Bounds on the growth of energy for particles on the torus with unbounded time dependent perturbations.

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Abstract

We prove a C^∞ version of Nekhoroshev theorem for time dependent Hamiltonians in $\mathbb{R}^d \times \mathbb{T}^d$. Precisely, we prove a result showing that for all times the energy of the system is bounded by a constant times $\langle t \rangle^\varepsilon$. We apply the result to the dynamics of a charged particle in $\mathbb{T}^d$ subject to a time dependent electromagnetic field.

Keywords: Nekhoroshev theorem, time dependent perturbations, growth of solutions, particles in an electromagnetic field.

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1 Introduction

In this paper we study the dynamics of a Hamiltonian system of the form

$$H = h_0(p) + P(p, x, t) \ , \quad h_0(p) := \sum_{j=1}^{d} \frac{p_j^2}{2} \ , \quad (p, x) \in \mathbb{R}^d \times \mathbb{T}^d \ , \tag{1.1}$$

where $P(p, x, t)$ is a C^∞ function bounded by $\langle p \rangle^{\mathtt{b}}$ with some $\mathtt{b} < 2$ (as usual $\langle p \rangle := \sqrt{1 + \|p\|^2}$). The main example we have in mind is that of a particle subject to a time dependent electromagnetic field, namely

$$H = \sum_{j=1}^{d} \frac{(p_j - A_j(x, t))^2}{2} + \varphi(x, t) \ , \tag{1.2}$$

with $A(x, t)$ and $\varphi(x, t)$ functions of class $C^\infty(\mathbb{T}^d \times \mathbb{R}; \mathbb{R})$.

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We are going to prove that $\forall \varepsilon > 0 \; \exists C_\varepsilon$, s.t. the solution of the system fulfils forever an estimate of the form

$$\|p(t)\| \leq C_\varepsilon \langle t \rangle^\varepsilon (1 + \|p_0\|) \ , \quad \forall t \in \mathbb{R} \tag{1.3}$$

$$\langle t \rangle := \sqrt{1 + |t|^2} \ ; \tag{1.4}$$

of course the estimate is meaningful for $0 < \varepsilon \ll 1$.

This is essentially a Nekhoroshev type theorem [1, 2] (see also [3, 4, 5, 6, 7, 8] and literature therein). We recall that the original Nekhoroshev theorem deals with perturbations of steep integrable systems and gives upper bounds on the drift of the actions for very long times. The original Nekhoroshev's theorem deals with an analytic context and the result it gives is valid over times which are exponentially long with the inverse of the size ϵ of the perturbation. Corresponding C^∞ versions have been obtained [9, 10, 11, 12, 13]: in the general case one gets bounds valid over a time scale of order ϵ^{-N} with an arbitrary N; specifying the class of C^∞ functions at study, namely using Gevrey functions, or more generally ultradifferentiable functions, one gets more precise estimates on the time scale over which the drift of actions is under control (see [9, 10, 11, 13]).

We emphasize that all the above results deal with the time independent case, but it is very easy to use them to deal with systems depending *in a periodic way* on time; however already the case of *quasiperiodic time dependence* requires a different approach [14] and the case of a general time dependence requires different techniques and different ideas as developped in [15].

The present paper is strongly related to [15]. In that paper the authors considered a system of the form (1.1) with an analytical *and globally bounded* perturbation P. They proved that if the initial datum is large enough, then it takes a time exponentially long with its size to possibly double the size of the solution. The mechanism underlying the result of [15] is that, if the solution has large energy, then the unperturbed dynamics involves very large frequencies, while the time scales related to the time dependence are bounded, due to the boundedness of the forcing term. Thus the "mechanical frequencies" decouple from the forcing frequencies making possible the development of perturbation theory. This is essentially the same mechanism uderlying the results [16, 17, 18, 19].

In the present paper we investigate the case where also the forcing term is unbounded: we obtain that, provided the growth at infinity of the perturbation ($\mathtt{b}$ in our notation) is slower than that of the unperturbed term (2 in the present case), then one can develop Nekhoroshev theory. We think that the result could fail for $\mathtt{b} = 2$. Technically the main step to get the result consists in changing the definition of resonance and of resonant region: the idea is that a point p is resonant, not if $|\omega(p) \cdot k| < \alpha$ with some $\alpha > 0$, but if $|\omega(p) \cdot k| < \|k\| \, \|p\|^\delta$ with some positive $\delta < 1$. We emphasize that in

[15] the definition of resonance was the standard one and the key technical ingredient of [15] was to take α large in order to make the effective perturbation small. In the case of unbounded perturbations this is not enough, since the size of the perturbation overcomes any possible fixed value of α as the size of the solution increases.

In order to exploit our definition of resonance we also have to modify the geometric part of the proof of Nekhoroshev's theorem: while we still use "resonant blocks" and "extended resonant blocks", we have to modify their shape with respect to the standard definitions. In particular we have to change the exponenti δ as the number of the resonances present in block increases. Actually the proofs and constructions we give here are a transposition to the classical context of those done in [20, 21, 22] in a quantum context. We remark that in the present context the so called analytic part of the proof turns out to be particularly simple and the tools of symbolic calculus developed in the framework of pseudodifferential calculus turn out to be very efficient in this context. In particular, as a difference with respect to the papers [9, 10, 11, 13] it allows to develop the analytic part without making any quantitative estimate and thus to deal with C^∞ functions without any control of the size of the derivatives of functions. This is similar to what has been done in [12], but we use here a different class of symbols *á la* Hörmander, characterized by their behaviour at infinity.

Our proof follows the classical Nekhoroshev's scheme consisting in developping an analytic part and a geometric part. Thus the results we get here can be easily extended to the case where h_0 is a homogeneous *steep* function of some degree $\mathtt{d} > 1$ and $\mathtt{b} < \mathtt{d}$, following the quantum construction of [22]. However, I am not aware of concrete and relevant examples of such a situation: in all the example I know the action variables have singularities accumulating at infinity and in order to deal with them one should develop a version of Nekhoroshev theorem independent of the set of coordinates: we leave this for future work.

Finally we remark that a different proof, by Lochak [23, 24], of Nekhoroshev's Theorem exists, a proof which is much simpler than the original one, and which has also proved to be suitable for the extension to some infinite dimensional systems [18] and in particular to PDEs [25, 26]. Unfortunately we have not been unable to adapt such a proof to the present time dependent case.

We emphasize that, with respect to classical Nekhoroshev's results, we add a global in time estimate, according to which the energy grows at most with a rate slower than any (small) power of time. We also remark that the quantum analogous results controlling the growth of the Sobolev norms of the wave function by $\langle t\rangle^{\varepsilon}$ are by now quite standard [27, 28, 29, 30, 31, 20, 32], and the present work arises from the curiosity of understanding if something similar is true in classical mechanics.

2 Main Result

Definition 2.1. *A function $f \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ is said to be a symbol of class S^m_δ, if it fulfils*

$$\left|\partial_x^\alpha \partial_p^\beta f(x,p)\right| \leq C_{\alpha,\beta} \langle p \rangle^{m-\delta|\beta|} \ , \quad \forall \alpha, \beta \in \mathbb{N}^n \ , \quad \forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d \ . \quad (2.1)$$

The best constants s.t. (2.1) hold are a family of seminorms for the space of symbols. In this way the space of symbols becomes a Fréchet space.

In order to deal with time dependent perturbations, we have to consider also function taking value in the spaces of symbols.

Definition 2.2. *If $\mathcal{F}$ is a Fréchet space, we denote by $C^k_b(\mathbb{R}; \mathcal{F})$ the space of the functions $f \in C^k(\mathbb{R}; \mathcal{F})$, such that all the seminorms of $\partial_t^j f$ are bounded uniformly over $\mathbb{R}$ for all $j \leq k$. If this is true for all k we write $f \in C^\infty_b(\mathbb{R}; \mathcal{F})$.*

Theorem 2.3. *Assume that $P \in C^\infty_b(\mathbb{R}; S^{\mathtt{b}}_1)$ with $\mathtt{b} < 2$, then $\forall \varepsilon > 0$ $\exists R_\varepsilon$, s.t., if the initial datum fulfills $\|p_0\| \geq R_\varepsilon$ then along the solutions of the Cauchy problem for the Hamilton equation of* (1.1) *one has*

$$\|p(t)\| \leq 16 \|p_0\| \left\langle \frac{t}{\|p_0\|} \right\rangle^\varepsilon \ , \quad \forall t \in \mathbb{R} \ . \quad (2.2)$$

Then, by compactness of the ball of radius R_ε, it is easy to obtain the following corollary

Corollary 2.4. *Assume that $P \in C^\infty_b(\mathbb{R}; S^{\mathtt{b}}_1)$ with $\mathtt{b} < 2$, then $\forall \varepsilon > 0$ $\exists C_\varepsilon$, s.t. along the solutions of the Cauchy problem for the Hamilton equation of* (1.1) *with initial datum p_0 one has*

$$\|p(t)\| \leq C_\varepsilon \langle p_0 \rangle \langle t \rangle^\varepsilon \ , \quad \forall t \in \mathbb{R} \ , \quad (2.3)$$

and therefore $\forall p_0 \in \mathbb{R}^d$

$$\lim_{t \to \infty} \frac{\ln \langle p(t) \rangle}{\ln |t|} = 0 \ . \quad (2.4)$$

Remark 2.5. *We expect that, by a technique similar to the one that allows to deduce Theorem 2.3 from Theorem 4.20, it should be possible to use the main result of [15] to prove that in the case of analytic bounded perturbations one would have an estimate of the form*

$$\|p(t)\| \leq C \langle p_0 \rangle \ln \langle t \rangle \ ,$$

however we did not work out the details. Furthermore, it could be interesting to understand the form such an estimate would take in the case of unbounded analytic perturbations.

Such results have to be confronted with the corresponding quantum result, in which one considers the Schrödinger equation with Hamiltonian given by the quantisation of H. In this case it was proved in [21, 22] that the wave function $\psi(t)$ fulfils

$$\|\psi(t)\|_{H^s} \le C_{\varepsilon,s} \|\psi_0\|_{H^s} \langle t\rangle^{\varepsilon} , \quad \forall t \in \mathbb{R} .$$

Actually the question of the validity of a similar estimate in the classical case was the main motivation for the present work.

As anticipated above, the main example we have in mind is that of a particle in an electromagnetic field with Hamiltonian (1.2), in which one has

$$P(p,x,t) = -p \cdot A(x,t) + \frac{\|A(x,t)\|^2}{2} + \varphi(x,t) .$$

3 Analytic Part

We start by fixing some notations and definitions that will be used in the rest of the paper. Given two real valued functions f and g, sometimes we will use the notation $f \lesssim g$ to mean that there exists a constant $C > 0$, independent of all the relevant quantities, such that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we will write $f \simeq g$.

We will denote by $B_R(p)$ the open ball of radius R centered p.

Given a function g, we will denote by X_g the corresponding Hamiltonian vector field and by

$$\{f; g\} := df X_g \equiv \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial x_j}\frac{\partial f}{\partial p_j} \tag{3.1}$$

the Poisson bracket of two functions. Remark that if $f \in S^{m_1}_{\delta}$ and $g \in S^{m_2}_{\delta}$, then $\{f; g\} \in S^{m_1+m_2-\delta}_{\delta}$. The gain of δ is fundamental for the construction of the regularising transformation.

Given $\mu > 0$ and $0 < \delta < 1$ (typically $\mu \ll 1$ and $\delta \simeq 1$), we give the following definitions:

Definition 3.1. *We say that a point $p \in \mathbb{R}^d$ is resonant with $k \in \mathbb{Z}^d \setminus \{0\}$ if*

$$|p \cdot k| \le \|p\|^{\delta} \|k\| \quad \text{and} \quad \|k\| \le \|p\|^{\mu} . \tag{3.2}$$

Definition 3.2. *[Normal form] We say that a function $Z(p,x) = \sum_{k\in\mathbb{Z}^d} \hat{Z}(p) e^{ik\cdot x}$ is* in normal form *if all the points in supp$(\hat{Z}_k(.))$ are resonant with k.*
We say that a function $Z(p,x,t)$ is in resonant normal form if this is true in the above sense, for any fixed time t.

In order to characterise the properties of the Lie transform we need to introduce also the following class of functions

Definition 3.3. *A function R will be said to be a remainder of order N and we will write $R \in \mathcal{R}^N$ if*

$$\left|\partial_x^\alpha \partial_p^\beta R(x,p)\right| \leq C_{\alpha,\beta} \langle p \rangle^{-N} \ , \tag{3.3}$$

$$\forall \alpha, \beta \in \mathbb{N}^n \ , \ |\alpha| + |\beta| \leq 2 \ , \forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d \ . \tag{3.4}$$

In the following we will use time dependent transformations $\Phi(p,x,t)$ with the property that for any fixed t they are canonical. In this case it is easy to see that the change of coordinates $(p,x) = \Phi(p',x',t)$ transforms the equations of motions of a Hamiltonian H to the equations of motion of a new Hamiltonian H'. In this case we will say that Φ conjugates H and H'.

We are going to prove the following normal form theorem

Theorem 3.4. *[Normal Form] Let H be given by (1.1), with $P \in C_b^\infty\left(\mathbb{R}; S_1^{\mathtt{b}}\right)$, $\mathtt{b} < 2$. Fix $N \gg 1$, then there exists $0 < \delta_* < 1$, $\mu_* > 0$ such that, if $\delta_* < \delta < 1$, $0 < \mu < \mu_*$, then there exists a time dependent canonical transformation $\mathcal{T}$ which conjugates H to*

$$H^{(N)} := h_0 + Z_N(t) + R^{(N)}(t) \, , \tag{3.5}$$

with $Z_N \in C_b^\infty\left(\mathbb{R}; S_\delta^{\mathtt{b}}\right)$ in normal form, while $R^{(N)} \in C_b^\infty\left(\mathbb{R}; \mathcal{R}^N\right)$ is a remainder of order N.

Denoting $(p,x) = \mathcal{T}(p',x',t)$, one has $\|p - p'\| \leq C \|p\|^{\mathtt{b}-\delta}$.

The rest of this section is devoted to the proof of this theorem.

The idea is to perform a sequence of canonical transformations conjugating the original system to a normal form plus a remainder whose growth at infinity decreases at each step. Eventually it becomes a function arbitrarily decreasing in the action space.

3.1 Time dependent Lie transform

The canonical transformations will be constructed as Lie transforms generated by time dependent symbols. So, first we recall the main definitions and properties.

Consider a family of time dependent Hamiltonians $g(p,x,t)$, but think of t as an external parameter. Denote by $\Phi_g^\tau(p,x,t)$ the time τ flow it generates, namely the solution of

$$\frac{dx}{d\tau} = \frac{\partial g}{\partial p}(p,x,t) \ , \quad \frac{dp}{d\tau} = -\frac{\partial g}{\partial x}(p,x,t) \ . \tag{3.6}$$

In the case $g \in S_\delta^m$ with $m \leq 1$ the flow of (3.6) is globally defined. This is the only situation we will encounter.

Definition 3.5. *The time dependent coordinate transformation*

$$(p,x) = \Phi_g(p',x',t) \equiv \Phi_g^1(p',x',t) := \Phi_g^\tau(p',x',t)\big|_{\tau=1} \tag{3.7}$$

is called the time dependent Lie transform generated by g.

Remark 3.6. *The time dependent Lie transform Φ_g conjugates a Hamiltonian f to the Hamilton*

$$f'(p',x',t) := f(\Phi_g(p',x',t)) - \Psi_g(p',x',t) \ , \tag{3.8}$$

$$\Psi_g(p',x',t) := \int_0^1 \frac{\partial g}{\partial t}\left(\Phi_g^\tau(p',x',t)\right) d\tau \ . \tag{3.9}$$

This can be easily seen by working in the extended phase space in which time is added as a new variable.

Given a function $f \in S_\delta^m$, we study $f \circ \Phi_g$. We start by the time independent case

Lemma 3.7. *Let $g \in S_\delta^\eta$ be a time independent function, and $f \in S_\delta^m$, with $\eta < \delta$, then, for any positive N, one has*

$$f \circ \Phi_g^1 = \sum_{l=0}^N \frac{f_l}{l!} + R_{m-(N+1)(\delta-\eta)} \ , \tag{3.10}$$

with $f_l \in S_\delta^{m-l(\delta-\eta)}$, precisely given by

$$f_0 := f \ , \quad f_l := \{f_{l-1}; g\} \equiv \left.\frac{d^l}{dt^l}\right|_{t=0} f \circ \Phi_g^t \ , \quad l \geq 1 \ , \tag{3.11}$$

and $\mathcal{R}_{m-(N+1)(\delta-\eta)}$ a remainder of order $(N+1)(\delta-\eta) - m$.
Furthermore, if one denotes $(p,x) = \Phi_g(p',x',t)$, one has

$$\left\|p - p'\right\| \leq C \left\|p\right\|^\eta \ . \tag{3.12}$$

Proof. Just use the formula for the remainder of the Taylor series (in time) which gives

$$f \circ \Phi_g^1 = \sum_{l=0}^N \frac{f_l}{l!} + \frac{1}{N!}\int_0^1 (1+s)^N f_{N+1} \circ \Phi_g^s ds \ ,$$

from which the thesis immediately follows. □

In particular we have the following corollary which covers the case of the time dependent Lie transform and which is proved by simply remarking that $\Psi_g \in C^\infty(\mathbb{R}; S_\delta^\eta)$ up to a remainder of arbitrary order.

Corollary 3.8. *Let $g \in C_b^\infty(\mathbb{R}; S_\delta^\eta)$ with $\eta < \delta$; denote by Φ_g the time dependent Lie transform generated by g. Let $f \in S_\delta^m$, then, for any N, one has*

$$f \circ \Phi_g = f + \{f; g\} + C^\infty(\mathbb{R}; S_\delta^{m-2(\delta-\eta)}) + C^\infty(\mathbb{R}; S_\delta^\eta) + C^\infty(\mathbb{R}; \mathcal{R}^N) \ . \quad (3.13)$$

By $C^\infty(\mathbb{R}; S_\delta^\eta)$ in the above formula, we mean a function belonging to such a space and similarly for the other terms.

From now on, time will only play the role of a parameter, so we will omit to write explicitly this variable and omit to specify the dependence on it, which will always be of class C_b^∞.

We are now ready for the construction of the normal form transformation. Before starting we change the family of seminorms that we will use for symbols. Actually we will use them explicitly only in the proof of Lemma 3.11.

Remark 3.9. *One has that $f \in S_\delta^m$ if and only if for all integers N_1 and N_2 there exists a positive constant C_{δ,N_1,N_2}^m such that*

$$\wp_{\delta,N_1,N_2}^m(f) := \sup_{\substack{p\in\mathbb{R}^d,\ k\in\mathbb{Z}^d, \\ \alpha\in\mathbb{N}^d,\ |\alpha|=N_1}} \left|\partial_p^\alpha \hat{f}_k(p)\right| |k|^{N_2} \langle p\rangle^{-(m-\delta|\alpha|)} < \infty \,, \quad (3.14)$$

with $\hat{f}$ the Fourier coefficients of f.

As anticipated in the notation of equation (3.14), in the following we will use the constants $\wp_{\delta,N_1,N_2}^m$ as seminorms.

We come to the normal form procedure: we look for a generating function g that we want to use to transform H to a normal form plus a remainder decaying at infinity faster than $|p|^{\mathtt{b}}$. If $g \in S_\delta^\eta$, with a suitable η (as it will occur), the Lie transform Φ_g conjugates H to

$$h_0 + P + \{h_0; g\} + \text{lower order terms} \ . \quad (3.15)$$

So we look for a symbol g s.t. $P + \{h_0; g\}$ is in normal form. Actually we will construct a symbol g with the property that $P + \{h_0; g\}$ consists of a part in normal form plus a part decaying at infinity faster than any inverse power of $|p|$.

3.2 Solution of the Cohomological equation

In this subsection we are going to prove the lemma of solution of the Cohomological equation

$$\{h_0, g\} + f - Z \in S_\delta^{-\infty} \ . \quad (3.16)$$

It is a small variant of Lemma 5.8 of [33], we give the proof for the sake of completeness.

Lemma 3.10. *Let $\frac{2}{3} < \delta < 1$, then the following holds true: $\forall f \in S^m_\delta$, there exist $g \in S^{m-\delta}_\delta$, $Z \in S^m_\delta$, with Z in normal form, s.t.* (3.16) *holds.*

First, following [33] we split f in a resonant, a nonresonant and a smoothing part. This will be done with the help of suitable cutoffs, so let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a symmetric cutoff function which is equal to 1 in $[-\frac{1}{2}, \frac{1}{2}]$ and has support in $[-1, 1]$. With its help we define,

$$\tilde{\chi}_k(p) := \chi\left(\frac{\|k\|}{\|p\|^\mu}\right) , \qquad \chi_k(p) := \chi\left(\frac{p \cdot k}{\|p\|^\delta \|k\|}\right) , \tag{3.17}$$

$$d_k(p) := \frac{1}{p \cdot k}\left(1 - \chi\left(\frac{a \cdot k}{\|p\|^\delta \|k\|}\right)\right) . \tag{3.18}$$

By a simple computation one verifies that such functions are symbols, precisely (for more details see Lemma 5.4 of [33]) $\chi_k,\ \tilde{\chi}_k \in S^0_\delta$, and $d_k \in S^{-\delta}_\delta$. We use the above cutoffs to decompose any function $f \in S^m_\delta$:

$$f = f^{(nr)} + f^{(res)} + f^{(S)} . \tag{3.19}$$

with

$$f^{(nr)}(p, x) := \sum_{k \in Z^d \setminus \{0\}} (1 - \chi_k(p))\tilde{\chi}_k(p)\hat{f}_k(p)e^{ikx} , \tag{3.20}$$

$$f^{(res)}(p, x) := \sum_{k \in Z^d} \chi_k(p)\tilde{\chi}_k(p)\hat{f}_k(p)e^{ikx} , \tag{3.21}$$

$$f^{(S)}(p, x) := \sum_{k \in Z^d \setminus \{0\}} (1 - \tilde{\chi}_k(p))\hat{f}_k(a)e^{ikx} . \tag{3.22}$$

Furthermore, one has that, if $f \in S^m_\delta$, then $f^{(res)}, f^{(nr)} \in S^m_\delta$ and $f^{(res)}$ is in normal form. Concerning $f^{(S)}$, by a variant of Lemma 5.6 of [33] one has the following Lemma.

Lemma 3.11. *Assume $f \in S^m_\delta$, then $f^{(S)} \in S^{-\infty}_\delta$.*

Proof. Consider the k-th Fourier coefficient of $f^{(S)}$: keeping into account that $(1 - \tilde{\chi}_k(p))$ is supported in the region $\|k\| \geq \|p\|^\mu$, one has

$$\left|(1 - \tilde{\chi}_k(p))\hat{f}_k(p)\right| \leq \frac{\left|(1 - \tilde{\chi}_k(p))\hat{f}_k(p)\right|}{\|k\|^N} \|k\|^N$$

$$\lesssim \frac{\left|\hat{f}_k(p)\right|}{\|p\|^{\mu N}} \|k\|^N \leq \wp^m_{\delta,0,N}(f)\langle p\rangle^m \frac{1}{\langle p\rangle^{N\mu}} ,$$

which, provided N is large enough decreases at infinity as much as desired. The control of the other seminorms is done similarly and is omitted. □

Proof of Lemma 3.10. Define

$$g := i \sum_{k \neq 0} d_k(p) \hat{f}_k(a) \ , \tag{3.23}$$

then it is immediate to verify that

$$\{h_0; g\} \equiv -p \cdot \frac{\partial}{\partial x} g = f^{(nr)} \ .$$

□

3.3 End of the proof of Theorem 3.4

In this subsection we prove the following iterative lemma from which Theorem 3.4 immediately follows.

Lemma 3.12. *Fix M, let H be as in equation* (1.1)*. There exists* $0 < \delta_* < 1$ *and* $\mu_* > 0$ *such that, if* $\delta_* < \delta < 1$*, and* $0 < \mu < \mu_*$*, define*

$$\mathtt{a} := \min \{2\delta - \mathtt{b}; \delta\} \ ; \tag{3.24}$$

then $\mathtt{a} > 0$ *and the following holds. For any* $\forall n \in \mathbb{N}$ *with* $M \geq n \geq 0$ *there exists a time dependent canonical transformations* $\mathcal{T}_n$ *conjugating* H *to*

$$H_n = h_0 + Z_n + R_n + \widetilde{R}_n \,, \tag{3.25}$$

where $Z_n \in S^{\mathtt{b}}_\delta$ *is in normal form;* $R_n \in S^{\mathtt{b}-n\mathtt{a}}_\delta$*,* $\widetilde{R}_n \in \mathcal{R}^N$*. Furthermore, denoting as before* $(p,x) = \mathcal{T}_n(p', x', t)$*, one has* $\|p - p'\| \leq C_n \|p\|^{\mathtt{b}-\delta}$*.*

Proof. We prove the theorem by induction. In the case $n = 0$, the claim is trivially true.

We consider now the case $n > 0$. Denote $m := \mathtt{b} - n\mathtt{a}$; we determine $g_{n+1} \in S^\eta_\delta$, $\eta = \mathtt{b} - n\mathtt{a} - \delta < 1$, according to Lemma 3.10 with f replaced by R_n. Then one uses $\Phi_{g_{n+1}}$ to conjugate H_n to H'_n given by

$$\begin{aligned} H'_n &= H - \{H_n; g_{n+1}\} + S^{m+2(\eta-\delta)}_\delta + S^\eta_\delta + \widetilde{\widetilde{R}}_{n+1} + \tilde{R}_n \circ \Phi_{g_{n+1}} \\ &= h_0 + Z_n + R_n - \{h_0; g_{n+1}\} + S^{\mathtt{b}+\eta-\delta}_\delta + S^{m+2(\eta-\delta)}_\delta + S^\eta_\delta + \widetilde{\widetilde{R}}_{n+1} + \tilde{R}_n \circ \Phi_{g_{n+1}} \\ &= h_0 + Z_n + R^{(res)}_n + S^{m-(2\delta+\delta-2)}_\delta + S^{\mathtt{b}+\eta-\delta}_\delta + S^{m+2(\eta-\delta)}_\delta + S^\eta_\delta \\ &\qquad + \widetilde{\widetilde{R}}_{n+1} + \tilde{R}_n \circ \Phi_{g_{n+1}} \ , \end{aligned} \tag{3.26}$$

where the term $\widetilde{\widetilde{R}}_{n+1}$ contains the remainder of the expansion of the Lie transforms of the different functions. Define now $Z_{n+1} := Z_n + R^{(res)}_n$, $\tilde{R}_{n+1} := \widetilde{\widetilde{R}}_{n+1} + \tilde{R}_n \circ \Phi_{g_{n+1}}$ and R_{n+1} to be the sum of the remaining terms. Writing explicitly the different exponents of the classes S of the terms composing R_{n+1}, we get that they are given by

$$\begin{aligned} e_1 &:= \mathtt{b} - n\mathfrak{a} - (3\delta - 2) = \mathtt{b} - n\mathfrak{a} - \mathfrak{a}_1 \ , \quad \mathfrak{a}_1 := 3\delta - 2 \\ e_2 &:= \mathtt{b} + \mathtt{b} - n\mathfrak{a} - 2\delta = \mathtt{b} - n\mathfrak{a} - \mathfrak{a}_2 \ , \quad \mathfrak{a}_2 := 2\delta - \mathtt{b} \ , \\ e_3 &:= \mathtt{b} - n\mathfrak{a} + 2(\mathtt{b} - n\mathfrak{a} - 2\delta) = \mathtt{b} - n\mathfrak{a} - \mathfrak{a}_3 \ , \quad \mathfrak{a}_3 := 2(n\mathfrak{a} + 2\delta - \mathtt{b}) \\ e_4 &:= \mathtt{b} - n\mathfrak{a} - \delta = \mathtt{b} - n\mathfrak{a} - \mathfrak{a}_4 \ , \quad \mathfrak{a}_4 := \delta \ . \end{aligned}$$

Remarking that $\mathfrak{a}_3 \geq \mathfrak{a}_2$ and taking the smallest $\mathfrak{a}$ one immediately gets the thesis. □

To conclude the proof of Theorem 3.4 just take $M = [N/\mathfrak{a}] + 1$.

4 Geometric Part

4.1 The partition

Following Nekhoroshev, in this section we partition the action space $\mathbb{R}^d$ according to the resonance relations fulfilled in each region. We adapt the construction to our setting. We will point out during the construction the main differences with the standard construction. The construction is very similar to the one developed in a quantum context in [20, 22]. As in original Nekhoroshev's construction, the sub moduli of $\mathbb{Z}^d$ play a fundamental role in this construction, so, we first recall their definition.

Definition 4.1. *A subgroup $M \subseteq \mathbb{Z}^d$ will be called a module if $\mathbb{Z}^d \cap span_{\mathbb{R}} M = M$. Given a module M, we will denote $M_{\mathbb{R}}$ the linear subspace of $\mathbb{R}^d$ generated by M. Furthermore, given a vector $p \in \mathbb{R}^d$ we will denote by p_M its orthogonal projection on $M_{\mathbb{R}}$.*

In order to perform our construction we take positive parameters δ, μ, $\mathtt{C}_1, \dots, \mathtt{C}_d$, $\mathtt{D}_1, \dots, \mathtt{D}_d$, $\mathtt{R}$ fulfilling

$$\begin{aligned} &\frac{d(d+1)}{2}\mu < 1 - \delta \, , \\ &1 = \mathtt{C}_1 < \mathtt{C}_2 \cdots < \mathtt{C}_d \, , \\ &1 = \mathtt{D}_1 < \mathtt{D}_2 \cdots < \mathtt{D}_d \, , \end{aligned} \tag{4.1}$$

and define

$$\delta_s := \delta + \frac{s(s-1)}{2}\mu \, (= \delta_{s-1} + (s-1)\mu) \ , \tag{4.2}$$

while $\mathtt{R}$ will be assumed to be large enough.

We start by the following definition

Definition 4.2 (Resonant zones)**.** *Let M be a module of $\mathbb{Z}^d$ of dimension s.*

(i) *If $s = 0$, namely $M = \{0\}$, we say that $p \in \mathcal{Z}_M^{(0)}$ if either $\|p\| < \mathtt{R}$ or*

$$|p \cdot k| \geq \|k\| \, \|p\|^{\delta} \ , \qquad \forall k \ : \quad \|k\| \leq \|p\|^{\mu} \ . \tag{4.3}$$

$\mathcal{Z}_{\{0\}}^{(0)}$ *will be called the* non resonant zone*.*

(ii) *If $s \geq 1$, for any set of linearly independent vectors $\{k_1, \dots, k_s\}$ in M, we say that $p \in \mathcal{Z}_{k_1,\dots,k_s}$ if $\|p\| \geq \mathtt{R}$ and $\forall j = 1, ..., s$ one has*

$$\|p \cdot k_j\| \leq \mathtt{C}_j \, \|k\| \, \|p\|^{\delta_j} \qquad \text{and} \qquad \|k\| \leq \mathtt{D}_j \, \|p\|^{\mu} \ . \tag{4.4}$$

then we put

$$\mathcal{Z}_M^{(s)} := \bigcup_{\substack{k_1,\dots,k_s \\ lin.\ ind.\ in\ M}} \mathcal{Z}_{k_1,\dots,k_s} \, . \tag{4.5}$$

The sets $\mathcal{Z}_M^{(s)}$ are called resonant zones*.*

In the standard case the resonant zones are strips whose width changes as the order of the resonance increases. Here also the shape of the resonant zone changes as the number of resonances incresases, since the exponents δ_s depend on s. Furthermore, the fact of defining a zone using any possible choice independent vectors in the resonance modulus gives a shape slightly different from the standard one.

The sets $\mathcal{Z}_M^{(s)}$ contain points p which are in resonance with *at least* s linearly independent vectors in M.

Remark 4.3. *Fix $r, s \in \{1, \dots, d\}$ with $1 \leq r < s$, then for any M with dim $M = s$, one has*

$$\mathcal{Z}_M^{(s)} \subseteq \bigcup_{\substack{M' \subset M \\ dim M' = r}} \mathcal{Z}_{M'}^{(r)} \, .$$

Following Nekhoroshev we now define the *resonant blocks*, which are composed by the points which are resonant with the vectors in a module M, but are non-resonant with the vectors $k \notin M$ and the *extended blocks* which will turn out to be invariant under the dynamics of $h_0 + Z$.

Definition 4.4 (Resonant blocks)**.** *We first define, for $M = \mathbb{Z}^d$, the set $\mathcal{B}_{\mathbb{Z}^d}^{(0)} := \mathcal{Z}_{\mathbb{Z}^d}^{(0)}$. Then, we proceed iteratively: for $s < d$ let M be a module of dimension s, we define the resonant block*

$$\mathcal{B}_M^{(s)} = \mathcal{Z}_M^{(s)} \setminus \left(\bigcup_{\substack{s' > s \\ \dim M' = s'}} \mathcal{B}_{M'}^{(s')} \right) \, .$$

Definition 4.5 (Extended blocks and fast drift planes)**.** *For any module M of dimension s, we define*

$$\widetilde{E}_M^{(s)} = \left\{ \mathcal{B}_M^{(s)} + M_{\mathbb{R}} \right\} \cap \mathcal{Z}_M^{(s)}$$

and the extended blocks

$$E_M^{(s)} = \widetilde{E}_M^{(s)} \setminus \left(\bigcup_{\substack{s'<s \\ \dim M'=s'}} E_{M'}^{(s')} \right) ,$$

where $A + B = \{a + b \mid a \in A, b \in B\}$. Moreover, for all $p \in E_M^{(s)}$ we define the fast drift plain

$$\Pi_M^{(s)}(p) = \{p + M_{\mathbb{R}}\} \cap \mathcal{Z}_M^{(s)} \quad .$$

4.2 Properties of the partition

A useful technical tool is given by the following remark:

Remark 4.6. *If $p, b \in \mathbb{R}^d$, fulfil $\|p\|, \|b\| \geq 1$, and*

$$\|p - b\| \leq C\|b\|^{\tilde{\delta}} ,$$

with some constants $C > 0$ and $0 < \tilde{\delta} < 1$, then one has

$$\|p - b\| \lesssim \|p\|^{\tilde{\delta}} .$$

We start now to study the properties of the partition.

Remark 4.7. *By the very definition of $\mathcal{Z}_M^{(s)}$, for any $s \geq 1$, one has $\mathcal{Z}_M^{(s)} \cap B_{\mathtt{R}}(0) = \emptyset$.*

Lemma 4.8. *Provided $\mathtt{R}$ is large enough, the resonant zone $\mathcal{Z}_{\mathbb{Z}^d}^{(d)}$ is empty.*

The proof requires the use of the following Lemma from [6]. For the proof we refer to [6].

Lemma 4.9. *[Lemma 5.7 of [6]] Let $s \in \{1, \dots, d\}$ and let $\{u_1, \dots u_s\}$ be linearly independent vectors in $\mathbb{R}^d$. Let $w \in \operatorname{span}\{u_1, \dots u_s\}$ be any vector. If α, N are such that*

$$\|u_j\| \leq N \quad \forall j = 1, \dots s ,$$
$$|w \cdot u_j| \leq \alpha \quad \forall j = 1, \dots s ,$$

then

$$\|w\| \leq \frac{sN^{s-1}\alpha}{Vol\{u_1 | \cdots |u_s\}} .$$

Proof of Lemma 4.8. Assume that $\mathcal{Z}^{(d)}_{\mathbb{Z}^d}$ is not empty and take $p \in \mathcal{Z}^{(d)}_{\mathbb{Z}^d}$, then there exist $\{k_1, \ldots, k_d\} \subset \mathbb{Z}^d$ linear independent vectors such that (4.4) is fulfilled by p with the given k_j's. Using Lemma 4.9 we deduce

$$\|p\| \le d(\mathtt{D}_d)^d \mathtt{C}_d \|p\|^{\delta_d + \mu d} .$$

By (4.1), one has that $\delta_d + d\mu < 1$. So, provided $\mathtt{R}$ is large enough this is in contradiction $\|p\| < \mathtt{R}$. □

Lemma 4.10. *There exists a constant C s.t. if $\Pi^{(s)}_M(p)$ is a fast drift plane, then*

$$diam(\Pi^{(s)}_M(p)) \le C\|p\|^{\delta_{s+1}} . \tag{4.6}$$

Proof. First, by definition of resonant zones, for $a \in \mathcal{Z}^{(s)}_M$, there exist $k_1, ..., k_s \in M$ s.t. $|p \cdot k_j| \le \mathtt{C}_s \|k_j\| \, \|p\|^{\delta_s}$, $\forall j = 1, ..., s$, so that, by Lemma 4.9

$$\|\Pi_M p\| \lesssim \|p\|^{\delta_s + s\mu} . \tag{4.7}$$

If $p' \in \Pi^{(s)}_M(p)$ then the same holds for p'. So we have

$$\left\|p - p'\right\| = \left\|\Pi_M(p - p')\right\| \le \|\Pi_M p\| + \left\|\Pi_M p'\right\| \lesssim \left(\|p\|^{\delta_{s+1}} + \left\|p'\right\|^{\delta_{s+1}}\right) .$$

By Remark 4.6 this implies the thesis. □

In particular we have the following Corollary

Corollary 4.11. *If $p \in E^{(s)}_M$ there exists $p' \in \mathcal{B}^{(s)}_M$ s.t.*

$$\left\|p - p'\right\| \le C \, \|p\|^{\delta_{s+1}} \quad . \tag{4.8}$$

Indeed, by definition of extended block $\exists p' \in \Pi^{(s)}_M(p) \cap \mathcal{B}^{(s)}_M$, and therefore, by (4.6) the corollary holds.

The next lemma ensures that, if the parameters $\mathtt{C}_j, \mathtt{D}_j$ are suitably chosen, an extended block $E^{(s)}_{M,j}$ is separated from every resonant zone associated to a module M' with $\dim(M) = s' \le s$, which is not contained in M. This is the extension to our context of the classical property of separation of resonances.

Lemma 4.12. *[Separation of resonances] Take $K > 0$. There exist positive constants $\mathtt{R}$, $\tilde{\mathtt{C}}_{s+1}$ and $\tilde{\mathtt{D}}_{s+1}$ depending only on $\mu, \delta_s, \mathtt{C}_s, \mathtt{D}_s, K$ such that, if*

$$\mathtt{C}_{s+1} > \tilde{\mathtt{C}}_{s+1} , \quad \mathtt{D}_{s+1} > \tilde{\mathtt{D}}_{s+1}, \quad \mathtt{R} > \bar{\mathtt{R}} ,$$

then the following holds true. Let $p \in E^{(s)}_M$ for some M of dimension $s = 1, \ldots, d-1$, and let $p' \in \mathbb{R}^d$ be such that

$$\left\|p - p'\right\| \le K\|p\|^{\delta_{s+1}} ,$$

then $\forall M' \not\subset M$ s. t. $s' := \dim M' \le s$ one has

$$p' \notin \mathcal{Z}^{(s')}_{M'} .$$

Proof. Assume by contradiction that $p' \in \mathcal{Z}^{(s')}_{M'}$ for some $M' \neq M$. It follows that there exist s' integer vectors, $k_1, ..., k_{s'} \in M'$ *among which at least one does not belong to* M, s.t.

$$\left|p' \cdot k_j\right| \leq \mathtt{C}_j \|p'\|^{\delta_j} \|k_j\| \ , \quad \|k_j\| \leq \mathtt{D}_j \|p'\|^{\mu} \ . \tag{4.9}$$

Let $k_{\bar{\jmath}}$ be the vector which does not belong to M. By Corollary 4.11, there exists $b \in \mathcal{B}^{(s)}_M$ s.t. $\|p-b\| \lesssim \|p\|^{\delta_{s+1}}$ and thus also $\|p'-b\| \lesssim \|p'\|^{\delta_{s+1}}$ (of course with a different constant). Thus it follows that there exist constants $\tilde{\mathtt{C}}_{s+1}$, $\tilde{\mathtt{D}}_{s+1}$ s.t.

$$|b \cdot k_{\bar{\jmath}}| \leq \tilde{\mathtt{C}}_{s+1} \|b\|^{\delta_{s+1}} \|k_{\bar{\jmath}}\| \ , \quad \|k_{\bar{\jmath}}\| \leq \tilde{\mathtt{D}}_{s+1} \|b\|^{\mu} \ .$$

But, if $\mathtt{C}_{s+1} > \tilde{\mathtt{C}}_{s+1}$, $\mathtt{D}_{s+1} > \tilde{\mathtt{D}}_{s+1}$, this means that b is also resonant with $k_{\bar{\jmath}} \notin M$, and this contradicts the fact that $b \in \mathcal{B}^{(s)}_{M,j}$. □

In order to take into account the effects of the remainder in the normal form theorem and to conclude the proof we need to extend the resonant planes.

Definition 4.13. *We define*

$$\left[\Pi^{(s)}_M(p)\right]^{ext} := \bigcup_{p' \in \Pi^{(s)}_M(p)} B_{\|p'\|^{\delta_{s+1}}}(p') \ , \tag{4.10}$$

$$\left[\Pi^{(s)}_M(p)\right]^{ext}_{tr} := \left[\Pi^{(s)}_M(p)\right]^{ext} \cap \mathcal{Z}^{(s)}_M \ , \tag{4.11}$$

$$\tag{4.12}$$

where, ad before $B_R(p)$ is the ball of radius R and centre p.

Remark 4.14. *One has*

$$diam(\left[\Pi^{(s)}_M(p)\right]^{ext}) \leq C_s \|p\|^{\delta_{s+1}} \ , \tag{4.13}$$

for some C_s.

Remark 4.15. *If the constants $\mathtt{R}$, $\tilde{\mathtt{C}}_{s+1}$ and $\tilde{\mathtt{D}}_{s+1}$ are chosen suitably, then $\forall p' \in [\Pi^{(s)}_M(p)]^{ext}$ and all p'' s.t.*

$$\left\|p''-p'\right\| \leq \left\|p'\right\|^{\delta_{s+1}} \ ,$$

one has

$$p'' \notin \mathcal{Z}^{s'}_{M'} \ , \quad \forall (s' \leq s \ , \ M' : M' \not\subseteq M) \ . \tag{4.14}$$

Remark 4.16. *By Lemma 4.12, it follows that if $p'' \in \partial \Pi^{(s)}_M(p')$, $p' \in [\Pi^{(s)}_M(p)]^{ext}_{tr}$, then $p'' \in \partial E^{(s')}_{M'}$ with $M \subset M'$ and $s' < s$.*

Lemma 4.17. *If the constants* $\mathtt{R}$*,* $\tilde{\mathtt{C}}_{s+1}$ *and* $\tilde{\mathtt{D}}_{s+1}$ *are chosen suitably, then,* $\forall p \in [\Pi_M^{(s)}(p')]^{ext}$ *and* $\forall k \notin M$ *one has*

$$\|k\| \leq \|p\|^{\mu} \implies |p \cdot k| \geq \|p\|^{\delta} \|k\| \ . \tag{4.15}$$

Proof. Following the proof of Lemma 4.12, assume by contradiction that $|p \cdot k| \leq \|p\|^{\delta} \|k\|$, then $\exists p' \in E_M^{(s)}$ s.t. $|p - p'| \lesssim \|p\|^{\delta_{s+1}}$, and therefore $\exists p'' \in \mathcal{B}_M^{(s)}$ s.t. $|p - p''| \lesssim \|p\|^{\delta_{s+1}}$. It follows

$$\left|p'' \cdot k\right| \leq \left|p - p''\right| \|k\| + |p \cdot k| \lesssim |p|^{\delta_{s+1}} \|k\| \lesssim \left|p''\right|^{\delta_{s+1}} \|k\| \ ,$$

but if $\mathtt{C}_{s+1}$ is chosen large enough, this means that p'' fulfils (4.4) with $j = s+1$, against $p'' \in \mathcal{B}_M^{(s)}$. □

Corollary 4.18. *Consider the normal form* Z_N *obtained by Theorem 3.4. Define* $Z(p, x, t) := Z_N(p, x, t)(1 - \chi\left(\frac{\|p\|^2}{\mathtt{R}}\right))$ *(which is supported outside* $B_{\mathtt{R}}(0)$*). For* $p' \in [\Pi_M^{(s)}(p)]^{ext}$*,*

$$Z(p', x, t) = \sum_{k \in M} Z_k(p', t) e^{\mathrm{i}k\cdot x} \ , \tag{4.16}$$

namely the sum is restricted to $k \in M$*.*

Remark 4.19. $\forall p \in \mathbb{R}^d$ $\exists! M$ *s.t.* $p \in E_M^{(s)}$*. The important point is the unicity.*

We are now ready to prove the following Theorem, giving a control on the dynamics over long times. This is the typical Nekhoroshev type theorem adapted to our C^{∞} context.

Theorem 4.20. *There exist positive* $K_1 < K_2 < ... < K_d$ *s.t. the following holds true: consider the Cauchy problem for the Hamiltonian system* (3.5) *with initial datum* p_0*. Let* M *with* $dim M = s$ *be s.t.* $p_0 \in E_M^{(s)}$*. Then one has*

$$p(t) \in [\Pi_M^{(s)}(p_0)]^{ext} \ , \quad \forall |t| \leq \frac{1}{K_s} \|p_0\|^{N+\delta} \ , \tag{4.17}$$

and thus, in particular, for the same times one has

$$\|p(t)\| \leq 2 \|p_0\| \ . \tag{4.18}$$

Proof. First we remark that this is true when $s = 0$ and thus p_0 is in the nonresonant region. Indeed, in this case one has that the equations for p reduce to $\dot{p} = -\frac{\partial R^{(N)}}{\partial x} = \mathcal{O}(\left\|p_0^{-N}\right\|)$.

Following Nekhoroshev we proceed by induction on s. So, assume $p_0 \in E_M^{(s)}$ with $s \geq 1$. Assume that the result has been proved for $s-1$ and we prove it for s. Introduce in $[\Pi_M^{(s)}(p_0)]^{ext}$ coordinates $p = (p_M, p_\perp)$ with

$p_M \in M_{\mathbb{R}}$ and $p_\perp \in (M_{\mathbb{R}})^\perp$. In these coordinates one has that $\Pi_M^{(s)}(p)$ is the set of $\tilde{p} = (\tilde{p}_M, \tilde{p}_\perp)$ with $\tilde{p}_\perp = p_\perp$ and $\tilde{p}_M$ belonging to some domain D (which depends also on $p_\perp$, but this is not important in the following). Then one has that $\tilde{p} \in \partial\Pi_M^s(p)$ is equivalent to $\tilde{p}_M \in \partial D$. It follows that $\tilde{p} \in \partial[\Pi_M^{(s)}(p_0)]_{tr}^{ext}$ implies that either $\|p_\perp\| = \|p_0\|^{\delta_{s+1}}$ or $\tilde{p} \in \partial\Pi_M^{(s)}(p')$ for some $p' \in [\Pi_M^{(s)}(p_0)]_{tr}^{ext}$.

We are now ready to conclude the proof: assume that $\exists 0 < \bar{t} \leq \|p_0/K_s\|$ s.t. $p(\bar{t}) \in \partial[\Pi_M^{(s)}(p_0)]_{tr}^{ext}$. If it does not exists, then there is nothing to prove. Then either $\|p_{0,\perp} - p_\perp(\bar{t})\| = \|p_0\|^{\delta_{s+1}}$ or $p(\bar{t}) \in \partial\Pi_M^{(s)}(p')$ for some $p' \in [\Pi_M^{(s)}(p_0)]_{tr}^{ext}$. Now the first possibility is ruled out by the fact that in $[\Pi_M^{(s)}(p_0)]^{ext}$ the equations for $p_\perp$ reduce to $\dot{p}_\perp = \mathcal{O}(\|p_0\|^N)$. So, assume that $p(\bar{t}) \in \partial\Pi_M^{(s)}(p')$, then, by Remark 4.16 one has $p(\bar{t}) \in \partial E_{M'}^{(s')}$ with $s' < s$. Then, by induction, considering $p(\bar{t}^+)$ as an initial datum, one gets that (by Remark 4.14)

$$\|p_0 - p(\bar{t})\| \leq C_s \|p(\bar{t})\|^{\delta_s} \ , \quad \|p(t) - p(\bar{t})\| \leq C_s \|p(\bar{t})\|^{\delta_s} \ , \quad |t - \bar{t}| \leq \frac{\|p(\bar{t})\|^{N+\delta}}{K_{s-1}} \ . \tag{4.19}$$

But, actually, by the bound on time (4.17), the times fulfil

$$|t - \bar{t}| \leq |t| + |\bar{t}| \leq \frac{2\|p_0\|^{N+\delta}}{K_s} \leq C\frac{2\|p(\bar{t})\|^{N+\delta}}{K_s} \ ,$$

and therefore, if $2C/K_s < K_{s-1}$, the estimates (4.19) hold for the times we are interested in. Concerning the distance from $\Pi_M^s(p_0)$, the above estimates imply

$$\|p(t) - p_0\| \leq 2\tilde{C}_s \|p(\bar{t})\|^{\delta_s} \lesssim \|p_0\|^{\delta_s} \ ,$$

so the left hand side is smaller than $\|p_0\|^{\delta_{s+1}}$ provided p_0 is large enough. For p_0 in a compact set the estimate is trivial. □

Proof of Theorem 2.3. Assume that there exists a solution with

$$\limsup_{t\to+\infty} \|p(t)\| = +\infty \ ,$$

otherwise the result holds trivially. Let $R_k := R_0 2^k$ with $R_0 := \|p_0\|$, then there exists a sequence of times t_k s.t.

$$\sup_{|t|\leq t_k} \|p(t_k)\| = 2R_k \ .$$

Applying Theorem (4.20) with initial datum $p(t_k)$, one gets

$$t_{k+1} > \frac{1}{K_d} R_k^N + t_k$$

(where we redefined $N+\delta \to N$). So, taking $t_0 = 0$, one gets

$$t_{L+1} \geq \sum_{k=0}^{L} \frac{1}{K_d}(R_0 2^k) = \frac{1}{K_d} R_0^N \frac{2^{(L+1)N}-1}{2^N-1} \geq \frac{1}{2K_d} R_0^N 2^{LN} \ . \tag{4.20}$$

Thus, defining

$$\tau_0 := 0 \ , \quad \tau_{k+1} := \frac{1}{2K_d}(R_0 2^k)^N \ , \quad k \geq 0 \ ,$$

we have

$$\sup_{|t|\leq \tau_k} \|p(t)\| \leq 2R^k \ . \tag{4.21}$$

To write a global in time formula consider the function

$$\widetilde{\theta}(t) := \prod_{k=0}^{\infty} \left(2\theta(t-\tau_k)\right) \ ,$$

with $\theta(t)$ the standard Heaviside step function. Remark that $\widetilde{\theta}$ is well defined since, for any time only a finite number of factors is different from zero. Thus we have the global estimate

$$\|p(t)\| \leq R_0 \widetilde{\theta}(t) \ .$$

Consider now the function $f(t) := \frac{R_0 \widetilde{\theta}(t)}{t}$. We consider it only for $t \geq 1$. Such a function has positive jumps at τ_k and in all the other intervals it is monotonically decreasing like t^{-1}. In order to bound such a function we look for a function interpolating the peaks. One has, for $k \geq 1$

$$f(\tau_k^+) = 2K_d \frac{R_0 2^{k+1}}{R_{k-1}^N} = 2K_d \frac{4R_{k-1}}{R_{k-1}^N} = \frac{2\left(2K_d \tau_k\right)^{1/N}}{\tau_k} \ .$$

So it is clear that an interpolating function is

$$\tilde{f}(t) := 4\left(2K_d\right)^{1/N} \frac{t^{1/N}}{t} \ , \tag{4.22}$$

and in this way, $\forall t \geq \tau_1$, one has $f(t) \leq \tilde{f}(t)$. It follows

$$\|p(t)\| \leq 4\left(2K_d\right)^{1/N} t^{1/N} \ , \quad t \geq \tau_1 \ . \tag{4.23}$$

We now manipulate such an expression to get the thesis. Taking into account that at τ_1 the r.h.s. of (4.23) is equal to $4R_0$, one has, $\forall t > 0$

$$\|p(t)\| \leq \max\left\{4R_0, 4\left(2K_d\right)^{1/N} t^{1/N}\right\} = \max\left\{4R_0, 4\left(2K_d\right)^{1/N} \tau_1^{1/N} \left(\frac{t}{\tau_1}\right)^{1/N}\right\} \tag{4.24}$$

$$= 4R_0 \max\left\{1, \left(\frac{t}{\tau_1}\right)^{1/N}\right\} \leq 4R_0 \left\langle \frac{t}{\tau_1} \right\rangle^{1/N} \ . \tag{4.25}$$

Still we have to take into account the change of coordinates. In this way we get that there exists $\bar{R}_0$, s.t., if the initial datum fulfills $\|p_0\| \geq \bar{R}_0$, then one has

$$\|p(t)\| \leq 16R_0 \left\langle \frac{t}{\tau_1} \right\rangle^{1/N} .$$

□

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References

[1] N. N. Nekhoroshev, "An exponential estimate of the time of stability of nearly integrable Hamiltonian systems," *Uspehi Mat. Nauk*, vol. 32, no. 6(198), pp. 5–66, 287, 1977.

[2] N. N. Nekhoroshev, "An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. II," *Trudy Sem. Petrovsk.*, no. 5, pp. 5–50, 1979.

[3] G. Benettin, L. Galgani, and A. Giorgilli, "A proof of Nekhoroshev's theorem for the stability times in nearly integrable Hamiltonian systems," *Celestial Mech.*, vol. 37, no. 1, pp. 1–25, 1985.

[4] G. Benettin and G. Gallavotti, "Stability of motions near resonances in quasi-integrable Hamiltonian systems," *J. Statist. Phys.*, vol. 44, no. 3-4, pp. 293–338, 1986.

[5] J. Pöschel, "Nekhoroshev estimates for quasi-convex Hamiltonian systems," *Math. Z.*, vol. 213, no. 2, pp. 187–216, 1993.

[6] A. Giorgilli, "Exponential stability of Hamiltonian systems," in *Dynamical systems. Part I*, Pubbl. Cent. Ric. Mat. Ennio Giorgi, pp. 87–198, Scuola Norm. Sup., Pisa, 2003.

[7] M. Guzzo, L. Chierchia, and G. Benettin, "The steep Nekhoroshev's theorem," *Comm. Math. Phys.*, vol. 342, no. 2, pp. 569–601, 2016.

[8] M. Kunze and D. M. A. Stuart, "Nekhoroshev type stability results for Hamiltonian systems with an additional transversal component," *J. Math. Anal. Appl.*, vol. 419, no. 2, pp. 1351–1386, 2014.

[9] J.-P. Marco and D. Sauzin, "Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems," *Publ. Math. Inst. Hautes Études Sci.*, no. 96, pp. 199–275 (2003), 2002.

[10] J.-P. Marco and D. Sauzin, "Wandering domains and random walks in Gevrey near-integrable systems," *Ergodic Theory Dynam. Systems*, vol. 24, no. 5, pp. 1619–1666, 2004.

[11] A. Bounemoura, "Nekhoroshev estimates for finitely differentiable quasi-convex Hamiltonians," *J. Differential Equations*, vol. 249, no. 11, pp. 2905–2920, 2010.

[12] D. Bambusi and B. Langella, "A C^∞ Nekhoroshev theorem," *Math. Eng.*, vol. 3, no. 2, pp. Paper No. 019, 17, 2021.

[13] A. Bounemoura and J. Féjoz, "Hamiltonian perturbation theory for ultra-differentiable functions," *Mem. Amer. Math. Soc.*, vol. 270, no. 1319, pp. v+89, 2021.

[14] A. Bounemoura, "Nekhoroshev's estimates for quasi-periodic time-dependent perturbations," *Comment. Math. Helv.*, vol. 91, no. 4, pp. 653–703, 2016.

[15] A. Giorgilli and E. Zehnder, "Exponential stability for time dependent potentials," *Z. Angew. Math. Phys.*, vol. 43, no. 5, pp. 827–855, 1992.

[16] G. Benettin, L. Galgani, and A. Giorgilli, "Realization of holonomic constraints and freezing of high frequency degrees of freedom in the light of classical perturbation theory. I," *Comm. Math. Phys.*, vol. 113, no. 1, pp. 87–103, 1987.

[17] G. Benettin, L. Galgani, and A. Giorgilli, "Realization of holonomic constraints and freezing of high frequency degrees of freedom in the light of classical perturbation theory. II," *Comm. Math. Phys.*, vol. 121, no. 4, pp. 557–601, 1989.

[18] D. Bambusi and A. Giorgilli, "Exponential stability of states close to resonance in infinite-dimensional Hamiltonian systems," *J. Statist. Phys.*, vol. 71, no. 3-4, pp. 569–606, 1993.

[19] G. Benettin and F. Fassò, "Fast rotations of the rigid body: a study by Hamiltonian perturbation theory. I," *Nonlinearity*, vol. 9, no. 1, pp. 137–186, 1996.

[20] D. Bambusi, B. Langella, and R. Montalto, "Spectral asymptotics of all the eigenvalues of Schrödinger operators on flat tori," *Nonlinear Anal.*, vol. 216, pp. Paper No. 112679, 37, 2022.

[21] D. Bambusi, B. Langella, and R. Montalto, "Growth of Sobolev norms for unbounded perturbations of the Schrödinger equation on flat tori," *J. Differential Equations*, vol. 318, pp. 344–358, 2022.

[22] D. Bambusi and B. Langella, "Growth of sobolev norms in quasi integrable quantum systems," *Arxiv:2202.04505*, 2022.

[23] P. Lochak, "Canonical perturbation theory: an approach based on joint approximations," *Uspekhi Mat. Nauk*, vol. 47, no. 6(288), pp. 59–140, 1992.

[24] P. Lochak and A. I. Neĭshtadt, "Estimates of stability time for nearly integrable systems with a quasiconvex Hamiltonian," *Chaos*, vol. 2, no. 4, pp. 495–499, 1992.

[25] D. Bambusi, "Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations," *Math. Z.*, vol. 230, no. 2, pp. 345–387, 1999.

[26] D. Bambusi and P. Gérard, "A Nekhoroshev theorem for some perturbations of the Benjamin-Ono equation with initial data close to finite gap tori," *Math. Z.*, vol. 307, no. 3, p. Paper No. 54, 2024.

[27] J. Bourgain, "Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential," *Communications in Mathematical Physics*, vol. 204, no. 1, pp. 207–247, 1999.

[28] J. Bourgain, "On growth in time of Sobolev norms of smooth solutions of nonlinear Schrödinger equations in $\mathbf{R}^D$," *J. Anal. Math.*, vol. 72, pp. 299–310, 1997.

[29] W.-M. Wang, "Logarithmic bounds on Sobolev norms for time dependent linear Schrödinger equations," *Communications in Partial Differential Equations*, vol. 33, no. 12, pp. 2164–2179, 2008.

[30] J.-M. Delort, "Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds," *Int. Math. Res. Not. IMRN*, no. 12, pp. 2305–2328, 2010.

[31] D. Bambusi, B. Grébert, A. Maspero, and D. Robert, "Growth of sobolev norms for abstract linear schrödinger equations," *Journal of the European Mathematical Society*, vol. 23, no. 2, pp. 557–583, 2020.

[32] M. Berti and A. Maspero, "Long time dynamics of Schrödinger and wave equations on flat tori," *J. Differential Equations*, vol. 267, no. 2, pp. 1167–1200, 2019.

[33] D. Bambusi, B. Langella, and R. Montalto, “On the spectrum of the Schrödinger operator on T^d: a normal form approach,” *Communications in Partial Differential Equations*, pp. 1–18, 2019.