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Research article

Time almost-periodic solutions of the incompressible Euler equations†

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Abstract: We construct time almost-periodic solutions (global in time) with finite regularity to the incompressible Euler equations on the torus $\mathbb{T}^d$, with $d = 3$ and $d \in \mathbb{N}$ even.

Keywords: fluid dynamics; Euler equations; almost-periodic solutions

1. Introduction

The goal of this paper is to construct *time almost-periodic* solutions (infinite dimensional invariant tori) of the Euler equations

$$\begin{aligned} &\partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0, \\ &u : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^d, \quad p : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}, \end{aligned} \tag{1.1}$$

on the d-dimensional torus $\mathbb{T}^d$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, where either $d = 3$ or $d \geq 2$ is any even positive integer. These solutions extend the works ot Crouseille & Faou [16] (in dimension 2) and Enciso, Peralta Salas & Torres de Lizaur [17] (in dimensions 3 or even) from time quasi-periodic to time almost-periodic solutions. In fact, the construction here follows closely the one in [17].

We need to specify how a smooth solution of the Euler equation (1.1) is called *almost-periodic* in this paper. We need some preliminaries.

Let $C^s_{\rm div}(\mathbb{T}^d, \mathbb{R}^d)$, with $s \in \mathbb{N} \cup \{+\infty\}$, be the space of C^s-smooth, divergence free d-dimensional vector fields on $\mathbb{T}^d$. This space is a Banach space if $s < \infty$ and a Fréchet space when $s = \infty$. We endow it with the system of seminorms $(\|\cdot\|_{n,\infty})_{n\in\{0,1,...,s\}}$ defined by

$$\|f\|_{n,\infty} := \sup_{x\in\mathbb{T}^d} \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|=n}} |\partial_x^\alpha f(x)|, \quad n = 0, 1, ..., s\,, \tag{1.2}$$

throughout the paper, for sake of simplicity in the notation, $|\cdot|$ denotes the standard Euclidean norm, without specifying the dimension of the evaluated object, which will be clear from the context each time. We denote by $\ell^\infty(\mathbb{N},\mathbb{N})$ the set of sequences in $\mathbb{N}$ that are bounded. Let $(J_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N},\mathbb{N}) \setminus \{0\}$ be given and, for a fixed $m \in \{1, ..., d-1\}$, we define the sequence

$$(N_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N},\mathbb{N}), \quad N_k := (d-m)J_k \in \mathbb{N}, \quad k \in \mathbb{N}. \tag{1.3}$$

We define the infinite dimensional torus $(\mathbb{T}^{N_k})_{k\in\mathbb{N}}$ and its "tangent space" $(\mathbb{R}^{N_k})_{k\in\mathbb{N}}$ as

$$\begin{aligned} (\mathbb{T}^{N_k})_{k\in\mathbb{N}} &:= \{\theta = (\theta_k)_{k\in\mathbb{N}} \,:\, \theta_k \in \mathbb{T}^{N_k},\ |\theta|_\infty < \infty\}, \\ (\mathbb{R}^{N_k})_{k\in\mathbb{N}} &:= \{\nu = (\nu_k)_{k\in\mathbb{N}} \,:\, \nu_k \in \mathbb{R}^{N_k},\ |\nu|_\infty < \infty\}, \end{aligned} \tag{1.4}$$

where we defined $|\nu|_\infty := \sup_{k\in\mathbb{N}} |\nu_k|$. Note that, since $(N_k)_{k\in\mathbb{N}}$ is bounded, then $|\theta|_\infty \leq (2\pi)^{\|(N_k)\|_{\ell^\infty}} < \infty$ for any sequence $\theta = (\theta_k)_{k\in\mathbb{N}}$.

Definition 1.1. Let $s \in \mathbb{N} \cup \{+\infty\}$. We say that $u(t, x)$ is *time almost-periodic* if there exists a sequence of vectors $\nu \in (\mathbb{R}^{N_k})_{k\in\mathbb{N}}$ and a C^1-smooth embedding $U : (\mathbb{T}^{N_k})_{k\in\mathbb{N}} \to C^s_{\rm div}(\mathbb{T}^d, \mathbb{R}^d)$ such that the velocity field $u(t, x)$ can be written as

$$u(t,\,\cdot\,) = U(\vartheta)|_{\vartheta=\theta+\nu t}, \qquad \text{for some} \quad \theta \in (\mathbb{T}^{N_k})_{k\in\mathbb{N}}, \tag{1.5}$$

and the sequence of frequency vectors $\nu = (\nu_k)_{k\in\mathbb{N}}$ is *non-resonant*, meaning that

$$\sum_{k\in\mathbb{N}} \nu_k \cdot \ell_k \neq 0, \quad \forall\, \ell = (\ell_k)_{k\in\mathbb{N}} \in (\mathbb{Z}^{N_k})_{k\in\mathbb{N}} \quad \text{with} \quad 0 < |\ell|_\eta < \infty, \tag{1.6}$$

where, for a fixed $\eta > 0$, we define $|\ell|_\eta := \sum_{k\in\mathbb{N}} k^\eta |\ell_k|$. Note that $|\ell|_\eta < \infty$ implies that $\ell_k \neq 0 \in \mathbb{Z}^{N_k}$ only for finitely many $k \in \mathbb{N}$.

Definition 1.2. By saying that the map $U : (\mathbb{T}^{N_k})_{k\in\mathbb{N}} \to C^s_{\rm div}(\mathbb{T}^d, \mathbb{R}^d)$ is C^1_b (C^1 and bounded), we mean that, U is a Frechet-differentiable map with continuous Frechet derivative and for any $n \in \{0, 1, \ldots, s\}$, there exists a constant $C_n > 0$ such that

$$\begin{aligned} &\sup_{\vartheta\in(\mathbb{T}^{N_k})_{k\in\mathbb{N}}} \|U(\vartheta)\|_{n,\infty} \leq C_n, \\ &\sup_{\vartheta\in(\mathbb{T}^{N_k})_{k\in\mathbb{N}}} \|{\rm d}_\vartheta U(\vartheta)[\widehat{\vartheta}]\|_{n,\infty} \leq C_n |\widehat{\vartheta}|_\infty \quad \forall\, \widehat{\vartheta} \in (\mathbb{R}^{N_k})_{k\in\mathbb{N}}, \end{aligned} \tag{1.7}$$

where the linear operator ${\rm d}_\vartheta U(\vartheta) : (\mathbb{R}^{N_k})_{k\in\mathbb{N}} \to C^s_{\rm div}(\mathbb{T}^d, \mathbb{R}^d)$ is the Fréchet differential of the embedding $U(\vartheta)$ at the point $\vartheta \in (\mathbb{T}^{N_k})_{k\in\mathbb{N}}$.

With this definition of a C^1_b embedding, we have that the function $u(t,\,\cdot\,)$ is C^1 with respect to $t \in \mathbb{R}$, by (1.5) and

$$\partial_t u(t,\,\cdot\,) = {\rm d}_\vartheta U(\theta + \nu t)[\nu] \in C^s_{\rm div}(\mathbb{T}^d, \mathbb{R}^d).$$

We will look for solutions where the embedding U is *non-symmetric*, or *non-traveling*, in the sense that, for any $\vartheta \in (\mathbb{T}^{N_k})_{k\in\mathbb{N}}$, the divergence-free vector field $U(\vartheta)$ is not invariant under any 1-parameter group of translations on $\mathbb{T}^d$. In this way, we ensure that the solution $u(t,x)$ depends effectively on all d coordinates and we do not have any reduction to solutions of lower dimensions by traveling directions.

The statement of the main result is the following.

Theorem 1.3 (Time almost-periodic solutions of the Euler equations)**.** *Assume that the dimension d is either 3 or even. Let $S \in \mathbb{N}$ be fixed. There exists $\varepsilon_0 \in (0,1)$ small enough such that, for any $\varepsilon \in (0,\varepsilon_0)$ and for any sequence of frequencies $\nu \in (\mathbb{R}^{N_k})_{k\in\mathbb{N}} \setminus \{0\}$ satisfying*

$$\sup_{k\in\mathbb{N}} \varepsilon^{-(S+1)(k-1)} |\nu_k| < \infty, \tag{1.8}$$

there exists a non-symmetric C^1_b embedding $U : (\mathbb{T}^{N_k})_{k\in\mathbb{N}} \to C^S_{\rm div}(\mathbb{T}^d,\mathbb{R}^d)$ and a family of initial data $u_\theta \in C^S_{\rm div}(\mathbb{T}^d,\mathbb{R}^d)$, $\theta \in (\mathbb{T}^{N_k})_{k\in\mathbb{N}}$, such that $u(t,\,\cdot\,) = U(\theta+\nu t)$, with $u(0,\,\cdot\,) = u_\theta$, is a solution of (1.1) *with pressure $p(t,\,\cdot\,) = P(\theta+\nu t)$, where*

$$P(\vartheta) := (-\Delta)^{-1}\big[\mathrm{div}(U(\vartheta)\cdot\nabla U(\vartheta))\big] : (\mathbb{T}^{N_k})_{k\in\mathbb{N}} \to C^S(\mathbb{T}^d). \tag{1.9}$$

As a consequence, if the sequence $\nu = (\nu_k)_{k\in\mathbb{N}}$ is non-resonant, namely it satisfies (1.6)*, then the solution $u(t,x)$ is time almost-periodic.*

Remark 1.4. As it will be clear from the construction in the following section, the embedding $U(\vartheta)$ is determined as a combination of infinitely many embedding $U_k(\vartheta_k)$, with $\vartheta_k \in \mathbb{T}^{N_k}$, which coincides with the embedding constructed in [17], with the size of the embedding U_k becoming smaller and smaller as $k \to \infty$. The major difference in the analysis with respect to [17] is that we have to effectively prove the smoothness of the embedding and the regularity of the vector field. This is not trivial.

Remark 1.5. The condition (1.6) of irrationality for the sequence of frequencies $\nu \in (\mathbb{R}^{N_k})_{k\in\mathbb{N}} \setminus \{0\}$ is not necessary in the construction of the embedding U. Depending on relations between all the frequencies, we may obtain embedding for lower dimensional tori, either finite dimensional (quasi-periodic or periodic) or still infinite dimensional (that is, almost-periodic). On the other hand, the control on the frequency vectors in (1.8) is required to ensure that the solution $u(t,x)$ is indeed a finitely smooth vector field and a simpler control on the norm $|\nu|_\infty$ is not enough. At the physical level, is also implies that we obtain solutions whose leading order frequencies of oscillations are only finitely many and the almost-periodicity in time is due to the presence of infinitely oscillations with smaller and smaller frequencies.

Related results. In the last years, there has been a discrete surge of works proving the existence of time quasi-periodic waves for PDEs arising in fluid dynamics. With the exception of the aforementioned works [16, 17], there type of results in literature are proved by means of KAM for PDEs techniques, to deal with the presence of small divisors issues and consequent losses of regularity. For the two dimensional water waves equations, we mention Berti and Montalto [7], Baldi et al. [2] for time quasi-periodic standing waves and Berti, Maspero and Franzoi [4, 5], Feola and Giuliani [18] for time quasi-periodic traveling wave solutions. Recently, the existence of time quasi-periodic solutions was proved for the contour dynamics of vortex patches in active scalar equations. We mention Berti, Hassainia and Masmoudi [6] for vortex patches of the Euler equations close to Kirchhoff ellipses, Hmidi and Roulley [26] for the quasi-geostrophic shallow water equations, Hassainia, Hmidi and Masmoudi [23] and Gómes-Serrano, Ionescu and Park [22] for generalized surface quasi-geostrophic equations, Roulley [31] for Euler-α flows, Hassainia and Roulley [25] for Euler equations in the unit disk close to Rankine vortices, and Hassainia, Hmidi and Roulley [24] for 2D Euler annular vortex patches. Time quasi-periodic solutions were also constructed for the 3D Euler equations with time quasi-periodic external force [3] and for the forced 2D Navier-Stokes equations [19] approaching in the zero viscosity limit time quasi-periodic solutions of the 2D Euler equations for all times.

The existence of other non-trivial invariant structures is also a topic of interest in fluid dynamics. In particular, for the Euler equations in two dimension close to shear flows, we mention the works by Lin and Zeng [27], and Castro and Lear [10] for periodic traveling waves close the Couette flow, by Coti Zelati, Elgindi and Widmayer [15] for stationary waves around non-monotone shears, by Franzoi, Masmoudi and Montalto [20] for quasi-periodic traveling waves close to the Couette flow, and the recent work by Castro and Lear [11] for time periodic rotating solutions close to the Taylor-Couette flow.

Concerning the existence of almost periodic solutions by means of KAM methods, we mention Pöschel [30], Bourgain [9], Biasco, Massetti and Procesi [8,28] and Corsi, Gentile and Procesi [13]. In all these results the authors consider semilinear NLS type equations with external parameters. For PDEs with unbounded perturbations (with external parameters as well) we mention Montalto and Procesi [29] and Corsi, Montalto and Procesi [14].

We remark that our result is the first one concerning existence of almost-periodic solutions for an autonomous quasi-linear PDEs in higher space dimension and it is obtained with non-KAM techniques.

Notations. In this paper, we use the following notations:

- $B_{d,\rho}(\mathtt{p}) := \{x \in \mathbb{R}^d \,:\, |x - \mathtt{p}| < \rho\}$, with $\mathtt{p} \in \mathbb{R}^d$ and $\rho > 0$;
- $C^s(\mathbb{T}^{m_1}, \mathbb{R}^{m_2}) := \{f : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2} \,:\, \|f\|_{n,\infty} < \infty \;\; \forall\, n \in \mathbb{N} \cup \{0\},\, n \le s\}$, $s \in \mathbb{N} \cup \{\infty\}$;
- $C^\infty(X, \mathbb{R}) := C^\infty(X)$, with $X = \mathbb{T}^d, \mathbb{R}^d$;
- $a \lesssim b$ stands for $a \le Cb$, for some constant $C > 0$;
- $a \lesssim_n b$ stands for $a \le C_n b$, for some constant $C_n > 0$ depending on n.

2. Proof of Theorem 1.3

The scheme follows essentially the one proposed in [17], with the required adaptations. The key starting point is the existence of smooth, compactly supported stationary solutions of the Euler equations. In $d = 3$, this is celebrated result by Gravilov [21] (see also [12]), whereas in even dimension it has been proved in [17]. We recall the statement of the result of the latter.

Proposition 2.1 (Smooth stationary Euler flows with compact support– Proposition 2, [17])**.** *If $d = 3$ or $d \in \mathbb{N}$ is even, there exists a smooth, compactly supported solution $v(x) \in C^\infty_{\rm div}(\mathbb{R}^d, \mathbb{R}^d)$, with pressure $p_v(x) \in C^\infty(\mathbb{R}^d)$, of*

$$v \cdot \nabla v + \nabla p_v = 0, \quad \operatorname{div} v = 0, \quad x \in \mathbb{R}^d. \tag{2.1}$$

Remark 2.2. In [1], Baldi studies the fluid particle dynamics with vector field given exactly by the compactly supported solutions found in [21] in dimension $d = 3$, and proved the existence of periodic and quasi-periodic motions.

Without any loss of generality, we assume that $\mathrm{spt}(v), \mathrm{spt}(p_v) \subseteq B_{d,1}(0) \subset \mathbb{R}^d$. Then, given $S \in \mathbb{N}$ and for any $k \in \mathbb{N}$, we define the rescaled functions, for any $\varepsilon \in (0,1)$ small enough,

$$v_k(x) := \varepsilon^{(S+1)(k-1)} v(\varepsilon^{-k}x), \quad p_{v_k}(x) := \varepsilon^{2(S+1)(k-1)} p_v(\varepsilon^{-k}x). \tag{2.2}$$

A straightforward computation shows that $v_k(x) \in C^\infty_{\mathrm{div}}(\mathbb{R}^d, \mathbb{R}^d)$ is also a solution of (2.1) with pressure $p_{v_k}(x) \in C^\infty(\mathbb{R}^d)$ with compact support

$$\mathrm{spt}(v_k), \mathrm{spt}(p_{v_k}) \subseteq B_{d,\varepsilon^k}(0) \subset \mathbb{R}^d, \tag{2.3}$$

and we have the control on the seminorms, for any $n \in \mathbb{N}_0$,

$$\|v_k\|_{n,\infty} \leq C_n \varepsilon^{k(S+1-n)-S-1}, \quad \|p_{v_k}\|_{n,\infty} \leq C_n \varepsilon^{k(2S+2-n)-2S-2}, \tag{2.4}$$

for some constant $C_n > 0$ independent of $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$. We remark that, as soon as $n > S+1$, the seminorms $\|v_k\|_{n,\infty}$ start to diverge with respect to $k \to \infty$ as $\varepsilon^{-(n-S-1)k}$ for $\varepsilon \in (0,1)$, whereas the seminorms $\|p_{v_k}\|_{n,\infty}$ start to diverge when $n > 2S+2$.

Moreover, for $\varepsilon \in (0,1)$ small enough and independent of $k \in \mathbb{N}$, we define the periodicized versions

$$\overline{v_k}(x) := \sum_{q \in \mathbb{Z}^d} v_k(x + 2\pi q), \quad \overline{p_{v_k}}(x) := \sum_{q \in \mathbb{Z}^d} p_{v_k}(x + 2\pi q), \quad x \in \mathbb{T}^d. \tag{2.5}$$

We recall the sequence $(N_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathbb{N})$ in (1.3) is determined by a fixed $m \in \{1, ..., d-1\}$ and a fixed sequence $(J_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathbb{N})$. The choice of $m \in \{1, ..., d-1\}$ induces the splitting $\mathbb{T}^d = \mathbb{T}^m \times \mathbb{T}^{d-m}$ and we write

$$\mathbb{T}^d \ni x = (x', x'') \in \mathbb{T}^m \times \mathbb{T}^{d-m}, \quad \nabla = (\nabla', \nabla'') := (\nabla_{x'}, \nabla_{x''}). \tag{2.6}$$

We select a sequence of points $(\mathrm{y}_{k,j})_{k\in\mathbb{N},\, j=1,..,J_k} \subset \mathbb{T}^m$ with the properties that:

(A) For any $k_1, k_2 \in \mathbb{N}$, $j_1 = 1, ..., J_{k_1}$, $j_2 = 1, ..., J_{k_2}$, with $(k_1, j_1) \neq (k_2, j_2)$, we have

$$\overline{B_{m,2\varepsilon^{k_1}}(\mathrm{y}_{k_1,j_1})} \cap \overline{B_{m,2\varepsilon^{k_2}}(\mathrm{y}_{k_2,j_2})} = \emptyset; \tag{2.7}$$

(B) We have

$$\Big|\mathbb{T}^m \setminus \big(\bigcup_{k\in\mathbb{N}} \bigcup_{j=1}^{J_k} \overline{B_{m,2\varepsilon^k}(\mathrm{y}_{k,j})}\big)\Big| \geq \frac{2}{3}|\mathbb{T}^m| > 0.$$

The existence of such sequence of points with these desired properties is proved in the following lemma:

Lemma 2.3. *There exist $\varepsilon_0 = \varepsilon_0(m, \|(J_k)\|_{\ell^\infty}) \in (0,1)$ small enough and a choice of infinitely many distinct points $(\mathrm{y}_{k,j})_{k\in\mathbb{N},\, j=1,..,J_k} \subset \mathbb{T}^m$, such that the following holds. For $\varepsilon > 0$, we define iteratively the sets*

$$E_0 := \emptyset, \quad E_k := E_{k-1} \cup \bigcup_{j=1}^{J_k} \overline{B_{m,2\varepsilon^k}(\mathrm{y}_{k,j})}, \quad k \in \mathbb{N}. \tag{2.8}$$

Then, for any $\varepsilon \in (0, \varepsilon_0)$ and for any $k \in \mathbb{N}$, we have:

(i) $E_{k-1} \cap \bigcup_{j=1}^{J_k} \overline{B_{m,2\varepsilon^k}(\mathrm{y}_{k,j})} = \emptyset$*;*

(ii) $\overline{B_{m,2\varepsilon^k}(\mathsf{y}_{k_1,j_1})} \cap \overline{B_{m,2\varepsilon^k}(\mathsf{y}_{k_2,j_2})} = \emptyset$ *for any* $j_1, j_2 = 1, ..., J_k$*;*
(iii) $\mathbb{T}^m \setminus E_k$ *is open and* $|\mathbb{T}^m \setminus E_k| \geq \left(1 - \sum_{n=1}^k 4^{-n}\right)|\mathbb{T}^m|$.

As a consequence, conditions **(A)** *and* **(B)** *are satisfied.*

Proof. In the following, we use that $|B_{m,2r}(\mathsf{y})| = C_m r^m |\mathbb{T}^m|$, where the explicit constant

$$C_m := \left(\pi^{m/2}\Gamma(\tfrac{m}{2}+1)\right)^{-1} \in (0,1)$$

depends only on the dimension $m \in \mathbb{N}$, where Γ is the Euler Gamma function.

We argue by induction. Let $k = 1$. We pick an arbitrary choice of distinct points $\mathsf{y}_{1,1}, ..., \mathsf{y}_{1,J_1} \in \mathbb{T}^m$ and, for $\varepsilon > 0$, we define the set $E_1 := \bigcup_{j=1}^{J_1} \overline{B_{m,2\varepsilon}(\mathsf{y}_{1,j})}$. By (2.8), item (i) is automatically satisfied for $k = 1$. We define

$$\varepsilon_{1,1} := \tfrac{1}{4}\min\{|\mathsf{y}_{1,j_1} - \mathsf{y}_{1,j_2}| \,:\, 1 \leq j_1 < j_2 \leq J_1\}.$$

Then, for any $\varepsilon \in (0, \varepsilon_{1,1})$, the closed balls $\left(\overline{B_{m,2\varepsilon}(\mathsf{y}_{1,j})}\right)_{j=1,...,J_1}$ are pairwise disjoint, that is, item (ii) is satisfied when $k = 1$. Clearly, we also have that $\mathbb{T}^m \setminus E_1$ is open, since E_1 is a finite union of closed sets. Moreover, using that $C_m \in (0,1)$, we compute

$$|\mathbb{T}^m \setminus E_1| = (1 - C_m J_1 \varepsilon^m)|\mathbb{T}^m| \geq (1 - J_1\varepsilon^m)|\mathbb{T}^m| \geq \tfrac{3}{4}|\mathbb{T}^m|, \tag{2.9}$$

as soon as $\varepsilon < \varepsilon_{1,2} := (4J_1)^{-1/m}$. Therefore, choosing $\varepsilon_0 \leq \min\{\varepsilon_{1,1}, \varepsilon_{1,2}\}$, we conclude that (i)–(iii) are satisfied for $k = 1$.

We now assume that the claims (i)–(iii) are satisfied for some $k \in \mathbb{N}$ and we prove them for $k + 1$. We set

$$\varepsilon_0 := \min\{\varepsilon_{1,1}, (4\|(J_k)\|_{\ell^\infty})^{-1/m}\} \leq \min\{\varepsilon_{1,1}, \varepsilon_{1,2}\}. \tag{2.10}$$

By (2.10), there exist J_{k+1} distinct points

$$\mathsf{y}_{k+1,1}, ..., \mathsf{y}_{k+1,J_{k+1}} \in \mathbb{T}^m \setminus E_k, \text{ with } J_{k+1} \leq \|(J_k)\|_{\ell^\infty},$$

such that, for any $\varepsilon \in (0, \varepsilon_0)$ we have that the J_{k+1} balls $\overline{B_{m,2\varepsilon^{k+1}}(\mathsf{y}_{k+1,1})}, ..., \overline{B_{m,2\varepsilon^{k+1}}(\mathsf{y}_{k+1,J_{k+1}})}$ are contained in $\mathbb{T}^m \setminus E_k$ and they are disjoint, namely they satisfy items (i) and (ii) at the step $k + 1$. This follows from the fact that, by the induction assumption on (iii), we have that $\mathbb{T}^m \setminus E_k$ is open with measure $|\mathbb{T}^m \setminus E_k| > \frac{2}{3}|\mathbb{T}^m|$, whereas the measure of the finite union of closed disjoint balls is estimated, for any $\varepsilon \in (0, \varepsilon_0)$ with ε_0 as in (2.10), by

$$\begin{aligned}\left|\bigcup_{j=1}^{J_{k+1}} \overline{B_{m,2\varepsilon^{k+1}}(\mathsf{y}_{k+1,j})}\right| &= C_m J_{k+1} \varepsilon^{(k+1)m}|\mathbb{T}^m| \\ &\leq \frac{1}{4^{k+1}} \frac{J_{k+1}}{\|(J_k)\|_{\ell^\infty}^{k+1}}|\mathbb{T}^m| \leq \frac{1}{4^{k+1}}|\mathbb{T}^m| < \frac{2}{3}|\mathbb{T}^m|,\end{aligned} \tag{2.11}$$

which implies the existence of the J_{k+1} points $\mathsf{y}_{k+1,1}, ..., \mathsf{y}_{k+1,J_{k+1}}$ in the open and bounded set $\mathbb{T}^m \setminus E_k$ with the desired properties. Therefore, let E_{k+1} be defined as in (2.8). Clearly, E_{k+1} is closed, which also implies that $\mathbb{T}^m \setminus E_{k+1}$ is open. By (2.11) and item (ii) at the step $k + 1$, we also deduce that,

$$|\mathbb{T}^m \setminus E_{k+1}| = |\mathbb{T}^m \setminus E_k| - \left|\bigcup_{j=1}^{J_{k+1}} \overline{B_{m,2\varepsilon^{k+1}}(\mathsf{y}_{k+1,j})}\right| \geq \left(1 - \sum_{n=1}^{k} 4^{-n} - 4^{-(k+1)}\right)|\mathbb{T}^m|, \tag{2.12}$$

which is indeed the estimate in item (*iii*) at the step $k+1$. This closes the induction argument and concludes the proof. □

As a last preliminary, we take a sequence of frequency vectors $\nu = (\nu_k)_{k\in\mathbb{N}} \in (\mathbb{R}^{N_k})_{k\in\mathbb{N}}$, where $\nu_k = (\nu_{k,1}, ..., \nu_{k,J_k}) \in \mathbb{R}^{N_k}$, with $\nu_{k,j} \in \mathbb{R}^{d-m}$, recalling (1.3). We now define the vector field

$$u(t,x) := \sum_{k=1}^{\infty} u_k(t,x), \quad u_k(t,x) := \sum_{j=1}^{J_k} \overline{v_{k,j}}(t,x) + w_k(x) \tag{2.13}$$

with pressure

$$p_u(t,x) := \sum_{k=1}^{\infty} p_{u_k}(t,x), \quad p_{u_k}(t,x) := \sum_{j=1}^{J_k} \overline{p_{k,j}}(t,x), \tag{2.14}$$

where

$$\begin{aligned} \overline{v_{k,j}}(t,x) &:= \overline{v_k}(x' - \mathrm{y}_{k,j}, x'' - \nu_{k,j}t), \\ \overline{p_{k,j}}(t,x) &:= \overline{p_{v_k}}(x' - \mathrm{y}_{k,j}, x'' - \nu_{k,j}t), \end{aligned} \tag{2.15}$$

and

$$w_k(x) = (0, F_k(x')), \quad F_k : \mathbb{T}^m \to \mathbb{R}^{d-m}. \tag{2.16}$$

Note that, no matter the choice of $F_k(x')$ sufficiently smooth is, the vector field $w_k(x)$ is a stationary solution with constant pressure of the Euler equation (1.1), namely we have

$$w_k \cdot \nabla w_k = 0, \quad \operatorname{div} w_k = 0. \tag{2.17}$$

To make sure that $u(t,x)$ is indeed a solution of (1.1), we need to specify the functions $F_k(x')$. In particular, we choose

$$F_k(x') := \sum_{j=1}^{J_k} \nu_{k,j}\chi_k(|x' - \mathrm{y}_{k,j}|), \tag{2.18}$$

where $\chi_k(r) \in C^\infty(\mathbb{R})$ is an even cut-off function satisfying

$$\begin{aligned} &\chi_k(r) = 1 \quad \text{when } |r| < \varepsilon^k, \quad \chi_k(r) = 0 \quad \text{when } |r| > 2\varepsilon^k, \quad \chi_k \in [0,1], \\ &|\partial_r^n \chi_k(r)| \leq C_n \varepsilon^{-kn}, \quad \forall\, r \in \mathbb{R}, \quad n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{2.19}$$

for some constant $C_n > 0$. For each $k \in \mathbb{N}$, we have that $F_k \in C^\infty(\mathbb{T}^m, \mathbb{R}^{d-m})$ and that the vector field $w_k(x)$ is locally equal to $(0, \nu_{k,j})$ when $x \in \operatorname{spt}(\overline{v_{k,j}})$. Note that each pair $(u_k(t,x), p_{u_k}(t,x))$ defined above by (2.13)–(2.18) has actually the form of a quasi-periodic solution of the Euler equation (1.1) as provided in [17], which has been reproduced here on supports of scale ε^k. Moreover, by construction and by **(A)**, the support in space of $(u_k(t,x), p_{u_k}(t,x))$ is in $\big(\bigcup_{j=1}^{J_k} B_{m,2\varepsilon^k}(\mathrm{y}_{k,j})\big) \times \mathbb{T}^{d-m}$ and it is disjoint from the one of $(u_{k'}(t,x), p_{u_{k'}}(t,x))$ for any $k' \neq k$. We use these properties to check that the pair $(u(t,x), p_u(t,x))$ in (2.13) and (2.14) is indeed a solution of (1.1) as well.

First, we prove that each pair $(u_k(t,x), p_{u_k}(t,x))$ is a solution of (1.1) and we provide estimates on the seminorms.

Lemma 2.4. *Assume that $v = (v_k)_{k\in\mathbb{N}}$ satisfies* (1.8)*. For each $k \in \mathbb{N}$, the vector field $u_k(t,x)$ is in $C^\infty_{\mathrm{div}}(\mathbb{T}^d,\mathbb{R}^d)$, with pressure $p_{u_k}(t,x)$ in $C^\infty(\mathbb{T}^d)$, is a solution of the Euler equation* (1.1)*, namely*

$$\partial_t u_k + u_k\cdot\nabla u_k + \nabla p_{u_k} = 0, \quad \operatorname{div} u_k = 0, \tag{2.20}$$

compactly supported in space in $\bigcup_{j=1}^{J_k} B_{m,2\varepsilon^k}(\mathrm{y}_{k,j})\times\mathbb{T}^{d-m}$. Moreover, we have the estimates, for any integer $n\geq 0$,

$$\begin{aligned}\sup_{t\in\mathbb{R}}\|u_k(t,\cdot)\|_{n,\infty} &\leq C_n\varepsilon^{k(S+1-n)-S-1},\\ \sup_{t\in\mathbb{R}}\|\partial_t u_k(t,\cdot)\|_{n,\infty} &\leq C_n\varepsilon^{k(2S+2-(n+1))-2S-2},\\ \sup_{t\in\mathbb{R}}\|p_{u_k}(t,\cdot)\|_{n,\infty} &\leq C_n\varepsilon^{k(2S+2-n)-2S-2},\end{aligned} \tag{2.21}$$

for some constant $C_n>0$ independent of $\varepsilon\in(0,1)$ and of $k\in\mathbb{N}$.

Proof. By (2.13)–(2.18), Proposition 2.1 and by **(A)**, we compute

$$\partial_t u_k = -\sum_{j=1}^{J_k} v_{k,j}\cdot\nabla''\overline{v_k}(x'-\mathrm{y}_{k,j}, x''-v_{k,j}t), \tag{2.22}$$

and, using (2.22),

$$\begin{aligned} u_k\cdot\nabla u_k &= \Big(\sum_{j=1}^{J_k}\overline{v_{k,j}} + w_k\Big)\cdot\Big(\sum_{j=1}^{J_k}\nabla\overline{v_{k,j}} + \nabla w_k\Big)\\ &= \sum_{j=1}^{J_k}\overline{v_{k,j}}\cdot\nabla\overline{v_{k,j}} + \sum_{j=1}^{J_k}(\overline{v_{k,j}}\cdot\nabla w_k + w_k\cdot\nabla\overline{v_{k,j}}) + w_k\cdot\nabla w_k\\ &= -\sum_{j=1}^{J_k}\nabla\overline{p_{k,j}} + \sum_{j=1}^{J_k}(0 + v_{k,j}\cdot\nabla''\overline{v_k}(x'-\mathrm{y}_{k,j}, x''-\widetilde{v}_{k,j}t)) + 0\\ &= -\nabla p_{u_k} - \partial_t u_k.\end{aligned} \tag{2.23}$$

This, together with the fact that $\operatorname{div}\overline{v_{k,j}} = 0$ for any $j = 1,...,J_k$ by (2.15) and Proposition 2.1, concludes the proof of (2.20).

To prove the estimates (2.21), we first note that, by (2.19), we have $\|\chi_k\|_{n,\infty}\leq C_n\varepsilon^{-nk}$ for any integer $n\geq 0$. Moreover, by (1.8), we have that $|v_k|\leq C\varepsilon^{(S+1)(k-1)}$, for some constant $C>0$ independent of $k\in\mathbb{N}$. Therefore, we deduce, that $w_k(x)$ in (2.16)–(2.18) satisfies $\|w_k\|_{n,\infty}\leq C_n\varepsilon^{(S+1-k)n-S-1}$ for any integer $n\geq 0$, recalling that $(J_k)_{k\in\mathbb{N}}\in\ell^\infty(\mathbb{N},\mathbb{N})$. Furthermore, by (2.22) and using the fact that each $\overline{v_{k,j}}(t,x)$ is supported in space on the cylinder $B_{m,2\varepsilon^k}(\mathrm{y}_{k,j})\times\mathbb{T}^{d-m}$, disjoint for any $j'\neq j$ from the cylinder $B_{m,2\varepsilon^k}(\mathrm{y}_{k,j'})\times\mathbb{T}^{d-m}$ supporting $\overline{v_{k,j'}}(t,x)$, we estimate, for any integer $n\geq 0$ and uniformly in $t\in\mathbb{R}$,

$$\begin{aligned}\|\partial_t u_k(t,\cdot)\|_{n+1,\infty} &\sup_{j=1,\dots,J_k}|v_{k,j}|\|\nabla''\overline{v_k}(x'-\mathrm{y}_{k,j}, x''-\widetilde{v}_{k,j}t)\|_{n,\infty}\\ &\lesssim \varepsilon^{(S+1)(k-1)}\|v_k\|_{n+1,\infty}\\ &\lesssim_n \varepsilon^{(S+1)(k-1)}\varepsilon^{k(S+1-(n+1))-S-1}.\end{aligned} \tag{2.24}$$

Collecting together (2.13), (2.14), (2.16) and estimates (2.4), (2.24), we obtain the estimates (2.21) and the proof is concluded. □

We now show that $u(t,x)$ in (2.13) solves the Euler system (1.1) and that it has the desired regularity and estimates.

Proposition 2.5. *The vector field $u(t,x)$ in* (2.13)*, with pressure $p_u(t,x)$ as in* (2.14)*, is a solution of the Euler equation* (1.1)*. Moreover, assuming that $\nu = (\nu_k)_{k\in\mathbb{N}}$ satisfies* (1.8)*, we have $u(t,\cdot) \in C^{S+1}_{\rm div}(\mathbb{T}^d,\mathbb{R}^d)$, $\partial_t u(t,\cdot) \in C^{2S+1}_{\rm div}(\mathbb{T}^d,\mathbb{R}^d)$ and $p_u(t,\cdot) \in C^{2S+2}(\mathbb{T}^d)$, with estimates,*

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{n,\infty} &\leq C_n \varepsilon^{-S-1}, \quad \forall\, n = 0,1,...,S+1, \\ \sup_{t\in\mathbb{R}} \|\partial_t u(t,\cdot)\|_{n,\infty} &\leq C_n \varepsilon^{-2S-2}, \quad \forall\, n = 0,1,...,2S+1, \\ \sup_{t\in\mathbb{R}} \|p_u(t,\cdot)\|_{n,\infty} &\leq C_n \varepsilon^{-2S-2}, \quad \forall\, n = 0,1,...,2S+2. \end{aligned} \tag{2.25}$$

Proof. By Lemma 2.4 and by **(A)**, each vector field of the sequence $(u_k(t,x))_{k\in\mathbb{N}}$ is compactly supported in space and all these supports are pairwise disjoint. We use this properties and the fact that each $u_k(t,x)$ in solves (2.20) to compute, with $u(t,x)$ and $p_u(t,x)$ as in (2.13), (2.14),

$$\begin{aligned} u\cdot\nabla u &= \sum_{k=1}^\infty u_k \cdot \sum_{k=1}^\infty \nabla u_k = \sum_{k=1}^\infty u_k\cdot\nabla u_k = \sum_{k=1}^\infty \big(-\nabla p_{u_k} - \partial_t u_k\big) = -\nabla p_u - \partial_t u, \\ \operatorname{div} u &= \sum_{k=1}^\infty \operatorname{div} u_k = 0, \end{aligned} \tag{2.26}$$

which indeed proves (1.1). It remains to prove the finite regularity of the solution. By (2.13), (2.21), since the support in space are pairwise disjoint, we have that, for any $n = 0,1,...,S+1$,

$$\sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{n,\infty} = \sup_{t\in\mathbb{R}}\sup_{k\in\mathbb{N}} \|u_k(t,\cdot)\|_{n,\infty} \leq C_n \varepsilon^{-S-1} \sup_{k\in\mathbb{N}} \varepsilon^{k(S+1-n)} \leq C_n\varepsilon^{-S-1}. \tag{2.27}$$

The estimates for $\partial_t u(t,\cdot)$, $p_u(t,\cdot)$ can be proved similarly and we omit them. Hence (2.25) follows. This concludes the proof. □

In order to conclude the proof of Theorem 1.3, it remains to show the existence of the embedding $U : (\mathbb{T}^{N_k})_{k\in\mathbb{N}} \to C^S_{\rm div}(\mathbb{T}^d,\mathbb{R}^d)$. We define the claimed family of (almost-periodic) solutions as $\theta \to U(\theta+\nu t)$, where the embedding $U : (\mathbb{T}^{N_k})_{k\in\mathbb{N}} \to C^S_{\rm div}(\mathbb{T}^d,\mathbb{R}^d)$ is given by

$$u(t,x) := U(\theta+\nu t)(x) = \sum_{k\in\mathbb{N}} u_k(t,x) = \sum_{k\in\mathbb{N}} U_k(\theta_k+\nu_k t)(x), \tag{2.28}$$

with each

$$U_k : \mathbb{T}^{N_k} \to C^\infty_{\rm div}(\mathbb{T}^d,\mathbb{R}^d) \subset C^S_{\rm div}(\mathbb{T}^d,\mathbb{R}^d)$$

for $k\in\mathbb{N}$, given by

$$\begin{aligned} U_k(\theta_k+\nu_k t)(x) &:= \sum_{j=1}^{J_k} \overline{v_k}(x' - {\rm y}_{k,j}, x'' - \theta_{k,j} - \nu_{k,j} t) + \sum_{j=1}^{J_k} (0, \nu_{k,j}\chi_k(|x' - {\rm y}_{k,j}|)), \\ \theta_k &= (\theta_{k,j})_{j=1,\dots,J_k} \in \mathbb{T}^{N_n}, \quad \theta_{k,j}\in\mathbb{T}^{d-m}, \quad \forall\, k\in\mathbb{N}, \ \ j = 1,\dots,J_k, \end{aligned} \tag{2.29}$$

(recall that $N_k = (d-m)J_k$, see (1.3)) and with initial data

$$u_\theta(x) := U(\theta)(x) = \sum_{k\in\mathbb{N}} u_{\theta_k}(x) = \sum_{k\in\mathbb{N}} U_k(\theta_k)(x), \tag{2.30}$$

$$U_k(\theta_k)(x) := \sum_{j=1}^{J_k} \overline{v_k}(x' - \mathrm{y}_{k,j}, x'' - \theta_{k,j}) + \sum_{j=1}^{J_k} (0, \nu_{k,j}\chi_k(|x' - \mathrm{y}_{k,j}|)). \tag{2.31}$$

As last step, we prove the estimate on the continuity and the differentiability of the embedding U.

Proposition 2.6. *Assume that* $\nu = (\nu_k)_{k\in\mathbb{N}}$ *satisfies* (1.8). *Then the embedding* $U : (\mathbb{T}^{N_k})_{k\in\mathbb{N}} \to C^S_{\mathrm{div}}(\mathbb{T}^d, \mathbb{R}^d)$ *is* C^1_b *according to Definition 1.2, with estimates*

$$\sup_{\vartheta\in(\mathbb{T}^{N_k})_{k\in\mathbb{N}}} \|U(\vartheta)\|_{n,\infty} \le C_n \varepsilon^{-S-1}, \quad 0 \le n \le S, \tag{2.32}$$

$$\sup_{\vartheta\in(\mathbb{T}^{N_k})_{k\in\mathbb{N}}} \|\mathrm{d}_\vartheta U(\vartheta)[\widehat{\vartheta}]\|_{n,\infty} \le C_n \varepsilon^{-S-1} |\widehat{\vartheta}|_\infty \quad \forall\, \widehat{\vartheta} \in (\mathbb{R}^{N_k})_{k\in\mathbb{N}}, \quad 0 \le n \le S. \tag{2.33}$$

Proof. The first estimate in (2.32) follows by (2.30) and (2.25). We now prove the second estimate in (2.32). By (2.30), we compute, for any $\vartheta \in (\mathbb{T}^{N_k})_{k\in\mathbb{N}}$ and $\widehat{\vartheta} \in (\mathbb{R}^{N_k})_{k\in\mathbb{N}}$,

$$\mathrm{d}_\vartheta U(\vartheta)[\widehat{\vartheta}] = -\sum_{k\in\mathbb{N}} \sum_{j=1}^{J_k} \widehat{\vartheta}_k \cdot \nabla'' \overline{v_k}(x' - \mathrm{y}_{k,j}, x'' - \vartheta_{k,j}). \tag{2.34}$$

Therefore, by (2.4), using the fact that each term in the series in (2.34) is supported in space on the cylinder $B_{m,2\varepsilon^k}(\mathrm{y}_{k,j}) \times \mathbb{T}^{d-m}$, that these supports are disjoint one from the other, and that $|\widehat{\vartheta}_k| \le |\widehat{\vartheta}|_\infty$, we obtain, for any $\vartheta \in (\mathbb{T}^{N_k})_{k\in\mathbb{N}}$, $\widehat{\vartheta} \in (\mathbb{R}^{N_k})_{k\in\mathbb{N}}$ and for any $n = 0, 1, ..., S$,

$$\begin{aligned} \|\mathrm{d}_\vartheta U(\vartheta)[\widehat{\vartheta}]\|_{n,\infty} &\le \sup_{k\in\mathbb{N}} \sup_{j=1,\dots,J_k} \|\nabla'' \overline{v_k}(\,\cdot\, - \mathrm{y}_{k,j}, \,\cdot\, - \vartheta_{k,j})\|_{n,\infty} |\widehat{\vartheta}|_\infty \\ &\le \sup_{k\in\mathbb{N}} \sup_{j=1,\dots,J_k} \|\overline{v_k}(\,\cdot\, - \mathrm{y}_{k,j}, \,\cdot\, - \vartheta_{k,j})\|_{n+1,\infty} |\widehat{\vartheta}|_\infty \\ &\le C_n \sup_{k\in\mathbb{N}} \varepsilon^{k(S+1-(n+1))-S-1} |\widehat{\vartheta}|_\infty \le C_n \varepsilon^{-S-1} |\widehat{\vartheta}|_\infty. \end{aligned} \tag{2.35}$$

This implies the claimed estimate and concludes the proof. □

3. Conclusions

In Section 2, we succeeded in proving Theorem 1.3. An interesting and challenging question is the analysis of the stability both for the quasi-periodic solutions in [16, 17] and for the almost-periodic solutions in Theorem 1.3 for long times, or in general for invariant motions in fluids. Further perspectives include the adaptation of this construction of quasi-periodic and almost-periodic solutions to other non-forced models in Fluid Dynamics, for instance for the equations of the magneto-hydrodynamics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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