Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 23012–23026.
DOI: 10.3934/math.20241118
Received: 21 May 2024
Revised: 12 July 2024
Accepted: 15 July 2024
Published: 26 July 2024

Research article

A couple of BO equations as a normal form for the interface problem

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Abstract: We consider two fluids in a 2-dimensional region: The lower fluid occupies an infinitely depth region, while the upper fluid occupies a region with a fixed upper boundary. We study the dynamics of the interface between the two fluids (interface problem) in the limit in which the interface has a space periodic profile, is close to horizontal, and has a "long wave profile". We use a Hamiltonian normal form approach to show that up to corrections of second order, the equations are approximated by two decoupled Benjamin-Ono equations.

Keywords: water waves; Benjamin-Ono; Hamiltonian partial differential equations; normal form
Mathematics Subject Classification: 35Q35, 37K55, 76B55

1. Introduction

In this paper, we study the dynamics of the interface between two fluids lying below a horizontal surface. One of the fluids, with density ρ, occupies, at rest, the region $-\infty < z < 0$, and the other, with density ρ_1, occupies, at rest, the region $0 < z < h_1$. Precisely, we study the evolution of the interface $\eta(x,t)$ between the fluids. We are interested in studying space periodic interfaces, with a very long space period, say of order ϵ^{-1} and a small amplitude of order ϵ. This is obtained by passing to the scaled profile $\tilde{\eta}$ defined by

$$\eta(x,t) = \epsilon\tilde{\eta}(\epsilon x,t)\ , \quad \tilde{\eta}(y+2\pi,t) = \tilde{\eta}(y,t) \tag{1.1}$$

(actually, in the following we will make a slightly different scaling leading to cleaner equations, but here for the sake of simplicity we omit inserting such a further scaling). We show that, at the first nontrivial order, the dynamics is described by two non-interacting Benjamin Ono (BO) equations which describe respectively right and left moving waves. This is the analogue of the well known situation of long small amplitude surface waves of a fluid which are described by two non-interacting KdV equations [3, 17].

Our result is based on the procedure of Birkhoff normal form in the sense that the two BO equations appear as the first order normal form of the equations of the interface problem. The result we prove is "semiformal" in the sense that we show that, given any sufficiently smooth initial datum, there exists a solution of the couple of Benjamin Ono equations, which provides an approximate solution of the interface problem. Precisely, the approximate solution fulfills the equations of the interface problem up to an error whose norm is of order ϵ^2 for all times. Instead, we do not prove that the solution of the interface problem corresponding to the given initial datum remains close to the solution constructed through BO for times of order ϵ^{-3}. Since the interface problem is a quasilinear one, this would require some nontrivial amount of work.

As the use of the term "Birkhoff normal form" suggests, we take here a Hamiltonian point of view: We start from the Hamiltonian of the interface problem as constructed in [10] (which follows [9, 12]), we insert the small amplitude-long wave Ansatz and, after passing to a scaled time $t' \simeq \epsilon^2 t$, we get a Hamiltonian of the form

$$H_0 + \epsilon H_1 \,,$$

where

$$H_0 = \int_{\mathbb{T}} \frac{\eta^2 + \xi_y^2}{2} dy \tag{1.2}$$

and the precise form of H_1 is not important now (it will be given below). In (1.2), the function η is the rescaled interface profile previously denoted by $\tilde{\eta}$, and ξ is its conjugated variable. The important fact is that the Hamilton equations of (1.2) are

$$\dot{\eta} = -\xi_{yy} \,, \quad \dot{\xi} = -\eta \,, \tag{1.3}$$

which are essentially a wave equation. Now, it is well known that the general solution of the wave equation consists of the superposition of a right going rigid wave and a left going rigid wave. Thus, following [9] it is particularly useful to introduce the so called characteristic variables*

$$r = \frac{\eta + \xi_y}{\sqrt{2}} \,, \quad s = \frac{\eta - \xi_y}{\sqrt{2}} \,, \tag{1.4}$$

in which the Eq (1.3) take the trivial form

$$\dot{r} = -r_y \quad \dot{s} = s_y \,. \tag{1.5}$$

This was done in [9] for the standard Water Wave problem, and in [10] for the interface problem. The result of [10] is that if one restricts $H_0 + \epsilon H_1$ to the surface $r \equiv 0$, one gets the Hamiltonian of the BO equation in translating (to the left) coordinates. Similarly, if one restricts $H_0 + \epsilon H_1$ to the manifold $s \equiv 0$, one gets the BO equation in right translating coordinates. We remark that *this corresponds to two particular choices of initial data*, however, for general initial data, neither r nor s vanish.

The problem that we address here is that of understanding the dynamics corresponding to general initial data.

*This is a true coordinate transformation, since we will restrict to the space of functions with zero space average, so that the operator ∂_y is invertible.

The method that we use is that of [3] (see also [8, 14]) of looking for a canonical transformation which eliminates from the Hamiltonian the terms coupling left going waves to right going waves. As a result, we construct a map T with the property that

$$H \circ T = H_0 + \epsilon \langle H_1 \rangle + O(\epsilon^2) ,$$

where $\langle H_1 \rangle$ is the average of H_1 with respect to the flow generated by H_0, namely the flow of (1.5). An explicit computation shows that $\langle H_1 \rangle$ is the Hamiltonian of two decoupled BO equations in translating coordinates.

From a technical point of view, there is a serious difficulty in performing the above procedure; this is due to the fact that the expansion in ϵ of the Hamiltonian is a singular one, namely the terms one neglects contain derivatives of higher order with respect to those contained in H_1, and furthermore, H_1 contains terms with more derivatives than H_0. This also causes a further difficulty, namely that the transformation T used in order to put the system in normal form is not well defined as a coordinate transformation.

Here, we use the technique developed in [3] to solve such problems. The idea is that of approximating all the quantities one has to compute by some truncation, which always involves a finite number of derivatives. As a consequence, one gets that the normalizing transformation is well defined as a map from a Sobolev space to a Sobolev space of lower regularity, and this also allows control of the reminders in the expansions by losing derivatives.

Before closing this introduction, we recall that, following [4–6, 16] and many others, the deduction of the BO equation as a modulation equation for the interface problem is by now standard. However, as already emphasized, this has always been done considering BO as an equation governing the dynamics of waves propagating in one direction. On the contrary, we prove here that two BO equations are needed to describe solutions corresponding to arbitrary initial data. The need of two BO equations is stated in [11]; however, in that paper, the fact that they should or should not be coupled is not discussed.

We also emphasize that in the present paper we work in the case of periodic boundary conditions: in such a case the right going wave and the left going wave interact forever, so it is quite surprising that the interactions averages away and, as a result, the dynamics is described by two non interacting BO equations.

The rest of the paper is organized as follows: In Section 2, we present in a precise way our main result, which is Theorem 2.2. In Section 3, we recall the method of normal form that we use here. In particular, in Subsection 3.2 we recall the method introduced in [3] to construct the normal form in the case of singular expansions. In Section 4, we compute the expansion of the Hamiltonian of the interface problem, use the normal form method to compute the normal form, and conclude the proof of our main result. Finally, we add an appendix with a couple of technical lemmas.

2. Main result

Consider a fluid occupying the domain

$$S := \{(x, z) \in \mathbb{T}_L \times \mathfrak{R} \ : \ -\infty < z < h_1\} ,$$

where $\mathbb{T}_L := \mathfrak{R}/L\mathbb{Z}$, L is a large parameter, and $h_1 > 0$. This domain is divided into 2 regions determined by a function $\eta(x)$:

$$\Omega := \{(x, z) \in \mathbb{T}_L \times \mathfrak{R} \ : \ -\infty < z < \eta(x)\} , \tag{2.1}$$

$$\Omega_1 := \{(x,z) \in \mathbb{T}_L \times \mathfrak{R} \ : \ \eta(x) < z < h_1\}\,; \tag{2.2}$$

the region Ω is occupied by a fluid with density ρ, while the region Ω_1 is occupied by a fluid with density $\rho_1 < \rho$. The two fluids are assumed to be immiscible.

We assume the function η to have zero integral over the torus.

We restrict to irrotational motions so that there exist two velocity potentials: φ and φ_1, which are harmonic functions in the region Ω and in the region Ω_1, respectively, and whose gradient gives the velocity of the fluid in the corresponding regions.

It is known [10] that in this situation the equations governing the interface evolution are Hamiltonian, with the Hamiltonian given by

$$H(\eta,\xi) = \frac{1}{2}\int_{\mathbb{T}_L} \xi G_1(\eta)[\rho_1 G(\eta) + \rho G_1(\eta)]^{-1} G(\eta)\xi dx + \frac{1}{2}\int_{\mathbb{T}_L} g(\rho-\rho_1)\eta^2 dx, \tag{2.3}$$

where $G(\eta)$, $G_1(\eta)$ are the Dirichlet-Neumann operators for the domains Ω and Ω_1, respectively (see Definitions 4.1, 4.2 below), and the variable ξ conjugated to η is given by $\xi(x) = \rho\varphi(x,\eta(x)) + \rho_1\varphi_1(x,\eta(x))$. By this we mean that the equations of motion for the system are given by

$$\dot\eta = \nabla_\xi H(\eta,\xi)\,, \quad \dot\xi = -\nabla_\eta H(\eta,\xi)\,, \tag{2.4}$$

where ∇_η is the L^2 gradient defined by $\langle \nabla_\eta H; h\rangle_{L^2} = d_\eta H h,\ \forall h \in \mathcal{S}$.

We study the case where η is small and is a "long wave"; this is achieved by taking $L := \frac{2\pi}{\mu}$ and making the scaling

$$\eta(x) = \epsilon\tilde\eta(\mu x)\,, \quad \xi(x) = \alpha\tilde\xi(\mu x)\,, \quad 0 < \mu \simeq \epsilon \ll 1. \tag{2.5}$$

The scaling of ξ has been introduced only for future convenience, and we will take $\alpha \simeq 1$. In the following, we will denote by

$$y = \mu x\,, \quad \mu := \epsilon\frac{\rho_1}{\rho h_1} \tag{2.6}$$

the scaled space variable.

In these variables, the Hamiltonian turns out to have the form

$$\frac{1}{2}a\epsilon^2\left[\int\left(\tilde\eta^2 + \tilde\xi_y^2\right)dy - \epsilon\int\left(\tilde\xi_y^2\tilde\eta + \tilde\xi_y|D|\tilde\xi_y\right)dy\right] \tag{2.7}$$

where

$$a := \left(\frac{\rho_1}{\rho h_1}\right)^2\sqrt{gh_1\frac{\rho-\rho_1}{\rho_1}}\,, \tag{2.8}$$

and $|D|$ is the Fourier multiplier by $|k|$, namely,

$$(|D|\tilde\xi)(y) := \sum_{k\neq 0}|k|\hat\xi_k e^{iky},$$

where $\hat\xi_k := \frac{1}{2\pi}\int_0^{2\pi}\tilde\xi(y)e^{-iky}dy$ are the Fourier coefficients of $\tilde\xi$. Here and in all the rest of the paper, integrals will be over a period in the y variable, namely from 0 to 2π. Expanding the Hamiltonian in powers of ϵ and making the following scaling of time

$$\tilde t := \frac{t}{a\epsilon^2} \tag{2.9}$$

the Hamiltonian takes the form (see Lemma 4.3)

$$H_{Int} := H_0 + \epsilon H_1 + O(\epsilon^2) \tag{2.10}$$

where (dropping the tildes)

$$H_0 = \int \frac{\eta^2 + \xi_y^2}{2} dy \tag{2.11}$$

$$H_1 = -\frac{1}{2} \int \left({\xi_y}^2 \eta + \xi_y |D| \xi_y \right) dy. \tag{2.12}$$

Then, as anticipated in the introduction, following [9], it is convenient to introduce the characteristic variables defined by (1.4), in terms of which the Poisson tensor essentially becomes the Gardner Poisson tensor (see Lemma A.2 below). Precisely, the Hamilton equations of a Hamiltonian $H(r, s)$ turn out to be given by

$$\dot{r} = -\partial_y \nabla_r H \ , \quad \dot{s} = \partial_y \nabla_s H. \tag{2.13}$$

In particular, one has that H_0 takes the form

$$H_0 = \int \frac{r^2 + s^2}{2} dy, \tag{2.14}$$

whose equations of motion are (1.5), and H_1 is given by

$$H_1 = -\int \left(\frac{(r-s)|D|(r-s)}{4} + \frac{(r-s)^2(r+s)}{4\sqrt{2}} \right) dy. \tag{2.15}$$

We remark that (1.4) is just a change of variables, so if a solution is written in terms of the variables $r = r(y, t)$ and $s = s(y, t)$, then one can go back to the original variables, (η, ξ). Following [3], it is convenient to give the following definition.

Definition 2.1. *Given the two functions* $r(y, t)$ *and* $s(y, t)$*, we say that*

$$u(y, t) := (\eta(y, t), \xi(y, t)) \ , \tag{2.16}$$

with η, ξ *constructed inverting* (1.4)*, is called* the corresponding function in scaled physical variables.

As anticipated in the introduction, our goal is to put the system in normal form at first order. In order to state the result, we recall that the first two Hamiltonians of the BO hierarchy are

$$K_0(w) = \frac{1}{2} \int w^2 dy \ , \tag{2.17}$$

$$K_1(w) = -\frac{1}{4} \int \left(w|D|w + \frac{1}{\sqrt{2}} w^3 \right) dy \ . \tag{2.18}$$

We will obtain that the first order normal form of the interface problem is given by the Hamiltonian

$$H_{NF}(r, s) := K_0(r) + \epsilon K_1(r) + K_0(s) + \epsilon K_1(s) \ , \tag{2.19}$$

and remark that its Hamilton equations are two decoupled BO equations in translating coordinates, one for r and one for s.

In the statement of Theorem 2.2, which is our main theorem, we denote by $H^{\mathtt{s}}$ the standard Sobolev spaces of L^2 functions with $\mathtt{s}$ derivatives in L^2. More precisely, we work in the space of functions with zero average, namely s.t.

$$\int_{\mathbb{T}} \eta(y)dy = 0 \, , \quad \int_{\mathbb{T}} \xi(y)dy = 0 \, .$$

Such a space will be denoted by $H^{\mathtt{s}}_0$. We also denote by

$$\mathcal{H}^{\mathtt{s}} := H^{\mathtt{s}}_0 \times H^{\mathtt{s}}_0 \ni (r, s)$$

the phase space, and by $B^{\mathtt{s}}_{\mathtt{R}}$ the ball of radius $\mathtt{R}$ and center 0 in $\mathcal{H}^{\mathtt{s}}$.

Theorem 2.2. *For any* $\mathtt{s}$ *there exists* $\epsilon_* > 0$*,* $\mathtt{R} > 0$*,* $\mathtt{s}'$*, and* $\mathtt{s}''$ *fulfilling* $\mathtt{s}' < \mathtt{s}'' < \mathtt{s}$*, s.t., if* $0 < \epsilon < \epsilon_*$*, then there exists a map* $T_\epsilon : B^{\mathtt{s}''}_1 \to \mathcal{H}^{\mathtt{s}'}$ *with the following properties:*

(i) $\sup_{(r,s)\in B^{\mathtt{s}''}_1} \|T_\epsilon(r, s) - (r, s)\|_{\mathcal{H}^{\mathtt{s}''}} \le C\epsilon.$

(ii) For any initial datum $u_0 \equiv (r_0, s_0) \in B^{\mathtt{s}}_{\mathtt{R}} \subset \mathcal{H}^{\mathtt{s}}$ *for the scaled interface problem* (2.10)*, there exists a solution* $u_{BO}(t) \equiv (r_{BO}(t), s_{BO}(t))$*,* $u_{BO}(.) \in C^0(\mathbb{R}; B^{\mathtt{s}''}_1)$ *of the Hamilton equations of* (2.19) *s.t., defining*

$$u_a(.) \equiv (r_a(.), s_a(.)) := T_\epsilon(u_{BO}(.)) \in C^0(\mathbb{R}; \mathcal{H}^{\mathtt{s}'}) \, , \tag{2.20}$$

one has

$$\|u_a(0) - u_0\|_{\mathcal{H}^{\mathtt{s}'}} \le C\epsilon^2 \tag{2.21}$$

$$\dot{u}_a(t) = J\nabla H_{Int}(u_a(t)) + \epsilon^2 R(t) \, , \quad \forall t \in \mathbb{R} \, , \tag{2.22}$$

where H_{Int} *is the Hamiltonian* (2.10) *of the interface problem rewritten in the variables* (r, s) *and* $R(.) \in C^0(\mathbb{R}, \mathcal{H}^{\mathtt{s}'})$ *is bounded together with its time derivative.*

Remark 2.3. *Since the scaled variables are achieved through the scaling* (2.5)*, the condition that the* $\mathcal{H}^{\mathtt{s}}$ *norm of the initial datum is smaller than* $\mathtt{R}$ *is essentially a condition of smallness of* $\mathtt{R}\epsilon$*.*

We end this section by stating a conjecture that would become a theorem if a result similar to Theorem 4.18 of [15] were available for the interface problem. Indeed a theorem of that kind would allow one to deduce the following.

Conjecture 2.1. *With the same notations of Theorem 2.2, for any* $T > 0$ *there exist* $\epsilon_* > 0$ *and* $\mathtt{s} > 0$ *s.t., if the non scaled initial datum* $u_0 = (\eta_0, \xi_0)$ *has the form* (2.5) *with* $\left\|(\tilde{\eta}_0, \tilde{\xi}_0)\right\|_{H^{\mathtt{s}}_0 \times H^{\mathtt{s}+1}_0} \le 1$*, and* $\epsilon < \epsilon_*$*, denoting by* $u(t)$ *the solution of the interface problem* (2.3) *with initial datum* u_0*, then one has*

$$\|u_a(t) - u(t)\|_{L^\infty} \le C\epsilon^2 \, , \quad \forall t \in \left[-\frac{T}{\epsilon^3}, \frac{T}{\epsilon^3}\right] \, . \tag{2.23}$$

We leave the investigation of such a result to future work.

3. First order Birkhoff normal form when the generating flow does not exist

3.1. Standard Birkhoff normal form

In this subsection, we recall the construction of the first order Birkhoff normal form in the standard smooth case.

We first introduce some notations. We assume that the phase space is endowed by a scalar product $\langle .;. \rangle$; we denote by J the Poisson tensor[†] and define the Hamiltonian vector field X_χ of a Hamiltonian χ by $X_\chi := J\nabla\chi$, where ∇ is the gradient with respect to the scalar product of the space. Furthermore, given a function F, we denote by

$$\{F;\chi\} := \mathcal{L}_{X_\chi} F := dFJ\nabla\chi \equiv \langle \nabla F; J\nabla\chi \rangle$$

its Poisson bracket where χ is just its Lie derivative with respect to the vector field of χ.

Consider a Hamiltonian system depending on a small parameter ϵ and suppose we are just interested in its first order development so that we write

$$H = H_0 + \epsilon H_1 + O(\epsilon^2) . \tag{3.1}$$

We study the case where H_0 is a quadratic form and its Hamiltonian vector field generates a flow $\Phi^t_{H_0}$ which is periodic of period $\mathtt{T}$. We are now going to recall the proof of Theorem 3.1 below.

Theorem 3.1. *There exists a canonical transformation T (defined in a neighborhood of the origin) such that*

$$H \circ T = H_0 + \epsilon \langle H_1 \rangle + O(\epsilon^2) \tag{3.2}$$

where

$$\langle H_1 \rangle := \frac{1}{\mathtt{T}} \int_0^{\mathtt{T}} H_1 \circ \Phi^t_{H_0} dt . \tag{3.3}$$

Before recalling the proof, we anticipate a remark simplifying the explicit computation of $\langle H_1 \rangle$.

Remark 3.2. *If F is a function with the property that*

$$\{H_0; F\} = 0 , \tag{3.4}$$

then one has

$$\langle F \rangle \equiv F . \tag{3.5}$$

The proof that we are going to recall is based on the technique of the Lie transform, namely, we look for an auxiliary Hamiltonian χ whose Hamiltonian vector field generates the canonical transformation.

Let χ be a smooth function, consider the corresponding Hamilton equations, namely $\dot{u} = X_\chi(u)$, and denote by Φ^t_χ the corresponding flow.

Definition 3.3. *The map Φ^ϵ_χ will be called the Lie transform generated by χ.*

[†] In our case, this will be a constant invertible skewsymmetric linear operator. In the general situation, which however we will not meet in this paper, a Poisson tensor is not asked to be neither invertible, nor constant: it is only asked to fulfill a condition leading to the validity of the Jacobi identity for the Poisson brackets.

It is well known that Φ_{χ}^{ϵ} is a canonical transformation.

Then, exploiting the definition of Poisson brackets, one immediately gets that for any smooth function F

$$F \circ \Phi_{\chi}^{\epsilon} = F + \epsilon \{F;\chi\} + O(\epsilon^2) \,, \tag{3.6}$$

so that

$$\begin{aligned} H \circ \Phi_{\chi}^{\epsilon} &= (H_0 + \epsilon H_1) \circ \Phi_{\chi}^{\epsilon} + O(\epsilon^2) \\ &= H_0 + \epsilon \{H_0;\chi\} + \epsilon H_1 + O(\epsilon^2). \end{aligned} \tag{3.7}$$

We want to determine χ in such a way that the terms of order ϵ coincide with the average of H_1. To this end we recall the following lemma (see Lemma 5.3 of [7]).

Lemma 3.4. *Assume that the flow $\Phi_{H_0}^t$ is periodic of period $\mathtt{T}$. Define*

$$\langle H_1 \rangle(u) := \frac{1}{\mathtt{T}} \int_0^{\mathtt{T}} H_1(\Phi_{H_0}^{\tau}(u)) d\tau, \tag{3.8}$$

and $W := H_1 - \langle H_1 \rangle$. Then,

$$\chi(u) := \frac{1}{\mathtt{T}} \int_0^{\mathtt{T}} \tau W(\Phi_{H_0}^{\tau}(u)) d\tau \tag{3.9}$$

solves the homological equation

$$\{H_0;\chi\} + W = 0. \tag{3.10}$$

Then, using the function χ just constructed to generate the normalizing canonical transformation, one gets Theorem 3.1.

3.2. The case with vector field that does not generate a flow

To generalize the above construction to the case where the vector field of the function χ does not generate a flow, we exploit the fact that, in our construction, all the terms of order ϵ^2 are neglected. We are now going to make this precise, following [3] and [2].

We will work in a scale of Hilbert spaces $\mathcal{H} \equiv \{\mathcal{H}^{\mathtt{s}}\}$. The kind of maps that we are going to use have the property of being smooth maps from any Sobolev space to some Sobolev space of lower regularity. This is captured by the following definition.

Definition 3.5. *A map X will be said to be almost smooth if, $\forall \mathtt{r}, \mathtt{s}' \geq 0$, there exist $\mathtt{s}$ and an open neighborhood of the origin $\mathcal{U}_{\mathtt{rss}'} \subset \mathcal{H}^{\mathtt{s}}$ such that*

$$X \in C^{\mathtt{r}}(\mathcal{U}_{\mathtt{rss}'};\mathcal{H}^{\mathtt{s}'}) \,. \tag{3.11}$$

Furthermore, we will also deal with maps which depend on a small parameter ϵ. We will say that a family of maps $X(\epsilon, u)$ is almost smooth if it fulfills the above definition as a map from the scale of Hilbert spaces $\mathbb{R} \times H^{\mathtt{s}}$ to the scale $H^{\mathtt{s}}$. *In this case, we will assume that the domain $\mathcal{U}_{\mathtt{rss}'}$ of* (3.11) *has the form $\mathcal{U}_{\mathtt{rss}'} = I_{\mathtt{rss}'} \times \mathcal{V}_{\mathtt{rss}'}$ with $\mathcal{V}_{\mathtt{rss}'} \subset \mathcal{H}^{\mathtt{s}}$ and $I_{\mathtt{rss}'}$ an interval.* The important point is that the size of the open set $\mathcal{V}_{\mathtt{rss}'}$ does not depend on ϵ.

In the following, the width of open sets does not play any role, so we will avoid specifying it. In particular, we will often consider maps from a Hilbert space to some other space, and *by this we* **always** *mean a map defined in an open neighborhood of the origin.*

Definition 3.6. *In this context, we will write*

$$A = B + O(\epsilon^{r+1})$$

if

$$\frac{A - B}{\epsilon^{r+1}}$$

is an almost smooth map.

In this language, we can reformulate the construction of the preceding subsection simply by substituting the map Φ_χ^ϵ with the map

$$T_\epsilon := 1 + \epsilon X_\chi \ .$$

Then, in particular, defining

$$\mathcal{T}_\epsilon := 1 - \epsilon X_\chi \ , \tag{3.12}$$

one has

$$T_\epsilon \circ \mathcal{T}_\epsilon = 1 + O(\epsilon^2) \ , \quad \mathcal{T}_\epsilon \circ T_\epsilon = 1 + O(\epsilon^2) \ , \tag{3.13}$$

and, according to Theorem 4.9 of [3], one has that the following theorem holds.

Theorem 3.7. *Assume that H_1 has an almost smooth vector field, denote by χ the function* (3.9) *and define*

$$T_\epsilon := 1 + \epsilon X_\chi \ , \tag{3.14}$$

$$H_{NF} := H_0 + \epsilon \langle H_1 \rangle \tag{3.15}$$

Fix $\mathtt{s}'$, then there exist $\mathtt{s}''$ and $\mathcal{U}_{\mathtt{s}'\mathtt{s}''} \subset \mathcal{H}^{\mathtt{s}'}$ with the following property: Let $\zeta \in C^0([-\mathtt{T}_0, \mathtt{T}_0]; \mathcal{U}_{\mathtt{s}'\mathtt{s}''})$, $0 < \mathtt{T}_0 \leq \infty$, be a solution of

$$\dot{\zeta} = J \nabla H_{NF}(\zeta) \ .$$

Then, there exists $R \in C^0([-\mathtt{T}_0, \mathtt{T}_0]; \mathcal{H}^{\mathtt{s}''})$ s.t. $u(.) := T(\zeta(.)) \in C^1([-\mathtt{T}_0, \mathtt{T}_0]; \mathcal{H}^{\mathtt{s}''})$ fulfills the equation

$$\dot{u} = J \nabla H(u) + \epsilon^2 R(t) \ .$$

4. The Hamiltonian, its expansion and its normal form

To apply the above theory to the interface problem, we make precise the Hamiltonian formulation of the equations of the interface and compute the first two terms of its expansion in the small parameter.

First, we define the Dirichlet Neumann operators G and G_1.

Definition 4.1. *Given a function $\psi(x)$, consider the boundary value problem*

$$\Delta \phi = 0 \qquad (x, z) \in \Omega \tag{4.1}$$

$$\lim_{z \to -\infty} \phi_z(x, z) = 0 \tag{4.2}$$

$$\phi\Big|_{z=\eta(x)} = \psi \quad , \tag{4.3}$$

and let ϕ be its solution. Then, the linear operator $G(\eta)$ defined by

$$G(\eta)\psi = (-\eta_x, 1) \cdot \nabla \phi\Big|_{z=\eta(x)} \tag{4.4}$$

is called the Dirichlet Neumann operator for the domain Ω.

Definition 4.2. *Given a function $\psi_1(x)$, consider the boundary value problem*

$$\Delta\phi_1 = 0 \qquad (x, z) \in \Omega_1 \tag{4.5}$$

$$(\phi_1)_z\Big|_{z=h_1} = 0 \tag{4.6}$$

$$\phi_1\Big|_{z=\eta(x)} = \psi_1 \tag{4.7}$$

and let ϕ_1 be its solution. Then, the linear operator $G_1(\eta)$ defined by

$$G_1(\eta)\psi_1 = -(-\eta_x, 1) \cdot \nabla\phi\Big|_{z=\eta(x)} \tag{4.8}$$

is the Dirichlet Neumann operator for the domain Ω_1.

These definitions give a precise meaning to the Hamiltonian (2.3).

Then, in order to proceed in the expansion we need to know the expansion of the Dirichlet Neumann operators computed in [10] (see formulas A.7 and A.9). It turns out that G has a Taylor expansion in powers of the variable η, whose only relevant term for our construction is the first one, which is given by

$$G^{(0)} = |D| \tag{4.9}$$

where we used the standard notation $D := -i\partial_x$ (and with a little abuse of notation we are going to use the same symbol even when, after rescaling, the independent variable will be y, i.e., we will have $D := -i\partial_y$ when needed).

Concerning G_1, it also admits a Taylor expansion whose important terms (see again [10], formula A.10) are the first two given by

$$G_1^{(0)} = D\tanh(h_1 D) \,, \tag{4.10}$$

$$G_1^{(1)} = -D\eta D + G_1^{(0)}\eta G_1^{(0)} \,. \tag{4.11}$$

Computing the expansion of Hamiltonian (2.3) in terms of the scaled quantities $\tilde{\eta}$ and $\tilde{\xi}$ defined by

$$\eta(x) = \epsilon_1\tilde{\eta}(\mu x) \,, \qquad \xi(x) = \alpha\tilde{\xi}(\mu x) \,, \tag{4.12}$$

and the scaled variable

$$y = \mu x$$

one gets (omitting tildes) the form of the Hamiltonian summarized in the following lemma.

Lemma 4.3. *In the scaled physical coordinates, the Hamiltonian takes the form $H = a\epsilon^2(H_0 + \epsilon H_1 + O(\epsilon^2))$ with*

$$H_0 = \int_{\mathbb{T}} \frac{\xi_y^2 + \eta^2}{2} dy \tag{4.13}$$

$$H_1 = -\frac{1}{2}\int_{\mathbb{T}} (\xi_y^2\eta + \xi_y|D|\xi_y) dy \tag{4.14}$$

where we define ϵ and choose the other parameters as follows:

$$\epsilon_1 =: \epsilon h_1 , \qquad \mu = \epsilon \frac{\rho_1}{\rho h_1} , \tag{4.15}$$

and

$$\alpha = \rho h_1 \sqrt{g h_1 \frac{\rho - \rho_1}{\rho_1}} , \qquad a = \left(\frac{\rho_1}{\rho h_1}\right)^2 \sqrt{g h_1 \frac{\rho - \rho_1}{\rho_1}} . \tag{4.16}$$

Proof. We start by inserting the scaling (4.12) into the expansions of the Dirichlet Neumann operators:

$$\begin{aligned} G &= \mu|D| + O(\mu^2\epsilon_1) \\ G_1 &= \mu^2 h_1 D^2 - \mu^2 \epsilon_1 D\eta D + O(\mu^4) . \end{aligned} \tag{4.17}$$

Then, in order to get the expansion of the Hamiltonian, we need to expand

$$\begin{aligned} G_1[\rho_1 G + \rho G_1]^{-1} G &= \frac{1}{\rho_1} G_1 - \frac{\rho}{\rho_1} G_1 [\rho_1 G + \rho G_1]^{-1} G_1 \\ &= \mu^2 \frac{h_1 D^2}{\rho_1} - \mu^2 \epsilon_1 \frac{1}{\rho_1} D\eta D - \mu^3 \frac{h_1^2 \rho}{\rho_1^2} D^2 |D| + O(\epsilon_1 \mu^3 + \mu^4) . \end{aligned}$$

Then, according to Lemma A.1 of the Appendix, one has that, in terms of the scaled variables, the Hamiltonian is given by the integral of

$$\frac{1}{2} \frac{\mu}{\epsilon_1 \alpha} \left[\mu^2 \alpha^2 \xi \frac{h_1 D^2}{\rho_1} \xi - \mu^2 \alpha^2 \epsilon_1 \frac{1}{\rho_1} \xi D\eta D \xi \right. \tag{4.18}$$

$$\left. - \mu^3 \alpha^2 \frac{h_1^2 \rho}{\rho_1^2} \xi D^2 |D| \xi + g(\rho - \rho_1) \epsilon_1^2 \eta^2 + O(\epsilon_1 \mu^3) \right] . \tag{4.19}$$

Choosing

$$\frac{\mu^2 \alpha^2}{\epsilon_1^2} = \frac{g(\rho - \rho_1)}{h_1} \rho_1$$

one gets that the coefficients of the two main terms of the Hamiltonian (i.e., the first and the fourth in the above expression) become equal. Choosing

$$\frac{\mu}{\epsilon_1} = \frac{\rho_1}{\rho h_1^2}$$

the coefficients of the first order part of the Hamiltonian also become equal. In particular, this gives

$$\alpha = \rho h_1 \sqrt{g h_1 \frac{\rho - \rho_1}{\rho_1}} ,$$

and the relation between μ and the newly introduced ϵ. □

Finally, we pass to the characteristic variables (1.4), getting the expansion (2.10) for the Hamiltonian. Applying Theorem 3.7, we get the following result.

Theorem 4.4. *There exist two almost smooth maps $T_\epsilon = 1 + O(\epsilon)$ and $\mathcal{T}_\epsilon = 1 + O(\epsilon)$ such that, defining H_{NF} by* (2.19)*, the following holds true. Fix $\mathtt{s}'$, then there exist $\mathtt{s}''$ and $\mathcal{U}_{\mathtt{s}'\mathtt{s}''} \subset \mathcal{H}^{\mathtt{s}''}$ with the following property: Let $\zeta \in C^0([-\mathtt{T}_0, \mathtt{T}_0]; \mathcal{U}_{\mathtt{s}'\mathtt{s}''})$, $0 < \mathtt{T}_0 \le \infty$, be a solution of*

$$\dot{\zeta} = J\nabla H_{NF}(\zeta) \ .$$

Then, there exists $R \in C^0([-\mathtt{T}_0, \mathtt{T}_0]; \mathcal{H}^{\mathtt{s}'})$ s.t. $u(.) := T(\zeta(.)) \in C^0([-\mathtt{T}_0, \mathtt{T}_0]; \mathcal{H}^{\mathtt{s}'})$ fulfills the equation

$$\dot{u} = J\nabla H_{Int}(u) + \epsilon^2 R(t) \ .$$

Proof. We just have to show that the average of H_1 as written in (2.15) is given by (2.18). To this end, we show that the terms depending only on r, respectively only on s, coincide with their average, while the terms involving the product of r and s have zero average, and therefore disappear from $\langle H_1 \rangle$.

In order to prove that the terms depending only on r coincide with their average, we use Remark 3.2. We concentrate on the most non-standard term, so we compute

$$\left\{ H_0; \int_{\mathbb{T}} \frac{r|D|r}{2} \right\} = \langle \nabla H_0; J\nabla \int_{\mathbb{T}} r|D|r \rangle = -\int_{\mathbb{T}} r\partial(|D|r) = \int_{\mathbb{T}} r(|D|r_y) \ . \tag{4.20}$$

It is easy to see that the operator $|D|\partial$ is skew symmetric, and thus we have

$$\int_{\mathbb{T}} r(|D|r_y) = \int_{\mathbb{T}} r(|D|\partial r) = -\int_{\mathbb{T}} (|D|\partial r) r = -\int_{\mathbb{T}} r(|D|r_y) \ .$$

From this equality, it immediately follows that (4.20) vanishes. The other terms are easier and the verification of the fact that they vanish is left to the reader. Thus, we have that

$$\left\langle \int (r|D|r + s|D|s + r^3 + s^3) dy \right\rangle = \int (r|D|r + s|D|s + r^3 + s^3) dy \ .$$

In order to show that the remaining terms of $\langle H_1 \rangle$ are indeed zero, the idea is that, in the actual calculation, the integral of the relevant products is factorized in the product of integrals, one of which is the average of r (or of s), and that is zero by hypothesis. For example, we have

$$\begin{aligned} \left\langle \int r^2 s \, dy \right\rangle &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{T}} r^2(\Phi^t_{H_0}(y)) s(\Phi^t_{H_0}(y)) dy dt = \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\pi}^{+\pi} r^2(y+t) s(y-t) dy dt \end{aligned}$$

which can be rewritten in a computable way using the variables $\alpha = y + t$ and $\beta = y - t$. Indeed, making the change of variables and exploiting the periodicity of the functions r and s, one gets that this is equal to

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} r^2(\alpha) s(\beta) d\alpha d\beta \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} r^2(\alpha) d\alpha \int_{-\pi}^{+\pi} s(\beta) d\beta = 0 \end{aligned}$$

since s has zero average. In a similar way, all the other terms depending linearly on s and r (i.e., rs^2, $r|D|s$ and $s|D|r$) have zero average. □

End of the proof of Theorem 2.2. We proceed as follows: First, taking an initial datum $u_0 \equiv (r_0, s_0) \in \mathcal{H}^{s}$, we transform it through the "pseudoinverse" $\mathcal{T}$ of Eq (3.12), getting

$$\zeta_0 := \mathcal{T}(u_0) \in \mathcal{H}^{s''} .$$

Then, by the theory of [1] (see also [13]), there exists a solution $\zeta_{BO}(.) \in C^0(\mathbb{R}; \mathcal{H}^{s''})$ of the BO equation, and therefore of the normal form equation, with the property that

$$\sup_{t\in\mathbb{R}} \|\zeta_{BO}(t)\|_{\mathcal{H}^{s''}} \leq C \|\zeta_0\|_{\mathcal{H}^{s''}} .$$

In particular, it follows that, if R is chosen small enough, the solution $\zeta(t)$ always lies in the ball of radius 1 which is the domain of T, and one can apply Theorem 4.4 to get the estimate (2.22). Furthermore, by the approximate inverse result Eq (3.13), one gets (2.21). □

5. Conclusions

We analyzed the interface problem for a two-layers fluid in the limit where BO equations are involved. Previous results where valid for solutions which where only right moving or left moving. We showed that, for generic albeit sufficiently smooth initial data, a couple of non-interacting BO equations describe the dynamics at a first order in the sense of the Birkhoff normal form theory. In a forthcoming work we aim at extending such a result to higher order approximations.

Appendix

A. On linear changes of coordinates

First, we recall that given a linear change of variables $u = B\zeta$, the equations of any Hamiltonian system remain Hamiltonian possibly with a different Poisson tensor. Precisely, one has that the equation $\dot{u} = J\nabla H(u)$ becomes $\dot{\zeta} = \tilde{J}\nabla\widehat{H}(\zeta)$, where $\widehat{H}(\zeta) := H(B\zeta)$ and $\tilde{J} := B^{-1}JB^{-*}$, where B^{-*} is the adjoint (with respect to the L^2 metric) of the inverse of B.

In particular, if B is a scaling of the form

$$u = B\zeta \qquad \Longleftrightarrow \qquad \begin{cases} \eta(x) = \epsilon_1 \tilde{\eta}(\mu x) \\ \xi(x) = \alpha\tilde{\xi}(\mu x) \end{cases} , \tag{A.1}$$

one has the following.

Lemma A.1. *The transformation* (A.1) *transforms the Hamilton's equations of H into the Hamilton's equations*

$$\tilde{H}(\zeta) := \frac{\mu}{\epsilon_1 \alpha} H(B\zeta) . \tag{A.2}$$

Concerning the characteristic variables, one has the following lemma.

Lemma A.2. *In the variables* (r, s) *defined by* (1.4)*, the Poisson tensor takes the form*

$$J = \begin{bmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{bmatrix}$$

so that the Hamilton's equations of an arbitrary Hamiltonian take the form (2.13).

Author contributions

Dario Bambusi and Simone Paleari: Conceptualization, formal analysis, investigation, validation and writing. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We thank Gene Wayne for pointing to our attention the paper [11] and Patrick Gerard for some suggestions on the problem of existence and uniqueness of BO. We also had some interesting discussions with Antonio Ponno. This project is supported by INDAM/GNFM and by PRIN 2020 (2020XB3EFL001) "Hamiltonian and dispersive PDEs".

Conflict of interest

The authors declare no conflict of interest.

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