(q_\infty) - x^1_{\lambda,j}(q_j)| \\ &\to 0 \qquad \text{as } j \to \infty,\end{aligned}$$

contradiction. Similarly to (25), for the tangent cone $M_k^\mu \to M_\infty$ and for $\eta > 0$ we obtain

$$\max_{q \in B_{\mu_k}} |u(q) - b_1 x^1(q) - b_2 x^2(q)| < \eta\mu_k \qquad \text{for } k \ge k_0(\eta). \tag{26}$$

Consider for each $p > 2$ and j the solution $u_{p,j}$ to

$$\begin{cases} \Delta_p u_{p,j} = 0 & \text{on } B_{2\lambda_j}, \\ u_{p,j} = u & \text{on } \partial B_{2\lambda_j}. \end{cases}$$

As recalled in Section 5, $u_{p,j} \to u$ uniformly on $\overline{B}_{2\lambda_j}$ as $p \to \infty$. Moreover, by (25) we get

$$\ell \ge \lambda_j^{-1} \|u_{p,j} - x^1\|_{L^\infty(B_{\lambda_j})} \qquad \text{for } p \ge p_j \text{ large}. \tag{27}$$

Therefore, since Sec ≥ 0 and u is 1-Lipschitz, Theorems 5.1 and 5.3 with $p \ge \ell^{-1}$ guarantee the existence of a constant $C = C(m, u(o))$ such that

$$|\nabla u_{p,j}| \le C, \qquad |\nabla u_{p,j}|^2 \le \partial_1 u_{p,j} + C\ell^{\frac{1}{8}}(1+\ell)^{\frac{5}{4}} \quad \text{on } B_{\lambda_j/2}.$$

We now choose ℓ, η. First, define

$$\theta = 1 - b_1 \in (0,1),$$

and let $\ell, \eta > 0$ small enough to satisfy

$$C\ell^{\frac{1}{8}}(1+\ell)^{\frac{5}{4}} < \frac{\theta}{4}, \qquad \eta < \frac{3\theta}{4 \cdot 28}.$$

Therefore,

$$|\nabla u_{p,j}|^2 \le \partial_1 u_{p,j} + \frac{\theta}{4} \qquad \forall\, j \ge j_0(\ell). \tag{28}$$

For $k_0 = k_0(\eta)$ as in (26), choose $j_1 = j_1(\eta)$ such that $\lambda_{j_1} \ge 2\mu_{k_0}$ and let

$$j_2(\ell, \eta) = \max\{j_0(\ell), j_1(\eta)\}.$$

We choose $j = j_2$ and write $u_p = u_{p,j_2}$ for ease of notation. From $B_{\mu_{k_0}} \subset B_{\lambda_{j_2}/2}$ and the uniform convergence $u_p \to u$ as $p \to \infty$, and from (25), (26), we infer the existence of $p_1 = p_1(\ell, \eta)$ for which

$$\max_{q \in B_{\mu_{k_0}}} |u_p(q) - b_1 x^1(q) - b_2 x^2(q)| < \eta\mu_{k_0} \qquad \text{for } p \ge p_1. \tag{29}$$

Using Lemma 4.1, we get the existence of $q_0 \in B_{\mu_{k_0}}$ such that

$$|\partial_1 u_p(q_0)| \le b_1 + 4\eta,$$
$$|\nabla u_p(q_0)| \ge 1 - |(b_1 - \partial_1 u_p)\partial_{x^1} + (b_2 - \partial_2 u_p)\partial_{x^2} - \nabla^{N'} u_p(q_0)| \ge 1 - 12\eta.$$

On the other hand, (28) and $B_{\mu_{k_0}} \subset B_{\lambda_{j_2}/2}$ give $|\nabla u_p(q_0)|^2 \le \partial_1 u_p(q_0) + \theta/4$. Putting together the estimates we conclude

$$1 - 24\eta \le (1 - 12\eta)^2 \le |\nabla u_p(q_0)|^2 \le b_1 + 4\eta + \frac{\theta}{4} = 1 - \frac{3\theta}{4} + 4\eta,$$

contradicting our choice for η. □

We conclude this section with a sufficient condition for the function u in Theorem 4.3 to depend only on the variable x. The result below is inspired by the proof of the "Half-space theorem" in [16, Thm. 4.1], which guarantees that a solution to $\Delta_\infty u \ge 0$ on $\mathbb{R}^m$ is affine provided that it lies below an affine function. However, our statement is significantly different: on the one hand, it only applies to ∞-harmonic functions, while on the other hand the extra condition we require is only localised on a single ray.

Theorem 4.4. *Let M be a complete manifold with* Sec ≥ 0*, and let $u \in C(M)$ be an ∞-harmonic function such that*

$$\limsup_{r(q)\to\infty} \frac{u(q)}{r(q)} = 1, \tag{30}$$

where r is the distance from a fixed origin. Assume that there exist a ray γ and a constant C such that either

$$u\big(\gamma(t)\big) \ge t - C \qquad or \qquad u\big(\gamma(t)\big) \le -t + C$$

for each $t \in \mathbb{R}^+$. Then, there exists a splitting $M = (\mathbb{R} \times N, \mathrm{d}x^2 + g_N)$ for some complete manifold (N, g_N) such that $u(x, y) = x$ for each $(x, y) \in \mathbb{R} \times N$.

Proof. In our assumptions, we know that $\mathrm{Lip}(u, M) = 1$. Define $o = \gamma(0)$. By Theorem 4.3, M splits as $\mathbb{R} \times N$ with metric $\mathrm{d}x^2 + g_N$ in such a way that (24) is satisfied. In particular, (30) holds both for u and for $-u$. Up to changing the sign of u, we can therefore assume that

$$u\big(\gamma(t)\big) \le -t + C \qquad \forall\, t \in \mathbb{R}^+. \tag{31}$$

We first claim that in coordinates (x, y) we have $\gamma(t) = (-t, o')$. Indeed, let $M_\infty = \mathbb{R} \times N_\infty$ be the blow-down of M at o, $o_\infty = (0, o'_\infty)$ its reference point and u_∞, γ_∞ the associated blowdowns of u and γ. We know by Theorem 3.1 that $u_\infty(x, y) = x$, with x the arclength of the $\mathbb{R}$-factor properly oriented. Blowing down (31) and using that u, hence u_∞, is 1-Lipschitz we deduce $u_\infty(\gamma_\infty(t)) = -t$ for each $t \in \mathbb{R}^+$. Writing $\gamma_\infty(t) = (b_1 t, \sigma_\infty(t)) \in \mathbb{R} \times N_\infty$ with $b_1^2 + |\sigma'_\infty|^2 = 1$ we get $\gamma_\infty(t) = (-t, o'_\infty)$. Our claim follows by the uniqueness part in Lemma 4.1. We therefore proved that

$$u(x, o') - x \le C \qquad \forall\, x \in (-\infty, 0]. \tag{32}$$

Since $\mathrm{Lip}(u, M) = 1$, the function $x \mapsto \delta(x, y) \doteq u(x, y) - x$ is non-increasing on $\mathbb{R}$ (thus, (32) holds for each $x \in \mathbb{R}$). Let us call $\delta(-\infty, y)$ its limit as $x \to -\infty$. By assumption, $\delta(-\infty, y_0)$ is finite. We prove that $\delta(-\infty, y)$ does not depend on y. First, since u is 1-Lipschitz we have

$$|\delta(x, y) - \delta(x, y_0)| = |u(x, y) - u(x, y_0)| \le \mathrm{d}_N(y, y_0) \qquad \forall\, x \in \mathbb{R},$$

whence $\delta(-\infty, y)$ is finite for each y. Again since u is 1-Lipschitz,

$$\begin{aligned}|x + \delta(x, y) - \lambda - \delta(\lambda, y_0)|^2 &= |u(x, y) - u(\lambda, y_0)|^2 \\ &\leq (x - \lambda)^2 + \mathrm{d}_N(y, y_0)^2.\end{aligned}$$

Expanding the squares and rearranging,

$$\left[\delta(x, y) - \delta(\lambda, y_0)\right]^2 + 2(x - \lambda)(\delta(x, y) - \delta(\lambda, y_0)) \leq \mathrm{d}_N(y, y_0)^2.$$

Discarding the first term on the left hand side, putting $x = 2\lambda < 0$, dividing by λ and letting $\lambda \to -\infty$ gives $\delta(-\infty, y) - \delta(-\infty, y_0) \geq 0$. On the other hand, letting $x = \lambda/2 < 0$, dividing by λ and letting $\lambda \to -\infty$ gives $\delta(-\infty, y) - \delta(-\infty, y_0) \leq 0$. Hence, $\delta(-\infty, y) = \delta(-\infty, y_0)$.

Up to translating u, we can therefore assume

$$\delta(-\infty, y) = 0 \qquad \text{for each } \ y \in N$$

so that

$$u(x, y) \leq x \quad \forall\, (x, y) \in M, \qquad \text{and} \qquad \lim_{x\to-\infty} (x - u(x, y)) = 0.$$

By contradiction, we assume that $u(x_0, y_0) < x_0$ for some $(x_0, y_0) \in M$. Defining $v(x, y) \doteq u(x + x_0, y) - x_0$, we observe that v is an ∞-harmonic function satisfying

$$\lim_{x\to-\infty} (x - v(x, y)) = 0 \quad \text{and} \quad v(x, y) \leq x, \quad \text{with} \quad v(0, y_0) < 0. \tag{33}$$

Fix $\mu > 0$ such that $v(0, y_0) \leq -\mu$. Since $|\nabla v| \leq 1$ a.e. in M, we get

$$v(x, y) \leq \sqrt{|x|^2 + d_N(y, y_0)^2} - \mu. \tag{34}$$

Next, for $R > r$ we consider the sphere and ball of radius R in $\mathbb{R} \times N$ centered at $(-r, y_0)$:

$$S_R = \left\{(x, y) : \ \sqrt{|x + r|^2 + d_N(y, y_0)^2} = R\right\},$$

and

$$\overline{B}_R = \left\{(x, y) : \ \sqrt{|x + r|^2 + d_N(y, y_0)^2} \leq R\right\}.$$

In order to obtain an upper bound for $v(x, y)$ on S_R, on the one hand $v(x, y) \leq x$, while on the other hand, by (34),

$$\begin{aligned}v(x, y) &\leq -\mu + \sqrt{|x|^2 + d_N(y, y_0)^2} \\ &= -\mu + \sqrt{(x + r)^2 - 2rx - r^2 + d_N(y, y_0)^2} \\ &= -\mu + \sqrt{R^2 - 2rx - r^2} \qquad \text{on } S_R.\end{aligned} \tag{35}$$

Whence, for each $(x, y) \in S_R$ we have

$$v(x, y) \leq \max_{x\in[-R-r,R-r]} \min\left\{x, -\mu + \sqrt{R^2 - 2rx - r^2}\right\}.$$

Since the function $\sqrt{R^2 - 2rx - r^2}$ is decreasing in x, the maximum is attained when

$$x \in [-R - r, R - r] \qquad \text{solves} \qquad x = -\mu + \sqrt{R^2 - 2rx - r^2},$$

that is, $x = -\mu - r + \sqrt{R^2 + 2\mu r}$. Concluding,

$$v(x, y) \le \sqrt{R^2 + 2\mu r} - \mu - r = -r + \frac{\sqrt{R^2 + 2\mu r} - \mu}{R} \sqrt{|x + r|^2 + d_N(y, y_0)^2} \qquad \text{on } S_R.$$

The same inequality is also satisfied at the vertex $(-r, y_0)$. Therefore, by the comparison with cone property,

$$v(x, y) \le -r + \frac{\sqrt{R^2 + 2\mu r} - \mu}{R} \sqrt{|x + r|^2 + d_N(y, y_0)^2} \qquad \text{on } B_R.$$

To conclude the proof, for $0 \le s < r$ we choose $x = -s$ and $y = y_0$ to deduce

$$u(-s, y_0) - (-s) \le \left(-1 + \frac{\sqrt{R^2 + 2\mu r} - \mu}{R} \right) (r - s).$$

Taking $R = 2r$ and letting $r \to \infty$, we infer

$$u(-s, y_0) - (-s) \le -\frac{\mu}{4} < 0,$$

which implies by letting $s \to \infty$ that $\delta(-\infty, y_0) \le -\mu/4$, a contradiction. □

5 On the approximation via p-harmonic functions

Let $u \in C(M)$ solve $\Delta_\infty u = 0$ on M. In this section, we prove the relevant gradient and one-side gradient estimates for p-harmonic approximations of u that are used in the proof of Theorem 4.3. In Euclidean setting, the result is due to Evans and Smart [22], where it is used to infer the uniqueness of the blowup of u at a given point and, consequently, the everywhere differentiability of u. Therein, for $\varepsilon > 0$ the authors choose to approximate Δ_∞ with the operator

$$\Delta_{\infty,\varepsilon} \phi = \varepsilon e^{-\frac{|\nabla \phi|^2}{2\varepsilon}} \operatorname{div} \left(e^{\frac{|\nabla \phi|^2}{2\varepsilon}} \nabla \phi \right) = \Delta_\infty \phi + \varepsilon \Delta \phi$$

and, for each $R > 0$, the function u with the solution u_ε to the problem

$$\begin{cases} \Delta_{\infty,\varepsilon} u_\varepsilon = 0 & \text{on } B_R, \\ u_\varepsilon = u & \text{on } \partial B_R. \end{cases}$$

While their arguments can be adapted to the manifold setting, we prefer to approximate via the p-Laplace operator, in the hope that its peculiar features (notably its homogeneity) may be further exploited to get even sharper estimates in the direction of those required in [21].

Fix a smooth, relatively compact set Ω, and for $p \in (1, \infty)$ consider the solution u_p to

$$\begin{cases} \Delta_p u_p \doteq \operatorname{div} \left(|\nabla u_p|^{p-2} \nabla u_p \right) = 0 & \text{on } \Omega, \\ u_p = u & \text{on } \partial\Omega. \end{cases} \tag{36}$$

It is known by [8] and the uniqueness result in [4, 27] that $u_p \to u$ uniformly on $\overline{\Omega}$ as $p \to \infty$. Moreover, $u_p \in C^{1,\alpha}(\Omega)$ and it is C^∞ on the open set $\{|\nabla u_p| > 0\}$. In a manifold setting, a sharp local gradient estimate for p-harmonic functions was obtained by Kotschwar and Ni in [30, Thm. 1.1], see also [42]. While they were interested in the limit $p \to 1$, in our setting their estimate implies the following simpler one, which is enough for our purposes.

Theorem 5.1. *Assume that u_p is p-harmonic on a ball $B_{2R}(o) \Subset M^m$, and that* $\mathrm{Sec} \ge -\kappa^2$ *on $B_{2R}(o)$. Then,*

$$\sup_{B_R(o)} |\nabla \log u_p|^2 \le \frac{C_m}{(p-1)^2}\left(\frac{p^2}{R^2} + \frac{p\kappa}{R} + \kappa^2\right),$$

where C_m only depends on m.

As a consequence, if $R \ge 1$ and u_p solves (36) on $\Omega = B_{2R}(o)$, then for $p \ge R$ we have

$$|\nabla u_p(x)| \le C_m (1+\kappa)\frac{u(x)}{R} \qquad \forall\, x \in B_R(o).$$

Theorem 5.1 is obtained by a careful application of the improved version of Cheng-Yau's technique to a suitable Bochner formula for the p-Laplacian, which we now recall. Formally differentiating the p-Laplacian at u_p we get the linearized operator

$$\phi \longmapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Delta_p(u_p + t\phi) = \mathrm{div}\,\left(A(\nabla u_p)\nabla\phi\right)$$

where, for a nonzero vector $X \in T_xM$, $A(X) : T_xM \to T_xM$ is the endomorphism

$$A(X) = |X|^{p-2}\left((p-2)\langle \frac{X}{|X|}, \cdot\rangle \frac{X}{|X|} + \mathrm{id}\right).$$

Notice that $A(X)$ has eigenvalues $(p-1)|X|^{p-2}$ in direction X and $|X|^{p-2}$ on $X^\perp$. For each $x \in \{|\nabla u_p| > 0\}$ we consider a local orthonormal frame $\{\nu, e_j\}$ with $\nu = \nabla u_p/|\nabla u_p|$ and $\{e_j\}$, $2 \le j \le m$ tangent to the level sets of u_p. Then the following Bochner formula in [36, Prop. 2.14] holds:

$$\begin{aligned}
&\frac{1}{2}\mathrm{div}\,\left(A(\nabla u_p)\nabla|\nabla u_p|^2\right) = \\
&= |\nabla u_p|^{p-2}\Big\{(p-1)u_{\nu\nu}^2 + p\sum_j u_{\nu j}^2 + \sum_{i,j} u_{ij}^2 + \mathrm{Ric}(\nabla u_p, \nabla u_p)\Big\},
\end{aligned} \tag{37}$$

where $u_{\nu\nu}, u_{\nu j}, u_{ij}$ are the components of $\nabla^2 u_p$. For convenience, we also consider the normalized linearization

$$\mathscr{L}_p\phi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{\Delta_p(u_p + t\phi)}{|\nabla(u_p + t\phi)|^{p-2}} = |\nabla u_p|^{2-p}\,\mathrm{div}\,\left(A(\nabla u_p)\nabla\phi\right).$$

So that the above Bochner formula simplifies to

$$\begin{aligned}
\tfrac{1}{2}\mathscr{L}_p|\nabla u_p|^2 &= (p-1)u_{\nu\nu}^2 + p\sum_j u_{\nu j}^2 + \sum_{i,j} u_{ij}^2 + \mathrm{Ric}(\nabla u_p, \nabla u_p) \\
&= (p-2)|\nabla^2 u_p(\nu)|^2 + |\nabla^2 u_p|^2 + \mathrm{Ric}(\nabla u_p, \nabla u_p) \\
&\ge (p-1)|\nabla^2 u_p(\nu)|^2 + \mathrm{Ric}(\nabla u_p, \nabla u_p).
\end{aligned} \tag{38}$$

We shall rewrite $\mathscr{L}_p$ in trace form. Let $\{e^a\}$, $1 \le a, b \le m$ be the dual coframe of $\{e_a\}$. The components of $A(\nabla u_p)$ satisfy

$$A^a_b = |\nabla u|^{p-2}\left[\delta^a_b + (p-2)\frac{u^a u_b}{|\nabla u|^2}\right],$$

and expanding $\Delta_p u_p = 0$ we get

$$\Delta u = -(p-2)u_{\nu\nu}.$$

Therefore, a computation gives

$$[\mathrm{div}\ A(\nabla u_p)]_b e^b = A^a_{b,a} e^b = 2(p-2)|\nabla u|^{p-3} u_{\nu j} e^j.$$

It follows that in components

$$\begin{aligned} \mathscr{L}_p \phi &= |\nabla u|^{2-p} \left(A^a_b \phi^b\right)_a = |\nabla u|^{2-p} A^a_b \phi^b_a + 2(p-2)|\nabla u|^{-1} u_{\nu j} \phi^j \\ &= \left[\delta^a_b + (p-2)\frac{u^a u_b}{|\nabla u|^2}\right] \phi^b_a + 2(p-2)|\nabla u|^{-1} u_{\nu j} \phi^j. \end{aligned} \tag{39}$$

Assume that M splits as $\mathbb{R} \times N$ with coordinates (x, y) and metric $\mathrm{d}x^2 + g_N$, and consider the function $\partial_x u_p = \langle \nabla u_p, \partial_x \rangle$. Since ∂_x is a Killing field,

$$\mathscr{L}_p(\partial_x u_p) = 0 \quad \text{on } \{|\nabla u_p| > 0\}. \tag{40}$$

Furthermore, by using (39) we get

$$|\mathscr{L}_p x| \le 2|p-2||\nabla u|^{-1}|\nabla^2 u(\nu)|. \tag{41}$$

We next estimate $\mathscr{L}_p \varphi^2$ when φ is a cut-off depending on the distance from a fixed point.

Lemma 5.2. *Assume that* Sec $\ge -\kappa^2$ *in a ball* $B_{2R} \Subset M$, *for some* $\kappa \in \mathbb{R}^+_0$. *Then, there exists a function* $\varphi \in \mathrm{Lip}_c(B_{2R})$ *such that* $0 \le \varphi \le 1$, $\varphi \equiv 1$ *on* B_R *and, for* $p \ge m$,

$$|\nabla \varphi| \le \frac{C}{R}, \qquad \mathscr{L}_p \varphi^2 \ge -Cp\varphi\big(C_R + R^{-1}|\nabla u|^{-1}|\nabla^2 u(\nu)|\big)$$

in the barrier sense, where C *is an absolute constant and* $C_R \doteq R^{-2}(1+\kappa R)$.

Proof. Let $\eta \in C^\infty_c([0,2))$ satisfy

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } [0,1], \quad \eta' \le 0, \quad |\eta'| + |\eta''| \le C,$$

and let $\varphi(x) = \eta(r(x)/R)$ where r is the distance from the center of the ball. Setting

$$\mathrm{tn}_\kappa(t) = \begin{cases} \kappa \coth(\kappa t) & \text{if } \kappa > 0, \\ 1/t & \text{if } \kappa = 0, \end{cases}$$

by the Hessian comparison theorem

$$\nabla^2 r \le \mathrm{tn}_\kappa(r)\Big(\langle\ ,\ \rangle - \mathrm{d}r^2\Big)$$

in the barrier (i.e. support) sense, see [38, Lem. 12.2.4]. Noting that $\eta' = 0$ on $[0,1]$, we have

$$\begin{aligned} \nabla^2 \varphi &= \eta'' R^{-2} \mathrm{d}r^2 + \eta' R^{-1} \nabla^2 r \ge -CR^{-2}\mathrm{d}r^2 + \eta' R^{-1} \mathrm{tn}_\kappa(R)\big(\langle\ ,\ \rangle - \mathrm{d}r^2\big) \\ &\ge -CR^{-2}(1 + R\,\mathrm{tn}_\kappa(R))\langle\ ,\ \rangle \ge -CC_R\langle\ ,\ \rangle \end{aligned} \tag{42}$$

for suitable C, where we used that $R\mathrm{tn}_\kappa(R) \le 1 + \kappa R$. Whence, using (39) and for $p \ge m$,

$$\begin{aligned} \mathscr{L}_p \varphi &= \left[\delta^a_b + (p-2)\frac{u^a u_b}{|\nabla u|^2}\right]\varphi^b_a + 2(p-2)|\nabla u|^{-1} u_{\nu j}\varphi^j \\ &= (p-1)\varphi_{\nu\nu} + \varphi_{jj} + 2(p-2)|\nabla u|^{-1} u_{\nu j}\varphi^j \\ &\ge -CC_R p - 2(p-2)|\nabla u|^{-1}|\nabla^2 u(\nu)||\nabla \varphi| \\ &\ge -Cp\big(C_R + R^{-1}|\nabla u|^{-1}|\nabla^2 u(\nu)|\big) \end{aligned} \tag{43}$$

and

$$\mathscr{L}_p \varphi^2 \ge 2\varphi \mathscr{L}_p \varphi, \tag{44}$$

from which the thesis follows. □

We now prove the one-side gradient estimate.

Theorem 5.3. *Let $M^m = \mathbb{R} \times N$ be a complete manifold with metric $\mathrm{d}x^2 + g_N$, and assume that N has $\mathrm{Sec} \ge -\kappa^2$. Let $u_p \in C^1(\overline{B}_{2R})$ solve $\Delta_p u_p = 0$ on B_{2R}. Then, for*

$$\ell \ge R^{-1}\|u - x\|_{L^\infty(B_{2R})}, \qquad A \ge 1 + \|\nabla u_p\|_{L^\infty(B_{2R})},$$

and $p \ge 2$ there exists an absolute constant C such that

$$|\nabla u_p|^2 \le \partial_x u_p + CmA^2 \ell^{\frac{1}{8}}(\ell+1)^{\frac{5}{4}}\left[1 + \kappa R + \frac{1}{\ell p}\right] \qquad \text{on } B_R.$$

Proof. Let φ be a cut-off function. For convenience, we suppress the subscript p and simply write u. We consider

$$\Phi = |\nabla u|^2 - \partial_x u.$$

Because of (38), (40) and our assumptions on the sectional curvature, on the set $\{\Phi > 0\}$ and setting $\nu = \nabla u/|\nabla u|$ it holds

$$\mathscr{L}_p \Phi^2 \ge 2\Phi \mathscr{L}_p \Phi \ge 4(p-1)\Phi|\nabla^2 u(\nu)|^2 - 4(m-1)\kappa^2 \Phi |\nabla u|^2.$$

We compute on $\{\Phi > 0\}$ the following expression:

$$\begin{aligned} \mathscr{L}_p(\varphi^2\Phi^2) &\ge \varphi^2 \mathscr{L}_p \Phi^2 + \Phi^2 \mathscr{L}_p \varphi^2 + 2(p-2)\langle \nu, \nabla\varphi^2\rangle\langle \nu, \nabla \Phi^2\rangle + 2\langle \nabla \varphi^2, \nabla \Phi^2\rangle \\ &\ge 4(p-1)\varphi^2\Phi|\nabla^2 u(\nu)|^2 - 4(m-1)\kappa^2\varphi^2\Phi|\nabla u|^2 + \Phi^2 \mathscr{L}_p \varphi^2 \\ &\quad -8(p-1)\Phi\varphi|\nabla\varphi||\langle \nu, \nabla\Phi\rangle| - 8\Phi\varphi|\nabla\varphi||\nabla\Phi|. \end{aligned}$$

Notice that $\nabla\Phi = 2\nabla^2 u(\nabla u) - \nabla^2 u(e_1)$, whence

$$|\langle \nu, \nabla\Phi\rangle| \le 2|\nabla^2 u(\nu)|(1 + |\nabla u|), \qquad |\nabla\Phi| \le 2|\nabla^2 u|(1 + |\nabla u|).$$

Inserting into the above we get

$$\begin{aligned} \mathscr{L}_p(\varphi^2\Phi^2) &\ge 4(p-1)\varphi^2\Phi|\nabla^2 u(\nu)|^2 - 4(m-1)\kappa^2\varphi^2\Phi|\nabla u|^2 + \Phi^2\mathscr{L}_p\varphi^2 \\ &\quad -16(p-1)\Phi\varphi|\nabla\varphi||\nabla^2 u(\nu)|(1 + |\nabla u|) - 16\Phi\varphi|\nabla\varphi||\nabla^2 u|(1 + |\nabla u|). \end{aligned}$$

On the other hand, we compute

$$\begin{aligned}\mathscr{L}_p(u-x)^2 &\geq 2(u-x)\mathscr{L}_p(u-x) + 2(p-1)\langle \nu, \nabla(u-x)\rangle^2 \\ &\geq -2|u-x||\mathscr{L}_p x| + 2(p-1)|\nabla u|^{-2}\left(|\nabla u|^2 - \partial_x u\right)^2 \\ &\geq -4|p-2||u-x||\nabla u|^{-1}|\nabla^2 u(\nu)| + 2(p-1)|\nabla u|^{-2}\Phi^2.\end{aligned}$$

Let us define

$$w \doteq \varphi^2\Phi^2 + \beta R^{-2}(u-x)^2 + \ell|\nabla u|^2,$$

for some $\beta > 0$ to be determined later. Notice that β, ℓ are invariant under the natural scaling $\langle\, ,\,\rangle' = R^{-2}\langle\, ,\,\rangle$ and $u' = u/R$. Let us assume that w attains its maximum at an interior point p_0. If $\Phi(p_0) > 0$, then

$$\begin{aligned}0 \geq \mathscr{L}_p w &= \mathscr{L}_p(\varphi^2\Phi^2) + \beta R^{-2}\mathscr{L}_p(u-x)^2 + \ell\mathscr{L}_p|\nabla u|^2 \\ &\geq 4(p-1)\varphi^2\Phi|\nabla^2 u(\nu)|^2 - 4(m-1)\kappa^2\varphi^2\Phi|\nabla u|^2 + \Phi^2\mathscr{L}_p\varphi^2 \\ &\quad - 16(p-1)\Phi\varphi|\nabla\varphi||\nabla^2 u(\nu)|(1+|\nabla u|) - 16\Phi\varphi|\nabla\varphi||\nabla^2 u|(1+|\nabla u|) \\ &\quad - 4|p-2|\beta R^{-2}|u-x||\nabla u|^{-1}|\nabla^2 u(\nu)| + 2(p-1)\beta R^{-2}|\nabla u|^{-2}\Phi^2 \\ &\quad + 2(p-2)\ell|\nabla^2 u(\nu)|^2 + 2\ell|\nabla^2 u|^2 - 2(m-1)\kappa^2\ell|\nabla u|^2.\end{aligned}$$

Using that

$$\begin{aligned}&16\Phi\varphi|\nabla\varphi||\nabla^2 u(\nu)|(1+|\nabla u|) \leq 2\varphi^2\Phi|\nabla^2 u(\nu)|^2 + 32\Phi|\nabla\varphi|^2(1+|\nabla u|)^2, \\ &16\Phi\varphi|\nabla\varphi||\nabla^2 u|(1+|\nabla u|) \leq 2\ell|\nabla^2 u|^2 + 32\ell^{-1}\Phi^2\varphi^2|\nabla\varphi|^2(1+|\nabla u|)^2, \\ &4\beta R^{-2}|u-x||\nabla u|^{-1}|\nabla^2 u(\nu)| \leq 2\ell|\nabla^2 u(\nu)|^2 + 2\beta^2 R^{-4}\ell^{-1}(u-x)^2|\nabla u|^{-2},\end{aligned}$$

and the definition of Φ we get

$$\begin{aligned}0 &\geq 2(p-1)\varphi^2\Phi|\nabla^2 u(\nu)|^2 - 4(m-1)\kappa^2\varphi^2\Phi|\nabla u|^2 + \Phi^2\mathscr{L}_p\varphi^2 \\ &\quad - 32(p-1)\Phi|\nabla\varphi|^2(1+|\nabla u|)^2 - 32\ell^{-1}\Phi^2\varphi^2|\nabla\varphi|^2(1+|\nabla u|)^2 \\ &\quad - 2\beta^2 R^{-4}\ell^{-1}(u-x)^2|\nabla u|^{-2} + 2(p-1)\beta R^{-2}|\nabla u|^{-2}\Phi^2 \\ &\quad - 2(m-1)\kappa^2\ell|\nabla u|^2.\end{aligned}$$

We hereafter denote with $C_1, C_2, \ldots$ absolute constants. Define φ as in Lemma 5.2, so that $|\nabla\varphi|^2 \leq CR^{-2}$ and, since $p \geq 2$,

$$\begin{aligned}\mathscr{L}_p\varphi^2 &\geq -Cp\varphi C_R - Cp\varphi R^{-1}|\nabla u|^{-1}|\nabla^2 u(\nu)| \\ &\geq -Cp\varphi C_R - 2(p-1)\varphi^2\Phi^{-1}|\nabla^2 u(\nu)|^2 - C_1 p\Phi R^{-2}|\nabla u|^{-2}.\end{aligned}$$

Inserting into the above and multiplying by $|\nabla u|^2$ we infer

$$\begin{aligned}0 \geq &- 4(m-1)\kappa^2\varphi^2\Phi|\nabla u|^4 - Cp\varphi\Phi^2 C_R|\nabla u|^2 - C_1 p\Phi^3 R^{-2} \\ &- 32(p-1)\Phi|\nabla\varphi|^2(1+|\nabla u|)^2||\nabla u|^2 - 32\ell^{-1}\Phi^2\varphi^2|\nabla\varphi|^2(1+|\nabla u|)^2|\nabla u|^2 \\ &- 2\beta^2 R^{-4}\ell^{-1}(u-x)^2 + 2(p-1)\beta R^{-2}\Phi^2 - 2(m-1)\kappa^2\ell|\nabla u|^4.\end{aligned}$$

Using $|\varphi| \leq 1$, $|\nabla\varphi| \leq C_2/R$, $|u - x| \leq \ell R$, $|\nabla u| \leq A$ together with the inequality $\Phi^3 \leq A^2\Phi^2$, and rearranging, we obtain

$$\begin{aligned} 0 \geq & -4(m-1)\kappa^2\Phi A^4 - Cp\Phi^2 C_R A^2 - C_1 p A^2 \Phi^2 R^{-2} \\ & - 32(p-1)\Phi C_2^2 R^{-2} A^4 - 32\ell^{-1}\Phi^2 C_2^2 R^{-2} A^4 \\ & - 2\beta^2 R^{-2}\ell + 2(p-1)\beta R^{-2}\Phi^2 - 2(m-1)\kappa^2 \ell A^4 \\ = & \ (E\Phi^2 - B\Phi - F)R^{-2}, \end{aligned}$$

where we set

$$\begin{aligned} E &\doteq 2(p-1)\beta - CpR^2 C_R A^2 - C_1 p A^2 - 32\ell^{-1} C_2^2 A^4 \\ B &\doteq 4(m-1)\kappa^2 R^2 A^4 + 32(p-1)C_2^2 A^4, \\ F &\doteq 2\beta^2\ell + 2(m-1)\kappa^2 \ell A^4 R^2. \end{aligned}$$

Whence,

$$E\Phi^2 \leq B\Phi + F \leq BA^2 + F \qquad \text{at } p_0.$$

It follows that, at p_0,

$$\begin{aligned} w &\leq \Phi^2 + \beta R^{-2}(u-x)^2 + \ell|\nabla u|^2 \\ &\leq E^{-1}(BA^2 + F) + \beta\ell^2 + \ell A^2. \end{aligned}$$

Therefore, for each $x \in B_R$,

$$\Phi^2(x) \leq w(x) \leq w(p_0) \leq E^{-1}(BA^2 + F) + \ell(\beta\ell + A^2). \tag{45}$$

By the definition of C_R

$$E \geq p\beta - C_3 p A^2 \left[1 + \kappa R + \frac{A^2}{\ell p}\right] \geq \frac{p\beta}{2},$$

where in the latter inequality we have chosen

$$\beta = 2C_3 \left(\frac{\ell+1}{\ell}\right)^{\frac{1}{4}} A^4 \left[1 + \kappa R + \ell^{-1}p^{-1}\right].$$

Estimating B, F for such a choice of β we obtain:

$$B \leq C_5 mpA^4(1 + \kappa^2 R^2), \qquad F \leq C_6 m\sqrt{\ell(\ell+1)}(1 + \kappa^2 R^2 + \ell^{-2}p^{-2})A^8,$$

which gives

$$\begin{aligned} \Phi^2(x) &\leq C_7 m A^4 \ell^{\frac{1}{4}}(\ell+1)^{\frac{3}{4}}(1 + \kappa R + \ell^{-1}p^{-1}) \\ &\leq C_7 m A^4 \ell^{\frac{1}{4}}(\ell+1)^{\frac{5}{2}}(1 + \kappa R + \ell^{-1}p^{-1}) \end{aligned}$$

and the conclusion follows by taking square roots. If $p_0 \in \partial B_{2R}$ we have for each $x \in B_R$

$$\begin{aligned} \Phi^2(x) &\leq w(x) \leq w(p_0) = \left(\beta R^{-2}(u-x)^2 + \ell|\nabla u|^2\right)(p_0) \leq \beta\ell^2 + \ell A^2 \\ &\leq C_8 A^4 \ell^{\frac{7}{4}}(\ell+1)^{\frac{3}{4}}\left[1 + \kappa R + \ell^{-1}p^{-1}\right] \\ &\leq C_8 A^4 \ell^{\frac{1}{4}}(\ell+1)^{\frac{5}{2}}\left[1 + \kappa R + \ell^{-1}p^{-1}\right], \end{aligned}$$

from which the desired inequality follows as well. □

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