A LOG-WEIGHTED MOSER INEQUALITY ON THE PLANE

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ABSTRACT. We establish a sharp Moser type inequality with logarithmic weight in the non-radial mass-weighted Sobolev spaces, on the whole plane $\mathbb{R}^2$. We identify the sharp threshold for the uniform boundedness of the weighted Moser functional, which is still given by 4π: further, we prove the validity of the inequality also at the limiting sharp value 4π. Even if the increasing nature of the log weight prevents the application of any symmetrization tool, we prove our inequality in the general framework of Sobolev space, and not on radial subspaces, as in the available literature. The main strategy is a careful analysis of the behaviour of the normalized maximizing sequences.
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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be an open domain with finite measure. It is well known that

$$\begin{cases} H_0^1(\Omega) \hookrightarrow L^p(\Omega) & \text{for } p \in [1, \infty); \\ H_0^1(\Omega) \not\hookrightarrow L^\infty & \text{for } p = \infty. \end{cases}$$

A counterexample is given by the function

$$u(x) = (-\log|\log|x||)_+,$$

when Ω is the unit ball. The maximal degree of summability for functions in $H_0^1(\Omega)$ was established independently by Pohožaev [32] and Trudinger [37] (see also [38]) and is of quadratic exponential type. Several years later, Moser [28] was able to simplify Trudinger's proof, and to determine the optimal threshold; more precisely,

$$\sup_{\|\nabla u\|_2 \le 1} \int_\Omega e^{\alpha u^2}\, dx \le M(\alpha)|\Omega|, \qquad \alpha \le 4\pi \tag{1}$$

where the constant $M(\alpha)$ stays bounded provided $\alpha \le 4\pi$ and the supremum becomes infinity when $\alpha > 4\pi$. While the proof of the validity of (1) for $\alpha < 4\pi$ is not difficult, it becomes very delicate when $\alpha = 4\pi$. This is usually done by showing, after reducing by symmetrization to the radial case, that if $\{v_k\}$ is a maximizing sequence, it can not be "too far" from the so-called Moser sequence. The same sequence is also used to prove the failure of (1) for $\alpha > 4\pi$.

Clearly, as the measure $|\Omega| \to +\infty$ no uniform bound can be retained in (1), and the exponential needs to be suitably regularized when u is near 0. The standard way to do this is to consider the reduced exponential $e^t - 1$. Then, by restricting to smooth functions such that $\|\nabla u\|_2 \le 1$ and $\|u\|_2 \le K$, $K > 0$, Cao [11] proved that

$$\sup_{\|\nabla u\|_2 \le 1,\, \|u\|_2 \le K} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx \le C(\alpha, K) < \infty \quad \text{ if } \alpha \le 4\pi(1-\delta), \tag{2}$$

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where $\delta \in (0,1)$ (see also [31]). A further result in this direction was obtained by Adachi-Tanaka in [1] which reads as follows: for all $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ one has

$$\int_{\mathbb{R}^2} \left(e^{\alpha \frac{u^2}{\|\nabla u\|_2^2}} - 1 \right) dx \le C(\alpha) \frac{\|u\|_2^2}{\|\nabla u\|_2^2}\,, \tag{3}$$

where

$$C(\alpha) < \infty \quad \text{as long as} \quad \alpha < 4\pi.$$

Inequalities (2) and (3) are usually named as *subcritical TM inequalities*, since they involve only values of the parameter $\alpha < 4\pi$. The *critical* Moser case in which $\alpha = 4\pi$ remained uncovered until Ruf in [33] established the following inequality

$$\sup_{\|\nabla u\|_2^2 + \|u\|_2^2 \le 1} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1 \right) dx < \infty \quad \text{ if } \alpha \le 4\pi, \tag{4}$$

which is sharp, namely the supremum becomes infinity as $\alpha > 4\pi$. Note that Ruf's inequality yields, as a byproduct, a uniform bound for the Moser functional which is *independent of* the measure of the domain when dealing with bounded domains Ω, provided that one considers the *complete* Sobolev norm instead of the Dirichlet norm, as in Moser's result.

Starting from these pioneering results, a very large amount of literature has been developed, and many interesting phenomena have been discovered: one of the most surprising is the attainability of the supremum in (1), in contrast with the classical Sobolev case, both in the bounded ([12, 21]) and in the unbounded ([22, 27, 33]) setting. This striking difference highlights the peculiarity of the 2-dimensional framework, which motivates a still very active line of research.

From the point of view of PDEs, differently from the Sobolev case, the exponent α in (1) does not play any role: the critical growth, in terms of threshold between existence and nonexistence of solutions, is represented by the quadratic exponential growth retained by the Orlicz class of functions underlying (1) (see [2, 20]). Motivated by the study of a Schrödinger-Poisson system in the plane which can be reduced to a Choquard type equation with logarithmic kernel and exponential nonlinearity, we recently proved in [7, 14] a new log-weighted version of the Trudinger inequality in the whole plane:

$$\sup_{\|u\|_w^2 \le 1} \int_{\mathbb{R}^2} \left(e^{2\pi u^2} - 1 \right) \log(e + |x|) dx < +\infty \tag{5}$$

where $\|u\|_w^2 = \int (|\nabla u|^2 + \log(e + |x|) u^2)$ is the Sobolev mass-weighted norm. It seems now natural to investigate the validity of a corresponding log-weighted Moser type inequality. Weighted Moser type inequalities have been already considered in the literature, starting from the pioneering work of Adimurthi and Sandeep [3] where the authors established the following singular Moser-Hardy inequality

$$\sup_{\|\nabla u\|_2 \le 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta}\, dx < +\infty, \qquad \text{if and only if} \quad \frac{\alpha}{4\pi} + \frac{\beta}{2} \le 1$$

on bounded domains containing the origin in $\mathbb{R}^2$, which has been generalized on the whole plane in [35]. Note that here the Hardy type weight is *singular and decreasing*: if, on one hand, the singularity affects the value of the sharp exponent, on the other hand it allows the usual reduction to radial functions. If, on the contrary, we aim at Moser inequalities involving *increasing* radial weights, as in (6), the application of standard symmetrization tools is avoided. For this reason, these kind of inequalities have been established, till now, only in the framework of *radial* Sobolev spaces (see, e.g., [23, 25, 29] and references therein). Up to our knowledge, even in the subcritical Trudinger setting, only Albuquerque [4] and the author [36] have addressed the question in the general framework of mass weighted Sobolev spaces, that is, without any a-priori restriction to radial functions (see also [17] for a close result).

The main goal of this paper is to prove a sharp, Moser type result in the whole plane in the special case of the log-weight, in the framework of log-mass weighted Sobolev spaces. More precisely, we will prove the following

Theorem 1. *Let $H^1_w(\mathbb{R}^2)$ be the weighted Sobolev space, endowed with the norm*

$$\|u\|_w^2 = \|\nabla u\|_2^2 + \|u\|_{2,w}^2 = \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} u^2 \log(e + |x|) dx.$$

Then

$$\sup_{\|u\|_w^2 \le 1} \int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1 \right) \log(e + |x|) dx < +\infty. \tag{6}$$

Inequality (6) *is sharp, that is, for any $\alpha > 4\pi$*

$$\sup_{\|u\|_w^2 \le 1} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1 \right) \log(e + |x|) dx = +\infty. \tag{7}$$

Remark 1. *We choose the weight $w = \log(e + |x|)$ instead of $\log(1 + |x|)$ because we want to avoid any role of the weight, except for its unboundedness at infinity. Indeed, the zero value of the function $\log(1 + |x|) \sim |x|$ at the origin may affect the Moser inequality, as in the Hénon cases, see* [10].

Remark 2. *As motivated above, we focus here on the effect of an increasing weight in the mass term. A complementary interest that has been recently developing in the literature is devoted to the impact of a weight w in the Dirichlet norm. The presence of a power weight in the gradient term affects the sharp exponent in the* radial *Moser inequality, as proved in* [10] *in the bounded domain framework. A much more relevant improvement can be obtained by considering a logarithmic weight, as established by Calanchi and Ruf (see e.g.* [8, 9]*). Note that one needs to restrict attention to* radial *functions in order to obtain any actual improvement of these limiting inequalities: otherwise, suitable translation and dilation of the Moser's sequence in a region far from the origin and far from the boundary, where the presence of can be "neglected", shows that the sharp exponent 4π cannot be enhanced. Roughly speaking, the presence of a radial weight in the Dirichlet norm may produce an improvement of the Moser inequalities if it can affect all the concentrating sequences: and this is possible only if the concentration occurs at the origin.*
The whole plane case have been recently considered in [5]*: the authors in Theorem 1.9 prove that the presence of a weight unbounded at infinity (independently from the growth rate) in the Dirichlet norm does not affect the sharp exponent in the Moser functional, even in the radial framework.*

Our proof is partially inspired to [18] where the authors deal with monomial increasing weights (even if the Moser type result, stated there, is affected by an erroneous application of rearrangement results). The main tool will be a transformation which relates functions in the weighted space $H^1_w(\mathbb{R}^2)$ to functions in the unweighted space $H^1(\mathbb{R}^2)$, based upon a change of variables acting only on the radial part of x: the price to pay is a *dilation* term in the Dirichlet norm, whose effect is *lower and lower* as $|x| \to \infty$ due to the logarithmic growth of the weight. This property is the key tool that allows to retain the sharp threshold 4π. Nevertheless, the discussion of the critical case $\alpha = 4\pi$ requires a deeper insight, which relies on a careful analysis of the behaviour of the maximizing sequences. Inspired by the concentration-compactness principle for Moser type inequalities on the whole plane stated by Černý in [15], we will deal with the following phenomena: no concentration (which includes compactness and vanishing), concentration on bounded domains, partial concentration at infinity and concentration at infinity. The last two are new phenomena which are not visible in the unweighted case, due to the standard reduction to the radial decreasing setting, first described by Chabrowski [16]: we will further split the analysis of sequences concentrating at infinity into concentration at points that diverges at infinity and spread concentration at infinity. The key tools to deal with these last two cases, the most delicate, will be a careful application of an improved version of Adachi-Tanaka inequality due to the author together with Cassani and Sani ([13], see subsection 3.3.1) and the Vitali Covering Lemma, together with suitable cut-off arguments.

We hope that the strategy adopted to deal with this special log-weight may be applied to more general weights, such as the *slowly varying* ones; we recall that a measurable function $w : (0, \infty) \to (0, \infty)$ is said to be slowly varying if

$$\lim_{r\to+\infty} \frac{w(ar)}{w(r)} = 1 \quad \forall\, a > 0,$$

such as $w(r) = \log^{\beta}(1 + r)$, for all $\beta > 0$. Since our proof relies on the lower effect of the dilation term near ∞ when applying the transformation $\mathcal{T}$, due to the logarithmic growth of the weight, we suspect that a similar phenomenon could be observed in the more general framework of slowly varying weights.
Finally, we believe that the higher dimensional limiting Sobolev case $W^{1,N}(\mathbb{R}^N)$ can be handled with similar tools. We hope that the analysis performed in the last section may be useful to approach other related and challenging questions, such as the attainability of log-weighted Moser type inequalities.

2. Sharp subcritical inequalities

In this section we aim at identifying the sharp threshold between uniform boundedness and unboundedness of the log-weighted Moser functional, so improving the result stated in [14, 36] up to 4π.
Let $H^1_w(\mathbb{R}^2)$ be the space of measurable functions defined as

$$H^1_w(\mathbb{R}^2) := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2 \log(e + |x|)dx < +\infty \right\} \tag{8}$$

equipped with the norm

$$\|u\|_w^2 := \|\nabla u\|_2^2 + \|u\|_{2,w}^2 = \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} u^2 \log(e + |x|))dx. \tag{9}$$

With the notations of [24], H^1_w is nothing that $W^{1,2}(\mathbb{R}^2, S)$ where S is the set of weights given by $S = \{\log(e + |x|), 1\}$. Since the weight $w = \log(e + |x|)$ satisfies the condition $w^{-1} \in L^1_{loc}(\mathbb{R}^2)$, it turns out that $H^1_w(\mathbb{R}^2)$ is a Banach space ([24, Theorem 1.11]), and further, it is a Hilbert space, endowed with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^2} \nabla u \nabla v dx + \int_{\mathbb{R}^2} uv \log(e + |x|)dx.$$

Further, its dual can be characterized thanks to the Hahn Banach Theorem as

$$H^{-1}_w(\mathbb{R}^2) = \left(H^1(\mathbb{R}^2) \cap L^2_w(\mathbb{R}^2)\right)' = H^{-1}(\mathbb{R}^2)|_{H^1_w} + (L^2_w)'(\mathbb{R}^2)|_{H^1_w}$$

(see Theorem 14.9 in [34]).
We briefly recall the proof of inequality (6) given in [14], since its main tools will be used frequently in what follows; we also slightly modify the change of variable originally introduced.

Proposition 1. *[Theorem 1.1 in* [14]*] For any $\alpha \leq 2\pi$*

$$\sup_{\|u\|_w^2 \leq 1} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \log(e + |x|)dx < +\infty. \tag{10}$$

Furthermore,

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \log(e + |x|) dx < \infty \tag{11}$$

for any $u \in H^1_w(\mathbb{R}^2)$ *and any* $\alpha > 0$.

Proof of Proposition 1. Let us perform a change of variable which acts only on the radial part of any point in $\mathbb{R}^2$. To shorten the notation we will write

$$s = T(r) = \sqrt{2 \int_0^r \rho \log(e + \rho)\ d\rho}, \text{ where } \ r = |x|,\ s = |y|, \quad x \mapsto y := \frac{x}{|x|} T(|x|).$$

Note that $T(r)$ can be explicitly determined, even if not strictly necessary; since

$$2 \int_0^r \rho \log(e + \rho)\ d\rho = r^2 \log(e + r) - \int_0^r \frac{\rho^2}{e + \rho}\ d\rho \quad \text{and} \quad 0 \le \frac{\rho^2}{e + \rho} \le \rho,$$

we easily get

$$r\sqrt{\log(e + r) - \frac{1}{2}} \le T(r) \le r\sqrt{\log(e + r)} \quad \text{and} \quad \frac{T(r)}{r\sqrt{\log(e + r)}} \longrightarrow 1 \text{ as } \ r \to +\infty \tag{12}$$

where $r\sqrt{\log(e + r)}$ is the former change of variable suggested in [14]. The transformation T is invertible on $\mathbb{R}^2$, even if its inverse map is not explicit. Let us define

$$v(y) := u(x), \quad \text{that is,} \quad v(y) = u\left(T^{-1}(|y|)\cos\theta, T^{-1}(|y|)\sin\theta\right)$$

and denote by

$$w(r, \theta) := u(r\cos\theta, r\sin\theta), \quad \widetilde{w}(s, \theta) := v\left(s\cos\theta, s\sin\theta\right).$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla v|^2 dy_1 dy_2 &= \int_0^{2\pi} \int_0^{+\infty} \left[\widetilde{w}_s^2 + \frac{\widetilde{w}_\theta^2}{s^2}\right] s ds d\theta \\ &= \int_0^{2\pi} \int_0^{+\infty} \left[w_r^2(r, \theta) \cdot \frac{1}{[T'(r)]^2} + \frac{w_\theta^2(r, \theta)}{r^2} \cdot \frac{r^2}{T^2(r)}\right] T'(r) T(r) dr d\theta.\end{aligned}$$

Now,

$$T'(r) = \frac{r \log(e + r)}{T(r)} \implies \frac{1}{2\log(e + r)} \le \frac{1}{[T'(r)]^2} \le \frac{1}{\log(e + r)} \tag{13}$$

whereas

$$\frac{1}{\log(e + r)} \le \frac{r^2}{T^2(r)} \le \frac{1}{\log(e + r) - \frac{1}{2}} \le \frac{2}{\log(e + r)}, \tag{14}$$

so that, at the end,

$$\frac{1}{4} \frac{r^2}{T^2(r)} < \frac{1}{[T'(r)]^2} < \frac{r^2}{T^2(r)}. \tag{15}$$

Then,

$$\frac{1}{4} \int_0^{2\pi} \int_0^{+\infty} \left[w_r^2 + \frac{w_\theta^2}{r^2}\right] \frac{r^2 T'(r)}{T(r)} dr d\theta \le \int_{\mathbb{R}^2} |\nabla v|^2 dy_1 dy_2 \le \int_0^{2\pi} \int_0^{+\infty} \left[w_r^2 + \frac{w_\theta^2}{r^2}\right] \frac{r^2 T'(r)}{T(r)} dr d\theta.$$

By (13), (14)

$$\frac{r^2 T'(r)}{T(r)} = \frac{r^2}{[T(r)]^2} \cdot r \log(e + r) \implies r < \frac{r^2 T'(r)}{T(r)} < 2r$$

which implies

$$\frac{1}{4}\int_{\mathbb{R}^2} |\nabla u|^2 dx_1 dx_2 < \int_{\mathbb{R}^2} |\nabla v|^2 dy_1 dy_2 < 2\int_{\mathbb{R}^2} |\nabla u|^2 dx_1 dx_2.$$

On the other hand, thanks to (13)

$$\int_{\mathbb{R}^2} v^2 dy = \int_0^{2\pi}\int_0^{\infty} \widetilde{w}^2(s,\theta) s ds d\theta = \int_0^{2\pi}\int_0^{\infty} w^2(r,\theta)T'(r)T(r) dr d\theta$$
$$= \int_0^{2\pi}\int_0^{\infty} w^2(r,\theta) r\log(e+r) dr d\theta = \int_{\mathbb{R}^2} u^2 \log(e+|x|)dx,$$

so that, finally,

$$\frac{1}{4}\|u\|_w^2 < \|v\|^2 = \|\nabla v\|_2^2 + \|v\|_2^2 < 2\|u\|_w^2. \tag{16}$$

We have then proved that the map

$$\begin{array}{rcl}\mathcal{T} : H^1_w(\mathbb{R}^2) & \to & H^1(\mathbb{R}^2) \\ u & \mapsto & v\end{array}$$

is an invertible, continuous map, with continuous inverse map too. As before, it is easy to verify that

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2}-1\right)\log(e+|x|)dx = \int_{\mathbb{R}^2} \left(e^{\alpha v^2}-1\right)dx < +\infty$$

by [33], for any $\alpha > 0$, that is (11). The uniform bound (10) follows directly by combining Ruf's inequality (4) with the norm's estimate (16). □

Note that, thanks to (12) and (13), the scaling factor appearing when changing variable in the Dirichlet norm has an interesting property:

$$\frac{rT'(r)}{T(r)} = \frac{r^2\log(e+r)}{T^2(r)} \longrightarrow 1 \quad \text{as} \quad r\to+\infty. \tag{17}$$

This observation is essentially related to the nature of the weight, and it is the key tool which allows us to improve the previous inequality up to the (sharp) exponent 4π, as stated in the following.

Proposition 2. *For any* $\alpha < 4\pi$

$$\sup_{\|u\|_w^2\le 1}\int_{\mathbb{R}^2} \left(e^{\alpha u^2}-1\right)\log(e+|x|)dx < +\infty. \tag{18}$$

The inequality is sharp, that is,

$$\sup_{\|u\|_w^2\le 1}\int_{\mathbb{R}^2} \left(e^{\alpha u^2}-1\right)\log(e+|x|)dx = +\infty \tag{19}$$

for any $\alpha > 4\pi$.

Proof. The main idea of the proof is to take advantage of the property (17): since α is strictly less then 4π, we have room enough to split any function $u \in H^1_w$ as the sum of two functions, the first one compactly supported in a uniform bound domain and the second one supported far from the origin, where we will perform the change of variable T.
Let us consider a smooth, radial cut-off function $\chi(|x|)$, such that

$$\chi(|x|) = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{if } |x|>2\end{cases}$$

and $0 \le \chi \le 1$ with bounded derivative. Set $\xi(|x|) := 1 - \chi(|x|))$; then scale ξ as follows:

$$\xi_\eta(|x|) := \xi(\eta|x|),$$

so that

$$\xi_\eta(|x|) = \begin{cases} 0 & \text{if } |x| < 1/\eta \\ 1 & \text{if } |x| > 2/\eta, \end{cases}$$

and $|\nabla \xi_\eta(|x|)| = \eta |\nabla \xi(\eta|x|)| = \eta |\nabla \chi(\eta|x|)|$. For any $u \in H^1_w(\mathbb{R}^2)$ with $\|u\|^2_w \le 1$ let us define

$$u_\eta := u \cdot \xi_\eta$$

whose support is contained in $B^{\complement}_{1/\eta}$. We have

$$\int_{\mathbb{R}^2} |\nabla u_\eta|^2 \le \|\nabla u\|_2^2 + 2\eta \|\nabla \chi\|_\infty \|u\|_2 \|\nabla u\|_2 + \eta^2 \|\nabla \chi\|^2_\infty |\|u\|_2^2,$$

so that

$$\|u_\eta\|^2_w \quad \le \quad (1 + c\eta)\|u\|^2_w \le (1 + c\eta) \tag{20}$$

for some positive fixed constant c independent of η. Let $\alpha < 4\pi$ fixed. Obviously,

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \log(e + |x|) \le \int_{B_{2/\eta}} \left(e^{\alpha u^2} - 1\right) \log(e + |x|) + \int_{\mathbb{R}^2} \left(e^{\alpha u_\eta^2} - 1\right) \log(e + |x|).$$

By Ruf's inequality

$$\int_{B_{2/\eta}} \left(e^{\alpha u^2} - 1\right) \log(e + |x|) \le \log(e + \frac{2}{\eta}) \int_{B_{\frac{2}{\eta}}} \left(e^{\alpha u^2} - 1\right) \le$$

$$\le \log(e + \frac{2}{\eta}) \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \le \frac{C}{\eta^2}$$

since $\|\nabla u\|_2^2 + \|u\|_2^2 \le \|u\|^2_w \le 1$. To bound the second term, let us apply the change of variable introduced in the proof of Proposition 1 to u_η, taking advantage of the fact that u_η is supported far from the origin. Using the same notations as in Proposition 1, let v_η be defined as

$$u_\eta(r\cos\theta, r\sin\theta) = v_\eta\left(T(r)\cos\theta, T(r)\sin\theta\right).$$

Then

$$\int_{\mathbb{R}^2} |\nabla v_\eta|^2 dy_1 dy_2 \;=\; \int_0^{2\pi} \int_{T(\frac{1}{\eta})}^{+\infty} \left[\widetilde{w}_s^2 + \frac{\widetilde{w}_\theta^2}{s^2}\right] s ds d\theta \;\le\; \int_0^{2\pi} \int_{\frac{1}{\eta}}^{+\infty} \left[w_r^2 + \frac{w_\theta^2}{r^2}\right] \frac{r^2 T'(r)}{T(r)} dr d\theta.$$

If η is small enough and $r > 1/\eta$, by (13), (14)

$$r \le \frac{r^2 T'(r)}{T(r)} \le r \frac{\log(e + r)}{\log(e + r) - \frac{1}{2}} \Longrightarrow r \le \frac{r^2 T'(r)}{T(r)} < \left(1 + \frac{1}{2\log(1 + \frac{1}{\eta})}\right) r,$$

so that, at the end,

$$\int_{\mathbb{R}^2} |\nabla v_\eta|^2 dy_1 dy_2 < \left(1 + \frac{1}{2\log(1 + \frac{1}{\eta})}\right) \int_{\mathbb{R}^2} |\nabla u_\eta|^2 dx_1 dx_2. \tag{21}$$

Since

$$\int_{\mathbb{R}^2} v_\eta^2 dy = \int_{\mathbb{R}^2} |u_\eta|^2 \log(e + |x|) dx$$

we conclude that

$$\|v_\eta\|^2 = \|\nabla v_\eta\|_2^2 + \|v_\eta\|_2^2 < \left(1 + \frac{1}{2\log(1 + \frac{1}{\eta})}\right) \|u_\eta\|^2_w \le \left(1 + \frac{1}{2\log(1 + \frac{1}{\eta})}\right)(1 + c\eta)$$

by (20). Now, let us fix η such that

$$\left(1+\frac{1}{2\log(1+\frac{1}{\eta})}\right)(1+c\eta)<\frac{4\pi}{\alpha},$$

so that $\alpha\|v_\eta\|^2<4\pi$. Then, since $T'T=r\log(e+r)$, we have

$$\int_{\mathbb{R}^2}\left(e^{\alpha u_\eta^2}-1\right)\log(e+|x|)=\int_0^{2\pi}\int_{\frac{1}{\eta}}^{+\infty}\left(e^{\alpha u_\eta^2(r\cos\theta,r\sin\theta)}-1\right)\log(e+r)rdrd\theta$$

$$=\int_{\mathbb{R}^2}\left(e^{\alpha\|v_\eta\|^2v_\eta^2/\|v_\eta\|^2}-1\right)dx<C(\alpha)$$

by Ruf's inequality (4), as $\alpha\|v_\eta\|^2<4\pi$.

The sharpness of 4π can be verified, as usually, by means of a suitable Moser type sequence. Let

$$v_n(x):=\frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\delta_n\log n} & 0<|x|\le\frac{1}{n}\\ \frac{\sqrt{\delta_n}}{\sqrt{\log n}}\log\frac{1}{|x|} & \frac{1}{n}<|x|<1\\ 0 & |x|\ge 1\end{cases}\tag{22}$$

where $\delta_n\in(0,1)$ will be fixed later. Then

$$\|\nabla v_n\|_2^2=\frac{\delta_n}{\log n}\int_0^{1/n}\frac{1}{r}dr=\delta_n,$$

whereas

$$\int_{\mathbb{R}^2}v_n^2\log(e+|x|)dx=\int_{B_1}v_n^2\log(e+|x|)dx\le 2\int_{B_1}v_n^2dx$$

$$=\frac{\log n}{n^2}\delta_n+\frac{2\delta_n}{\log n}\int_{1/n}^1 r\log^2\frac{1}{r}\,dr=\left(\frac{1}{2\log n}+\frac{2\log n}{n^2}-\frac{1}{n^2}-\frac{1}{2n^2\log n}\right)\delta_n,$$

so that

$$\|v_n\|_{2,w}^2\le\left(1+\frac{1}{2\log n}+\frac{2\log n}{n^2}-\frac{1}{n^2}-\frac{1}{2n^2\log n}\right)\delta_n.$$

If we choose

$$\delta_n:=\left(1+\frac{1}{2\log n}+\frac{2\log n}{n^2}-\frac{1}{n^2}-\frac{1}{2n^2\log n}\right)^{-1}=1-\frac{1}{2\log n}+\mathrm{O}(n^{-2}\log n)\tag{23}$$

we have $\|v_n\|_w\le 1$. Thanks to (23), for any $\alpha>4\pi$ there is an n_α such that if $n>n_\alpha$

$$\frac{\alpha}{2\pi}\delta_n>\frac{\alpha}{2\pi}\left(1-\frac{1}{\log n}\right)>2+\frac{\alpha-4\pi}{4\pi},$$

so that

$$\int_{\mathbb{R}^2}\left(e^{\alpha v_n^2}-1\right)\log(e+|x|)dx\ge 2\pi\int_0^{1/n}\left(e^{\frac{\alpha}{2\pi}\delta_n\log n}-1\right)\log(e+r)rdr$$

$$\ge\pi\int_0^{1/n}n^2e^{\frac{\alpha-4\pi}{4\pi}\log n}\log(e+r)rdr\ge\frac{\pi}{2}n^{\frac{\alpha-4\pi}{4\pi}}\to+\infty\text{ as }n\to+\infty,$$

which ends the proof. ∎

We end this section by proving a variant of Cao's inequality (2) available in our log-weighted setting. As before, since the parameter α involved is always less than 4π, there will have again 'room enough' to perform a change of variable when x is large, obtaining a new function whose Dirichlet norm is larger than the former, but still less than 1.

Proposition 3. *For any $\delta \in (0,1)$ and $M > 0$*

$$\sup_{\substack{u \in H^1_w(\mathbb{R}^2) \\ \|\nabla u\|_2 \le 1-\delta,\, \|u\|_{2,w} \le M}} \int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right) \log(e+|x|)dx = C(\delta, M) < +\infty. \tag{24}$$

The inequality is sharp, that is,

$$\sup_{\substack{u \in H^1_w(\mathbb{R}^2) \\ \|\nabla u\|_2 < 1,\, \|u\|_{2,w} \le M}} \int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right) \log(e+|x|)dx = +\infty. \tag{25}$$

Proof. The proof follows arguments similar to the previous one. Let $M > 0$ and $0 < \delta < 1$ be fixed. For any $\eta > 0$, $u \in H^1_w(\mathbb{R}^2)$ with $\|\nabla u\|_2 \le 1-\delta$, $\|u\|_{2,w} \le M$ let $u_\eta := u \cdot \xi_\eta$ as before, and, again, v_η

$$u_\eta(r\cos\theta, r\sin\theta) = v_\eta\left(T(r)\cos\theta, T(r)\sin\theta\right)$$

as in Proposition 1. Then, for any $\eta > 0$,

$$\int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right) \log(e+|x|) = \int_{B_{2/\eta}} \left(e^{4\pi u^2} - 1\right) \log(e+|x|) + \int_{B^C_{2/\eta}} \left(e^{4\pi u^2} - 1\right) \log(e+|x|).$$

Let's estimate the two terms separately. By Cao's inequality (2),

$$\begin{aligned} \int_{B_{2/\eta}} \left(e^{4\pi u^2} - 1\right) \log(e+|x|) &\le \log(e+\frac{2}{\eta}) \int_{B_{\frac{2}{\eta}}} \left(e^{4\pi u^2} - 1\right) \le \\ &\le \log(e+\frac{2}{\eta}) \int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right) \le C(\delta, M)|\log\eta|. \end{aligned} \tag{26}$$

On the other hand, if $\eta < 1/M$,

$$\begin{aligned} \|\nabla u_\eta\|_2^2 &\le \|\nabla u\|_2^2 + 2\eta\|\nabla\chi\|_\infty\|u\|_2\|\nabla u\|_2 + \eta^2\|\nabla\chi\|_\infty^2\|u\|_2^2 \\ &\le (1+cM\eta)\|\nabla u\|_2^2 + c\eta^2M^2 \\ &\le (1+cM\eta)(1-\delta) + c\eta^2M^2 \le (1+cM\eta)(1-\delta) + c\eta M \quad \text{since} \quad \eta < 1/M \\ &\le 1-\delta+cM\eta(2-\delta) \le 1-\frac{\delta}{2} \end{aligned}$$

if $\eta < \min(\frac{1}{M}, \frac{\delta}{2cM(1-\delta)})$. Further, eventually choosing η smaller (depending on δ and M)

$$\|\nabla v_\eta\|_2^2 < \left(1 + \frac{1}{2\log(1+\frac{1}{\eta})}\right) \cdot \|\nabla u_\eta\|_2^2 \le \left(1 + \frac{c}{|\log(\eta)|}\right)\left(1-\frac{\delta}{2}\right) < 1 - \frac{\delta}{4},$$

whereas

$$\|v_\eta\|_2^2 \le \left(1+\frac{c}{|\log\eta|}\right)\|u_\eta\|_{2,w}^2 \le \left(1+\frac{c}{|\log\eta|}\right)\|u\|_{2,w}^2 \le \left(1+\frac{c}{|\log\eta|}\right)M^2.$$

In what follows, recall that $u_\eta(x) = v_\eta(x)$ for any x s.t. $|x| \ge 2/\eta$, and that supp $u_\eta = B^C_{1/\eta}$; further, as in the proof of Proposition 1, denote by

$$w(r,\theta) := u(r\cos\theta, r\sin\theta), \quad \widetilde{w}(s,\theta) := v\left(s\cos\theta, s\sin\theta\right).$$

Then, for any η small enough, depending only on M and δ, we get

$$\begin{aligned}
(27)\quad \int_{B^C_{2/\eta}} \left(e^{4\pi u^2}-1\right)\log(e+|x|)\,dx &\le \int_{B^C_{2/\eta}} \left(e^{4\pi u_\eta^2}-1\right)\log(e+|x|)\,dx \\
&\le \int_{B^C_{1/\eta}} \left(e^{4\pi u_\eta^2}-1\right)\log(e+|x|)\,dx = \int_{\mathbb{R}^2} \left(e^{4\pi u_\eta^2}-1\right)\log(e+|x|)\,dx \\
&= \int_0^{2\pi}\int_0^{+\infty} \left(e^{4\pi w_\eta^2}-1\right)\log(e+r)r\,drd\theta = \int_0^{2\pi}\int_0^{+\infty} \left(e^{4\pi \widetilde{w}_\eta^2}-1\right) s\,dsd\theta \\
&= \int_{\mathbb{R}^2} \left(e^{4\pi v_\eta^2}-1\right)\,dx = \int_{B^C_{T(1/\eta)}} \left(e^{4\pi v_\eta^2}-1\right)dx \le C(\delta, M, \eta)
\end{aligned}$$

by Cao's inequality (2). Combining the two estimates (26), (27) with a fixed η (small enough) yields the statement.

To prove the sharpness of inequality (24) we use the sequence of functions introduced in [13]. Since we cannot rely on dilation argument in our weighted framework, we have to perform estimates directly by hand. Let B_{R_n} be the ball of radius R_n, where

$$(28)\qquad R_n := \frac{\sqrt{\log n}}{\log\log n} \longrightarrow \infty, \quad \delta_n := 1 - \frac{\log\log n}{4\log n} \longrightarrow 1^- \quad \text{as } n \to \infty,$$

and consider the sequence of radial functions

$$u_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \dfrac{\sqrt{\delta_n}}{\sqrt{\log n}} \log\left(\dfrac{R_n}{|x|}\right), & \dfrac{R_n}{n} < |x| \le R_n \\[2ex] \sqrt{\delta_n \log n}, & 0 \le |x| \le \dfrac{R_n}{n}. \end{cases}$$

Then

$$\|\nabla u_n\|_2^2 = \delta_n \longrightarrow 1^- \quad \text{as } n \to \infty,$$

whereas, integrating by parts and erasing all the negative terms, we have

$$\begin{aligned}
\|u_n\|_{2,w}^2 &= \delta_n \log n \int_0^{R_n/n} r\log(e+r)dr + \frac{\delta_n}{\log n}\int_{R_n/n}^{R_n} r\log^2\left(\frac{R_n}{r}\right)\log(e+r)dr \\
&\le \delta_n \frac{\log n}{2n^2} R_n^2 \log(e+\frac{R_n}{n}) + \frac{\delta_n}{\log n}\int_{R_n/n}^{R_n} r\log\left(\frac{R_n}{r}\right)\log(e+r)dr \\
&\le \delta_n \frac{\log n}{n^2} R_n^2 + \frac{\delta_n}{\log n}\int_{R_n/n}^{R_n} \frac{r}{2}\log(e+r)dr \\
&\le \delta_n \frac{\log n}{n^2} R_n^2 + \frac{\delta_n}{\log n}\frac{R_n^2}{4}\log(e+R_n) \to 0
\end{aligned}$$

by (28). On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}^2} \left(e^{4\pi u_n^2}-1\right)\log(e+|x|)\,dx &\ge 2\pi \int_0^{R_n/n} \left(e^{2\delta_n \log n}-1\right) r\log(e+r)\,dr \\
&\ge \pi e^{2\delta_n \log n}\int_0^{R_n/n} r\,dr = \frac{\pi}{2}R_n^2 e^{-2(1-\delta_n)\log n} = \frac{\pi}{2}R_n^2 e^{-\frac{1}{2}\log\log n} \\
&= \frac{\pi}{2}\frac{R_n^2}{\sqrt{\log n}} \longrightarrow +\infty \quad \text{as } n\to\infty.
\end{aligned}$$

□

Remark 3. *Note that, in the classical, unweighted setting, Adachi-Tanaka inequality* (3) *can be interpreted as a scale-invariant form of Moser's type inequalities; indeed, if we set*

$$J_\alpha(u) := \frac{1}{\|u\|_2^2}\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)dx, \quad u_\lambda(x) := u(\lambda x),$$

then we have

$$\|\nabla u_\lambda\|_2 = \|\nabla u\|_2, \quad J_\alpha(u_\lambda) = J_\alpha(u), \quad \forall\,\lambda\in\mathbb{R} \tag{29}$$

and Adachi-Tanaka inequality (3) *can be written in a scaling invariant form as*

$$\sup_{\substack{u\in H^1(\mathbb{R}^2)\setminus\{0\},\\ \|\nabla u\|_2\le 1}} J_\alpha(u) < \infty \quad \textit{if and only if} \quad \alpha < 4\pi.$$

The same scaling can be applied when proving Cao's inequality (2)*, also noting that* $J_\alpha(bu) = J_{b^2\alpha}(u)$ *for all* $b\in\mathbb{R}$ *(see* [13]*). This scaling have a key role when describing the lack of compactness phenomena of sequences in* $H^1(\mathbb{R}^2)$*, as clearly described by Ishiwata in* [22]*. By means of the transformation* $\mathcal{T}: H^1_w \to H^1$ *introduced in this Section, one may identify a nonlinear replacement of the standard scaling* $u\to u_\lambda$ *in the weighted framework, namely*

$$\tilde u_\lambda(x) = \mathcal{T}^{-1}\left((\mathcal{T}u)_\lambda\right)(x)$$

which preserves the nature *of the sequences (vanishing or concentrating) and the invariance property*

$$\frac{1}{\|\tilde u_\lambda\|_{2,w}^2}\int_{\mathbb{R}^2}\left(e^{\alpha\tilde u_\lambda^2}-1\right)\log(e+|x|)\,dx = \frac{1}{\|u\|_{2,w}^2}\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)\log(e+|x|)\,dx;$$

nevertheless, the Dirichlet norm does not enjoy any invariance property. Note, finally, that our weighted setting seems not enjoy any (at least evident) analogous property as (29).

3. The critical case $\alpha = 4\pi$: proof of Theorem 1

The aim of this section is to prove our main theorem, which deals with the critical case $\alpha = 4\pi$. Let us first note that, if the supremum in (6) is attained, then the statement is a direct consequence of (11). Therefore, it remains to consider the opposite case: in the following we will analyse the behaviour of any maximizing normalized sequence $\{u_n\}$, in the spirit of [15] (where we say that a sequence u_n is normalized if $\|u_n\|_w = 1$ for any $n\in\mathbb{N}$). As recalled in the Introduction, we will deal with a new phenomenon, first described by Chabrowski in [16] and then studied by Černý in [15]: the *concentration at infinity*. Compared to these two former papers, our analysis will be more detailed; more precisely, we classify the behaviour of any normalized maximizing sequence $\{u_n\}$ as follows (up to subsequences):

- *no concentration*: for some $\delta\in(0,1)$, $\|\nabla u_n\|_2^2 \le 1-\delta$ for any n;
- *concentration on a bounded domain*: for some $R>0$
$$\int_{B_R(0)}|\nabla u_n|^2 \to 1 \quad\text{as}\quad n\to+\infty;$$
- *partial concentration at infinity*:
$$\|\nabla u_n\|_2^2\to 1 \quad\text{and}\quad \lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{|x|>R}|\nabla u_n|^2 \in (0,1);$$

- *concentration at infinity*:

$$\|\nabla u_n\|_2^2 \to 1 \quad \text{and} \quad \lim_{R\to+\infty} \limsup_{n\to+\infty} \int_{|x|>R} |\nabla u_n|^2 = 1.$$

We will further specify this last class in two different sub-classes, see section 3.3.

The first case is the easiest one: if no concentration phenomenon occurs, then the statement follows directly from Proposition 3. So, let us consider the remaining ones.

3.1. **Concentration on a bounded domain.** We consider here maximizing sequence concentrating on a bounded domain. That is, we suppose that there exists $R > 0$ such that

$$\int_{B_R(0)} |\nabla u_n|^2 \to 1 \quad \text{as} \quad n \to +\infty.$$

This implies that

$$\int_{|x|>R} |\nabla u_n|^2 \to 0, \qquad \|u_n\|_{2,w} \to 0 \quad \text{as} \quad n \to +\infty.$$

Obviously,

(30)

$$\begin{aligned}\int_{\mathbb{R}^2} \left(e^{4\pi u_n^2} - 1\right) \log(e+|x|) &\le \log(e+2R) \int_{B_{2R}} \left(e^{4\pi u_n^2} - 1\right) + \int_{|x|>2R} \left(e^{4\pi u_n^2} - 1\right) \log(e+|x|) \\ &\le C \cdot \log(e+2R) + \int_{|x|>2R} \left(e^{4\pi u_n^2} - 1\right) \log(e+|x|)\end{aligned}$$

by Ruf's inequality, since $\|u_n\|_{H^1}^2 = \|\nabla u_n\|_2^2 + \|u_n\|_2^2 \le \|u\|_w^2 \le 1$.
To bound the second term in (30), let us consider a $\mathcal{C}^1$-piecewise radial monotone cut-off function

$$\chi(|x|) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 2 \end{cases}, \qquad \|\nabla \chi\|_\infty \le 1/2$$

as in the proof of Proposition 2, and the associated function $\xi(|x|) := 1 - \chi(|x|))$, together with $\xi_R(|x|) := \xi(|x|/R)$. Set $u_n^R = u_n \cdot \xi_R$: the function u_n^R has support in $\{|x| > R\}$, and for any $\varepsilon > 0$ there is n_ε such that for $n \ge n_\varepsilon$

$$\|u_n\|_2^2 < \varepsilon, \qquad \int_{|x|>R} |\nabla u_n|^2 < \varepsilon,$$

which yields directly

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u_n^R|^2 \le \int_{|x|>R} |\nabla u_n|^2 + \frac{2}{R} \|\nabla \chi\|_\infty \int_{R<|x|<2R} |u_n||\nabla u_n| + \frac{1}{R^2} \|\nabla \chi\|_\infty^2 \| |u_n\|_2^2 \\ \le \varepsilon + \frac{\varepsilon}{R} + \frac{\varepsilon}{4R^2} \le \frac{1}{2}\end{aligned}$$

for a proper choice of ε. Since u_n^R is supported on $\{|x| > R\}$, if we consider the change of variable T^{-1} introduced in the proof of Proposition 1 we obtain a sequence of functions v_n such that

$$u_n^R(x) := v_n(T(x)), \quad \|v_n\|_2^2 = \|u_n^R\|_{2,w} \le \|u_n\|_{2,w}^2 = \mathrm{o}(1),$$

$$\|\nabla v_n\|_2^2 \le \left(1 + \frac{1}{2\log(1+R)}\right) \|\nabla u_n^R\|_2^2 \le \frac{1}{2},$$

which yields $\|v_n\|_w^2 < 1$. Applying again Ruf's inequality we obtain

$$\int_{|x|>2R} \left(e^{4\pi u_n^2} - 1\right) \log(e+|x|) < \int_{|x|>R} \left(e^{4\pi (u_n^R)^2} - 1\right) \log(e+|x|) = \int_{\mathbb{R}^2} \left(e^{4\pi v_n^2} - 1\right) < C$$

which yields the statement, thanks to (30).

3.2. **Partial concentration at infinity.** We deal here with maximizing sequence such that

$$\|\nabla u_n\|_2^2 \to 1 \quad \text{and} \quad \lim_{R\to+\infty} \limsup_{n\to+\infty} \int_{|x|>R} |\nabla u_n|^2 := A_\infty \in (0,1).$$

Eventually considering subsequences, we assume that the lim sup in the previous condition is actually a limit. Hence, for any $\varepsilon > 0$ there are R_ε such that for any $R > R_\varepsilon$ there is a $n_{R,\varepsilon}$ with

$$\int_{|x|>R} |\nabla u_n|^2 \in (A_\infty - \varepsilon, A_\infty + \varepsilon) \qquad \forall\, n > n_{R,\varepsilon}.$$

Fix now ε_0 such that $A_\infty - 2\varepsilon_0 > 0, A_\infty + 2\varepsilon_0 < 1$. Since $\|\nabla u_n\|_2^2 \to 1$ we have also

$$\int_{|x|<R} |\nabla u_n|^2 < 1 + \varepsilon_0 - \int_{|x|>R} |\nabla u_n|^2 < 1 - A_\infty + 2\varepsilon_0 < 1, \quad \text{if } n > \max(n_0, n_{R,\varepsilon_0}).$$

Applying the same argument used to prove the previous case, we have

$$\int_{B_{2R}} \left(e^{4\pi u_n^2} - 1\right) \log(e + |x|) \le \log(e + 2R) \int_{B_{2R}} \left(e^{4\pi u_n^2} - 1\right) < C \cdot \log(e + 2R)$$

and

$$\int_{|x|>2R} \left(e^{4\pi u_n^2} - 1\right) \log(e + |x|) < \int_{|x|>R} \left(e^{4\pi (u_n^R)^2} - 1\right) \log(e + |x|) = \int_{\mathbb{R}^2} \left(e^{4\pi v_n^2} - 1\right),$$

where the last term on the right hand side is uniformly bounded if, for some $\delta < 1$

$$\|\nabla v_n\|_2^2 \le \left(1 + \frac{1}{2\log(1+R)}\right) \|\nabla u_n^R\|_2^2 \le \delta.$$

But this estimate can be easily verified choosing R large enough and $n > n_R$, since

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u_n^R|^2 \le \int_{|x|>R} |\nabla u_n|^2 + \frac{2}{R}\|\nabla\chi\|_\infty \int_{R<|x|<2R} |u_n||\nabla u_n| + \frac{1}{R^2}\|\nabla\chi\|_\infty^2 \||u_n\|_2^2 \\ \le A_\infty + \varepsilon_0 + \frac{1}{R} + \frac{1}{4R^2} \le A_\infty + 2\varepsilon_0 < 1\end{aligned}$$

by assumption.

3.3. **Concentration at infinity.** We deal here with maximizing sequences such that

$$\|\nabla u_n\|_2^2 \to 1 \quad \text{and} \quad \lim_{R\to+\infty} \limsup_{n\to+\infty} \int_{|x|>R} |\nabla u_n|^2 = 1.$$

We further distinguish between sequences *concentrating at points running at infinity or not*, that is, between the two following behaviours:

- for any $\rho, \delta < 1$ (small) there are a subsequence u_{n_k} and a sequence of points x_{δ,ρ,n_k} such that
$$\int_{B_\rho(x_{\rho,\delta,n_k})} |\nabla u_{n_k}|^2 > \delta;$$
- there are two constants $\rho, \delta \in (0,1)$ such that for any $x \in \mathbb{R}^2$ and for any n
$$\int_{B_\rho(x)} |\nabla u_n|^2 \le \delta.$$

3.3.1. *Concentration at points $x_n \to \infty$.* Let us suppose that the maximizing sequence satisfy the first condition: by choosing $\rho = \frac{1}{2}, \delta = \frac{1}{2}$, up to subsequence, for any n there is a point x_n such that

$$\int_{B_{\frac{1}{2}}(x_n)} |\nabla u_n|^2 > \frac{1}{2}$$

and $x_n \to \infty$. To get a uniform bound for the Moser functional, we split the integral as the sum of two terms:

$$\int_{\mathbb{R}^2} \left(e^{4\pi u_n^2} - 1\right) \log(e + |x|) = \int_{B_1(x_n)} \left(e^{4\pi u_n^2} - 1\right) \log(e + |x|) + \int_{B_1^{\complement}(x_n)} \left(e^{4\pi u_n^2} - 1\right) \log(e + |x|). \tag{31}$$

In the first term, since the domain of integration is $B_1(x_n)$, the log-weight behaves like $\log |x_n|$: the uniform bound will follow by applying a refinement of Adachi-Tanaka inequality proved by the author together with Cassani and Sani in [13], that we recall here for the reader's convenience, in a slightly different, but equivalent statement:

Theorem 2 (Theorem 1.2 in [13])**.** *There exists $C > 0$ such that the following inequality holds for all $u \in H^1(\mathbb{R}^2)$ with $\|\nabla u\|_2 < 1$*

$$\int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right) \, dx \le C \frac{\|u\|_2^2}{1 - \|\nabla u\|_2^2}. \tag{32}$$

The second term in (31), instead, presents an unbounded domain of integration where, nevertheless, the Dirichlet energy is not concentrating: the bound will be, then, a consequence of the subcritical inequality proved in the previous section. Note that in both the two cases we have to perform first a suitable cut-off in order to assure that the supports of the functions lie in the proper domains, and checking that this operation do not increase to much the energy.
Let us start by performing a smooth cut off on the ball centered in x_n with radius 2, with a very careful choice of the cut off function. Let us consider

$$\chi(|x|) = \begin{cases} 1 & \text{if } |x| < 1 \\ 1 - (1 - |x|)^2 & \text{if } 1 < |x| < 2 \\ 0 & \text{if } |x| > 2 \end{cases}$$

and set $w_n := u_n \cdot \chi_n(x - x_n)$, whose support is $B_2(x_n)$. Then, applying the inequality $2ab < 4a^2 + \frac{b^2}{4}$,

$$\begin{aligned} \|\nabla w_n\|_2^2 \le& \int_{B_1(x_n)} |\nabla u_n|^2 + \int_{1<|x-x_n|<2} |\nabla u_n|^2 \chi^2 + 2 \int_{1<|x-x_n|<2} u_n |\nabla u_n| |\nabla \chi| \chi \\ &+ \int_{1<|x-x_n|<2} |\nabla \chi|^2 u_n^2 \\ \le& \int_{B_1(x_n)} |\nabla u_n|^2 + \int_{1<|x-x_n|<2} |\nabla u_n|^2 \chi^2 + 4 \int_{1<|x-x_n|<2} u_n^2 \\ &+ \frac{1}{4} \int_{1<|x-x_n|<2} |\nabla u_n|^2 |\nabla \chi|^2 \chi^2 + \int_{1<|x-x_n|<2} |\nabla \chi|^2 u_n^2 \\ \le& 1 - \|u_n\|_{2,w}^2 + \int_{1<|x-x_n|<2} |\nabla u_n|^2 \left[\chi^2 \left(\frac{|\nabla \chi|^2}{4} + 1\right) - 1\right] \\ &+ 4 \int_{1<|x-x_n|<2} u_n^2 + \int_{1<|x-x_n|<2} |\nabla \chi|^2 u_n^2. \end{aligned} \tag{33}$$

Now, if $r = |x| \in (1, 2)$

$$\chi^2\left(\frac{|\nabla\chi|^2}{4}+1\right) = \left[1-(1-r)^2\right]^2\left((1-r)^2+1\right) = \left[1-(1-r)^2\right]\left(1-(1-r)^4\right) \leq 1,$$

so that

$$\int_{1<|x-x_n|<2} |\nabla u_n|^2\left[\chi^2\left(\frac{|\nabla\chi|^2}{4}+1\right)-1\right] \leq 0,$$

which yields, thanks to (33),

$$\begin{aligned}\|\nabla w_n\|_2^2 &\leq 1-\|u_n\|_{2,w}^2+8\int_{1<|x-x_n|<2} u_n^2 \\ &\leq 1-\|u_n\|_{2,w}^2+\frac{c}{\log|x_n|}\int_{1<|x-x_n|<2} u_n^2\log(e+|x|) \leq 1-\frac{\|u_n\|_{2,w}^2}{2}\end{aligned}$$

for large n, since $x_n \to +\infty$. Therefore, for large n,

$$\begin{aligned}\int_{B_1(x_n)}\left(e^{4\pi u_n^2}-1\right)\log(e+|x|) &\leq c\log|x_n|\int_{B_1(x_n)}\left(e^{4\pi w_n^2}-1\right) \\ &\leq c\log|x_n|\int_{B_2(x_n)}\left(e^{4\pi w_n^2}-1\right) \leq c\log|x_n|\frac{\|w_n\|_2^2}{1-\|\nabla w_n\|_2^2} \\ &\leq 2c\log|x_n|\frac{\int_{B_2(x_n)} u_n^2}{\|u_n\|_{2,w}^2} \leq 2c,\end{aligned} \tag{34}$$

which yields the first uniform bound.
To estimate the second term in (31), note that, by assumption

$$\int_{|x-x_n|>\frac{1}{2}} |\nabla u_n|^2 < \frac{1}{2}.$$

Let us perform a piecewise linear cut off as follows:

$$\xi(x) := \begin{cases} 0 & \text{if } |x| < 1/2 \\ 2|x|-1 & \text{if } 1/2 < |x| < 1 \\ 1 & \text{if } |x| > 1 \end{cases}, \qquad \bar{w}_n := u_n \cdot \xi(x-x_n).$$

The functions $\bar{w}_n$ are supported on $B_{1/2}^{\complement}(x_n)$, and

$$\begin{aligned}\|\nabla\bar{w}_n\|_2^2 &\leq \int_{|x-x_n|>\frac{1}{2}} |\nabla u_n|^2+4\int_{|x-x_n|>\frac{1}{2}} u_n|\nabla u_n|+4\int_{|x-x_n|>\frac{1}{2}} u_n^2 \\ &\leq \frac{1}{2}+2\sqrt{2}\|u_n\|_2+4\|u_n\|_2^2 \leq \frac{1}{2}+2\sqrt{2}\|u_n\|_{2,w}+4\|u_n\|_{2,w}^2 < \frac{3}{4} \quad \text{as } n \to +\infty,\end{aligned}$$

so that, by Proposition 3

$$\int_{B_1^{\complement}(x_n)}\left(e^{4\pi u_n^2}-1\right)\log(e+|x|) \leq \int_{\mathbb{R}^2}\left(e^{3\pi\frac{\bar{w}_n^2}{\|\nabla\bar{w}_n\|_2^2}}-1\right)\log(e+|x|) < C. \tag{35}$$

The statement now follows combining (31), (34) and (35).

3.3.2. *Spread concentration at infinity.* Let us suppose that the maximizing sequences satisfy the second condition: there are two constants $\rho, \delta \in (0,1)$ such that for any $x \in \mathbb{R}^2$ and for any n

$$\int_{B_\rho(x)} |\nabla u_n|^2 \leq \delta.$$

The main tool here will be the Vitali Covering Lemma (see e.g.[19]): we are inspired by the paper [4], which, however, deals only with the much more easier case of vanishing bounded weights and subcritical inequalities. The idea is to split the log-weighted Moser integral as the (infinite) sum of integrals whose domains are balls with the same fixed radius: on each ball, as before, the log-weight behaves like a constant, which allows us to apply inequality (32) and to conclude the estimate. Again, we have to perform a suitable cut off which has not to increase too much the energy: this is possible since we are analysing sequences whose energy 'is spreading' away.
More precisely, by Vitali Covering Lemma, for any fixed radius R there exists a (countable) covering of $\mathbb{R}^2$ of balls $B(x_i, R)$ such that each point of $\mathbb{R}^2$ belongs to at most 5 balls. Let us now fix $R = \rho/2$ and perform a piecewise linear cut off as follows

$$\phi(x) := \begin{cases} 1 & \text{if } |x| < \rho/2 \\ 2\,(1 - |x|/\rho) & \text{if } \rho/2 < |x| < \rho\,, \\ 0 & \text{if } |x| > \rho \end{cases} \qquad u_{n,i} := u_n \cdot \phi(x - x_i).$$

Then each $u_{n,i}$ is supported in the ball $B_\rho(x_i)$, and

$$\begin{aligned} (36) \quad \|\nabla u_{n,i}\|_2^2 &\leq \int_{|x-x_i|<\rho} |\nabla u_n|^2 + 4\int_{|x-x_i|<\rho} u_n |\nabla u_n| + 4\int_{|x-x_i|<\rho} u_n^2 \\ &\leq \delta + 2\sqrt{\delta}\|u_n\|_2 + 4\|u_n\|_2^2 \leq \delta + 2\sqrt{\delta}\|u_n\|_{2,w} + 4\|u_n\|_{2,w}^2 < \frac{\delta+1}{2} < 1 \quad \text{as } n \to +\infty. \end{aligned}$$

Then, by (32)

$$\begin{aligned} \int_{\mathbb{R}^2} \left(e^{4\pi u_n^2} - 1\right) \log(e + |x|) &\leq C_2 \int_{\bigcup_i B_{\rho/2}(x_i)} \left(e^{4\pi u_{n,i}^2} - 1\right) \log(e + |x|) \\ &\leq C_2 \sum_i \int_{B_\rho(x_i)} \left(e^{4\pi u_{n,i}^2} - 1\right) \log(e + |x|) \\ &\leq C_2 \sum_i \log(e + |x_i| + \rho) \int_{B_\rho(x_i)} \left(e^{4\pi u_{n,i}^2} - 1\right) \\ &\leq C_2 \sum_i \frac{\log(e + |x_i| + \rho) \int_{B_\rho(x_i)} u_{n,i}^2}{1 - \|\nabla u_{n,i}\|_2^2}. \end{aligned}$$

Now we have to 'bring' again the log-weight inside the integrals: note that if $|x_i| > 3\rho$ then $\log(e + |x_i| + \rho) < 2\log(e + |x_i| - \rho)$, and $\log(e + |x_i| - \rho) < \log(e + |x|)$ for any $x \in B_\rho(x_i)$.

Hence,

$$
\begin{aligned}
C_2 \sum_i \frac{\log(e+|x_i|+\rho)\int_{B_\rho(x_i)} u_{n,i}^2}{1-\|\nabla u_{n,i}\|_2^2} &\le 2C_2 \sum_{|x_i|>3\rho} \frac{\log(e+|x_i|-\rho)\int_{B_\rho(x_i)} u_{n,i}^2}{1-\|\nabla u_{n,i}\|_2^2} + \\
&\quad + C_2 \sum_{|x_i|\le 3\rho} \frac{\log(e+4\rho)\int_{B_\rho(x_i)} u_{n,i}^2}{1-\|\nabla u_{n,i}\|_2^2} \\
\le 2C_2 \sum_{|x_i|>3\rho} \frac{\int_{B_\rho(x_i)} u_{n,i}^2 \log(e+|x|)}{1-\|\nabla u_{n,i}\|_2^2} &+ C_2 \sum_{|x_i|\le 3\rho} \frac{\log(e+4\rho)\int_{B_\rho(x_i)} u_{n,i}^2 \log(e+|x|)}{1-\|\nabla u_{n,i}\|_2^2} \\
&\le C_3 \frac{2\log(4+\rho)}{1-\delta} \int_{\bigcup_i B_\rho(x_i)} u_{n,i}^2 \log(e+|x|) \le C_4 \frac{2\log(4+\rho)}{1-\delta} \|u_n\|_{2,w}^2
\end{aligned}
$$

where we have used (36) to bound the Dirichlet energy. This yields directly the statement.

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