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# COLLECTIVE DYNAMIC RISK MEASURES

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Abstract. We extend the framework introduced in "Collective Arbitrage and the Value of Cooperation" by F. Biagini, A. Doldi, J.-P. Fouque, M. Frittelli, and T. Meyer-Brandis (arXiv:2306.11599v2, 2024) in order to analyze *collective* dynamic risk measures. In segmented markets, we explore the implications of cooperation on dynamic risk measurement, focusing particularly on aggregation and time consistency.

1. **Introduction.** This paper is inspired by the recent work [6] on collective arbitrage and collective super-replication and extends the framework elaborated there to the analysis of collective dynamic risk measures. In order to describe our contribution we first provide a short summary of the setting and concepts developed in [6]. In the aforementioned paper, a No Arbitrage theory is established in a global market where several agents may trade in individual submarkets and are allowed to cooperate by engaging a risk exchange mechanism with no external capital injections or withdrawals.

As mentioned in [6], the decision to cooperate and participate in an exchange may be driven by factors such as: Individual agent rationality, mandates imposed by a parent organization (e.g., collaborating trading desks within a financial institution), regulatory directives compelling cross-subsidization among financial institutions, or taxation policies.

Markets segmentation can be justified in several ways. As pointed out in [10], one can think of a financial institution comprising multiple trading desks specialized in distinct financial products or market segments such as stocks, currencies, commodities, or bonds. As posited by [27], cognitive constraints necessitate traders to focus on a limited set of assets. While each desk operates independently, the institution's overall profit and loss is the aggregate of the profit and loss generated by individual desks.

To be more specific, consider $N \geq 1$ agents investing in a frictionless stochastic security market. Each agent is allowed to invest in a subset of the available risky

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assets and in a common riskless asset. Let $K_i$ be the market of agent $i$, that is the vector space of all the possible terminal time payoffs that agent $i$ can obtain by using admissible trading strategies in his/her allowed investments and having zero initial cost.

The $N$ agents not only participate in their respective markets but may also collaborate to optimize their positions by exploiting risk exchange opportunities. Consider the set of all zero-sum risk exchanges defined as

$$\mathcal{Y}_0 = \left\{ Y \in (L^0(\Omega, \mathcal{F}, P))^N \mid \sum_{i=1}^{N} Y^i = 0 \ P\text{-a.s.} \right\}, \tag{1}$$

and the convex cone $\mathcal{Y}$ of possible/allowed exchanges $\mathcal{Y} \subseteq \mathcal{Y}_0$.

Notably, while the aggregate sum of the components of $Y \in \mathcal{Y}$ is $P$-almost surely zero, individual components $Y^i$ are generally random variables. Positive values of $Y^i$ on a given event signify a capital inflow for agent $i$ from the collective, whereas negative values represent an outflow. Consequently, $Y \in \mathcal{Y}$ encapsulates the potential capital transfers among agents subject to the constraint of zero net transfer.

In [6] a *Collective Arbitrage* consists of vectors $(k^1, \dots, k^N) \in K_1 \times \cdots \times K_N$ and $Y = (Y^1, \dots, Y^N) \in \mathcal{Y}$ satisfying

$$\begin{aligned} & k^i + Y^i \geq 0 \ P\text{-a.s.} \ \ \forall\, i \in 1, \dots, N \\ \text{and } \ & P(k^j + Y^j > 0) > 0 \text{ for at least one } j \in 1, \dots, N. \end{aligned}$$

The interdependence among agents, induced by the exchange vector $Y \in \mathcal{Y}$, may generate a collective arbitrage even in the absence of individual arbitrage opportunities. Similar to the case of classical arbitrage, efficient markets with risk exchanges should not allow for the possibility of collective arbitrage, and one of the main objectives of [6] is the study and characterization of markets without collective arbitrage, thus providing a collective formulation of the Fundamental Theorem of Asset Pricing. We defer to the cited reference for more details on this topic and for the economic motivation behind the No Collective Arbitrage assumption.

Furthermore, the problem of *hedging* simultaneously $N$ given (suitably integrable) contingent claims $X = (X^1, \dots, X^N)$ was thoroughly analyzed in [6]. The (static) *collective super-replication* cost (or price) was defined as

$$\rho_0^{\mathcal{Y}}(X) := \inf \left\{ \sum_{i=1}^{N} m^i \mid m \in \mathbb{R}^N \ \exists k_i \in K_i, \exists Y \in \mathcal{Y} \text{ s.t. } m^i + k^i + Y^i \geq X^i \ \forall i \right\}. \tag{2}$$

Several of its properties were studied. Among these, it was proved that $\rho_0^{\mathcal{Y}}$ is well-posed under the No Collective Arbitrage assumption. Both $\rho_0^{\mathcal{Y}}(X)$ and the (static) classical super-replication price

$$\rho_0^{N}(X) := \inf \left\{ \sum_{i=1}^{N} m^i \mid m \in \mathbb{R}^N \ \exists k_i \in K_i \text{ s.t. } m^i + k^i \geq X^i \ \forall i \right\} \tag{3}$$

$$= \sum_{i=1}^{N} \inf \left\{ m \in \mathbb{R} \mid \exists k_i \in K_i \text{ s.t. } m + k^i \geq X^i \right\} := \sum_{i=1}^{N} \rho_{0,i}(X^i) \tag{4}$$

represents the minimal aggregate capital required to simultaneously super-replicate all claims $(X^1,\dots,X^N)=X$. However, $\rho_0^{\mathcal{Y}}(X)$ incorporates the possibility of inter-agent exchanges through $\mathcal{Y}$, leading to $\rho_0^{\mathcal{Y}}\leq\rho_0^N$. This implies that the collective super-replication $\rho_0^{\mathcal{Y}}$ is more cost-effective than the classical super-replication $\rho_0^N$, with the difference $\rho_0^N(X)-\rho_0^{\mathcal{Y}}(X)\geq 0$ quantifying the *value of cooperation* for the given claims $X$. One of the main message from [6] is that cooperation may help the system of the $N$ agents to save money when facing several risks.

The present paper builds upon this idea and investigates the consequences, in dynamic risk measurement, of the cooperation among several agents in the presence of segmented markets. In a companion paper we treat the case of static collective risk measures, while in the present paper we analyze dynamic collective risk measures with special focus on the properties of the aggregation of several individual conditional risk measurements $\rho^i$, $i=1,\dots,N$, with particular emphasis on the time consistency property that such resulting aggregation $\overline{\rho}$ may possess.

We now describe the main objectives and findings of this work, deferring the description of the precise setup, definitions and mathematically rigorous claims to the subsequent sections. For unexplained terminology concerning risk measures we defer the reader to [26]. Let $T>0$ be a fixed expiration date, $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},P)$ be a filtered probability space with $\mathcal{F}_T=\mathcal{F}$, $L$ be a subspace of $L^0(\Omega,\mathcal{F},P)$ and set $L(\mathcal{F}_t):=L\cap L^0(\Omega,\mathcal{F}_t,P)$ and $L(\mathcal{F}):=L(\mathcal{F}_T)$. Consider $N$ families $\rho^i=(\rho^i_t)_{t\in[0,T]}$, $i=1,\dots N$, where each $\rho^i_t:L(\mathcal{F})\to L^0(\mathcal{F}_t)$, is an individual conditional risk measure. We are interested in maps $\overline{\rho}=(\overline{\rho}_t)_{t\in[0,T]}$, with $\overline{\rho}_t:(L(\mathcal{F}))^N\to L^0(\mathcal{F}_t)$, that aggregate the individual risk measures $\rho^i$, $i=1,\dots N$, and assess the overall risk of $(X^1,\dots,X^N)=X$ with the property of being conservative, namely

$$\overline{\rho}_t(X^1,\dots,X^N)\leq\sum_{i=1}^N\rho^i_t(X^i),\quad X\in(L(\mathcal{F}))^N.$$

Conservativity reads: The overall risk expressed by $\overline{\rho}_t$ does not exceed the sum of the individual risks. One obvious but extreme choice would be to take $\overline{\rho}$ as the sum of each individual map. Just to mention a simple example where this simplest aggregation makes sense, when one considers as (static) risk measures the individual super-replication costs $\rho_{0,i}(X^i)$ defined in (4), in the previously described segmented markets setup, the (static) classical super-replication price $\rho_0^N(X)$ defined in (3) is precisely equal to $\sum_{i=1}^N\rho_{0,i}(X^i)$.

As observed in Remark 2.2 below, if each $\rho^i=(\rho^i_t)_{t\in[0,T]}$ is cash additive and time consistent then the only conservative, cash additive and time consistent aggregator (see Definition 2.1 item 4) is

$$\overline{\rho}_t(X)=\sum_{i=1}^N\rho^i_t(X^i).$$

However, by introducing cooperation among the agents, as e.g. in the above definition of the *collective* super-replication cost $\rho_0^{\mathcal{Y}}(X)$, the question on which aggregation functionals one could consider is not any more so trivial. Indeed, admissible exchanges $Y\in\mathcal{Y}$ can be exploited, as already stressed before, to reduce the overall risk while keeping some natural features inherited by each $\rho^i$, such as cash additivity and time consistency.

We thus introduce a properly defined notion (see Definition 2.4) of time $\mathcal{Y}$-consistency for maps $\overline{\rho}=(\overline{\rho}_t)_{t\in[0,T]}$ defined on $(L(\mathcal{F}))^N$ and show (see Propositions

2.6 and 3.6) that if each individual $\rho^i = (\rho^i_t)_{t\in[0,T]}$ is cash additive and time consistent, then the only cash additive, time $\mathcal{Y}$-consistent, conservative and compatible (see Definition 2.3) aggregator $\overline{\rho}$ of the $N$ maps $\rho^i$ is

$$\overline{\rho}_t(X) = \operatorname{ess\,inf}\left\{\sum_{i=1}^N \rho^i_t(X^i + Y^i) \mid Y \in \mathcal{Y}\right\}, \quad X \in (L(\mathcal{F}))^N.$$

The relevance of the map in the right hand side of the above equation, which will be denoted by $\rho^1_t \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N_t(X)$ (see Definition 2.7), can be understood also from the following considerations.

First, recall that if $\mathcal{A}^i \subseteq L(\mathcal{F})$ is a monotone acceptance set of agent $i$ then the monetary conditional risk measures induced by $\mathcal{A}^i$ is $\rho_{t,\mathcal{A}^i}(X^i) := \operatorname{ess\,inf}\{\alpha \in L(\mathcal{F}_t) \mid \alpha + X^i \in \mathcal{A}^i\}$. Now, as a natural extension of the static collective super-replication price $\rho^{\mathcal{Y}}_0$ defined in (2), we introduce the concept of a conditional *collective risk measure*, namely the map

$$\begin{aligned}&\rho_{t,\mathbb{A},\mathcal{Y}}(X^1,\dots,X^N)\\ &:= \operatorname{ess\,inf}\left\{\sum_{i=1}^N \alpha^i \mid \alpha^i \in L(\mathcal{F}_t) \text{ and } \exists Y \in \mathcal{Y} \text{ s.t. } \alpha^i + Y^i + X^i \in \mathcal{A}^i \quad \forall i\right\},\end{aligned}$$

where $\mathbb{A} := \mathcal{A}^1 \times \cdots \times \mathcal{A}^N$. Some example of collective risk measures arising in financial markets are presented in Example 4.5. In analogy to the discussion on the collective super-replication price, cooperation among the agents contributes to reduce the capital requirement of the overall risk, that is

$$\rho_{t,\mathbb{A},\mathcal{Y}}(X) \le \sum_{i=1}^N \rho_{t,\mathcal{A}^i}(X^i),$$

so that $\rho_{t,\mathbb{A},\mathcal{Y}}$ is a conservative aggregator. Furthermore, as showed in Section 4,

$$\rho_{t,\mathbb{A},\mathcal{Y}}(X) = \rho_{t,\mathcal{A}^1} \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_{t,\mathcal{A}^N}(X)$$

and thus the properties of $\rho_{t,\mathbb{A},\mathcal{Y}}$ can be obtained from the corresponding ones of the $\rho^1_t \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N_t$ operator (see Proposition 2.8 and 3.4), whenever $\rho^i_t := \rho_{t,\mathcal{A}^i}$.

For any $s \in [0,T]$, a relevant example of allowed exchanges is the vector subspace of $(L(\mathcal{F}_s))^N$, assigned by

$$\mathcal{Y}(s) = \left\{(Y^1,\dots,Y^N) \in (L(\mathcal{F}_s))^N \mid \sum_{i=1}^N Y^i = 0 \ P\text{-a.s.}\right\}. \tag{5}$$

In this case, the zero-sum risk exchanges are additionally constrained to be $\mathcal{F}_s$ measurable. The choice $\mathcal{Y}(s)$ for admissible exchanges models the fact that agents might only be allowed to cooperate/exchange up to a certain fixed date prior to the "horizon" of the underlying risky positions $(X^1,\dots,X^N)$, say by regulatory constraints or by modelling choices determined by partial common information known to each of the agents in the system.

For this specific selection $\mathcal{Y} = \mathcal{Y}(s)$ of the set of exchanges, we additionally prove:

1. Time consistency properties of $(\rho^1_t \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N_t)_{t\in[0,T]}$, see Proposition 3.4 item 4.
2. The dual representation of $\rho^1_t \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N_t$, see Corollary 3.10.
3. In Section 3.2, the characterization of the time $\mathcal{Y}$-consistency of $(\rho^1_t \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N_t)_{t\in[0,T]}$ in terms of
   - Decompositions of the acceptance sets

- Cocycle type properties for the penalty functions in the dual representation.

4. In Proposition 3.4 item 3, the formula

$$\rho_t^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_t^N (X^1, \dots, X^N) = \begin{cases} \rho_t^1 \square_s \dots \square_s \rho_t^N \Big( - \sum_{i=1}^N \rho_s^i(X^i) \Big), & \text{if } t \le s \\ \sum_{i=1}^N \rho_t^i(X^i) & \text{if } s \le t, \end{cases}$$

   which enlights the connection between the $\Diamond_{\mathcal{Y}}$ operator and the $s$-inf-convolution operator $\square_s$ introduced in (12).

5. In Section 4.1, an explicit formula for the dynamic collective entropic risk measure.

As pointed out in item 4 above, there is an evident link between the newly defined aggregators and well known inf-convolutions. Indeed the former can be seen as an extension/generalization of the latter to a larger space.

Keeping in mind the measurability constraint enforced by the set $\mathcal{Y}(s)$, we conclude from item 4 above that the valuation $\rho_t^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_t^N (X^1, \dots, X^N)$ at time $t$ of the overall risk is simply the sum of the time $t$ individual risks $\rho_t^i(X^i)$ if no risk sharing is yet possible in $[t, T]$, that is if $s \le t$. Instead, if risk sharing is possible in $[t, T]$ (that is if $t \le s$), it corresponds to the inf-convolution of (minus) the sum of the time $s$ individual risks $\rho_s^i(X^i)$.

In the non conditional setup, the seminal paper by [4], which introduced inf-convolutions in the context of (convex) risk measures, has spawned a substantial body of literature. Existence of optima, among several other properties, was studied e.g. in the law-invariant case by [30]. For instance, [1] and [25] investigated cases without monotonicity assumptions on the underlying risk measures, while [11] addressed multivariate risks. Further analysis was carried over in [34], [21], [22], [33], [9]. For an overview of the early developments of the topic, see [38]. The conditional case was considered in [31] which is a main reference for the present work, especially on technical aspects. In our paper we mainly consider the theory of risk measures (convex, monetary) originated from the seminal paper [2] but many other ramifications can be found in the literature, e.g. conic finance and acceptability indices as in [12], [35], [7]. Our work is also related to conditional systemic risk measures. We refer to [26] Chapter 11 for a good overview on univariate dynamic/conditional Risk Measures, to [18] for further references and discussion, and to [19] linking convolution operations to systemic risk measures. Time consistency for risk measure has long been studied in the literature, with a particular focus on the BSDE approach. To this end, we cite [3] where inf-convolutions are also considered, and [37], [15] investigating time consistency of convex risk measures in a BSDE approach, with the entropic risk measure as a special case. [36] explored cash-subadditive and quasi-convex functionals, and [17] has further developed the BSDE approach. Existence of conditional means, related to scalarization and time consistency properties, was also explored in [20] and [5]. Regarding (time) consistency properties in the systemic framework, the papers [28], [29], as well as [32], consider conditional extension of (static) Systemic Risk Measures of the “first aggregate, then allocate” type and study related consistency issues. Multivariate/Systemic and set-valued conditional Risk Measures, and related time consistency aspects, have also been analyzed in [23], [24], [13].

## 2. **Dynamic aggregation rules.**

2.1. **Notations and definitions.** Let $T > 0$ be a fixed expiration date and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a filtered probability space, with $\mathcal{F}_0$ the trivial sigma-algebra and $\mathcal{F}_T = \mathcal{F}$. We let $L$ be a subspace of $L^0(\Omega, \mathcal{F}, P)$ and set $L(\mathcal{F}_t) := L \cap L^0(\Omega, \mathcal{F}_t, P)$ and $L(\mathcal{F}) := L(\mathcal{F}_T)$. Some typical examples for $L$ are the $L^p(\Omega, \mathcal{F}, P)$ spaces, $p \in [1, \infty]$, the Orlicz spaces $L^\psi(\Omega, \mathcal{F}, P)$ and Orlicz Heart $M^\psi(\Omega, \mathcal{F}, P)$, for a Young function $\psi$. For any $N \geq 1$ we denote the cartesian product $L(\mathcal{F}) \times \cdots \times L(\mathcal{F})$ by $(L(\mathcal{F}))^N$. We will denote with $\mathbb{E}[Z]$ the expectation of some random variable $Z$ under the probability $P$ , while we explicitly write $\mathbb{E}_Q[Z]$ for any probability $Q$ different from $P$. It is also understood that all inequalities among random variables holds $P$-a.s..

**Definition 2.1.** Let $t, u, s \in [0, T]$, $N \geq 1$ and $i \in \{1, \dots, N\}$.

1. A family $\rho = (\rho_t)_{t\in[0,T]}$, with $\rho_t : (L(\mathcal{F}))^N \to L^0(\mathcal{F}_t)$ is *cash additive* on $[0, s]$ if for each $t \in [0, s]$
$$\rho_t\left(X^1 + W^1, \dots, X^N + W^N\right) = \rho_t(X^1, \dots, X^N) - \sum_{i=1}^N W^i$$
for $W \in (L(\mathcal{F}_t))^N$ and $X \in (L(\mathcal{F}))^N$.
We write $\rho(0) = 0$ if $\rho_t(0) = 0$ for all $t \in [0, T]$.
2. A family $\rho = (\rho_t)_{t\in[0,T]}$, with $\rho_t : (L(\mathcal{F}))^N \to L^0(\mathcal{F}_t)$ is:
*Convex* on $[0, s]$ if for each $t \in [0, s]$
$$\rho_t(\lambda X + (1-\lambda)W) \leq \lambda\rho_t(X) + (1-\lambda)\rho_t(W)$$
for $0 \leq \lambda \leq 1$, $\lambda \in L(\mathcal{F}_t)$ and $X, W \in (L(\mathcal{F}))^N$;
*Monotone decreasing* if so is each functional $\rho_t$ (w.r.t. the componentwise a.s. order).
3. A family $\rho = (\rho_t)_{t\in[0,T]}$, with $\rho_t : L(\mathcal{F}) \to L^0(\mathcal{F}_t)$ is *time consistent* on $[0, s]$ if, for all $t \leq u \leq s$, $\mathrm{Range}(\rho_u) \subseteq L(\mathcal{F})$ and
$$\rho_t(Z) = \rho_t(-\rho_u(Z)), \quad Z \in L(\mathcal{F}).$$
4. Consider $N$ families $\rho^i = (\rho^i_t)_{t\in[0,T]}$, $i = 1, \dots N$, with $\rho^i_t : L(\mathcal{F}) \to L^0(\mathcal{F}_t)$. We say that $\overline{\rho} = (\overline{\rho}_t)_{t\in[0,T]}$, with $\overline{\rho}_t : (L(\mathcal{F}))^N \to L^0(\mathcal{F}_t)$, is:
A *conservative aggregation* of the N maps $\rho^1, \dots, \rho^N$ on $[0, s]$ if for all $t \leq s$
$$\overline{\rho}_t(X^1, \dots, X^N) \leq \sum_{i=1}^N \rho^i_t(X^i), \quad X \in (L(\mathcal{F}))^N;$$
A *time consistent aggregation* of the N maps $\rho^1, \dots, \rho^N$ on $[0, s]$ if, for all $t \leq u \leq s$ and for all $i$, $\mathrm{Range}(\rho^i_u) \subseteq L(\mathcal{F})$ and
$$\overline{\rho}_t(X^1, \dots, X^N) = \overline{\rho}_t(-\rho^1_u(X^1), \dots, -\rho^N_u(X^N)),\ X \in (L(\mathcal{F}))^N. \tag{6}$$

A conservative aggregator always reduces the overall risk, compared to the sum of the individual risks. The notion of time consistent aggregation is an immediate multivariate counterpart to the time consistency of maps $\rho_t : L(\mathcal{F}) \to L^0(\mathcal{F}_t)$.

Observe that if $\rho = (\rho_t)_{t\in[0,T]}$, with $\rho_t : (L(\mathcal{F}))^N \to L^0(\mathcal{F}_t)$, is cash additive on $[0, T]$ and $\rho(0) = 0$ then
$$\rho_T(X) = -\sum_{i=1}^N X^i,\ X \in (L(\mathcal{F}))^N. \tag{7}$$

In the rest of the paper, we denote with $\rho^i = (\rho^i_t)_{t\in[0,T]}$, $i = 1,\dots N$, $N$ families of maps with $\rho^i_t : L(\mathcal{F}) \to L^0(\mathcal{F}_t)$.

**Remark 2.2.** Suppose that each $\rho^i$ is cash additive, time consistent and satisfies $\rho^i(0) = 0$. The only cash additive and time consistent aggregator $\overline{\rho} = (\overline{\rho}_t)_{t\in[0,T]}$, $\overline{\rho}_t : (L(\mathcal{F}))^N \to L^0(\mathcal{F}_t)$, satisfying $\overline{\rho}(0) = 0$, is the aggregator

$$\overline{\rho}_t(X) = \sum_{i=1}^N \rho^i_t(X^i),$$

which is also conservative. Indeed, the sum of the components inherits cash additivity and time consistency from each component. Moreover, if $\overline{\rho}$ is cash additive and time consistent then

$$\overline{\rho}_t(X^1,\dots,X^N) = \overline{\rho}_t(-\rho^1_t(X^1),\dots,-\rho^N_t(X^N)) = \overline{\rho}_t(0) + \sum_{i=1}^N \rho^i_t(X^i).$$

Thus in this case $\overline{\rho}$ is nothing else but the sum of the individual risk.

As stated in the Introduction, the key feature in this paper is the possibility that agents may cooperate through the exchange of the random amounts described by vectors $Y$ belonging to a *convex cone* $\mathcal{Y} \subseteq (L(\mathcal{F}))^N$. When we introduce exchanges in the picture, some more interesting features can be analyzed.

First, it is reasonable to assume that it is not possible to decrease the overall risk $\rho_t(\cdot)$ by adding an exchange element $Y \in \mathcal{Y}$. Thus we introduce the following

**Definition 2.3.** The class of functionals $\rho = (\rho_t)_{t\in[0,T]}$, with $\rho_t : (L(\mathcal{F}))^N \to L^0(\mathcal{F}_t)$ is *compatible* with the risk exchange set $\mathcal{Y} \subseteq (L(\mathcal{F}))^N$ if

$$\rho_t(X) \le \rho_t(X+Y), \quad \forall Y \in \mathcal{Y}, \quad \forall X \in (L(\mathcal{F}))^N, \quad \forall t \in [0,T]. \tag{8}$$

When $\rho(0) = 0$, the condition (8) in particular implies that the risk of each exchange vector is non negative: $\rho_t(Y) \ge 0$ for all $Y \in \mathcal{Y}$ and $t \in [0,T]$.

Second, a more elaborated type of time consistency can be considered.

**Definition 2.4.** For a given exchange set $\mathcal{Y} \subseteq (L(\mathcal{F}))^N$, we say that $\overline{\rho} = (\overline{\rho}_t)_{t\in[0,T]}$, with $\overline{\rho}_t : (L(\mathcal{F}))^N \to L(\mathcal{F}_t)$, is a *time $\mathcal{Y}$-consistent aggregation* of the N maps $\rho^1,\dots,\rho^N$ if for every $X \in (L(\mathcal{F}))^N$ and every pair of times $0 \le t \le u \le T$ there exists $Y \in \mathcal{Y}$ (possibly depending on $X$) satisfying

$$\overline{\rho}_t(X^1,\dots,X^N) = \overline{\rho}_t\left(-\rho^1_u(X^1+Y^1),\dots,-\rho^N_u(X^N+Y^N)\right). \tag{9}$$

**Remark 2.5.** If $\mathcal{Y}$ is a vector space, then (9) is equivalent to:

$$\overline{\rho}_t(X^1+\widehat{Y}^1,\dots,X^N+\widehat{Y}^N) = \overline{\rho}_t\left(-\rho^1_u(X^1+Y^1),\dots,-\rho^N_u(X^N+Y^N)\right), \tag{10}$$

for some $Y \in \mathcal{Y}$ and $\widehat{Y} \in \mathcal{Y}$. The interpretation of a time $\mathcal{Y}$-consistent aggregation is then immediate: The rationale rests on the possibility for the agents to find allowable exchanges $Y \in \mathcal{Y}$, $\widehat{Y} \in \mathcal{Y}$ for which the aggregation is time consistent.

**Proposition 2.6.** *Suppose that each $\rho^i = (\rho^i_t)_{t\in[0,T]}$ is cash additive and time consistent. If an aggregator $\overline{\rho} = (\overline{\rho}_t)_{t\in[0,T]}$, with $\overline{\rho}_t : (L(\mathcal{F}))^N \to L^0(\mathcal{F}_t)$ and $\overline{\rho}(0) = 0$, is cash additive, time $\mathcal{Y}$-consistent, conservative and compatible then*

$$\overline{\rho}_t(X) = \operatorname{ess\,inf}\left\{\sum_{i=1}^N \rho^i_t(X^i+Y^i) \mid Y \in \mathcal{Y}\right\}, \quad X \in (L(\mathcal{F}))^N,$$

*and the* ess inf *is attained.*

*Proof.* Suppose that $\overline{\rho}$ satisfies all the assumptions. For notational simplicity we take $N = 2$. Applying time $\mathcal{Y}$-consistency, letting in (9) $u = t$, we have for any $X \in (L(\mathcal{F}))^N$ and for some $\widehat{Y} \in \mathcal{Y}$

$$\begin{aligned}\overline{\rho}_t(X^1, X^2) &= \overline{\rho}_t\left(-\rho_t^1(X^1 + \widehat{Y}^1), -\rho_t^2(X^2 + \widehat{Y}^2)\right) = \rho_t^1(X^1 + \widehat{Y}^1) + \rho_t^2(X^2 + \widehat{Y}^2)\\ &\geq \operatorname*{ess\,inf}_{Y \in \mathcal{Y}} \left(\rho_t^1(X^1 + Y^1) + \rho_t^2(X^2 + Y^2)\right),\end{aligned}$$

where we used in the second equality the cash additivity of $\overline{\rho}$ and $\overline{\rho}(0) = 0$. On the other hand

$$\operatorname*{ess\,inf}_{Y \in \mathcal{Y}} \left(\rho_t^1(X^1 + Y^1) + \rho_t^2(X^2 + Y^2)\right) \geq \operatorname*{ess\,inf}_{Y \in \mathcal{Y}} \overline{\rho}_t(X^1 + Y^1, X^2 + Y^2) \geq \overline{\rho}_t(X^1, X^2)$$

where we used the assumption that $\overline{\rho}$ is conservative in the first inequality and the compatibility condition (8) in the second one. This also shows that the ess inf is attained at $\widehat{Y}$. □

Proposition 2.6 motivates introducing the following inf-convolution type functional on the product space $(L(\mathcal{F}))^N$ and in the following sections we will focus on exploring some of its key features.

**Definition 2.7.** Given the exchange set $\mathcal{Y} \subseteq (L(\mathcal{F}))^N$, we define on $(L(\mathcal{F}))^N$ the aggregator $\rho^1 \Diamond \dots \Diamond \rho^N = (\rho_t^1 \Diamond \dots \Diamond \rho_t^N)_{t \in [0,T]}$ of the $N$ families $\rho^i = (\rho_t^i)_{t \in [0,T]}$ as the $\mathcal{F}_t$-measurable random variable

$$\rho_t^1 \Diamond \dots \Diamond \rho_t^N(X) := \operatorname{ess\,inf} \left\{ \sum_{i=1}^N \rho_t^i(X^i + Y^i) \mid Y \in \mathcal{Y} \right\},$$

for $X = (X^1, \dots, X^N) \in (L(\mathcal{F}))^N$. We use the notation $\rho^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N$ when the reference to the set $\mathcal{Y}$ is relevant.

The functional $\rho^1 \Diamond \dots \Diamond \rho^N$ enjoys some rather natural properties, some of them directly inherited from the corresponding ones of the initial families $\rho^i$. We list them below, providing short proofs for the less evident ones.

**Proposition 2.8.** *Suppose that each $\rho^i = (\rho_t^i)_{t \in [0,T]}$, $i = 1, \dots, N$, is cash additive on $[0, T]$ and let $X \in (L(\mathcal{F}))^N$. Then*

1. *$\rho_t^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_t^N(X^1, \dots, X^N) \leq \sum_{i=1}^N \rho_t^i(X^i)$, for all $t \in [0, T]$, namely $\rho^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N$ is conservative.*
2. *Fix $s \in [0, T]$ and suppose that $\mathcal{Y} \subseteq (L(\mathcal{F}_s))^N$ and $\operatorname{ess\,sup}_{Y \in \mathcal{Y}} \{\sum_{i=1}^N Y^i\} \leq 0$. Then*
   *$\rho_t^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_t^N(X^1, \dots, X^N) = \sum_{i=1}^N \rho_t^i(X^i)$ for all $s \leq t \leq T$.*
3. *$\rho^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N$ is cash additive on $[0, T]$. If $\rho^i$ is monotone decreasing for each $i = 1, \dots, N$, so is $\rho^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N$ (w.r.t. the componentwise order).*
4. *$\rho_t^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_t^N(X^1 + Y^1, \dots, X^N + Y^N) = \rho_t^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_t^N(X^1, \dots, X^N)$, for all $t \in [0, T]$, if $(Y^1, \dots, Y^N) \in \mathcal{Y}$. Thus in particular $\rho^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho^N$ is a compatible aggregator.*
5. *Suppose in addition that $\rho^i$ is time consistent for each $i = 1, \dots, N$, take $s \in [0, T]$ and suppose that $\mathcal{Y} \subseteq (L(\mathcal{F}_s))^N$. If $u \geq s$, then for all $0 \leq t \leq u \leq T$ we have*

$$\rho_t^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_t^N(X^1, \dots, X^N) = \rho_t^1 \Diamond_{\mathcal{Y}} \dots \Diamond_{\mathcal{Y}} \rho_t^N(-\rho_u^1(X^1), \dots, -\rho_u^N(X^N)). \quad (11)$$

*Proof.* To simplify the exposition, we prove the properties in the case $N = 2$.

1. It follows from $0 \in \mathcal{Y}$.
2. Since $Y \in \mathcal{Y}$ is $\mathcal{F}_t$-measurable, by cash additivity $\rho_t^i(X^i + Y^i) = \rho_t^i(X^i) - Y^i$. The property follows by using the inequality in item 1 and the assumption.
3. By definition of $\rho_t^1 \Diamond_{\mathcal{Y}} \rho_t^2$.
4. $\rho_t^1 \Diamond_{\mathcal{Y}} \rho_t^2(X^1+Y^1, X^2+Y^2) = \operatorname{ess\,inf} \Big\{ \rho_t^1(X^1 + Y^1 + \widehat{Y}^1) + \rho_t^2(X^2 + Y^2 + \widehat{Y}^2) \mid \widehat{Y} \in \mathcal{Y} \Big\} = \rho_t^1 \Diamond_{\mathcal{Y}} \rho_t^2(X^1, X^2)$, as $Y + \widehat{Y} \in \mathcal{Y}$.
5. In the first equality below we apply the definition of $\Diamond_{\mathcal{Y}}$ and the time consistency of $\rho^i$, and in the second one the cash additivity of $\rho^i$, using the fact that $Y$ is $\mathcal{F}_u$-measurable.

$$\begin{aligned}\rho_t^1 \Diamond_{\mathcal{Y}} \rho_t^2(X^1, X^2) &= \operatorname{ess\,inf} \{ \rho_t^1 \left( -\rho_u^1(X^1 + Y^1) \right) + \rho_t^2 \left( -\rho_u^2(X^2 + Y^2) \right) \mid Y \in \mathcal{Y} \} \\ &= \operatorname{ess\,inf} \{ \rho_t^1 \left( -\rho_u^1(X^1) + Y^1 \right) + \rho_t^2 \left( -\rho_u^2(X^2) + Y^2 \right) \mid Y \in \mathcal{Y} \} \\ &= \rho_t^1 \Diamond_{\mathcal{Y}} \rho_t^2 \left( -\rho_u^1(X^1), -\rho_u^2(X^2) \right).\end{aligned}$$

□

**Remark 2.9.** Suppose that each $\rho^i$ is cash additive and time consistent. Proposition 2.6 shows that the only possible cash additive, time $\mathcal{Y}$-consistent, conservative and compatible aggregator $\overline{\rho} = (\overline{\rho}_t)_{t \in [0,T]}$, of the families $\rho^i$, with $\overline{\rho}(0) = 0$, is $\overline{\rho} = \rho^1 \Diamond_{\mathcal{Y}} \ldots \Diamond_{\mathcal{Y}} \rho^N$, even if the existence of such aggregator is not guaranteed. Moreover, from Proposition 2.8 we know that $\rho^1 \Diamond_{\mathcal{Y}} \ldots \Diamond_{\mathcal{Y}} \rho^N$ is cash additive, conservative and compatible. Under the assumption that $\rho^1 \Diamond_{\mathcal{Y}} \ldots \Diamond_{\mathcal{Y}} \rho^N(0) = 0$, it is desirable to identify conditions guaranteeing that $\rho^1 \Diamond_{\mathcal{Y}} \ldots \Diamond_{\mathcal{Y}} \rho^N$ is also time $\mathcal{Y}$-consistent. Given Proposition 2.6, a necessary condition is that $\rho^1 \Diamond_{\mathcal{Y}} \ldots \Diamond_{\mathcal{Y}} \rho^N$ is attained. For the choices $L = L^\infty$ and $\mathcal{Y} = \mathcal{Y}(s)$ as in (5) and a technical condition, as detailed in the following Section, this turns out to be also sufficient, thus showing existence and uniqueness of an aggregator having all the stated properties, see Proposition 3.6.

3. **The aggregation for a particular choice of $\mathcal{Y} = \mathcal{Y}(s)$.** In this section we choose the vector space $\mathcal{Y} = \mathcal{Y}(s)$, as in (5), we analyze the consequences of this choice and in particular we compare the diamond operator $\Diamond_{\mathcal{Y}(s)}$ with the $s$-convolution $\square_s$ of the maps $\rho^i$. Let us first recall the definition of the latter.

**Definition 3.1.** Let $s \in [0, T]$ and consider $N$ families $\rho^i = (\rho_t^i)_{t \in [0,T]}$ with $\rho_t^i : L(\mathcal{F}) \to L^0(\mathcal{F}_t)$. We define on $L(\mathcal{F}_s)$ the $s$-convolution $\rho^1 \square_s \ldots \square_s \rho^N = (\rho_t^1 \square_s \ldots \square_s \rho_t^N)_{t \in [0,T]}$ of the $N$ families $\rho^i$ as the $\mathcal{F}_t$-measurable random variable

$$\begin{aligned}&\rho_t^1 \square_s \ldots \square_s \rho_t^N(Z) \\ &= \operatorname{ess\,inf} \left\{ \sum_{i=1}^N \rho_t^i(Y^i) \mid Y^i \in L(\mathcal{F}_s),\ i = 1, \ldots, N,\ \text{ s.t. } \sum_{i=1}^N Y^i = Z \right\},\ Z \in L(\mathcal{F}_s).\end{aligned} \tag{12}$$

If $s = T$ then $\rho_t^1 \square_T \ldots \square_T \rho_t^N$ is the usual inf-convolution of the $N$ maps $\rho_t^i$.

Note that $\rho_t^1 \square_s \ldots \square_s \rho_t^N$ is defined on the space $(L(\mathcal{F}_s))$. This accounts for the fact that, when considering usual inf-convolutions, exchanges have (informally speaking) the same measurability as the initial positions. Moreover, such functional are defined on random variables rather than random vectors. This is motivated by the measurability requirement we just pointed out. Indeed, if one would define the $s$-convolution for a vector, then that would turn out to solely depend on the componentwise sum of that vector. A mathematically precise statement of what we just informally mentioned can be found in Proposition 3.4 item 1 and 2 below.

We list some simple properties of $\rho_t^1 \Box_s \dots \Box_s \rho_t^N$ which will be needed in the following.

**Proposition 3.2.** *We suppose $L = L^\infty$ and $\mathcal{Y} = \mathcal{Y}(s)$. We assume that each $\rho^i$ is cash additive, monotone decreasing and time consistent on $[0, s]$, and that for each time $t$, $\rho_t^j$ is continuous from below for some $j$. Finally, we assume that $\rho^i(0) = 0$, $i = 1, \dots, N$ and $\rho^1 \Box_s \dots \Box_s \rho^N(0) = 0$. Then $\rho^1 \Box_s \dots \Box_s \rho^N$ is monotone decreasing and*

1. *For $0 \le s \le t$, $\rho_t^1 \Box_s \dots \Box_s \rho_t^N(\xi) = -\xi$ if $\xi \in L(\mathcal{F}_s)$, while $\rho_t^1 \Box_s \dots \Box_s \rho_t^N(\xi) = +\infty$ otherwise.*
2. *$\rho^1 \Box_s \dots \Box_s \rho^N$ is time consistent on $[0, s]$, namely: For every $0 \le t \le u \le s \le T$ and $\xi \in L(\mathcal{F}_s)$ we have*
$$\rho_t^1 \Box_s \dots \Box_s \rho_t^N \left( -\rho_u^1 \Box_s \dots \Box_s \rho_u^N(\xi) \right) = \rho_t^1 \Box_s \dots \Box_s \rho_t^N(\xi).$$

*Proof.* We only provide details for the case $N = 2$ for the sake of simplicity. Monotonicity follows from the definition. Item 1 follows observing that if $\xi$ is not $\mathcal{F}_s$-mesaurable it cannot be written as the sum of two $\mathcal{F}_s$-measurable random variables, so that the defining infimum (over an empty set) is $+\infty$ by convention. Item 2 follows from [31] Theorem 4.1.(d), together with [26] Lemma 11.11 and Exercise 11.2.1. $\square$

For the particular choice $\mathcal{Y} = \mathcal{Y}(s)$ the definition of the $\Diamond_{\mathcal{Y}}$ aggregator becomes:

**Definition 3.3.** Let $s \in [0, T]$. We define on $(L(\mathcal{F}))^N$ the aggregator $\rho^1 \Diamond_s \dots \Diamond_s \rho^N = (\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N)_{t \in [0,T]}$, of the $N$ families $\rho^i = (\rho_t^i)_{t \in [0,T]}$ as the $\mathcal{F}_t$-measurable random variable

$$\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N(X) := \operatorname{ess\,inf} \left\{ \sum_{i=1}^N \rho_t^i(X^i + Y^i) \mid Y^i \in L(\mathcal{F}_s) \, \forall i \text{ s.t. } \sum_{i=1}^N Y^i = 0 \right\},$$

for $X = (X^1, \dots, X^N) \in (L(\mathcal{F}))^N$.

We begin by listing some features of such a functional, which are clearly more specific and detailed than those of the general $\Diamond_{\mathcal{Y}}$-aggregator, given the peculiar choice of the underlying set of exchanges. We emphasize that $\rho^1 \Diamond_s \dots \Diamond_s \rho^N$ extends to a wider space the convolution functional, as will become apparent once we detail the properties of the former which involve the latter at key points (see e.g. (13) below) .

**Proposition 3.4.** *Let $s, t, u \in [0, T]$ and $(X^1, \dots, X^N) \in (L(\mathcal{F}))^N$ and suppose that $\rho^i = (\rho_t^i)_{t \in [0,T]}$ is cash additive on $[0, T]$ for each $i = 1, \dots, N$. Then the properties in items 1 to 5 in Proposition 2.8 hold true. In addition:*

1. *$\rho_t^1 \Diamond_T \dots \Diamond_T \rho_t^N(X^1, \dots, X^N) = \rho_t^1 \Box_T \dots \Box_T \rho_t^N(X^1 + \dots + X^N)$ ($s = T$).*
2. *$\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N(X^1, \dots, X^N) = \rho_t^1 \Box_s \dots \Box_s \rho_t^N(X^1 + \dots + X^N)$ if $X^i$ is $\mathcal{F}_s$-measurable, $i = 1, \dots, N$.*

3. *Suppose in addition that $\rho^i$ is time consistent on $[0,T]$ for each $i$. We have*

$$\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N (X^1, \dots, X^N) = \begin{cases} \rho_t^1 \Box_s \dots \Box_s \rho_t^N \Big( -\sum_{i=1}^N \rho_s^i(X^i) \Big), & \text{if } t \le s \\ \sum_{i=1}^N \rho_t^i(X^i) & \text{if } t \ge s. \end{cases} \tag{13}$$

4. *Suppose $L = L^\infty$. Assume that each $\rho^i$ is cash additive, monotone decreasing and time consistent on $[0,T]$, and that for each time $t$, $\rho_t^j$ is continuous from below for some $j$. Finally, assume that $\rho^i(0) = 0$, $i = 1, \dots, N$ and $\rho^1 \Diamond_s \dots \Diamond_s \rho^N(0) = 0$. The following types of time consistency conditions hold:*

$$\begin{aligned} &\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N (X^1, \dots, X^N) \\ &= \rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N (-\rho_u^1(X^1), \dots, -\rho_u^N(X^N)) \quad \text{if } t \le s \le u. \end{aligned} \tag{14}$$

$$\begin{aligned} &\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N (X^1, \dots, X^N) \\ &= \rho_t^1 \Box_s \dots \Box_s \rho_t^N (-\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N (X^1, \dots, X^N)) \quad \text{if } t \le u \le s. \end{aligned} \tag{15}$$

$$\begin{aligned} &\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N (X^1, \dots, X^N) \\ &= \rho_u^1 \Box_s \dots \Box_s \rho_u^N (-\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N (X^1, \dots, X^N)) \quad \text{if } u \le t \le s. \end{aligned} \tag{16}$$

*Proof.* Again, we take $N = 2$ to simplify the notation. The assumptions stated on $\mathcal{Y}$ in Proposition 2.8 item 2 hold true when $\mathcal{Y} = \mathcal{Y}(s)$.

1. It follows directly from the definition, by replacing the $\mathcal{F}_T$-measurable random variable $X^i + Y^i$ with $Z^i$, and noticing that $Z^1 + Z^2 = X^1 + X^2$.
2. Same proof of item 1.
3. The case $s \le t$ is item 2 in Proposition 2.8. If on the other hand $t \le s$, then

$$\begin{aligned} &\rho_t^1 \Diamond_s \rho_t^2 (X^1, X^2) \\ &= \operatorname{ess\,inf} \{ \rho_t^1 (X^1 + Y^1) + \rho_t^2 (X^2 + Y^2) \mid Y^i \in L(\mathcal{F}_s),\, i = 1, 2,\, , Y^1 + Y^2 = 0 \} \\ &= \operatorname{ess\,inf} \{ \rho_t^1 \left( -\rho_s^1 (X^1 + Y^1) \right) + \rho_t^2 \left( -\rho_s^2 (X^2 + Y^2) \right) \mid Y^i \in L(\mathcal{F}_s), \\ &\qquad\qquad i = 1, 2,\, , Y^1 + Y^2 = 0 \} \\ &= \operatorname{ess\,inf} \{ \rho_t^1 \left( -\rho_s^1 (X^1) + Y^1 \right) + \rho_t^2 \left( -\rho_s^2 (X^2) + Y^2 \right) \mid Y^i \in L(\mathcal{F}_s), \\ &\qquad\qquad i = 1, 2,\, , Y^1 + Y^2 = 0 \} \\ &= \rho_t^1 \Box_s \rho_t^2 (-[\rho_s^1(X^1) + \rho_s^2(X^2)]). \end{aligned}$$

   The second equality above is a consequence of the time consistency of $\rho^i$ and the third one of the cash additivity of $\rho^i$, while the last one follows from the definition 3.1.
4. When $t \le s \le u$ this case was proven in (11). If $t \le u \le s$ we get from item 3 that $0 = \rho_t^1 \Diamond_s \rho_t^2(0) = \rho_t^1 \Box_s \rho_t^2(0)$, so that all the assumptions in Proposition 3.2 hold true. Then

$$\begin{aligned} &\rho_t^1 \Diamond_s \rho_t^2 (X^1, X^2) = \rho_t^1 \Box_s \rho_t^2 (-[\rho_s^1(X^1) + \rho_s^2(X^2)]) \\ &= \rho_t^1 \Box_s \rho_t^2 \left( -\rho_u^1 \Box_s \rho_u^2 (-[\rho_s^1(X^1) + \rho_s^2(X^2)]) \right) = \rho_t^1 \Box_s \rho_t^2 (-\rho_u^1 \Diamond_s \rho_u^2 (X^1, X^2)), \end{aligned}$$

   where we used in the first and third equality the formula (13), and the time consistency of the convolution in the second equality (see Proposition 3.2). The case $u \le t \le s$ is exactly as the case $t \le u \le s$, interchanging $u$ and $t$.

□

We are now ready to investigate some more specific and interesting properties of the aggregation functional we have just defined above, with a specific focus on the consistency properties previously introduced for aggregators.

**Assumption 3.5.** *We consider $L = L^\infty$ and $\mathcal{Y} = \mathcal{Y}(s)$ . We suppose that each $\rho^i = (\rho^i_t)_{t\in[0,T]}$ is monotone decreasing, convex, cash additive, time consistent on $[0,T]$ and that for each time $t$, $\rho^i_t$ is continuous from below for every $i$. We also assume that $\rho^i(0) = 0$, $i = 1, \dots, N$, $\rho^1 \diamondsuit_s \dots \diamondsuit_s \rho^N(0) = 0$ and that $\rho^1_t \diamondsuit_s \dots \diamondsuit_s \rho^N_t(X^1, \dots, X^N)$ is attained for every $t \in [0,T]$ and $(X^1, \dots, X^N) \in (L^\infty(\mathcal{F}))^N$.*

**Proposition 3.6.** *Under assumption 3.5, $\rho^1 \diamondsuit_s \dots \diamondsuit_s \rho^N$ is time $\mathcal{Y}(s)$-consistent.*

*Proof.* Equation (14) trivially covers the case $0 \le t \le s \le u \le T$ (take $Y = 0$ in the time consistency requirement (9)). For $0 \le t \le u \le s \le T$ instead use the hypothesis of attainment to guarantee that $\rho^1_u \diamondsuit_s \dots \diamondsuit_s \rho^N_u(X^1, \dots, X^N) = \rho^1_u(X^1 + Y^1) + \dots + \rho^N_u(X^N + Y^N)$ for some $Y \in \mathcal{Y}(s)$, and plug in (15) to obtain

$$\begin{aligned}
&\rho^1_t \diamondsuit_s \dots \diamondsuit_s \rho^N_t(X^1, \dots, X^N) \\
&= \rho^1_t \square_s \dots \square_s \rho^N_t(-\rho^1_u \diamondsuit_s \dots \diamondsuit_s \rho^N_u(X^1, \dots, X^N)) \\
&= \rho^1_t \square_s \dots \square_s \rho^N_t \left(-\left(\rho^1_u(X^1 + Y^1) + \dots + \rho^N_u(X^N + Y^N)\right)\right) \\
&= \rho^1_t \diamondsuit_s \dots \diamondsuit_s \rho^N_t \Big( -\rho^1_u(X^1 + Y^1), \dots, -\rho^N_u(X^N + Y^N)\Big),
\end{aligned}$$

where the last equality is due to item 2 in Proposition 3.4, since $\rho^i_u(X^i + Y^i) \in L(\mathcal{F}_u) \subseteq L(\mathcal{F}_s)$. □

In the following Lemma we describe an additional property of the $\diamondsuit_s$ operator that will be crucial to prove the equivalences in Proposition 3.11.

**Lemma 3.7.** *Under Assumption 3.5, $\overline{\rho} = \rho^1 \diamondsuit_s \dots \diamondsuit_s \rho^N$ satisfies for all $0 \le t \le u \le T$ and $X^1, \dots, X^N \in L^\infty(\mathcal{F})$*

$$\overline{\rho}_t(X^1, \dots, X^N) \le \overline{\rho}_t\left(-\rho^1_u(X^1 + Y^1), \dots, -\rho^N_u(X^N + Y^N)\right) \quad \forall Y \in \mathcal{Y}. \tag{17}$$

*Proof.* We start with the simplest case, namely $s \le u$. By the attainment hypothesis for some $\overline{Y} \in \mathcal{Y}$ we have

$$\overline{\rho}_t\left(-\rho^1_u(X^1 + Y^1), \dots, -\rho^N_u(X^N + Y^N)\right) = \sum_{i=1}^N \rho^i_t(-\rho^i_u(X^i + Y^i) + \overline{Y}^i).$$

Continuing from RHS of this equality, we first apply cash additivity (as $\overline{Y}^i \in L(\mathcal{F}_s) \subseteq L(\mathcal{F}_u)$) and then the time consistency of $\rho^i$ to obtain

$$\begin{aligned}
&\overline{\rho}_t\left(-\rho^1_u(X^1 + Y^1), \dots, -\rho^N_u(X^N + Y^N)\right) \\
&= \sum_{i=1}^N \rho^i_t(-\rho^i_u(X^i + Y^i + \overline{Y}^i)) = \sum_{i=1}^N \rho^i_t(X^i + Y^i + \overline{Y}^i) \ge \overline{\rho}_t(X),
\end{aligned}$$

by definition of $\overline{\rho} = \rho^1 \diamondsuit_s \dots \diamondsuit_s \rho^N$.

Now, for $u \le s$ instead, we have $\rho^i_t(X^i + Y^i) \in L(\mathcal{F}_t) \subseteq L(\mathcal{F}_u) \subseteq L(\mathcal{F}_s)$ so that

$$\overline{\rho}_t\left(-\rho^1_u(X^1 + Y^1), \dots, -\rho^N_u(X^N + Y^N)\right) = \rho^1_t \square_s \dots \square_s \rho^N_t \left(-\sum_{i=1}^N \rho^i_u(X^i + Y^i)\right)$$

$$\geq \rho_t^1 \Box_s \dots \Box_s \rho_t^N \left(-\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N(X)\right) \tag{18}$$

using Proposition 3.4 item 2 for the first equality, decreasing monotonicity of $\Box_s$ and the fact that $\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N(X) \leq \sum_{i=1}^N \rho_u^i(X^i + Y^i)$ for every $Y \in \mathcal{Y}$ for the last inequality. By selecting in (15) $t = u$, we have

$$\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N(X) = \rho_u^1 \Box_s \dots \Box_s \rho_u^N(-\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N(X)). \tag{19}$$

Thus from the inequality (18) we obtain

$$\begin{aligned}
&\overline{\rho}_t\left(-\rho_u^1(X^1+Y^1), \dots, -\rho_u^N(X^N+Y^N)\right) \geq \rho_t^1 \Box_s \dots \Box_s \rho_t^N \left(-\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N(X)\right) \\
&= \rho_t^1 \Box_s \dots \Box_s \rho_t^N \left(-\rho_u^1 \Box_s \dots \Box_s \rho_u^N(-\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N(X))\right) \\
&= \rho_t^1 \Box_s \dots \Box_s \rho_t^N \left(-\rho_u^1 \Diamond_s \dots \Diamond_s \rho_u^N(X)\right) \\
&= \rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N(X) = \overline{\rho}_t(X),
\end{aligned}$$

where we applied (19) in the first equality, the time consistency of the $\Box_s$ operator in the second equality (see Proposition 3.2 item 2) and (15) in the third equality. □

**Remark 3.8.** Suppose that $\rho^i$ are cash additive and $\rho^i(0) = 0$ for all $i$. Then any aggregator $\overline{\rho}$ satisfying the condition (17) is also compatible. To check the compatibility condition (equation (8)), just take $u = T$ in (17) and recall from (7) that $\rho_T^i(X^i) = -X^i$. Moreover, any cash additive aggregator $\overline{\rho}$ satisfying the condition (17) and $\overline{\rho}(0) = 0$ is also conservative. Indeed, take $Y = 0$ and $u = t$ in (17) and apply the cash additivity of $\overline{\rho}$.

3.1. **The dual representation of $\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N$.** As the title of this Section subtly suggest, we investigate how to specialize the conditional Fenchel-Moreau type dual representation of conditional (systemic) risk measures to the functional we are studying. This essentially revolves around two key points. On the one hand, we want to provide sufficient conditions to ensure that such a dual representation can actually be achieved. One expects intuitively that this involves some lower-semicontinuity type requirements. On the other hand, once a general duality result can be applied, we look for the specific shape taken by the penalty function and the set of (probability) measures appearing in the duality. **Throughout this Section we use the notation introduced in [18] Section 2 and 3. We assume that $(L(\mathcal{F}))^N \subseteq (L^1(\mathcal{F}))^N$ is $\mathcal{F}_t$-decomposable and that $L^* \subseteq (L^1(\mathcal{F}))^N$ is a vector space such that $\sum_j X^j Z^j \in L^1(\mathcal{F})$ whenever $X \in (L(\mathcal{F}))^N, Z \in L^*$.** We recall that a convex, monotone decreasing, cash additive, real valued map (i.e. a convex systemic risk measure) $\rho_0 : (L(\mathcal{F}))^N \to \mathbb{R}$ is nicely representable (with respect to the $\sigma(L(\mathcal{F}), L^*)$ topology) if

$$\rho_0(X) = \max_{Q \in \mathcal{Q}} \left( \sum_{i=1}^N \mathbb{E}_{Q_i}[-X^i] - \alpha_0(Q) \right), \quad X \in (L(\mathcal{F}))^N, \tag{20}$$

where for $Q = (Q^1, \dots, Q^N)$

$$\alpha_0(Q) := \rho_0^*\left(-\frac{\mathrm{d}Q}{\mathrm{d}P}\right) = \sup_{X \in (L(\mathcal{F}_T))^N} \left( \sum_{i=1}^N \mathbb{E}_{Q_i}[-X^i] - \rho_0(X) \right), \quad Q \in \mathcal{Q}, \tag{21}$$

$$\mathcal{Q} := \left\{ Q = (Q_1, \dots, Q_N) \ll P \;\middle|\; \frac{\mathrm{d}Q}{\mathrm{d}P} \in L^* \right\},$$

and $\rho_0^*$ is the convex conjugate of $\rho_0$. Sufficient conditions for nice representability involve continuity conditions (order upper semicontinuity or continuity from below) and structural properties of the vector spaces $L$ and $L^*$. These are, at least in the one dimensional case, well known and studied, see for example [18] Remark 2.3. With the additional assumptions stated in Corollary 3.10, the nice representability of the real valued map $\mathbb{E}\left[\rho_t^1\right]\Diamond_s \dots \Diamond_s \mathbb{E}\left[\rho_t^N\right]$ will be a sufficient condition for the dual representation of the conditional map $\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N$.

**Proposition 3.9.** *Suppose that $\rho^i$ is monotone decreasing, cash additive and convex on $[0,T]$, and that $\rho^i(0)=0$ for every $i=1,\dots,N$. Then*

$$\begin{aligned}&\mathbb{E}\left[\rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N(X^1,\dots,X^N)\right]\\ &=\mathbb{E}\left[\rho_t^1\right]\Diamond_s \dots \Diamond_s \mathbb{E}\left[\rho_t^N\right](X^1,\dots,X^N) \quad \forall X\in (L(\mathcal{F}))^N.\end{aligned} \tag{22}$$

*Proof.* We show the result for $N=2$. We start with a simple observation: Fix $X_1, X_2 \in L(\mathcal{F})$. Then for every $t\le s$ the set

$$\Theta_t^s := \left\{\rho_t^1(X^1+W^1)+\rho_t^2(X^2+W^2) \mid W^i\in L(\mathcal{F}_s),\ i=1,2,\ \text{ s.t. } W^1+W^2=0\right\}$$

is downward directed, that is whenever $Z_1,Z_2\in\Theta_t^s$ there exists an element $Z\in\Theta_t^s$ such that $\min(Z_1,Z_2)\ge Z$.

It is enough to show that $\forall\, Z_1,Z_2\in\Theta_t^s$ and $\forall\, A\in\mathcal{F}_t$ there exists $Z\in\Theta_t^s$ such that

$$Z_1 1_A + Z_2 1_{A^c} \ge Z. \tag{23}$$

Indeed, choosing $A:=\{Z_1\le Z_2\}\in\mathcal{F}_t$, (23) yields the desired property.

We have that $Z_i=\rho_t^1(X^1+W_i^1)+\rho_t^2(X^2+W_i^2), i=1,2$ where $W_1,W_2\in\mathcal{Y}(s)$. Now we compute $Z_1 1_A+Z_2 1_{A^c}$, for $A\in\mathcal{F}_t$, and apply the convexity property:

$$\begin{aligned}&Z_1 1_A + Z_2 1_{A^c}\\ &=[\rho_t^1(X^1+W_1^1)+\rho_t^2(X^2+W_1^2)]1_A+[\rho_t^1(X^1+W_2^1)+\rho_t^2(X^2+W_2^2)]1_{A^c}\\ &=[\rho_t^1(X^1+W_1^1)1_A+\rho_t^1(X^1+W_2^1)1_{A^c}]+[\rho_t^2(X^2+W_1^2)1_A+\rho_t^2(X^2+W_2^2)1_{A^c}]\\ &\ge \rho_t^1\Big(X^1+\big(W_1^1 1_A+W_2^1 1_{A^c}\big)\Big)+\rho_t^2\Big(X^2+\big(W_1^2 1_A+W_2^2 1_{A^c}\big)\Big)\\ &=\rho_t^1(X^1+W^1)+\rho_t^2(X^2+W^2)\in\Theta_t^s,\end{aligned}$$

where we defined $W:=W_1 1_A+W_2 1_{A^c}\in\mathcal{Y}(s)$.

We now prove (22). Observe that the claim is trivial if $t>s$ by Proposition 3.4 item 3. For $t\le s$, since $\Theta_t^s\neq\emptyset$ we have $\max(\rho_t^1\Diamond_s\rho_t^2(X^1,X^2),0)\in L^1$. It is also immediate to verify that $\mathbb{E}\left[\rho_t^1\Diamond_s\rho_t^2(X^1,X^2)\right]\le\mathbb{E}\left[\rho_t^1\right]\Diamond_s\mathbb{E}\left[\rho_t^2\right](X^1,X^2)$ since whenever $W\in\mathcal{Y}(s)$, $\mathbb{E}\left[\rho_t^1\Diamond_s\rho_t^2(X^1,X^2)\right]\le\mathbb{E}\left[\rho_t^1(X^1+W^1)\right]+\mathbb{E}\left[\rho_t^2(X^2+W^2)\right]$. At the same time by downward directedness $\rho_t^1\Diamond_s\rho_t^2(X^1,X^2)=\inf_n Z_n$ for a (a.s. monotone decreasing) sequence $(Z_n)_n\subseteq\Theta_t^s$. Observe that, since one sees $\mathbb{E}\left[\rho_t^1\right]\Diamond_s\mathbb{E}\left[\rho_t^2\right]$ $(X^1,X^2)=\inf_{Z\in\Theta_t^s}\mathbb{E}\left[Z\right]$ by definition, it holds that

$$\mathbb{E}\left[Z_n\right]\ge\mathbb{E}\left[\rho_t^1\right]\Diamond_s\mathbb{E}\left[\rho_t^2\right](X^1,X^2).$$

By monotone convergence we have

$$\mathbb{E}\left[\rho_t^1\Diamond_s\rho_t^2(X^1,X^2)\right]=\lim_n\mathbb{E}\left[Z_n\right]\ge\mathbb{E}\left[\rho_t^1\right]\Diamond_s\mathbb{E}\left[\rho_t^2\right](X^1,X^2)$$

which concludes the proof. □

**Corollary 3.10.** *Take $L\subseteq L^\infty$. Suppose that $\rho_t^1\Diamond_s\dots\Diamond_s\rho_t^N(0,\dots,0)=0$, that $\rho^i(0)=0$, and that on $[0,T]$ $\rho^i$ is cash additive, monotone decreasing, convex for* *each $i=1,\dots,N$. (i) If $\mathbb{E}\left[\rho_t^1\right]\Diamond_s\dots\Diamond_s\mathbb{E}\left[\rho_t^N\right]$ is $(L(\mathcal{F}),L^*)$-nicely representable, then*

$$\begin{aligned}&\rho_t^1\Diamond_s\dots\Diamond_s\rho_t^N(X_1,\dots,X^N)\\&=\operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t^s}\left(\sum_{i=1}^N\mathbb{E}_{Q^i}\left[-X^i\middle|\mathcal{F}_t\right]-\sum_{i=1}^N\alpha_i(Q^i)\right)\ \ \forall X\in(L(\mathcal{F}))^N\,,\end{aligned}\tag{24}$$

*for*

$$\mathcal{Q}_t^s:=\left\{Q=(Q^1,\dots,Q^N)\ll P\ \middle|\ \begin{array}{l}\frac{\mathrm{d}Q}{\mathrm{d}P}\in L^*,\ \mathbb{E}\left[\frac{\mathrm{d}Q^i}{\mathrm{d}P}\middle|\mathcal{F}_t\right]=1\ \textit{and}\\ \mathbb{E}\left[\frac{\mathrm{d}Q^i}{\mathrm{d}P}\middle|\mathcal{F}_s\right]=\mathbb{E}\left[\frac{\mathrm{d}Q^j}{\mathrm{d}P}\middle|\mathcal{F}_s\right]\ \textit{for all}\ i,j=1,\dots,N\end{array}\right\}\tag{25}$$

$$\alpha_t^i(Q^i):=\operatorname*{ess\,sup}\left\{\mathbb{E}_{Q^i}\left[-X^i\middle|\mathcal{F}_t\right]\mid X^i\in L(\mathcal{F}_T),\ \rho_t^i(X^i)\le 0\right\}\quad i=1,\dots,N\,.\tag{26}$$

*Moreover the essential supremum in (24) is attained. (ii) Under the Assumption 3.5, the functional $\mathbb{E}\left[\rho_t^1\right]\Diamond_s\dots\Diamond_s\mathbb{E}\left[\rho_t^N\right]$ is $(L^\infty(\mathcal{F}),L^1(\mathcal{F}))$-nicely representable.*

*Proof.* As usual we provide the proof for $N=2$. (i): note that $\rho_t^1\Diamond_s\rho_t^2(X^1,X^2)\in L^\infty$ for every $X^1,X^2\in L(\mathcal{F})$ since $\rho_t^1\Diamond_s\rho_t^2(0,0)=0$, $\rho_t^1\Diamond_s\rho_t^2$ is monotone and cash additive. One can apply [18] Theorem 3.9. This yields

$$\rho_t^1\Diamond_s\rho_t^2(X_1,X^2)=\operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t}\left(\sum_{j=1}^2\mathbb{E}_{Q^j}\left[-X^j\middle|\mathcal{F}_t\right]-\alpha(Q)\right)\ \ \forall X_1,X_2\in L(\mathcal{F}_T)$$

for

$$\alpha(Q):=\operatorname*{ess\,sup}\left\{\sum_{j=1}^2\mathbb{E}_{Q^j}\left[-X^j\middle|\mathcal{F}_t\right]\mid X_1,X_2\in L(\mathcal{F}_T),\rho_t^1\Diamond_s\rho_t^2(X_1,X^2)\le 0\right\}$$

$$\mathcal{Q}_t:=\left\{Q=(Q^1,Q^2)\ll P\mid\ \frac{\mathrm{d}Q}{\mathrm{d}P}\in L^*,\ \mathbb{E}\left[\frac{\mathrm{d}Q^j}{\mathrm{d}P}\middle|\mathcal{F}_t\right]=1\,\forall\,j=1,2\right\}\tag{27}$$

and the attainment in the dual representation. When $t>s$, $\mathcal{Q}_t=\mathcal{Q}_t^s$. When $t\le s$, whenever $\mathbb{E}\left[\frac{\mathrm{d}Q^j}{\mathrm{d}P}\middle|\mathcal{F}_t\right]=1\,\forall\,j=1,2$, we have

$$\begin{aligned}&\operatorname*{ess\,sup}\left\{\sum_{j=1}^2\mathbb{E}_{Q^j}\left[-X^j\middle|\mathcal{F}_t\right]\mid X_1,X_2\in L(\mathcal{F}_T),\rho_t^1\Diamond_s\rho_t^2(X_1,X^2)\le 0\right\}\\&\ge\operatorname*{ess\,sup}_{\lambda\in\mathbb{R}}\left(\mathbb{E}_{Q^1}\left[\lambda 1_A|\mathcal{F}_t\right]+\mathbb{E}_{Q^2}\left[-\lambda 1_A|\mathcal{F}_t\right]\right)\\&=\operatorname*{ess\,sup}_{\lambda\in\mathbb{R}}\mathbb{E}\left[\lambda 1_A\frac{\mathrm{d}Q^1}{\mathrm{d}P}\middle|\mathcal{F}_t\right]+\mathbb{E}\left[-\lambda 1_A\frac{\mathrm{d}Q^2}{\mathrm{d}P}\middle|\mathcal{F}_t\right]\\&=\operatorname*{ess\,sup}_{\lambda\in\mathbb{R}}\mathbb{E}\left[\lambda 1_A\left(\mathbb{E}\left[\frac{\mathrm{d}Q^1}{\mathrm{d}P}\middle|\mathcal{F}_s\right]-\mathbb{E}\left[\frac{\mathrm{d}Q^2}{\mathrm{d}P}\middle|\mathcal{F}_s\right]\right)\middle|\mathcal{F}_t\right]\end{aligned}$$

for every $A\in\mathcal{F}_s$ since for such an event $A$, $\rho_t^1\Diamond_s\rho_t^2(\lambda 1_A,-\lambda 1_A)=\rho_t^1\Diamond_s\rho_t^2(0,0)=0$. Then, $\alpha(Q)<+\infty$ a.s. implies $\mathbb{E}\left[\frac{\mathrm{d}Q^1}{\mathrm{d}P}\middle|\mathcal{F}_s\right]=\mathbb{E}\left[\frac{\mathrm{d}Q^2}{\mathrm{d}P}\middle|\mathcal{F}_s\right]$, so that again $\mathcal{Q}_t$ can be replaced with $\mathcal{Q}_t^s$. Finally we show that for $Q \in \mathcal{Q}_t^s$ we have $\alpha(Q) = \alpha_1(Q^1) + \alpha_2(Q^2)$. By inspection of the proof of [18] Theorem 3.9 (Step 5) we have that

$$\begin{aligned}
&\alpha(Q) \\
&= \operatorname*{ess\,sup}_{X^1,X^2 \in L(\mathcal{F})} \left( \sum_{j=1}^{2} \mathbb{E}_{Q^j}\left[-X^j \middle| \mathcal{F}_t\right] - \rho_t^1 \Diamond_s \rho_t^2(X_1, X^2) \right) \\
&= \operatorname*{ess\,sup}_{X^1,X^2 \in L(\mathcal{F})} \operatorname*{ess\,sup}_{Y \in \mathcal{Y}(s)} \left( \sum_{j=1}^{2} \mathbb{E}_{Q^j}\left[-X^j \middle| \mathcal{F}_t\right] - \rho_t^1(X^1+Y^1) - \rho_t^2(X^2+Y^2) \right) \\
&= \operatorname*{ess\,sup}_{X^1,X^2 \in L(\mathcal{F})} \operatorname*{ess\,sup}_{Y \in \mathcal{Y}(s)} \left( \sum_{j=1}^{2} \mathbb{E}_{Q^j}\left[-(X^j+Y^j) \middle| \mathcal{F}_t\right] - (\rho_t^1(X^1+Y^1) + \rho_t^2(X^2+Y^2)) \right) \qquad (28) \\
&= \operatorname*{ess\,sup}_{Z^1,Z^2 \in L(\mathcal{F})} \left( \sum_{j=1}^{2} \mathbb{E}_{Q^j}\left[-Z^j \middle| \mathcal{F}_t\right] - (\rho_t^1(Z^1) + \rho_t^2(Z^2)) \right) \\
&= \operatorname*{ess\,sup}_{Z^1 \in L(\mathcal{F})} \left( \mathbb{E}_{Q^1}\left[-Z^1 \middle| \mathcal{F}_t\right] - \rho_t^1(Z^1) \right) + \operatorname*{ess\,sup}_{Z^2 \in L(\mathcal{F})} \left( \mathbb{E}_{Q^2}\left[-Z^2 \middle| \mathcal{F}_t\right] - \rho_t^2(Z^2) \right) \qquad (29) \\
&= \alpha_1(Q^1) + \alpha_2(Q^2) \qquad (30)
\end{aligned}$$

where in (28) we used the fact that $\sum_{j=1}^{2} \mathbb{E}_{Q^j}\left[Y^j \middle| \mathcal{F}_t\right] = 0$ whenever $Q \in \mathcal{Q}_t^s$.

We come to item (ii). Since $\mathbb{E}\left[\rho_t^1\right] \Diamond_s \mathbb{E}\left[\rho_t^2\right](0) = 0$, the functional $\mathbb{E}\left[\rho_t^1\right] \Diamond_s \mathbb{E}\left[\rho_t^2\right]$ is cash additive, monotone, hence finite valued and norm continuous on $(L^\infty(\mathcal{F}))^2$. By the Fenchel-Moreau theorem it can be represented by standard arguments (see [26] Remark 4.18) as

$$\begin{aligned}
&\mathbb{E}\left[\rho_t^1\right] \Diamond_s \mathbb{E}\left[\rho_t^2\right](X) \\
&= \max_{\substack{(\xi_1,\xi_2) \in ((L^\infty(\mathcal{F}))^*)^2 \\ \xi^i \geq 0, \xi^i(1)=1, i=1,2}} \left\{ -\xi_1(X^1) - \xi_2(X^2) - (\mathbb{E}\left[\rho_t^1\right] \Diamond_s \mathbb{E}\left[\rho_t^2\right])^*(\xi_1,\xi_2) \right\} \qquad (31)
\end{aligned}$$

where $(\mathbb{E}\left[\rho_t^1\right] \Diamond_s \mathbb{E}\left[\rho_t^2\right])^*$ denotes the usual convex conjugate. A computation identical to the one in (30) shows that

$$(\mathbb{E}\left[\rho_t^1\right] \Diamond_s \mathbb{E}\left[\rho_t^2\right])^*(\xi^1,\xi^2) = (\mathbb{E}\left[\rho_t^1\right])^*(\xi^1) + (\mathbb{E}\left[\rho_t^2\right])^*(\xi^2) + \sup_{Y \in \mathcal{Y}} \left( -\xi^1(Y^1) - \xi^2(Y^2) \right)$$

for every $(\xi_1,\xi_2) \in ((L^\infty(\mathcal{F}))^*)^2$ with $\xi^i \geq 0, \xi^i(1) = 1, i = 1,2$. Since both $\mathbb{E}\left[\rho_t^1\right]$ or $\mathbb{E}\left[\rho_t^2\right]$, are continuous from below, by [26] Theorem 4.22 we have that $(\mathbb{E}\left[\rho_t^i\right])^*(\xi^i) < +\infty$ implies that $\xi^i \in L^1(\mathcal{F})$ with a self explaining abuse of notation. Hence, by adding (with no effect on the value) the additional constraint $(\mathbb{E}\left[\rho_t^1\right] \Diamond_s \mathbb{E}\left[\rho_t^2\right])^*(\xi_1,\xi_2) < +\infty$ in the maximum in RHS of (31) the $(L^\infty(\mathcal{F}), L^1(\mathcal{F}))$-nice representability follows. □

3.2. **Time consistency, acceptance sets and penalty functions.** In this Section we investigate the impact of time $\mathcal{Y}$-consistency on acceptance sets and penalty functions. For classical (non collective) dynamic risk measures we recall (see [14] and [8]) that time consistency is equivalent to a cocycle property for the penalty functions and to a decomposition property for the acceptance sets.

Set

$$\overline{\mathcal{A}}_t = \{X \in (L(\mathcal{F}_T))^N \mid \overline{\rho}_t(X) \leq 0\} \tag{32}$$

$$\overline{\mathcal{A}}_{t,u} = \{X \in (L(\mathcal{F}_u))^N \mid \overline{\rho}_t(X) \leq 0\} \tag{33}$$

the acceptance sets of $\overline{\rho}_t$ and similarly define $\mathcal{A}_t^i \subseteq L(\mathcal{F}_T), \mathcal{A}_{t,u}^i \subseteq L(\mathcal{F}_u)$ taking $\rho_t^i$ in place of $\overline{\rho}_t$.

Set also, for $Q = (Q^1, \dots, Q^N) \in \mathcal{Q}_t$ (see (27))

$$\overline{\alpha}_t(Q) = \operatorname*{ess\,sup}_{\overline{X}_t \in \overline{\mathcal{A}}_t} \sum_{i=1}^N \mathbb{E}_{Q^i}[-\overline{X}_t^i \mid \mathcal{F}_t] \tag{34}$$

$$\overline{\alpha}_{t,u}(Q) = \operatorname*{ess\,sup}_{\overline{X}_{t,u} \in \overline{\mathcal{A}}_{t,u}} \sum_{i=1}^N \mathbb{E}_{Q^i}[-\overline{X}_{t,u}^i \mid \mathcal{F}_t] \tag{35}$$

$$\sigma_t^{\mathcal{Y}}(Q) = \operatorname*{ess\,sup}_{Y \in \mathcal{Y}} \sum_{i=1}^N \mathbb{E}_{Q^i}[-Y^i \mid \mathcal{F}_t]. \tag{36}$$

**Proposition 3.11.** *Let Assumption 3.5 hold and assume that $\rho^i(0) = 0, i = 1, \dots, N$. Suppose that $L^* = L^1(\mathcal{F})$ and that for every $0 \leq t \leq T$ and $\mathcal{Q}_t$ defined in (27) it holds*

$$\overline{\rho}_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \left( \sum_{i=1}^N \mathbb{E}_{Q^i}\left[-X^i \middle| \mathcal{F}_t\right] - \overline{\alpha}_t(Q) \right) \ \forall X \in (L(\mathcal{F}))^N. \tag{37}$$

*The following are then equivalent:*

1. *$\overline{\rho}_t$ is time $\mathcal{Y}$-consistent and satisfies (17).*
2. *$\overline{\mathcal{A}}_t = \overline{\mathcal{A}}_{t,u} + \left(\prod_{i=1}^N \mathcal{A}_u^i + \mathcal{Y}\right)$ for every $0 \leq t \leq u \leq T$.*
3. *$\overline{\alpha}_t(Q) = \overline{\alpha}_{t,u}(Q) + \sum_{i=1}^N \mathbb{E}_{Q^i}\left[\alpha_u^i(Q^i) \mid \mathcal{F}_t\right] + \sigma_t^{\mathcal{Y}}(Q)$ for any $t \leq u$ and $Q \in \mathcal{Q}_t$.*
4. *$\overline{\rho}_t(X) = \rho_t^1 \Diamond_s \dots \Diamond_s \rho_t^N(X)$ for any $X \in (L(\mathcal{F}))^N$.*

We emphasize that equivalence among 1.-3. in Proposition 3.11 can be seen as a counterpart to [8] Theorem 2.5 (see also [26] Proposition 11.15) in a collective setting.

*Proof.* For notation simplicity we consider the case $N = 2$. The general case, however, can be driven similarly. We write $(X^i)_i$ for $X = (X^1, X^2) \in L(\mathcal{F}_T) \times L(\mathcal{F}_T)$ and $(\rho_t^i(X^i))_i$ for $((\rho_t^1(X^1), \rho_t^2(X^2)) \in L(\mathcal{F}_t) \times L(\mathcal{F}_t)$.
*1. $\iff$ 2.*

**Step 1:** We first show that

$$(X^i)_i \in \overline{\mathcal{A}}_{t,u} + \left(\mathcal{A}_u^1 \times \mathcal{A}_u^2\right) \iff -(\rho_u^i(X^i))_i \in \overline{\mathcal{A}}_{t,u}. \tag{38}$$

Indeed to see ($\Rightarrow$) take $X = \overline{X}_{t,u} + X_u$ with obvious notation. Then

$$\left(\rho_u^i(X^i)\right)_i = \left(\rho_u^i(\overline{X}_{t,u}^i + X_u^i)\right)_i = \left(\rho_u^i(X_u^i)\right)_i - \overline{X}_{t,u} \leq -\overline{X}_{t,u}$$

so that by monotonicity $\overline{\rho}_t(-\left(\rho_u^i(X^i)\right)_i) \leq \overline{\rho}_t(\overline{X}_{t,u}) \leq 0$, that is $-\left(\rho_u^i(X^i)\right)_i \in \overline{\mathcal{A}}_{t,u}$. Conversely to see ($\Leftarrow$) we note that $X = X + (\rho_u^i(X^i))_i - (\rho_u^i(X^i))_i$ and $X + (\rho_u^i(X^i))_i \in \mathcal{A}_u^1 \times \mathcal{A}_u^2$ by definition, while $-(\rho_u^i(X^i))_i \in \overline{\mathcal{A}}_{t,u}$ by assumption.

**Step 2:** We show that

$$\overline{\mathcal{A}}_t \subseteq \overline{\mathcal{A}}_{t,u} + \left(\mathcal{A}_u^1 \times \mathcal{A}_u^2 + \mathcal{Y}\right) \iff \overline{\rho}_t(X^1, X^2) \geq \overline{\rho}_t(-(\rho_u^i(X^i + Y^i))_i) \text{ for some } Y \in \mathcal{Y}.$$

To see ($\Rightarrow$), take any $\widehat{X} \in L(\mathcal{F}_t) \times L(\mathcal{F}_t)$ s.t. $\sum_i \widehat{X}^i = \overline{\rho}_t(X^1, X^2)$ so that $X + \widehat{X} \in \overline{\mathcal{A}}_t$. Hence, for some $Y \in \mathcal{Y}$ it must hold that $X + \widehat{X} + Y \in \overline{\mathcal{A}}_{t,u} + \mathcal{A}^1_u \times \mathcal{A}^2_u$. By (38)

$$-(\rho^i_u(X^i + Y^i))_i + \widehat{X} = -(\rho^i_u(X^i + \widehat{X}^i + Y^i))_i \in \overline{\mathcal{A}}_{t,u}.$$

Consequently

$$\overline{\rho}_t\Big( - (\rho^i_u(X^i + Y^i))_i\Big) - \sum_i \widehat{X}^i = \overline{\rho}_t\Big( - (\rho^i_u(X^i + Y^i))_i + \widehat{X}\Big) \leq 0$$

which yields the desired inequality. The converse ($\Leftarrow$) is seen taking $X \in \overline{\mathcal{A}}_t$ and observing that for some $Y \in \mathcal{Y}$, $0 \geq \overline{\rho}_t(X) \geq \overline{\rho}_t(-(\rho^i_u(X^i + Y^i))_i)$. Hence, $-(\rho^i_u(X^i+Y^i))_i \in \overline{\mathcal{A}}_{t,u}$. Now, using (38) $X+Y \in \overline{\mathcal{A}}_{t,u} + \mathcal{A}^1_u \times \mathcal{A}^2_u$ and easily (since $\mathcal{Y}$ is a vector space) $X \in \overline{\mathcal{A}}_{t,u} + \big(\mathcal{A}^1_u \times \mathcal{A}^2_u + \mathcal{Y}\big)$.

**Step 3:** We show that

$$\overline{\mathcal{A}}_t \supseteq \overline{\mathcal{A}}_{t,u} + \big(\mathcal{A}^1_u \times \mathcal{A}^2_u + \mathcal{Y}\big) \iff \overline{\rho}_t(X^1, X^2) \leq \overline{\rho}_t(-(\rho^i_u(X^i+Y^i))_i) \text{ for every } Y \in \mathcal{Y}.$$

We start with ($\Rightarrow$). Take any $Y \in \mathcal{Y}$ and $\widehat{X} \in L(\mathcal{F}_t) \times L(\mathcal{F}_t)$ s.t. $\sum_i \widehat{X}^i = \overline{\rho}_t(-(\rho^i_u(X^i + Y^i))_i)$. Then

$$X + \widehat{X} = \Big(\widehat{X} - (\rho^i_u(X^i + Y^i))_i\Big) + \Big(X + Y + (\rho^i_u(X^i + Y^i))_i\Big) + (-Y).$$

By definition of $\widehat{X}$, $\widehat{X} - (\rho^i_u(X^i + Y^i))_i \in \overline{\mathcal{A}}_{t,u}$, while by cash additivity $X + Y + (\rho^i_u(X^i + Y^i))_i \in \mathcal{A}^1_u \times \mathcal{A}^2_u$, and trivially $-Y \in \mathcal{Y}$. Thus, $X + \widehat{X} \in \overline{\mathcal{A}}_{t,u} + \big(\mathcal{A}^1_u \times \mathcal{A}^2_u + \mathcal{Y}\big)$. By assumption then

$$\overline{\rho}_t(X + \widehat{X}) = \overline{\rho}_t(X) - \overline{\rho}_t(-(\rho^i_u(X^i + Y^i))_i) \leq 0$$

which yields the desired inequality. Conversely to get ($\Leftarrow$), observe that for $X \in \overline{\mathcal{A}}_{t,u} + \big(\mathcal{A}^1_u \times \mathcal{A}^2_u + \mathcal{Y}\big)$ we have $X + \widehat{Y} \in \overline{\mathcal{A}}_{t,u} + \mathcal{A}^1_u \times \mathcal{A}^2_u$ for some $\widehat{Y} \in \mathcal{Y}$. Then using (38) $-(\rho^i_u(X^i + \widehat{Y}^i))_i \in \overline{\mathcal{A}}_{t,u}$. We conclude by assumption that $\overline{\rho}_t(X) \leq \overline{\rho}_t(-(\rho^i_u(X^i + \widehat{Y}^i))_i) \leq 0$, providing $X \in \overline{\mathcal{A}}_t$.

*2.* $\Longrightarrow$ *3.* By the decomposition of acceptance sets in 2., we note that any $X \in \overline{\mathcal{A}}_t$ can be decomposed as $X = \overline{X}_{t,u} + X_u + Y$ with $\overline{X}_{t,u} \in \overline{\mathcal{A}}_{t,u}$, $X_u \in \mathcal{A}^1_u \times \mathcal{A}^2_u$ and $Y \in \mathcal{Y}$. Then, for any $Q = (Q^1, Q^2) \in \mathcal{Q}_t$

$$\begin{aligned}
\overline{\alpha}_t(Q) &= \operatorname*{ess\,sup}_{X \in \overline{\mathcal{A}}_t} \sum_{i=1}^2 \mathbb{E}_{Q^i}[-X^i \mid \mathcal{F}_t] \\
&= \operatorname*{ess\,sup}_{\overline{X}_{t,u} \in \overline{\mathcal{A}}_{t,u}, X_u \in \mathcal{A}^1_u \times \mathcal{A}^2_u, Y \in \mathcal{Y}} \{\mathbb{E}_Q[-\overline{X}_{t,u} \mid \mathcal{F}_t] + \mathbb{E}_Q[-X_u \mid \mathcal{F}_t] + \mathbb{E}_Q[-Y \mid \mathcal{F}_t]\} \\
&= \operatorname*{ess\,sup}_{\overline{X}_{t,u} \in \overline{\mathcal{A}}_{t,u}} \mathbb{E}_Q[-\overline{X}_{t,u} \mid \mathcal{F}_t] + \operatorname*{ess\,sup}_{X_u \in \mathcal{A}^1_u \times \mathcal{A}^2_u} \mathbb{E}_Q[-X_u \mid \mathcal{F}_t] + \operatorname*{ess\,sup}_{Y \in \mathcal{Y}} \mathbb{E}_Q[-Y \mid \mathcal{F}_t] \\
&= \operatorname*{ess\,sup}_{\overline{X}_{t,u} \in \overline{\mathcal{A}}_{t,u}} \sum_{i=1}^2 \mathbb{E}_{Q^i}[-\overline{X}^i_{t,u} \mid \mathcal{F}_t] \\
&\quad + \operatorname*{ess\,sup}_{X_u \in \mathcal{A}^1_u \times \mathcal{A}^2_u} \sum_{i=1}^2 \mathbb{E}_{Q^i}[-X^i_u \mid \mathcal{F}_t] + \operatorname*{ess\,sup}_{Y \in \mathcal{Y}} \sum_{i=1}^2 \mathbb{E}_{Q^i}[-Y^i \mid \mathcal{F}_t] \\
&= \overline{\alpha}_{t,u}(Q) + \operatorname*{ess\,sup}_{X_u \in \mathcal{A}^1_u \times \mathcal{A}^2_u} \sum_{i=1}^2 \mathbb{E}_{Q^i}[-X^i_u \mid \mathcal{F}_t] + \sigma^{\mathcal{Y}}_t(Q).
\end{aligned} \tag{39}$$

The second term in the last line becomes

$$\begin{aligned}
&\operatorname*{ess\,sup}_{X_u\in\mathcal{A}_u^1\times\mathcal{A}_u^2}\sum_{i=1}^2\mathbb{E}_{Q^i}[-X_u^i\mid\mathcal{F}_t]=\operatorname*{ess\,sup}_{(X_u^1,X_u^2)\in\mathcal{A}_u^1\times\mathcal{A}_u^2}\sum_{i=1}^2\mathbb{E}_{Q^i}[\mathbb{E}_{Q^i}[-X_u^i\mid\mathcal{F}_u]\mid\mathcal{F}_t]\\
&=\operatorname*{ess\,sup}_{X_u^1\in\mathcal{A}_u^1}\mathbb{E}_{Q^1}[\mathbb{E}_{Q^1}[-X_u^1\mid\mathcal{F}_u]\mid\mathcal{F}_t]+\operatorname*{ess\,sup}_{X_u^2\in\mathcal{A}_u^2}\mathbb{E}_{Q^2}[\mathbb{E}_{Q^2}[-X_u^2\mid\mathcal{F}_u]\mid\mathcal{F}_t]\\
&=\mathbb{E}_{Q^1}\left[\operatorname*{ess\,sup}_{X_u^1\in\mathcal{A}_u^1}\mathbb{E}_{Q^1}[-X_u^1\mid\mathcal{F}_u]\mid\mathcal{F}_t\right]+\mathbb{E}_{Q^2}\left[\operatorname*{ess\,sup}_{X_u^2\in\mathcal{A}_u^2}\mathbb{E}_{Q^2}[-X_u^2\mid\mathcal{F}_u]\mid\mathcal{F}_t\right] \qquad (40)\\
&=\mathbb{E}_{Q^1}\left[\alpha_u^1(Q^1)\mid\mathcal{F}_t\right]+\mathbb{E}_{Q^2}\left[\alpha_u^2(Q^2)\mid\mathcal{F}_t\right], \qquad (41)
\end{aligned}$$

where (40) follows from the same arguments as in [16] Theorem 1, while the last equality is due to cash additivity of $\rho_t^i$ that guarantees that $\alpha_u^i(Q^i)=\operatorname{ess\,sup}_{X_u^i\in\mathcal{A}_u^i}\mathbb{E}_{Q^i}[-X_u^i\mid\mathcal{F}_u]=\operatorname{ess\,sup}_{X_u^i\in\mathcal{A}_u^i}\{\mathbb{E}_{Q^i}[-X_u^i\mid\mathcal{F}_u]-\rho_u^i(X^i)\}$ (see [26]). Item 3. then follows from (39).

*3.* $\Longrightarrow$ *4.* Taking item 3. with $t=u$, we get for $Q=(Q^1,Q^2)\in\mathcal{Q}_t$

$$\begin{aligned}
\overline{\alpha}_t(Q^1,Q^2)&=\overline{\alpha}_{t,t}(Q^1,Q^2)+\sum_{i=1}^2\mathbb{E}_{Q^i}\left[\alpha_t^i(Q^i)\mid\mathcal{F}_t\right]+\sigma_t^{\mathcal{Y}}(Q^1,Q^2)\\
&=\sum_{i=1}^2\alpha_t^i(Q^i)+\sigma_t^{\mathcal{Y}}(Q^1,Q^2).
\end{aligned}$$

We can then substitute in (37). Then, one observes that the supremum in such a dual representation can be restricted to those $Q\in\mathcal{Q}_t$ such that $\overline{\alpha}_t(Q)<+\infty$ a.s., which in turns yields that the supremum can be restricted to $\mathcal{Q}_t^s$. Then, one recovers the dual representation (24). Noticing that we are working under Assumption 3.5, Corollary 3.10 ensures that $\mathbb{E}\left[\rho_t^1\right]\Diamond_s\mathbb{E}\left[\rho_t^2\right]$ is $(L^\infty(\mathcal{F}),L^1(\mathcal{F}))$-nicely representable. Hence, (24) holds and we conclude $\overline{\rho}_t(X^1,X^2)=\rho_t^1\Diamond_s\rho_t^2(X^1,X^2)$ for any $X^1,X^2\in L^\infty(\mathcal{F})$.

*4.* $\Longrightarrow$ *1.* It follows directly from Proposition 3.6 and Lemma 3.7. □

4. **On collective dynamic risk measures.** We consider $N$ agents in the market and the risky vector $X:=(X^1,\dots,X^N)\in(L(\mathcal{F}))^N$. We study the *collective* version of conditional and dynamic risk measures. Let $\mathcal{A}^i\subseteq L(\mathcal{F})$ be the acceptance set of agent $i$, that we assume to be a monotone set, namely it satisfies $X^i\geq Y^i\in\mathcal{A}^i$ and $X^i\in L(\mathcal{F})$ imply $X^i\in\mathcal{A}^i$. We will use the notation $\mathbb{A}:=\mathcal{A}^1\times\cdots\times\mathcal{A}^N$. We recall the following well known (see [16]) definition of monetary conditional risk measures induced by acceptance sets

**Definition 4.1.** For each $t\in[0,T]$, each $i$ and each $\mathcal{A}^i\subseteq L(\mathcal{F})$, the time-t conditional risk measure $\rho_{t,\mathcal{A}^i}:L(\mathcal{F})\to L^0(\mathcal{F}_t)$ is given by

$$\rho_{t,\mathcal{A}^i}(X^i):=\operatorname{ess\,inf}\left\{\alpha\in L(\mathcal{F}_t)\mid\alpha+X^i\in\mathcal{A}^i\right\}.$$

Thus $\rho_{\mathcal{A}^i}=(\rho_{t,\mathcal{A}^i})_{t\in[0,T]}$ defines a dynamic risk measure induced by acceptance sets.

As easily checked, for each $i$ the dynamic risk measure $\rho_{\mathcal{A}^i}$ is cash additive and when each $\mathcal{A}^i$ is a (monotone) convex set then $\rho_{\mathcal{A}^i}$ is convex and monotone decreasing (see Definition 2.1).

**Assumption 4.2.** *Take $L = L^\infty(\mathcal{F})$. Set*

$$\widetilde{\mathcal{A}}^i_t = \{X \in L(\mathcal{F}_t) \mid \rho_{t,\mathcal{A}^i}(X) \leq 0\} \tag{42}$$

$$\widetilde{\mathcal{A}}^i_{t,u} = \{X \in L(\mathcal{F}_u) \mid \rho_{t,\mathcal{A}^i}(X) \leq 0\}. \tag{43}$$

*We assume that $\rho_{t,\mathcal{A}^i}(0) = 0$ and $\widetilde{\mathcal{A}}^i_t = \widetilde{\mathcal{A}}^i_u + \widetilde{\mathcal{A}}^i_{t,u}$ for every $t \in [0,T]$ and $i = 1,\dots,N$.*

**Proposition 4.3.** *Under the Assumption 4.2, for each $i$ the dynamic risk measure $\rho_{\mathcal{A}^i} = (\rho_{t,\mathcal{A}^i})_{t\in[0,T]}$ is time consistent.*

*Proof.* See [26] Lemma 11.14. □

As explained in the introduction we aim to extend such concept to the collective framework, where $N$ agents may cooperate via exchanges that are modeled via vectors $Y = (Y^1,\dots,Y^N) \in \mathcal{Y} \subseteq (L(\mathcal{F}))^N$, where $\mathcal{Y}$ is an assigned convex cone of vectors of allowable exchanges.

**Definition 4.4.** For any $t \in [0,T]$, the time-t collective conditional risk measure given $\mathbb{A}$ and $\mathcal{Y}$, is defined on $(L(\mathcal{F}))^N$ as (the extended real-valued, $\mathcal{F}_t$-measurable random variable)

$$\rho_{t,\mathbb{A},\mathcal{Y}}(X^1,\dots,X^N)$$
$$:= \operatorname{ess\,inf}\left\{\sum_{i=1}^N \alpha^i \mid \alpha^i \in L(\mathcal{F}_t) \text{ and } \exists Y \in \mathcal{Y} \text{ s.t. } \alpha^i + Y^i + X^i \in \mathcal{A}^i \quad \forall i\right\}$$

and we call $\rho_{\mathbb{A},\mathcal{Y}} = (\rho_{t,\mathbb{A},\mathcal{Y}})_{t\in[0,T]}$ a collective dynamic risk measure.

**Example 4.5** (Collective Conditional Risk Measures in Financial Markets)**.** As described in the Introduction, consider $N$ agents investing in a frictionless stochastic security market that are allowed to cooperate through suitable exchanges $Y \in \mathcal{Y}$. Let $K_i$ be the market of agent $i$, that is the vector space of all the possible time-$T$ payoffs that agent $i$ can obtain by using admissible trading strategies in his/her allowed investments and having zero initial cost. In this example we take $L = L^\infty(\Omega,\mathcal{F},P)$. We now consider the problem of *hedging* simultaneously $N$ given claims $X = (X^1,\dots,X^N) \in (L(\mathcal{F}))^N$.

1. **Collective super-replication price from dynamic trading only**. Let $\mathcal{A}^i := (L^0_+ - K_i)$ be the acceptance the agent $i$. The collective conditional risk measures $\rho_{t,\mathbb{A},\mathcal{Y}}$ evaluated in $-X$ and defined by

$$\rho_{t,\mathbb{A},\mathcal{Y}}(-X)$$
$$= \operatorname{ess\,inf}\left\{\sum_{i=1}^N \alpha^i \mid \exists \alpha^i \in L(\mathcal{F}_t),\, \exists Y \in \mathcal{Y} \text{ s.t. } \alpha^i + Y^i - X^i \in \mathcal{A}^i \quad \forall i\right\}$$
$$= \operatorname{ess\,inf}\left\{\sum_{i=1}^N \alpha^i \mid \exists \alpha^i \in L(\mathcal{F}_t),\, \exists Y \in \mathcal{Y}, \exists k^i \in K_i \text{ s.t. } \alpha^i + k^i + Y^i \geq X^i \quad \forall i\right\}$$

is thus the conditional version of the static collective super-replication cost from dynamic trading defined in (2). As discussed in [6] for the static case, when super-replicating the $N$ claims $(X^1,\dots,X^N)$, such collective notion allows to save money with respect to the classical super-replication without cooperation, namely

$$\rho_{t,\mathbb{A},\mathcal{Y}}(-X) \leq \operatorname{ess\,inf}\left\{\sum_{i=1}^{N}\alpha^i \mid \exists \alpha^i \in L(\mathcal{F}_t),\, \exists k^i \in K_i \text{ s.t. } \alpha^i + k^i \geq X^i \quad \forall i\right\}$$

and the strict inequality holds in many cases.

2. **Collective super-replication price from static and dynamic trading**. For each $i$ fix a pricing measure $\widehat{Q}^i$, that could be inferred by agent $i$ by several means. Consider the set $L_{\widehat{Q}^i} := \{\phi \in L(\mathcal{F}) \mid \mathbb{E}_{\widehat{Q}^i}[\phi] = 0\}$ of time-T contingent claims having, without loss of generality, price zero under the pricing measure $\widehat{Q}^i$. Agents $i$ acceptance set is $\mathcal{A}^i = (L^0_+ - K_i - L_{\widehat{Q}^i})$. The collective conditional risk measures $\rho_{t,\mathbb{A},\mathcal{Y}}$ evaluated in $-X$ and defined by

$$\begin{aligned}
&\rho_{t,\mathbb{A},\mathcal{Y}}(-X)\\
&= \operatorname{ess\,inf}\left\{\sum_{i=1}^{N}\alpha^i \mid \exists \alpha^i \in L(\mathcal{F}_t),\, \exists Y \in \mathcal{Y} :\, \alpha^i + Y^i - X^i \in \mathcal{A}^i \quad \forall i\right\}\\
&= \operatorname{ess\,inf}\Bigg\{ \sum_{i=1}^{N}\mathbb{E}_{\widehat{Q}^i}[\phi^i] \mid \phi^i \in L(\mathcal{F}) :\, \exists \alpha^i \in L(\mathcal{F}_t), \exists Y \in \mathcal{Y},\\
&\qquad \exists k^i \in K_i \text{ s.t. } k^i + Y^i + \phi^i \geq X^i \quad \forall i\Bigg\}
\end{aligned}$$

thus represents the conditional collective super replication from static and dynamic hedging.

3. **Collective Indifference Price**
For each agent $i$ consider a concave monotone increasing utility function $u_i : \mathbb{R} \to \mathbb{R}$, normalized by $u_i(0) = 0$, and the indirect utility function (from zero initial wealth), defined by

$$U_i(X^i) := \sup_{k\in K_i} \mathbb{E}[u_i(k + X^i)] = \sup_{f\in(K_i - L^0_+)\cap L^\infty} \mathbb{E}[u_i(f + X^i)], \quad X^i \in L(\mathcal{F}).$$

Let $\mathcal{A}^i = \{f \in L(\mathcal{F}) \mid U_i(f) \geq U_i(0)\}$ be the acceptance set of agent $i$. The collective conditional risk measures $\rho_{t,\mathbb{A},\mathcal{Y}}$ evaluated in $-X$ and defined by

$$\begin{aligned}
&\rho_{t,\mathbb{A},\mathcal{Y}}(-X)\\
&= \operatorname{ess\,inf}\left\{\sum_{i=1}^{N}\alpha^i \mid \exists \alpha^i \in L(\mathcal{F}_t),\, \exists Y \in \mathcal{Y} \text{ s.t. } \alpha^i + Y^i - X^i \in \mathcal{A}^i \quad \forall i\right\}\\
&= \operatorname{ess\,inf}\left\{\sum_{i=1}^{N}\alpha^i \mid \exists \alpha^i \in L(\mathcal{F}_t), \exists Y \in \mathcal{Y} \text{ s.t. } U_i(\alpha^i + Y^i - X^i) \geq U_i(0) \quad \forall i\right\}.
\end{aligned}$$

thus represents the (seller) conditional collective Indifference Price.

Observe that we may write $\rho_{t,\mathbb{A},\mathcal{Y}}$ in the following way

$$\begin{aligned}
\rho_{t,\mathbb{A},\mathcal{Y}}(X) &:= \operatorname*{ess\,inf}_{\alpha\in(L(\mathcal{F}_t))^N}\left\{\sum_{i=1}^{N}\alpha^i \mid \exists Y \in \mathcal{Y} \text{ s.t. } \alpha^i + Y^i + X^i \in \mathcal{A}^i \quad \forall i\right\}\\
&= \operatorname*{ess\,inf}_{Y\in\mathcal{Y}}\left\{\operatorname*{ess\,inf}_{\alpha\in(L(\mathcal{F}_t))^N}\left\{\sum_{i=1}^{N}\alpha^i \mid \alpha^i + Y^i + X^i \in \mathcal{A}^i \quad \forall i\right\}\right\}
\end{aligned}$$

$$= \operatorname*{ess\,inf}_{Y\in\mathcal{Y}} \left\{ \sum_{i=1}^{N} \operatorname*{ess\,inf}_{\alpha^i\in L(\mathcal{F}_t)} \left\{ \alpha^i \mid \alpha^i + Y^i + X^i \in \mathcal{A}^i \right\} \right\}$$
$$= \operatorname*{ess\,inf}_{Y\in\mathcal{Y}} \left\{ \sum_{i=1}^{N} \rho_{t,\mathcal{A}^i}(X^i + Y^i) \right\} := \rho_{t,\mathcal{A}^1} \lozenge_{\mathcal{Y}} \dots \lozenge_{\mathcal{Y}} \rho_{t,\mathcal{A}^N}(X).$$

Thus if we set $\rho_t^i := \rho_{t,\mathcal{A}^i}$ we conclude that collective dynamic risk measures $\rho_{\mathbb{A},\mathcal{Y}} = (\rho_{t,\mathbb{A},\mathcal{Y}})_{t\in[0,T]}$ are particular examples of the $\lozenge_{\mathcal{Y}}$ aggregator:

$$\rho_{t,\mathbb{A},\mathcal{Y}}(X) = \rho_t^1 \lozenge_{\mathcal{Y}} \dots \lozenge_{\mathcal{Y}} \rho_t^N(X), \quad \forall t \in [0,T], \tag{44}$$

and thus the study of collective risk measures $\rho_{\mathbb{A},\mathcal{Y}}$ reduces to the analysis of the $\lozenge_{\mathcal{Y}}$ aggregator that was developed in the previous sections. In particular,

**Proposition 4.6.** *Suppose that $\mathcal{Y}$ is a convex cone and each $\mathcal{A}^i$ is a (monotone) convex set. Then the collective dynamic risk measure $\rho_{\mathbb{A},\mathcal{Y}}$ is cash additive, convex, monotone decreasing, conservative and compatible. If the assumptions 4.2 and 3.5 hold true, then the collective dynamic risk measure $\rho_{\mathbb{A},\mathcal{Y}(s)}$ is also $\mathcal{Y}(s)$-time consistent.*

4.1. **On the collective entropic dynamic risk measure.** Consider two agents and two risky positions $X = (X^1, X^2) \in (L^\infty(\mathcal{F}))^2$. We now apply the collective setting with exchange set $\mathcal{Y} = \mathcal{Y}(s)$ to the case of two conditional entropic risk measures $\rho_t^i(X^i) = \gamma_i \ln \mathbb{E}[\exp(-\frac{X^i}{\gamma^i}) \mid \mathcal{F}_t]$, associated to the acceptance set $\mathcal{A}^i \subseteq L^\infty(\mathcal{F})$ generated by the exponential utility function with risk tolerance parameter $\gamma_i > 0$, $i = 1, 2$.

By the time consistency of any conditional entropic risk measure, by Proposition 3.4 item 3 and by (44) we get, for $\rho_t^i := \rho_{t,\mathcal{A}^i}$,

$$\rho_{t,\mathbb{A},\mathcal{Y}(s)}(X^1, X^2) = \rho_t^1 \lozenge_s \rho_t^2(X^1, X^2) = \begin{cases} \rho_t^1 \square_s \rho_t^2(-[\rho_s^1(X^1) + \rho_s^2(X^2)]), & \text{if } t \le s \\ \rho_t^1(X^1) + \rho_t^2(X^2) & \text{if } t \ge s. \end{cases} \tag{45}$$

Given the choice $\mathcal{Y} = \mathcal{Y}(s)$ and since $[\rho_s^1(X^1) + \rho_s^2(X^2)]$ is the argument of the $s$-convolution $\square_s$ and it is $\mathcal{F}_s$-measurable, the expression for $t \le s$ can be seen as a *classical* inf-convolution of conditional risk measures with time horizon $s$. Hence by (45) and [4] Theorem 3.9 we obtain the explicit formula for the collective entropic conditional risk measure

$$\rho_t^1 \lozenge_s \rho_t^2(X^1, X^2) =$$
$$= \begin{cases} (\gamma_1+\gamma_2) \ln \mathbb{E}\left[ \exp\left( \frac{\gamma_1}{\gamma_1+\gamma_2} \ln \mathbb{E}[\exp(-\frac{X^1}{\gamma^1})|\mathcal{F}_s] + \frac{\gamma_2}{\gamma_1+\gamma_2} \ln \mathbb{E}[\exp(-\frac{X^2}{\gamma^2})|\mathcal{F}_s] \right) \middle| \mathcal{F}_t \right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } t \le s \\ \gamma_1 \ln \mathbb{E}[\exp(-\frac{X^1}{\gamma^1})|\mathcal{F}_t] + \gamma_2 \ln \mathbb{E}[\exp(-\frac{X^2}{\gamma^2})|\mathcal{F}_t] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } t \ge s \end{cases}.$$

The previous expression, for $t \le s$, enchains the initial entropic risk measures via a new entropic one with risk tolerance parameter $(\gamma_1 + \gamma_2)$, while it reduces to a sum of entropic risk measures for $t \ge s$.

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