OPEN ACCESS
Reviews in Mathematical Physics
(2024) 2450029 (62 pages)
© The Author(s)
DOI: 10.1142/S0129055X24500296

World Scientific
www.worldscientific.com

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Extended state space for describing renormalized Fock spaces in QFT

Sascha Lill

Dipartimento di Matematica,
Università degli Studi di Milano,
Via Cesare Saldini 50, 20133 Milano, Italy
sascha.lill@unimi.it

Received 8 March 2023
Accepted 25 May 2024
Published 16 July 2024

We revisit the non-perturbative renormalization of a class of simple polaron models with resting fermions. The considered dispersion relations and form factors are allowed to be highly singular, such that infinite self-energies and wave function renormalizations may occur and the diagonalizing dressing transformations might not be implementable on Fock space. Instead of taking cutoffs, we rigorously interpret the self-energies and wave function renormalizations as elements of suitable vector spaces, as well as a field extension of the complex numbers. Moreover, we define two extended state vector spaces (ESSs) over this field extension, which contain a dense subspace of Fock space, but also incorporate non-square-integrable wave functions. Elements of these ESSs can be seen as states of virtual particles, described in a mathematically rigorous way. The Hamiltonian without cutoffs, formally infinite counterterms, and the dressing transformation can then be defined as linear operators between certain subspaces of the two Fock space extensions. This way, we obtain a renormalized Hamiltonian which can be realized as a densely defined self-adjoint operator on Fock space.

Keywords: Non-perturbative QFT; renormalization; divergent integrals; Van Hove model; Fock space; Hamiltonian formalism; dressing transformation; interior-boundary conditions (IBC).

Mathematics Subject Classification 2020: 81T16, 81R10, 81Q10, 12F05, 81R30

This is an Open Access article published by World Scientific Publishing Company. It is distributed under the terms of the Creative Commons Attribution 4.0 (CC BY) License which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

1. Introduction

In this work, we revisit the renormalization of a certain class of formal quantum field theory[a] (QFT) Hamiltonians, which is done by employing a new mathematical framework that permits rigorous direct manipulations of formally infinite quantities. These quantities include divergent self-energies and wave function renormalizations, which are ubiquitous in QFT. The renormalization process is non-perturbative and does not involve any cutoffs.

The considered QFT Hamiltonians read as

$$H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty, \tag{1}$$

which means the following: We consider a species of $M \in \mathbb{N}_0$ fermions (associated with an index x) and a species of $N \in \mathbb{N}_0$ bosons (associated with an index y). The operators $A^\dagger(v), A(v)$ make each fermion create/annihilate a boson with some form factor v (in momentum space), $H_{0,y}$ describes the free evolution of the bosons and E_∞ is an *infinite self-energy counterterm*.

The Hamiltonian (1) thus corresponds to a direct product of many Van Hove Hamiltonians [3, 4], for which non-perturbative renormalization procedures are well known to exist. But in contrast to preceding works, (1) will not just be a formal expression, but a rigorous relation between linear operators that map $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$, where $\overline{\mathscr{F}} \subset \overline{\mathscr{F}}_{\mathrm{ex}}$ are extensions of a dense subspace of Fock space. In particular, the multiplication by the (possibly infinite) self-energy becomes a well-defined operator $E_\infty : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$.

In Sec. 8, we will also encounter more general Hamiltonians of the form

$$H_0 + A^\dagger(v) + A(v) - E_\infty, \tag{2}$$

with $H_0 = H_{0,x} + H_{0,y}$ including both a fermionic and a bosonic dispersion relation.

Apart from the flat fermionic dispersion relation, our class of Hamiltonians (1) allows to include the same dispersion relations and form factors as are used in many non-relativistic QFT models, including the Nelson model [5–8], the Fröhlich polaron [9], as well as relativistic polarons [10, 11].

Our novel framework includes several mathematical tools that allow for rigorous manipulations with infinite quantities in QFT. These tools include the following:

- A *vector space of divergent integrals* $\mathfrak{r} \in \mathrm{Ren}_1$.
- A *field of* (*possibly infinite*) *wave function renormalizations* eRen, containing exponentials $e^{\mathfrak{r}}$ and fractions of linear combinations thereof. In particular, eRen is a field extension of $\mathbb{C}$.

[a]Here, the term "QFT" is to be understood as any quantum mechanical model whose description involves creation and annihilation operators. There is also a narrower sense in which a QFT is a model satisfying the Wightman axioms [1, Chap. 3–1] or the Haag–Kastler Axioms [2]. Our class of models in kept simple, as it serves for demonstrative purposes, and does not satisfy either of both axiom sets.

- An *extended state space* (ESS) $\overline{\mathscr{F}}$, which is an eRen-vector space over smooth, complex-valued functions on configuration space, which are not necessarily L^2-integrable. $\overline{\mathscr{F}}$ extends a dense subspace of the Fock space, $\mathscr{F}$.
- A *second ESS* $\overline{\mathscr{F}}_{\mathrm{ex}}$, which allows for multiplying $\Psi \in \overline{\mathscr{F}}$ by elements $\mathfrak{r} \in \mathrm{Ren}_1$ (and not only by elements of eRen).

Neither the space $\overline{\mathscr{F}}$ nor $\overline{\mathscr{F}}_{\mathrm{ex}}$ carries a scalar product or even a topology. Instead, we assign physical meaning to subspaces of them by a *dressing transformation* $W(s)$, which is a linear operator $W(s) : \overline{\mathscr{F}}_{\mathrm{ex}} \supset \mathcal{D}_W \to \overline{\mathscr{F}}_{\mathrm{ex}}$ in form of a Gross transformation. The form factor $s(\boldsymbol{k})$ does only depend on the boson momentum $\boldsymbol{k}$ and not on the fermion momentum $\boldsymbol{p}$, so one can fiber-decompose $W(s)$ into several Weyl transformations, one for each fermion configuration $\boldsymbol{X} \in \mathbb{R}^{Md}$. The renormalized Hamiltonian $\widetilde{H}$ will be defined without cutoffs by

$$W(s)\widetilde{H} = HW(s),$$

i.e. we require that the following diagram commutes:

$$\begin{array}{ccccc}
\overline{\mathscr{F}}_{\mathrm{ex}} \supset W(s)[\mathcal{D}_{\mathscr{F}}] \supset W(s)[\widetilde{\mathcal{D}}_{\mathscr{F}}] & & \xrightarrow{\quad H \quad} & & W(s)[\mathcal{D}_{\mathscr{F}}] \subset \overline{\mathscr{F}}_{\mathrm{ex}} \\
& \uparrow W(s) & & & \uparrow W(s) \\
\mathscr{F} \supset \mathcal{D}_{\mathscr{F}} \supset \widetilde{\mathcal{D}}_{\mathscr{F}} & & \xrightarrow{\quad \widetilde{H} \quad} & & \mathcal{D}_{\mathscr{F}} \subset \mathscr{F}
\end{array}$$

The large domain $\mathcal{D}_{\mathscr{F}}$, with the fermionic wave function being a Schwartz function, is defined in (135). It contains the small domain $\widetilde{\mathcal{D}}_{\mathscr{F}}$ defined in (165) via (164), with fermionic wave function in C_c^∞, and its support avoiding the set of collision configurations. It is necessary to introduce two domains, since $\widetilde{H}$ does neither map $\mathcal{D}_{\mathscr{F}}$ nor $\widetilde{\mathcal{D}}_{\mathscr{F}}$ into itself. Nevertheless, $\widetilde{H} : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$ can be defined. By Lemmas 5.6 and 7.1, both $\mathcal{D}_{\mathscr{F}}$ and $\widetilde{\mathcal{D}}_{\mathscr{F}}$ are dense in $\mathscr{F}$.

Our *main result* is Theorem 6.1, which states that the renormalized Hamiltonian $\widetilde{H}$ is indeed defined as a linear operator $\widetilde{H} : \mathscr{F} \supset \mathcal{D}_{WS} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ (with definition of the domain $\mathcal{D}_{WS}$ in (108)). By means of Lemma 7.1, under certain conditions, this $\widetilde{H}$ can indeed be interpreted as a Fock space operator $\widetilde{H} : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$, which allows for self-adjoint extensions by Corollary 7.1. More precisely, we have

$$\widetilde{H} = H_{0,y} + V, \tag{3}$$

with V being a pair potential interaction between fermions.

The result (3) after a dressing is not too surprising, as (1) just corresponds to a direct integral of Van Hove Hamiltonians. It is well known [3] that Van Hove Hamiltonians can be diagonalized by an algebraic Weyl transformation. That means, after replacing creation and annihilation operators like $a(v) \mapsto a(v) + \langle v, s\rangle$ and $a^\dagger(v) \mapsto a^\dagger(v) + \langle s, v\rangle$, as well as dropping an infinite self-energy constant, one arrives at a non-interacting Hamiltonian, which equals $H_{0,y}+V$ restricted to a fiber.

The main novelty in this paper is that we are able to define $W(s), s \notin L^2$ and certain products thereof, creation operators $A^\dagger(v), v \notin L^2$ and even divergent integrals

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

$\mathfrak{r} \in \mathrm{Ren}_1$, as linear operators on suitable domains. A similar ESS framework was recently successfully applied by the author to Bogoliubov transformations in place of $W(s)$ [12, 13]. The rigorous treatment of divergent integrals may become useful also in a purely algebraic approach, which does not refer to any Fock space extensions, but nevertheless produces divergent integrals when evaluating commutation relations.

Let us add some remarks about the dressing operator $W(s)$: Gross transformations are usually of the form $W(s) = e^{A^\dagger(s) - A(s)}$, where $A^\dagger(s) = \sum_{j=1}^M A_j^\dagger(s)$ describes a boson creation induced by all fermions $j \in \{1, \dots, M\}$. We will choose a slightly different dressing operator, which formally reads

$$W(s) = W_M(s) \dots W_1(s), \quad W_j(s) = e^{A_j^\dagger(s) - A_j(s)}. \tag{4}$$

The reason is that, when applying the Baker–Campbell–Hausdorff formula (as in (118)), formal calculations with $e^{A^\dagger(s) - A(s)}$ would produce exponential operators of the kind $e^{V_{jj'}}$, with $V_{jj'}$ being a potential interaction between fermions j and j'. Those expressions $e^{V_{jj'}}$ cannot be defined as operators on the ESS $\overline{\mathscr{F}}_{\mathrm{ex}}$, so we need to avoid their occurrence.

In general, we will define the extended dressing operators $W_j(s)$ such that their action for $s \in \mathfrak{h}$ slightly differs from that of the usual Fock space dressing operators $W_{\mathscr{F},j}(s) : \mathscr{F} \to \mathscr{F}$. That is, we drop any terms of the kind $e^{V_{jj'}}$ in our definition. This *ad hoc* modification is justified by the fact that, in a sufficiently regular case, the $V_{jj'}$-operators commute with $W(s)$ (Lemma 5.5). So even if we could define them as operators, they would just act as an independent factor that can be pulled to the left. So dropping all $e^{V_{jj'}}$-terms can essentially be seen as a simplification of the bookkeeping.

Nevertheless, it would be interesting for future works to define $e^{V_{jj'}}$ as an operator, mapping to some ESS $\overline{\mathscr{F}}'_{\mathrm{ex}}$, which is constructed differently than $\overline{\mathscr{F}}_{\mathrm{ex}}$. The current ESS $\overline{\mathscr{F}}_{\mathrm{ex}}$ also does not allow for defining general products of annihilation operators $A_{j_1}(v_1) \dots A_{j_n}(v_n)$ with $n \geq 2$. It is a further interesting question, how such products can be defined in the future by an alternative construction of some $\overline{\mathscr{F}}'_{\mathrm{ex}}$.

The ESS construction is currently at an early stage of development. We expect that the methods developed here may facilitate the renormalization of more involved formal Hamiltonians.

Although the spaces $\overline{\mathscr{F}}, \overline{\mathscr{F}}_{\mathrm{ex}}$ do not have a topological structure, we may define a *renormalized scalar product* on $W(s)[\mathcal{D}_{\mathscr{F}}]$ via

$$\langle W(s)\Psi, W(s)\Phi \rangle_{\mathrm{ren}} := \langle \Psi, \Phi \rangle \quad \forall\, \Psi,\ \Phi \in \mathcal{D}_{\mathscr{F}}. \tag{5}$$

The completion of $W(s)[\mathcal{D}_{\mathscr{F}}]$ with respect to $\langle \cdot, \cdot \rangle_{\mathrm{ren}}$ defines a Hilbert space $\mathscr{F}_{\mathrm{ren}}$, the *renormalized Fock space*. The map $W(s)$ then uniquely extends to an isometric isomorphism between $\mathscr{F}$ and $\mathscr{F}_{\mathrm{ren}}$.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

We remark that our result (3) is actually just the lowest-order approximation in a perturbation expansion within the weak-coupling regime, and completely decouples the fermions from the bosonic radiation field. The reason is that we both consider resting fermions (by excluding $H_{0,x}$ from H) and restrict to form factors $v(\boldsymbol{k})$, that only depend on the momentum $\boldsymbol{k}$ of the emitted boson, and not on the momentum $\boldsymbol{p}$ of the fermion emitting it. It is physically expected and confirmed for the Nelson model with UV-cutoff [14, 15], that $W_\Lambda^* H_\Lambda W_\Lambda$ contains interactions between fermions and the radiation field. When changing to $v(\boldsymbol{p}, \boldsymbol{k})$, one may formally use a so-called "Lie–Schwinger series" [16]

$$\tilde{H} = e^A B a^{-A} = \sum_{n=0}^{\infty} \frac{\mathrm{ad}^n(A)B}{n!},$$

where $A := -A^\dagger(s) + A(s)$, $B := H$ and with the n-fold commutator

$$\mathrm{ad}^n(A)B := [A, [A, \ldots, [A, B]\ldots]].$$

However, establishing well-definedness of $\mathrm{ad}^n(A)B$ for $H + H_{0,x}$ with v depending on $\boldsymbol{p}$ is a rather involved task, so we postpone it to future investigations.

Another interesting objective for future works would be to introduce a mass renormalization term δm. In constructive QFT (CQFT), where the Hamiltonian is of a form different from (1), mass renormalization terms can also be found, e.g., within the renormalization of Φ^4-theory in $2+1$ dimensions [17], the Yukawa model [18–20] or the Lee model [21, Chap. III]. Other CQFT renormalization procedures work without a mass renormalization, such as those for Φ^4- and $P(\Phi)$-theory in $1+1$ dimensions [22–26], for related models with exponential interactions [27], the Thirring model [28] or the Federbush model [29]. For a more extensive review on CQFT literature, see [30, 31].

Let us also mention that there has recently been some considerable interest in another cutoff-free renormalization technique via so-called "interior-boundary conditions" (IBC) [32–42]. In fact, IBC renormalization has been an important source of inspiration for the ESS construction: As we explain in Sec. 8, IBC renormalization provides an explicit formula for the domain of the renormalized Hamiltonian (181), which may formally contain vectors outside L^2, if the interaction becomes too singular. In fact, IBC renormalization is limited to models, where the "interacting vacuum is Fock", that is, one may use the same representation of creation and annihilation operators for both the free and the interacting Hamiltonian. A famous theorem of Haag [43; 44, Sec. II.1] states that this condition can never be fulfilled within a relativistic QFT satisfying the Haag–Kastler axioms. When passing over to relativistic dispersion relations and form factors, IBC renormalization therefore reaches a certain limit, which the ESS construction is designed to overcome. Even in cases where IBC renormalization is successful, ESSs allow for presenting the "cancellation of infinities" in a rigorous and particularly convenient way, as shown in Sec. 8.

The rest of this paper is structured as follows. After specifying some mathematical notation in Sec. 2, we conduct the ESS construction in Sec. 3, finally resulting in spaces $\overline{\mathscr{F}}, \overline{\mathscr{F}}_{\mathrm{ex}}$.

We then establish $H_0, A^\dagger, A$ and E_∞ as linear operators on $\overline{\mathscr{F}}$ or $\overline{\mathscr{F}}_{\mathrm{ex}}$ in Sec. 4.

In Sec. 5, we construct the dressing transformation $W(s)$. Further, we define an *extended Weyl algebra* $\overline{\mathcal{W}}$, which includes linear combinations of the operators $W_j(s) = e^{A_j^\dagger(s) - A_j(s)}$. $\overline{\mathcal{W}}$ allows for a multiplication with infinite wave function renormalization factors $\mathfrak{c} \in \mathrm{eRen}$, and the Weyl relations

$$\begin{aligned} W_j(s)^{-1} &= W_j(-s), \\ W_j(s_1) W_j(s_2) &= e^{\mathrm{Im}\langle s_1, s_2\rangle} W_j(s_1 + s_2), \end{aligned} \tag{6}$$

hold with $e^{\mathrm{Im}\langle s_1, s_2\rangle} \in \mathrm{eRen}$. However, note that $\overline{\mathcal{W}}$ is generated by $W_j(s)$ for a fixed j, instead of $W(s)$ and the dressing operator $W(s)$ is not contained in $\overline{\mathcal{W}}$.

In Sec. 6, we compute $\widetilde{H}$ such that $W(s)\widetilde{H} = HW(s)$. This section contains the main result, Theorem 6.1.

The proof that under certain conditions $\widetilde{H}$ is a Fock space operator and allows for self-adjoint extensions follows in Sec. 7. Thus, $\widetilde{H}$ generates well-defined quantum dynamics, although they are not necessarily unique.

Finally, in Sec. 8, we briefly review, using ESSs, how the cancellation of divergent quantities within IBC renormalization works. Essentially, IBC renormalization uses a non-unitary dressing transformation $W_{\mathrm{IBC}} = (1 + H_0^{-1} A^\dagger)^{-1}$, which we prove to be a bijective map $\overline{\mathscr{F}} \to \overline{\mathscr{F}}$ under relatively mild assumptions (Proposition 8.1). This includes cases, where W_{IBC} maps out of the Fock space. We also consider the non-unitary dressing transformation $T = e^{-\boldsymbol{\Lambda}/2} e^{-H_0^{-1} A^\dagger}$ with $e^{-\boldsymbol{\Lambda}/2}$ being an infinite wave function renormalization, which is a simplified version of CQFT dressings in [17, 21, 45]. Here, we define $T : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$.

2. The Mathematical Model

2.1. *Formal Hamiltonian*

In this paper, we consider a class of non-perturbative QFT models. These involve two species of particles:

- There is one species of spinless fermionic particles (x).
- These fermions interact by exchange of spinless bosons (y).

By M and N, we denote the number of x- and y-particles, respectively. In the particle-position representation, a single particle is described by a Hilbert space vector

$$\varphi \in \mathfrak{h} = L^2(\mathbb{R}^d, \mathbb{C}), \tag{7}$$

with some coordinates denoted by a boldface symbol $\boldsymbol{x} \in \mathbb{R}^d$. So the system has d space dimensions and $d + 1$ spacetime dimensions. The configuration of the entire

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

system is given by an element $(\boldsymbol{X}, \boldsymbol{Y})$ of the configuration space $\mathcal{Q}$

$$(\boldsymbol{X}, \boldsymbol{Y}) = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_N) \in \mathcal{Q}_x \times \mathcal{Q}_y =: \mathcal{Q}, \tag{8}$$

where the x-, and y-configuration spaces $\mathcal{Q}_x, \mathcal{Q}_y$ and its sectors $\mathcal{Q}_x^{(M)}, \mathcal{Q}_y^{(N)}, \mathcal{Q}^{(M,N)}$ are defined as

$$\begin{aligned} &\bigsqcup_{M=0}^{\infty} \mathbb{R}^{Md} =: \bigsqcup_{M=0}^{\infty} \mathcal{Q}_x^{(M)} =: \mathcal{Q}_x, \quad \bigsqcup_{N=0}^{\infty} \mathbb{R}^{Nd} =: \bigsqcup_{N=0}^{\infty} \mathcal{Q}_y^{(N)} =: \mathcal{Q}_y, \\ &\mathcal{Q}^{(M,N)} := \mathcal{Q}_x^{(M)} \times \mathcal{Q}_y^{(N)}. \end{aligned} \tag{9}$$

The state of the system at time t is described by a Fock space vector

$$\Psi \in L^2(\mathcal{Q}, \mathbb{C}). \tag{10}$$

Note that the term "state" is also used to describe a complex-valued map ω on the bounded linear operators on $\mathscr{F}$ (possibly a mixed state). We will only refer to state vectors when using the term "state".

For states, we assume the function $\Psi \in L^2(\mathcal{Q}, \mathbb{C})$ to be antisymmetric under coordinate exchange for fermions x and symmetric under coordinate exchange for bosons y. The corresponding symmetrization and antisymmetrization operators S_-, S_+ are defined on $L^2(\mathcal{Q}_x, \mathbb{C})$ and $L^2(\mathcal{Q}_y, \mathbb{C})$ as

$$\begin{aligned} (S_-\Psi)(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M) &= \frac{1}{M!} \sum_{\sigma \in S_M} \operatorname{sgn}(\sigma) \Psi(\boldsymbol{x}_{\sigma(1)}, \ldots, \boldsymbol{x}_{\sigma(M)}), \\ (S_+\Psi)(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N) &= \frac{1}{N!} \sum_{\sigma \in S_N} \Psi(\boldsymbol{x}_{\sigma(1)}, \ldots, \boldsymbol{x}_{\sigma(N)}), \end{aligned} \tag{11}$$

where S_N is the permutation group of $\{1, \ldots, N\}$. The *fermionic/bosonic/total Fock space* is then

$$\mathscr{F}_x := S_-[L^2(\mathcal{Q}_x, \mathbb{C})], \quad \mathscr{F}_y := S_+[L^2(\mathcal{Q}_y, \mathbb{C})], \quad \mathscr{F} := \mathscr{F}_x \otimes \mathscr{F}_y. \tag{12}$$

Each of these spaces decays into sectors according to (9)

$$\mathscr{F}_x = \bigoplus_{M=0}^{\infty} \mathscr{F}_x^{(M)}, \quad \mathscr{F}_y = \bigoplus_{N=0}^{\infty} \mathscr{F}_y^{(N)}, \quad \mathscr{F} = \bigoplus_{M,N=0}^{\infty} \mathscr{F}^{(M,N)}. \tag{13}$$

It is convenient to describe the action of the formal Hamiltonian H in the particle-momentum representation: For any $\Psi \in \mathscr{F}$ with particle-position representation $\Psi(\boldsymbol{X}, \boldsymbol{Y})$, we define the Fourier transform $\Psi(\boldsymbol{P}, \boldsymbol{K})$ with momentum configuration

$$(\boldsymbol{P}, \boldsymbol{K}) = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_M, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N) \in \mathbb{R}^{Md+Nd}, \tag{14}$$

via

$$\Psi(\boldsymbol{P}, \boldsymbol{K}) := (2\pi)^{-\frac{Md+Nd}{2}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^M} \Psi(\boldsymbol{X}, \boldsymbol{Y})\, e^{-i\boldsymbol{P}\cdot\boldsymbol{X} - i\boldsymbol{K}\cdot\boldsymbol{Y}}\, d^{Md}\boldsymbol{X} d^{Nd}\boldsymbol{Y}. \tag{15}$$

In the following, the set of variables plugged into $\Psi(\cdot,\cdot)$, i.e. $(\boldsymbol{X},\boldsymbol{Y})$ or $(\boldsymbol{P},\boldsymbol{K})$ will specify, which representation is meant.

As a formal Hamiltonian, we consider the following expression with zero fermion dispersion relation:

$$H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty. \tag{16}$$

In Sec. 8, we will also consider similar Hamiltonians of this kind that feature a nonzero fermion dispersion relation, i.e. where $H_{0,y}$ is replaced by $H_0 = H_{0,x} + H_{0,y}$. The formal definitions of the relevant expressions read as follows:

- The kinetic term H_0 is characterized by two dispersion relations, i.e. by real-valued functions $\theta(\boldsymbol{p}), \omega(\boldsymbol{k}) \in C^\infty(\mathbb{R}^d\backslash\{0\})$ for fermions and bosons, respectively. In Theorem 6.1 we assume $\theta \equiv 0$, but it can be nonzero in Proposition 4.1 and in Sec. 8. We decompose

$$\begin{aligned} H_0 &= H_{0,x} + H_{0,y}, \\ (H_{0,x}\Psi)(\boldsymbol{P},\boldsymbol{K}) &= \sum_{j=1}^M \theta(\boldsymbol{p}_j)\Psi(\boldsymbol{P},\boldsymbol{K}), \\ (H_{0,y}\Psi)(\boldsymbol{P},\boldsymbol{K}) &= \sum_{\ell=1}^N \omega(\boldsymbol{k}_\ell)\Psi(\boldsymbol{P},\boldsymbol{K}). \end{aligned} \tag{17}$$

  A shorthand notation is to use the symbols $d\Gamma_x(\cdot)$ and $d\Gamma_y(\cdot)$ for second quantization

$$H_{0,x} = d\Gamma_x(\theta), \quad H_{0,y} = d\Gamma_y(\omega), \tag{18}$$

  with multiplication operators

$$\begin{aligned} &\theta : \mathfrak{h} \supseteq \mathrm{dom}(\theta) \to \mathfrak{h}, \qquad && \omega : \mathfrak{h} \supseteq \mathrm{dom}(\omega) \to \mathfrak{h}, \\ &\phi(\boldsymbol{p}) \mapsto \theta(\boldsymbol{p})\phi(\boldsymbol{p}), && \phi(\boldsymbol{k}) \mapsto \omega(\boldsymbol{k})\phi(\boldsymbol{k}). \end{aligned} \tag{19}$$

- The creation part $A^\dagger(v)$ makes each fermion create a boson. It is specified by a form factor v. In this paper, we will restrict to $v \in C^\infty(\mathbb{R}^d\backslash\{0\})$ and put some further assumptions on the scalings of v for $|\boldsymbol{k}| \to 0$ (IR-regime) and $|\boldsymbol{k}| \to \infty$ (UV-regime). These assumptions are described in Sec. 2.2.

  We may write $A^\dagger(v)$ as a sum over operators $A_j^\dagger(v), j \in \{1,\dots,M\}$, which only make fermion j create a boson. The formal definition in particle-momentum representation reads

$$\begin{aligned} (A^\dagger(v)\Psi)(\boldsymbol{P},\boldsymbol{K}) &= \left(\sum_{j=1}^M A_j^\dagger \Psi\right)(\boldsymbol{P},\boldsymbol{K}) \\ &= \sum_{j=1}^M \frac{1}{\sqrt{N}} \sum_{\ell=1}^N v(\boldsymbol{k}_\ell)\Psi(\boldsymbol{P} + e_j\boldsymbol{k}_\ell, \boldsymbol{K}\backslash\boldsymbol{k}_\ell), \end{aligned} \tag{20}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

where

- $\boldsymbol{K}\backslash\boldsymbol{k}_\ell = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\ell-1}, \boldsymbol{k}_{\ell+1}, \ldots, \boldsymbol{k}_N)$ denotes $\boldsymbol{K}$ without $\boldsymbol{k}_\ell$.
- $\boldsymbol{P} + e_j\boldsymbol{k}_\ell = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{j-1}, \boldsymbol{p}_j + \boldsymbol{k}_\ell, \boldsymbol{p}_{j+1}, \ldots, \boldsymbol{p}_M)$ is the shifted fermion momentum.

Here, e_j is meant to denote the jth unit vector and, by slight abuse of notation, $e_j\boldsymbol{k}_\ell = (0, \ldots, \boldsymbol{k}_\ell, \ldots, 0) \in \mathbb{R}^{Md}$ is used to denote the assignment of an additional momentum $\boldsymbol{k}_\ell$ to fermion j, before boson ℓ is emitted.

The corresponding definition in particle-position representation uses the Fourier inverse $\check{v}$ of the form factor v and reads[b]

$$(A^\dagger(v)\Psi)(\boldsymbol{X}, \boldsymbol{Y}) = \left(\sum_{j=1}^{M} A_j^\dagger(v)\Psi\right)(\boldsymbol{X}, \boldsymbol{Y})$$
$$= \sum_{j=1}^{M} \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} \check{v}(\boldsymbol{y}_\ell - \boldsymbol{x}_j)\Psi(\boldsymbol{X}, \boldsymbol{Y}\backslash\boldsymbol{y}_\ell). \tag{21}$$

Without the cutoff, the form factor is $v \notin L^2$ in many physically relevant models. Its Fourier inverse $\check{v}$ is therefore $\notin L^2$, as well, if it is even defined.

Note that there is a more general class of physically interesting models, which we do not treat here: The form factor may depend on the fermion momentum $\boldsymbol{p}$, i.e. it may read $v(\boldsymbol{p}, \boldsymbol{k})$ with $v : \mathbb{R}^{2d} \to \mathbb{C}$, such as in [46–48].

- The annihilation part $A(v)$ in particle-momentum representation reads

$$(A(v)\Psi)(\boldsymbol{P}, \boldsymbol{K}) = \left(\sum_{j=1}^{M} A_j(v)\Psi\right)(\boldsymbol{P}, \boldsymbol{K})$$
$$= \sum_{j=1}^{M} \sqrt{N+1} \int v(\tilde{\boldsymbol{k}})^* \Psi(\boldsymbol{P} - e_j\tilde{\boldsymbol{k}}, \boldsymbol{K}, \tilde{\boldsymbol{k}}) d\tilde{\boldsymbol{k}}. \tag{22}$$

In particle-position representation

$$(A(v)\Psi)(\boldsymbol{X}, \boldsymbol{Y}) = \left(\sum_{j=1}^{M} A_j(v)\Psi\right)(\boldsymbol{X}, \boldsymbol{Y})$$
$$= \sum_{j=1}^{M} \sqrt{N+1} \int \check{v}(\tilde{\boldsymbol{y}} - \boldsymbol{x}_j)^* \Psi(\boldsymbol{X}, \boldsymbol{Y}, \tilde{\boldsymbol{y}})\, d\tilde{\boldsymbol{y}}. \tag{23}$$

Note that for $v \in \mathfrak{h}$, both $A^\dagger(v)$ and $A(v)$ can be defined as operators on a dense domain in $\mathscr{F}$, where $A^\dagger(v)$ is the adjoint of $A(v)$.

[b]We use the convention that $A_j^\dagger$ is defined on all (M)-sectors, where $(A_j^\dagger(v)\Psi)^{(M,N)} = 0$ whenever $j > M$.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

- The self-energy E_∞ is a formal multiplication operator of the form $E_\infty = d\Gamma_x(E_1)$ with $E_1 : \mathbb{R}^d \to \mathbb{R}$. In particle-momentum representation

$$(E_\infty \Psi)(\boldsymbol{P}, \boldsymbol{K}) = \sum_{j=1}^{M} E_1(\boldsymbol{p}_j)\Psi(\boldsymbol{P}, \boldsymbol{K}). \tag{24}$$

  We will consider the expression

$$E_1(\boldsymbol{p}) = \int -\frac{v(\boldsymbol{k})^* v(\boldsymbol{k})}{\omega(\boldsymbol{k})}\, d\boldsymbol{k}. \tag{25}$$

  With cutoffs σ, Λ applied to v, this integral would be finite in many physically interesting situations. However, without cutoffs it is typically divergent and hence the operator E_∞ becomes a formal expression. We will define it as a map $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ in Proposition 4.1.

2.2. *Scaling degrees*

The convergence of integrals appearing in formal calculations depends on how $\theta(\boldsymbol{p}), \omega(\boldsymbol{k})$ and $v(\boldsymbol{k})$ scale at $|\boldsymbol{p}|, |\boldsymbol{k}|$ going to ∞ or 0. We assume a polynomial scaling, of which we keep track using scaling degrees m (UV) and β (IR). The scaling degree m in the UV-regime is commonly used for symbols $s \in \mathcal{S}^m$. As we choose $\theta, \omega, v \in C^\infty(\mathbb{R}^d \backslash \{0\})$, an additional pole might appear at $|\boldsymbol{k}| \to 0$, which we assume to be of order β. Both scalings are assumed to be exact, i.e. polynomial bounds exist from above and below.

Definition 2.1 (Scaling). Consider $s \in C^\infty(\mathbb{R}^d \backslash \{0\})$. We say that s has a *polynomial scaling* if there are some scaling degrees $m_s \leq \beta_s \in \mathbb{R}$ and constants $C_1, C_2 \in \mathbb{R}$ such that

$$|s(\boldsymbol{k})| \leq C_1 |\boldsymbol{k}|^{\beta_s} + C_2 |\boldsymbol{k}|^{m_s} \quad \forall\, \boldsymbol{k} \in \mathbb{R}^d \backslash \{0\}. \tag{26}$$

The space of all *symbols with polynomial scaling* is denoted

$$\dot{\mathcal{S}}^\infty_1 := \big\{ s \in C^\infty(\mathbb{R}^d \backslash \{0\}) \,|\, (26) \text{ holds} \big\}. \tag{27}$$

Definition 2.2 (Exact Scaling). Consider a symbol $s \in C^\infty(\mathbb{R}^d \backslash \{0\})$. We say that the symbol has an *exact polynomial scaling* (see Fig. 1) if there are some $0 < \underline{\varepsilon} < \overline{\varepsilon} \in \mathbb{R}$ such that

- There are a UV scaling degree $m_s \in \mathbb{R}$ and $c_1, C_1 > 0$ with

$$c_1 |\boldsymbol{k}|^{m_s} \leq |s(\boldsymbol{k})| \leq C_1 |\boldsymbol{k}|^{m_s} \quad \forall\, |\boldsymbol{k}| > \overline{\varepsilon}. \tag{28}$$

- There are an IR scaling degree $\beta_s \in \mathbb{R}$ and $c_2, C_2 > 0$ with

$$c_2 |\boldsymbol{k}|^{\beta_s} \leq |s(\boldsymbol{k})| \leq C_2 |\boldsymbol{k}|^{\beta_s} \quad \forall\, 0 < |\boldsymbol{k}| < \underline{\varepsilon}. \tag{29}$$

- There is a constant $c_3 > 0$ with

$$c_3 \leq |s(\boldsymbol{k})| \quad \forall\, \underline{\varepsilon} \leq |\boldsymbol{k}| \leq \overline{\varepsilon}. \tag{30}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

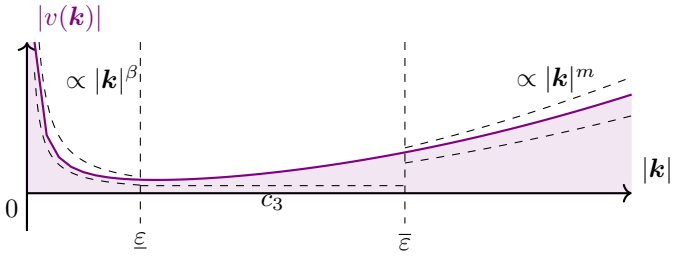

Fig. 1. Scaling degrees in the special case of a radially symmetric, positive function v with $|v(\boldsymbol{k})| > 0$.

The space of all *symbols with exact polynomial scaling* is denoted

$$\dot{\mathcal{S}}^{\infty}_{1,>} := \left\{ s \in C^{\infty}(\mathbb{R}^d \backslash \{0\}) \,|\, (28)\text{–}(30) \text{ hold} \right\} \subseteq \dot{\mathcal{S}}^{\infty}_{1}. \tag{31}$$

The ">" refers to the fact that $|s(\boldsymbol{k})| > 0$, so $\frac{1}{s}$ is defined everywhere except from $\boldsymbol{k} = 0$ and also scales polynomially.

Obviously, $C^{\infty}_{c}(\mathbb{R}^d) \subset \dot{\mathcal{S}}^{\infty}_{1}$, so $\dot{\mathcal{S}}^{\infty}_{1} \cap \mathfrak{h}$ is dense in $\mathfrak{h}$.

For obtaining a well-defined renormalized Hamiltonian $\widetilde{H}$ in Lemma 7.1, we will assume

$$\theta, \omega, v \in \dot{\mathcal{S}}^{\infty}_{1,>}, \tag{32}$$

(see Fig. 2) and denote their scaling degrees with m_θ, m_ω, m_v and $\beta_\theta, \beta_\omega, \beta_v$, respectively. Further, we will assume in Lemma 7.1

$$m_\theta, m_\omega, \beta_\theta, \beta_\omega \geq 0. \tag{33}$$

This implies, that for θ and ω, there is no pole at the origin. QFT models often use dispersion relations based on symbols with $m_\theta, m_\omega, \beta_\theta, \beta_\omega \in \{0, 1, 2\}$, which all satisfy this condition.

In order to obtain symmetric operators, we will also need to impose a symmetry condition in Lemma 7.1

$$\theta(\boldsymbol{p}) = \theta(-\boldsymbol{p}), \quad \omega(\boldsymbol{k}) = \omega(-\boldsymbol{k}), \quad v(\boldsymbol{k}) = v(-\boldsymbol{k}). \tag{34}$$

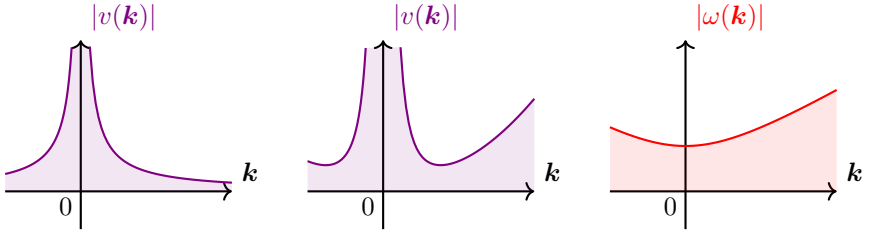

Fig. 2. Examples for absolute values of functions $v, \omega \in \dot{\mathcal{S}}^{\infty}_{1,>}$. The functions v, ω are complex-valued. Note that $\beta_\omega, m_\omega \geq 0$.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

3. Construction of the ESS

3.1. *Wave function renormalization factors*

The motivation behind the introduction of an ESS is to rigorously define the formal dressing operator

$$W(s) = W_M(s) \dots W_1(s) \quad \text{with } W_j(s) = e^{A_j^\dagger(s) - A_j(s)}, \quad s = -\frac{v}{\omega}, \tag{35}$$

in the case where $s \in \dot{\mathcal{S}}_1^\infty \subseteq C^\infty(\mathbb{R}^d \backslash \{0\})$, but $s \notin \mathfrak{h} = L^2(\mathbb{R}^d, \mathbb{C})$. Here, the number M corresponds to the fermion sector, so the second term in (35) is a rather symbolic expression requiring us to adapt M in dependence of the fermion sector (similar to (17)). Formally, if we apply $W(s)$ to a state vector Ψ_1 where the boson field is in the vacuum Ω_y and $\Psi_x \in \mathscr{F}_x^{(1)}$ is a one-fermion state

$$\Psi_1 = \Psi_x \otimes \Omega_y, \tag{36}$$

then we obtain the following expression in momentum space:

$$(W(s)\Psi_1)(\boldsymbol{p}, \boldsymbol{K}) = e^{-\frac{\|s\|^2}{2}} \frac{1}{\sqrt{N!}} \left(\prod_{\ell=1}^N s(\boldsymbol{k}_\ell) \right) \Psi_x \left(\boldsymbol{p} + \sum_{\ell=1}^N \boldsymbol{k}_\ell \right) =: e^{-\frac{\|s\|^2}{2}} \Psi_0(\boldsymbol{p}, \boldsymbol{K}). \tag{37}$$

For $s \in \mathfrak{h}$, the expression $W(s)\Psi_1$ is a Fock space vector with norm $\|W(s)\Psi_1\| = \|\Psi_1\|$. However, if we set $s \notin \mathfrak{h}$, two problems arise:

- $\|s\|^2 = \langle s, s \rangle$ is formally a divergent integral. So the wave function renormalization factor $e^{-\frac{\|s\|^2}{2}}$ is not well defined.
- $(\prod_{\ell=1}^N s(\boldsymbol{k}_\ell))$ (on every sector $N \geq 1$) is a non-square integrable function, so even without the infinite renormalization factor $e^{-\frac{\|s\|^2}{2}}$, (37) does not describe an element of $\mathscr{F}$.

The second problem is tackled by defining a space $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$ containing non-square integrable functions, including the above product. As $s \in \dot{\mathcal{S}}_1^\infty$, the product above is a smooth function apart from the zero boson momentum configuration set

$$\exists\,(\boldsymbol{k} = 0) := \big\{ q = (\boldsymbol{P}, \boldsymbol{K}) \in \mathcal{Q} \,|\, \exists\, \ell \in \{1, \dots, N\} : \ \boldsymbol{k}_\ell = 0 \big\}. \tag{38}$$

The restriction of $\exists\,(\boldsymbol{k} = 0)$ to any sector is a union of hyperplanes (see Fig. 3), which have Lebesgue measure 0 and hence, also $\exists\,(\boldsymbol{k} = 0) \subseteq \mathcal{Q}$ is null. Excluding this set from configuration space, we obtain

$$\dot{\mathcal{Q}} := \mathcal{Q} \backslash \exists\,(\boldsymbol{k} = 0), \tag{39}$$

and on $q \in \dot{\mathcal{Q}}$, the function Ψ_0 in (37) is smooth. For $|\boldsymbol{k}| \to 0$ or $|\boldsymbol{p}|, |\boldsymbol{k}| \to \infty$ we require the following scaling conditions to hold:

$$\lim_{|\boldsymbol{k}_\ell| \to 0} \frac{\Psi_0(\boldsymbol{P}, \boldsymbol{K})}{|\boldsymbol{k}_\ell|^\beta} = \lim_{|\boldsymbol{k}_\ell| \to \infty} \frac{\Psi_0(\boldsymbol{P}, \boldsymbol{K})}{|\boldsymbol{k}_\ell|^m} = \lim_{|\boldsymbol{p}_j| \to \infty} \frac{\Psi_0(\boldsymbol{P}, \boldsymbol{K})}{|\boldsymbol{p}_j|^m} = 0, \tag{40}$$

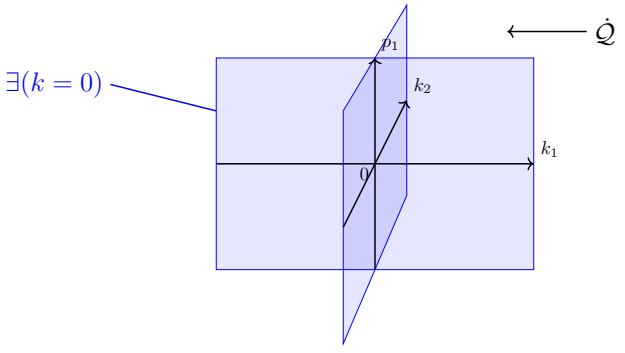

Fig. 3. The set $\exists\,(\boldsymbol{k}=0)$ on the $(M,N)=(1,2)$-sector in configuration space $\mathcal{Q}$ for $d=1$.

for all fixed $q=(\boldsymbol{P},\boldsymbol{K})\in\dot{\mathcal{Q}}$ and some $\beta,m\in\mathbb{R}$. We hence interpret Ψ_0 as an element of the following space:

$$\dot{\mathcal{S}}^{\infty}_{\mathscr{F}}:=\{\Psi\in C^{\infty}(\dot{\mathcal{Q}})\,|\,(40)\text{ holds}\}. \tag{41}$$

Note that $\dot{\mathcal{S}}^{\infty}_{\mathscr{F}}\cap\mathscr{F}$ contains $C^{\infty}_{c}(\mathcal{Q})$, which is a dense set in $\mathscr{F}$. So $\dot{\mathcal{S}}^{\infty}_{\mathscr{F}}\cap\mathscr{F}$ is dense in $\mathscr{F}$. The Fourier transform of some $\Psi_0\in\dot{\mathcal{S}}^{\infty}_{\mathscr{F}}$ can be taken, if it is an element of the space

$$\dot{\mathcal{S}}^{\infty}_{\mathscr{F},\mathrm{loc}}:=\dot{\mathcal{S}}^{\infty}_{\mathscr{F}}\cap L^{1}_{\mathrm{loc}}(\mathcal{Q}). \tag{42}$$

In order to address the first problem, it is necessary to assign mathematical meaning to the expression $\langle s,s\rangle$ (called *renormalization factor*). It may be convenient to take a Fourier transform of $s\in\dot{\mathcal{S}}^{\infty}_{1}$, for instance, in order to define a particle-position representation. This can be done whenever $s\in L^{1}_{\mathrm{loc}}(\mathbb{R}^d)$, since then $s\in\mathcal{S}'(\mathbb{R}^d)$, i.e. it is a tempered distribution. We may hence distinguish two interesting function spaces for s:

- The generic case is given by $s\in\dot{\mathcal{S}}^{\infty}_{1}$.
- The special case is given by $s\in\dot{\mathcal{S}}^{\infty}_{1}\cap L^{1}_{\mathrm{loc}}(\mathbb{R}^d)=:\dot{\mathcal{S}}^{\infty}_{1,\mathrm{loc}}$, which allows for taking the Fourier transform.

In the following, we will present a construction based on the generic case, since the special one can be treated by the same means. Clearly, $\mathfrak{h}\cap\dot{\mathcal{S}}^{\infty}_{1}\subseteq\dot{\mathcal{S}}^{\infty}_{1,\mathrm{loc}}$, so even $\dot{\mathcal{S}}^{\infty}_{1,\mathrm{loc}}$ contains a dense subspace of $\mathfrak{h}$. If $s\in\mathfrak{h}$, then

$$\langle s,s\rangle=\int|s(\boldsymbol{k})|^2\,d\boldsymbol{k}\in\mathbb{C}. \tag{43}$$

Otherwise, the renormalization factor $\langle s,s\rangle$ is a symbolic expression, corresponding to a divergent integral. We interpret it as an element of the following vector space:

Definition 3.1. Consider the free $\mathbb{C}$-vector space $F(\dot{\mathcal{S}}^{\infty}_{1}\times\dot{\mathcal{S}}^{\infty}_{1})$. By a free vector space, we mean the set of all finite $\mathbb{C}$-linear combinations of pairings, denoted

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

$\mathfrak{r} = \sum_{m=1}^{M} c_m \langle s_m, t_m \rangle$, with $s_m, t_m \in \dot{\mathcal{S}}_1^\infty$, $c \in \mathbb{C}$ and the sum $\sum_{m=1}^{M}$ being commutative. The *space of renormalization integrals* is defined as the quotient space

$$\mathrm{Ren}_{01} := F(\dot{\mathcal{S}}_1^\infty \times \dot{\mathcal{S}}_1^\infty)/_{\sim_{\mathrm{Ren}_{01}}}, \tag{44}$$

where the equivalence relation $\sim_{\mathrm{Ren}_{01}}$ of formal equality is given by

$$\begin{aligned} &\sum_{m=1}^{M} c_m \langle s_m, t_m \rangle \sim_{\mathrm{Ren}_{01}} \sum_{m=1}^{\tilde{M}} \tilde{c}_m \langle \tilde{s}_m, \tilde{t}_m \rangle \\ :\Leftrightarrow &\sum_{m=1}^{M} c_m \overline{s_m(\boldsymbol{k})} t_m(\boldsymbol{k}) = \sum_{m=1}^{\tilde{M}} \tilde{c}_m \overline{\tilde{s}_m(\boldsymbol{k})} \tilde{t}_m(\boldsymbol{k}) \quad \forall\, \boldsymbol{k} \in \mathbb{R}^d \backslash \{0\}. \end{aligned} \tag{45}$$

There is a natural one-to-one identification of renormalization integrals with functions $\mathrm{Ren}_{01} \cong \dot{\mathcal{S}}_1^\infty$: It is easy to see that the following map is an embedding (by definition of $\sim_{\mathrm{Ren}_{01}}$):

$$\begin{aligned} &\iota_1 : \mathrm{Ren}_{01} \to \dot{\mathcal{S}}_1^\infty, \\ &\sum_{m=1}^{M} c_m \langle s_m, t_m \rangle = \mathfrak{r} \mapsto r(\boldsymbol{k}) = \sum_{m=1}^{M} c_m \overline{s_m(\boldsymbol{k})} t_m(\boldsymbol{k}). \end{aligned} \tag{46}$$

Conversely, for a given $r \in \dot{\mathcal{S}}_1^\infty$, the element $\langle r, \chi_{\mathbb{R}^d \backslash \{0\}} \rangle \in \mathrm{Ren}_1$ (with χ being the indicator function) is identified with r, so ι_1 is indeed an isomorphism.

However, it can be misleading to think of $\mathfrak{r} \in \mathrm{Ren}_1$ as a function r. Renormalization integrals are formally $\mathfrak{r} = \int r(\boldsymbol{k})\, d\boldsymbol{k}$, so they can be added but not directly multiplied. Hence the notation $\langle \cdot, \cdot \rangle$, resembling an L^2 scalar product. By contrast, functions $r_1, r_2 \in \dot{\mathcal{S}}_1^\infty$ can be multiplied which leads again to a function in $\dot{\mathcal{S}}_1^\infty$.

In case $r \in L^1(\mathbb{R}^d)$, we can identify $\mathfrak{r} \in \mathrm{Ren}_{01}$ with the $\mathbb{C}$-number

$$\mathfrak{r} = \int_{\mathbb{R}^d} \sum_{m=1}^{M} c_m \overline{s_m(\boldsymbol{k})} t_m(\boldsymbol{k}) d\boldsymbol{k} = \int_{\mathbb{R}^d} r(\boldsymbol{k}) d\boldsymbol{k} \in \mathbb{C}. \tag{47}$$

Now, several $\mathfrak{r} \in \mathrm{Ren}_{01}$ get identified with the same $\mathbb{C}$-number, e.g., all $\mathfrak{r}$ corresponding to a function r with $\int r(\boldsymbol{k})\, d\boldsymbol{k} = 0$ are identified with 0. We remove this ambiguity by modding out another equivalence relation.

Definition 3.2. The renormalization factor space Ren_1 is defined as

$$\mathrm{Ren}_1 := \mathrm{Ren}_{01}/_{\sim_{\mathrm{Ren}_1}}, \tag{48}$$

where for $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathrm{Ren}_{01}$ we define

$$\begin{aligned} \mathfrak{r}_1 \sim_{\mathrm{Ren}_1} \mathfrak{r}_2 :\Leftrightarrow\ & \iota_1(\mathfrak{r}_1) - \iota_1(\mathfrak{r}_2) \in L^1 \quad \text{and} \\ & \int (\iota_1(\mathfrak{r}_1) - \iota_1(\mathfrak{r}_2))(\boldsymbol{k}) d\boldsymbol{k} = 0. \end{aligned} \tag{49}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Elements of Ren_1 will be denoted equally to a representative $\mathfrak{r}$. Note that we can identify Ren_1 with the quotient space

$$\mathrm{Ren}_1 \cong (\mathbb{C} \oplus \mathrm{Ren}_{01})/D \quad \text{with } D := \left\{ \left(-\int \iota_1(\mathfrak{r})(\boldsymbol{k})\, d\boldsymbol{k}, \mathfrak{r} \right) \,\middle|\, \iota_1(\mathfrak{r}) \in L^1(\mathbb{R}^d) \right\}, \tag{50}$$

where the isomorphism is given by identification of $(c, \mathfrak{r})$ with that class $[\mathfrak{r}'] \in \mathrm{Ren}_1$ where $\int(\iota_1(\mathfrak{r}')(\boldsymbol{k}) - \iota_1(\mathfrak{r})(\boldsymbol{k}))d\boldsymbol{k} = c$.

We will encounter the special case where $r(\boldsymbol{k})$ only takes values in $[0, \infty)$.

Definition 3.3. The *positive renormalization factor cone* Ren_{1+} is the set of all $[\mathfrak{r}] \in \mathrm{Ren}_1$, where at least one representative $\mathfrak{r} \in \mathrm{Ren}_{01}$ is identified with a positive-valued function $\iota_1(\mathfrak{r}) : \mathbb{R}^d \backslash \{0\} \to [0, \infty)$.

Scalar multiplication by $c \in [0, \infty)$ is well defined, making this indeed a cone.

It is also convenient to make sense of products and polynomials of factors $\mathfrak{r} \in \mathrm{Ren}_1$. We hence define the following vector spaces.

Definition 3.4. Consider the free $\mathbb{C}$-vector space of all finite linear combinations of products of up to $P \in \mathbb{N}_0$ renormalization factors (i.e. formal polynomials of degree P)

$$\mathrm{Pol}_P := \left\{ \mathfrak{R} = \sum_{m=1}^{M} c_m \mathfrak{r}_{m,1} \cdot \cdots \cdot \mathfrak{r}_{m,p_m} \,\middle|\, 0 \le p_m \le P,\ c_m \in \mathbb{C},\ \mathfrak{r}_{m,p} \in \mathrm{Ren}_1 \right\}, \tag{51}$$

with the sum $\sum_{m=1}^{M}$ and products being commutative. Then, the space of *renormalization factor polynomials* of order $\le P$ is the quotient space

$$\mathrm{Ren}_P = \mathrm{Pol}_P / {\sim_{\mathrm{Ren}_P}}, \tag{52}$$

with the equivalence relation $\sim_{\mathrm{Ren}_P}$ of formal equality generated by

$$\begin{aligned} \mathfrak{r}_1 \mathfrak{r}_2 \dots \mathfrak{r}_{p_1} + \mathfrak{R} &\sim_{\mathrm{Ren}_P} c\mathfrak{r}_2 \dots \mathfrak{r}_{p_1} + \mathfrak{R} \quad \text{if } \mathfrak{r}_1 = c \in \mathbb{C}, \\ (c_1 c_2)\mathfrak{r}_1 \dots \mathfrak{r}_{p_1} + \mathfrak{R} &\sim_{\mathrm{Ren}_P} c_1(c_2 \mathfrak{r}_1) \dots \mathfrak{r}_{p_1} + \mathfrak{R}, \end{aligned} \tag{53}$$

with $P_1 \le P$, $\mathfrak{R} \in \mathrm{Pol}_P$, $c_1, c_2 \in \mathbb{C}$ and $\mathfrak{r}_m \in \mathrm{Ren}_1$.

The bound P on the polynomial order is removed by taking the union over all orders.

Definition 3.5. The space of *renormalization factor polynomials* is given by

$$\mathrm{Ren} = \bigcup_{P=1}^{\infty} \mathrm{Ren}_P. \tag{54}$$

The spaces defined in this section follow the hierarchy

$$\mathrm{Ren}_{1+} \subseteq \mathrm{Ren}_1 \subseteq \mathrm{Ren}_2 \subseteq \cdots \subseteq \mathrm{Ren}_P \subseteq \cdots \subseteq \mathrm{Ren}. \tag{55}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

3.2. *Exponentials of renormalization factors and the field* eRen

The state vector $W(s)\Psi_1$ in (37) contains an exponential of a renormalization factor, i.e. an expression $\mathfrak{c} = e^{\mathfrak{r}}$ with $\mathfrak{r} \in \mathrm{Ren}_1$. More generally, we would like to give a meaning to sums of exponentials with different (perhaps infinite) renormalization factors $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathrm{Ren}_1$. Formally, we would like to identify

$$e^{\mathfrak{r}_1} + e^{\mathfrak{r}_2} = e^{\mathfrak{r}_1}(1 + e^{\mathfrak{r}_2 - \mathfrak{r}_1}). \tag{56}$$

The bracket is a $\mathbb{C}$-number, whenever $\mathfrak{r}_1 - \mathfrak{r}_2 \in \mathbb{C}$, which defines an equivalence relation

$$\mathfrak{r}_1 \sim_1 \mathfrak{r}_2 :\Leftrightarrow \mathfrak{r}_1 - \mathfrak{r}_2 \in \mathbb{C}. \tag{57}$$

Definition 3.6. The space of *renormalization factor classes* is then given by the quotient space

$$\mathrm{Clas}_1 = \mathrm{Ren}_1/_{\sim_1}. \tag{58}$$

Whenever two elements $\mathfrak{r}_1, \mathfrak{r}_2$ are of two different classes, we have $\mathfrak{c} = e^{\mathfrak{r}_1 - \mathfrak{r}_2} \notin \mathbb{C}$ and may think of $\mathfrak{c}$ as an "infinite constant".

We may also interpret Clas_1 as a subspace of Ren_1: The elements in the class of zero, $b \in \mathrm{Ren}_1 : b \sim_1 0$, form a subspace $V \subset \mathrm{Ren}_1$, $V \cong \mathbb{C}$. Now we can find a basis B (containing one element) of V and, by the axiom of choice [49], extend it to a basis $B \cup B'$ of Ren_1. By defining $W := \mathrm{span}(B')$, we obtain a decomposition

$$\mathrm{Ren}_1 = V \oplus W \Rightarrow W \cong \mathrm{Ren}_1/V. \tag{59}$$

Now, the following map is a bijection from W to Clas_1:

$$W \ni w \mapsto [w]_{\sim_{\mathrm{Ren}_1}} \in \mathrm{Clas}_1. \tag{60}$$

Note that this bijection is not unique, since it depends on the choice of B', i.e. of representatives within each class.

We now consider elements of the group algebra $\mathbb{C}[\mathrm{Ren}_1] \ni c_1 e^{\mathfrak{r}_1} + \cdots + c_M e^{\mathfrak{r}_M}$ of the additive group of the vector space Ren_1. The formal exponentials make an addition in the group $e^{\mathfrak{r}_1 + \mathfrak{r}_2} = e^{\mathfrak{r}_1} e^{\mathfrak{r}_2}$ appear as a multiplication. Here, we would like to consider two summands as equal, if "parts of the complex number can be pulled into the exponent", i.e. $ce^{c' + \mathfrak{r}} = (ce^{c'})e^{\mathfrak{r}}$. This is done by defining an ideal $\mathcal{I} \subset \mathbb{C}[\mathrm{Ren}_1]$ generated by all elements of the form

$$e^{c + \mathfrak{r}} - e^c e^{\mathfrak{r}} \quad \text{with } c \in \mathbb{C},\ \mathfrak{r} \in \mathrm{Ren}_1. \tag{61}$$

Note that this ideal gives rise to an equivalence relation

$$u \sim_{\mathcal{I}} v :\Leftrightarrow (u - v) \in \mathcal{I}. \tag{62}$$

Proposition 3.1.

$$\mathbb{C}[\mathrm{Ren}_1]/\mathcal{I} \cong \mathbb{C}[W]. \tag{63}$$

Proof. We define a map $\pi : \mathrm{Ren}_1 \to W$, which assigns to $\mathfrak{r} \in \mathrm{Ren}_1$ the vector w within the decomposition $\mathrm{Ren}_1 = V \oplus W$ above. So $\mathfrak{r} - \pi(\mathfrak{r}) \in \mathbb{C}$. Now within,

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

$c_1 e^{\mathfrak{r}_1} + \cdots + c_M e^{\mathfrak{r}_M} \in \mathbb{C}[\mathrm{Ren}_1]$, we can re-write each summand as

$$c_m e^{\mathfrak{r}_m} \sim_{\mathcal{I}} (c_m e^{\mathfrak{r}_m - \pi(\mathfrak{r}_m)}) e^{\pi(\mathfrak{r}_m)}. \tag{64}$$

This gives rise to a re-writing map

$$\begin{aligned} &\Pi : \mathbb{C}[\mathrm{Ren}_1] \to \mathbb{C}[\mathrm{Ren}_1], \\ &c_1 e^{\mathfrak{r}_1} + \cdots + c_M e^{\mathfrak{r}_M} \mapsto (c_1 e^{\mathfrak{r}_1 - \pi(\mathfrak{r}_1)}) e^{\pi(\mathfrak{r}_1)} + \cdots + (c_M e^{\mathfrak{r}_M - \pi(\mathfrak{r}_M)}) e^{\pi(\mathfrak{r}_M)}. \end{aligned} \tag{65}$$

Note that by the computation rules for group algebras, whenever $e^{\pi(\mathfrak{r}_m)} = e^{\pi(\mathfrak{r}_{m'})}$ for two summands, they can be combined into one.

The map Π is an algebra homomorphism: it is linear and respects the multiplication

$$\begin{aligned} &\Pi(c_1 e^{\mathfrak{r}_1} + \cdots + c_M e^{\mathfrak{r}_M}) \Pi(c'_1 e^{\mathfrak{r}'_1} + \cdots + c'_M e^{\mathfrak{r}'_M}) \\ &\quad = \Pi(c_1 e^{\mathfrak{r}_1} + \cdots + c_M e^{\mathfrak{r}_M} + c'_1 e^{\mathfrak{r}'_1} + \cdots + c'_M e^{\mathfrak{r}'_M}). \end{aligned} \tag{66}$$

The latter is a consequence of $\pi(\mathfrak{r}_1 + \mathfrak{r}_2) = \pi(\mathfrak{r}_1) + \pi(\mathfrak{r}_2)$ which does only hold true because we have chosen W as a vector space. Further, Π does not change the equivalence class with respect to $\sim_{\mathcal{I}}$. So $\Pi(I) = 0$ implies $I \in \mathcal{I}$. Conversely, if $I \in \mathcal{I}$ then $I = I_1 + \cdots + I_M$ with each $I_i = A_i B_i C_i$ with $A_i, C_i \in \mathbb{C}[\mathrm{Ren}_1]$ and $B_i = e^{c_i + \mathfrak{r}_i} - e^{c_i} e^{\mathfrak{r}_i}$, so $\Pi(B_i) = 0$ and $\Pi(I_i) = 0$ and hence $\Pi(I) = 0$. So the kernel of Π is exactly $\mathrm{Ker}(\Pi) = \mathcal{I}$.

Moreover, the image is $\Pi[\mathbb{C}[\mathrm{Ren}_1]] = \mathbb{C}[W]$, since only elements of $\mathbb{C}[W]$ appear in it and any element of $\mathbb{C}[W] \subset \mathbb{C}[\mathrm{Ren}_1]$ is mapped to itself.

By the isomorphism theorems, we now have that

$$\mathbb{C}[\mathrm{Ren}_1]/\mathrm{Ker}(\Pi) \cong \Pi[\mathbb{C}[\mathrm{Ren}_1]] :\Leftrightarrow \mathbb{C}[\mathrm{Ren}_1]/\mathcal{I} \cong \mathbb{C}[W], \tag{67}$$

as claimed. □

Proposition 3.2. *$\mathbb{C}[\mathrm{Ren}_1]/\mathcal{I}$ has no proper zero divisors.*

($a, b \in \mathbb{C}[\mathrm{Ren}_1]/\mathcal{I}, a, b \neq 0$ are called proper zero divisors if and only if $ab = 0$.)

Proof. By Proposition 3.1, it suffices to show that $\mathbb{C}[W]$ has no proper zero divisors.

Now, the additive group W is an Abelian group that is torsion-free, i.e. for $g \in W$ and $n \in \mathbb{N}$

$$\underbrace{g + g + \cdots + g}_{n \text{ times}} = 0 \Rightarrow g = 0. \tag{68}$$

Now, by [50, Lemma 26.6], the group $G = W$ is ordered, so by Lemma 26.4 it is "t.u.p.", so by Lemma 26.2 it is "u.p." and $K[G] = \mathbb{C}[W]$ has no proper zero divisors, i.e. it is an entire ring. □

Following [51, II, §3], it is a theorem that for every ring without proper zero divisors, the quotients form a field. So by Proposition 3.2 the following field extension of $\mathbb{C}$ is well defined.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Definition 3.7 (and Corollary). The *field of* (*exponential*) *wave function renormalizations* is given by all fractions of linear combinations

$$\mathrm{eRen} := \left\{ \mathfrak{c} = \frac{a_1}{a_2} \;\middle|\; a_1, a_2 \in \mathbb{C}[\mathrm{Ren}_1]/\mathcal{I} \right\}. \tag{69}$$

By using representatives of $\mathbb{C}[\mathrm{Ren}_1]/\mathcal{I}$, we can write any $\mathfrak{c} \in \mathrm{eRen}$ as

$$\mathfrak{c} = \frac{\sum_{m=1}^{M} c_m e^{\mathfrak{r}_m}}{\sum_{m'=1}^{M'} c_{m'} e^{\mathfrak{r}_{m'}}} \quad \text{with } c_m,\ c_{m'} \in \mathbb{C}, \quad \mathfrak{r}_m,\ \mathfrak{r}_{m'} \in \mathrm{Ren}_1. \tag{70}$$

In particular, we can view a wave function renormalization $\mathfrak{c} \in \mathrm{eRen}$ as an "extended complex number".

3.3. *First ESS*

With the above definitions, we are able to give meaning to expressions like

$$\Psi = e^{-\mathfrak{r}}\Psi_0 \quad \text{or even } \Psi = \mathfrak{c}\Psi_0, \tag{71}$$

with $\mathfrak{r} \in \mathrm{Ren}_1, \mathfrak{c} \in \mathrm{eRen}$ and $\Psi_0 \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$, as they appear in (37). We would like to take linear combinations of them and even handle expressions like $\mathfrak{c}\mathfrak{R}\Psi_0$ with $\mathfrak{R} \in \mathrm{Ren}$. This is done by defining eRen-vector spaces including such expression, either without $\mathfrak{R}$ (this will be the first ESS $\overline{\mathscr{F}}$) or with $\mathfrak{R}$ (this will be the second ESS $\overline{\mathscr{F}}_{\mathrm{ex}}$).

Definition 3.8. Consider the free eRen-vector space of all finite (commutative) sums of the form

$$\overline{\mathscr{F}}_0 := \left\{ \Psi = \sum_{m=1}^{M} \mathfrak{c}_m \Psi_m \;\middle|\; \Psi_m \in \dot{\mathcal{S}}^\infty_{\mathscr{F}},\ \mathfrak{c}_m \in \mathrm{eRen} \right\}. \tag{72}$$

Then, the *first ESS* is the eRen-quotient space

$$\overline{\mathscr{F}} = \overline{\mathscr{F}}_0 /_{\sim_{\mathrm{F}}}, \tag{73}$$

with equivalence relation $\sim_{\mathrm{F}}$ generated by

$$(c\mathfrak{c})\Psi_a + \Psi \sim_{\mathrm{F}} \mathfrak{c}(c\Psi_a) + \Psi \quad \text{if } c \in \mathbb{C}, \tag{74}$$

for any $\mathfrak{c} \in \mathrm{eRen}$, $\Psi \in \overline{\mathscr{F}}_0$ and $\Psi_a \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$.

The dense subspace $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap \mathscr{F}$ of the Fock space can be naturally embedded into $\overline{\mathscr{F}}$: Every $\Psi_{\mathscr{F}} \in \dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap \mathscr{F}$, can be identified with

$$\Psi_{\mathscr{F}} = e^0 \Psi_{\mathscr{F}} \in \overline{\mathscr{F}}. \tag{75}$$

Elements of $\overline{\mathscr{F}}$ do not necessarily satisfy any symmetry conditions. However, we can decompose $\overline{\mathscr{F}}$ into sectors, just as $\mathscr{F}$ in (13): Since $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$ consists of functions in

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

$C^\infty(\mathcal{Q})$, we may introduce the projections

$$P^{(M,N)} : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}, \quad (P^{(M,N)}\Psi)(q) = \begin{cases} \Psi(q) & \text{if } q \in \mathcal{Q}^{(M,N)}, \\ 0 & \text{else.} \end{cases} \tag{76}$$

By eRen-linearity, $P^{(M,N)}$ extends to $\overline{\mathscr{F}}$, which allows to define

$$\overline{\mathscr{F}}^{(M,N)} := P^{(M,N)}\overline{\mathscr{F}} \quad \text{so } \overline{\mathscr{F}} = \bigoplus_{M,N\in\mathbb{N}_0} \overline{\mathscr{F}}^{(M,N)}. \tag{77}$$

The coherent state in (37) can now be seen as an ESS element

$$(W(s)\Psi_1)(\boldsymbol{p},\boldsymbol{K}) = \underbrace{e^{-\frac{\|s\|^2}{2}}}_{=e^{\mathfrak{r}}} \underbrace{\frac{1}{\sqrt{N!}} \left(\prod_{\ell=1}^N s(\boldsymbol{k}_\ell)\right) \Psi_x\left(\boldsymbol{p}+\sum_\ell \boldsymbol{k}_\ell\right)}_{\in \dot{\mathcal{S}}^\infty_{\mathscr{F}}}.$$

For $s \in \dot{\mathcal{S}}^\infty_{1,\mathrm{loc}}$ and $\Psi_x \in C^\infty_c$, the second factor is even in $\dot{\mathcal{S}}^\infty_{\mathscr{F},\mathrm{loc}}$.

3.4. *Second ESS*

Later in this work, we will encounter expressions like

$$\Psi(q) = e^{-\mathfrak{r}}\mathfrak{R}(q)\Psi_0(q), \tag{78}$$

with $q \in \dot{\mathcal{Q}}$, $\mathfrak{R}(q) \in \mathrm{Ren}, \Psi_0(q) \in \mathbb{C}$, or linear combinations of them. We interpret $\Psi_{\mathfrak{R}} := \mathfrak{R}\Psi_0$ as a function in the function space

$$\Psi_{\mathfrak{R}} \in \mathrm{Ren}^{\dot{\mathcal{Q}}} := \big\{\Psi_{\mathfrak{R}} : \dot{\mathcal{Q}} \to \mathrm{Ren}\big\}. \tag{79}$$

The second ESS then covers expressions (78) and linear combinations of them.

Definition 3.9. Consider the free eRen-vector space of all finite (commutative) sums of the form

$$\overline{\mathscr{F}}_{\mathrm{ex},0} := \left\{ \Psi = \sum_{m=1}^M \mathfrak{c}_m \Psi_m \;\middle|\; \mathfrak{c}_m \in \mathrm{eRen}, \Psi_m \in \mathrm{Ren}^{\dot{\mathcal{Q}}} \right\}. \tag{80}$$

Then, the *second ESS* is the eRen-quotient space

$$\overline{\mathscr{F}}_{\mathrm{ex}} = \overline{\mathscr{F}}_{\mathrm{ex},0}/_{\sim_{\mathrm{Fex}}}, \tag{81}$$

with equivalence relation $\sim_{\mathrm{Fex}}$ generated by

$$(c\mathfrak{c})\Psi_a + \Psi \sim_{\mathrm{Fex}} \mathfrak{c}(c\Psi_a) + \Psi \quad \text{if } c \in \mathbb{C}, \tag{82}$$

where $\mathfrak{c} \in \mathrm{eRen}$, $\Psi \in \overline{\mathscr{F}}_{\mathrm{ex},0}$ and $\Psi_a \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

The first ESS can be embedded into the second ESS $\overline{\mathscr{F}} \hookrightarrow \overline{\mathscr{F}}_{\mathrm{ex}}$ by interpreting all $\Psi_m \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ as elements $\Psi_m \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

4. Operators on the ESS

We first prove that creation and annihilation terms $A^\dagger(v), A(v)$ as in (20) and (22) can be defined as operators using ESSs. Here, we even permit form factors $v(\boldsymbol{p}, \boldsymbol{k})$, that are allowed to depend on the fermion momentum $\boldsymbol{p}$. The momentum space definition reads

$$\begin{aligned}(A_j^\dagger(v)\Psi)(\boldsymbol{P}, \boldsymbol{K}) &= \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} v(\boldsymbol{p}_j, \boldsymbol{k}_\ell)\Psi(\boldsymbol{P} + e_j\boldsymbol{k}_\ell, \boldsymbol{K}\backslash\boldsymbol{k}_\ell),\\ (A_j(v)\Psi)(\boldsymbol{P}, \boldsymbol{K}) &= \sqrt{N+1} \int v(\boldsymbol{p}_j - \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}})^* \Psi(\boldsymbol{P} - e_j\tilde{\boldsymbol{k}}, \boldsymbol{K}, \tilde{\boldsymbol{k}}) d\tilde{\boldsymbol{k}}, \qquad (83)\\ A^\dagger(v) &= \sum_{j=1}^{M} A_j^\dagger(v), \quad A(v) = \sum_{j=1}^{M} A_j(v),\end{aligned}$$

which can be seen as a generalization of (20) and (22).

Lemma 4.1 ($A^\dagger, A$ are Well Defined for $v(\boldsymbol{p}, \boldsymbol{k})$). *Let $v : \mathbb{R}^d \times (\mathbb{R}^d\backslash\{0\}) \to \mathbb{C}$ be smooth and satisfying the scaling condition* (40)*. Then,* (83) *entails well-defined operators*

$$A_j^\dagger(v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}, \quad A_j(v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}, \qquad (84)$$

which may be restricted[c] *to*

$$A_j^\dagger(v) : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}, \quad A_j(v) : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \mathrm{Ren}^{\dot{\mathcal{Q}}}. \qquad (85)$$

We may even extend $A_j^\dagger(v) : \overline{\mathscr{F}}_{\mathrm{ex}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$.

Proof. Suppose $\Psi \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ and consider the expression $A_j^\dagger(v)\Psi$ in (83). This is a finite sum over products consisting of two factors. By definition of $\dot{\mathcal{S}}^\infty_1$, the first factor $v(\boldsymbol{p}_j, \boldsymbol{k}_\ell)$ in each product is smooth everywhere except where $\boldsymbol{k}_\ell = 0$. By definition of $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$, the second factor $\Psi(\boldsymbol{P} + e_j\boldsymbol{k}_\ell, \boldsymbol{K}\backslash\boldsymbol{k}_\ell)$ is smooth at all configurations, at which $\boldsymbol{K}\backslash\boldsymbol{k}_\ell$ contains no coordinate $\boldsymbol{k}_{\ell'} = 0$. So the product is smooth on $\dot{\mathcal{Q}}$. In addition, the factors v and Ψ scale polynomially as in (40), so $A^\dagger(v)\Psi \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$.

The expression for $A(v)\Psi$ in (22) is a (possibly divergent) integral for each fixed $(\boldsymbol{P}, \boldsymbol{K}) \in \dot{\mathcal{Q}}$. Since both the functions $\tilde{\boldsymbol{k}} \mapsto v(\boldsymbol{p}_j - \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}})$ and $\tilde{\boldsymbol{k}} \mapsto \Psi(\boldsymbol{P} - e_j\tilde{\boldsymbol{k}}, \boldsymbol{K}, \tilde{\boldsymbol{k}})$ are in $\dot{\mathcal{S}}^\infty_1$, the integral defines an element $\mathfrak{R} \in \mathrm{Ren}_1 \subset \mathrm{Ren}$ for each fixed $(\boldsymbol{P}, \boldsymbol{K})$. Thus, $A(v)\Psi \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

Both operators can be extended to $\overline{\mathscr{F}}$ by taking eRen-linear combinations.

For $\Psi \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$, the expression $A_j^\dagger(v)\Psi$ in (83) is again a sum of products, that are all in $\mathrm{Ren}^{\dot{\mathcal{Q}}}$. By eRen-linearity, we can then extend $A_j^\dagger(v)$ to $\overline{\mathscr{F}}_{\mathrm{ex}}$. □

We may also define the constituent operators of $H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty$, as well as $H_{0,x}$ on $\overline{\mathscr{F}}$.

[c]We use the same notation for all extended or restricted versions of operators, here.

Proposition 4.1 (Constituents of H are Well Defined). *Consider the momentum space definitions of H_0 (17), $A^\dagger(v)$ (20), $A(v)$ (22) and E_∞ (24), where we only assume $\theta, \omega, v \in \dot{\mathcal{S}}_1^\infty$ and an arbitrary self-energy function $E_1 : \mathbb{R}^d \to \mathrm{Ren}$. Then, the above four momentum space definitions entail well-defined operators*

$$\begin{aligned} H_0 &: \overline{\mathscr{F}} \to \overline{\mathscr{F}}, \qquad A^\dagger(v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}, \\ A(v) &: \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}, \qquad E_\infty : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}. \end{aligned} \tag{86}$$

It is also possible to restrict

$$\begin{aligned} H_0 &: \dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \dot{\mathcal{S}}_{\mathscr{F}}^\infty, \qquad A^\dagger(v) : \dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \dot{\mathcal{S}}_{\mathscr{F}}^\infty, \\ A(v) &: \dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \mathrm{Ren}^{\dot{\mathcal{Q}}}, \qquad E_\infty : \dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \mathrm{Ren}^{\dot{\mathcal{Q}}}, \end{aligned} \tag{87}$$

or to extend

$$H_0 : \overline{\mathscr{F}}_{\mathrm{ex}} \to \overline{\mathscr{F}}_{\mathrm{ex}}, \quad A^\dagger(v) : \overline{\mathscr{F}}_{\mathrm{ex}} \to \overline{\mathscr{F}}_{\mathrm{ex}}. \tag{88}$$

Further, the statements for H_0 equally hold true for $H_{0,x}$ and $H_{0,y}$.

Proof. Well-definedness and the mapping properties of $A^\dagger(v)$ and $A(v)$ are a consequence of Lemma 4.1: The function $v(\boldsymbol{k})$ in (20) and (22) can be seen as a special case of $v(\boldsymbol{p}, \boldsymbol{k})$ in (83). Taking the finite linear sum over $j \in \{1, \ldots, M\}$ sustains the mapping properties.

The operator H_0 as in (17) just multiplies with a function in $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$ in momentum space, so for $\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$, we also have $H_0\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$. The same holds true for $H_{0,x}$ and $H_{0,y}$. By an analogous argument, H_0 can be extended to $H_0 : \mathrm{Ren}^{\dot{\mathcal{Q}}} \to \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

Finally, we consider $E_\infty\Psi$ for $\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$. By (24), at each fixed $(\boldsymbol{P}, \boldsymbol{K}) \in \dot{\mathcal{Q}}$ the expression $E_\infty\Psi$ is defined as a finite sum over terms $E_1(\boldsymbol{p})\Psi(\boldsymbol{P}, \boldsymbol{K}) \in \mathrm{Ren}$. So indeed, $E_\infty\Psi \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

Extensions to $\overline{\mathscr{F}}$ or $\overline{\mathscr{F}}_{\mathrm{ex}}$ can again be done by eRen-linear combination. □

Thus, also the linear operator $H : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ is well defined.

Finally, we prove that the momentum space definition (83) indeed entails certain canonical commutation relations on the ESS.

Lemma 4.2 (Extended CCR). *For $\phi, \varphi \in \dot{\mathcal{S}}_1^\infty$, the definitions (20) and (22) imply the commutation relations*

$$[A_j^\dagger(\varphi), A_{j'}^\dagger(\phi)] = 0, \quad [A_j(\varphi), A_{j'}^\dagger(\phi)] = \begin{cases} \langle \varphi, \phi \rangle & \text{for } j = j', \\ V_{jj'}(\varphi^* \phi) & \text{for } j \neq j', \end{cases} \tag{89}$$

as a strong operator identity. That is, we have operators

$$[A_j^\dagger(\varphi), A_{j'}^\dagger(\phi)] : \overline{\mathscr{F}} \to \overline{\mathscr{F}}, \quad [A_j(\varphi), A_{j'}^\dagger(\phi)] : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}. \tag{90}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Here, the interaction potential $V_{jj'} : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ *for momentum transfer from fermion* j' *to* j *is given by*

$$V_{jj'}(\varphi^*\phi)(\boldsymbol{P},\boldsymbol{K}) := \int \varphi^*(\tilde{\boldsymbol{k}})\phi(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P}+(e_{j'}-e_j)\tilde{\boldsymbol{k}},\boldsymbol{K})d\tilde{\boldsymbol{k}}. \tag{91}$$

Proof. By Lemma 4.1, the products $A_j(\varphi)A^\dagger_{j'}(\phi)$ and $A^\dagger_{j'}(\phi)A_j(\varphi)$ are well defined as operators $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$. A momentum space evaluation renders

$$\begin{aligned}\big([A_j(\varphi),A^\dagger_{j'}(\phi)]\Psi\big)(\boldsymbol{P},\boldsymbol{K}) &= \int \varphi^*(\tilde{\boldsymbol{k}})\phi(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P}+(e_{j'}-e_j)\tilde{\boldsymbol{k}},\boldsymbol{K})d\tilde{\boldsymbol{k}}\\ &= \begin{cases}\langle\varphi,\phi\rangle\Psi(\boldsymbol{P},\boldsymbol{K}) & \text{for } j=j',\\ \displaystyle\int \varphi^*(\tilde{\boldsymbol{k}})\phi(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P}+(e_{j'}-e_j)\tilde{\boldsymbol{k}},\boldsymbol{K})d\tilde{\boldsymbol{k}} & \text{for } j\neq j'.\end{cases}\end{aligned} \tag{92}$$

Similarly, Lemma 4.1 establishes that $A^\dagger_j(\varphi)A^\dagger_{j'}(\phi)$ and $A^\dagger_{j'}(\phi)A^\dagger_j(\varphi)$ are well-defined operators $\overline{\mathscr{F}} \to \overline{\mathscr{F}}$ and a short momentum space calculation verifies that they are equal. □

The operator $V_{jj'}$ defined above can be seen as an interaction potential operator. Under an inverse Fourier transform $\mathcal{F}^{-1}$, it amounts to a multiplication operator in position space via

$$(V_{jj'}(\varphi^*\phi)\Psi)(\boldsymbol{X},\boldsymbol{Y}) = \mathcal{F}^{-1}(\varphi^*s)(\boldsymbol{x}_j-\boldsymbol{x}_{j'})\Psi(\boldsymbol{X},\boldsymbol{Y}), \tag{93}$$

provided that $\mathcal{F}^{-1}(\varphi^*\phi)$ exists (e.g., for exact scaling degrees $\beta_\varphi+\beta_s > -d \Rightarrow \varphi^*s \in \mathcal{S}'(\mathbb{R}^d)$).

The following property about $V_{jj'}$ will become useful in later proofs.

Lemma 4.3. *If either of the functions* $\phi,\varphi \in \dot{\mathcal{S}}^\infty_1$ *is an element of* $C^\infty_c(\mathbb{R}^d\backslash\{0\})$, *then we even have*

$$V_{jj'}(\varphi^*\phi) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}. \tag{94}$$

Proof. First, let us consider $\Psi \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$. Without loss of generality, assume that $\varphi \in C^\infty_c(\mathbb{R}^d\backslash\{0\})$, so φ is compactly supported, and the function

$$\tilde{\boldsymbol{k}} \mapsto \phi(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P}+(e_{j'}-e_j)\tilde{\boldsymbol{k}},\boldsymbol{K}),$$

is smooth everywhere on that support. So the function

$$\tilde{\boldsymbol{k}} \mapsto \varphi^*(\tilde{\boldsymbol{k}})\phi(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P}+(e_{j'}-e_j)\tilde{\boldsymbol{k}},\boldsymbol{K}) \quad \text{is in } C^\infty_c,$$

and the integral over it converges to a $\mathbb{C}$-number.

We now show that this number depends smoothly on $(\boldsymbol{P},\boldsymbol{K}) \in \dot{\mathcal{Q}}$: Consider any multi-index α corresponding to a derivative ∂^α composed of arbitrarily many

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

partial derivatives $\partial_{\boldsymbol{p}_j}, \partial_{\boldsymbol{k}_\ell}$ with $j, \ell \in \mathbb{N}$. Then, also $\partial^\alpha \Psi \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ and by the same arguments as above

$$\tilde{\boldsymbol{k}} \mapsto \varphi^*(\tilde{\boldsymbol{k}})\phi(\tilde{\boldsymbol{k}})\partial^\alpha \Psi(\boldsymbol{P} + (e_{j'} - e_j)\tilde{\boldsymbol{k}}, \boldsymbol{K}) \quad \text{is in } C^\infty_c. \tag{95}$$

So the integral converges absolutely, derivative and integral commute, and we obtain $\partial^\alpha(V_{jj'}(\varphi^*\phi)\Psi)(\boldsymbol{P}, \boldsymbol{K}) \in \mathbb{C}$ for any multi-index α. Hence, $(V_{jj'}(\varphi^*\phi)\Psi)(\boldsymbol{P}, \boldsymbol{K})$ is smooth at $(\boldsymbol{P}, \boldsymbol{K}) \in \dot{\mathcal{Q}}$.

Polynomial scaling of $V_{jj'}(\varphi^*\phi)\Psi$ can be seen as follows: Under the coordinate rotation $\boldsymbol{p}_+ := \boldsymbol{p}_j + \boldsymbol{p}_{j'}$ and $\boldsymbol{p}_- := \boldsymbol{p}_j - \boldsymbol{p}_{j'}$, the expression $V_{jj'}\Psi$ becomes a convolution in $\boldsymbol{p}_-$ of a polynomially scaling function with the function $\varphi^*\phi \in C^\infty_c$. Polynomial scaling bounds are neither affected by the coordinate rotation nor by the convolution with a C^∞_c-function. So indeed, $V_{jj'}(\varphi^*\phi)\Psi \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$, and we have that $V_{jj'}(\varphi^*\phi) : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$.

This mapping property extends to $V_{j'j}(\varphi^*\phi) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ by eRen-linearity. □

Remark

(1) In (89), we have not included the commutation relation for annihilation operators $[A_j(\varphi), A_{j'}(\phi)] = 0$. The reason is that products of two or more annihilation operators are not necessarily defined, since we only have $A_j(\varphi) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$. An arbitrary product $A_j(\varphi)A_{j'}(\phi)\Psi$ with $\Psi \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ contains a double integral

$$\begin{aligned}&\big(A_j(\varphi)A_{j'}(\phi)\Psi\big)(\boldsymbol{P}, \boldsymbol{K})\\ &\quad = \sqrt{(N+1)(N+2)} \iint \varphi(\tilde{\boldsymbol{k}})^* \phi(\tilde{\boldsymbol{k}}')^* \Psi(\boldsymbol{P} - e_j\tilde{\boldsymbol{k}} - e_{j'}\tilde{\boldsymbol{k}}', \boldsymbol{K}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}') d\tilde{\boldsymbol{k}} d\tilde{\boldsymbol{k}}',\end{aligned} \tag{96}$$

where the first integral produces a configuration space function $\dot{\mathcal{Q}} \to \mathrm{Ren}$. And a second integral over such a function can generally not be interpreted as an element in Ren.

A definition of such operator products would require a modification of $\overline{\mathscr{F}}_{\mathrm{ex}}$ such that it also accommodates general divergent integrals over multiple coordinates of a $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$-function as in (96). We postpone the investigation of such choices for $\overline{\mathscr{F}}_{\mathrm{ex}}$ to future investigations.

5. Dressing on the ESS

Our next step is to define a dressing operator $W(s)$ with $s \in \dot{\mathcal{S}}^\infty_1$. To do so, a naive approach would be to start from the expression $W(s) = e^{A^\dagger_M(s) - A_M(s)} \dots e^{A^\dagger_1(s) - A_1(s)}$ (with M depending on the fermion sector) and expand the exponentials into series

$$e^{A^\dagger_j(s) - A_j(s)} = \sum_{n \in \mathbb{N}_0} \frac{(A^\dagger_j(s) - A_j(s))^n}{n!},$$

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

which can be multiplied out. There are two difficulties with this approach:

- Some terms in the resulting sum contain two or more annihilation operators $A(s)$ (see Remark 1).
- There is an infinite number of such terms.

So with the current definition of $\overline{\mathscr{F}}_{\mathrm{ex}}$ we cannot simply define $W(s)$ as an operator $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$. Instead, we pursue a different approach and define $W(s) : \mathcal{D}_W \to \mathscr{F}_{\mathrm{ex}}$. Here, we choose $\mathcal{D}_W \subset \overline{\mathscr{F}}_{\mathrm{ex}}$ in (134), such that $\mathcal{D}_{\mathscr{F}}$ (which is a symmetrized version of $\mathcal{D}_W \cap L^2$) is dense in $\mathscr{F}$.

If we consider $W_{\mathscr{F}}(\varphi) : \mathscr{F} \to \mathscr{F}, \varphi \in \mathfrak{h}$ as a unitary operator on Fock space, together with some suitable $\Psi \in \mathscr{F}$, then there is a well-defined expression (similar to (37)) for $W_{\mathscr{F}}(\varphi)\Psi$ as an L^2-function on momentum–configuration space. For φ replaced by $s \in \dot{\mathcal{S}}_1^\infty$, we may then define $W(s)\Psi \in \overline{\mathscr{F}}$ based on the momentum space expression of $W_{\mathscr{F}}(\varphi)\Psi$.

The domain $\mathcal{D}_W$ in (134) is generated by vectors Ψ_C of the form

$$\Psi_C = W_1(\varphi)A_j^\dagger(v)\Psi_m \quad \text{or} \quad \Psi_C = XW_1(\varphi)\Psi_m,$$

where

- $\Psi_m = \Psi_{mx} \otimes \Omega_y \in \mathscr{F}$ with $\Psi_{mx} \in \mathcal{S}(\mathcal{Q}_x)$ (i.e. we have a Schwartz function) and Ω_y describing the boson field in the vacuum.
- $W_1(\varphi) = e^{A_1^\dagger(\varphi) - A_1(\varphi)}, \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ describes a dressing induced by only the first fermion.
- $A_j^\dagger(v), v \in \dot{\mathcal{S}}_1^\infty$ describes creation by only the fermion with number $j \in \{1, \ldots, M\}$.
- X is a linear combination of Ren_1-constants and operators $V_{jj'}(v^*s)$ that formally commute with $W(s)$.

When setting $X = 1$ (which formally commutes with $W(s)$), we see that $\mathcal{D}_W$ contains vectors of the kind $W_1(\varphi)\Psi_m$. We show in Lemma 5.4, that these are equal to $W_{\mathscr{F},1}(\varphi)\Psi_m$ and, after symmetrization, span a dense subspace of $\mathscr{F}$ (Lemma 5.1). This will later allow for a dense definition of $\widetilde{H}$.

The definition of $W(s)\Psi_C$ now exactly works as explained above: We establish a momentum space expression in case $s, v \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ using Lemma 5.2. Then, we generalize to $s, v \in \dot{\mathcal{S}}_1^\infty$ by a suitable definition. As discussed in the introduction, we remove certain exponential factors of the form $e^{V_{jj'}}$ in an *ad hoc* modification. Thus, $W_j(\varphi_j)$ differs from $W_{\mathscr{F},j}(\varphi_j)$ for $\varphi_j \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$. However, we show in Lemma 5.5 that any $V_{j'j''}$ commutes with $W_{\mathscr{F},j}(\varphi_j)$, so the omitted factor $e^{V_{jj'}}$ can heuristically be "pulled into any position". Heuristically, this factor disappears when performing a dressing, which justifies the omission within the computation of $\widetilde{H}$.

5.1. *Bosonic dressing* $W_y(\varphi)$

The upcoming proofs are based on some well-known facts about coherent states, where only bosons are present, i.e. $\Psi_y \in \mathscr{F}_y$. The momentum space representation

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

of the bosonic creation and annihilation operators $a^\dagger(v), a(v)$ with form factor $v \in \mathfrak{h}$ given by

$$\begin{aligned}(a^\dagger(v)\Psi_y)(\boldsymbol{K}) &:= \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} v(\boldsymbol{k}_\ell)\Psi_y(\boldsymbol{K}\backslash\boldsymbol{k}_\ell),\\ (a(v)\Psi_y)(\boldsymbol{K}) &:= \sqrt{N+1}\int v(\tilde{\boldsymbol{k}})^*\Psi_y(\boldsymbol{K},\tilde{\boldsymbol{k}})\, d\tilde{\boldsymbol{k}}.\end{aligned} \tag{97}$$

This definition implies that the commutation relations $[a(v_1), a^\dagger(v_2)] = \langle v_1, v_2\rangle$ hold as a strong operator identity on a dense domain in $\mathscr{F}_y$. These operators $a^\dagger(v), a(v)$ substantially differ from $A^\dagger(v), A(v)$ defined in (20), (22), which create one boson at each position of a fermion, whereas in $\mathscr{F}_y$, there are no fermions.

Using $a^\dagger(v), a(v)$, we may define a set of displacement operators

$$W_y(\varphi) = e^{a^\dagger(\varphi)-a(\varphi)}, \tag{98}$$

and coherent states $\Psi_y(\varphi) := W_y(\varphi)\Omega_y$. Indeed, $W_y(\varphi)$ is well defined, since for $\varphi \in \mathfrak{h}$, we have the bounds

$$\big\|a^\dagger(\varphi)\Psi_y\big\| \leq \big\|(N+1)^{1/2}\Psi_y\big\|\, \|\varphi\|, \quad \big\|a(\varphi)\Psi_y\big\| \leq \big\|N^{1/2}\Psi_y\big\|\, \|\varphi\|, \tag{99}$$

so the exponential series (98) converges in norm

$$\begin{aligned}&\Psi_y(\varphi) = W_y(\varphi)\Omega_y := \sum_{k=0}^{\infty} \frac{1}{k!}(a^\dagger(\varphi) - a(\varphi))^k\Omega_y\\ &\text{where } \left\|\frac{1}{k!}(a^\dagger(\varphi) - a(\varphi))^k\Omega_y\right\| \leq \frac{1}{k!}\big\|2^k(k!)^{1/2}\Omega_y\big\|\, \|\varphi\|^k = (k!)^{-1/2}\|2\varphi\|^k.\end{aligned} \tag{100}$$

Here, we used in the second line that $(a^\dagger(\varphi) - a(\varphi))^k\Omega_y$ occupies only sectors in Fock space with $\leq k$ particles, so we can set $N \leq (k-1)$ in (99). Subsequent application of (99) leads to the factor $(k!)^{1/2}$.

In momentum space representation and in terms of tensor products

$$\Psi_y(\varphi)(\boldsymbol{K}) = e^{-\frac{\|\varphi\|^2}{2}} \frac{1}{\sqrt{N!}} \left(\prod_{\ell=1}^{N} \varphi(\boldsymbol{k}_\ell)\right) \Leftrightarrow \Psi_y(\varphi) = \sum_{N=0}^{\infty} \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \underbrace{\varphi \otimes_S \cdots \otimes_S \varphi}_{N \text{ times}}. \tag{101}$$

A calculation similar to (100) verifies that $W_y(\varphi)$ can be defined on all Ψ_y with finite particle number, i.e. $\Psi_y \in \mathscr{F}_{\mathrm{fin},y}$ with

$$\mathscr{F}_{\mathrm{fin},y} := \big\{\Psi_y \in \mathscr{F}_y \,|\, \exists\, N_{\max} \in \mathbb{N} : \Psi_y^{(N)} = 0\ \forall\, N > N_{\max}\big\}. \tag{102}$$

And by continuity, we can thus define $W_y(\varphi)$ on all of $\mathscr{F}_y$.

Moreover, the $W_y(\varphi)$ are unitary, so $\big\|\Psi_y(\varphi)\big\| = 1$, and they satisfy the Weyl relations. Further, it is a well-known fact that the span of the set of coherent states $\{\Psi_y(\varphi)\,|\,\varphi \in \mathfrak{h}\}$ is dense in $\mathscr{F}_y$ [34, Proposition 12]. In addition

$$\langle \Psi_y(\varphi_1), \Psi_y(\varphi_2)\rangle_{\mathscr{F}_y} = e^{-\frac{\|\varphi_1\|^2+\|\varphi_2\|^2}{2}} e^{\langle\varphi_1,\varphi_2\rangle}, \tag{103}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

so

$$\big\|\Psi_y(\varphi_1)-\Psi_y(\varphi_2)\big\|^2=\big\|\Psi_y(\varphi_1)\big\|^2+\big\|\Psi_y(\varphi_2)\big\|^2-2e^{-\frac{\|\varphi_1\|^2+\|\varphi_2\|^2}{2}}\mathrm{Re}\big(e^{\langle\varphi_1,\varphi_2\rangle}\big). \tag{104}$$

As $\mathfrak{h}$ is separable, we can find a countable dense set $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathfrak{h}$, such that $(\Psi_y(\varphi_n))_{n\in\mathbb{N}}$ is also dense in the coherent states. So

$$\mathrm{span}\big\{\Psi_y(\varphi_n)\,|\,n\in\mathbb{N}\big\}\quad\text{is dense in }\mathscr{F}_y. \tag{105}$$

Since $\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty\supset C_c^\infty$ is dense in $\mathfrak{h}$ the above statements hold true, if we replace $\varphi_n\in\mathfrak{h}$ by $\varphi_n\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty$.

5.2. *Dressing induced by fermions* $W_1(\varphi)$

5.2.1. *Density*

Now, let us turn to the case with two particle species, i.e. $\mathscr{F}=\mathscr{F}_x\otimes\mathscr{F}_y$ and $A^\dagger, A$ instead of $a^\dagger, a$. In order to make an analogous statement to (105) work, we restrict from $A^\dagger(\varphi)=\sum_{j=1}^M A_j^\dagger(\varphi)$ to $A_1^\dagger(\varphi)$, i.e. creation by only the first fermion. Just as $W_y(\varphi)$, the Fock space operator $W_{\mathscr{F},1}(\varphi)=e^{A_1^\dagger(\varphi)-A_1^\dagger(\varphi)}$ can be defined in analogy to $W_y(\varphi)$. The operators $A_1^\dagger(\varphi)$ and $W_{\mathscr{F},1}(\varphi)$ break the fermionic symmetry, so they map $\mathscr{F}\to L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$ (instead of $\mathscr{F}\to\mathscr{F}$). We will therefore proceed by considering vectors $\Psi\in L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$, i.e. with only bosonic exchange symmetry.

As a "cyclic set" of vectors Ψ_m, used for generating further domains, we choose

$$\mathcal{C}_{WS}:=\mathcal{S}(\mathcal{Q}_x)\otimes\{\Omega_y\}\subset L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y. \tag{106}$$

Since the boson field is in the vacuum, $\mathcal{C}_{WS}$ is obviously not dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$. However, it generates a dense subspace by applying operators $W_{\mathscr{F},1}(\varphi)$ to it. The momentum representation after such an application is given by

$$(W_{\mathscr{F},1}(\varphi)\Psi_m)(\boldsymbol{P},\boldsymbol{K})=\frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\left(\prod_{\ell=1}^N\varphi(\boldsymbol{k}_\ell)\right)\Psi_{mx}\left(\boldsymbol{p}_1+\sum_{\ell=1}^N\boldsymbol{k}_\ell,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_M\right). \tag{107}$$

Definition 5.1. The *span of coherent states created by the first fermion* is given by

$$\mathcal{D}_{WS}:=\mathrm{span}\big\{W_{\mathscr{F},1}(\varphi)\Psi_m\,|\,\varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty,\ \Psi_m\in\mathcal{C}_{WS}\big\}. \tag{108}$$

With (108), the following is true.

Lemma 5.1. *$(S_-\otimes I)[\mathcal{D}_{WS}]$ is dense in $\mathscr{F}=\mathscr{F}_x\otimes\mathscr{F}_y$.*

The proof is based on denseness of coherent states in $\mathscr{F}_y$ and can be found in Appendix A.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

5.2.2. *Dressed one-boson states*

Just as $W_{\mathscr{F},1}(\varphi)$, we may define $W_{\mathscr{F},j}(\varphi), j \in \mathbb{N}$ and $W_{\mathscr{F}}(\varphi) = W_{\mathscr{F},M}(\varphi)\dots W_{\mathscr{F},1}(\varphi)$ with $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ on each M-fermion sector. These operators are all unitary on $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ and well defined on $\mathscr{F}$.

We will now establish some useful commutation relations on the dense subspace

$$\mathscr{F}_{\text{fin}} := \mathscr{F}_x \otimes \mathscr{F}_{\text{fin},y} \subset \mathscr{F}, \tag{109}$$

with $\mathscr{F}_{\text{fin},y}$ defined in (102).

Lemma 5.2 (Commutation Relations for $W_{\mathscr{F}}$). *For $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$, we have the following strong operator identities*[d] *on $\mathscr{F}_{\text{fin}}$:*

$$\begin{aligned}
W_{\mathscr{F},j}(\varphi)A^\dagger_{j'}(\phi) &= \begin{cases} \big(A^\dagger_{j'}(\phi) - \langle\varphi,\phi\rangle\big)W_{\mathscr{F},j}(\varphi) & \text{if } j = j', \\ \big(A^\dagger_{j'}(\phi) - V_{jj'}(\varphi^*\phi)\big)W_{\mathscr{F},j}(\varphi) & \text{if } j \neq j', \end{cases} \\
W_{\mathscr{F}}(\varphi)A^\dagger_{j'}(\phi) &= \big(A^\dagger_{j'}(\phi) - \langle\varphi,\phi\rangle - V_{\bullet j'}(\varphi^*\phi)\big)W_{\mathscr{F}}(\varphi), \\
W_{\mathscr{F},j}(\varphi)A^\dagger(\phi) &= \big(A^\dagger(\phi) - \langle\varphi,\phi\rangle - V_{j\bullet}(\varphi^*\phi)\big)W_{\mathscr{F},j}(\varphi), \\
W_{\mathscr{F}}(\varphi)A^\dagger(\phi) &= \big(A^\dagger(\phi) - M\langle\varphi,\phi\rangle - V(\varphi^*\phi)\big)W_{\mathscr{F}}(\varphi),
\end{aligned} \tag{110}$$

as well as

$$\begin{aligned}
W_{\mathscr{F},j'}(\varphi)A_j(\phi) &= \begin{cases} \big(A_j(\phi) - \langle\phi,\varphi\rangle\big)W_{\mathscr{F},j'}(\varphi) & \text{if } j = j', \\ \big(A_j(\phi) - V_{jj'}(\phi^*\varphi)\big)W_{\mathscr{F},j'}(\varphi) & \text{if } j \neq j', \end{cases} \\
W_{\mathscr{F}}(\varphi)A_j(\phi) &= \big(A_j(\phi) - \langle\phi,\varphi\rangle - V_{j\bullet}(\phi^*\varphi)\big)W_{\mathscr{F}}(\varphi), \\
W_{\mathscr{F},j'}(\varphi)A(\phi) &= \big(A(\phi) - \langle\phi,\varphi\rangle - V_{\bullet j'}(\phi^*\varphi)\big)W_{\mathscr{F},j'}(\varphi), \\
W_{\mathscr{F}}(\varphi)A(\phi) &= \big(A(\phi) - M\langle\phi,\varphi\rangle - V(\phi^*\varphi)\big)W_{\mathscr{F}}(\varphi),
\end{aligned} \tag{111}$$

with $V_{jj'}$ defined in (91) *and*[e]

$$\begin{aligned}
V_{\bullet j'}(\varphi^*\phi) &:= \sum_{j:j\neq j'} V_{jj'}(\varphi^*\phi), \quad V_{j\bullet}(\varphi^*\phi) := \sum_{j':j\neq j'} V_{jj'}(\varphi^*\phi), \\
V(\varphi^*\phi) &:= \sum_{j\neq j'} V_{jj'}(\varphi^*\phi).
\end{aligned} \tag{112}$$

The proof of Lemma 5.2 is straightforward by applying the CCR. We present it in Appendix B.

[d]By a strong operator identity $A = B$ for $A, B : \mathscr{F} \to L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$, we mean that $A\Psi = B\Psi \ \forall\, \Psi \in \mathscr{F}$, even if possibly $A\Psi, B\Psi \notin \mathscr{F}$.

[e]Here, $\sum_{j:j\neq j'}$ is to be understood as a sum over only j, while $\sum_{j\neq j'}$ is a sum over both j and j'. The second kind of sum will appear more often, so we give it a shorter notation.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

5.3. *Extended dressing $W(s)$*

We would now like the relations in (110) to also hold true if we replace $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ by $s, v \in \dot{\mathcal{S}}_1^\infty$. In that case, the Fock space operators $W_{\mathscr{F},j}$ turn into extended operators W_j. More precisely, it would be desirable to have

$$W_j(s)A_{j'}^\dagger(v)\Psi_m = \begin{cases} \big(A_{j'}^\dagger(v) - \langle s, v\rangle\big)W_j(s)\Psi_m & \text{if } j = j', \\ \big(A_{j'}^\dagger(v) - V_{jj'}(s^*v)\big)W_j(s)\Psi_m & \text{if } j \neq j', \end{cases} \tag{113}$$

for $\Psi_m \in \mathcal{C}_{WS}$. By Lemma 4.2, and since $\langle s, v\rangle \in \mathrm{Ren}_1$, we may obviously interpret

$$V_{jj'}(s^*v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}, \quad \langle s, v\rangle : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}. \tag{114}$$

So if we can define $W_j(s)\Psi_m \in \overline{\mathscr{F}}$, then the right-hand side of (113) serves as a definition for $W_j(s)A_{j'}^\dagger(v)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}$. And if we can further define products like $W(s)W_1(\varphi)\Psi_m = W_M(s)\dots W_1(s)W_1(\varphi)\Psi_m \in \overline{\mathscr{F}}$, then a generalization of (110) may even be used to define $W(s)W_1(\varphi)A_{j'}^\dagger(v)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}$.

However, before doing so, it is first necessary to specify what $W_1(s)W_1(\varphi)$ is, which we will do by introducing some "extended Weyl relations".

5.3.1. *Extended Weyl relations*

In order to treat products of factors $W_j(s), s \in \dot{\mathcal{S}}_1^\infty$, we introduce an extended Weyl algebra $\overline{\mathcal{W}}$ that is generated by all $W_j(s)$ and taken over the field eRen (as in Definition 3.7). Recall that $\mathfrak{c} \in \mathrm{eRen}$ is a fraction of linear combinations of exponentials $e^{\mathfrak{r}}$, with $\mathfrak{r} \in \mathrm{Ren}_1$ being a possibly divergent integral (see Definition 3.2). Multiplication on $\overline{\mathcal{W}}$ is defined by the Weyl relations

$$\begin{aligned} W_j(s)^{-1} &= W_j(-s), \\ W_j(s_1)W_j(s_2) &= e^{-\frac{i}{2}\sigma(s_1,s_2)}W_j(s_1+s_2), \end{aligned} \tag{115}$$

with symplectic form

$$\begin{aligned} \sigma = \dot{\mathcal{S}}_1^\infty \times \dot{\mathcal{S}}_1^\infty &\to \mathrm{Ren}_1, \\ (s_1, s_2) &\mapsto \langle s_1, s_2\rangle - \langle s_2, s_1\rangle. \end{aligned} \tag{116}$$

Note that $e^{-\frac{i}{2}\sigma(s_1,s_2)} = e^{\mathrm{Im}\langle s_1,s_2\rangle} \in \mathrm{eRen}$ is not necessarily a complex number.

This $\overline{\mathcal{W}}$ can be seen as an "almost-extension" of the usual Weyl algebra generated by $\{W_j(s) \,|\, s \in \mathfrak{h}\}$ (strictly speaking, it only extends some Weyl algebra $\mathcal{W}_0$ generated by $\{W_j(s) \,|\, s \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty\}$).

The definition of the extended Weyl algebra now allows us to write

$$\begin{aligned} W(s)W_1(\varphi) &= W_M(s)\dots W_2(s)W_1(s)W_1(\varphi) \\ &= e^{\mathrm{Im}\langle s,\varphi\rangle}W_M(s)\dots W_2(s)W_1(s+\varphi). \end{aligned} \tag{117}$$

So $W(s)W_1(\varphi)$ can be brought into the form $W_M(s_M)\dots W_1(s_1)$ times an eRen-factor, which is the same on each M-fermion sector.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

5.3.2. *Extended dressing on coherent states*

In order to define vectors of the kind $W_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}$, we make use of the momentum space definition of $W_{\mathscr{F},M}(\varphi_M)\dots W_{\mathscr{F},1}(\varphi_1)\Psi_m$ for $\varphi_\ell \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$. For two dressing operators with $j \neq j'$, the Baker–Campbell–Hausdorff formula implies

$$\begin{aligned}
&W_{\mathscr{F},j}(\varphi_j)W_{\mathscr{F},j'}(\varphi_{j'})\Psi_m \\
&\quad = e^{A_j^\dagger(\varphi_j)-A_j(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})-A_{j'}(\varphi_{j'})}\Psi_m \\
&\quad = e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}}e^{A_j^\dagger(\varphi_j)}e^{-A_j(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})}\underbrace{e^{-A_{j'}(\varphi_{j'})}\Psi_m}_{=\Psi_m} \\
&\quad = e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}-[A_j(\varphi_j),A_{j'}^\dagger(\varphi_{j'})]}e^{A_j^\dagger(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})}e^{-A_j(\varphi_j)}\Psi_m \\
&\quad = e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}-V_{jj'}(\varphi_j^*\varphi_{j'})}e^{A_j^\dagger(\varphi_j)+A_{j'}^\dagger(\varphi_{j'})}\Psi_m. \qquad (118)
\end{aligned}$$

Here, we used that $V_{jj'}$ commutes with all $A_{j''}^\dagger$ and $A_{j''}$, which follows by the same arguments as in the proof of Lemma 5.5. Thus, all double commutators between A- and $A^\dagger$-operators vanish. The generalization to arbitrarily many factors, $W_{\mathscr{F},M}(\varphi_M)\dots W_{\mathscr{F},1}(\varphi_1)\Psi_m$ is straightforward, and we obtain an exponential of constants and $V_{jj'}$-terms, followed by $e^{A_M^\dagger(\varphi_M)+\dots+A_1^\dagger(\varphi_1)}\Psi_m$.

We now define $W_M(\varphi_M)\dots W_1(\varphi_1)$ by dropping the $V_{jj'}$-terms in (118), which yields the following momentum space expression for $\Psi_m = \Psi_{mx}\otimes\Omega_y \in \mathcal{C}_{WS}$:

$$\begin{aligned}
&(W_M(\varphi_M)\dots W_1(\varphi_1)\Psi_m)(\boldsymbol{P},\boldsymbol{K}) \\
&\quad := \frac{1}{\sqrt{N!}}e^{-\sum_{j=1}^M\frac{\|\varphi_j\|^2}{2}}\sum_\sigma\left(\prod_{\ell=1}^N\varphi_{\sigma(\ell)}(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P}'), \qquad (119)
\end{aligned}$$

where the sum over σ runs over all M^N maps

$$\sigma : \{1,\dots,N\} \to \{1,\dots,M\}, \qquad (120)$$

assigning each boson ℓ to a fermion $j = \sigma(\ell)$. The shifted momentum, illustrated in Fig. 4, is then

$$\boldsymbol{P}' := \boldsymbol{P} + \sum_\ell e_{\sigma(\ell)}\boldsymbol{k}_\ell. \qquad (121)$$

Lemma 5.3 (Products of W are Well Defined). *Consider a sequence $(s_j)_{j\in\mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$. Then, the momentum space definition* (119) *renders a well-defined vector*

$$W_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}, \qquad (122)$$

where (122) *is to be interpreted as a sector-wise definition in $M \in \mathbb{N}$.*

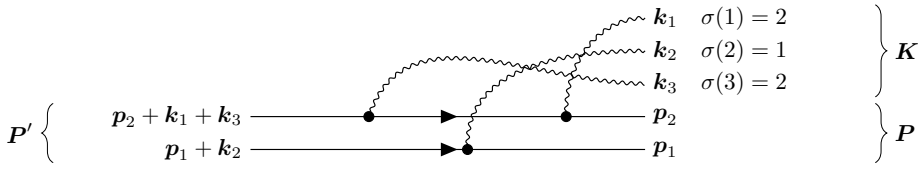

Fig. 4. Visual representation of the momentum shift within the dressing. The wiggly lines represent bosons that get attached to fermions (solid lines). $\sum_\sigma$ runs over all M^N possible ways to attach all bosons to the fermions. For a fixed σ, we obtain the entries of $\boldsymbol{P'}$ by summing up all particle momenta on the respective fermion line.

Proof. Copying the momentum space definition (119), we obtain

$$\begin{aligned}&(W_M(s_M)\dots W_1(s_1)\Psi_m)(\boldsymbol{P},\boldsymbol{K})\\ &\quad:= \frac{1}{\sqrt{N!}} e^{-\sum_{j=1}^M \frac{\|s_j\|^2}{2}} \sum_\sigma \left(\prod_{\ell=1}^N s_{\sigma(\ell)}(\boldsymbol{k}_\ell)\right) \Psi_{mx}(\boldsymbol{P'}).\end{aligned} \tag{123}$$

Obviously, $(\prod_{\ell=1}^N s_{\sigma(\ell)}(\boldsymbol{k}_\ell))\Psi_{mx}(\boldsymbol{P'})$ defines a function in $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$, which is still true after taking the finite sum over σ.

Further, we have $\|s_j\|^2 = \langle s_j, s_j\rangle \in \mathrm{Ren}_1$, so $e^{-\sum_{j=1}^M \frac{\|s_j\|^2}{2}} \in \mathrm{eRen}$. Therefore, the expression (119) defines an element of $\overline{\mathscr{F}}$. □

This already allows us to define $W_M(s_M)\dots W_1(s_1)$ on vectors $\Psi_m \in \mathcal{C}_{WS}$ with the boson field in the vacuum. In order to define $W_M(s_M)\dots W_1(s_1)$ also on a dense domain in $\mathscr{F}$, we extend the definition to vectors $W_{\mathscr{F},1}(\varphi)\Psi_m \in \mathcal{D}_{WS}$, whose symmetrized span, by Lemma 5.1, is dense in $\mathscr{F}$. This extension is done by assuming that $W_{\mathscr{F},1}(\varphi)$ can be merged into $W_1(s_1)$, just as $W_1(\varphi)$ in (117).

We will also allow for a treatment of state vectors by using the operator $(S_-\otimes I)$, which can obviously be extended to $(S_-\otimes I):\overline{\mathscr{F}}\to\overline{\mathscr{F}}$ or $(S_-\otimes I):\overline{\mathscr{F}}_{\mathrm{ex}}\to\overline{\mathscr{F}}_{\mathrm{ex}}$, using the momentum space definition (11).

Definition 5.2. Let $(s_j)_{j\in\mathbb{N}}\subset\dot{\mathcal{S}}^\infty_1$. Then, by Lemma 5.3, copying the momentum space definition (119) results in a well-defined *product of dressing operators*

$$\begin{aligned}&W_M(s_M)\dots W_1(s_1):\mathcal{D}_{WS}\to\overline{\mathscr{F}},\\ &W_M(s_M)\dots W_1(s_1)W_{\mathscr{F},1}(\varphi)\Psi_m := e^{\mathrm{Im}\langle s_1,\varphi\rangle}W_M(s_M)\dots W_1(s_1+\varphi)\Psi_m,\end{aligned} \tag{124}$$

where M is the respective fermion number on each sector. Further, we define the extension to *symmetrized vectors*

$$W_M(s_M)\dots W_1(s_1):(S_-\otimes I)[\mathcal{D}_{WS}]\cup\mathcal{D}_{WS}\to\overline{\mathscr{F}}, \tag{125}$$

by imposing that $W_M(s_M)\dots W_1(s_1)$ shall commute with the symmetrization operator $(S_-\otimes I)$.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

With this definition, it is true that

Lemma 5.4. *For all $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$, it holds that*

$$W_{\mathscr{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m, \tag{126}$$

in terms of momentum space functions.

Proof. Consider (118) with $j = 1$ and $\varphi_{j'} = 0$. Then, the $V_{jj'}$-term vanishes, so no $V_{jj'}$-terms are dropped when copying momentum space expressions in the transition $W_{\mathscr{F},1} \to W_1$ and indeed $W_{\mathscr{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m$. □

Remarks

(2) It may seem natural to extend Definition 5.2 to a general $\Psi \in \mathscr{F}$. By Lemma 5.1, we can write Ψ as a symmetrized version of $\Psi' = \sum_{nn'} \times W_1(\varphi_n)\Psi_{n'}$ with $\varphi_n \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and $\Psi_{n'} \in \mathcal{C}_{WS}$. In that case, $W(s)\Psi = \sum_{nn'} W(s)W_1(\varphi_n)\Psi_{n'}$ contains a possibly infinite sum over functions $\dot{\mathcal{Q}} \to \mathbb{C}$, which may not converge.

However, our aim is to give a dense definition of $\widetilde{H} : \mathscr{F} \supset \mathrm{dom}(\widetilde{H}) \to \mathscr{F}$, so it suffices to consider the action of $W(s)$, and $HW(s)$ on a dense subset of $\mathscr{F}$, such as $(S_- \otimes I)[\mathcal{D}_{WS}]$.

(3) Concerning the renormalization classes: Two ESS vectors $W(s)W_1(\varphi)\Psi_m$ and $W(s)W_1(\tilde{\varphi})\Psi_{m'}$ with Ψ_m and $\Psi_{m'}$ concentrated on the same M-fermion sector can be added if the wave function renormalizations $\mathfrak{c} = e^{\mathfrak{r}}, \mathfrak{r} = -\frac{\|s\|^2}{2}$ belong to the same renormalization factor class, i.e.

$$\begin{aligned}
\mathfrak{r} - \tilde{\mathfrak{r}} \in \mathbb{C} &\Leftrightarrow |\|s+\varphi\|^2 - \|s+\tilde{\varphi}\|^2 + 2\mathrm{Im}\langle s, \varphi - \tilde{\varphi}\rangle| < \infty \\
&\Leftrightarrow \left| 2\mathrm{Re}\langle s, \varphi - \tilde{\varphi}\rangle + 2\mathrm{Im}\langle s, \varphi - \tilde{\varphi}\rangle + \underbrace{\|\varphi\|^2}_{<\infty} - \underbrace{\|\tilde{\varphi}\|^2}_{<\infty} \right| < \infty \\
&\Leftarrow |\langle s, \varphi - \tilde{\varphi}\rangle| < \infty.
\end{aligned} \tag{127}$$

That means, convergence of the integral $\int s(\boldsymbol{k})^*(\varphi(\boldsymbol{k}) - \tilde{\varphi}(\boldsymbol{k}))d\boldsymbol{k}$ ensures that the renormalization classes coincide. Note that both Re and Im above may be infinite, but cancel each other out.

5.3.3. *Extended dressing on one-boson states*

Now, as announced, when replacing $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ by $v, s \in \dot{\mathcal{S}}_1^\infty$ in (110), we obtain a well-defined right-hand side. This allows for the following extension of dressing operator products.

Definition 5.3. Let $v \in \dot{\mathcal{S}}_1^\infty, (s_j)_{j\in\mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$. We extend the *product of dressing operators* to one-boson states via

$$W_M(s_M)\dots W_1(s_1)A_{j'}^\dagger(v)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}, \tag{128}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

where M is the respective fermion number on each sector, via

$$W_M(s_M)\dots W_1(s_1)A^\dagger_{j'}(v)\Psi_m := \left(A^\dagger_{j'}(v) - \sum_{j=1}^{M} X_j\right) W_M(s_M)\dots W_1(s_1)\Psi_m$$

$$\text{with } X_j = \begin{cases} \langle s, v\rangle & \text{if } j = j', \\ V_{jj'}(s^*v) & \text{if } j \neq j'. \end{cases} \tag{129}$$

This operator can further be extended to *symmetrized vectors* by imposing that $W_M(s_M)\dots W_1(s_1)$ shall commute with the symmetrization operator $(S_- \otimes I)$.

It is easy to see that the right-hand side of (129) makes sense: By Lemma 5.3, we have $W_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}$. Lemma 4.1 implies that $A^\dagger_{j'}(v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ and by Lemma 4.2 and $\langle s, v\rangle \in \mathrm{Ren}_1$, we have that $X_j : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$.

Heuristically, the factors X_j now commute with $W_{j'}(s)$, since the following.

Lemma 5.5. *For $\varphi, \varphi', \phi \in \mathfrak{h}$ it is true that*

$$[W_{\mathscr{F},j}(\varphi'), V_{j'j''}(\varphi^*\phi)] = 0 \quad \textit{and} \quad [W_{\mathscr{F},j}(\varphi'), \langle\varphi, \phi\rangle] = 0, \tag{130}$$

as a strong operator identity on $\mathscr{F}$.

Proof. Since $\varphi, \phi \in L^2$, we have $\varphi^*\phi \in L^1$, so after a Fourier transform, the operator $V_{j'j''}(\varphi^*\phi)$ amounts to a multiplication by an L^∞-function, and is hence bounded. Further, $W_{\mathscr{F},j}(\varphi')$ is unitary on $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ (and hence bounded). So the commutator is defined on all of $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ and hence $\mathscr{F}$.

Now, in position space, both $A^\dagger_j(\varphi')$ and $A_j(\varphi')$ can be decomposed into a fiber integral by fiber-decomposing $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y = \int_{\mathcal{Q}_x} \mathscr{F}_y \, d\boldsymbol{X}$ (see (21) and (23)). So we can also decompose $W_{\mathscr{F},j}(\varphi') = e^{A^\dagger_j(\varphi') - A_j(\varphi')}$ into a fiber integral. And by (93), the operator $V_{j'j''}(\varphi^*\phi)$ just amounts to a multiplication by a complex constant on each fiber Hilbert space. So the fiber operators commute on all fibers and hence the original operators commute on $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ and $\mathscr{F}$.

The expression $\langle\varphi, \phi\rangle$ is just a constant, so it trivially commutes with $W_{\mathscr{F},j}(\varphi')$. □

Mathematically, if we replace $\varphi, \varphi', \phi \in \mathfrak{h}$ by $s, s', v \in \dot{\mathcal{S}}^\infty_1$, then the commutation relations (130) are not *a priori* valid, since $W_j(s)$ is not necessarily defined on vectors of the kind $V_{j'j''}(\varphi^*\phi)\Psi_m$ or $\langle\varphi, \phi\rangle\Psi_m$. We enforce their validity by taking (130) as a definition for an extension of $W_j(s)$.

Definition 5.4. Let $(s_j)_{j\in\mathbb{N}} \subset \dot{\mathcal{S}}^\infty_1$, $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}^\infty_1$ and $\Psi_m \in \mathcal{C}_{WS}$, and let X be an element of the set of operators

$$\mathcal{X} := \mathrm{span}_{\mathrm{eRen}}\big\{\langle s, v\rangle,\ V_{jj'}(s^*v) \,\big|\, s, v \in \dot{\mathcal{S}}^\infty_1\big\}, \tag{131}$$

so X formally commutes with all $W_j(s_j)$. Then, we extend the *product of dressing operators* via

$$\begin{aligned} W_M(s_M)\dots W_1(s_1)X\Psi_m &:= XW_M(s_M)\dots W_1(s_1)\Psi_m, \\ W_M(s_M)\dots W_1(s_1)XW_{\mathscr{F},1}(\varphi)\Psi_m &:= XW_M(s_M)\dots W_1(s_1)W_{\mathscr{F},1}(\varphi)\Psi_m \\ &= XW_M(s_M)\dots W_1(s_1)W_1(\varphi)\Psi_m, \end{aligned} \tag{132}$$

with M being the respective fermion number on each sector and where the last equality in (132) holds by Lemma 5.5. Again, we may extend the definition to *symmetrized vectors* by imposing that $W_M(s_M)\dots W_1(s_1)$ shall commute with the symmetrization operator $(S_-\otimes I)$.

Again, it is easy to see that this definition makes sense: By Lemma 5.3, we have $W_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}$. And since $X\in\mathcal{X}$ maps $\overline{\mathscr{F}}\to\overline{\mathscr{F}}_{\mathrm{ex}}$, indeed

$$XW_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}, \tag{133}$$

so the right-hand sides of (132) are well defined.

Remark

(4) It seems natural to define (132) for all operators X which commute with $A_j^\dagger(s')$ in a sufficiently regular case and $A_j(s')$. However, since we have only defined $A_j(s'):\overline{\mathscr{F}}\to\overline{\mathscr{F}}_{\mathrm{ex}}$ and $V_{j'j''}(s^*v):\overline{\mathscr{F}}\to\overline{\mathscr{F}}_{\mathrm{ex}}$, it is not clear how to interpret the commutator $[A_j(s'),V_{j'j''}(s^*v)]$. So $V_{j'j''}(s^*v)$ would then not be a valid X-operator, although it commutes with $A_j^\dagger(s')$ and $A_j(s')$ for $s,s',v\in\mathfrak{h}$.

If one succeeded to modify the definition of $\overline{\mathscr{F}},\overline{\mathscr{F}}_{\mathrm{ex}}$ such that commutators as $[A_j(s'),V_{jj'}(s^*v)]$ are well-defined operators, then it seems reasonable to change the set of allowed X in Definition 5.4 to all X with $[A_j^\dagger(s'),X]=[A_j(s'),X]=0$.

5.3.4. *Final definition of $W(s)$*

With Definitions 5.3 and 5.4, we may now provide the final domains for the product $W_M(s_M)\dots W_1(s_1)$: The *extended dressing domain* $\mathcal{D}_W$ is defined as

$$\begin{aligned} \mathcal{D}_W := \mathrm{span}_{\mathrm{eRen}}\big\{ & W_1(\varphi)A_j^\dagger(v)\Psi_m,\ XW_1(\varphi)\Psi_m \,|\, \varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty,\ v\in\dot{\mathcal{S}}_1^\infty, \\ & X\in\mathcal{X},\ \Psi_m\in\mathcal{C}_{WS}\big\}, \end{aligned} \tag{134}$$

with $\dot{\mathcal{S}}_1^\infty$ defined in (27), $\mathcal{X}$ defined in (131) and $\mathcal{C}_{WS}$ defined in (106). Well-definedness of $W(s)$ on $\mathcal{D}_W$ can be seen by combining (117) with Definitions 5.3 and 5.4. By imposing that $W_M(s_M)\dots W_1(s_1)$ shall commute with $(S_-\otimes I)$, we extend $W_M(s_M)\dots W_1(s_1)$ to $(S_-\otimes I)[\mathcal{D}_W]\cup\mathcal{D}_W$.

The maximal domain of $W(s)$ in Fock space is now given by the *large domain*

$$\mathcal{D}_{\mathscr{F}} := (S_-\otimes I)\big[\mathcal{D}_W\cap(L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y)\big]. \tag{135}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

The symmetrization operator $(S_- \otimes I)$ ensures that indeed $\mathcal{D}_{\mathscr{F}} \subset \mathscr{F}$. With this definition, it holds true that

Lemma 5.6. *We have the inclusion*

$$\mathcal{D}_{WS} \subset \mathcal{D}_W \cap (L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y), \tag{136}$$

and in particular, $\mathcal{D}_{\mathscr{F}}$ is dense in $\mathscr{F}$.

Proof. Setting $X = 1$ and using Lemma 5.4, we see that $W_{\mathscr{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m \in \mathcal{D}_{WS}$ with $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$ is also an element of $\mathcal{D}_W$. Further, $W_{\mathscr{F},1}(\varphi)\Psi_m \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$, which yields the inclusion relation (136).

Hence, the symmetrized version $(S_- \otimes I)[\mathcal{D}_{WS}]$ is included in $\mathcal{D}_{\mathscr{F}}$. And since the former is dense in $\mathscr{F}$ (Lemma 5.1), also the latter is. □

In order to define the renormalized Hamiltonian $\widetilde{H} = W(s)^{-1}HW(s)$, we also need to have a well-defined inverse $W(s)^{-1}$.

Lemma 5.7. *$W(s)$ with $s \in \dot{\mathcal{S}}_1^\infty$ is invertible on $\mathcal{D}_{\mathscr{F}}$.*

In the proof of this Lemma we employ another general statement about coherent states.

Lemma 5.8. *For $k \in \{1, \dots, K\}, K \in \mathbb{N}$, consider $\Psi'_{m,k} \in L^2(\mathcal{Q}_x) \otimes \{\Omega_y\}$ and $\varphi_k \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$. Further, choose any partition $\{1, \dots, K\} = \mathcal{K}_{WA} \cup \mathcal{K}_W$, as well as $v_k \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and $j_k \in \mathbb{N}$ for $k \in \mathcal{K}_{WA}$, and define*

$$\Psi_k := \begin{cases} W_{\mathscr{F},1}(\varphi_k)A^\dagger_{j_k}(v_k)\Psi'_{m,k} & \text{if } k \in \mathcal{K}_{WA}, \\ W_{\mathscr{F},1}(\varphi_k)\Psi'_{m,k} & \text{if } k \in \mathcal{K}_W, \end{cases} \tag{137}$$

such that $\Psi_k \neq 0$. Further, assume that $\varphi_k \neq \varphi_{k'}$ whenever $k \neq k'$ both belong to either $\mathcal{K}_{WA}$ or $\mathcal{K}_W$. Then, the set

$$\big\{\Psi_k \,|\, k \in \{1, \dots, K\}\big\} \subset L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y, \tag{138}$$

is linearly independent.

Heuristically speaking, the proof relies on the argument, that there is a "largest φ_k", for which the term $\varphi_k^{\otimes N}$, occurring in a coherent state, eventually grows "too large to be canceled by the $K-1$ other terms" as $N \to \infty$. The proof itself is rather technical and can be found in Appendix C.

Proof of Lemma 5.7. We need to show that $W(s)$ is injective on $\mathcal{D}_{\mathscr{F}}$. That is, there is no $\Psi \in \mathcal{D}_{\mathscr{F}}, \Psi \neq 0$ with $W(s)\Psi = 0$.

First of all, note that by definition of $\mathcal{D}_W$ (134) and $\mathcal{D}_{\mathscr{F}}$ (135), any $\Psi \in \mathcal{D}_{\mathscr{F}}$ can be written as a finite sum $\Psi = \sum_{k=1}^K \Psi_k$ with

$$\Psi_k = W_1(\varphi_k)A^\dagger_{j_k}(v_k)\Psi_{m,k} \quad \text{or} \quad \Psi_k = X_k W_1(\varphi_k)\Psi_{m,k}, \tag{139}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

where $0 \neq \Psi_{m,k} = \Psi_{mx,k} \otimes \Omega_y$. Since $\Psi_k \in L^2$, we have $\varphi_k, v_k \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and without loss of generality we may assume that $v_k \neq 0$. And since X_k just multiplies by a function depending on the fermion momenta, X_k commutes with $W_1(\varphi_k)$ so it can be absorbed into $\Psi_{m,k}$. That is, we may re-define $X_k\Psi_{m,k}$ to be the new $\Psi_{m,k}$ and obtain that, without loss of generality, we could have chosen

$$\Psi_k = W_1(\varphi_k)A^\dagger_{j_k}(v_k)\Psi_{m,k} \quad \text{or} \quad \Psi_k = W_1(\varphi_k)\Psi_{m,k}, \tag{140}$$

with $\Psi_{m,k} \in L^2$. So we may define a disjoint union $\{1,\dots,K\} = \mathcal{K}_{WA} \cup \mathcal{K}_W$, such that

$$\Psi = \sum_{k\in\mathcal{K}_{WA}} W_1(\varphi_k)A^\dagger_{j_k}(v_k)\Psi_{m,k} + \sum_{k\in\mathcal{K}_W} W_1(\varphi_k)\Psi_{m,k}. \tag{141}$$

Now assume there was some $\Psi \neq 0$ with $W(s)\Psi = 0$. We define a "compression operator" B which "compresses" $W(s)\Psi$ into L^2. For this purpose, let m_s and β_s be the UV/IR-scaling degrees of s, respectively, and pick some real numbers $m_b < -m_s - d/2$ and $\beta_b > -\beta_s - d/2$. Choose a function $b \in \dot{\mathcal{S}}^\infty_{1,>}$ (so $b : \mathbb{R}^d\backslash\{0\} \to \mathbb{C}$ is invertible) which has exact UV/IR-scaling degrees m_b and β_b. With that choice, the "compressed" product function $\boldsymbol{k} \mapsto s(\boldsymbol{k})b(\boldsymbol{k})$ is in $\mathfrak{h}\cap\dot{\mathcal{S}}^\infty_1$, as is $\boldsymbol{k} \mapsto \varphi_k(\boldsymbol{k})b(\boldsymbol{k})$. Now, we define the compression operator $B : \overline{\mathscr{F}}_{\rm ex} \to \overline{\mathscr{F}}_{\rm ex}$ as

$$(B\Psi)(\boldsymbol{P},\boldsymbol{K}) = \left(\prod_{\ell=1}^N b(\boldsymbol{k}_\ell)\right)\Psi(\boldsymbol{P},\boldsymbol{K}). \tag{142}$$

It is easy to see that B maps $\overline{\mathscr{F}} \to \overline{\mathscr{F}}$, $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ and $B^{-1} : \overline{\mathscr{F}}_{\rm ex} \to \overline{\mathscr{F}}_{\rm ex}$ exists with

$$(B^{-1}\Psi)(\boldsymbol{P},\boldsymbol{K}) = \left(\prod_{\ell=1}^N \frac{1}{b(\boldsymbol{k}_\ell)}\right)\Psi(\boldsymbol{P},\boldsymbol{K}). \tag{143}$$

So $W(s)\Psi = 0$, if and only if $BW(s)\Psi = 0$.

Further, for $\Psi_k = W_1(\varphi_k)\Psi_{m,k}$, a momentum space calculation renders the following identity:

$$\begin{aligned}
&(BW(s)W_1(\varphi_k)\Psi_{m,k})(\boldsymbol{P},\boldsymbol{K})\\
&\quad\overset{(117)}{=} (Be^{{\rm Im}\langle s,\varphi_k\rangle}W_M(s)\dots W_2(s)W_1(s+\varphi_k)\Psi_{m,k})(\boldsymbol{P},\boldsymbol{K})\\
&\quad\overset{(119)}{=} \frac{1}{\sqrt{N!}}e^{{\rm Im}\langle s,\varphi_k\rangle - \frac{(M-1)\|s\|^2+\|s+\varphi_k\|^2}{2}}\sum_\sigma\left(\prod_{\ell=1}^N b(\boldsymbol{k}_\ell)s_{k,\sigma(\ell)}(\boldsymbol{k}_\ell)\right)\Psi_{mx,k}(\boldsymbol{P}')\\
&\quad= \big(e^{{\rm Im}\langle s,\varphi_k\rangle - \frac{(M-1)\|s\|^2+\|s+\varphi_k\|^2}{2}}e^{A^\dagger(bs)+A^\dagger_1(b\varphi_k)}\Psi_{m,k}\big)(\boldsymbol{P},\boldsymbol{K}),
\end{aligned} \tag{144}$$

with $s_{k,1} := s + \varphi_k$, as well as $s_{k,2} = \dots = s_{k,M} := s$. As above, the sum is running over all maps $\sigma : \{1,\dots,N\} \to \{1,\dots,M\}$ and $\boldsymbol{P}' = \boldsymbol{P} + \sum_{\ell=1}^N e_{\sigma(\ell)}\boldsymbol{k}_\ell$. On the

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

other hand, for unitary Fock space operators $W_{\mathscr{F}}, W_{\mathscr{F},j}$, the Weyl relations yield

$$\begin{aligned} &W_{\mathscr{F}}(bs)W_{\mathscr{F},1}(b\varphi_k)\Psi_{m,k} \\ &\quad = e^{\mathrm{Im}\langle bs, b\varphi_k\rangle} W_{\mathscr{F},M}(bs)\dots W_{\mathscr{F},2}(bs)W_{\mathscr{F},1}(b(s+\varphi_k))\Psi_{m,k} \\ &\quad \overset{(118)}{=} e^{\mathrm{Im}\langle bs, b\varphi_k\rangle - \frac{(M-1)\|bs\|^2 + \|b(s+\varphi_k)\|^2}{2}} e^{-\sum_{j>j'} V_{jj'}(s_{k,j}^* b^* b s_{k,j'})} \\ &\qquad \times e^{A^\dagger(bs) + A_1^\dagger(b\varphi_k)}\Psi_{m,k}. \end{aligned} \tag{145}$$

Since $bs_{k,j} \in L^2$, the operators $V_{jj'}(s_{k,j}^* b^* b s_{k,j'})$ amount to a convolution with an L^1-function, which, after a Fourier transformation, is equivalent to a multiplication by a bounded function. So the $V_{jj'}$-operators are all bounded and likewise, the exponential $e^{-\sum_{j>j'} V_{jj'}(s_{k,j}^* b^* b s_{k,j'})}$ is bounded. Further, this bounded exponential commutes with $e^{A^\dagger(bs)+A_1^\dagger(b\varphi_k)}$ by a similar fiber decomposition argument as in the proof of Lemma 5.5. So comparing (144) with (145), we obtain

$$BW(s)W_1(\varphi_k)\Psi_{m,k} = \mathfrak{c}_k \cdot W_{\mathscr{F}}(bs)W_{\mathscr{F},1}(b\varphi_k) e^{-\sum_{j>j'} V_{jj'}(s_{k,j}^* b^* b s_{k,j'})}\Psi_{m,k}, \tag{146}$$

for some $\mathfrak{c}_k \in \mathrm{eRen}, \mathfrak{c}_k \neq 0$. An analogous momentum space calculation yields

$$\begin{aligned} &BW(s)W_1(\varphi_k)A_{j_k}^\dagger(v_k)\Psi_{m,k} \\ &\quad = \mathfrak{c}_k \cdot W_{\mathscr{F}}(bs)W_{\mathscr{F},1}(b\varphi_k)A_{j_k}^\dagger(bv_k) e^{-\sum_{j>j'} V_{jj'}(s_{k,j}^* b^* b s_{k,j'})}\Psi_{m,k}. \end{aligned} \tag{147}$$

Thus, we have the following chain of implications:

$$\begin{aligned} 0 &= W(s)\Psi \\ \Leftrightarrow\ 0 &= BW(s)\Psi \\ \overset{(141)}{\Leftrightarrow}\ 0 &= \sum_{k\in\mathcal{K}_{WA}} BW(s)W_1(\varphi_k)A_{j_k}^\dagger(v_k)\Psi_{m,k} + \sum_{k\in\mathcal{K}_W} BW(s)W_1(\varphi_k)\Psi_{m,k} \\ \Rightarrow\ 0 &= \sum_{k\in\mathcal{K}_{WA}} \mathfrak{c}_k W_{\mathscr{F}}(bs)W_{\mathscr{F},1}(b\varphi_k)A_{j_k}^\dagger(bv_k) e^{-\sum_{j>j'} V_{jj'}(s_{k,j}^* b^* b s_{k,j'})}\Psi_{m,k} \\ &\quad + \sum_{k\in\mathcal{K}_W} \mathfrak{c}_k W_{\mathscr{F}}(bs)W_{\mathscr{F},1}(b\varphi_k) e^{-\sum_{j>j'} V_{jj'}(s_{k,j}^* b^* b s_{k,j'})}\Psi_{m,k} \\ \Leftrightarrow\ 0 &= \sum_{k\in\mathcal{K}_{WA}} \mathfrak{c}_k W_{\mathscr{F},1}(b\varphi_k)A_{j_k}^\dagger(bv_k) e^{-\sum_{j>j'} V_{jj'}(s_{k,j}^* b^* b s_{k,j'})}\Psi_{m,k} \\ &\quad + \sum_{k\in\mathcal{K}_W} \mathfrak{c}_k W_{\mathscr{F},1}(b\varphi_k) e^{-\sum_{j>j'} V_{jj'}(s_{k,j}^* b^* b s_{k,j'})}\Psi_{m,k}, \end{aligned} \tag{148}$$

where we exploited the unitarity of $W_{\mathscr{F}}(bs)$ in the last step.

We may now partition the indices k into several equivalence classes according to the relation

$$k \sim k' :\Leftrightarrow \mathfrak{c}_k = c\mathfrak{c}_{k'} \quad \text{for some } c \in \mathbb{C}. \tag{149}$$

The last equation of (148) now implies that for each equivalence class $[k']$, the following sum with $c_k := \frac{\mathfrak{c}_k}{\mathfrak{c}'_k}$ must vanish:

$$0 = \sum_{k\in[k']\cap\mathcal{K}_{WA}} c_k W_{\mathscr{F},1}(b\varphi_k) A^\dagger_{j_k}(bv_k) e^{-\sum_{j>j'} V_{jj'}(s^*_{k,j}b^*bs_{k,j'})}\Psi_{m,k} + \sum_{k\in[k']\cap\mathcal{K}_W} c_k W_{\mathscr{F},1}(b\varphi_k) e^{-\sum_{j>j'} V_{jj'}(s^*_{k,j}b^*bs_{k,j'})}\Psi_{m,k}. \tag{150}$$

This is exactly a linear combinations of vectors of the form (137) in Lemma 5.8 with $b\varphi_k, bv_k \in \mathfrak{h}\cap\dot{\mathcal{S}}^\infty_1$ where all $b\varphi_k$ are distinct, and with $\Psi'_{m,k} := e^{-\sum_{j>j'} V_{jj'}(s^*_{k,j}b^*bs_{k,j'})}\Psi_{m,k}$. Since $\mathfrak{c}_k \neq 0$, we also have $c_k \neq 0$. Only the premise $\Psi'_{m,k} \neq 0$ is missing in order for Lemma 5.8 to apply. However, if this premise was true, then Lemma 5.8 would imply that $c_k = 0$ for all k, which we ruled out above. Thus, we conclude that $\Psi'_{m,k} = 0$ for at least one k, and by repeatedly excluding this k from the linear combination and applying the argument to exclude further k, we arrive at

$$\Psi'_{m,k} = e^{-\sum_{j>j'} V_{jj'}(s^*_{k,j}b^*bs_{k,j'})}\Psi_{m,k} = 0 \quad \forall\, k\in\{1,\dots,K\}. \tag{151}$$

By boundedness of the operator $V_{jj'}(s^*_{k,j}b^*bs_{k,j'})$, we can now conclude that the exponential is semibounded from below, that is

$$\big\|e^{-\sum_{j>j'} V_{jj'}(s^*_{k,j}b^*bs_{k,j'})}\Psi'\big\| \geq c'\|\Psi'\| \quad \forall\,\Psi'\in L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y \quad \text{for some } c'>0. \tag{152}$$

So $e^{-\sum_{j>j'} V_{jj'}(s^*_{k,j}b^*bs_{k,j'})}\Psi_{m,k} = 0$ implies $\Psi_{m,k} = 0$ for all k, which contradicts $\Psi = 0$ and concludes the proof. □

Remark

(5) In essence, we just transferred the commutation relations (110) for creation operators $A^\dagger_j$ from Lemma 5.2 in a certain sense to extended dressing operators $W_j(s)$. This was done by imposing definitions such that these commutation relations still hold true. What about the commutation relations (111) for annihilation operators A_j?

In fact, these relations cannot be imposed by definition, but one may show that they are an immediate consequence of Definition 5.2. This is proved in the following lemma.

Lemma 5.9. *Let $s, v \in \dot{\mathcal{S}}^\infty_1$ and $\Psi_m \in \mathcal{C}_{WS}$. Then, we have the commutation relations*

$$W_{j'}(s)A_j(v)\Psi_m = \begin{cases} \big(A_j(v) - \langle v, s\rangle\big)W_{j'}(s)\Psi_m & \text{if } j = j', \\ \big(A_j(v) - V_{jj'}(v^*s)\big)W_{j'}(s)\Psi_m & \text{if } j \neq j', \end{cases}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$
\begin{aligned}
W(s)A_j(v)\Psi_m &= \big(A_j(v) - \langle v, s\rangle - V_{j\bullet}(v^*s)\big)W(s)\Psi_m,\\
W_{j'}(s)A(v)\Psi_m &= \big(A(v) - \langle v, s\rangle - V_{\bullet j'}(v^*s)\big)W_{j'}(s)\Psi_m,\\
W(s)A(v)\Psi_m &= \big(A(v) - M\langle v, s\rangle - V(v^*s)\big)W(s)\Psi_m.
\end{aligned}
\tag{153}
$$

Proof. First, note that $A_j(v)\Psi_m = 0$. The first line in (153) then follows by momentum space definitions (22) and (123)

$$
\begin{aligned}
&\big(A_j(v)W_{j'}(s)\Psi_m\big)(\boldsymbol{P},\boldsymbol{K})\\
&\quad= \frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}}\int v(\tilde{\boldsymbol{k}})^* s(\tilde{\boldsymbol{k}})\left(\prod_{\ell=1}^{N} s(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P}' + (e_{j'} - e_j)\tilde{\boldsymbol{k}})\, d\tilde{\boldsymbol{k}}\\
&\quad= \begin{cases} \langle v, s\rangle\Psi_m(\boldsymbol{P},\boldsymbol{K}) & \text{if } j = j',\\ \big(V_{jj'}(v^*s)\Psi_m\big)(\boldsymbol{P},\boldsymbol{K}) & \text{if } j \neq j', \end{cases}
\end{aligned}
\tag{154}
$$

with $\boldsymbol{P}' = \boldsymbol{P} + e_{j'}\sum_{\ell=1}^{N}\boldsymbol{k}_\ell$.

The second line in (153) is established similarly. We use again (22) and (123), yielding

$$
\begin{aligned}
\big(A_j(v)W(s)\Psi_m\big)(\boldsymbol{P},\boldsymbol{K}) = \frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}}\sum_{\tilde\sigma}\int v(\tilde{\boldsymbol{k}})^* s(\tilde{\boldsymbol{k}})\left(\prod_{\ell=1}^{N} s(\boldsymbol{k}_\ell)\right)\\
\times\Psi_{mx}(\boldsymbol{P}' + (e_{\tilde\sigma(N+1)} - e_j)\tilde{\boldsymbol{k}})d\tilde{\boldsymbol{k}},
\end{aligned}
\tag{155}
$$

where the sum runs over all $\tilde\sigma : \{1,\dots,N+1\} \to \{1,\dots,M\}$ and we have set $\boldsymbol{P}' = \boldsymbol{P} + \sum_{\ell=1}^{N} e_{\tilde\sigma(\ell)}\boldsymbol{k}_\ell$, as well as $\boldsymbol{k}_{N+1} = \tilde{\boldsymbol{k}}$. We can split this sum into a sum over (σ, j) with $\sigma : \{1,\dots,N\} \to \{1,\dots,M\}, \sigma(\ell) = \tilde\sigma(\ell)$ and $j' \in \{1,\dots,M\}, j' = \tilde\sigma(N+1)$

$$
\begin{aligned}
\big(A_j(v)W(s)\Psi_m\big)(\boldsymbol{P},\boldsymbol{K}) = \frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}}\sum_{j'}\sum_{\sigma}\int v(\tilde{\boldsymbol{k}})^* s(\tilde{\boldsymbol{k}})\left(\prod_{\ell=1}^{N} s(\boldsymbol{k}_\ell)\right)\\
\times\Psi_{mx}(\boldsymbol{P}' + (e_{j'} - e_j)\tilde{\boldsymbol{k}})d\tilde{\boldsymbol{k}}.
\end{aligned}
\tag{156}
$$

Now, the term with $j' = j$ renders the contribution $\langle s, v\rangle\Psi_m$ and all other $M-1$ terms add up to $\sum_{j':j\neq j'} V_{jj'}(v^*s)\Psi_m = V_{j\bullet}(v^*s)\Psi_m$, which is exactly the desired contribution.

Lines three and four of (153) just follow by summing over $j \in \{1,\dots,M\}$ in the first two lines. □

6. Pulling Back the Hamiltonian

This section concerns taking a formal Hamiltonian

$$
H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty,
$$

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

and pulling it back under the dressing transformation $W(s)$, i.e. we compute

$$\widetilde{H} : \mathscr{F}_{\rm ex} \supset \mathcal{D}_{WS} \to \overline{\mathscr{F}}_{\rm ex} \quad \text{with } W(s)\widetilde{H} = HW(s).$$

The computation is split into two steps. In Sec. 6.1, we compute the pullback of $(A(v) - E_\infty)$. Pulling back only $A(v)$ will result in divergences which are canceled by E_∞.

The pullback of $(H_{0,y} + A^\dagger(v))$ is then computed in Sec. 6.2. Combining $H_{0,y}$ and $A^\dagger(v)$ yields a particularly easy result.

Our main theorem is the following.

Theorem 6.1. *Let $s = -\frac{v}{\omega}$ with $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$. Then, the pullback of the self-energy renormalized Hamiltonian*

$$\widetilde{H} := H_{0,y} + V(v^* s) \quad \textit{satisfies } W(s)\widetilde{H} = \big(H_{0,y} + A^\dagger(v) + A(v) - E_\infty\big)W(s), \tag{157}$$

which holds as a strong operator identity on $\mathcal{D}_{WS}$ (defined in (108)*), as well as on $(S_- \otimes I)[\mathcal{D}_{WS}]$.*

Note that, the potential interaction V defined in (112) via (91) acts as

$$(V\Psi)(\boldsymbol{P}, \boldsymbol{K}) = \sum_{j \neq j'} \int v(\tilde{\boldsymbol{k}})^* s(\tilde{\boldsymbol{k}}) \Psi(\boldsymbol{P} + (e_j - e_{j'})\tilde{\boldsymbol{k}}, \boldsymbol{K}) d\tilde{\boldsymbol{k}}.$$

6.1. *Pulling back $A - E_\infty$*

We recall that by Proposition 4.1, one can define $E_\infty : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\rm ex}$ with

$$(E_\infty \Psi)(\boldsymbol{P}, \boldsymbol{K}) = M\langle v, s\rangle \ \Psi(\boldsymbol{P}, \boldsymbol{K}) = \sum_{j=1}^{M} \int -\frac{v(\boldsymbol{k})^* v(\boldsymbol{k})}{\omega(\boldsymbol{k})} \, d\boldsymbol{k} \ \Psi(\boldsymbol{P}, \boldsymbol{K}), \tag{158}$$

even if $\langle v, s\rangle \notin \mathbb{C}$, but $\langle v, s\rangle \in \mathrm{Ren}_1$.

Lemma 6.1. *Let $\Psi_m \in \mathcal{C}_{WS}$ and $s = -\frac{v}{\omega}$ with $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$. Then, for $\varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$*

$$\big(A(v) - E_\infty\big)W(s)W_{\mathscr{F},1}(\varphi)\Psi_m = W(s)\big(\mathrm{res}_1(\varphi) + V(v^* s)\big)W_{\mathscr{F},1}(\varphi)\Psi_m \in \overline{\mathscr{F}}_{\rm ex}, \tag{159}$$

where the residual operator

$$\mathrm{res}_1(\varphi) = \underbrace{\langle v, \varphi\rangle}_{\in \mathrm{Ren}_1} + V_{\bullet 1}(v^* \varphi), \tag{160}$$

is by Lemma 4.2 *a well-defined mapping $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\rm ex}$.*

The proof of Lemma 6.1 is given in Appendix D.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

6.2. *Pulling back $H_{0,y} + A^\dagger$*

Lemma 6.2. *Let $\Psi_m \in \mathcal{C}_{WS}$ and $s = -\frac{v}{\omega}$ with $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$. Then, for $\varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$*

$$\big(H_{0,y} + A^\dagger(v)\big)W(s)W_{\mathscr{F},1}(\varphi)\Psi_m = W(s)\big(H_{0,y} - \mathrm{res}_1(\varphi)\big)W_{\mathscr{F},1}(\varphi)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}, \tag{161}$$

with the same residual operator $\mathrm{res}_1 = \langle v, \varphi\rangle + V_{\bullet 1}(v^*\varphi)$ as in Lemma 6.1.

As for Lemma 6.1, the proof of Lemma 6.2 is rather technical. It can be found in Appendix E. With both lemmas at hand, Theorem 6.1 can directly be proved.

Proof of Theorem 6.1. This is a simple consequence of Lemmas 6.1 and 6.2. We put together (159) and (161) which yields

$$\begin{aligned} W(s)\widetilde{H} &= \big(H_{0,y} + A^\dagger(v) + A(v) - E_\infty\big)W(s) \\ &= W(s)\big(H_{0,y} + V(v^*s) + \mathrm{res}_1(\varphi) - \mathrm{res}_1(\varphi)\big) \\ &= W(s)\big(H_{0,y} + V(v^*s)\big), \end{aligned} \tag{162}$$

as a strong operator identity on all $\Psi = W_{\mathscr{F},1}(\varphi)\Psi_m, \varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$. And these Ψ span $\mathcal{D}_{WS}$.

Since we imposed in Definitions 5.2–5.4 that symmetrization $(S_- \otimes I)$ shall commute with $W(s)$, the strong operator identity is also valid on $(S_- \otimes I)[\mathcal{D}_{WS}]$. □

7. Self-Adjointness

In this section, we prove that in certain cases, $\widetilde{H}$ can indeed be defined as a self-adjoint operator $\widetilde{H} : \mathscr{F} \supset \mathrm{dom}(\widetilde{H}) \to \mathcal{D}_{\mathscr{F}}$. So far we have by Theorem 6.1 that $\widetilde{H} : (S_- \otimes I)[\mathcal{D}_{WS}] \to \overline{\mathscr{F}}_{\mathrm{ex}}$ is well defined.

In order for the image of $\widetilde{H}$ to be in $\mathcal{D}_{\mathscr{F}}$, we need to restrict the domain of $\widetilde{H}$ even further to some subspace $\widetilde{\mathcal{D}}_{\mathscr{F}} \subset \mathcal{D}_{\mathscr{F}}$, defined in (165), and prove well-definedness of $\widetilde{H} : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$ (Lemma 7.1). The existence of a self-adjoint extension on some $\mathrm{dom}(\widetilde{H}) \supset \widetilde{\mathcal{D}}_{\mathscr{F}}$ is then a simple consequence (Corollary 7.1).

7.1. *Existence of self-adjoint extensions*

First, we verify that $\widetilde{H} = H_{0,y} + V$ is well defined and symmetric on a dense domain in $\mathscr{F}$.

Definition 7.1. Let $\mathcal{Q}_{\mathrm{col},x}$ be the set of *collision configurations*, i.e. all fermion position space configurations

$$\mathcal{Q}_{\mathrm{col},x} := \big\{\boldsymbol{X} \in \mathcal{Q}_x \,|\, \exists\, j \neq j' : \boldsymbol{x}_j = \boldsymbol{x}_{j'}\big\}. \tag{163}$$

Denote by

$$\widetilde{\mathcal{C}}_{WS} := \big\{\Psi_m = \Psi_{mx} \otimes \Omega_y \,|\, \mathcal{F}^{-1}(\Psi_{mx}) \in C_c^\infty(\mathcal{Q}_x \backslash \mathcal{Q}_{\mathrm{col},x})\big\}, \tag{164}$$

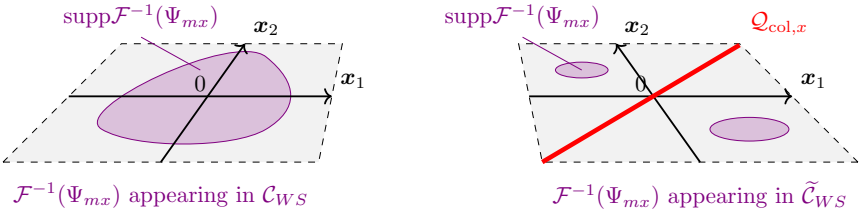

Fig. 5. Within $\mathcal{C}_{WS}$, the fermionic wave functions must be in $\mathcal{S}$, allowing for any support (including compact ones). Within $\widetilde{\mathcal{C}}_{WS}$, only C_c^∞-functions are allowed with support avoiding the collision configurations $\mathcal{Q}_{\mathrm{col},x}$.

the "cyclic set" of functions whose support avoids the collision configurations (where $\mathcal{F}^{-1}$ is the inverse Fourier transform). We define the *small domain*, on which $\widetilde{H}$ is initially defined as a Fock space operator as

$$\widetilde{\mathcal{D}}_{\mathscr{F}} := (S_- \otimes I)\big[\mathrm{span}\big\{W_{\mathscr{F},1}(\varphi)\Psi_m \,|\, \varphi \in C_c^\infty(\mathbb{R}^d),\ \Psi_m \in \widetilde{\mathcal{C}}_{WS}\big\}\big], \tag{165}$$

see Fig. 5. It is easy to see that $\widetilde{\mathcal{D}}_{\mathscr{F}} \subseteq \mathcal{D}_{\mathscr{F}}$ and $\widetilde{\mathcal{D}}_{\mathscr{F}} \subseteq (S_- \otimes I)[\mathcal{D}_{WS}]$.

In the following lemma, we will use that if $\omega, v \in \dot{\mathcal{S}}_1^\infty$ satisfy (32) and (33) (so they scale polynomially), then the potential function

$$\hat{V} := v^* s = -\tfrac{v^* v}{\omega}, \tag{166}$$

also scales polynomially with

$$m_V = 2m_v - m_\omega, \quad \beta_V = 2\beta_v - \beta_\omega. \tag{167}$$

Further, if $\beta_V > -d$, then the inverse Fourier transform $V = \mathcal{F}^{-1}(\hat{V}) \in S'(\mathbb{R}^d)$ also exists, so we can make statements about the singular support of V.

Lemma 7.1 ($\widetilde{H}$ is Densely Defined and Symmetric). *The set $\widetilde{\mathcal{D}}_{\mathscr{F}}$ is dense in $\mathscr{F}$.*

Assume, that $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$ with ω, v satisfying (32) *and* (33)*, as well as $\beta_V > -d$. If now*

$$\mathrm{singsupp}(V) \subseteq \{0\}, \tag{168}$$

then $\widetilde{H}$ maps $\widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$ and is thus densely defined. If in addition the symmetry condition (34) *holds, then $\widetilde{H}$ is symmetric.*

Proof. Density of $\widetilde{\mathcal{D}}_{\mathscr{F}}$ in $\mathscr{F}$ is established as density of $(S_- \otimes I)\mathcal{D}_{WS}$ in Lemma 5.1 (proof in Appendix A). We recall that by the last line of (A.4)

$$D_1 W_{\mathscr{F},1}(\varphi)\Psi_m = \Psi_{mx} \otimes W_y(\varphi)\Omega_y.$$

In the proof of Lemma 5.1, we argue that $(S_- \otimes I)\mathcal{D}_{WS}$ is dense in $\mathscr{F}$, since $\Psi_{mx} \otimes W_y(\varphi)\Omega_y$ approximates any $\Psi \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ arbitrarily well. The transition from $(S_- \otimes I)\mathcal{D}_{WS}$ to $\widetilde{\mathcal{D}}_{\mathscr{F}}$ is achieved by a restriction to $\Psi_{mx} \in \mathcal{F}[C_c^\infty(\mathcal{Q}_x \backslash \mathcal{Q}_{\mathrm{col},x})]$.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

The set $\mathcal{Q}_{\mathrm{col},x}$ is a union of hyperplanes on each sector in the fermionic configuration space, so $C_c^\infty(\mathcal{Q}_x \backslash \mathcal{Q}_{\mathrm{col},x})$ is dense in $L^2(\mathcal{Q}_x)$. The Fourier transform $\mathcal{F}$ is an isometry, so the allowed set for Ψ_{mx} is dense in $L^2(\mathcal{Q}_x)$. Thus, we can still approximate any $\Psi \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ arbitrarily well by $\Psi_{mx} \otimes W_y(\varphi)\Omega_y$ and by the same arguments as in the proof of Lemma 5.1, $\widetilde{\mathcal{D}}_{\mathscr{F}}$ is dense in $\mathscr{F}$.

Now we verify that $H_{0,y}$ maps $\widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$. By linearity, it suffices to show well-definedness on all vectors of the form $W_{\mathscr{F},1}(\varphi)\Psi_m, \varphi \in C_c^\infty(\mathbb{R}^d)$. Denote by $P_y^{(N)}$ the projection of $\mathscr{F}_y$ to the N-boson sector $\mathscr{F}_y^{(N)}$, so $\sum_{N\in\mathbb{N}_0} P_y^{(N)} = 1$. The L^2-norm squares of $W_{\mathscr{F},1}(\varphi)\Psi_m$ are Poisson-distributed over N, i.e.

$$\left\|P_y^{(N)} W_{\mathscr{F},1}(\varphi)\Psi_m\right\|^2 = e^{-\|\varphi\|^2} \frac{\|\varphi\|^{2N}}{N!}, \tag{169}$$

so for any $0 < q < 1$ they decay faster than q^N in N-direction. Now, define

$$\lambda := \max_{\boldsymbol{k}\in\mathrm{supp}(\varphi)} |\omega(\boldsymbol{k})|. \tag{170}$$

Then

$$\left\|P_y^{(N)} H_{0,y} W_{\mathscr{F},1}(\varphi)\Psi_m\right\|^2 \leq N^2\lambda^2 \left\|P_y^{(N)} W_{\mathscr{F},1}(\varphi)\Psi_m\right\|^2, \tag{171}$$

which still decays faster than q^N in N-direction. Thus, $\left\|H_{0,y} W_{\mathscr{F},1}(\varphi)\Psi_m\right\| < \infty$ and we have that $H_{0,y} W_{\mathscr{F},1}(\varphi)\Psi_m \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$.

It remains to be shown that $V = V(v^*s)$ is well defined, which amounts to proving that

$$(V\Psi)(\boldsymbol{P},\boldsymbol{K}) = \sum_{j\neq j'} \int \hat{V}(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P} + (e_j - e_{j'})\tilde{\boldsymbol{k}}, \boldsymbol{K}) d\tilde{\boldsymbol{k}},$$

defines an L^2-function on $\mathcal{Q}$. Since $\beta_V > -d$, we have that $\hat{V} \in L^1_{\mathrm{loc}} \Rightarrow \hat{V} \in \mathcal{S}'$, so we can take the Fourier transform as in (93)

$$(V\Psi)(\boldsymbol{X},\boldsymbol{Y}) = \sum_{j\neq j'} V(\boldsymbol{x}_j - \boldsymbol{x}_{j'})\Psi(\boldsymbol{X},\boldsymbol{Y}),$$

with $V(\boldsymbol{x}) = \mathcal{F}^{-1}(\hat{V})(\boldsymbol{x})$.

Now, for $\Psi = W_{\mathscr{F},1}(\varphi)\Psi_m$ we obtain the position space representation by Fourier-transforming (107)

$$\Psi(\boldsymbol{X},\boldsymbol{Y}) = \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \left(\prod_{\ell=1}^N \check{\varphi}(\boldsymbol{y}_\ell - \boldsymbol{x}_1)\right) \Psi_{mx}(\boldsymbol{X}), \tag{172}$$

where $\check{\varphi} = \mathcal{F}^{-1}(\varphi)$ is a Schwartz function, as $\varphi \in C_c^\infty$ is Schwartz. So as $\Psi_{mx}(\boldsymbol{X})$ is a smooth function with compact support apart from collision configurations in $\mathcal{Q}_x$, also Ψ is smooth, and it is zero at fermion collision configurations in $\mathcal{Q}$. Since the singular support of $V(\boldsymbol{x})$ is at most $\{0\}$, the multiplication function $\sum_{j\neq j'} V(\boldsymbol{x}_j - \boldsymbol{x}_{j'})$ is smooth on $\mathrm{supp}(\Psi_{mx})$ (which excludes collision configurations). And as

$\mathrm{supp}(\Psi_{mx})$ is compact, there is some $C_\Psi \in \mathbb{R}$ with

$$\max_{\boldsymbol{X}\in \mathrm{supp}(\Psi_{mx})} \left| \sum_{j\neq j'} V(\boldsymbol{x}_j - \boldsymbol{x}_{j'}) \right| \leq C_\Psi. \tag{173}$$

Further, by compactness of support, a maximum occupied fermion sector $\overline{M}$ exists. So

$$\|V\Psi\|^2 \leq \overline{M}^4 C_\Psi^2 \|\Psi\|^2 < \infty,$$

for $\Psi \in \widetilde{\mathcal{D}}_{\mathscr{F}}$. Thus, $V\Psi \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$.

Symmetrization for fermions by $(S_- \otimes I)$ is preserved by $H_{0,y}$ and V. Hence, indeed $\widetilde{H}\Psi \in \mathscr{F}$. And by Theorem 6.1, we have that $\widetilde{H}\Psi \in (S_-\otimes I)[\mathcal{D}_W]$ (otherwise, we could not apply $W(s)$ to it).

Symmetry of $\widetilde{H}$ is an obvious consequence of the symmetry condition (34). And since $\widetilde{H}$ preserves symmetry, it maps $\widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}} = (S_- \otimes I)[\mathcal{D}_W \cap (L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y)]$ (compare (135)). □

Corollary 7.1 (Existence of a Self-Adjoint Extension). $\widetilde{H} : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$ *as in Lemma* 7.1 *allows for a self-adjoint extension.*

Proof. This is a direct consequence of [52, Theorem X.3] (von Neumann's theorem): For a symmetric operator $\widetilde{H}$ (called A within [52]), this theorem asserts that there is a self-adjoint extension, provided that a conjugation operator $C : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \widetilde{\mathcal{D}}_{\mathscr{F}}$ can be found, such that

$$C\widetilde{H} = \widetilde{H}C. \tag{174}$$

As a conjugation, we choose $(C\Psi)(\boldsymbol{K}) = \Psi(-\boldsymbol{K})^*$, which amounts to complex conjugation in particle-position representation. By symmetry (34) and since ω is real-valued, $\hat{V}(\boldsymbol{k}) = \hat{V}(-\boldsymbol{k})^*$, so $VC = CV$. And analogously, $CH_{0,y} = H_{0,y}C$. Thus, (174) holds, and we have at least one self-adjoint extension. □

Remark

(6) *Essential self-adjointness*: We cannot expect $\widetilde{H}$ to be generally self-adjoint on $\widetilde{\mathcal{D}}_{\mathscr{F}}$: The $N = 0$-sector of $\widetilde{\mathcal{D}}_{\mathscr{F}}$ (165) is

$$\bigoplus_{M=0}^{\infty} \widetilde{\mathcal{D}}_{\mathscr{F}} \cap \mathscr{F}^{(M,0)} = \mathcal{F}^{-1}[C_c^\infty(\mathcal{Q}_x \backslash \mathcal{Q}_{\mathrm{col},x})]. \tag{175}$$

The removed diagonal $\mathcal{Q}_{\mathrm{col},x}$ may now allow for putting various boundary conditions that result in different self-adjoint extensions. An easy example is the case $\theta(\boldsymbol{p}) = |\boldsymbol{p}|^2, \omega = 0, v = 0$ for $d = 3$ (so $V = 0$). In the position representation of the $N = 2$-sector,

$$\mathscr{F}[\widetilde{\mathcal{D}}_{\mathscr{F}} \cap \mathscr{F}^{(2,0)}] = C_c^\infty(\mathbb{R}^6 \backslash \{\boldsymbol{x}_1 = \boldsymbol{x}_2\}). \tag{176}$$

Now, $\widetilde{H} = H = H_0$ acts as $(-\Delta_{\boldsymbol{x}_1} - \Delta_{\boldsymbol{x}_2})$ on this sector, which, in center-of-mass coordinates $(\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2)$ reduces to $(-2\Delta_{\boldsymbol{x}})$ acting on $C_c^\infty(\mathbb{R}^3 \backslash \{0\})$.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

The latter Hamiltonian is well known to have many self-adjoint extensions [53, Chap. I.1] that correspond to either a free particle or a particle in a δ-potential of arbitrary strength. Including all sectors $N \in \mathbb{N}_0$, we even obtain more boundaries, which allow for further choices of boundary conditions.

However, in case $\omega \geq 0$ there is a distinguished self-adjoint extension of $\widetilde{H}$ as in Lemma 7.1: We may decompose $\mathscr{F}$ and $\widetilde{H}$ into fibers as

$$\mathscr{F} = \int_{\boldsymbol{X} \in \mathcal{Q}_x}^{\oplus} \mathscr{F}_y, \quad \widetilde{H} = \int_{\boldsymbol{X} \in \mathcal{Q}_x}^{\oplus} \widetilde{H}_{\boldsymbol{X}}, \tag{177}$$

where

$$\widetilde{H}_{\boldsymbol{X}} = d\Gamma(\omega) + \sum_{j \neq j'} V(\boldsymbol{x}_j - \boldsymbol{x}_{j'}) \quad \text{for } \boldsymbol{X} \in \mathcal{Q}_x \backslash \mathcal{Q}_{\mathrm{col},x}. \tag{178}$$

Since the diagonal $\mathcal{Q}_{\mathrm{col},x}$ is a null set, $\widetilde{H}_{\boldsymbol{X}}$ is defined for almost every $\boldsymbol{X} \in \mathcal{Q}_x$, so the fiber integral (177) makes sense. Since $\widetilde{H}_{\boldsymbol{X}} = d\Gamma(\omega)$ up to an $\boldsymbol{X}$-dependent constant, for $\omega \geq 0$, each $\widetilde{H}_{\boldsymbol{X}}$ is positive and can be realized as a self-adjoint Friedrichs extension on $\mathscr{F}_y$. The fiber integral (177) then readily defines a self-adjoint operator $\widetilde{H}$.

8. Further Applications of the ESS

Employing an ESS construction, the cancellation of infinities within IBC renormalization mentioned in the introduction can be presented in a particularly convenient way: Formally, one would re-arrange divergent terms such that they cancel and consider only the steps after the cancellation as mathematically rigorous. Using the ESS framework, we can turn the formal manipulations before and including the "cancellation of infinities" into a rigorous statement.

Effectively, the IBC construction uses a non-unitary dressing operator $W = W_{\mathrm{IBC}}^{-1} = (1 + H_0^{-1} A^\dagger)^{-1}$ defined on a subspace of $\mathscr{F}$, which replaces the $W(s)$ from the above sections. We prove that under rather general assumptions, W_{IBC} can be defined as a bijection $W_{\mathrm{IBC}} : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ (Proposition 8.1) and can thus be used to construct a renormalized Fock space.

Another example for (non-unitary) dressing transformations comes from constructive QFT. These transformations depend on the model and can be quite involved. We will use the ESS to define a prototypical dressing operator $T = e^{-\boldsymbol{\Lambda}/2} e^{-H_0^{-1} A^\dagger} : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$, resembling typical CQFT dressing operators [45, Chap. 3; 17, §3]. The question, how to define concrete CQFT dressing operators and if they allow for a cutoff-free renormalization, is an interesting topic for future research.

8.1. *Rigorous cancellation of infinities within IBC renormalization*

Let us briefly sketch the IBC renormalization technique used in [35–39] and show how the ESS framework serves for making cancellations of infinities both explicit

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

and rigorous. We assume that $\theta, \omega, v \in \dot{\mathcal{S}}_1^\infty$ with $\theta(\boldsymbol{p}), \omega(\boldsymbol{k}) > 0$. For any self-energy operator $E : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ we can define $H_{\mathrm{IBC}} = H_0 + A^\dagger + A - E$ as $H_{\mathrm{IBC}} : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ by Proposition 4.1. The following (usually formal) manipulation now becomes a rigorous statement about equality of operators $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$:

$$\begin{aligned} H_{\mathrm{IBC}} &= H_0 + A^\dagger + A - E \\ &= H_0 + A^\dagger + A + AH_0^{-1}A^\dagger - AH_0^{-1}A^\dagger - E \\ &= \underbrace{(1 + AH_0^{-1})H_0^{1/2}}_{=:S^*} \underbrace{H_0^{1/2}(1 + H_0^{-1}A^\dagger)}_{=:S} \underbrace{-AH_0^{-1}A^\dagger - E}_{=:T} \\ &= S^*S + T. \end{aligned} \tag{179}$$

That is because H_0 is just a multiplication on momentum configuration space by a strictly positive function, so by the same argument as in the proof of Proposition 4.1, $H_0^{-1}, H_0^{1/2} : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ are well defined.

In (179), S^* is just a formal expression and not the adjoint of S. However, in sufficiently regular cases, S^*S can be densely defined on $\mathscr{F}$ as a closed operator, with S^* being the adjoint of S, so it is self-adjoint by [52, Theorem X.25]. Now, if $T : \mathscr{F} \supset \mathcal{D}(T) \to \mathscr{F}$ is a densely defined Kato perturbation of S^*S, that is

$$\|T\Psi\| \leq a\|S^*S\Psi\| + b\|\Psi\| \quad \forall\, \Psi \in \mathrm{dom}(S^*S), \tag{180}$$

with $a < 1$, then H_{IBC} is self-adjoint on $\mathrm{dom}(S^*S)$ by the Kato–Rellich theorem [52, X.12]. In [35–39], this argument was successfully employed in order to establish self-adjointness of IBC Hamiltonians $S^*S + T$ on the domain

$$\mathrm{dom}(S^*S) = \big\{\Psi \in \mathscr{F} \,|\, (1 + H_0^{-1}A^\dagger)\Psi \in \mathrm{dom}(H_0)\big\}. \tag{181}$$

The condition $(1 + H_0^{-1}A^\dagger)\Psi \in \mathrm{dom}(H_0)$ is called *IBC* or *abstract boundary condition*.

We remark that earlier works on IBC [32–34] consider models where the integral $\int \frac{v^2}{\omega}$ and thus E is finite. In essence, these works consider the same formal Hamiltonian as we do in (1) and explicitly diagonalize it by a Weyl transformation as in Theorem 6.1, but for much less singular v and ω.

In the most singular case, $\int \frac{v^2}{\omega^2} = \infty \Leftrightarrow \frac{v}{\omega} \notin L^2$, for any $\Psi \in \mathrm{dom}(S^*S)$, $\Psi^{(0)} \neq 0$, we formally have

$$((1 + H_0^{-1}A^\dagger)\Psi)^{(1)}(\boldsymbol{k}) = \Psi^{(1)}(\boldsymbol{k}) + \frac{v(\boldsymbol{k})}{\omega(\boldsymbol{k})}\Psi^{(0)}, \tag{182}$$

which is required to define a function in $\mathrm{dom}(\omega) \subset L^2$. However, for $\frac{v}{\omega} \notin L^2$, this would imply $\Psi^{(1)} \notin L^2$, so Ψ cannot be an element of Fock space. Similar problems appear if $\Psi^{(0)} = 0$ but $\Psi \neq 0$, when replacing $\Psi^{(0)}$ with the lowest occupied sector of Ψ. So for $\frac{v}{\omega} \notin L^2$, the only Fock space vector in $\mathrm{dom}(S^*S)$ is 0, and we expect IBC renormalization as above to fail. Indeed, there even exists a no-go theorem [40], excluding any interactions mediated by boundary conditions at a point source for ω corresponding to a Dirac particle.

Getting beyond the threshold where $\mathrm{dom}(S^*S)$ becomes $\{0\}$ as vectors "move out of Fock space" was a central motivation for introducing the ESS framework. Indeed, with Proposition 8.1, $W_{\mathrm{IBC}} = (1 + H_0^{-1}A^\dagger)$ can be defined as a bijective operator on the ESS $\overline{\mathscr{F}}$, so we are able to define a generalized IBC domain

$$\big\{\Psi \in \overline{\mathscr{F}} \,|\, (1 + H_0^{-1}A^\dagger)\Psi \in \mathrm{dom}(H_0)\big\} = W_{\mathrm{IBC}}^{-1}[\mathrm{dom}(H_0) \cap \dot{\mathcal{S}}^\infty_{\mathscr{F}}], \tag{183}$$

which is identified by W_{IBC} with the dense subspace $\mathrm{dom}(H_0) \cap \dot{\mathcal{S}}^\infty_{\mathscr{F}} \subset \mathscr{F}$.

There are also recent works on IBC, which use strategies different from the one above to obtain quantum dynamics. In [41], a self-adjoint Hamiltonian is reconstructed from resolvents. By contrast, in [42] quantum dynamics are formulated in terms of a multi-time integral equation with boundary conditions and solutions are constructed by a fixed point argument.

Further, there is a recent series of works about direct renormalization on spin-boson type models [54–56], which recover the IBC domain (181), but use an explicit computation of deficiency spaces to prove self-adjointness. These works cover cases where $v \notin L^2$ and even $\int \frac{v^2}{\omega} = \infty$, but still $\int \frac{v^2}{\omega^2} < \infty$. We hope that the ESS framework allows progressing also to $\int \frac{v^2}{\omega^2} = \infty$ in models of this kind.

8.2. *Renormalized Fock spaces for IBC*

In order for (183) to be a valid characterization of the generalized IBC domain, we need W_{IBC}^{-1} to be well defined, which is true under fairly mild assumptions.

Proposition 8.1. *Let $\theta, \omega, v \in \dot{\mathcal{S}}^\infty_1$ with $\theta > 0$, $\omega(\boldsymbol{k}) > 0$ for $\boldsymbol{k} \neq 0$, and $\omega(\boldsymbol{k}) \geq c|\boldsymbol{k}|^\beta$ in some neighborhood of $\boldsymbol{k} = 0$ for some $c, \beta > 0$. Define H_0 and $A^\dagger = A^\dagger(v)$ as in Proposition* 4.1 *(via* (17) *and* (20)).

Then, the operator $W_{\mathrm{IBC}} = (1 + H_0^{-1}A^\dagger) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ is bijective.

We remark that the technical condition $\omega(\boldsymbol{k}) \geq c|\boldsymbol{k}|^\beta$ near $\boldsymbol{k} = 0$ is only necessary to make H_0^{-1} well defined. This would also be true for the stronger condition $\omega > 0$, which, however, excludes the physically interesting cases $\omega(\boldsymbol{k}) = |\boldsymbol{k}|$ and $\omega(\boldsymbol{k}) = |\boldsymbol{k}|^2$.

Proof. We show that $W_{\mathrm{IBC}} : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ is bijective by constructing $W_{\mathrm{IBC}}^{-1} : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$. Bijectivity of W_{IBC} on $\overline{\mathscr{F}}$ then readily follows by linearity with respect to the field eRen.

First, note that $H_0^{-1} : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ is well defined. That is because H_0 is a multiplication on configuration space by the function

$$H_0(\boldsymbol{P}, \boldsymbol{K}) = \sum_{j=1}^{M} \theta(\boldsymbol{p}_j) + \sum_{\ell=1}^{N} \omega(\boldsymbol{k}_\ell), \tag{184}$$

which is nonzero on all configurations $(\boldsymbol{P}, \boldsymbol{K})$ that do not contain a zero boson momentum, i.e. everywhere apart from $\exists\,(\boldsymbol{k} = 0)$ (defined in (38)). So $H_0(\boldsymbol{P}, \boldsymbol{K})^{-1}$ is also defined apart from $\exists\,(\boldsymbol{k} = 0)$. Now, in the vicinity of $\exists\,(\boldsymbol{k} = 0)$, $H_0(\boldsymbol{P}, \boldsymbol{K})$ is

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

bounded from below by $c\sum_{\ell=1}^{N}|\boldsymbol{k}_\ell|^\beta$, so $H_0(\boldsymbol{P},\boldsymbol{K})^{-1}$ blows up at most polynomially in each $|\boldsymbol{k}_\ell|$. Thus, the multiplication by $H_0(\boldsymbol{P},\boldsymbol{K})^{-1}$ is a map $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ that defines the operator H_0^{-1}.

On $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$, we now define W^{-1}_{IBC} by the Neumann series

$$W^{-1}_{\mathrm{IBC}} = (1 + H_0^{-1}A^\dagger)^{-1} := \sum_{k=0}^{\infty}(-H_0^{-1}A^\dagger)^k. \tag{185}$$

Claim. (185) *defines an operator* $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$.

Proof of the Claim Each $-H_0^{-1}A^\dagger$ increases the boson number by 1. So $(-H_0^{-1}A^\dagger)^k\Psi$ is only supported on configuration space sectors with $N \geq k$. Hence, on each $(\boldsymbol{P},\boldsymbol{K}) \in \dot{\mathcal{Q}}$ with $\boldsymbol{K} \in \mathbb{R}^{Nd}$, we have that

$$\begin{aligned} &\big((-H_0^{-1}A^\dagger)^k\Psi\big)(\boldsymbol{P},\boldsymbol{K}) = 0 \quad \text{for } k > N \\ \Rightarrow &\left(\sum_{k=0}^{\infty}(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P},\boldsymbol{K}) = \left(\sum_{k=0}^{N}(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P},\boldsymbol{K}). \end{aligned} \tag{186}$$

Since $H_0^{-1}A^\dagger$ maps $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$ to itself, also all sums $\sum_{k=0}^{n}(-H_0^{-1}A^\dagger)^k$ with $n \in \mathbb{N}$ map $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$. So the Neumann series is defined on each N-boson sector and hence on all of $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$. ◊

Claim. *The Neumann series* (185) *is the inverse of* $(1 + H_0^{-1}A^\dagger)$.

Proof of the Claim We use the first line of (186) and perform a sector-wise verification

$$\begin{aligned} &(1 + H_0^{-1}A^\dagger)\left(\sum_{k=0}^{\infty}(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P},\boldsymbol{K}) \\ &\quad= \left(\sum_{k=0}^{\infty}(-H_0^{-1}A^\dagger)^k\Psi - \sum_{k=1}^{\infty}(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P},\boldsymbol{K}) \\ &\quad\overset{(186)}{=} \left(\sum_{k=0}^{N}(-H_0^{-1}A^\dagger)^k\Psi - \sum_{k=1}^{N}(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P},\boldsymbol{K}) \\ &\quad= \Psi(\boldsymbol{P},\boldsymbol{K}). \end{aligned} \tag{187}$$

So indeed $(1 + H_0^{-1}A^\dagger)(\sum_{k=0}^{\infty}(-H_0^{-1}A^\dagger)^k) = 1$. ◊

So $W^{-1}_{\mathrm{IBC}} : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ defined in (185) is the (well defined) inverse of W_{IBC} on $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$, which establishes the proposition. □

With Proposition 8.1, we have a well-defined linear space $W^{-1}_{\mathrm{IBC}}[\dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap \mathscr{F}]$ that can be equipped with a scalar product

$$\langle W^{-1}_{\mathrm{IBC}}\Psi, W^{-1}_{\mathrm{IBC}}\Phi\rangle_{\mathrm{renI}} := \langle\Psi,\Phi\rangle \quad \text{for } \Psi,\ \Phi \in \dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap \mathscr{F}. \tag{188}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

The completion of $W_{\mathrm{IBC}}^{-1}[\dot{\mathcal{S}}_{\mathscr{F}}^{\infty} \cap \mathscr{F}]$ with respect $\langle \cdot, \cdot \rangle_{\mathrm{renI}}$ is a Hilbert space $\mathscr{F}_{\mathrm{renI}}$, which we call the *IBC-renormalized Fock space*. H_{IBC} is then defined on $\mathscr{F}_{\mathrm{renI}}$. The pullback to $\mathscr{F}$ reads

$$\widetilde{H}_{\mathrm{IBC}} = W_{\mathrm{IBC}} H_{\mathrm{IBC}} W_{\mathrm{IBC}}^{-1}. \tag{189}$$

Whenever the expression (189) extends to a self-adjoint operator, H_{IBC} extends to a self-adjoint operator on $\mathscr{F}_{\mathrm{renI}}$.

Remark

(7) Another IBC-renormalized Hamiltonian would formally be given by the expression $(W_{\mathrm{IBC}}^{*})^{-1} H_{\mathrm{IBC}} W_{\mathrm{IBC}}^{-1}$ with $W_{\mathrm{IBC}}^{*} = 1 + A H_0^{-1}$. In view of (179), this expression would even be more desirable, since $(W_{\mathrm{IBC}}^{*})^{-1}$ would eliminate the factor of $(1 + A H_0^{-1})$ inside S^{*}. However, we only know of A that it maps $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$, so $A H_0^{-1} A$ and hence a Neumann series for $(W_{\mathrm{IBC}}^{*})^{-1}$ is generally ill-defined. If one succeeds to refine the definitions of $\overline{\mathscr{F}}$ and $\overline{\mathscr{F}}_{\mathrm{ex}}$ in such a way that also higher powers of A are well defined, then the expression $(W_{\mathrm{IBC}}^{*})^{-1} H_{\mathrm{IBC}} W_{\mathrm{IBC}}^{-1}$ might be used to define a renormalized Hamiltonian. Still, such a refinement is beyond the scope of this paper.

8.3. *The $e^{-H_0^{-1}A^{\dagger}}$-transformation inspired by CQFT*

In the CQFT literature [17, 21, 45], operators of a form similar to $T \propto e^{-H_0^{-1}A^{\dagger}}$ appear. Further, within [17], a renormalized scalar product of the kind

$$\langle T\Psi, T\Phi \rangle_{\mathrm{ren}} := \lim_{\Lambda \to \infty} \langle T_\Lambda \Psi, T_\Lambda \Phi \rangle e^{-\mathbf{\Lambda}_\Lambda} \quad \forall\, T\Psi,\ T\Phi \in T[\mathcal{D}], \quad \mathcal{D} \subset \mathscr{F} \tag{190}$$

is constructed, where the operators T, T_Λ in [17] are, however, much more involved. Here, $\mathbf{\Lambda}_\Lambda = 4! \|\gamma^{-1} v_{4,\Lambda}\|_2^2$, where $v_4 : \mathbb{R}^{4d} \to \mathbb{C}$ is a cutoff form factor of the Φ^4-interaction and $\gamma(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4) = \sum_{\ell=1}^{4} \omega(\boldsymbol{k}_\ell)$. The renormalized Hamiltonian is then constructed by a limiting procedure [17, Theorem 4.1.1]

$$\langle T\Psi, H_{\mathrm{ren}} T\Phi \rangle_{\mathrm{ren}} = \lim_{\Lambda \to \infty} \langle T_\Lambda \Psi, H_{\mathrm{ren}}(\Lambda) T_\Lambda \Phi \rangle e^{-\mathbf{\Lambda}_\Lambda}, \tag{191}$$

with $H_{\mathrm{ren}}(\Lambda)$ containing cutoff counterterms. The limit $\Lambda \to \infty$ formally leads to an infinite wave function renormalization of the kind $e^{-\mathbf{\Lambda}/2}$.

As a simplified divergent term in our prototypical transformation, we choose

$$\mathbf{\Lambda} = 4! \|\omega^{-1} v\|_2^2. \tag{192}$$

The ESS construction then allows to interpret $\mathbf{\Lambda} \in \mathrm{Ren}_1$. In the simplified case $T \propto e^{-H_0^{-1}A^{\dagger}}$, we can even define the dressing transformation directly on $\overline{\mathscr{F}}$:

Proposition 8.2. *Under the assumptions of Proposition* 8.1, *for* $\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^{\infty}$, *we have*

$$T\Psi := e^{-\mathbf{\Lambda}/2} e^{-H_0^{-1}A^{\dagger}} \Psi \in \overline{\mathscr{F}}, \tag{193}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

with $e^{-\mathbf{\Lambda}/2} \in$ eRen. *In particular,* $e^{-H_0^{-1}A^\dagger}$ *and* T *are well-defined linear operators*

$$e^{-H_0^{-1}A^\dagger} : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}, \quad T : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \overline{\mathscr{F}},$$

that extend by eRen*-linearity to* $\overline{\mathscr{F}} \to \overline{\mathscr{F}}$.

Proof. As argued in the proof of Proposition 8.1, $H_0^{-1}A^\dagger$ is well defined on $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$ and maps Fock space vectors supported on the N-particle sector to those supported on the $N+1$-particle sector. We can now write the exponential series as

$$e^{-H_0^{-1}A^\dagger} = \sum_{k=0}^{\infty} \frac{(-H_0^{-1}A^\dagger)^k}{k!} = 1 - H_0^{-1}A^\dagger + \frac{1}{2} H_0^{-1}A^\dagger H_0^{-1}A^\dagger - \cdots . \tag{194}$$

Each $-H_0^{-1}A^\dagger$ maps $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$ to itself and strictly increases the sector number. So the N-sector of $e^{-H_0^{-1}A^\dagger}\Psi$ may only depend on at most $N+1$ terms of the series (194). All terms are elements of $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \subset \overline{\mathscr{F}}$. Hence, $e^{-H_0^{-1}A^\dagger}$ maps $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$ into itself, as claimed. Clearly, the factor $e^{-\mathbf{\Lambda}/2}$ is an element of eRen, so $T\Psi \in \overline{\mathscr{F}}$. □

Of course, the transformation T_Λ appearing in [17] is much more involved, and it remains an interesting and highly nontrivial question whether a corresponding T can be defined using an ESS construction adapted to that model.

Appendix A. Proof of Lemma 5.1

Proof of Lemma 5.1. We show that $\mathcal{D}_{WS}$ is dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$. Since $(S_-\otimes I) : L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y \to \mathscr{F}$ is surjective and bounded with norm 1, we then immediately get density of $(S_-\otimes I)[\mathcal{D}_{WS}]$ within $\mathscr{F}$.

If $A_1^\dagger(\varphi)$ created a boson without giving a recoil to a fermion, we would be done: In this case, $W_{\mathscr{F},1}(\varphi)\Psi_m$ would be of the form

$$\Psi_{mx} \otimes W_y(\varphi)\Omega_y. \tag{A.1}$$

Now, $\Psi_{mx} \in \mathcal{S}(\mathcal{Q}_x)$, which is dense in $L^2(\mathcal{Q}_x)$. Further, we have that $\operatorname{span}\{W_y(\varphi)\Omega_y \,|\, \varphi \in \mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty\}$ is dense in $\mathscr{F}_y$, so

$$\operatorname{span}\big\{\Psi_{mx}\otimes W_y(\varphi)\Omega_y \,|\, \varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty,\ \Psi_{mx}\in\mathcal{S}(\mathcal{Q}_x)\big\}, \tag{A.2}$$

is dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$.

However, $W_{\mathscr{F},1}(\varphi)\Psi_m$ is not of the form (A.1), since $W_{\mathscr{F},1}(\varphi)$ shifts the momentum $\boldsymbol{p}_1$ by $\sum_{\ell=1}^N \boldsymbol{k}_\ell$, as in (37) (the first fermion gets a recoil). The same recoil occurs when applying $A_1^\dagger(v)$. In other words, creation and dressing "entangle" the fermion with the created boson by giving the fermion a recoil. In order to solve this problem, we introduce a *disentangling operator* $D_1 : L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y \to L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$, which removes all recoils

$$(D_1\Psi)(\boldsymbol{P},\boldsymbol{K}) := \Psi\left(\boldsymbol{P} - \sum_{\ell=1}^N e_1\boldsymbol{k}_\ell, \boldsymbol{K}\right). \tag{A.3}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Clearly, D_1 is unitary. Now

$$\begin{aligned}
(W_{\mathscr{F},1}(\varphi)\Psi_m)(\boldsymbol{P},\boldsymbol{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\left(\prod_{\ell=1}^{N}\varphi(\boldsymbol{k}_\ell)\right)\Psi_{mx}\left(\boldsymbol{P}+\sum_{\ell=1}^{N}e_1\boldsymbol{k}_\ell\right)\\
\Rightarrow (D_1W_{\mathscr{F},1}(\varphi)\Psi_m)(\boldsymbol{P},\boldsymbol{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\left(\prod_{\ell=1}^{N}\varphi(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P})\\
\Leftrightarrow D_1W_{\mathscr{F},1}(\varphi)\Psi_m &= \Psi_{mx}\otimes\sum_{N=0}^{\infty}\frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\underbrace{\varphi\otimes\cdots\otimes\varphi}_{N\text{ times}}\\
&= \Psi_{mx}\otimes W_y(\varphi)\Omega_y.
\end{aligned}\tag{A.4}$$

So by density of (A.2), we have that

$$\operatorname{span}\big\{D_1W_{\mathscr{F},1}(\varphi)\Psi_m\,|\,\varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty,\ \Psi_m\in\mathcal{C}_{WS}\big\},$$

is dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$. And since D_1 is unitary, its preimage

$$\mathcal{D}_{WS}=\operatorname{span}\big\{W_{\mathscr{F},1}(\varphi)\Psi_m\,|\,\varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty,\ \Psi_m\in\mathcal{C}_{WS}\big\},$$

is dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$, as well. So $(S_-\otimes I)[\mathcal{D}_{WS}]$ is dense in $\mathscr{F}$. □

Appendix B. Proof of Lemma 5.2

Proof of Lemma 5.2. For evaluating $[A^\dagger,W]$, we first consider the simple case where Ψ is replaced by a boson-only vector $\Psi_y\in\mathscr{F}_y$. A vector with one boson can be written as

$$\phi=a^\dagger(\phi)\Omega_y,\tag{B.1}$$

with $\phi\in\mathfrak{h}=\mathscr{F}_y^{(1)}$. A coherent displacement can then be described using

$$W_y(\varphi)a^\dagger(\phi)=a^\dagger(\phi)W_y(\varphi)-[a^\dagger(\phi),W_y(\varphi)],\tag{B.2}$$

where $\varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty$. The first expression is easily computed in momentum space

$$\begin{aligned}
(a^\dagger(\phi)\Psi_y)(\boldsymbol{K}) &= \frac{1}{\sqrt{N}}\sum_{\ell=1}^{N}\phi(\boldsymbol{k}_\ell)\Psi_y(\boldsymbol{K}\backslash\boldsymbol{k}_\ell)\\
\Rightarrow (a^\dagger(\phi)W_y(\varphi)\Omega_y)(\boldsymbol{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\sum_{\ell=1}^{N}\phi(\boldsymbol{k}_\ell)\left(\prod_{\ell'\neq\ell}\varphi(\boldsymbol{k}_{\ell'})\right).
\end{aligned}\tag{B.3}$$

The commutator $[a^\dagger(\phi),W_y(\varphi)]$ is computed using

$$\begin{aligned}
&(*)\quad [a^\dagger(\varphi),a^\dagger(\phi)]=0,\\
&(**)\quad [a(\varphi),a^\dagger(\phi)]=\langle\varphi,\phi\rangle.
\end{aligned}\tag{B.4}$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

We have

$$\begin{aligned}
[a^\dagger(\phi), W_y(\varphi)] &= [a^\dagger(\phi), e^{a^\dagger(\varphi)-a(\varphi)}] = \sum_{k=0}^{\infty} \frac{1}{k!}[a^\dagger(\phi), (a^\dagger(\varphi)-a(\varphi))^k] \\
&\overset{(*)}{=} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \big([a^\dagger(\phi), a(\varphi)](a^\dagger(\varphi)-a(\varphi))^{k-1} \\
&\quad + (a^\dagger(\varphi)-a(\varphi))[a^\dagger(\phi), a(\varphi)](a^\dagger(\varphi)-a(\varphi))^{k-2} \\
&\quad + \cdots + (a^\dagger(\varphi)-a(\varphi))^{k-1}[a^\dagger(\phi), a(\varphi)] \big) \\
&\overset{(**)}{=} -\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} k\langle\varphi,\phi\rangle (a^\dagger(\varphi)-a(\varphi))^{k-1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle\varphi,\phi\rangle (a^\dagger(\varphi)-a(\varphi))^{k} \\
&= \langle\varphi,\phi\rangle W_y(\varphi). \qquad \text{(B.5)}
\end{aligned}$$

It is easy to check that all above formulas hold as strong operator identities on the space of finite-boson states $\mathscr{F}_{\mathrm{fin},y}$ (defined in (102)). By (99), an application of k operators of the kind[f] $a^\sharp(\varphi) \in \{a(\varphi), a^\dagger(\varphi)\}$ to $\Psi_y \in \mathscr{F}_{\mathrm{fin},y}$ is bounded by

$$\big\| \big(a(\varphi)^\sharp\big)^k \Psi_y \big\| \leq \sqrt{\frac{(N_{\max}+k)!}{N_{\max}!}} \|\Psi_y\| \, \|\varphi\|^n. \qquad \text{(B.6)}$$

This allows to estimate

$$\big\| a^\sharp(\phi) W_y(\varphi)\Psi \big\| \leq \sum_{k\in\mathbb{N}_0} \frac{1}{k!} \sqrt{\frac{(N_{\max}+k+1)!}{N_{\max}!}} \|\Psi_y\| \, \|2\varphi\|^n \, \|\phi\| < \infty, \qquad \text{(B.7)}$$

and an analogous estimate shows that $W_y(\varphi)a^\sharp(\phi)$ is well defined.

Putting together (B.2) and (B.5), we obtain the action of $W_y(\varphi)$ on single-boson states

$$W_y(\varphi)a^\dagger(\phi)\Omega_y = (a^\dagger(\phi) - \langle\varphi,\phi\rangle) W_y(\varphi)\Omega_y. \qquad \text{(B.8)}$$

Now, we turn to state vectors with many fermions and one boson, $A_1^\dagger(\phi)\Psi_m \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$. Further, we go over from dressings by $W_y(\varphi)$ to $W_{\mathscr{F},j}(\varphi)$, which is done replacing $a(\phi), a^\dagger(\phi)$ by $A_j(\phi), A_j^\dagger(\phi)$. Note that $A_j(\phi), A_j^\dagger(\phi)$ are no longer merely creating and annihilating bosons, but they also shift a fermion's momentum. Computations in (B.5) run through in almost the same manner. We have to replace $(**)$ by the CCR (89). If $j \neq j'$, we further use that $V_{jj'}(\varphi^*\phi)$ (which replaces $\langle\varphi,\phi\rangle$ in (B.5)) commutes with[g] $A_{j''}(\varphi)$ and $A_{j''}^\dagger(\varphi)$, so we can still pull it to the left.

[f]Here, we allow $\sharp$ to be a different superscript in each factor. For example, $(a^\sharp)^4$ would also represent $aa^\dagger aa^\dagger$.

[g]This is shown by a similar fiber decomposition argument, as used in the proof of Lemm 5.5.

The final result is

$$W_{\mathscr{F},j}(\varphi)A^\dagger_{j'}(\phi) = \begin{cases} (A^\dagger_{j'}(\phi) - \langle\varphi,\phi\rangle)W_{\mathscr{F},j}(\varphi) & \text{if } j = j', \\ (A^\dagger_{j'}(\phi) - V_{jj'}(\varphi^*\phi))W_{\mathscr{F},j}(\varphi) & \text{if } j \neq j'. \end{cases} \tag{B.9}$$

This is one of the four identities claimed in (110). The other three identities follow by summation over j or j'.

The four identities in (111) are obtained analogously. In place of $(**)$, we use the CCR (89) together with $[A_j(\varphi), A_{j'}(\phi)] = 0$ and keep in mind that the factor $(-1)^k$ drops out in (B.5), which yields the desired result.

It is easy to check that all identities above hold as strong operator identities on the domain $\mathscr{F}_{\text{fin}}$ defined in (109). The momentum space definitions of $A^\dagger_j(\varphi), A_j(\varphi)$ in (20) and (22) directly yield the well-known estimates

$$\big\|A^\dagger_j(\varphi)\Psi\big\| \leq \big\|(N+1)^{1/2}\Psi\big\|\,\|\varphi\|, \quad \big\|A_j(\varphi)\Psi\big\| \leq \big\|N^{1/2}\Psi\big\|\,\|\varphi\|, \tag{B.10}$$

which are analogous to (99) and allow for employing the same arguments as below (102). Thus, all expressions in (110) and (111) are well defined on $\mathscr{F}_{\text{fin}}$. □

Remark

(8) If we replace the form factors $\varphi(\boldsymbol{k})$ and $\phi(\boldsymbol{k})$ in $A^\dagger(\phi)$ by $\varphi(\boldsymbol{p},\boldsymbol{k})$ and $\phi(\boldsymbol{p},\boldsymbol{k})$ (as they appear in more realistic QFT models), then $[A^\dagger_j(\varphi), A^\dagger_{j'}(\phi)] = 0$ will no longer hold true. The operators $A^\dagger_j(\phi)$ change the momentum of the fermion which is emitting a boson. So when one fermion creates two bosons with different form factors, which depend on the fermion momentum, then it can make a difference, which boson is created first. In addition, $V_{j'j''}$ will no longer commute with A_j and $A^\dagger_j$. In this case, several multi-commutators of the form $[A^\sharp_{j_1}, [A^\sharp_{j_2}, \ldots, [A^\sharp_{j_n}, A^\sharp_{j'}]]]$ with $A^\sharp_j \in \{A_j, A^\dagger_j\}$ appear.

Appendix C. Proof of Lemma 5.8

Proof of Lemma 5.8. First, note that $\Psi'_{m,k} \neq 0$ and $v_k \neq 0$, since otherwise at least one vector Ψ_k would be 0.

We now establish the linear independence by an induction over K.

For $K = 1$, linear independence is an obvious fact. Also for $|\mathcal{K}_{WA}| = |\mathcal{K}_W| = 1$, linear independence is easy to see: The (N)-sectors have the norms

$$\begin{aligned} \big\|(W_{\mathscr{F},1}(\varphi_1)\Psi'_{m,1})^{(N)}\big\|^2 &= \frac{e^{-\|\varphi_1\|^2}}{N!}\|\varphi_1\|^{2N}\|\Psi'_{m,1}\|^2, \\ \big\|(W_{\mathscr{F},1}(\varphi_2)A^\dagger_{j_2}(v_2)\Psi'_{m,2})^{(N)}\big\|^2 &= \frac{e^{-\|\varphi_2\|^2}}{(N-1)!}\|\varphi_2\|^{2(N-1)}\|v_2\|^2\|\Psi'_{m,2}\|^2. \end{aligned} \tag{C.1}$$

These cannot agree for all $N \in \mathbb{N}_0$ simultaneously, as $\frac{N!}{(N-1)!}\frac{\|\varphi_1\|^{2N}}{\|\varphi_2\|^{2(N-1)}}$ is never constant in N.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Now assume (induction assumption), we have shown linear independence for any set containing $\leq K-1$ vectors Ψ_k of the kind (137) and consider any other set of K vectors of the kind (137), where $|\mathcal{K}_{WA}| \geq 2$ or $|\mathcal{K}_W| \geq 2$, so we have at least two distinct form factors φ_k. Suppose there was a linear combination

$$\Psi := \sum_{k=1}^{K} c_k \Psi_k = \sum_{k\in\mathcal{K}_{WA}} c_k W_{\mathscr{F},1}(\varphi_k) A^\dagger_{j_k}(v_k)\Psi'_{m,k} + \sum_{k\in\mathcal{K}_W} c_k W_{\mathscr{F},1}(\varphi_k)\Psi'_{m,k} = 0, \tag{C.2}$$

with $c_k \in \mathbb{C}$ not being all zero. Without loss of generality, we may assume $c_k \neq 0$ for all k.

Now, by premise of the lemma, each φ_k can appear at most twice in (C.2). We group equal form factors together by introducing the partition

$$\{1,\ldots,K\} = \bigcup_{z=1}^{Z} \mathcal{K}_z, \quad \mathcal{K}_z \cap \mathcal{K}_{z'} = \emptyset \quad \text{for } z \neq z', \tag{C.3}$$

such that $\varphi_k = \varphi_{k'}$ if and only if k and k' belong to the same index set $\mathcal{K}_z$. In particular, $K/2 \leq Z \leq K$ and $|\mathcal{K}_z| \leq 2$. Since there are at least two distinct φ_k, we also have

$$\max_k \|\varphi_k\| > 0. \tag{C.4}$$

We pick an index $\overline{z}$, such that $\|\varphi_{\overline{k}}\|$ attains this maximum for $\overline{k} \in \mathcal{K}_{\overline{z}}$. Regarding $\mathcal{K}_{\overline{z}}$, there are now two cases which can occur as follows.

Case 1. $\mathcal{K}_{\overline{z}} \cap \mathcal{K}_{WA} = \emptyset$. For $z' \in \{1,\ldots,Z\}$ with the unique associated form factor $\varphi_{k'}, k' \in \mathcal{K}_{z'}$, consider the "$(N)$-sector trial state"

$$\Phi^{(N)}_{z'}(\boldsymbol{K}) = \prod_{\ell=1}^{N} \varphi_{k'}(\boldsymbol{k}_\ell), \quad \Phi^{(N)}_{z'} \in L^2(\mathbb{R}^{Nd}). \tag{C.5}$$

We choose a fixed representative function for L^2-elements, and use the following abbreviations:

$$\begin{aligned} &\Psi''_k := e^{-\frac{\|\varphi_k\|^2}{2}} c_k \Psi'_{m,k}, \quad w_{k,\boldsymbol{P}}(\boldsymbol{k}) := v_k(\boldsymbol{k})\Psi''_k(\boldsymbol{P} + (e_{j_k} - e_1)\boldsymbol{k}), \\ &\xi_{z'z} := \langle \varphi_{k'}, \varphi_k \rangle, \end{aligned} \tag{C.6}$$

where $w_{k,\boldsymbol{P}}$ is only defined for $k \in \mathcal{K}_{WA}$. Integrating $\Phi^{(N)}_{z'}$ against the (N)-boson sector of $\sum_{k\in\mathcal{K}_z} \Psi_k$ (which contains at most two terms), we now obtain for almost all[h] $\boldsymbol{P} \in \mathcal{Q}_x$

$$\int \Phi^{(N)}_{z'}(\boldsymbol{K})^* \left(\sum_{k\in\mathcal{K}_z} c_k \Psi_k \left(\boldsymbol{P} - e_1 \sum_{\ell=1}^{N} \boldsymbol{k}_\ell, \boldsymbol{K} \right) \right) d\boldsymbol{K}$$

[h]There is no guarantee that $w_{k,\boldsymbol{P}} \in L^2$, so integrals involving $w_{k,\boldsymbol{P}}$ might be ill-defined. However, since $\Psi, v \in L^2$, we have $w_{k,\boldsymbol{P}} \notin L^2$ only for $\boldsymbol{P}$ within some null set in $\mathcal{Q}_x$, so (C.7) holds almost everywhere.

$$
\begin{aligned}
&= \sum_{k\in\mathcal{K}_z\cap\mathcal{K}_{WA}} (N!)^{-1/2} N\langle\varphi_{k'},\varphi_k\rangle^{N-1}\langle\varphi_{k'},w_{k,\boldsymbol{P}}\rangle \\
&\quad + \sum_{k\in\mathcal{K}_z\cap\mathcal{K}_W} (N!)^{-1/2}\langle\varphi_{k'},\varphi_k\rangle^N \Psi_k''(\boldsymbol{P}) \\
&= N(N!)^{-1/2}\xi_{z'z}^{N-1}\widetilde{\Psi}_{z'z,1}(\boldsymbol{P}) + (N!)^{-1/2}\xi_{z'z}^{N-1}\widetilde{\Psi}_{z'z,2}(\boldsymbol{P}),
\end{aligned}
\tag{C.7}
$$

where $\widetilde{\Psi}_{z'z,1}, \widetilde{\Psi}_{z'z,2}$ do not depend on N and are given by

$$
\widetilde{\Psi}_{z'z,1}(\boldsymbol{P}) := \sum_{k\in\mathcal{K}_z\cap\mathcal{K}_{WA}} \langle\varphi_{k'},w_{k,\boldsymbol{P}}\rangle, \quad \widetilde{\Psi}_{z'z,2}(\boldsymbol{P}) := \sum_{k\in\mathcal{K}_z\cap\mathcal{K}_W} \xi_{z'z}\Psi_k''(\boldsymbol{P}). \tag{C.8}
$$

Note that each of the sums over $\mathcal{K}_z\cap\mathcal{K}_{WA}$ or $\mathcal{K}_z\cap\mathcal{K}_W$ contains at most one term, so the k associated with z in (C.8) is unique. Now, consider the ratio

$$
R := \operatorname*{ess\,sup}_{\boldsymbol{P}\in\mathcal{Q}_x\setminus\mathcal{N}_x} \frac{|\widetilde{\Psi}_{\overline{z}z,2}(\boldsymbol{P})|}{\sum_{z\neq\overline{z}}|\widetilde{\Psi}_{\overline{z}z,1}(\boldsymbol{P})| + \sum_{z\neq\overline{z}}|\widetilde{\Psi}_{\overline{z}z,2}(\boldsymbol{P})|} \geq 0, \tag{C.9}
$$

where $\mathcal{N}_x$ denotes the set of all $\boldsymbol{P}$, where the above fraction amounts to $\frac{0}{0}$. Since $\mathcal{K}_{\overline{z}}\neq\emptyset$, the set $\mathcal{K}_{\overline{z}}\cap\mathcal{K}_W$ contains exactly one element $\overline{k}$, with $\Psi_{\overline{k}}''(\boldsymbol{P})\neq 0$ on a set of positive measure in $\boldsymbol{P}\in\mathcal{Q}_x$. And as $\varphi_{\overline{k}}\neq 0$, we have $\widetilde{\Psi}_{\overline{z}\overline{z},2}(\boldsymbol{P})\neq 0$ for those $\boldsymbol{P}$, so $R>0$. Hence, the set

$$
U = \left\{ \boldsymbol{P}\in\mathcal{Q}_x \,|\, |\widetilde{\Psi}_{\overline{z}\overline{z},2}(\boldsymbol{P})| > \frac{R}{2}\left(\sum_{z\neq\overline{z}}|\widetilde{\Psi}_{\overline{z}z,1}(\boldsymbol{P})| + \sum_{z\neq\overline{z}}|\widetilde{\Psi}_{\overline{z}z,2}(\boldsymbol{P})|\right) \text{ and (C.7) holds} \right\}, \tag{C.10}
$$

has positive measure. Further, as $\|\varphi_{\overline{k}}\|$ is maximal and $\varphi_k\neq\varphi_{\overline{k}}$ for $k\neq\overline{k}$, we conclude that

$$
\langle\varphi_{\overline{k}},\varphi_{\overline{k}}\rangle > |\langle\varphi_{\overline{k}},\varphi_k\rangle| \Leftrightarrow |\xi_{\overline{z}z}| > |\xi_{\overline{z}z}| \quad \text{for } z\neq\overline{z}. \tag{C.11}
$$

Hence, there is an $N\in\mathbb{N}$ with

$$
|\xi_{\overline{z}\overline{z}}^{N-1}| > \frac{4}{R}|N\xi_{\overline{z}z}^{N-1}| \quad \text{for } z\neq\overline{z}. \tag{C.12}
$$

Now, for this N and $\boldsymbol{P}\in U$, the term $\widetilde{\Psi}_{\overline{z}\overline{z},2}$ is dominant in the following sense:

$$
\begin{aligned}
&\left|\int \Phi_{\overline{k}}^{(N)}(\boldsymbol{K})^*\left(\sum_{k=1}^K c_k\Psi_k\left(\boldsymbol{P}-e_1\sum_{\ell=1}^N \boldsymbol{k}_\ell,\boldsymbol{K}\right)\right)d\boldsymbol{K}\right| \\
&\quad \overset{(C.7)}{=} (N!)^{-1/2}\left|\sum_{z=1}^Z N\xi_{\overline{z}z}^{N-1}\widetilde{\Psi}_{\overline{z}z,1}(\boldsymbol{P}) + \sum_{z=1}^Z \xi_{\overline{z}z}^{N-1}\widetilde{\Psi}_{\overline{z}z,2}(\boldsymbol{P})\right|
\end{aligned}
$$

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$\geq (N!)^{-1/2}\left(|\xi_{\overline{z}\overline{z}}^{N-1}\widetilde{\Psi}_{\overline{z}\overline{z},2}(\boldsymbol{P})| - \sum_{z\neq\overline{z}}|N\xi_{\overline{z}z}^{N-1}\widetilde{\Psi}_{\overline{z}z,1}(\boldsymbol{P})| - \sum_{z\neq\overline{z}}|\xi_{\overline{z}z}^{N-1}\widetilde{\Psi}_{\overline{z}z,2}(\boldsymbol{P})|\right)$$

$$\overset{(C.12)}{\geq} (N!)^{-1/2}|\xi_{\overline{z}\overline{z}}|^{N-1}\left(|\widetilde{\Psi}_{\overline{z}\overline{z},2}(\boldsymbol{P})| - \frac{R}{4}\left(\sum_{z\neq\overline{z}}|\widetilde{\Psi}_{\overline{z}z,1}(\boldsymbol{P})| + \sum_{z\neq\overline{z}}|\widetilde{\Psi}_{\overline{z}z,2}(\boldsymbol{P})|\right)\right)$$

$$\overset{(C.10)}{\geq} (N!)^{-1/2}|\xi_{\overline{z}\overline{z}}|^{N-1}\left(|\widetilde{\Psi}_{\overline{z}\overline{z},2}(\boldsymbol{P})| - \frac{1}{2}|\widetilde{\Psi}_{\overline{z}\overline{z},2}(\boldsymbol{P})|\right) > 0. \tag{C.13}$$

So the L^2-function $\boldsymbol{K}\mapsto\Psi(\boldsymbol{P}-e_1\sum_{\ell=1}^N\boldsymbol{k}_\ell,\boldsymbol{K})$ cannot be zero for $\boldsymbol{P}\in U$. Since U has positive measure, we conclude $\Psi\neq 0$, which contradicts (C.2).

Case 2. $\mathcal{K}_{\overline{z}}\cap\mathcal{K}_{WA}\neq\emptyset$. Since $\mathcal{K}_{\overline{z}}\cap\mathcal{K}_{WA}$ contains at most one element, we can set $\mathcal{K}_{\overline{z}}\cap\mathcal{K}_{WA}=:\{\overline{k}\}$. We copy the notations (C.6) and (C.8) from Case 1 and choose as a trial state for $N\geq 1, k'\in\mathcal{K}_{z'}\cap\mathcal{K}_{WA}$ and $\boldsymbol{P}\in\mathcal{Q}_x$

$$\Xi_{z',\boldsymbol{P}}^{(N)}(\boldsymbol{K}) = w_{k',\boldsymbol{P}}(\boldsymbol{k}_1)\left(\prod_{\ell=2}^N\varphi_{k'}(\boldsymbol{k}_\ell)\right). \tag{C.14}$$

For almost all $\boldsymbol{P}\in\mathcal{Q}_x$, we then obtain

$$\begin{aligned}
&\int\Xi_{z',\boldsymbol{P}}^{(N)}(\boldsymbol{K})^*\left(\sum_{k\in\mathcal{K}_z}c_k\Psi_k\left(\boldsymbol{P}-e_1\sum_{\ell=1}^N\boldsymbol{k}_\ell,\boldsymbol{K}\right)\right)d\boldsymbol{K}\\
&\quad=\sum_{k\in\mathcal{K}_z\cap\mathcal{K}_{WA}}(N!)^{-1/2}\langle\varphi_{k'},\varphi_k\rangle^{N-1}\langle w_{k',\boldsymbol{P}},w_{k,\boldsymbol{P}}\rangle\\
&\qquad+\sum_{k\in\mathcal{K}_z\cap\mathcal{K}_{WA}}(N!)^{-1/2}(N-1)\langle\varphi_{k'},\varphi_k\rangle^{N-2}\langle w_{k',\boldsymbol{P}},\varphi_k\rangle\langle\varphi_{k'},w_{k,\boldsymbol{P}}\rangle\\
&\qquad+\sum_{k\in\mathcal{K}_z\cap\mathcal{K}_W}(N!)^{-1/2}\langle\varphi_{k'},\varphi_k\rangle^{N-1}\langle w_{k',\boldsymbol{P}},\varphi_k\rangle\Psi_k''(\boldsymbol{P})\\
&\quad=(N!)^{-1/2}\big((N-1)\xi_{z'z}^{N-2}\widetilde{\Psi}'_{z'z,1}(\boldsymbol{P})+\xi_{z'z}^{N-2}\widetilde{\Psi}'_{z'z,2}(\boldsymbol{P})\big),
\end{aligned}\tag{C.15}$$

with

$$\begin{aligned}
\widetilde{\Psi}'_{z'z,1}(\boldsymbol{P}) &:= \sum_{k\in\mathcal{K}_z\cap\mathcal{K}_{WA}}\langle w_{k',\boldsymbol{P}},\varphi_k\rangle\langle\varphi_{k'},w_{k,\boldsymbol{P}}\rangle,\\
\widetilde{\Psi}'_{z'z,2}(\boldsymbol{P}) &:= \sum_{k\in\mathcal{K}_z\cap\mathcal{K}_{WA}}\xi_{z'z}\langle w_{k',\boldsymbol{P}},w_{k,\boldsymbol{P}}\rangle+\sum_{k\in\mathcal{K}_z\cap\mathcal{K}_W}\xi_{z'z}\langle w_{k',\boldsymbol{P}},\varphi_k\rangle\Psi_k''(\boldsymbol{P}).
\end{aligned}\tag{C.16}$$

First, consider the sub-case where $\langle w_{\overline{k},\boldsymbol{P}},\varphi_{\overline{k}}\rangle\neq 0$ holds for all $\boldsymbol{P}$ inside some subset of $\mathcal{Q}_x$ with positive measure. So $\widetilde{\Psi}'_{\overline{z}\overline{z},1}(\boldsymbol{P})\neq 0$ for those $\boldsymbol{P}$. In this case, the term

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

with $\widetilde{\Psi}'_{\overline{z}z,1}$ will be dominant. Here, the ratio

$$R' := \operatorname*{ess\,sup}_{\boldsymbol{P}\in\mathcal{Q}_x\setminus\mathcal{N}'_x} \frac{|\widetilde{\Psi}'_{\overline{z}z,1}(\boldsymbol{P})|}{\sum_{z\neq\overline{z}}|\widetilde{\Psi}'_{\overline{z}z,1}(\boldsymbol{P})|+\sum_{z=1}^{Z}|\widetilde{\Psi}'_{\overline{z}z,2}(\boldsymbol{P})|}, \tag{C.17}$$

is strictly positive. Here, $\mathcal{N}'_x$ denotes the set of all $\boldsymbol{P}$ with the above fraction amounting to $\frac{0}{0}$. Therefore, the set

$$U' = \left\{\boldsymbol{P}\in\mathcal{Q}_x \,|\, |\widetilde{\Psi}'_{\overline{z}z,1}(\boldsymbol{P})| > \frac{R'}{2}\left(\sum_{z\neq\overline{z}}|\widetilde{\Psi}'_{\overline{z}z,1}(\boldsymbol{P})|+\sum_{z=1}^{Z}|\widetilde{\Psi}'_{\overline{z}z,2}(\boldsymbol{P})|\right) \text{ and (C.15) holds}\right\}, \tag{C.18}$$

has positive measure. In this case, we can find some $N\in\mathbb{N}$, such that

$$(N-1) > \frac{4}{R'} \quad \text{and} \quad |\xi_{\overline{z}z}^{N-2}| > \frac{4}{R'}|\xi_{\overline{z}z}^{N-2}| \quad \text{for } z\neq\overline{z}. \tag{C.19}$$

Hence, for this N and $\boldsymbol{P}\in U$, we obtain

$$\begin{aligned}
&\left|\int \Xi^{(N)}_{\overline{z},\boldsymbol{P}}(\boldsymbol{K})^*\left(\sum_{k=1}^{K}c_k\Psi_k\left(\boldsymbol{P}-e_1\sum_{\ell=1}^{N}\boldsymbol{k}_\ell,\boldsymbol{K}\right)\right)d\boldsymbol{K}\right| \\
&\overset{(C.15)}{=} (N!)^{-1/2}\left|\sum_{z=1}^{Z}(N-1)\xi_{\overline{z}z}^{N-2}\widetilde{\Psi}'_{\overline{z}z,1}(\boldsymbol{P})+\sum_{z=1}^{Z}\xi_{\overline{z}z}^{N-2}\widetilde{\Psi}'_{\overline{z}z,2}(\boldsymbol{P})\right| \\
&\geq (N!)^{-1/2}\left(\left|(N-1)\xi_{\overline{z}\overline{z}}^{N-2}\widetilde{\Psi}'_{\overline{z}\overline{z},1}(\boldsymbol{P})\right|-\sum_{z\neq\overline{z}}\left|(N-1)\xi_{\overline{z}z}^{N-2}\widetilde{\Psi}'_{\overline{z}z,1}(\boldsymbol{P})\right|\right. \\
&\qquad \left.-\sum_{z=1}^{Z}\left|\xi_{\overline{z}z}^{N-2}\widetilde{\Psi}'_{\overline{z}z,2}(\boldsymbol{P})\right|\right) \\
&\overset{(C.19)}{\geq} (N-1)(N!)^{-1/2}|\xi_{\overline{z}\overline{z}}|^{N-2} \\
&\qquad \times\left(\left|\widetilde{\Psi}'_{\overline{z}\overline{z},1}(\boldsymbol{P})\right|-\frac{R'}{4}\left(\sum_{z\neq\overline{z}}\left|\widetilde{\Psi}'_{\overline{z}z,1}(\boldsymbol{P})\right|+\sum_{z=1}^{Z}\left|\widetilde{\Psi}'_{\overline{z}z,2}(\boldsymbol{P})\right|\right)\right) \\
&\overset{(C.18)}{\geq} (N-1)(N!)^{-1/2}|\xi_{\overline{z}\overline{z}}|^{N-2}\left(|\widetilde{\Psi}'_{\overline{z}\overline{z},1}(\boldsymbol{P})|-\frac{1}{2}\left|\widetilde{\Psi}'_{\overline{z}\overline{z},1}(\boldsymbol{P})\right|\right) \\
&> 0.
\end{aligned} \tag{C.20}$$

By the same argument as that below (C.13), we conclude $\Psi\neq 0$ which contradicts (C.2).

It remains to establish a contradiction in the sub-case where $\langle w_{\overline{k},\boldsymbol{P}}, \varphi_{\overline{k}}\rangle = 0$ for almost all $\boldsymbol{P} \in \mathcal{Q}_x$. In that case, the dominant term is no longer $\widetilde{\Psi}'_{\overline{zz},1}(\boldsymbol{P})$ (which is zero almost everywhere), but instead

$$\widetilde{\Psi}'_{\overline{zz},2}(\boldsymbol{P}) = \xi_{\overline{zz}}\langle w_{\overline{k},\boldsymbol{P}}, w_{\overline{k},\boldsymbol{P}}\rangle. \tag{C.21}$$

Since $\Psi'_{m,\overline{k}} \neq 0$, we have $w_{\overline{k},\boldsymbol{P}} \neq 0$ for all $\boldsymbol{P}$ inside some subset of $\mathcal{Q}_x$ with positive measure, which implies $\widetilde{\Psi}'_{\overline{zz},2}(\boldsymbol{P}) \neq 0$ for these $\boldsymbol{P}$. So we are in a similar situation as in Case 1 meaning that $\widetilde{\Psi}'_{\overline{zz},1}$ is zero almost everywhere while $\widetilde{\Psi}'_{\overline{zz},2}$ is not. We may thus copy the definitions of R and U and employ the same arguments as in (C.11)–(C.13), replacing N by $N-1$ in the required positions. This yields the desired contradiction. □

Appendix D. Proof of Lemma 6.1

Proof of Lemma 6.1. We evaluate the pullbacks of $A(v)$ and E_∞, keeping in mind that by Lemma 5.4, $\Psi = W(s)W_{\mathscr{F},1}(\varphi)\Psi_m = W(s)W_1(\varphi)\Psi_m$.

- $A(v)$: The commutator $[W(s)W_1(\varphi), A(v)]$ is evaluated using the extended commutation relations established in Lemma 5.9

$$\begin{aligned} A(v)\Psi &= A(v)W(s)W_1(\varphi)\Psi_m \\ &= [A(v), W(s)W_1(\varphi)]\Psi_m + W(s)W_1(\varphi)\underbrace{A(v)\Psi_m}_{=0} \\ &= \big(M\langle v, s\rangle + \langle v, \varphi\rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s)\big)\Psi. \end{aligned} \tag{D.1}$$

- E_∞: The self-energy operator (158) now exactly removes the term $M\langle v, s\rangle$, so we have

$$(A(v) - E_\infty)\Psi = \big(\langle v, \varphi\rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s)\big)\Psi. \tag{D.2}$$

Now, by Definition 5.4, we may commute $\langle v, \varphi\rangle, V$ and $V_{\bullet 1}$ with $W(s)$ yielding

$$\begin{aligned} W^{-1}(s)\big(A(v) - E_\infty\big)W(s)W_1(\varphi)\Psi_m &= \big(\langle v, \varphi\rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s)\big)W_1(\varphi)\Psi_m \\ &= (\mathrm{res}_1(\varphi) + V)W_1(\varphi)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}. \end{aligned} \tag{D.3}$$

Now, Lemma 5.4 allows replacing W_1 by $W_{\mathscr{F},1}$, which renders the desired result. □

Remark

(9) If the form factor v depended on the fermion momentum $\boldsymbol{p}$, then $s = -\frac{v}{\omega}$ would also depend on $\boldsymbol{p}$. In that case, $W^{-1}(s)VW(s) = V$ will no longer hold true in general, as the function $s(\boldsymbol{k}, \boldsymbol{p})$ in the dressing operator $W(s)$ then depends on $\boldsymbol{p}$ and fermion momenta are changed by V. If $(A(v) - E_\infty)W(s)W_{\mathscr{F},1}(\varphi)\Psi_m$ is then expressed in terms of dressed coherent states, multiple commutators appear.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Appendix E. Proof of Lemma 6.2

Proof of Lemma 6.2. Again, we may use Lemmas 5.3 and 5.4, to write

$$\Psi = W(s)W_{\mathscr{F},1}(\varphi)\Psi_m = W(s)W_1(\varphi)\Psi_m \in \overline{\mathscr{F}}.$$

Following Proposition 4.1 we further have $H_{0,y}\Psi, A^\dagger(v)\Psi \in \overline{\mathscr{F}}$. We first evaluate $(H_{0,y} + A^\dagger(v))W(s)W_1(\varphi)\Psi_m$ in momentum space and then use Definition 5.3 to pull $W(s)$ to the left.

- $H_{0,y}$: The important point is that the expression $(H_{0,y} + A^\dagger(v))W(s)$ cancels terms in $W(s)H_{0,y}$. We therefore investigate the commutator expression $[H_{0,y}, W(s)]W_1(\varphi)\Psi_m$ and compare it to $A^\dagger(v)W(s)W_1(\varphi)\Psi_m$.

  First, we note that the application of $H_{0,y}$ to a dressed state just changes one single photon dispersion relation. Therefore, it is equivalent to applying a creation operator to the dressed state

$$\begin{aligned}(H_{0,y}W_1(\varphi)\Psi_m)(\boldsymbol{P},\boldsymbol{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\left(\sum_{\ell'=1}^N \omega(\boldsymbol{k}_{\ell'})\right)\left(\prod_{\ell=1}^N \varphi(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P}') \\ &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\sum_{\ell=1}^N \omega(\boldsymbol{k}_\ell)\varphi(\boldsymbol{k}_\ell)\left(\prod_{\ell'\neq\ell}^N \varphi(\boldsymbol{k}_{\ell'})\right)\Psi_{mx}(\boldsymbol{P}') \\ &= (A_1^\dagger(\omega\varphi)W_1(\varphi)\Psi_m)(\boldsymbol{P},\boldsymbol{K}), \end{aligned} \tag{E.1}$$

  with $\boldsymbol{P}' = \big(\boldsymbol{p}_1 + \sum_{\ell'=1}^N \boldsymbol{k}_{\ell'}, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_M\big)$. Replacing the dressing $W_1(\varphi)$ by $W(s)W_1(\varphi)$ and applying the commutation relations from Definitions 5.3 and 5.4, we obtain

$$\begin{aligned} H_{0,y}W(s)W_1(\varphi)\Psi_m &= (A^\dagger(\omega s) + A_1^\dagger(\omega\varphi))W(s)W_1(\varphi)\Psi_m \\ &= -A^\dagger(v)W(s)W_1(\varphi)\Psi_m + W(s)W_1(\varphi)\big(A_1^\dagger(\omega\varphi) - \langle v,\varphi\rangle \\ &\quad - V_{\bullet 1}(v^*\varphi) + \langle\varphi,\omega\varphi\rangle\big)\Psi_m. \end{aligned} \tag{E.2}$$

  Here, we used $s^*\omega = \omega s = -v$ and Definition 5.4, which allows us to pull the formal scalar products and the $V_{\bullet 1}$-term past the dressing operators.

  By means of (E.1) and the commutation relations from Definitions 5.3 and 5.4, we also have

$$W(s)H_{0,y}W_1(\varphi)\Psi_m = W(s)W_1(\varphi)A_1^\dagger(\omega\varphi)\Psi_m + W(s)W_1(\varphi)\langle\varphi,\omega\varphi\rangle\Psi_m. \tag{E.3}$$

  Combining (E.2) and (E.3), we finally obtain

$$\begin{aligned}&[H_{0,y}, W(s)]W_1(\varphi)\Psi_m \\ &\quad = -A^\dagger(v)W(s)W_1(\varphi)\Psi_m - W(s)W_1(\varphi)\big(\langle v,\varphi\rangle + V_{\bullet 1}(v^*\varphi)\big)\Psi_m. \end{aligned} \tag{E.4}$$

- $A^\dagger(v)$: Here, we do not need to perform any calculations. The appearing term simply cancels the $-A^\dagger(v)W(s)W_1(\varphi)\Psi_m$ from (E.4).

Now, adding both terms and using Definition 5.4, (E.4) amounts to

$$\big([H_{0,y}, W(s)] + A^\dagger(v)W(s)\big)W_1(\varphi)\Psi_m = W(s)W_1(\varphi)\big(-\langle v, \varphi\rangle - V_{\bullet 1}(v^*\varphi)\big)\Psi_m$$
$$= W(s)\big(-\mathrm{res}_1(\varphi)\big)W_1(\varphi)\Psi_m. \qquad \text{(E.5)}$$

This is equivalent to

$$\big(H_{0,y} + A^\dagger(v)\big)W(s)W_1(\varphi)\Psi_m = W(s)\big(H_{0,y} - \mathrm{res}_1(\varphi)\big)W_1(\varphi)\Psi_m. \qquad \text{(E.6)}$$

Lemma 5.4 allows again to replace W_1 by $W_{\mathscr{F},1}$, which yields the final result. □

Acknowledgments

The idea of introducing a Fock space extension arose within numerous discussions about mathematical QFT. Important information and/or relevant literature was pointed out by Stefan Teufel, Jonas Lampart, Gianluca Panati, Gian Michele Graf, Jürg Fröhlich, Volker Bach, Wojciech Dybalski, Michał Wrochna, Kasia Rejzner, Holger Gies, Jan Mandrysch, Joscha Henheik, Martin Oelker, as well as some further members of the LQP2 network and at Tübingen. In particular, I am grateful to Roderich Tumulka for valuable suggestions and discussions concerning the Extended state space construction.

This work was financially supported by the Wilhelm Schuler-Stiftung Tübingen, by Deutscher Akademischer Austauschdienst (DAAD), by the Basque Government through the BERC 2018–2021 program and by the Ministry of Science, Innovation and Universities: BCAM Severo Ochoa accreditation SEV-2017-0718, as well as by the European Union (ERC FermiMath, grant agreement nr. 101040991). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them. Moreover, it was partially supported by Gruppo Nazionale per la Fisica Matematica in Italy.

ORCID

Sascha Lill https://orcid.org/0000-0002-9474-9914

References

[1] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, New York, 1964).

[2] R. Haag and D. Kastler, An algebraic approach to quantum field theory, *J. Math. Phys.* **5**(7) (1964) 848–861.

[3] C. J. Fewster and K. Rejzner, Algebraic quantum field theory — An introduction, in *Progress and Visions in Quantum Theory in View of Gravity*, eds. F. Finster *et al.* (Springer, 2020), pp. 1–61, arXiv:1904.04051.

[4] J. Dereziński, Van Hove Hamiltonians — Exactly solvable models of the infrared and ultraviolet problem, *Ann. Henri Poincaré* **4** (2003) 713–738.

Rev. Math. Phys. Downloaded from www.worldscientific.com by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

[5] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964) 1190–1197.
[6] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, *Ann. Henri Poincaré* **19** (1973) 1–103.
[7] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschr. Phys.* **22** (1974) 159–198.
[8] M. Griesemer and A. Wünsch, On the domain of the Nelson Hamiltonian, *J. Math. Phys.* **59** (2018) 042111, arXiv:1711.10916.
[9] M. Griesemer and A. Wünsch, Self-adjointness and domain of the Fröhlich Hamiltonian, *J. Math. Phys.* **57** (2016) 021902, arXiv:1508.02533.
[10] L. Gross, The relativistic polaron without cutoffs, *Comm. Math. Phys.* **31** (1973) 25–73.
[11] A. D. Sloan, The polaron without cutoffs in two space dimensions, *J. Math. Phys.* **15** (1974) 190–201.
[12] S. Lill, Implementing Bogoliubov transformations beyond the Shale–Stinespring condition, preprint (2022), arXiv:2204.13407.
[13] S. Lill, Bogoliubov transformations beyond Shale–Stinespring: Generic v^*v for bosons, in *Proc. Intensive Period "INdAM Quantum Meetings (IQM22)" at Politecnico di Milano* (Springer, 2023), arXiv:2208.03487.
[14] S. Teufel, Effective N-body dynamics for the massless Nelson model and adiabatic decoupling without spectral gap, *Ann. Henri Poincaré* **3** (2002) 939–965, arXiv:math-ph/0203046.
[15] L. Tenuta and S. Teufel, Effective dynamics for particles coupled to a quantized scalar field, *Comm. Math. Phys.* **280** (2008) 751–805, arXiv:math-ph/0702067.
[16] J. Fröhlich, Application of commutator theorems to the integration of representations of Lie algebras and commutation relations, *Comm. Math. Phys.* **54** (1977) 135–150.
[17] J. Glimm, Boson fields with the $:\Phi^4:$ interaction in three dimensions, *Comm. Math. Phys.* **10** (1968) 1–47.
[18] J. Glimm and A. Jaffe, Self-adjointness of the Yukawa$_2$, Hamiltonian, *Ann. Phys.* **60** (1970) 321–383.
[19] J. Glimm and A. Jaffe, The Yukawa$_2$, quantum field theory without cutoffs, *J. Funct. Anal.* **7** (1971) 323–357.
[20] R. Schrader, Yukawa quantum field theory in two space-time dimensions without cutoffs, *Ann. Phys.* **70** (1972) 412–457.
[21] K. Hepp, *Théorie de la Renormalisation* (Springer-Verlag, 1969).
[22] J. Glimm and A. Jaffe, The $\lambda(\phi)_2^4$ quantum field theory without cutoffs. I, *Phys. Rev.* **176** (1968) 1945–1951.
[23] J. Glimm and A. Jaffe, The $\lambda(\phi)_2^4$ quantum field theory without cutoffs. II. The field operators and the approximate vacuum, *Ann. of Math. (2)* **91**(2) (1970) 362–401.
[24] J. Glimm and A. Jaffe, The $\lambda(\phi)_2^4$ quantum field theory without cutoffs. III. The physical vacuum, *Acta Math.* **125** (1970) 203–267.
[25] J. Glimm and A. Jaffe, Positivity and self adjointness of the $P(\phi)_2$ Hamiltonian, *Comm. Math. Phys.* **22** (1971) 253–258.
[26] J. Glimm and A. Jaffe, The $\lambda(\phi)_2^4$ quantum field theory without cutoffs. IV. Perturbations of the Hamiltonian, *J. Math. Phys.* **13** (1972) 1568–1584.
[27] R. Høegh-Krohn, A general class of quantum fields without cut-offs in two space-time dimensions, *Comm. Math. Phys.* **21** (1971) 244–255.
[28] W. Thirring, A soluble relativistic field theory, *Ann. Phys.* **3**(1) (1958) 91–112.
[29] S. N. M. Ruijsenaars, The Wightman axioms for the fermionic Federbush model, *Comm. Math. Phys.* **87** (1982) 181–228.

[30] S. J. Summers, A perspective on constructive quantum field theory, preprint (2016), arXiv:1203.3991.
[31] M. Dedushenko, Snowmass white paper: The quest to define QFT, preprint (2022), arXiv:2203.08053.
[32] S. Teufel and R. Tumulka, Hamiltonians without ultraviolet divergence for quantum field theories, *Quantum Stud.*, *Math. Found.* **8** (2020) 17–35, arXiv:1505.04847.
[33] S. Teufel and R. Tumulka, Avoiding ultraviolet divergence by means of interior–boundary conditions, in *Quantum Mathematical Physics*, eds. F. Finster *et al.* (Birkhäuser, Cham, 2016), pp. 293–311, arXiv:1506.00497.
[34] J. Lampart, J. Schmidt, S. Teufel and R. Tumulka, Particle creation at a point source by means of interior-boundary conditions, *Math. Phys. Anal. Geom.* **21** (2018) 12, arXiv:1703.04476.
[35] J. Lampart and J. Schmidt, On the domain of Nelson-type Hamiltonians and abstract boundary conditions, *Comm. Math. Phys.* **367** (2019) 629–663, arXiv:1803.00872.
[36] J. Lampart, A nonrelativistic quantum field theory with point interactions in three dimensions, *Ann. Henri Poincaré* **20** (2019) 3509–3541, arXiv:1804.08295.
[37] J. Schmidt, On a direct description of pseudorelativistic Nelson Hamiltonians, *J. Math. Phys.* **60** (2019) 102303, arXiv:1810.03313.
[38] J. Schmidt, The massless Nelson Hamiltonian and its domain, in *Mathematical Challenges of Zero-Range Physics*, ed. A. Michelangeli (Springer, 2021), pp. 57–80, arXiv:1901.05751.
[39] J. Lampart, The renormalized Bogoliubov–Fröhlich Hamiltonian, *J. Math. Phys.* **61** (2020) 101902, arXiv:1909.02430.
[40] J. Henheik and R. Tumulka, Interior-boundary conditions for the Dirac equation at point sources in three dimensions, *J. Math. Phys.* **63** (2022) 122302, arXiv:2006.16755.
[41] A. Posilicano, On the self-adjointness of $\mathrm{H} + \mathrm{A}^* + \mathrm{A}$, *Math. Phys. Anal. Geom.* **23** (2020) 37, arXiv:2003.05412.
[42] M. Lienert and L. Nickel, Multi-time formulation of particle creation and annihilation via interior-boundary conditions, *Rev. Math. Phys.* **32**(2) (2020) 2050004, arXiv:1808.04192.
[43] R. Haag, On quantum field theories, *Kgl. Danske Videnskab. Selakab*, *Mat.-Fys. Medd.* **29**(12) (1955) 1–37, https://cds.cern.ch/record/212242.
[44] R. Haag, *Local Quantum Physics* (Springer, 1996).
[45] K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space* (American Mathematical Society, 1965).
[46] J. P. Eckmann, A model with persistent vacuum, *Comm. Math. Phys.* **18** (1970) 247–264.
[47] V. Bach, J. Fröhlich and I. M. Sigal, Quantum electrodynamics of confined nonrelativistic particles, *Adv. Math.* **137**(2) (1998) 299–395.
[48] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Photons and Atoms* (Wiley-VCH, 1997).
[49] T. Jech, *The Axiom of Choice* (North-Holland Publishing Company, 1973).
[50] D. S. Passman, *Infinite Group Rings* (Marcel Dekker, 1971).
[51] S. Lang, *Algebra* (Addison-Wesley, 1984).
[52] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II*: *Fourier Analysis*, *Self-Adjointness* (Academic Press, 1975).
[53] S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics* (Springer, 1988).

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.

[54] D. Lonigro, Generalized spin-boson models with non-normalizable form factors, *J. Math. Phys.* **63** (2022) 072105, arXiv:2111.06121.
[55] D. Lonigro, Renormalization of spin–boson interactions mediated by singular form factors, in *Proc. Intensive Period INdAM Quantum Meetings (IQM22) at Politecnico di Milano* (Springer, 2022), arXiv:2210.15267.
[56] D. Lonigro, Self-adjointness of a class of multi-spin-boson models with ultraviolet divergences, *Math. Phys. Anal. Geom.* **26** (2023) 15, arXiv:2301.10694.

Rev. Math. Phys. Downloaded from www.worldscientific.com
by 151.41.0.13 on 08/26/24. Re-use and distribution is strictly not permitted, except for Open Access articles.