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Quantitative stability via the method of moving planes: approximate symmetry for overdetermined and rigidity problems

MAT/05

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Chapter 1

Introduction

Since its introduction thanks to the work of Alexandrov [Ale62], the method of moving planes has seen a widespread use in various applications of geometric analysis. Brought to the attention of the PDE community with the seminal work of Serrin [Ser71], it has been a useful tool to prove a large variety of results, including symmetry results for overdetermined and rigidity problems. This thesis investigates three such problems from a quantitative viewpoint, by employing the method of moving planes and developing tools and techniques to prove symmetry and approximate symmetry results.

1.1 Overdetermined & Rigidity Problems

Overdetermined problems are boundary value problems on which an additional condition is imposed. The boundary value problem taken into account is usually well posed; any further hypothesis makes it overdetermined, forcing a symmetry on the problem if a solution still exists. The study of overdetermined problems started with the aforementioned [Ser71], in which the author considers the solution u in a bounded domain $\Omega \subset \mathbb{R}^n$ of the torsion problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Together with the Dirichlet condition in (1.1), a Neumann condition is also prescribed

$$\partial_\nu u = c \quad \text{on } \partial\Omega, \tag{1.2}$$

where ν is the outer normal vector to $\partial\Omega$. Under some regularity hypothesis on the set Ω (namely, asking for $\partial\Omega$ to be of class C^2), Serrin proves that a solution u of (1.1)-(1.2) exists if and only if the set Ω is actually a ball, and the function u is then radial and radially decreasing about the center of the ball. We stress out that the symmetry of the problem is double-faceted: on the one end the underlying domain turns out to be symmetrical; on the other end, the solution u inherits the symmetry of the domain. The study of problem (1.1) has two physical motivations: it arises both in fluid dynamics and in the linear theory of torsion. Regarding the latter, as Serrin himself points out in his paper, his result "*states that, when a solid straight bar is subject to torsion, the* *magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section*".

The research field started with the work of Serrin is still very active: in the literature it is possible to find several results regarding symmetry of overdetermined problems involving more general elliptic operators, problems in which a different overdetermined condition is considered or overdetermined problems on manifolds: we highlight some of them in what follows, heading towards the main theorems presented in this work.

A somewhat discrete analogue of the Serrin's problem can be considered by replacing the overdetermined condition (1.2) with the so called parallel surface condition. Namely, we actually start by considering a smooth and bounded open set $G \subset \mathbb{R}^n$, the ball B_R of radius $R > 0$ centered at the origin and we let $\Omega := G + B_R$ be the Minkowski sum of G and B_R, where we recall that for every two sets $X, Y \subset \mathbb{R}^n$ we have

$$X + Y := \{ \ x + y \mid x \in X, \ y \in Y \ \}.$$

The overdetermined condition is then given by

$$u = c \quad \text{on} \quad \partial G. \tag{1.3}$$

With enough regularity hypotheses, ∂G is a surface parallel to $\partial\Omega$ (for further details, see the introduction of Chapter 4). Problem (1.1)-(1.3) was first studied by Shahgolian in [Sha12] in which the author, just like in the thesis of Serrin's theorem, proves that a solution u satisfying (1.1) and (1.3) exists if and only if Ω is a ball; u is then radial and radially decreasing about the center of the ball. We can see why problem (1.1)-(1.3) can be seen as the discrete analogue of Serrin's problem by considering a smooth solution u of 1.1 such that $u = c_k > 0$ on a family Γ_k of surfaces which are at distance $1/k$ from the boundary of Ω. Then, the sequence kc_k converges to some constant c by regularity and u satisfies (1.2). This was also noted in [CM14]. The parallel surface torsion problem (1.1)-(1.3) arises in the study of invariant isothermic surfaces for fast diffusion equations (we refer to [MS10], [MS13], [CM14] and [CMS15] for further details) - for its nonlocal counterpart, which is presented shortly after, we also give an interpretation in terms of population dynamics in the last section of Chapter 4.

An exterior variant of problem (1.1)-(1.2) comes from capacity and was considered in [Rei97], in which the author proves symmetry for the bounded domain Ω provided that there exists a solution u_Ω of

$$\begin{cases} \Delta u_\Omega = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u_\Omega = 1 & \text{on } \partial\Omega, \\ u_\Omega(x) \to 0 & \text{as } |x| \to +\infty\,, \end{cases} \tag{1.4}$$

that also satisfies (1.2). This problem arises from potential theory and it is the Euler-Lagrange equation associated to the capacitary problem for a set Ω, where we set the *capacity* of Ω as

$$\operatorname{cap}(\Omega) := \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^n} |\nabla v|^2 \;\middle|\; v \in C_c^\infty(\mathbb{R}^n), \ v|_\Omega \geq 1 \right\},$$

From a physical point of view, $\operatorname{cap}(\Omega)$ represents the capacitance of the set Ω, that is the total charge that the set can hold while embedded in a dielectric medium and maintaining a given potential energy with respect to an idealized ground at infinity (see [Rei97] and references therein for more details). A similar result can be formulated for the relative capacity, that is for annular domains of the form $A := E \setminus D$, where $D \subset E \subset \mathbb{R}^n$ are both bounded domains. In this setting, the decay condition in (1.4) is replaced with the homogeneous Dirichlet boundary condition on ∂E.

Among the several generalizations of these two overdetermined problems, in this thesis we consider their nonlocal counterparts. In the last couple of decades there has been a growing interest in the study of PDEs involving nonlocal operators, also in the context of overdetermined problems. In particular, Fall & Jarohs [FJ15] and Soave & Valdinoci [SV19] prove symmetry results for the nonlocal counterpart of problems (1.1) and (1.4), respectively. The differential operator taken into account is the fractional Laplacian, which is defined in (1.15) and is properly introduced in Chapter 2.

Together with Serrin's overdetermined results, the celebrated work of Gidas, Ni & Nirenberg [GNN79] is another milestone where the method of moving planes has been employed. In this paper, the authors consider the solution u of a semilinear elliptic problem in the unit ball B_1,

$$\begin{cases} -\Delta u = f(u) & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{1.5}$$

Under minimal hypotheses on the regularity of the function f (namely, only asking for f to be locally Lipschitz), they are able to prove that the solution u is radial and radially decreasing. This result stemmed several generalizations: for a brief review, we refer to the beginning of Chapter 3.

1.2 Quantitative stability

The scope of the present work is to investigate the problems just showcased in a quantitative way, starting from the following question regarding overdetermined problems: given that the overdetermined condition implies symmetry, is it possible to show that if said condition is *almost* satisfied the problem turns out to be *almost* symmetrical? A similar question can be posed for rigidity problems: is it possible to perform a perturbation so that the problem itself is *almost* symmetrical?

Naturally, part of the work is giving a mathematically accurate definition of what the word "almost" means. Regarding symmetry results for the underlying domain, a way to measure how much Ω is close to a ball is given by the quantity

$$\rho(\Omega) := \inf\{\ t - s \mid \exists p \in \Omega \text{ such that } B_s(p) \subset E \subset B_t(p)\ \}. \tag{1.6}$$

From the definition it is clear that Ω gets closer to a ball as $\rho(\Omega)$ approaches zero. When investigating the symmetry of the solution instead, a good way to measure how much a function $u : B_1 \to \mathbb{R}$ is close to being radial might be estimating the quantity

$$\varsigma(u) := \sup\{\ |u(x) - u(y)| \mid x, y \in B_1, \text{ with } |x| = |y|\ \}. \tag{1.7}$$

Giving stability results for overdetermined problems means perturbing the overdetermined condition or the problem, identifying a deficit and providing a quantitative estimate of the distance of the domain or the solution from the symmetrical configuration, namely giving estimates for (1.6) or

(1.7) in terms of the perturbation. It is worth pointing out that (1.6) and (1.7) are just two of the possible ways to measure proximity to symmetry, which arise from the proof of the specific problem taken into account once a proper perturbation is performed. Some more examples are introduced in what follows.

The research for quantitative versions of already established results has been present in the literature for some decades and has typically involved geometrical or functional inequalities, with the two main examples being the isoperimetric inequality and the Sobolev inequality. The isoperimetric inequality states that balls are minimizers for the perimeter functional in the class of Borel sets with fixed (and finite) Lebesgue measure; in particular, if E is a Borel set in $\mathbb{R}^n$ with $|E| < +\infty$ and $n \geq 2$ we have that

$$n\,\omega_n^{1/n}|E|^{(n-1)/n} \leq P(E), \tag{1.8}$$

where $P(E)$ denotes the distributional perimeter of E and ω_n is the measure of the unit ball of dimension n. It was De Giorgi [DG58] who first proved (1.8) in the general framework of sets of finite perimeter. A very natural question is then the following: is it possible to measure how close a general set E is to being a ball depending on how close (1.8) is to being an equality? For a complete review on the history of the problem we refer to Fusco, Maggi & Pratelli [FMP08], who also proved a sharp quantitative result. Namely, by setting

$$\lambda(E) := \min_{x\in\mathbb{R}^n} \frac{|E \triangle B_r(x)|}{r^n} \quad \text{and} \quad \mathcal{I}(E) := \frac{P(E)}{n\,\omega_n^{1/n}|E|^{(n-1)/n}} - 1,$$

where $r > 0$ is such that $|B_r| = |E|$ and $A \triangle B$ denotes the symmetric difference between sets $A, B \subset \mathbb{R}^n$, the authors show that

$$\lambda(E) \leq C\,\sqrt{\mathcal{I}(E)},$$

where $C > 0$ is a dimensional constant. Notice how the *Fraenkel asymmetry* $\lambda(E)$ is a measure of closeness to the symmetry configuration different from $\rho(\Omega)$ and suited for the problem taken into exam.

The Sobolev inequality with exponent 2 states that, for any $n \geq 3$ and any $u \in H^1(\mathbb{R}^n)$, it holds that

$$S\,\|u\|_{L^{2^*}} \leq \|\nabla u\|_{L^2}, \tag{1.9}$$

where $2^* = 2n/(n-2)$ and $S > 0$ is a dimensional constant. Thanks to the work of Aubin [Aub76] and Talenti [Tal76], the optimal value of S is known together with the optimizers of the Sobolev inequality: the functions that satisfy the equality in (1.9) are the so called *Talenti bubbles*

$$U[c, z, \lambda](x) := \frac{c}{(1 + \lambda^2|x - z|^2)^{(n-2)/2}} \quad \text{for all } x \in \mathbb{R}^n,$$

where $c \in \mathbb{R}$, $\lambda \in (0, +\infty)$ and $z \in \mathbb{R}^n$. Once (1.9) is established, it is natural to look for a quantitative version, asking if a solution u which almost satisfies the Sobolev inequality is close to a Talenti bubble, as suggested by Brezis & Lieb [BL85, Question (c) on p. 75]. Bianchi & Egnell [BE91] give a positive answer to this question by proving that for any $u \in H^1(\mathbb{R}^n)$ it holds that

$$\inf_{z\in\mathbb{R}^n,\,\lambda>0,\,c\in\mathbb{R}} \|\nabla(u - U[c, z, \lambda])\|_{L^2}^2 \leq C\left(\|\nabla u\|_{L^2}^2 - S^2\,\|u\|_{L^{2^*}}^2\right),$$

where $C > 0$ is again a dimensional constant. Tackling the problem from a different point of view, Struwe [Str84], Ding [Din86], Ciraolo, Figalli & Maggi [CFM18] and Figalli & Glaudo [FG20] deal with solutions and almost solutions to the corresponding Euler-Lagrange equation associated to (1.9), namely (up to a suitable scaling)

$$\Delta u + u|u|^{2^*-2} \sim 0.$$

Regarding quantitative stability results for overdetermined problem, the first instance is due to Aftalion, Busca & Reichel [ABR99], where the authors prove approximate radial symmetry for the overdetermined Serrin problem. Under suitable assumptions, they prove the stability estimate

$$\rho(\Omega) \leq C \, | \log \left(\|\partial_\nu u - d\|_{C^1(\partial\Omega)} \right) |^{1/n} \tag{1.10}$$

for smooth solutions u of (1.1) with $f(u)$ on the right-hand side, where $C > 0$ is a constant depending on the data and d a suitable constant. Their estimate is improved in [BNST08a] only for the case of the torsion problem in two ways: the logarithmic dependence on the deficit in (1.10) is replaced with a Hölder-type estimate and the C^1-norm is replaced by the L^1-norm. A version of the result for Monge-Ampère equations is also available in [BNST09] (see also [BNST08b]). Inequality (1.10) is also improved by Ciraolo, Magnanini & Vespri [CMV16] with a Hölder-type estimate in terms of the Lipschitz seminorm of the derivative on the boundary at the cost of restricting the class of admissible sets Ω (which include convex sets). More recently, Feldman [Fel18] provides an approximate symmetry result for the Serrin problem by giving a linear estimate for the Fraenkel asymmetry of the domain in terms of the L^2-norm of the derivative at the boundary. Building up from the techniques first showcased in [BNST08a], Magnanini & Poggesi [MP20] provide an estimate on $\rho(\Omega)$ in terms of the L^2-norm of $\partial_\nu u$ on the boundary (see also [MP19] and [MP20]). Lastly, Pacella, Poggesi & Roncoroni [PPR23] investigate the problem in convex cones.

It is worth pointing out that the "bubbling" phenomenon was observed in [BNST08a]: [BNST08a, Theorem 1] shows that the set Ω could be close to a finite number of balls joined together by long thin tentacles even assuming that the set is connected. Bubbling is a classical phenomenon in PDEs and geometric analysis (see for instance [CM17] and [DMMN18] for results regarding Alexandrov's Soap Bubble Theorem). Later works on approximate symmetry for the aforementioned problems include additional hypotheses (i.e. uniform interior sphere condition) to ensure that the underlying domain Ω is actually close to just one ball.

For the local parallel surface torsion problem Ciraolo, Magnanini & Sakaguchi [CMS16] prove approximate symmetry with a linear estimate in terms of the Lipschitz seminorm of the solution on the parallel surface. Looking at the nonlocal setting, Ciraolo, Figalli, Maggi & Novaga [CFMN18] prove the nonlocal counterpart to the Alexandrov's Soap Bubble Theorem. For nonlocal overdetermined PDEs, as far as we know [CDP$^+$23] is the first instance of a quantitative stability result. The work has been very recently improved in [DPTV23a]; later in [DPTV23b] the authors prove approximate symmetry for the overdetermined nonlocal Serrin problem.

Taking into account problem (1.5), we point out that, while [GNN79] stemmed several generalizations (some of which are discussed in the introduction of Chapter 3), to the best of our knowledge, no other quantitative result for the rigidity problem (1.14) is available with the only exception being the work of Rosset [Ros94], which deals with the problem in a small perturbation of the unit ball. Therefore, the results presented in Chapter 3 (from which we can also recover [Ros94]), represent quite a novelty in this regard.

1.3 Main results

As stated at the beginning, the bulk of this thesis revolves around quantitative stability for three different problems, one local rigidity problem and two nonlocal overdetermined problems. Each one of them has a devoted chapter. We will present them starting with the qualitative results from which they stem and give a loose description of the quantitative versions before stating the main theorems. A more in-depth analysis of the specific problem is of course left to the corresponding chapter.

The first is a quantitative version of the aforementioned result by Gidas, Ni & Nirenberg, [GNN79]: in Chapter 3 we investigate a perturbed version of problem (1.5) where the right hand side of the differential equation is multiplied by a function $\kappa : B_1 \to \mathbb{R}$. We then measure the almost radiality and almost monotonicity of the solutions in terms of a deficit depending on the perturbation κ. Namely, we consider

$$\begin{cases} -\Delta u = \kappa\, f(u) & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.11}$$

for some continuously differentiable function $\kappa : B_1 \to [0, +\infty]$. When f is non-negative and κ is radially symmetric and decreasing, solutions of (1.11) are also radially symmetric and decreasing. This was already observed by Gidas, Ni & Nirenberg—see [GNN79, Theorem 1′]. In order to prove a quantitative version of the result, we introduce the deficit

$$\operatorname{def}(\kappa) := \|\nabla^T \kappa\|_{L^\infty(B_1)} + \|\partial_r^+ \kappa\|_{L^\infty(B_1)},$$

where ∂_r^+ denotes the positive part of the radial derivative $\partial_r := \frac{x}{|x|} \cdot \nabla$ (i.e., $\partial_r^+ \kappa := \max\{0, \partial_r \kappa\}$), while $\nabla^T := \nabla - \frac{x}{|x|}\, \partial_r$ indicates the angular gradient. Observe that the deficit of κ vanishes if and only if κ is radially symmetric and non-decreasing. The main result regarding this problem is the following

Theorem 1.3.1. *Let $f : [0, +\infty) \to \mathbb{R}$ be a non-negative locally Lipschitz continuous function and $\kappa \in C^1(\overline{B_1})$ be a non-negative function. Let $u \in C^2(B_1) \cap C^0(\overline{B}_1)$ be a solution of* (1.11) *satisfying*

$$\frac{1}{C_0} \leq \|u\|_{L^\infty(B_1)} \leq C_0, \tag{1.12}$$

for some constant $C_0 \geq 1$. Then,

$$|u(x) - u(y)| \leq C \operatorname{def}(\kappa)^\alpha \quad \textit{for all } x, y \in B_1 \textit{ such that } |x| = |y| \tag{1.13}$$

and

$$\partial_r u(x) \leq C \operatorname{def}(\kappa)^\alpha \quad \textit{for all } x \in B_1 \setminus \{0\}, \tag{1.14}$$

for some constants $\alpha \in (0, 1]$ and $C > 0$ depending only on n, $\|f\|_{C^{0,1}([0,C_0])}$, $\|\kappa\|_{L^\infty(B_1)}$, and C_0.

Formulas (1.13) and (1.14) embody the approximate radial symmetry and the approximate monotonicity of the solution, respectively. If $\operatorname{def}(\kappa)$ vanishes, we recover the symmetry result of [GNN79, Theorem 1′]. In order to obtain the result we need quantitative informations on the positivity and on the boundedness of the solution u, thus the assumption (1.12). This makes the constants α and C in Theorem 1.3.1 dependent on C_0. In Chapter 3 we also prescribe conditions on the function f for the growth estimate in (1.12) to hold, therefore removing the dependence on the size of u from estimates (1.13)-(1.14) - see Corollary 3.0.3 for a precise statement. Moreover, in Chapter 3 we also prove approximate symmetry for problem (1.11) with a more general right-hand side in Theorem 3.0.5. The results presented are all contained in [CCPP23].

The other problems that we consider in this thesis both have a nonlocal setting as the operator taken into account is the fractional Laplacian, defined for a function $u : \mathbb{R}^n \to \mathbb{R}$ as

$$(-\Delta)^s u(x) := c_{n,s}\, P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}} dz, \tag{1.15}$$

where $s \in (0, 1)$, the integral is intended in the principal value sense (see (2.6)) and $c_{n,s} > 0$ is a constant depending on the dimension n and the parameter s. This operator has its roots in various disciplines spanning from harmonic analysis to probability. It found its way to the PDE community with the seminal paper of Caffarelli & Silvestre [CS07] and has been intensively studied in the last few decades. Its popularity is also due to its ability to model phenomena in which long-term interactions between particles occur, leading to its various applications in physics, population dynamics and finance among others. A proper introduction to the fractional Laplacian is given in Chapter 2.

The first nonlocal overdetermined problem starts by considering the so called fractional torsion problem

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.16}$$

Note that due to the nonlocal nature of the operator the Dirichlet condition in (1.1) is replaced with a condition on $\mathbb{R}^n \setminus \Omega$. For the overdetermined condition, we ask for the solution u of (1.16) to be constant on a surface parallel to the boundary.

As already seen for the local case, we consider a smooth and bounded open set $G \subset \mathbb{R}^n$, the ball B_R of radius $R > 0$ centered at the origin and we let $\Omega := G + B_R$ be the Minkowski sum of G and B_R. The overdetermined condition is then given by (1.3). Just like in Serrin's problem, a solution u satisfying (1.16) and (1.3) exists if and only if Ω is a ball; u will then be radial and radially decreasing about the center of the ball. The precise statement is the following.

Theorem 1.3.2. *Let G be an open bounded set of $\mathbb{R}^n$ with ∂G of class C^1 and set $\Omega := G + B_R$, for some $R > 0$. There exists a solution $u \in C^s(\overline{\Omega})$ of* (1.16) *satisfying the additional condition* (1.3) *if and only if G (and therefore Ω) is a ball.*

Perturbing condition (1.3) means asking for the function u to be *almost* constant on ∂G. The quantity $\rho(\Omega)$ will then be estimated in terms of the Lipschitz seminorm of the solution on the parallel surface, which we define as

$$[u]_\Gamma := \sup_{x,y\in\Gamma,\, x\neq y} \frac{|u(x) - u(y)|}{|x - y|}$$

where Γ is a surface in Ω. We have the following result.

Theorem 1.3.3. *Let G be an open and bounded set of $\mathbb{R}^n$ with ∂G of class C^1 and let $\Omega := G + B_R$. Assume that $\partial\Omega$ is of class C^2. Let $u \in C^2(\Omega) \cap C(\mathbb{R}^n)$ be a solution of* (1.16)*. Then, we have that*

$$\rho(\Omega) \le C_* \, [u]_{\partial G}^{\frac{1}{s+2}} \,,$$

where $C_* > 0$ is an explicit constant only depending on n, s, R, and the diameter $\operatorname{diam}(\Omega)$ *of Ω.*

Chapter 4 is devoted to the discussion and proof of Theorem 1.3.2 and Theorem 1.3.3 - the results presented are all contained in [CDP$^+$23].

The last class of problems we take into account are the capacitary counterpart of problem (1.16) and (1.3) in exterior and annular sets, which are the main topic of Chapter 5. We consider the solution u_Ω of

$$\begin{cases} (-\Delta)^s u_\Omega = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u_\Omega = 1 & \text{in } \overline{\Omega}, \\ u_\Omega(x) \to 0 & \text{as } |x| \to +\infty \,, \end{cases} \tag{1.17}$$

where Ω is a bounded domain and we again ask for u_Ω to be constant on a surface parallel to the boundary. The symmetry result is the following.

Theorem 1.3.4. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $R > 0$ and assume that $G := \Omega + B_R$ is such that ∂G of class C^1. Then, there exists a solution $u \in H^s(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ of* (1.17) *such that* (1.3) *holds for for some constant c if and only if G and Ω are concentric balls and u is radially symmetric.*

Please note that in Theorem 1.3.3 it holds $\Omega \subset G$, while in Theorem 1.3.1 the roles are interchanged. This is rather natural since the equation for u is now set in the complement of Ω.

Once symmetry is established we tackle the quantitative stability for the problem in the same way as we did with the fractional torsion, namely giving an estimate of the quantity $\rho(\Omega)$ in terms of the Lipschitz seminorm of the solution u_Ω on the parallel surface. We have the following

Theorem 1.3.5. *Let Ω be a bounded domain of $\mathbb{R}^n$ with $\partial\Omega$ of class C^2. Let $R > 0$ and let $G = \Omega + B_R$ be such that ∂G is of class C^2. Let $u \in C^s(\mathbb{R}^n)$ be a solution of* (1.17)*. Then, we have that*

$$\rho(\Omega) \le C_* \, [u]_{\partial G}^{\frac{1}{s+2}} \,,$$

with $C_* = C_*(n, s, R, \operatorname{diam}(\Omega), |\Omega|, \mathfrak{r}_\Omega^e) > 0$, where $\operatorname{diam}(\Omega)$ and $|\Omega|$ denote the diameter and the volume of Ω, respectively, and $\mathfrak{r}_\Omega^e$ is the radius of the exterior touching ball condition at Ω.

By employing the same technique we can prove that symmetry and approximate symmetry results hold for solutions u_A of

$$\begin{cases} (-\Delta)^s u_A = 0 & \text{in } A, \\ u_A = 1 & \text{in } \overline{D}, \\ u_A = 0 & \text{in } \mathbb{R}^n \setminus E, \end{cases}$$

for some $D \subset E \subset \mathbb{R}^n$, with $A := E \setminus D$ being an annular set. We refer to Theorem 5.0.3 and Theorem 5.0.4 for precise statements. The results shown in Chapter 5 are present in [CP23].

1.4 Proof techniques

Since its original publication, a variety of techniques has been devised to give different proofs of Serrin's theorem, which have been extended to various degrees to some of its generalizations; a nice easy-to-read introduction on the topic is given by [NT18], while the survey [Mag17] also offers an overview on the more recent stability results for both the Serrin problem and the Alexandrov's Soap Bubble Theorem. The most well-known techniques either rely on the method of moving planes or carry on the proof via integral identities. The first instance of the latter is [Wei71], which appears in the same issue of the journal in which [Ser71] is published and makes use of integration by parts, the Cauchy-Schwarz inequality and the Pohozaev identity for an auxiliary function (the so called *P-function*) related to the solution. It is worth noting that Reilly, inspired by [Wei71], finds a proof of the Soap Bubble Theorem via integral identities in [Rei82]. Other alternative proofs for the Serrin problem are presented in [PS89] and [CH98].

Likewise, the symmetry result for problem (1.5) was first proved in [GNN79] through the method of moving planes. As we will also point out in Chapter 3, proofs of generalized versions of the result through integral inequalities are then given by Lions [Lio81], Kesavan & Pacella [KF94], and Serra [Ser13]. All the results present in this work stay true to the originals, employing the moving planes' procedure as the primary component for the proofs - for a thorough review of the various applications of the method we refer to [CR18] and to the references therein.

Another key ingredient in the proof of the main theorems are maximum principles, which are a family of results involving subsolutions and supersolutions of elliptic differential equations: their common trait is that they recover informations on a given function (its sign, its derivatives etc.) inside a domain with hypotheses on the operator. Strong and weak maximum principles, comparison principles and Hopf-type lemmas all belong to this family. These results need to be tailored on the differential operator taken into account and while being a required step in the proofs of these problems, they are standalone results of independent interest. In particular, when handling problems which involve the fractional Laplacian, the right tools needed seem to be maximum principles for antisymmetric functions. They were firstly introduced in [JW16]; later in [FJ15] the authors use fractional maximum principles and a Hopf type lemma for antisymmetric functions to prove symmetry for the fractional Serrin problem (we also refer to [CLL17] for an overview on the nonlocal symmetry results).

When dealing with approximate symmetry, qualitative maximum principles need to be replaced with their quantitative counterparts, that is, some versions of Harnack-type inequalities. The quantitative tools for Theorem 1.3.1 were already present in literature: we employ a version of the ABP estimate due to Cabré [Cab95] and a weak Harnack inequality (available i.e. in [GT01, Theorem 9.22]). The case of nonlocal operators is quite different: quantitative maximum principles for antisymmetric functions are starting to appear in literature very recently. Starting from a result present in [FJ15], a quantitative version of a Hopf type lemma is proved in Chapter 4 together with a new boundary Harnack inequality. After their publication in [CDP$^+$23], the authors in [DPTV23b] generalized these results proving approximate symmetry for the Serrin problem.

Once the quantitative tools are available, there is a somewhat common strategy in the proof of approximate symmetry, which follows the lines of the one showcased in Section 4 of [CR18]. We start by perturbing the overdetermined condition or the rigidity problem and by identifying a deficit related to the perturbation: we then use it to control the quantities which estimate the symmetry of our problem (in our cases, either $\rho(\Omega)$ or $\varsigma(u)$). We then proceed to prove approximate symmetry with respect to one specific direction: this is typically the bulk of the proof, in which maximum principles come into play. Once this is done, the last step is to obtain approximate symmetry with respect to any direction. This is immediate for Theorem 1.3.1 (with only a slightly more careful examination needed for the general operator in Theorem 3.0.4), while for Theorem 1.3.3 and 1.3.5 a more thorough but standard proof is needed, which is presented in Chapter 4.5. This follows the lines of [CFMN18, Lemma 4.1], which in turn improves the technique first presented in [AB98].

This thesis is organized as follows. Chapter 2 gives some preliminary notions and results which represent the theoretical framework for the following chapters. In particular, we present the method of moving planes and introduce its notationì and give an overview on the fractional Laplacian. Chapter 3 is devoted to the quantitative stability for the perturbed problem (1.5) and its generalization; these results are present in [CCPP23]. Chapter 4 deals with approximate symmetry for the parallel surface fractional torsion problem in bounded domains - the results are present in [CDP$^+$23]. Chapter 5 deals with the parallel surface fractional capacitary problem in exterior and annular sets, whose results are present in [CP23].

Chapter 2

Preliminaries

This chapter is devoted to preliminary notions and basic properties which serve as background for the results presented in Chapter 3, 4 and 5. In particular, Section 2.1 introduces the method of moving planes with its notation and, for the convenience of the reader, proofs of both the Serrin's problem and the Gidas, Ni & Nirenberg result in the unit ball, which serve as a road-map for their various generalizations and, in particular, their quantitative counterparts. Section 2.2 focuses instead on the fractional Laplacian, providing some definitions and basic properties. Because of its introductory nature, this chapter can be skipped to a first read.

2.1 The method of moving planes

As already mentioned in Chapter 1, the method of moving planes was first introduced by Alexandrov [Ale62] to prove what is nowadays called Alexandrov's Soap Bubble Theorem, which states that spheres are the only connected closed embedded hypersurfaces with constant mean curvature (the method of moving planes was then called *Reflection Principle*). Some years later, Serrin [Ser71] employed the same method to prove a symmetry result in potential theory, which gave rise to the research field of the overdetermined problems.

Both results originated a great interest in the geometric analysis and PDE communities. In particular, one of the most influencing applications of the method of moving planes is the approach of Gidas, Ni & Nirenberg [GNN79], [GNN81]. For a review on the influence and many applications of the method of moving planes, we refer to the survey [CR18]. In what follows, we describe how the method works and show symmetry for both problem (1.14) and for problem (1.1)-(1.2), which will be the backbone for the main theorems presented in Chapter 3 and Chapter 4 & 5, respectively.

We introduce some notation which is useful for the application of the method of moving planes.

Given an arbitrary set $E \subset \mathbb{R}^n$, a unit vector $e \in \mathbb{S}^{n-1}$ and a parameter $\lambda \in \mathbb{R}$, we define

$T_\lambda = T_\lambda^e = \{x \in \mathbb{R}^n \mid x \cdot e = \lambda\}$	a hyperplane orthogonal to e,
$H_\lambda = H_\lambda^e = \{x \in \mathbb{R}^n \mid x \cdot e > \lambda\}$	the "positive" half space with respect to T_λ,
$E_\lambda = E \cap H_\lambda$	the "positive" cap of E,
$x'_\lambda = x - 2(x \cdot e - \lambda)\, e$	the reflection of x with respect to T_λ,
$\mathcal{Q}_\lambda = \mathcal{Q}_\lambda^e : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto x'_\lambda$	the reflection with respect to T_λ.

When there is no chance of ambiguity, the dependence on the unit vector e in the notation will be promptly dropped. The method of moving planes works as follows. We now choose E bounded and smooth enough and for a fixed direction e we consider the family of hyperplanes $\{T_\lambda\}_{\lambda \in \mathbb{R}}$. Since E is bounded, for λ large enough the plane T_λ will not intersect E. We can then decrease the value of λ (which correspond to a sliding of T_λ towards E, thus explaining the word *moving* in the name of the method) until T_λ and E are touching. Then, since ∂E is smooth (∂E of class C^1 is enough, see [Fra00]), we can keep decreasing λ and, at least for some values, the reflection $\mathcal{Q}_\lambda(E_\lambda)$ of the cap will still be contained in the set E itself.

Formally, for an open bounded set $E \subset \mathbb{R}^n$ with boundary of class C^1 we define

$$\Lambda_e := \sup\{\ x \cdot e \mid x \in E\ \}$$

and

$$\lambda_e = \inf\{\ \lambda \in \mathbb{R} \mid \mathcal{Q}_{\tilde\lambda}(E_{\tilde\lambda}) \subset E, \text{for all } \tilde\lambda \in (\lambda, \Lambda_e)\ \}.$$

From this point on, given a direction $e \in \mathbb{S}^{n-1}$, we refer to $T_{\lambda_e} = T^e$ and $E_{\lambda_e} = \widehat{E}$ as the *critical hyperplane* and the *critical cap* with respect to e, respectively, and call λ_e the *critical value* in the direction e. We now recall from [Ser71] that, for any given direction e, at least one of the following two conditions holds:

Case 1 - The boundary of the reflected cap $\mathcal{Q}(\widehat{E})$ becomes internally tangent to the boundary of E at some point $P \notin T$;

Case 2 - the critical hyperplane T becomes orthogonal to the boundary of E at some point $Q \in T$.

This is the point where maximum principles come into play to prove symmetry and approximate symmetry. The rough idea for both the Serrin's problem and the Gidas, Ni & Nirenberg result is to compare the solution u of the problem taken into account with its reflection, as we show below.

Now that the method of moving planes has been properly introduced, we are ready to tackle the proof of Serrin's theorem [Ser71, Theorem 1].

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ a bounded domain with $\partial\Omega$ of class C^2. A solution $u \in C^2(\overline{\Omega})$ of*

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the overdetermined condition

$$\partial_\nu u = c \quad \text{on } \partial\Omega$$

exists if and only if Ω is a ball.

Together with Weak and Strong Maximum Principles and the Hopf's Lemma (for a reference, see for example [Eva10, Chapter 6.5]), we also need a refinement of Hopf's Lemma which is [Ser71, Lemma 1] and is known in literature as *Serrin's corner lemma.* We present it here and refer to the original paper by Serrin for its proof.

Lemma 2.1.2 (Serrin's Corner Lemma). *Let $\Omega \subset \mathbb{R}^n$ be a C^2 domain of $\mathbb{R}^n$ and let ξ be a direction such that $\xi \cdot \nu(y) = 0$ for some $y \in \partial\Omega$. Let $H(\nu)$ be an open half space with outer normal ν, $\Omega(H) := \Omega \cap H(\nu)$ and let $w \in C^2(\overline{\Omega(H)})$ satisfy*

$$\begin{cases} -\Delta w \geq 0 & \text{in } \Omega(H), \\ w \geq 0 & \text{on } \Omega(H), \\ w(y) = 0. \end{cases}$$

If θ is a direction in y entering $\Omega(H)$ such that $\theta \cdot \nu \neq 0$, then either

$$\partial_\theta w(y) > 0 \quad \text{or} \quad \partial_\theta^2 w(y) > 0, \tag{2.1}$$

unless $w \equiv 0$.

Proof of Theorem 2.1.1. We apply the method of moving planes to the set Ω. Let $e \in \mathbb{S}^{n-1}$ be a fixed direction and λ_e the corresponding critical value. Without loss of generality, we can assume $e = e_1$ and that the critical hyperplane T goes through the origin (that is, $\lambda_e = 0$). We consider the function

$$v(x) := u(x) - u(x') \quad \text{for } x \in \mathcal{Q}(\widehat{\Omega}),$$

where $\mathcal{Q}(\widehat{\Omega})$ is the reflection of the critical cap. We notice that

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{Q}(\widehat{\Omega}), \\ v \geq 0 & \text{on } \partial\mathcal{Q}(\widehat{\Omega}). \end{cases}$$

Therefore, by the weak maximum principle we know that $v \geq 0$ in $\mathcal{Q}(\widehat{\Omega})$ and then the strong maximum principle actually tells us that either $v \equiv 0$ in $\mathcal{Q}(\widehat{\Omega})$ or $v > 0$ in $\mathcal{Q}(\widehat{\Omega})$. We want to show that the latter cannot occur. Indeed, let us assume that $v > 0$ and show that we reach a contradiction.

Case 1 - Let $P \in \partial Q(\widehat{\Omega}) \setminus T$ be the point at the intersection of the boundary of the critical cap with the boundary of Ω.

Of course we have that $v(P) = 0$ and thus applying the Hopf's Lemma we have that $\partial_\nu v(P) < 0$. At the same time though we have that

$$\partial_\nu v(P) = \partial_\nu u(P) - \partial_\nu u(P') = c - c = 0,$$

which is a contradiction.

Case 2 - The critical hyperplane T becomes orthogonal to the boundary of E at some point $Q \in T$.

Hopf's Lemma cannot be applied in this case; we therefore need a refinement of the result, which is Lemma 2.1.2. The goal is to show that v has a second order zero in Q. In order to do this, we fix a coordinate system with the origin at Q, the x_n axis in the direction of the inward normal to $\partial\Omega$ at Q and the x_1 axis normal to T. In this coordinate system the boundary of Ω is locally given by

$$x_n = \phi(x_1; \ldots, x_{n-1}) \quad \text{for some } \phi \text{ of class } C^2.$$

Since u is of class C^2, differentiating twice and making use of the boundary conditions yields

$$\partial^2_{ij} u + c\,\partial^2_{ij}\phi = 0 \quad \text{for } i, j = 1, \dots, n-1,$$

$$\partial^2_{ni} u(Q) = 0 \quad \text{for } i = 1, \dots, n-1$$

and

$$\partial^2_{nn} u(Q) = -\sum_{i=1}^{n-1} \partial^2_i u(Q) - 1 = c\Delta\phi(Q) - 1.$$

By construction $\mathcal{Q}(\widehat{\Omega}) \subset \Omega$ and $\partial^2_{ij} u(Q) = 0$ for $j = 2, \dots, n-1$, because $\partial_1 \phi$ has an extremum point at Q with respect to all but the first coordinates directions. Since $u'(x) = u(x') = u(-x_1, x_2, \dots, x_n)$ by the previous identities we have that all the first and second derivatives of u and u' coincide at Q, hence

$$\nabla v(Q) = 0 \quad \text{and} \quad D^2 v(Q) = 0. \tag{2.2}$$

On the other hand, v satisfies

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{Q}(\widehat{\Omega}), \\ v > 0 & \text{in } \mathcal{Q}(\widehat{\Omega}), \\ v(Q) = 0. \end{cases}$$

The contradiction is then obtained by applying Lemma 2.1.2. Indeed, let θ be any direction not parallel to ν. Lemma 2.1.2 ensures that (2.1) holds, which is a contradiction with (2.2).

We have then proved that $v \equiv 0$ and that Ω is actually symmetric with respect to direction e_1. By performing the same proof with respect to any other direction $e \in \mathbb{S}^{n-1}$ we obtain the result. $\square$

We now turn our attention to the result by Gidas, Ni & Nirenberg. The approach here is slighlty different: since we are working in the unit ball, the critical value for the method of moving planes with respect to any direction will always be $\lambda = 0$, where both Case 1 and Case 2 occur. The idea here is still to compare the solution with its reflection in the cap by proving a strict inequality and keep sliding the plane T_λ as long as the inequality hold.

Theorem 2.1.3. *Let $u \in C^2(\overline{B_1})$ be a solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{2.3}$$

with f locally Lipschitz. Then, u is radial and radially decreasing.

The proof, which follows the one in [Eva10, Chapter 9.5], also makes use of boundary estimate for solutions of (2.3) ([Eva10, Lemma 2 in Chapter 9.5], which we present below.

Lemma 2.1.4. *Let $u \in C^2(B_1)$ satisfy 2.3. Then at each point $\tilde{x} \in \partial B_1 \cap \{x_1 > 0\}$, either*

$$\partial_1 u(\tilde{x}) < 0$$

or else

$$\partial_1 u(\tilde{x}) = 0, \; \partial^2_1 u(\tilde{x}) > 0.$$

In either case, u is strictly decreasing as a function of x_1 near $\tilde{x}$.

Proof of Theorem 2.1.3. We apply the method of moving planes by fixing a direction $e = e_1 \in \mathbb{S}^{n-1}$, by letting $\lambda \in [0,1]$ and $T_\lambda := \{x_1 = \lambda\}$, $\Sigma_\lambda := B_1 \cap \{x_1 > \lambda\}$.

We consider the set

$$\Lambda := \Big\{\lambda \in (0,1) : u_\mu \geq u \text{ in } \Sigma_\mu \text{ for all } \mu \in [\lambda, 1)\Big\}.$$

An application of Lemma 2.1.4 on u tells us that $u_\lambda > u$ in Σ_λ for $\lambda < 1$ at least if $1-\lambda$ is small enough; that is, Λ is non-empty. We set set $\lambda_\star := \inf \Lambda$ and will show that $\lambda_\star = 0$.

Assume instead that $\lambda_\star > 0$. We set $v(x) := u(x_{\lambda_\star}) - u(x)$ for $x \in \Sigma_{\lambda_\star}$ and see that

$$\begin{cases} -\Delta v + cv = 0 & \text{in} \Sigma_{\lambda_\star}, \\ v \leq 0 & \text{in} \Sigma_{\lambda_\star}, \end{cases}$$

where $c(x) := -\int_0^1 f'(su(x_{\lambda_\star}) + (1-s)u(x))ds$. From the Maximum Principle and Hopf's Lemma we then deduce that $v > 0$ in $\Sigma_{\lambda_\star}$ and that $\partial_1 v > 0$ on $T_{\lambda_\star} \cap B_1$. Thus, $u(x_{\lambda_\star}) > u(x)$ in $\Sigma_{\lambda_\star}$ and $\partial_1 u > 0$ on $T_{\lambda_\star} \cap B_1$. Using again Lemma 2.1.4 we conclude that for a small enough $\varepsilon_0 > 0$ we have that

$$u(x_{\lambda_\star - \varepsilon}) > u(x) \quad \text{in } \Sigma_{\lambda_\star - \varepsilon} \quad \text{for all } 0 \leq \varepsilon \leq \varepsilon_0,$$

which contradicts our choice of $\lambda_\star > 0$.

Since $\lambda_\star = 0$, we see that $u(-x_1, x_2, \ldots, x_n) \geq u(x_1, x_2, \ldots, x_n)$ for all $x \in \Sigma_0$. A similar argument shows the opposite inequality. Thus u is symmetric with respect to the plane $T_0 = \{x_1 = 0\}$. The same argument applies after any rotation of coordinates and so the theorem follows. □

2.2 The fractional Laplacian

The fractional Laplacian is an integro-differential operator that is, in some sense, the non-local equivalent to the classical Laplacian. Although nonlocal equations and the fractional Sobolev spaces to which they're tied to are a topic that has been present in the literature for quite some time (see for example [Ste70] and [Lan72]), they saw a huge increase in popularity in the last couple of decades especially in the PDE community starting from the celebrated work of Cafarelli & Silvestre [CS07].

Thanks to its ability to model phenomena with long-term interactions between objects and particles, it sees widespread applications in different fields spanning from physics, population dynamics and finance among others. To name a few, Bucur & Valdinoci in [BV16] show how the one-dimensional fractional Laplacian models random walks with arbitrarily long jumps and is present in a payoff model related to finance. Later in the same work, the authors use it to model a problem arising in crystal dislocation. We also refer to [DNPV12] for a more detailed description of the problems and models which involve the fractional Laplacian.

The aim of this section is to present the definition and some basic properties related to the fractional Laplacian with a focus on maximum principles. While not trying to be exhaustive by any means, it represents the baseline knowledge needed in order to deal with the problems presented in Chapter 4 and Chapter 5. Our main references for this section are [BV16] and [DNPV12].

We start by recalling the definition of the Fourier transform and inverse Fourier transforms for smooth functions.

Definition 2.2.1 (Fourier transform)**.** *Given a function $f : \mathbb{R} \to \mathbb{R}$ regular enough, denoting the* space variable $x \in \mathbb{R}^n$ *and the* frequency variable $\xi \in \mathbb{R}^n$, *we define the* Fourier transform *and the* inverse Fourier transform *respectively as*

$$\widehat{f}(\xi) \coloneqq \mathcal{F}[f](\xi) \coloneqq \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx \quad \textit{and} \quad \mathcal{F}^{-1}[\widehat{f}](x) \coloneqq \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

We can now give the definition of the fractional Laplacian.

Definition 2.2.2 (Fractional Laplacian)**.** *Given $s \in (0,1)$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ regular enough, we define the* fractional Laplacian of u *in $x \in \mathbb{R}^n$ as*

$$(-\Delta)^s u(x) := -\frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}dy, \tag{2.4}$$

where $c_{n,s}$ is a positive constant depending only on n and s.

There are some alternative ways to define the operator (actually, at least ten, as you can see here [Kwa17]): we present in what follows the one given via an integral in the principal value sense, quite similar to Definition 2.2.2 and often handier in applications, and the one given via Fourier transform, which helps us understand why the operator $(-\Delta)^s$ is the fractional version of the Laplacian. The equivalence between (2.4) and the definition below follows from an easy computation which can be found for instance in [BV16, Chapter 3.1].

Definition 2.2.3 (Fractional Laplacian, via P.V.)**.** *Given $s \in (0,1)$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ regular enough, we define the* fractional Laplacian in the P.V. (principal value) sense of u in $x \in \mathbb{R}^n$ *as*

$$(-\Delta)^s u(x) \coloneqq c_{n,s}\, P.V. \int_{\mathbb{R}^n} \frac{u(z) - u(x)}{|x-z|^{n+2s}}dz, \tag{2.5}$$

where the integral in (2.5) *is intended in the principal value sense, that is*

$$P.V. \int_{\mathbb{R}^n} \frac{u(z) - u(x)}{|x-z|^{n+2s}}dz = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(z) - u(x)}{|x-z|^{n+2s}}dz. \tag{2.6}$$

The similarity between the classical Laplacian and the fractional one are highlighted thanks to next definition of the operator, which uses the inverse Fourier transform.

Definition 2.2.4 (Fractional Laplacian, via Fourier Transform)**.** *Given $s \in (0,1)$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ regular enough, we define the* fractional Laplacian via the Fourier transform *of u in $x \in \mathbb{R}^n$ as*

$$(-\Delta u)^s_{\mathcal{F}}(x) \coloneqq \mathcal{F}^{-1}((2\pi|\cdot|)^{2s}\widehat{u})(x) = \int_{\mathbb{R}^n} |\xi|^{2s}\widehat{u}(\xi)e^{i\xi x}d\xi$$

(the subscript is only used to point out which specific definition we are currently using and will be promptly removed as soon as we show their equivalence).

A computation shows the equivalence of (2.4) and 2.2.4.

Proposition 2.2.5. *Given $u \in \mathbb{S}(\mathbb{R}^n)$, then for every s in $(0,1)$ and for x in $\mathbb{R}^n$*

$$(-\Delta u)^s(x) = (-\Delta u)^s_{\mathcal{F}}(x).$$

Proof. Let us apply the Fourier transform to (2.4):

$$\begin{aligned}\mathcal{F}[(-\Delta u)^s](\xi) &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{\mathcal{F}[2u(\cdot)-u(\cdot+y)-u(\cdot-y)](\xi)}{|y|^{n+2s}}dy\\ &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\widehat{u}(\xi)\frac{2-e^{2\pi i y\cdot\xi}-e^{-2\pi i y\cdot\xi}}{|y|^{n+2s}}dy\\ &= c_{n,s}\,\widehat{u}(\xi)\int_{\mathbb{R}^n}\frac{1-cos(2\pi\xi\cdot y)}{|y|^{n+2s}}dy.\end{aligned}$$

We now operate the change of variables $z=|\xi|\,y$ in the above integral and obtain that

$$J(\xi) := \int_{\mathbb{R}^n}\frac{1-cos(2\pi\xi\cdot y)}{|y|^{n+2s}}dy = |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-cos(\frac{2\pi\xi}{|\xi|}\cdot z)}{|z|^{n+2s}}dz.$$

Let us now consider a rotation R (with R^T its transpose) such that $Re_1=\xi/|\xi|$ and apply the change of variables $\omega=R^Tz$ obtaining

$$\begin{aligned}J(\xi) &= |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-cos(2\pi Re_1\cdot z)}{|z|^{n+2s}}dz = |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-cos(2\pi R^Tz\cdot e_1)}{|R^Tz|^{n+2s}}dz\\ &= |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-cos(2\pi\omega_1)}{|\omega|^{n+2s}}d\omega = (2\pi|\xi|)^{2s}\int_{\mathbb{R}^n}\frac{1-cos(\eta_1)}{|\eta|^{n+2s}}d\eta,\end{aligned}$$

where we used that $\eta=2\pi\omega$ in the last equality. We now notice that the latter integral is finite, since

$$\int_{\mathbb{R}^n\setminus B_1}\frac{1-cos(\eta_1)}{|\eta|^{n+2s}}d\eta \le \int_{\mathbb{R}^n\setminus B_1}\frac{2}{|\eta|^{n+2s}}d\eta < +\infty$$

and by looking at the Taylor expansion inside the unit ball we have

$$\int_{B_1}\frac{1-cos(\eta_1)}{|\eta|^{n+2s}}d\eta \le \int_{B_1}\frac{|\eta|^2}{|\eta|^{n+2s}}d\eta \le \int_{B_1}\frac{d\eta}{|\eta|^{n+2s-2}} < +\infty.$$

Therefore, by choosing

$$c_{n,s} := \left(\int_{\mathbb{R}^n}\frac{1-cos(\eta_1)}{|\eta|^{n+2s}}d\eta\right)^{-1} \tag{2.7}$$

we can then write

$$J(\xi) = \frac{(2\pi|\xi|)^{2s}}{c_{n,s}}.$$

By putting all together we obtain that

$$\mathcal{F}[(-\Delta u)^s](\xi) = c_{n,s}\,\widehat{u}(\xi)J(\xi) = (2\pi|\xi|^{2s})\widehat{u}(\xi).$$

□

Remark 2.2.6. *Throughout the proof we were also able to compute the normalization constant* $c_{n,s}>0$ *introduced in* (2.4)*, which is showcased in* (2.7)*.*

In studying the asymptotics as s goes to 0^+ or to 1^- of the fractional Laplacian, it is useful to study the behaviour of the constant $c_{n,s}$ just introduced, for which the following proposition holds ([DNPV12, Corollary 4.2], to which we refer for its proof).

Proposition 2.2.7. *It holds that*

$$\lim_{s\to 1^-} \frac{c_{n,s}}{s(1-s)} = \frac{4n}{\omega_{n-1}} \quad \textit{and} \quad \lim_{s\to 0^+} \frac{c_{n,s}}{s(1-s)} = \frac{2}{\omega_{n-1}}.$$

The following simple remark together with Proposition 2.2.5 shows why (2.2.2) is the fractional version of the classical Laplace operator, as proved rigorously in Proposition 2.2.9.

Remark 2.2.8. *We can rewrite the classical Laplacian via the Fourier transform:*

$$\begin{aligned}-\Delta u(x) &= -\Delta(\mathcal{F}^{-1}(\widehat{u}))(x) = -\Delta \int_{\mathbb{R}^n} \widehat{u}(\xi) e^{2\pi i x\cdot\xi} d\xi \\ &= \int_{\mathbb{R}^n} (2\pi|\xi|)^2 \widehat{u}(\xi) e^{2\pi i x\cdot\xi} d\xi = \mathcal{F}^{-1}((2\pi|\xi|)^2\widehat{u}(\xi))(x).\end{aligned}$$

Proposition 2.2.9. *Given $u \in \mathcal{S}(\mathbb{R}^n)$, then for every x in $\mathbb{R}^n$*

$$\lim_{s\to 0^+} (-\Delta)^s u(x) = u(x) \quad \textit{and} \quad \lim_{s\to 1^-} (-\Delta)^s u(x) = -\Delta u(x). \tag{2.8}$$

Proof. We start by proving the first identity. Fix $x \in \mathbb{R}^n$, $R_0 > 0$ such that $\operatorname{supp} u \subset B_{R_0}(0)$ and set $R = R_0 + |x| + 1$. First,

$$\begin{aligned}&\left| \int_{B_R} \frac{2u(x) - u(x+y) - u(x-y)}{|x-y|^{n+2s}} dy \right| \le \|u\|_{C^2(\mathbb{R}^n)} \int_{B_R} \frac{|y|^2}{|y|^n + 2s} dy \\ &\le \omega_{n-1} \|u\|_{C^2(\mathbb{R}^n)} \int_0^R \frac{1}{\rho^{2s-1}} d\rho = \frac{\omega_{n-1}\|u\|_{C^2(\mathbb{R}^n)} R^{2-2s}}{2(1-s)}.\end{aligned}$$

By Proposition 2.2.7 we then get that

$$\lim_{s\to 0^+} \frac{c_{n,s}}{2} \int_{B_R} \frac{2u(x) - u(x+y) - u(x-y)}{|x-y|^{n+2s}} dy = 0.$$

On the other hand, we see that $|y| \ge R$ yelds $|x \pm y| \ge |y| - |x| \ge R - |x| > R_0$ and consequently $u(x \pm y) = 0$. Therefore,

$$\begin{aligned}&\frac{1}{2} \int_{\mathbb{R}^n \setminus B_R} \frac{2u(x) - u(x+y) - u(x-y)}{|x-y|^{n+2s}} dy = u(x) \int_{\mathbb{R}^n \setminus B_R} \frac{1}{|y|^{n+2s}} dy \\ &= \omega_{n-1}\, u(x) \int_R^{+\infty} \frac{1}{\rho^{2s+1}} d\rho = \frac{\omega_{n-1} R^{-2s}}{2s} u(x).\end{aligned}$$

From the previous computations and making again use of Proposition 2.2.7 we obtain

$$\lim_{s\to 0^+} (-\Delta u)^s = \lim_{s\to 0^+} \frac{c_{n,s}}{2} \int_{B_R} \frac{2u(x) - u(x+y) - u(x-y)}{|x-y|^{n+2s}} dy = \lim_{s\to 0^+} \frac{c_{n,s}\, \omega_{n-1} R^{-2s}}{2s} u(x).$$

Now turning our attention to the second identity in (2.8), we start by computing the contribution outside the unit ball, which turns out to be zero. Indeed,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n\setminus B_1}\frac{2u(x)-u(x+y)-u(x-y)}{|x-y|^{n+2s}}dy\right|\le 4\|u\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_1}\frac{1}{|y|^{n+2s}}dy\\&\le \omega_{n-1}\|u\|_{L^\infty(\mathbb{R}^n)}\int_1^{+\infty}\frac{1}{\rho^{2s+1}}d\rho=\frac{2\omega_{n-1}}{s}\|u\|_{L^\infty(\mathbb{R}^n)},\end{aligned}$$

which, making again use of Proposition 2.2.7, yields

$$\lim_{s\to 1^-}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n\setminus B_1}\frac{2u(x)-u(x+y)-u(x-y)}{|x-y|^{n+2s}}dy=0$$

By looking at what happens in the unit ball we have

$$\begin{aligned}&\left|\int_{B_1}\frac{2u(x)-u(x+y)-u(x-y)-D^2u(x)y\cdot y}{|y|^{n+2s}}dy\right|\le \|u\|_{C^3(\mathbb{R}^n)}\int_{B_1}\frac{|y|^3}{|y|^{n+2s}}dy\\&\le \omega_{n-1}\|u\|_{C^3(\mathbb{R}^n)}\int_0^1\frac{1}{\rho^{2s-2}}d\rho=\frac{\omega_{n-1}\|u\|_{C^3(\mathbb{R}^n)}}{3-2s}\end{aligned}$$

from which we get that

$$\lim_{s\to 1^-}\frac{c_{n,s}}{2}\int_{B_1}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}dy=\lim_{s\to 1^-}\frac{c_{n,s}}{2}\int_{B_1}\frac{D^2u(x)y\cdot y}{|y|^{n+2s}}dy.$$

Now notice that if $i,j=1,\dots,n$ with $i\neq j$ then

$$\int_{B_1}\partial^2_{ij}u(x)y_j\cdot y_j dy=-\int_{B_1}\partial^2_{ij}u(x)\tilde{y}_j\cdot\tilde{y}_j d\tilde{y}$$

where we set $\tilde{y}_k=y_k$ for any $k\neq j$ and $\tilde{y}_j=-y_j$; therefore

$$\int_{B_1}\partial^2_{ij}u(x)y_j\cdot y_j dy=0.$$

Moreover, we see that

$$\begin{aligned}\int_{B_1}\frac{\partial^2_i u(x)y_i^2}{|y|^{n+2s}}dy&=\partial^2_i u(x)\int_{B_1}\frac{y_i^2}{|y|^{n+2s}}dy=\partial^2_i u(x)\int_{B_1}\frac{y_1^2}{|y|^{n+2s}}dy\\&=\frac{\partial^2_i u(x)}{n}\sum_{j=1}^n\int_{B_1}\frac{y_j^2}{|y|^{n+2s}}dy=\frac{\partial^2_i u(x)}{n}\int_{B_1}\frac{|y|^2}{|y|^{n+2s}}dy.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\lim_{s\to 1^-}(\Delta)^s u(x)&=\lim_{s\to 1^-}\frac{c_{n,s}}{2}\int_{B_1}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}dy\\&=\lim_{s\to 1^-}\frac{c_{n,s}}{2}\int_{B_1}\frac{D^2u(x)y\cdot y}{|y|^{n+2s}}dy=\frac{c_{n,s}}{2}\sum_{j=1}^n\int_{B_1}\frac{\partial^2_i u(x)y_i^2}{|y|^{n+2s}}dy\\&=\lim_{s\to 1^-}\frac{c_{n,s}\omega_{n-1}}{4n(1-s)}\sum_{j=1}^n\partial^2_i u(x)=-\Delta u(x).\end{aligned}$$

□

So far we've been elusive regarding the regularity assumptions in the definitions and properties of the fractional Laplacian. What follows, while highlighting the connection between the operator and fractional Sobolev spaces, should explain why we only dealt with smooth functions.

First, given an open set $\Omega \subset \mathbb{R}^n$, for $s \in (0,1)$ and $p \in (1,+\infty)$ we define the *Gagliardo seminorm* of a function u as

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy \right)^p .$$

We can then introduce the fractional Sobolev spaces.

Definition 2.2.10. *Let $\Omega \subset \mathbb{R}^n$ be an open set. For $s \in (0,1)$ and $p \in (1,+\infty)$ we define*

$$W^{s,p}(\Omega) := \left\{ \; u \in L^p(\Omega) \mid [u]^p_{W^{s,p}(\Omega)} < +\infty \right\}$$

$W^{s,p}(\Omega)$ will then be a Banach space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)} \right)^p .$$

As in the classical case, when $\Omega = \mathbb{R}^n$, any function in the fractional Sobolev space can be approximated with smooth functions of compact support.

Theorem 2.2.11. *For any $s \in (0,1)$ and $p \in (1,+\infty)$, the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.*

When $p = 2$ we set $H^s(\Omega) := W^{s,2}(\Omega)$ and define the scalar product for two functions $u, v \in H^s(\mathbb{R}^n)$ as

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dx\, dy\, .$$

the following result shows the relationship between H^s and the fractional Laplace operator. It can be found with proof in [DNPV12, Propsition 3.6]

Proposition 2.2.12. *Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$. Then*

$$[u]^2_{H^s(\mathbb{R}^n)} = \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^n)}.$$

In the second part of this section we focus on maximum principles for the fractional Laplacian, key ingredients in order to perform the proofs of the results showcased in Chapter 1.3.

We start with a proposition which serves the purpose of highlighting a major difference with the classical case: it is not enough to prescribe the sign of a function inside the set in which it is s-harmonic in order to have a Harnack-type result. The proposition, with the sketch of the proof that we present below, can be found in [BV16, Theorem 2.3.1]

Proposition 2.2.13. *There exists a bounded function u which is s-harmonic in B_1, non-negative in B_1, but such that $\inf_{B_1} u = 0$.*

Proof. Let $M \geq 0$ and let u_M be the function satisfying

$$\begin{cases} (-\Delta)^s u_M = 0 & \text{in } B_1, \\ u_M = 1 - M & \text{in } B_3 \setminus B_2, \\ u_M = 1 & \text{in } \mathbb{R}^n \setminus \big((B_3 \setminus B_2) \cup B_1 \big). \end{cases} \tag{2.9}$$

When $M = 0$, the function u_M is identically 1, while we expect u_M to "bend down" when $M > 0$, since the fact that the fractional Laplacian vanishes in B_1 forces the second order quotient to vanish in average. Indeed, we claim that there exists $M_* > 0$ such that $u_{M_*} \geq 0$ in B_1 with $\inf_{B_1} u_{M_*} = 0$. Then, Proposition 2.2.13 would be proved by choosing $u := u_{M_*}$.

To check the existence of such M_*, we show that $\inf_{B_1} u_M \to -\infty$ as $M \to +\infty$. Indeed, we argue by contradiction and suppose this cannot happen. Then, for any $M \geq 0$ we would have that

$$\inf_{B_1} u_M \geq -a, \tag{2.10}$$

for some fixed $a \in \mathbb{R}$. We then set

$$v_M := \frac{u_M + M - 1}{M}.$$

Then, by (2.9) we have

$$\begin{cases} (-\Delta)^s v_M = 0 & \text{in } B_1, \\ v_M = 0 & \text{in } B_3 \setminus B_2, \\ v_M = 1 & \text{in } \mathbb{R}^n \setminus \big((B_3 \setminus B_2) \cup B_1\big). \end{cases}$$

Also, by (2.10), for any $x \in B_1$

$$v_M(x) \geq \frac{-a + M - 1}{M}.$$

As $m \to +\infty$, the function v_M approaches a function v_∞ which satisfies

$$\begin{cases} (-\Delta)^s v_\infty = 0 & \text{in } B_1, \\ v_\infty = 0 & \text{in } B_3 \setminus B_2, \\ v_\infty = 1 & \text{in } \mathbb{R}^n \setminus \big((B_3 \setminus B_2) \cup B_1\big). \end{cases}$$

and, for any $x \in B_1$,

$$v_\infty(x) \geq 1.$$

In particular, the maximum of v_∞ is attained at some point $x_* \in B_1$, with $v_\infty(x_*) \geq 1$. Accordingly,

$$0 = P.V. \int_{\mathbb{R}^n} \frac{v_\infty(x_*) - v_\infty(y)}{|x_* - y|^{n+2s}} \geq P.V. \int_{B_3 \setminus B_2} \frac{1 - 0}{|x_* - y|^{n+2s}} > 0,$$

which is a contradiction. □

An inspection of the proof of Proposition 2.2.13 hints at the right hypotheses needed to prove qualitative and quantitative fractional maximum principles: by making no assumptions on the sign of the function u on the outside of the unit ball, we are allowed to choose it as negative as we want. The nonlocality of the operator lets then the bending described in the proof take place. Prescribing the sign of the function on the whole space should fix the issue, as shown in the following propositions.

Proposition 2.2.14 (Fractional Weak Maximum Principle). *If $(-\Delta)^s u \geq 0$ in B_1 and $u \geq 0$ in $\mathbb{R}^n \setminus B_1$, then $u \geq 0$ in B_1.*

Proof. Suppose by contradiction that in the minimal point $x_\star \in B_1$ we have $u(x_\star) < 0$. Since u is positive outside of B_1, $x_\star$ is a minimum point for u in the whole space $\mathbb{R}^n$. Therefore, for $y \in B_2$ we have $2u(x_\star) - u(x_\star + y) - u(x_\star - y) \leq 0$. On the other hand, for $y \in \mathbb{R}^n \setminus B_2$, we have that $(x_\star \pm y) \in \mathbb{R}^n \setminus B_1$, hence $u(x_\star \pm y) \geq 0$. We thus have

$$0 \leq \int_{\mathbb{R}^n} \frac{2u(x_\star) - u(x_\star + y) - u(x_\star - y)}{|y|^{n+2s}} dy \leq \int_{\mathbb{R}^n \setminus B_2} \frac{2u(x_\star) - u(x_\star + y) - u(x_\star - y)}{|y|^{n+2s}} dy$$
$$\leq \int_{\mathbb{R}^n \setminus B_2} \frac{2u(x_\star)}{|y|^{n+2s}} dy < 0,$$

which is a contradiction. □

Proposition 2.2.15 (Fractional Strong Maximum Principle)**.** *If $(-\Delta)^s u \geq 0$ in B_1 and $u \geq 0$ in $\mathbb{R}^n \setminus B_1$, then $u > 0$ in B_1 unless u vanishes identically.*

Proof. Thanks to Proposition 2.2.14 we know that $u \geq 0$ in the whole space. Assuming u is not strictly positive in the unit ball, there exists $x_0 \in B_1$ such that $u(x_0) = 0$. Then,

$$0 \leq \int_{\mathbb{R}^n} \frac{2u(x_0) - u(x_0 + y) - u(x_0 - y)}{|y|^{n+2s}} dy = -\int_{\mathbb{R}^n} \frac{u(x_0 + y) + u(x_0 - y)}{|y|^{n+2s}} dy.$$

Now, both $u(x_0 + y)$ and $u(x_0 - y)$ are non-negative; therefore, the latter integral is less or equal than zero, and so it must vanish identically, meaning that u also must vanish identically. □

As pointed out in the Chapter 1.4 and in [CLL17], applying the method of moving planes for nonlocal problems will result in having to deal with antisymmetric functions for which proper maximum principles are proved in the devoted chapters.

Lastly, throughout Chapter 4 and Chapter 5 we will make repeated use of the explicit solution $\psi_{B_r(x_0)}$ of the s-torsion problem in a ball $B_r(x_0)$ of radius $r > 0$ and centered in $x_0 \in \mathbb{R}^n$. Dyda [Dyd12] shows that the function $\psi_{B_r(x_0)}$ satisfying

$$\begin{cases} (-\Delta)^s \psi_{B_r(x_0)} = 1 & \text{in } B_r(x_0) \\ \psi_{B_r(x_0)} = 0 & \text{in } \mathbb{R}^n \setminus B_r(x_0)\,, \end{cases}$$

has the explicit expression

$$\psi_{B_r(x_0)}(x) := \gamma_{n,s}(r^2 - |x - x_0|^2)_+^s \tag{2.11}$$

for any $x \in \mathbb{R}^n$, where where

$$\gamma_{n,s} := \frac{4^{-s}\Gamma(n/2)}{\Gamma(n/2 + s)\Gamma(1 + s)}.$$

Chapter 3

The Gidas-Ni-Nirenberg result in the Unit Ball

This chapter is devoted to the quantitative counterpart of a famous result by Gidas, Ni & Nirenberg involving solutions to semilinear elliptic problems inside the unit ball B_1. We start by recalling the original symmetry result.

Theorem 3.0.1 ([GNN79]). *Let $f : [0, +\infty) \to \mathbb{R}$ be a locally Lipschitz continuous function and $u \in C^2(B_1) \cap C^0(\overline{B_1})$ be a solution of*

$$\begin{cases} -\Delta u = f(u) & \textit{in } B_1, \\ u > 0 & \textit{in } B_1, \\ u = 0 & \textit{on } \partial B_1. \end{cases} \tag{3.1}$$

Then,u is radially symmetric and strictly decreasing in the radial direction.

After its publication, Theorem 3.0.1 has been extended in several directions. To name just a few of these generalizations, Berestycki & Nirenberg [BN91] and Dancer [Dan92] weakened the notion of solution and provided a simplified proof, Damascelli & Pacella [DP98] and Damascelli & Sciunzi [DS04] extended the result to the case of the p-Laplace operator and Berchio, Gazzola & Weth [BGW08] dealt with the problem for polylaplacians; more recently, Jarohs & Weth [JW16] and Chen, Li & Li [CLL17] treated generalized the result for nonlocal operators. On the other hand, different proofs have been devised in order to deal with other kinds of nonlinearities. In this regard, Brock [Bro98] dealt with continuous, non-negative f by using the continuous Steiner symmetrization, while Lions [Lio81], Kesavan & Pacella [KF94], and Serra [Ser13] even allowed for a class of discontinuous f through the aid of integral methods based on the isoperimetric inequality and Pohozaev's identity.

In what follows we consider a perturbed version of problem (3.1), namely

$$\begin{cases} -\Delta u = \kappa f(u) & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{3.2}$$

for some continuously differentiable function $\kappa : B_1 \to [0, +\infty]$. As already noted in [GNN79, Theorem 1′], when f is non-negative and κ is radially symmetric and decreasing, solutions of (3.2) are also radially symmetric and decreasing. In order to establish a quantitative version of this result, we introduce the *deficit*

$$\operatorname{def}(\kappa) := \|\nabla^T \kappa\|_{L^\infty(B_1)} + \|\partial_r^+ \kappa\|_{L^\infty(B_1)},$$

where ∂_r^+ denotes the positive part of the radial derivative $\partial_r := \frac{x}{|x|} \cdot \nabla$ (i.e., $\partial_r^+ \kappa := \max\{0, \partial_r \kappa\}$), while $\nabla^T := \nabla - \frac{x}{|x|}\,\partial_r$ indicates the angular gradient. Observe that the deficit of κ vanishes if and only if κ is radially symmetric and non-decreasing. Our first result is the following.

Theorem 3.0.2. *Let $f : [0, +\infty) \to \mathbb{R}$ be a non-negative locally Lipschitz continuous function and $\kappa \in C^1(\overline{B_1})$ be a non-negative function. Let $u \in C^2(B_1) \cap C^0(\overline{B_1})$ be a solution of (3.2) satisfying*

$$\frac{1}{C_0} \leq \|u\|_{L^\infty(B_1)} \leq C_0, \tag{3.3}$$

for some constant $C_0 \geq 1$. Then,

$$|u(x) - u(y)| \leq C \operatorname{def}(\kappa)^\alpha \quad \textit{for all } x, y \in B_1 \textit{ such that } |x| = |y| \tag{3.4}$$

and

$$\partial_r u(x) \leq C \operatorname{def}(\kappa)^\alpha \quad \textit{for all } x \in B_1 \setminus \{0\}, \tag{3.5}$$

for some constants $\alpha \in (0, 1]$ and $C > 0$ depending only on n, $\|f\|_{C^{0,1}([0,C_0])}$, $\|\kappa\|_{L^\infty(B_1)}$, and C_0.

Theorem 3.0.2 is a quantitative version of the classical result by Gidas, Ni & Nirenberg. We point out that, from another point of view, a different quantitative variant has been obtained by Rosset [Ros94], who considered space-independent semilinear equations set in small perturbations of the unit ball. In Corollary 3.0.5 below we also give a result in this direction.

In Theorem 3.0.2, the proximity of the solution u to a radial configuration is measured through the deficit $\operatorname{def}(\kappa)$. Clearly, if $\operatorname{def}(\kappa) = 0$ then u is radially symmetric and decreasing, so that [GNN79, Theorem 1′] is recovered. We point out that, if one is only interested in the statement concerning the almost radial symmetry of u, then weaker notions of deficit can be considered. Indeed, as confirmed by a careful analysis of the proof of Theorem 3.0.2, estimate (3.4) continues to hold if $\operatorname{def}(\kappa)$ is replaced by the zero-th order quantity

$$\sup_{r \in (0,1)} \operatorname*{osc}_{\partial B_r} \kappa + \sup_{e \in \partial B_1} \sup_{0 \leq \rho < r < 1} \big(\kappa(re) - \kappa(\rho e)\big)_+.$$

That is, the presence in $\operatorname{def}(\kappa)$ of a first order quantity such as the gradient of κ is only required to obtain the almost monotonicity statement (3.5).

We also point out that we required the nonlinearity f to be non-negative for the sole purpose of recovering exact symmetry when the right-hand side is radially symmetric and decreasing with respect to the space variable x. It is readily checked that, if f changes sign, our proof can still be applied in its essence. It yields estimates (3.4)-(3.5) with $\|\nabla \kappa\|_{L^\infty(B_1)}$ in place of $\operatorname{def}(\kappa)$ and provided assumption (3.3) is replaced by the stronger

$$\frac{1}{C_0}\big(1 - |x|\big) \leq u(x) \leq C_0 \quad \text{for all } x \in B_1. \tag{3.6}$$

The original proof of Theorem 3.0.1 is done via the method of moving planes, which in turns heavily relies on the maximum principle. Our proof of Theorem 3.0.2 is based on a quantitative version of this method. Hence, in it we replace the maximum principle with quantitative counterparts, such as the weak Harnack inequality and the ABP estimate.

The deficit of κ appears in estimates (3.4) and (3.5) raised to some unspecified power α. Its value could be explicitly computed following the proof of Theorem 3.0.2, but it does not feel of particular significance. We believe it could be an interesting future line of investigation to determine the sharpest value of α, at least for some specific choice of the nonlinearity f.

In order to obtain quantitative information on the asymmetry of the solution u, we need to understand its boundedness and positivity in a quantitative way. We do this through the imposition of assumption (3.3). As a consequence, the constants α and C appearing in estimates (3.4)-(3.5) depend on the constant C_0. Our next result provides conditions on the nonlinearity f which ensure the possibility of removing assumption (3.3) and thus making estimates (3.4)-(3.5) independent of the size of u.

Corollary 3.0.3. *Let $f : [0, +\infty) \to \mathbb{R}$ be a non-negative locally Lipschitz function satisfying the following two conditions:*

(a) Either $f(0) > 0$ or $f(s) \leq A s^p$ for all $s \in [0, s_0]$ and for some $A, s_0 > 0$, $p > 1$;

(b) Either $f(s) \leq B s^{q_1}$ for all $s \geq s_1$ and for some $B_1, s_1 > 0$, $q_1 \in (0,1)$, or the limit $\lim\limits_{s \to +\infty} \dfrac{f(s)}{s^{q_2}}$ exists finite and positive for some $q_2 \in (1, 2^\star - 1)$.[1]

Let $\kappa \in C^1(\overline{B_1})$ be a function satisfying $\kappa \geq \kappa_0$ in B_1, for some constant $\kappa_0 > 0$. Then, there exist two constants $C > 0$ and $\alpha \in (0,1)$, depending only on n, f, $\|\kappa\|_{L^\infty(B_1)}$, and κ_0, such that any solution $u \in C^2(B_1) \cap C^0(\overline{B_1})$ of (3.2) satisfies (3.4) and (3.5).

Corollary 3.0.3 follows almost immediately from Theorem 3.0.2. To obtain it, we simply verify that, under conditions (a) and (b), any solution of (3.2) satisfies the bounds (3.3) for some uniform constant C_0.

Condition (a) prescribes the growth of the nonlinearity f near the origin and is therefore the main tool needed to establish the lower bound on the L^∞ norm of u in (3.6). We point out that (a) allows virtually all possible behaviors for Lipschitz nonlinearities besides linear growth. On the other hand, condition (b) concerns the behavior of f at infinity and is thus connected with the upper bound in (3.3). Again, linear growth is excluded—both by the strict sublinearity assumption and by the Gidas-Spruck type asymptotic superlinearity/subcriticality prescription. We stress that this exception is to be expected, since for, say, $f(s) = \lambda_1 s$, with λ_1 being the first Dirichlet eigenvalue of B_1, no bound like (3.3) can hold with uniform constant C_0.

Theorem 3.0.2 is a particular case of a broader result, in which a perturbation of the Laplace operator is considered alongside a more general space-dependent nonlinearity. We introduce this framework here below.

Let L be a second order elliptic operator of the type

$$L[v] := -\mathrm{Tr}\big(A\, D^2 v\big) + b \cdot \nabla v, \tag{3.7}$$

[1] With the understanding that, if $n = 2$, the exponent q_2 can be any number strictly larger than 1.

where $A \in C^{1,\theta}(\overline{B}; \mathrm{Mat}_n(\mathbb{R}))$ and $b \in C^{1,\theta}(\overline{B}; \mathbb{R}^n)$, for some $\theta \in (0,1)$. We assume that $A(x)$ is symmetric for every $x \in B$ and that

$$\begin{gathered} \|A\|_{C^{1,\theta}(B_1)} + \|b\|_{C^{1,\theta}(B_1)} \leq \Lambda, \\ \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \geq \frac{1}{\Lambda}\,|\xi|^2 \quad \text{for all } x \in B_1 \text{ and } \xi \in \mathbb{R}^n, \end{gathered} \tag{3.8}$$

for some constant $\Lambda \geq 1$. Our main goal is to provide a quantitative symmetry result for the problem

$$\begin{cases} L[u] = g(\cdot, u) & \text{in } B_1 \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{3.9}$$

where $g \in C^{1,\theta}(\overline{B_1} \times [0, +\infty))$ is a non-negative function. In order to do this, we introduce a new deficit which quantifies how much the differential operator L differs from the Laplacian and how much h is far from a nonlinearity which assures radial symmetry of the solution. More precisely, given any real number $U \geq 0$ we define

$$\operatorname{def}(L, g, U) := \|A - I_n\|_{C^{0,1}(B_1)} + \|b\|_{C^{0,1}(B_1)} + G(g, U), \tag{3.10}$$

where $I_n \in \mathrm{Mat}_n(\mathbb{R})$ is the identity matrix and

$$G(g, U) := \sup_{s\in[0,U]} \|\nabla_x^T g(\cdot, s)\|_{L^\infty(B_1)} + \sup_{s\in[0,U]} \|\partial_r^+ g(\cdot, s)\|_{L^\infty(B_1)}.$$

Theorem 3.0.4. *Given $\theta \in (0,1)$, let $A \in C^{1,\theta}(\overline{B_1}; \mathrm{Mat}_n(\mathbb{R}))$ and $b \in C^{1,\theta}(\overline{B_1}; \mathbb{R}^n)$ be satisfying (3.8), for some $\Lambda \geq 1$, and let $g \in C^{1,\theta}_{\mathrm{loc}}\big(\overline{B_1} \times [0, +\infty)\big)$ be a non-negative function. Let $u \in C^2(B_1) \cap C^0(\overline{B_1})$ be a solution of (3.9), with L given by (3.7).*

Given any constant $C_0 \geq 1$, there exist two other constants $\alpha \in (0,1)$ and $C > 0$, depending only on n, θ, Λ, C_0, and on an upper bound on $\|g\|_{C^{1,\theta}(\overline{B}_1\times[0,C_0])}$, such that if u satisfies (3.3), then

$$|u(x) - u(y)| \leq C \operatorname{def}(L, g, C_0)^\alpha \quad \textit{for all } x, y \in B_1 \textit{ such that } |x| = |y| \tag{3.11}$$

and

$$\partial_r u(x) \leq C \operatorname{def}(L, g, C_0)^\alpha \quad \textit{for all } x \in B_1. \tag{3.12}$$

We mention that Theorem 3.0.4 can be applied for instance when one considers semilinear problems in a small normal perturbation of the ball, as done in [Ros94]. Indeed, we have the following corollary.

Corollary 3.0.5. *Given $0 < \epsilon \leq \epsilon_0$ and $\theta \in (0,1)$, let $\Psi_\epsilon : \overline{B_1} \to \mathbb{R}^n$ be an invertible map of class $C^{3,\theta}$ such that $\|\Psi_\epsilon - \mathrm{Id}\,\|_{C^{3,\theta}(B_1)} + \|\Psi_\epsilon^{-1} - \mathrm{Id}\,\|_{C^{3,\theta}(\Omega_\epsilon)} \leq \epsilon$, where $\Omega_\epsilon := \Psi_\epsilon(B_1)$. Let $u \in C^2(\Omega_\epsilon) \cap C^0(\overline{\Omega}_\epsilon)$ be a solution of*

$$\begin{cases} -\Delta u = f(u) & \textit{in } \Omega_\epsilon, \\ u > 0 & \textit{in } \Omega_\epsilon, \\ u = 0 & \textit{on } \partial\Omega_\epsilon, \end{cases} \tag{3.13}$$

where $f \in C^{1,\theta}_{\mathrm{loc}}([0,+\infty))$ is a nonnegative function, and assume that u satisfies

$$\frac{1}{C_0} \leq \|u\|_{L^\infty(\Omega_\epsilon)} \leq C_0, \tag{3.14}$$

for some constant $C_0 \geq 1$. Then, there exists a constant $C > 0$, depending only on n, θ, f, ϵ_0, and C_0, such that

$$|u(x) - u(y)| \leq C\, \epsilon^\alpha \quad \text{for all } x, y \in \Omega_\epsilon \text{ such that } \left|\Psi_\epsilon^{-1}(x)\right| = \left|\Psi_\epsilon^{-1}(y)\right|.$$

At the end of Section 3.4 we will exploit Corollary 3.0.5 to show approximate symmetry results for semilinear problems set in small perturbations of the unit ball, by means of a couple of examples.

This chapter is organized as follows. In Section 3.1 we recall some well-known facts and prove some preliminary results. Section 3.2 is devoted to the proof of Theorem 3.0.2, while that of Corollary 3.0.3 occupies the subsequent Section 3.3. The chapter is closed by Section 3.4, which contains the proofs of Theorem 3.0.4 and Corollary 3.0.5.

3.1 ABP-type estimate and weak Harnack inequality

In this section we collect some known results that will be used later. We begin by recalling the following version of the ABP estimate, due to Cabré [Cab95].

Lemma 3.1.1 ([Cab95])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $c : \Omega \to \mathbb{R}$ be a measurable function such that $c \geq 0$ a.e. in Ω, and $h \in L^n(\Omega)$. If $v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies*

$$-\Delta v + cv \leq h \quad \text{in } \Omega,$$

then

$$\sup_\Omega v \leq \sup_{\partial\Omega} v_+ + C\, |\Omega|^{\frac{1}{n}}\, \|h_+\|_{L^n(\Omega)},$$

for some dimensional constant $C \geq 1$.

Next, we have the following weak Harnack inequality. Given $\delta > 0$ and $\Omega \subset \mathbb{R}^n$, we write Ω_δ to indicate the set of points at distance more than δ from the boundary of Ω, that is

$$\Omega_\delta := \Big\{ x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta \Big\}.$$

Lemma 3.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Denote by $\mathfrak{r}^i_\Omega$ the inradius of Ω—i.e., the radius of the largest ball contained in Ω—and assume that*

$$\mathfrak{r}^i_\Omega \geq c_\sharp\, \mathrm{diam}(\Omega), \tag{3.15}$$

for some constant $c_\sharp \in (0, 1]$. Let $\delta \in \Big(0, \frac{\mathfrak{r}^i_\Omega}{3}\Big]$, $c \in L^\infty(\Omega)$, and $h \in L^n(\Omega)$. If $v \in C^2(\Omega)$ is a non-negative function satisfying

$$-\Delta v + cv \geq h \quad \text{in } \Omega,$$

then, it holds

$$\sup_{p \in \Omega_\delta} \left(\fint_{B_{\frac{\delta}{2}}(p)} v^s\, dx \right)^{\frac{1}{s}} \leq C \left(\frac{\mathfrak{r}^i_\Omega}{\delta} \right)^\beta \left(\inf_{\Omega_\delta} v + \|h_-\|_{L^n(\Omega)} \right), \tag{3.16}$$

for some positive constants s, β, and C depending only on n, $c_\sharp$, and on upper bounds on $\|c_+\|_{L^\infty(\Omega)}$, $\mathrm{diam}(\Omega)$.

The key point here is the polynomial dependence on δ of the constant appearing in (3.16). This occurs thanks to the *controlled* convexity of Ω, in the sense of (3.15)—in a general bounded domain the optimal dependence might instead be exponential. This phenomenon is possibly known to the expert reader and has been observed in [CMV16] for the full Harnack inequality.

Proof of Lemma 3.1.2. First of all, we observe that v satisfies $-\Delta v + c_+ v \geq -h_-$ in Ω. Hence, the standard weak Harnack inequality—see, e.g., [GT01, Theorem 9.22]—, yields that, if $B_r(q)$ is a ball such that $B_{2r}(q) \subset \Omega$, then

$$\left(\fint_{B_r(q)} v^s \, dx \right)^{\frac{1}{s}} \leq C_\star \left(\inf_{B_r(q)} v + r \| h_- \|_{L^n(B_{2r}(q))} \right), \tag{3.17}$$

for some constants $s > 0$ and $C_\star \geq 2$ depending only on n and on upper bounds on $\|c_+\|_{L^\infty(\Omega)}$, $\operatorname{diam}(\Omega)$.

Let $p_0 \in \Omega$ be a point for which $\operatorname{dist}(p_0, \partial\Omega) = \mathfrak{r}^i_\Omega$. Since $\delta < \mathfrak{r}^i_\Omega$, we have that $p_0 \in \Omega_\delta$. For any $p \in \Omega_\delta$, we claim that

$$\left(\fint_{B_{\frac{\mathfrak{r}^i_\Omega}{2}}(p_0)} v^s \, dx \right)^{\frac{1}{s}} \leq C_\flat \left(\frac{\mathfrak{r}^i_\Omega}{\delta} \right)^{\frac{\beta}{2}} \left(\inf_{B_{\frac{\delta}{2}}(p)} v + \| h_- \|_{L^n(\Omega)} \right), \tag{3.18}$$

$$\left(\fint_{B_{\frac{\delta}{2}}(p)} v^s \, dx \right)^{\frac{1}{s}} \leq C_\flat \left(\frac{\mathfrak{r}^i_\Omega}{\delta} \right)^{\frac{\beta}{2}} \left(\inf_{B_{\frac{\mathfrak{r}^i_\Omega}{2}}(p_0)} v + \| h_- \|_{L^n(\Omega)} \right), \tag{3.19}$$

for some constants $C_\flat \geq 1$ and $\beta > 0$ depending only on n, $c_\sharp$, and on upper bounds on $\|c_+\|_{L^\infty(\Omega)}$, $\operatorname{diam}(\Omega)$. It is clear that these estimates immediately lead to (3.16).

We only prove the validity of (3.18), since (3.19) can be established in a completely analogous fashion. To establish (3.18), we suppose, after a rigid movement, that $p_0 = 0$ and $p = \ell e_n$, where $\ell := |p - p_0|$. We initially assume that $2\ell \geq \mathfrak{r}^i_\Omega$. Denote by $\mathscr{C}$ and $\mathscr{C}'$ respectively the convex hulls of $B_{\mathfrak{r}^i_\Omega}(p_0) \cup B_\delta(p)$ and $B_{\mathfrak{r}^i_\Omega/2}(p_0) \cup B_{\delta/2}(p)$, i.e.,

$$\mathscr{C} = \bigcup_{t \in [0,1]} B_{(1-t)\mathfrak{r}^i_\Omega + t\delta}(t\ell e_n) \quad \text{and} \quad \mathscr{C}' = \bigcup_{t \in [0,1]} B_{\frac{(1-t)\mathfrak{r}^i_\Omega + t\delta}{2}}(t\ell e_n).$$

By the convexity of Ω, we have that $\mathscr{C}' \subset \mathscr{C} \subset \Omega$. Consider now the recursive sequence

$$\begin{cases} t_k := \dfrac{\mathfrak{r}^i_\Omega}{2\ell} + \dfrac{2\ell - \mathfrak{r}^i_\Omega + \delta}{2\ell} \, t_{k-1} & \text{for } k \in \mathbb{N}, \\ t_0 = 0, \end{cases}$$

as well as the balls

$$B_k := B_{r_k}(p_k) \quad \text{and} \quad B'_k := B_{\frac{r_k}{2}}(p_k),$$

where $r_k := (1 - t_k)\mathfrak{r}^i_\Omega + t_k \delta$ and $p_k := t_k \ell e_n$, for all $k \in \mathbb{N} \cup \{0\}$. It is easy to see that $\{t_k\}$ is a sequence of non-negative numbers, strictly increasing to $\frac{\mathfrak{r}^i_\Omega}{\mathfrak{r}^i_\Omega - \delta}$ as $k \to +\infty$. Clearly, $B_k \subset \mathscr{C}$ and

$B'_k \subset \mathscr{C}'$ for every $k \in \mathbb{N}\cup\{0\}$ such that $t_k \in [0,1]$. Furthermore, one checks that $B_{\frac{r_k}{4}}\left(p_k - \frac{t_k}{4}e_n\right) \subset B'_{k-1} \cap B'_k$ for every $k \in \mathbb{N}$ and thus

$$\frac{|B'_k|}{|B'_{k-1} \cap B'_k|} \leq 2^n \quad \text{for all } k \in \mathbb{N}. \tag{3.20}$$

Let $N \in \mathbb{N}$ be the largest integer for which $t_N \in [0,1)$. We claim that

$$N \leq C \log\left(\frac{\mathfrak{r}^i_\Omega}{\delta}\right), \tag{3.21}$$

for some constant $C > 0$ depending only on $c_\sharp$. To obtain (3.21), we observe that t_k is explicitly given by

$$t_k = \frac{\mathfrak{r}^i_\Omega}{2\ell} \sum_{j=0}^{k-1} \left(\frac{2\ell - \mathfrak{r}^i_\Omega + \delta}{2\ell}\right)^j = \frac{\mathfrak{r}^i_\Omega}{\mathfrak{r}^i_\Omega - \delta}\left[1 - \left(\frac{2\ell - \mathfrak{r}^i_\Omega + \delta}{2\ell}\right)^k\right],$$

for all $k \in \mathbb{N} \cup \{0\}$. Hence, the condition $t_N < 1$ is equivalent to the inequality

$$N < \frac{\log\left(\frac{\mathfrak{r}^i_\Omega}{\delta}\right)}{\log\left(1 + \frac{\mathfrak{r}^i_\Omega - \delta}{2\ell - \mathfrak{r}^i_\Omega + \delta}\right)}.$$

By (3.15) and the fact that $\delta \leq \frac{\mathfrak{r}^i_\Omega}{3}$, we see that $\frac{\mathfrak{r}^i_\Omega - \delta}{2\ell - \mathfrak{r}^i_\Omega + \delta} \geq \frac{c_\sharp}{3}$, from which (3.21) follows.

We now use the fundamental weak Harnack inequality (3.17) to compare the L^s norms of v over two consecutive balls in the chain B'_{k-1} and B'_k—recall that $B_k \subset \Omega$. By also taking into account (3.20), we compute

$$\begin{aligned}
\left(\fint_{B'_{k-1}} v^s\, dx\right)^{\frac{1}{s}} &\leq C_\star \left(\inf_{B'_{k-1}} v + r_{k-1}\|h_-\|_{L^n(B_{k-1})}\right) \leq C_\star \left(\inf_{B'_{k-1}\cap B'_k} v + \mathfrak{r}^i_\Omega\, \|h_-\|_{L^n(\Omega)}\right) \\
&\leq C_\star \left\{\left(\fint_{B'_{k-1}\cap B'_k} v^s\, dx\right)^{\frac{1}{s}} + \mathfrak{r}^i_\Omega\, \|h_-\|_{L^n(\Omega)}\right\} \\
&\leq C_\star \left\{2^n \left(\fint_{B'_k} v^s\, dx\right)^{\frac{1}{s}} + \mathfrak{r}^i_\Omega\, \|h_-\|_{L^n(\Omega)}\right\},
\end{aligned}$$

for every $k \in \{1, \ldots, N\}$. By chaining these estimates, we find that

$$\left(\fint_{B_{\frac{\mathfrak{r}^i_\Omega}{2}}(p_0)} v^s\, dx\right)^{\frac{1}{s}} \leq (2^n C_\star)^N \max\{2\mathfrak{r}^i_\Omega, 1\} \left\{\left(\fint_{B'_N} v^s\, dx\right)^{\frac{1}{s}} + \|h_-\|_{L^n(\Omega)}\right\}. \tag{3.22}$$

Arguing as before—using now that $\left|B'_N \cap B_{\delta/2}(p)\right| \geq \left|B_{\delta/4}\big(p - \frac{\delta}{4}e_n\big)\right| \geq 2^{-n}\left|B_{\delta/2}(p)\right|$ and estimate (3.17) a couple of times—we obtain

$$\left(\fint_{B'_N} v^s\, dx\right)^{\frac{1}{s}} \leq 2^n C_\star^2\, \max\{2\mathfrak{r}^i_\Omega, 1\}\left(\inf_{B_{\frac{\delta}{2}}(p)} v + \|h_-\|_{L^n(\Omega)}\right). \tag{3.23}$$

By combining this with (3.22) and recalling the upper bound (3.21) on N, estimate (3.18) readily follows, under the assumption that $2\ell \geq \mathfrak{r}_\Omega^i$.

When $2\ell < \mathfrak{r}_\Omega^i$, the computation is less involved, as the balls $B_{\mathfrak{r}_\Omega^i/2}(p_0)$ and $B_{\delta/2}(p)$ have large intersection—of measure comparable to $\left|B_{\delta/2}(p)\right|$. In view of this, (3.18) can be deduced at once by arguing exactly as for (3.23). The proof is thus complete. □

As an immediate consequence of the above lemma, we deduce the following Harnack inequality in the spherical dome

$$\Sigma_\lambda := B_1 \cap \{x_n > \lambda\}, \tag{3.24}$$

for $\lambda \in [0, 1)$. Given $\delta > 0$, we also consider the set

$$\Sigma_{\lambda,\delta} := \Big\{x \in \Sigma_\lambda : \operatorname{dist}(x, \partial\Sigma_\lambda) > \delta\Big\}. \tag{3.25}$$

Corollary 3.1.3. *Let $0 \leq \lambda \leq \lambda_0 < 1$, $\delta \in \big(0, \frac{1-\lambda_0}{6}\big]$, $c \in L^\infty(\Sigma_\lambda)$, and $h \in L^n(\Sigma_\lambda)$. Let $v \in C^2(\Sigma_\lambda)$ be a non-negative function satisfying*

$$-\Delta v + cv \geq h \quad \text{in } \Sigma_\lambda.$$

Then,

$$\sup_{p \in \Sigma_{\lambda,\delta}} \left(\fint_{B_{\frac{\delta}{2}}(p)} v^s \, dx \right)^{\frac{1}{s}} \leq \frac{C}{\delta^\beta} \left(\inf_{\Sigma_{\lambda,\delta}} v + \|h_-\|_{L^n(\Sigma_\lambda)} \right),$$

for some positive constants s, β, and C depending only on n, λ_0, and on an upper bound on $\|c_+\|_{L^\infty(\Sigma_\lambda)}$.

3.2 Proof of the main theorem

Our proof of Theorem 3.0.2 is via the method of moving planes. Before getting into the argument, we make a few preliminary observations.

Step 1: Preliminary remarks

First of all, as $0 \leq u \leq C_0$, κ is bounded, and f is locally bounded, by standard elliptic estimates there exists a constant $C_1 > 0$, depending only on n, C_0, $\|\kappa\|_{L^\infty(B_1)}$, and $\|f\|_{L^\infty([0,C_0])}$, for which

$$\|\nabla u\|_{L^\infty(B_1)} + [\nabla u]_{C^{\frac{9}{10}}(B_1)} \leq C_1. \tag{3.26}$$

Next, we claim that

$$u(x) \geq \frac{1}{C_2}\big(1 - |x|\big) \quad \text{for all } x \in B_1, \tag{3.27}$$

for some constant $C_2 \geq 1$ depending only on n, C_0, $\|\kappa\|_{L^\infty(B_1)}$, and $\|f\|_{L^\infty([0,C_0])}$. To see this, we first remark that, taking advantage of estimates (3.3) and (3.26), one easily obtains the existence of a point $p \in B_1$ and of a constant $r \in \big(0, \frac{1}{8}\big]$ such that

$$u \geq \frac{1}{2C_0} \text{ in } B_{2r}(p) \quad \text{and} \quad \operatorname{dist}\big(B_{2r}(p), \partial B_1\big) \geq 2r.$$

Note that the constant r only depends on n, C_0, $\|\kappa\|_{L^\infty(B_1)}$, and $\|f\|_{L^\infty([0,C_0])}$. We now seek a lower bound for u over $\overline{B_r}$. Clearly, we have that

$$B_{2r}(p) \subset B_{1-2r} \subset B_{1-r}(x) \subset B_1 \quad \text{for all } x \in \overline{B_r}.$$

As f and k are non-negative, the function u is superharmonic B_1. Hence, by the mean value theorem and the non-negativity of u in B_1 we obtain that

$$u(x) \geq \fint_{B_{1-r}(x)} u(y)\, dy \geq \frac{1}{|B_{1-r}|} \int_{B_{2r}(p)} u(y)\, dy \geq \frac{2^{n-1} r^n}{C_0} =: c_\sharp \quad \text{for all } x \in \overline{B_r}. \tag{3.28}$$

From this and the weak maximum principle we get that u is larger than the unique continuous functions which is harmonic in $B_1 \setminus \overline{B_r}$, vanishes on ∂B, and is equal to $c_\sharp$ on ∂B_r. Since this function is explicit—it is an appropriate affine transformation of the fundamental solution for the Laplacian in $\mathbb{R}^n$—, we easily deduce that

$$u(x) \geq c_\flat (1 - |x|) \quad \text{for all } x \in B_1 \setminus \overline{B_r},$$

for some constant $c_\flat > 0$ depending only on n, C_0, $\|\kappa\|_{L^\infty(B_1)}$, and $\|f\|_{L^\infty([0,C_0])}$. Claim (3.27) then readily follows from this estimate and (3.28).

Finally, we suppose without loss of generality that

$$\operatorname{def}(\kappa) \leq \gamma, \tag{3.29}$$

for some small $\gamma \in (0,1)$ to be chosen in dependence of n, $\|f\|_{C^{0,1}([0,C_0])}$, $\|\kappa\|_{L^\infty(B_1)}$, and C_0 only. Indeed, if $\operatorname{def}(\kappa) > \gamma$, then

$$|u(x) - u(y)| \leq |u(x)| + |u(y)| \leq 2C_0 \leq \frac{2C_0}{\gamma^\alpha} \operatorname{def}(\kappa)^\alpha \quad \text{for all } x, y \in B_1$$

and

$$\partial_r u(x) \leq |\nabla u(x)| \leq C_1 \leq \frac{C_1}{\gamma^\alpha} \operatorname{def}(\kappa)^\alpha \quad \text{for all } x \in B_1 \setminus \{0\},$$

for any $\alpha \in (0,1]$. Hence, claims (3.4) and (3.5) are trivially verified in this case.

Assuming (3.29) to hold true, we proceed with the proof. Clearly, claim (3.4) is equivalent to showing that, given any unit vector $e \in \partial B_1$, it holds

$$u(x) - u(x^{(e)}) \leq C \operatorname{def}(\kappa)^\alpha \quad \text{for all } x \in B_1 \text{ such that } x \cdot e > 0, \tag{3.30}$$

for some constants $C \geq 1$ and $\alpha \in (0,1)$ depending only on n, $\|f\|_{C^{0,1}([0,C_0])}$, $\|\kappa\|_{L^\infty(B_1)}$, and C_0, and where we write $x^{(e)} := x - 2(x \cdot e)e$ to indicate the symmetric point of x with respect to the hyperplane orthogonal to e passing through the origin. Up to a rotation, we may assume that $e = e_n$—note that the rotation of u solves an equation for a possibly different κ which, however, still satisfies (3.29). Under this assumption, (3.30) becomes

$$u(x', x_n) - u(x', -x_n) \leq C \operatorname{def}(\kappa)^\alpha \quad \text{for all } x \in B_1 \text{ such that } x_n > 0. \tag{3.31}$$

We shall establish (3.31) in the next four steps. In a further step we will then tackle the almost radial monotonicity statement (3.5).

Step 2: Starting the moving planes procedure.

Let $\lambda \in (0,1)$. Recalling definition (3.24), we consider the function

$$w_\lambda(x) := u(x^\lambda) - u(x) \quad \text{for } x \in \Sigma_\lambda,$$

where, for $x = (x', x_n)$, we define

$$x^\lambda := (x', 2\lambda - x_n).$$

Notice that w_λ is a solution of

$$-\Delta w_\lambda + c_\lambda w_\lambda = f_\lambda \quad \text{in } \Sigma_\lambda, \tag{3.32}$$

where

$$c_\lambda(x) := \begin{cases} -\kappa(x)\, \dfrac{f(u(x^\lambda)) - f(u(x))}{u(x^\lambda) - u(x)} & \text{if } u(x^\lambda) \neq u(x), \\ 0 & \text{if } u(x^\lambda) = u(x), \end{cases} \tag{3.33}$$

and

$$f_\lambda(x) := \big(\kappa(x^\lambda) - \kappa(x)\big) f(u(x^\lambda)), \tag{3.34}$$

for all $x \in \Sigma_\lambda$.

In view of equation (3.32), we see that $v := -w_\lambda$ satisfies

$$\begin{cases} -\Delta v + (c_\lambda)_+ v = -f_\lambda + (c_\lambda)_- v & \text{in } \Sigma_\lambda, \\ v \leq 0 & \text{on } \partial\Sigma_\lambda. \end{cases}$$

Hence, Lemma 3.1.1 gives that

$$\sup_{\Sigma_\lambda} v \leq C_3 |\Sigma_\lambda|^{\frac{1}{n}} \Big(\|(-f_\lambda)_+\|_{L^n(\Sigma_\lambda)} + \|(c_\lambda)_- v_+\|_{L^n(\Sigma_\lambda)} \Big), \tag{3.35}$$

for some dimensional constant $C_3 \geq 1$. Clearly,

$$\begin{aligned} \|(c_\lambda)_- v_+\|_{L^n(\Sigma_\lambda)} &\leq |\Sigma_\lambda|^{\frac{1}{n}} \|c_\lambda\|_{L^\infty(\Sigma_\lambda)} \|v_+\|_{L^\infty(\Sigma_\lambda)} \\ &\leq |\Sigma_\lambda|^{\frac{1}{n}} [f]_{C^{0,1}([0,C_0])} \|\kappa\|_{L^\infty(B_1)} \sup_{\Sigma_\lambda} v_+. \end{aligned} \tag{3.36}$$

On the other hand, considering the auxiliary point $\tilde{x}^\lambda := \frac{|x^\lambda|}{|x|}\, x$ and observing that

$$x \text{ and } \tilde{x}^\lambda \text{ belong to the same ray coming out of the origin} \tag{3.37}$$

and

$$|\tilde{x}^\lambda| = |x^\lambda| < |x|, \tag{3.38}$$

for every $x \in \Sigma_\lambda$ we estimate

$$\begin{aligned} -f_\lambda(x) &= \Big(\big(\kappa(x) - \kappa(\tilde{x}^\lambda)\big) + \big(\kappa(\tilde{x}^\lambda) - \kappa(x^\lambda)\big) \Big) f(u(x^\lambda)) \\ &\leq \|f\|_{L^\infty([0,C_0])} \left(\sup_{e \in \partial B_1} \sup_{0 \leq \rho < r < 1} \big(\kappa(re) - \kappa(\rho e)\big)_+ + \sup_{r \in (0,1)} \operatorname*{osc}_{\partial B_r} \kappa \right) \\ &\leq \pi \|f\|_{L^\infty([0,C_0])} \operatorname{def}(\kappa). \end{aligned}$$

Note that we exploited the non-negativity of f. Consequently,

$$\|(-f_\lambda)_+\|_{L^n(\Sigma_\lambda)} \le \pi |\Sigma_\lambda|^{\frac{1}{n}} \|f\|_{L^\infty([0,C_0])} \operatorname{def}(\kappa) \tag{3.39}$$

and thus, recalling (3.35) and (3.36),

$$\sup_{\Sigma_\lambda} v \le \frac{C_4}{2} |\Sigma_\lambda|^{\frac{2}{n}} \Big(\operatorname{def}(\kappa) + \sup_{\Sigma_\lambda} v_+ \Big),$$

with $C_4 := 8C_3 \left(1 + \|\kappa\|_{L^\infty(B)}\right) \|f\|_{C^{0,1}([0,C_0])}$. Now, if $\sup_{\Sigma_\lambda} v = \sup_{\Sigma_\lambda} v_+ > 0$, then rearranging terms in the previous inequality we deduce that

$$\sup_{\Sigma_\lambda} v \le \operatorname{def}(\kappa), \tag{3.40}$$

provided $C_4 |\Sigma_\lambda|^{\frac{2}{n}} \le 1$—which holds true, for instance, if $\lambda \ge 1 - \Big(C_4^{\frac{n}{2}} |B'| \Big)^{-1}$. Since (3.40) is also trivially satisfied when $\sup_{\Sigma_\lambda} v \le 0$, recalling the definition of v we conclude that

$$w_\lambda \ge -\operatorname{def}(\kappa) \quad \text{in } \Sigma_\lambda.$$

Consider the set

$$\Lambda := \Big\{ \lambda \in (0,1) : w_\mu \ge -\operatorname{def}(\kappa) \text{ in } \Sigma_\mu \text{ for all } \mu \in [\lambda, 1) \Big\}. \tag{3.41}$$

We just proved that Λ contains the interval $[\lambda_0, 1)$, where

$$\lambda_0 := \max \left\{ 1 - \Big(C_4^{\frac{n}{2}} |B'| \Big)^{-1}, \frac{1}{2} \right\}.$$

Its infimum

$$\lambda_\star := \inf \Lambda \tag{3.42}$$

is thus a well-defined real number lying in the interval $[0, \lambda_0]$.

Step 3: Reaching an intermediate position.

We first claim that

$$\lambda_\star \le \frac{1}{4}.$$

We argue by contradiction, and assume instead that $\lambda_\star \in \left(\frac{1}{4}, \lambda_0 \right]$. It is immediate to see that $\lambda_\star \in \Lambda$ and therefore that

$$w_{\lambda_\star} \ge -\operatorname{def}(\kappa) \quad \text{in } \Sigma_{\lambda_\star}.$$

Let now $v := w_{\lambda_\star} + \operatorname{def}(\kappa)$. Clearly, v satisfies

$$\begin{cases} -\Delta v + c_{\lambda_\star} v = f_{\lambda_\star} + c_{\lambda_\star} \operatorname{def}(\kappa) & \text{in } \Sigma_{\lambda_\star}, \\ v \ge 0 & \text{in } \Sigma_{\lambda_\star}. \end{cases} \tag{3.43}$$

Therefore, we may apply to it Corollary 3.1.3 and, by taking advantage of estimate (3.39) with $\lambda = \lambda_\star$, deduce that

$$\sup_{p \in \Sigma_{\lambda_\star,\delta}} \left(\fint_{B_{\frac{\delta}{2}}(p)} v^s \, dx \right)^{\frac{1}{s}} \leq \frac{C_5}{\delta^\beta} \left(\inf_{\Sigma_{\lambda_\star,\delta}} v + \operatorname{def}(\kappa) \right), \tag{3.44}$$

for every $\delta \in \left(0, \frac{1-\lambda_0}{6}\right]$, for three constants $s > 0$, $\beta > 0$, and $C_5 \geq 1$ depending only on n, $\|f\|_{C^{0,1}([0,C_0])}$, $\|\kappa\|_{L^\infty(B_1)}$, and with $\Sigma_{\lambda_\star,\delta}$ as in (3.25). Recalling the linear growth estimate (3.27), the gradient bound in (3.26), and the fact that u vanishes on the boundary of B_1, we now observe that

$$\begin{aligned}
\sup_{p \in \Sigma_{\lambda_\star,\delta}} \left(\fint_{B_{\frac{\delta}{2}}(p)} v^s \, dx \right)^{\frac{1}{s}} &\geq \left(\fint_{B_{\frac{\delta}{2}}((1-2\delta)e_n)} v^s \, dx \right)^{\frac{1}{s}} \geq \inf_{B_{\frac{\delta}{2}}((1-2\delta)e_n)} v \\
&= \inf_{x \in B_{\frac{\delta}{2}}((1-2\delta)e_n)} \big(u(x^{\lambda_\star}) - u(x)\big) + \operatorname{def}(\kappa) \\
&\geq \frac{1}{C_2} \min \left\{ 1 - \left| 2\lambda_\star - 1 + \frac{5}{2}\,\delta \right|, 1 - \left| 2\lambda_\star - 1 + \frac{3}{2}\,\delta \right| \right\} - 3\|\nabla u\|_{L^\infty(B_1)}\delta \\
&\geq \frac{1-\lambda_0}{2C_2} - 3C_1\delta
\end{aligned} \tag{3.45}$$

As a result,

$$\inf_{\Sigma_{\lambda_\star,\delta}} w_{\lambda_\star} \geq \frac{\delta^\beta}{C_5} \left(\frac{1-\lambda_0}{2C_2} - 3C_1\delta \right) - 2\operatorname{def}(\kappa). \tag{3.46}$$

Take now $\delta := \min\left\{ \frac{1-\lambda_0}{12C_1C_2}, \frac{(4C_4)^{-\frac{n}{2}}}{n+2} \right\} \in \left(0, \frac{1-\lambda_0}{6}\right]$ and assume that

$$\gamma \leq \frac{(1-\lambda_0)\delta^\beta}{16C_2C_5}.$$

By virtue of these choices, we easily deduce that $w_{\lambda_\star} \geq \frac{(1-\lambda_0)\delta^\beta}{8C_2C_5}$ in $\Sigma_{\lambda_\star,\delta}$. Consequently,

$$w_{\lambda_\star-\varepsilon}(x) \geq w_{\lambda_\star}(x) - 2\|\nabla u\|_{L^\infty(B_1)}\varepsilon \geq \frac{(1-\lambda_0)\delta^\beta}{8C_2C_5} - 2C_1\varepsilon \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_\star,\delta},$$

provided $\varepsilon > 0$ is sufficiently small. Hence, $v := -w_{\lambda_\star-\varepsilon}$ is a solution of

$$\begin{cases} -\Delta v + (c_{\lambda_\star-\varepsilon})_+ v = -f_{\lambda_\star-\varepsilon} + (c_{\lambda_\star-\varepsilon})_- v & \text{in } \Sigma', \\ v \leq 0 & \text{on } \partial\Sigma', \end{cases}$$

where $\Sigma' := \Sigma_{\lambda_\star-\varepsilon} \setminus \Sigma_{\lambda_\star,\delta}$. Taking advantage of Lemma 3.1.1, we then get that

$$\sup_{\Sigma'} v \leq \frac{C_4}{2}|\Sigma'|^{\frac{2}{n}} \left(\operatorname{def}(\kappa) + \sup_{\Sigma'} v_+ \right). \tag{3.47}$$

Observe that $\Sigma' \subset \left(\Sigma_{\lambda_\star-\varepsilon} \setminus \Sigma_{\lambda_\star+\delta}\right) \cup \left((B_1 \setminus B_{1-\delta}) \cap \{x_n \geq \lambda_\star + \delta\}\right)$ and thus that, recalling the definition of δ,

$$C_4|\Sigma'|^{\frac{2}{n}} \leq C_4\left(|B'|(\delta+\varepsilon) + |B_1 \setminus B_{1-\delta}|\right)^{\frac{2}{n}} \leq 4C_4\big((n+2)\delta\big)^{\frac{2}{n}} \leq 1, \tag{3.48}$$

provided $\varepsilon \leq \delta$. As a result, we easily infer from inequality (3.47) that

$$w_{\lambda_\star - \varepsilon} \geq -\mathrm{def}(\kappa) \quad \text{in } \Sigma_{\lambda_\star - \varepsilon} \quad \text{for all } \varepsilon \in [0, \varepsilon_0],$$

for some small $\varepsilon_0 > 0$, contradicting the fact that $\lambda_\star$ is the infimum of Λ.

Step 4: Going almost all the way.

We claim that

$$\lambda_\star \leq \left(3C_2C_5C_6^\beta \,\mathrm{def}(\kappa)\right)^{\frac{1}{1+\beta}}, \tag{3.49}$$

where $C_6 := \max\left\{3C_1C_2, (4C_4)^{\frac{n}{2}}(n+2)\right\} \geq 2$. Notice that

$$\left(3C_2C_5C_6^\beta \,\mathrm{def}(\kappa)\right)^{\frac{1}{1+\beta}} < \frac{1}{4},$$

provided we take

$$\gamma \leq \frac{1}{4^{2+\beta}C_2C_5C_6^\beta}.$$

To establish (3.49), we argue once again by contradiction and suppose that

$$\lambda_\star \in \left(\left(3C_2C_5C_6^\beta \,\mathrm{def}(\kappa)\right)^{\frac{1}{1+\beta}}, \frac{1}{4}\right]. \tag{3.50}$$

As before, we have that the function $v := w_{\lambda_\star} + \mathrm{def}(\kappa)$ satisfies (3.43) and therefore estimate (3.44) for every $\delta \in \left(0, \frac{1}{8}\right]$, by Corollary 3.1.3. Computing as for (3.45)-(3.46), we find that

$$\inf_{\Sigma_{\lambda_\star,\delta}} w_{\lambda_\star} \geq \frac{\delta^\beta}{C_5}\left(\frac{2\lambda_\star}{C_2} - 3C_1\delta\right) - 2\,\mathrm{def}(\kappa).$$

Taking $\delta := \min\left\{\frac{\lambda_\star}{C_6}, \frac{(4C_4)^{-\frac{n}{2}}}{n+2}\right\} \in \left(0, \frac{1}{8}\right]$, by recalling (3.50) and the definition of C_6 we get $w_{\lambda_\star} \geq \frac{\lambda_\star^{1+\beta}}{3C_2C_5C_6^\beta}$ in $\Sigma_{\lambda_\star,\delta}$. As a result, $w_{\lambda_\star-\varepsilon} \geq 0$ in $\Sigma_{\lambda_\star,\delta}$ if $\varepsilon > 0$ is small enough and, arguing as before,

$$\sup_{\Sigma'}(-w_{\lambda_\star-\varepsilon}) \leq \frac{C_4}{2}|\Sigma'|^{\frac{2}{n}}\left(\mathrm{def}(\kappa) + \sup_{\Sigma'}(-w_{\lambda_\star-\varepsilon})_+\right), \tag{3.51}$$

where $\Sigma' := \Sigma_{\lambda_\star-\varepsilon} \setminus \Sigma_{\lambda_*,\delta}$. Since we still have the bound (3.48) on the measure of Σ'—thanks to the definitions of δ and C_6, and provided we take $\varepsilon \leq \delta$—, it easily follows from inequality (3.51) that

$$w_{\lambda_\star - \varepsilon} \geq -\mathrm{def}(\kappa) \quad \text{in } \Sigma_{\lambda_\star - \varepsilon} \quad \text{for all } \varepsilon \in [0, \varepsilon_0],$$

for some small $\varepsilon_0 > 0$. This contradicts the definition of $\lambda_\star$ and thus (3.49) holds true.

Step 5: Almost radial symmetry in one direction.

Thus far, we have proved that

$$u(x', x_n) - u(x', 2\lambda - x_n) \le \operatorname{def}(\kappa) \quad \text{for all } (x', x_n) \in \Sigma_\lambda \text{ and } \lambda \in [\lambda_1, 1),$$

with

$$\lambda_1 := \Big(3C_2C_5C_6^\beta \operatorname{def}(\kappa)\Big)^{\frac{1}{1+\beta}}. \tag{3.52}$$

By choosing $\lambda = \lambda_1$ and recalling the gradient bound (3.26), we get that

$$u(x', x_n) - u(x', -x_n) \le C_7 \operatorname{def}(\kappa)^{\frac{1}{1+\beta}} \quad \text{for all } (x', x_n) \in \Sigma_{\lambda_1}.$$

for some constant $C_7 \ge 1$ depending only on n, C_0, $\|\kappa\|_{L^\infty(B_1)}$ and $\|f\|_{C^{0,1}([0,C_0])}$. On top of this, we also have that, for $(x', x_n) \in \Sigma_0 \setminus \Sigma_{\lambda_1}$,

$$\begin{aligned} u(x', x_n) - u(x', -x_n) &\le |u(x', x_n) - u(x', 0)| + |u(x', -x_n) - u(x', 0)| \\ &\le 2\|\nabla u\|_{L^\infty(B)} x_n \le 2C_1\lambda_1 \le C_7 \operatorname{def}(\kappa)^{\frac{1}{1+\beta}}, \end{aligned}$$

up to possibly taking a larger C_7. The last two inequalities yield the validity of (3.31).

Step 6: Almost monotonicity in the radial direction.

Our goal is to show that

$$\partial_n u(x) \le C \operatorname{def}(\kappa)^{\frac{1}{1+\beta}} \quad \text{for all } x \in B_1 \text{ such that } x_n > 0, \tag{3.53}$$

for some constant $C > 0$ depending only on n, C_0, $\|\kappa\|_{L^\infty(B_1)}$, and $\|f\|_{C^{0,1}([0,C_0])}$. It is clear that, by specializing this to the points $x = (0, x_n)$ with $x_n \in (0, 1)$ and up to a rotation, this yields (3.5).

Let $\lambda \in [\lambda_1, 1)$ with λ_1 defined as in (3.52) and let $\varepsilon > 0$ to be soon chosen small. Setting $v := -w_\lambda$ it holds

$$L_\lambda v := -\Delta v + c_\lambda v = -f_\lambda \quad \text{in } N_{\lambda,\varepsilon},$$

where c_λ and f_λ are as in (3.33) and (3.34), while $N_{\lambda,\varepsilon} := \Sigma_\lambda \setminus \Sigma_{\lambda+\varepsilon}$. Note that, by definition (3.24), if

$$\varepsilon \ge 1 - \lambda, \tag{3.54}$$

then $\Sigma_{\lambda+\varepsilon}$ is empty, in which case we simply have $N_{\lambda,\varepsilon} = \Sigma_\lambda$. We plane to achieve (3.53) by constructing a supersolution for the operator L_λ in $N_{\lambda,\varepsilon}$. In order to do this, we need appropriate estimates on the coefficient c_λ and the right-hand side $-f_\lambda$.

Definition (3.33) immediately yields

$$\|c_\lambda\|_{L^\infty(\Sigma_\lambda)} \le [f]_{C^{0,1}([0,C_0])} \|\kappa\|_{L^\infty(B_1)} =: Z. \tag{3.55}$$

From this it follows that there exists a constant $\varepsilon_0 > 0$, depending only on $[f]_{C^{0,1}([0,C_0])}$ and $\|\kappa\|_{L^\infty(B_1)}$, such that

$$\text{the weak maximum principle holds for } L_\lambda \text{ in } N_{\lambda,\varepsilon}, \tag{3.56}$$

for every $\varepsilon \in (0, \varepsilon_0]$, thanks to the maximum principle for narrow domains—see, e.g., [BNSV94] or [GT01, Section 3.3].

The estimate of $-f_\lambda$ from above requires a bit more work. Given $x \in \Sigma_\lambda$, let $\tilde{x}^\lambda := \frac{|x^\lambda|}{|x|}\, x$ and recall that (3.37) and (3.38) hold true. We also have that

$$|x| - |\tilde{x}^\lambda| \leq 2(x_n - \lambda) \quad \text{and} \quad \operatorname{dist}_{\partial B_{|x^\lambda|}}\big(\tilde{x}^\lambda, x^\lambda\big) \leq 2\pi(x_n - \lambda), \tag{3.57}$$

where $\operatorname{dist}_{\partial B_r}$ denotes the geodesic distance on the sphere of radius $r > 0$—we can disregard the case $|x^\lambda| = 0$ since, if this occurs, then $\tilde{x}^\lambda = x^\lambda = 0$. The first inequality in (3.57) follows right away from the definition of $\tilde{x}^\lambda$, while the second can be obtained noticing that $\operatorname{dist}_{\partial B_r}(p, q) \leq \frac{\pi}{2}|p - q|$ for all $p, q \in \partial B_r$ and computing as follows:

$$\begin{aligned}
|\tilde{x}^\lambda - x^\lambda|^2 &= 2\left(|x^\lambda|^2 - \tilde{x}^\lambda \cdot x^\lambda\right) = 2\,\frac{|x^\lambda|}{|x|}\left(|x||x^\lambda| - x \cdot x^\lambda\right) \leq 2\,\frac{|x|^2|x^\lambda|^2 - (x \cdot x^\lambda)^2}{|x||x^\lambda| + x \cdot x^\lambda} \\
&= 2\,\frac{\left(|x'|^2 + x_n^2\right)\left(|x'|^2 + (2\lambda - x_n)^2\right) - \left(|x'|^2 + x_n(2\lambda - x_n)\right)^2}{\sqrt{|x'|^2 + x_n^2}\sqrt{|x'|^2 + (2\lambda - x_n)^2} + |x'|^2 + x_n(2\lambda - x_n)} \\
&\leq 8\,\frac{|x'|^2\,(x_n - \lambda)^2}{x_n|2\lambda - x_n| + |x'|^2 + x_n(2\lambda - x_n)} \leq 8(x_n - \lambda)^2.
\end{aligned}$$

By virtue of (3.37), (3.38), and (3.57), recalling definition (3.34) we obtain

$$\begin{aligned}
-f_\lambda(x) &= \Big(\big(\kappa(x) - \kappa(\tilde{x}^\lambda)\big) + \big(\kappa(\tilde{x}^\lambda) - \kappa(x^\lambda)\big)\Big) f(u(x^\lambda)) \\
&\leq 2\pi\|f\|_{L^\infty([0,C_0])}\left(\|\partial_r^+\kappa\|_{L^\infty(B_1)} + \|\nabla^T\kappa\|_{L^\infty(B_1)}\right)(x_n - \lambda) \\
&= 2\pi\|f\|_{L^\infty([0,C_0])}\operatorname{def}(\kappa)(x_n - \lambda).
\end{aligned} \tag{3.58}$$

Note that here we also took advantage of the fact that f is non-negative.

Now that we understood the sizes of c_λ and $-f_\lambda$, we are in position to construct an upper barrier $\bar{v}$ for the function v in $N_{\lambda,\varepsilon}$. For $M, \mu > 0$, let

$$\bar{v}(x) := M \sin\big(\mu\,(x_n - \lambda)\big).$$

Recalling (3.55), for every $x \in N_{\lambda,\varepsilon}$ we have

$$L_\lambda \bar{v}(x) = (\mu^2 + c_\lambda)\,\bar{v}(x) \geq (\mu^2 - Z)\,\bar{v}(x) \geq \bar{v}(x) \geq \frac{2M\mu}{\pi}(x_n - \lambda),$$

if we take $\mu^2 \geq Z + 1$ and $\mu\,\varepsilon \leq \pi/2$. Going back to (3.58), this gives that

$$L_\lambda \bar{v} \geq -f_\lambda \quad \text{in } N_{\lambda,\varepsilon}, \tag{3.59}$$

provided $M\mu \geq \pi^2\|f\|_{L^\infty([0,C_0])}\operatorname{def}(\kappa)$.

To deal with the boundary condition, we decompose $\partial N_{\lambda,\varepsilon}$ as $\partial N_{\lambda,\varepsilon} = D \cup T_\lambda \cup T_{\lambda+\varepsilon}$, where $D := \{\lambda \leq x_n \leq \lambda + \varepsilon\} \cap \partial B_1$ is the round part, while $T_\lambda := \{x_n = \lambda\} \cap B_1$ and $T_{\lambda+\varepsilon} := \{x_n = \lambda + \varepsilon\} \cap B_1$ are the flat parts—note that, if (3.54) is satisfied, then $T_{\lambda+\varepsilon} = \varnothing$ and $\partial N_{\lambda,\varepsilon}$ is only made up of the round part D and a single flat part T_λ. Observe that

$$\begin{cases} v = 0 = \bar{v} & \text{on } T_\lambda, \\ v < 0 < \bar{v} & \text{on } D. \end{cases}$$

When (3.54) does not hold, recalling definitions (3.41) and (3.42), as well as the fact that $\lambda \geq \lambda_\star$, thanks to (3.49) and the way we took λ, we also have

$$v - \bar{v} \leq \mathrm{def}(\kappa) - \frac{2M\mu\,\varepsilon}{\pi} \leq 0 \quad \text{on } T_{\lambda+\varepsilon},$$

provided $M\mu\,\varepsilon \geq \frac{\pi}{2}\,\mathrm{def}(\kappa)$. Thus, whether (3.54) is satisfied of not, we get that $v \leq \bar{v}$ on $\partial N_{\lambda,\varepsilon}$.

Thanks to this, (3.56), and (3.59), by setting

$$\mu := \sqrt{Z+1}, \quad \varepsilon := \min\left\{\frac{\varepsilon_0}{2}, \frac{\pi}{2\mu}\right\}, \quad M := \frac{\pi}{\mu}\max\left\{\pi\|f\|_{L^\infty([0,C_0])}, \frac{1}{2\varepsilon}\right\}\mathrm{def}(\kappa)$$

and applying the weak maximum principle we find that

$$u(x) - u(x^\lambda) = v(x) \leq \bar{v}(x) \leq 2\,C_8\,\mathrm{def}(\kappa)\,(x_n - \lambda) \quad \text{for all } x \in N_{\lambda,\varepsilon},$$

for some constant $C_8 > 0$ depending only on n, $\|f\|_{C^{0,1}([0,C_0])}$, and $\|\kappa\|_{L^\infty(B_1)}$. Letting $x_n \to \lambda^+$ in the above relation, we get that $\partial_n u(x) \leq C_8\,\mathrm{def}(\kappa)$ for all $x \in N_{\lambda,\varepsilon}$ and $\lambda \in [\lambda_1, 1)$, that is

$$\partial_n u(x) \leq C_8\,\mathrm{def}(\kappa) \quad \text{for all } x \in \Sigma_{\lambda_1}. \tag{3.60}$$

If instead $x \in \Sigma_0 \setminus \Sigma_{\lambda_1}$, taking advantage of (3.26) we get

$$\begin{aligned} \partial_n u(x', x_n) &\leq \partial_n u(x', \lambda_1) + (\lambda_1 - x_n)^{\frac{9}{10}}\,[\nabla u]_{C^{\frac{9}{10}}(B_1)} \\ &\leq C_8\,\mathrm{def}(\kappa) + C_1\lambda_1^{\frac{9}{10}} \leq C_8\,\mathrm{def}(\kappa)^{\frac{9}{10}\frac{1}{1+\beta}}, \end{aligned} \tag{3.61}$$

where $C_8 > 0$ only depends on n, $\|f\|_{C^{0,1}([0,C_0])}$, $\|\kappa\|_{L^\infty(B_1)}$, and C_0. By putting together (3.60) and (3.61), we are led to (3.53). The proof of Theorem 3.0.2 is thus complete.

3.3 Proof of the main corollary

In order to establish Corollary 3.0.3, it suffices to show that, under conditions (a) and (b), any solution $u \in C^2(B_1) \cap C^0(\overline{B_1})$ of (3.2) satisfies the bounds (3.3) for some constant $C_0 \geq 1$ depending only on n, f, $\|\kappa\|_{L^\infty(B_1)}$, and κ_0. Indeed, once this is obtained, the result immediately follows from Theorem 3.0.2.

Step 1: Uniform bound from above

We begin by showing that (a) yields the upper bound in (3.3). Note that this is an immediate consequence of the classical a priori estimate of Gidas & Spruck [GS81] when f is asymptotically subcritical and superlinear at infinity.

If, on the other hand, f is strictly sublinear at infinity, then $f(s) \leq C'(1 + s^{q_1})$ for all $s \geq 0$, for some constants $C' > 0$ and $q_1 \in (0, 1)$. The claim now easily follows by testing the equation against u. Indeed, by doing this we get

$$\int_{B_1} |\nabla u|^2\,dx = \int_{B_1} \kappa f(u) u\,dx \leq C'\|\kappa\|_{L^\infty(B_1)}\left(\int_{B_1} u\,dx + \int_{B_1} u^{q_1+1}\,dx\right).$$

By combining this estimate with the Hölder's, Young's, and Poincaré's inequalities, we easily get that $\|u\|_{H^1(B_1)}$ is bounded by some constant depending only on n, q_1, C', and $\|\kappa\|_{L^\infty(B_)}$. By virtue of this estimate and, again, the sublinearity of f, a uniform $L^\infty(B_1)$ bound on u is obtained for instance by bootstrapping Calderón-Zygmund estimates.

Step 2: Uniform bound from below

We now address the validity of the lower bound in (3.3) under assumption (a).

When $f(0) > 0$, the result follows by comparison with an appropriate multiple of the torsion function. Indeed, by the continuity of f and the non-negativity of both f and κ, there exists $\epsilon \in (0,1]$ such that $\kappa(x) f(u) \geq \epsilon\, \chi_{[0,\epsilon]}(u)$ for every $x \in B_1$ and $u \geq 0$. Consequently, u is a supersolution of $-\Delta u = \epsilon \chi_{[0,\epsilon]}(u)$ in B_1. On the other hand, the function $\underline{u}(x) := \frac{\epsilon}{2n}(1 - |x|^2)$ is a solution of $-\Delta \underline{u} = \epsilon\, \chi_{[0,\epsilon]}(\underline{u})$ in B_1. From the comparison principle—note that the nonlinearity $\epsilon\chi_{[0,\epsilon]}$ is non-increasing in $[0, +\infty)$—we deduce that $u \geq \underline{u}$ in B_1, which gives the lower bound in (3.3).

Suppose now that the alternative assumption holds in (a), namely that $f(s) \leq A s^p$ for all $s \in [0, s_0]$, for some constants $A, s_0 > 0$ and $p > 1$. Without loss of generality, we may assume that $s_0 \in (0,1)$ and $p \in (1, 2^\star - 1)$ when $n \geq 3$. In order to get the bound from below in (3.3), we first observe that either $\|u\|_{L^\infty(B_1)} > s_0$—in which case we are done—or $0 \leq u(x) \leq s_0$ for every $x \in B_1$. Assuming the latter, we test the equation for u against u itself, obtaining

$$\int_{B_1} |\nabla u|^2\, dx = \int_{B_1} \kappa f(u) u \, dx \leq A \|\kappa\|_{L^\infty(B)} \int_{B_1} u^{p+1}\, dx.$$

By the Poincaré-Sobolev and Hölder inequalities, we deduce from this that

$$\|u\|^2_{L^{p+1}(B_1)} \leq C_\sharp A \|\kappa\|_{L^\infty(B_1)} \|u\|^{p+1}_{L^{p+1}(B_1)},$$

for some constant $C_\sharp > 0$ depending only on n and p. Since $p > 1$, we can reabsorb to the right the $L^{p+1}(B_1)$ norm appearing on the left-hand side. By doing this, we get that $\|u\|_{L^{p+1}(B_1)} \geq \big(C_\sharp A \|\kappa\|_{L^\infty(B_1)}\big)^{\frac{1}{1-p}}$. Our claim immediately follows from this and the proof is thus complete.

3.4 Proof of the generalized results

This section is devoted to the proof of Theorem 3.0.4 and Corollary 3.0.5. At the end of it, we will add a couple of examples of application of this last result.

In Theorem 3.0.4 we provide a quantitative symmetry result for problem (3.9), measured in terms of the deficit defined in (3.10). Its proof is analogous to that of Theorem 3.0.2. For this reason, in what follows we only highlight the main differences.

Sketch of the proof of Theorem 3.0.4. As we just mentioned, the proof follows the lines of that of Theorem 3.0.2. Here we only outline the modifications in what corresponds there to Steps 1, 2, and 6—with the understanding that the remaining parts follow by analogous arguments.

Let $g^\star > 0$ be such that $\|g\|_{C^{1,\theta}(\overline{B_1} \times [0, C_0])} \leq g^\star$. We begin by observing that, since u satisfies (3.3), by standard elliptic estimates (see, e.g., [GT01, Theorem 6.19]) there exists a constant $C_{11} > 0$, depending only on n, θ, Λ, $g^\star$, and C_0, for which

$$\|\nabla u\|_{L^\infty(B_1)} + \|D^2 u\|_{L^\infty(B_1)} + \|D^3 u\|_{L^\infty(B_1)} \leq C_{11}. \tag{3.62}$$

Furthermore, we assume without loss of generality that

$$\operatorname{def}(L, g, C_0) \leq \gamma$$

for some small $\gamma \in (0, 1/2]$ to be chosen in dependence of n, θ, Λ, $g^\star$, and C_0 only.

The main goal of the first step is to show that

$$u(x) \geq \frac{1}{C_{12}}(1 - |x|) \quad \text{for all } x \in B_1, \tag{3.63}$$

for some constant $C_{12} \geq 1$ depending only on n, θ, Λ, C_0, and g_*. Similarly to what we did when proving (3.27), we start by noticing that, thanks to (3.3) and (3.62), there exists a point $p_0 \in B$ and a constant $r \in (0, \frac{1}{4}]$ such that

$$u \geq \frac{1}{2C_0} \text{ in } B_r(p_0) \quad \text{and} \quad \operatorname{dist}(B_r(p_0), \partial B_1) \geq r. \tag{3.64}$$

From this we easily get a lower bound on the function u in a ball centered at the origin. To do it, one can use a version of the weak Harnack inequality of Lemma 3.1.2 for the operator L. Eventually, we get that $u \geq \frac{1}{C_\sharp}$ in $B_{1/2}$ for some constant $C_\sharp \geq 1$ depending only on n, θ, Λ, $g^\star$, and C_0. In order to prove (3.63), it then suffices to build a lower barrier for u inside the annulus $B_1 \setminus B_{1/2}$. For instance, the function

$$\varphi(x) := \delta \left(e^{M(1-|x|^2)} - 1 \right),$$

for some small enough $\delta \in (0, 1)$ and large enough $M \geq 1$, both in dependence of n, θ, Λ, $g^\star$, and C_0 only, satisfies

$$\begin{cases} L\varphi \leq 0 & \text{in } B_1 \setminus \overline{B_1}_{\frac{1}{2}}, \\ \varphi = 0 & \text{on } \partial B_1, \\ \varphi \leq \frac{1}{C_\sharp} & \text{on } \partial B_{\frac{1}{2}}. \end{cases}$$

Claim (3.63) then follows from the maximum principle and estimate (3.64).

We proceed towards the proof of statements (3.11) and (3.12). By rotating the coordinate frame, they will be established if we prove that

$$u(x', x_n) - u(x', -x_n) \leq C \operatorname{def}(L, g, C_0)^\alpha \quad \text{for all } x \in B \text{ such that } x_n > 0 \tag{3.65}$$

and

$$\partial_n u(x) \leq C \operatorname{def}(L, g, C_0)^\alpha \quad \text{for all } x \in B_1 \text{ such that } x_n > 0, \tag{3.66}$$

for some constants $C \geq 1$ and $\alpha \in (0, 1)$ depending only on n, θ, Λ, $g^\star$, and C_0. Note that a rotation typically leads to an equation for a different operator $\hat{L}$ and right-hand side $\hat{g}$. However, $\operatorname{def}(\hat{L}, \hat{g}, C_0) = \operatorname{def}(L, g, C_0)$ and the coefficients of $\hat{L}$ still satisfy assumption (3.8). For this reason, in what follows we still indicate these two objects by L and g.

To prove (3.65), we use the moving planes method. We do not reproduce here the full argument displayed in Steps 2-5 of the proof of Theorem 3.0.2, but only outline why the method works in this setting as well. Let $\lambda \in (0, 1)$ and consider the function $w_\lambda(x) = u(x^\lambda) - u(x)$ for $x \in \Sigma_\lambda$. A straightforward computation yields that w_λ is a solution of

$$-\Delta w_\lambda + \tilde{c}_\lambda w_\lambda = g_\lambda + \mathscr{R}_A + \mathscr{R}_b \quad \text{in } \Sigma_\lambda,$$

where

$$
\begin{aligned}
\tilde{c}_\lambda(x) &:= \begin{cases} -\dfrac{g\big(x, u(x^\lambda)\big) - g\big(x, u(x)\big)}{u(x^\lambda) - u(x)} & \text{if } u(x^\lambda) \neq u(x), \\ 0 & \text{if } u(x^\lambda) = u(x), \end{cases} \\
g_\lambda(x) &:= g\big(x^\lambda, u(x^\lambda)\big) - g\big(x, u(x^\lambda)\big), \\
\mathscr{R}_A(x) &:= \operatorname{Tr}\big((A(x^\lambda) - I_n) D^2 u(x^\lambda)\big) - \operatorname{Tr}\big((A(x) - I_n) D^2 u(x)\big), \\
\mathscr{R}_b(x) &:= b(x) \cdot \nabla u(x) - b(x^\lambda) \cdot \nabla u(x^\lambda),
\end{aligned}
$$

for $x \in \Sigma_\lambda$. A careful inspection of Steps 2-5 in the proof of Theorem 3.0.2 shows that the argument goes through almost verbatim provided that the two remainder terms $\mathscr{R}_A$ and $\mathscr{R}_b$ can be controlled by a multiple of the deficit—one deals with g_λ exactly as we did with f_λ in (3.39). This is indeed the case, since, recalling (3.62),

$$
\begin{aligned}
\|\mathscr{R}_A\|_{L^\infty(\Sigma_\lambda)} + \|\mathscr{R}_b\|_{L^\infty(\Sigma_\lambda)} &\leq 2\Big(\|A - I_n\|_{L^\infty(B_1)} \|D^2 u\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \|\nabla u\|_{L^\infty(B_1)} \Big) \\
&\leq 2C_1 \operatorname{def}(L, g, C_0).
\end{aligned}
$$

Hence, we conclude that estimate (3.65) holds true.

In order to achieve (3.66), we proceed as in Step 6 of the proof of Theorem 3.0.2. For the argument to work, we need $\mathscr{R}_A$ and $\mathscr{R}_b$ to be linearly growing away from $\{x_n = \lambda\}$. This follows from the estimates

$$
\begin{aligned}
|\mathscr{R}_a(x)| &\leq \big|\operatorname{Tr}\big((A(x^\lambda) - A(x)) D^2 u(x^\lambda)\big)\big| + \big|\operatorname{Tr}\big((A(x) - I_n)(D^2 u(x^\lambda) - D^2 u(x))\big)\big| \\
&\leq \big([A]_{C^{0,1}(B_1)} \|D^2 u\|_{L^\infty(B_1)} + \|A - I_n\|_{L^\infty(B_1)} \|D^3 u\|_{L^\infty(B_1)}\big)\, |x^\lambda - x| \\
&\leq 2C_1 \operatorname{def}(L, g, C_0)\, (x_n - \lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathscr{R}_b(x)| &\leq \big|\big(b(x) - b(x^\lambda)\big) \cdot \nabla u(x)\big| + \big|b(x^\lambda) \cdot \big(\nabla u(x) - \nabla u(x^\lambda)\big)\big| \\
&\leq \big([b]_{C^{0,1}(B_1)} \|\nabla u\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \|D^2 u\|_{L^\infty(B_1)}\big)\, |x^\lambda - x| \\
&\leq 2C_1 \operatorname{def}(L, g, C_0)\, (x_n - \lambda),
\end{aligned}
$$

which hold true for all $x \in \Sigma_\lambda$, thanks to (3.62). Once this is established, one obtains (3.66) by means of a barrier, precisely as in Step 6 of the proof of Theorem 3.0.2. □

As we mentioned at the beginning of the chapter, Theorem 3.0.4 can be used to provide almost symmetry results for semilinear problems set in a small normal perturbation of the ball. This is the claim of Corollary 3.0.5, which we establish here.

Proof of Corollary 3.0.5. Let $v := u \circ \Psi_\epsilon$. Writing $\Phi_\epsilon := \Psi_\epsilon^{-1}$, it is clear that v satisfies

$$
\begin{cases} L[v] = f(v) & \text{in } B_1 \\ v > 0 & \text{in } B_1 \\ v = 0 & \text{in } \partial B_1\,, \end{cases}
$$

where L is of the form (3.7) with

$$A_{ij} = \sum_{k=1}^{n} \left(\partial_k \Phi^i_\epsilon \circ \Psi_\epsilon\right) \left(\partial_k \Phi^j_\epsilon \circ \Psi_\epsilon\right) \quad \text{and} \quad b_i = -\Delta \Phi^i_\epsilon \circ \Psi_\epsilon,$$

for $i, j = 1, \dots, n$. Clearly, $\operatorname{def}(L, f, C_0) \leq C\epsilon$, for some constant $C > 0$ depending only on n. Consequently, the assertion of the corollary follows by a direct application of Theorem 3.0.4. □

We conclude the section with a couple of examples containing possible applications of Corollary 3.0.5.

Example 3.4.1. Let $\Omega_\epsilon \subset \mathbb{R}^n$ be an ellipsoid with small eccentricity. A simple computation gives that the solution of the torsional problem—i.e., (3.13) with $f \equiv 1$—is explicit and its level sets coincide with dilations of $\partial\Omega_\epsilon$. This last fact is no longer true for a general nonlinearity f. However, Corollary 3.0.5 can be used to recover an approximate symmetry result. To see it, consider for simplicity the ellipsoid

$$\Omega_\epsilon = \left\{ (x', x_n) \in \mathbb{R}^n \;\middle|\; |x'|^2 + \frac{|x_n|^2}{a^2} < 1 \right\},$$

with $a = 1 + \epsilon$ and $\epsilon \in (0, 1)$. By letting $\Psi_\epsilon : \overline{B_1} \to \overline{\Omega}_\epsilon$ be the smooth diffeomorphism given by

$$\Psi_\epsilon(y) = (y', a y_n) \quad \text{for } y \in \overline{B_1},$$

we clearly have that $|\Psi_\epsilon^{-1}(x)| = r$ if and only if $x \in r\partial\Omega_\epsilon$. In view of Corollary 3.0.5, we then infer that any solution $u \in C^2(\Omega_\epsilon) \cap C^0(\overline{\Omega}_\epsilon)$ of (3.13) fulfilling assumption (3.14) for some $C_0 \geq 1$ satisfies

$$|u(p) - u(q)| \leq C\epsilon^\alpha \quad \text{for every } p, q \in r\partial\Omega_\epsilon \text{ and } r \in (0, 1], \tag{3.67}$$

for some constants $C \geq 1$ and $\alpha \in (0, 1)$ depending only on n, f, and C_0.

Example 3.4.1 can be modified to treat a general smooth perturbation Ω_ϵ of the unit ball, as we show here below.

Example 3.4.2. Let $\Omega_\epsilon \subset \mathbb{R}^n$ be a small $C^{3,\theta}$-perturbation of the unit ball, that is

$$\Omega_\epsilon = \left\{ \, r\big(1 + \epsilon\varphi(x)\big)x \;\middle|\; x \in \partial B_1, \, r \in [0, 1) \, \right\},$$

for some $\varphi \in C^{3,\theta}(\partial B_1)$ with $\|\varphi\|_{C^{3,\theta}(\partial B_1)} \leq 1$ and $\epsilon \in \left(0, \frac{1}{2}\right]$. As in Example 3.4.1, we shall show that, if $u \in C^2(\Omega_\epsilon) \cap C^0(\overline{\Omega}_\epsilon)$ is a solution of problem (3.13) which satisfies (3.14) for some $C_0 \geq 1$, then (3.67) holds true for some constants $C \geq 1$ and $\alpha \in (0, 1)$ depending only on n, θ, f, and C_0, provided ϵ is sufficiently small. To deduce this from Corollary 3.0.5, we need to construct a diffeomorphism $\widetilde{\Psi}_\epsilon$ mapping spheres centered at the origin onto dilations of $\partial\Omega_\epsilon$. Naturally,

$$r\partial\Omega_\epsilon = \left\{ \, r\big(1 + \epsilon\varphi(x)\big)x \;\middle|\; x \in \partial B_1 \, \right\} \quad \text{for every } r \in (0, 1].$$

Hence, we are led to setting $\widetilde{\Psi}_\epsilon(x) := \big(1 + \epsilon\varphi\big(x/|x|\big)\big)x$ for $x \in \overline{B_1}$. If ϵ is small, this is a Lipschitz diffeomorphism of $\overline{B_1}$ onto $\overline{\Omega}_\epsilon$, which however fails to be more regular at the origin. In order to smooth things out, we consider a monotone non-decreasing function $\eta \in C^\infty([0,+\infty))$ satisfying $\eta = 0$ in $\left[0, \frac{1}{4}\right]$, $\eta = 1$ in $\left[\frac{1}{2}, +\infty\right)$, and define $\Psi_\epsilon : \overline{B_1} \to \overline{\Omega}_\epsilon$ by

$$\Psi_\epsilon(x) := \Big(1 + \epsilon\, \eta(|x|)\, \varphi\big(x/|x|\big)\Big)x \quad \text{for } x \in \overline{B_1}.$$

Now, Ψ_ϵ is a $C^{3,\theta}$-diffeomorphism satisfying $\|\Psi_\epsilon - \mathrm{Id}\,\|_{C^{3,\theta}(B_1)} + \|\Psi_\epsilon^{-1} - \mathrm{Id}\,\|_{C^{3,\theta}(\Omega_\epsilon)} \le C\epsilon$, for some dimensional constant C. As Ψ_ϵ agrees with $\widetilde{\Psi}_\epsilon$ in $\overline{B_1} \setminus B_{\frac{1}{2}}$, we immediately infer that (3.67) holds true for every $r \in \left[\frac{1}{2}, 1\right]$. Let then $r \in \left(0, \frac{1}{2}\right)$ and $p, q \in r\partial\Omega_\epsilon$. Consider the points $\hat{p} := \Psi_\epsilon\big(\widetilde{\Psi}_\epsilon^{-1}(p)\big)$ and $\hat{q} := \Psi_\epsilon\big(\widetilde{\Psi}_\epsilon^{-1}(q)\big)$. By construction, $\big|\Psi_\epsilon^{-1}(\hat{p})\big| = \big|\Psi_\epsilon^{-1}(\hat{q})\big| = r$. Hence, by Corollary 3.0.5 we have that $|u(\hat{p}) - u(\hat{q})| \le C\epsilon^\alpha$. Furthermore, by the regularity of u and the fact that both Ψ_ϵ and $\widetilde{\Psi}_\epsilon^{-1}$ are ϵ-close to the identity, we get that $|u(p) - u(\hat{p})| + |u(q) - u(\hat{q})| \le C\epsilon$. Accordingly, (3.67) is true for $r \in \left(0, \frac{1}{2}\right)$ as well.

Chapter 4

The Parallel Surface Fractional Torsion Problem

This chapter deals with the proof of symmetry and quantitative stability for the so called parallel surface fractional torsion problem. We start by recalling the Minkowski sum of two sets $X, Y \in \mathbb{R}^n$,

$$X + Y := \{\ x + y \mid x \in X,\ y \in Y\ \}.$$

Given G a smooth and bounded open set and B_R the ball of radius $R > 0$ centered at the origin, we let

$$\Omega := G + B_R \tag{4.1}$$

and consider solutions of

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.2}$$

with the overdetermined condition

$$u = c \quad \text{on } \partial G. \tag{4.3}$$

The overdetermined problem (4.2)-(4.3) was firstly studied in [MS10] for the classical Laplace operator and it was motivated by the study of invariant isothermic surfaces of a nonlinear nondegenerate fast diffusion equation. Later, in [CMS15] and [CMS16] symmetry and quantitative approximate symmetry results were studied for more general operators. See also [Sha12] for related symmetry results regarding the parallel surface problem.

In what follows we consider the nonlocal counterpart of this setting. Namely, on the one hand, by Lax-Milgram Theorem, problem (4.2) admits a solution. On the other, it is not clear whether or not a solution of (4.2) exists that also satisfies (4.3). In this context, our first main result is the following.

Theorem 4.0.1. *Let G be an open bounded set of $\mathbb{R}^n$ with ∂G of class C^1 and set $\Omega := G + B_R$, for some $R > 0$. There exists a solution $u \in C^s(\overline{\Omega})$ of* (4.2) *satisfying the additional condition* (4.3) *if and only if G (and therefore Ω) is a ball.*

It is clear that one implication of Theorem 4.0.1 is trivial. Indeed we recall from Chapter 2.2 that for a ball $B = B_r(x_0)$ of radius $r > 0$ and center $x_0 \in \mathbb{R}^n$, the explicit solution ψ_B of (4.2) with $\Omega = B$ is given by

$$\psi_B(x) = \gamma_{n,s}(r^2 - |x - x_0|^2)_+^s, \tag{4.4}$$

where

$$\gamma_{n,s} := \frac{4^{-s}\Gamma(n/2)}{\Gamma(n/2+s)\Gamma(1+s)}. \tag{4.5}$$

Since ψ_B is radial, then condition (4.3) is automatically satisfied for any $G = B_\rho(x_0)$, with $\rho < r$. Therefore, in order to prove Theorem 4.0.1 it is enough to show that if u is a solution to (4.2) satisfying (4.3) then Ω is a ball. In other words, we prove that if a solution of the torsion problem (4.2) has a level set which is parallel to $\partial\Omega$ then the domain is a ball and the solution is radially symmetric. Here we notice that the regularity assumptions required on ∂G are the minimal ones in order to be able to start the moving planes procedure.

Once the symmetry result for problem (4.2)-(4.3) is achieved, one can ask for its quantitative stability counterpart (as done in [CMS16] for the classical Laplacian case). We recall the Lipschitz seminorm $[u]_\Gamma$ of u on a surface Γ

$$[u]_\Gamma := \sup_{x,y\in\Gamma,\, x\neq y} \frac{|u(x) - u(y)|}{|x - y|}$$

and the parameter

$$\rho(\Omega) := \inf\{\ |t - s| \mid \exists p \in \Omega \text{ such that } B_s(p) \subset \Omega \subset B_t(p)\ \}, \tag{4.6}$$

which controls how much the set Ω differs from a ball (clearly, $\rho(\Omega) = 0$ if and only if Ω is a ball).

Our main goal is to obtain quantitative bounds on $\rho(\Omega)$ in terms of $[u]_{\partial G}$. In particular, our second main result[1] is the following.

Theorem 4.0.2. *Let G be an open and bounded set of $\mathbb{R}^n$ with ∂G of class C^1 and let $\Omega := G + B_R$. Assume that $\partial\Omega$ is of class C^2. Let $u \in C^2(\Omega) \cap C(\mathbb{R}^n)$ be a solution of (4.2). Then, we have that*

$$\rho(\Omega) \le C_*\, [u]_{\partial G}^{\frac{1}{s+2}}, \tag{4.7}$$

where $C_* > 0$ is an explicit constant only depending on n, s, R, and the diameter $\operatorname{diam}(\Omega)$ *of Ω.*

Hence, Theorem 4.0.2 asserts that the quantity $[u]_{\partial G}$ bounds from above a pointwise measure of closeness of Ω to a ball, namely $\rho(\Omega)$. The closer $[u]_{\partial G}$ is to zero, the closer the domain Ω is to a ball (in a pointwise sense). Of course, when $[u]_{\partial G} = 0$, estimate (4.7) reduces to $\rho(\Omega) = 0$, and therefore (4.6) gives that Ω is a ball: in this sense, Theorem 4.0.2 recovers Theorem 4.0.1.

We notice that the quantitative estimate (4.7) is of Hölder type and may be not optimal since we do not recover the optimal linear bound at the limit for $s \to 1$ which was obtained in [CMS16]. The main reason for the exponent $\frac{1}{s+2}$ in (4.7) is due to the technique used to obtain our quantitative

[1] We observe that there exist sets G which are C^∞ but such that $\Omega := G + B_R$ is not even C^1, see e.g. Figure 4.1. Moreover, we recall that well known properties of the distance function (see e.g. [GT01, Lemma 14.16] or [DZ94, Theorem 5.7]) guarantee that a certain amount of regularity of Ω suffices for the regularity of its parallel sets $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R\}$ if R is small enough, but in general Ω can be even C^∞ and its parallel sets may fail to be C^1, see e.g. Figure 4.2.

These observations justify the regularity assumptions on G and Ω in Theorem 4.0.2.

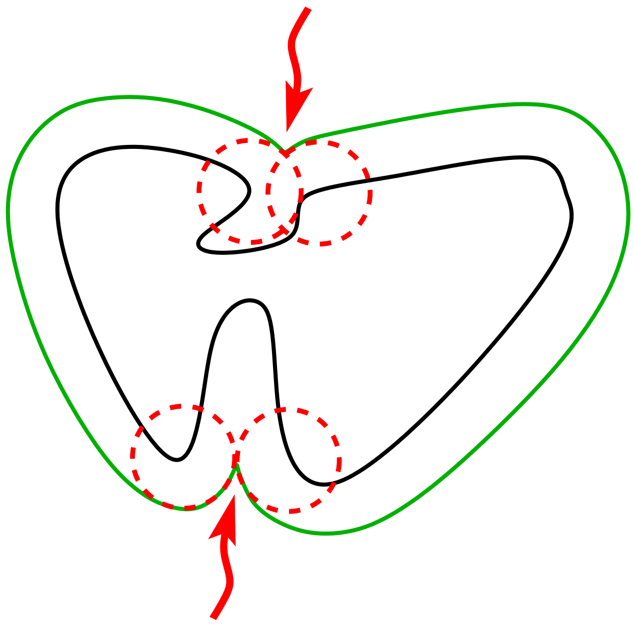

Figure 4.1: An example in which G is C^∞ but Ω is not C^1.

estimates, which are significantly different from the local case and rely on detecting "useful mass" of the functions involved in suitable regions of the domain.

We stress that the assumption that the constant C_* in Theorem 4.0.2 depends on the diameter of Ω is essential and cannot be removed: an explicit example will be presented in Section 4.6.

We finally notice that we do not have to make any assumption on connectedness on G. This is a remarkable difference with respect to the classical local case [CMS16]. In this direction it is not difficult to see that Theorems 4.0.1 and 4.0.2 hold under weaker assumptions, in particular by assuming that the value c in (4.3) may be different on each connected component of G. In Section 4.7 we give further and more precise details on this result.

This chapter is organized as follows. In Section 4.1 we present a new boundary Harnack result on a half ball for antisymmetric s-harmonic functions. Section 4.2 is devoted to the proof of the symmetry result; we make use of weak and strong maximum principles, as well as the boundary Harnack that we have established in Section 4.1.

In Section 4.3 we present a quantitative version of the fractional Hopf lemma introduced in [FJ15, Proposition 3.3]. Section 4.4 uses the previous results in order to get a quantitative stability estimate in one direction. Lastly, in Section 4.5 we complete the proof of Theorem 4.0.2 by passing from the approximate symmetry in one direction to the desired quantitative symmetry result following an idea used in [CFMN18].

Section 4.6 presents an example that shows that the dependence of the constant C_* in Theorem 4.0.2 upon the diameter of the domain cannot be removed. In Section 4.7 we describe some possible generalization of Theorems 4.0.1 and 4.0.2. A technical observation of geometric type is placed in Section 4.8.

In terms of applications, in addition to the classical motivations in the study of invariant isother-

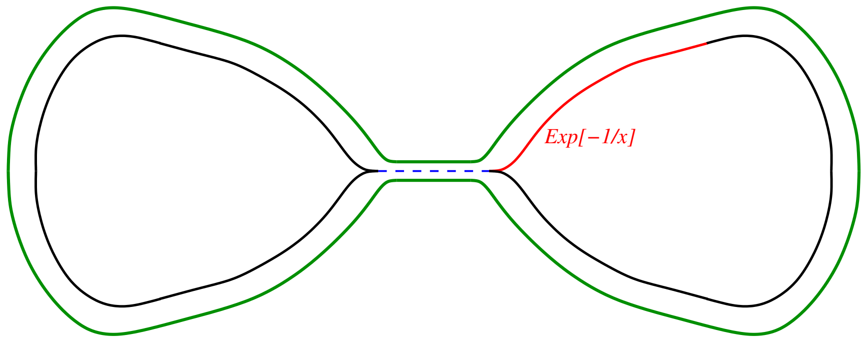

Figure 4.2: An example in which a parallel set of Ω is not C^1 even though Ω is C^∞.

mic surfaces [MS10], we mention that the overdetermined problem in (4.2) and (4.3) can be inspired by questions related to population dynamics and specifically to the determination of optimal rural-urban fringes: in this context, our results would detect that the fair shape for an urban settlement is the circular one, as detailed in Section 4.9.

4.1 Boundary Harnack inequality

We present here a new boundary Harnack inequality for antisymmetric s-harmonic functions. From now on, we will employ the notation $H^+ := \{x_1 > 0\}$, $H^- := \{x_1 < 0\}$ and $T := \{x_1 = 0\}$. We define $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n$, with $y \mapsto y' = (-y_1, y_2, \ldots, y_n)$, the reflection with respect to T. Moreover, for $R > 0$ we call $B_R^+ := B_R \cap H^+$ and $B_R^- := B_R \cap H^-$.

The main result towards the boundary Harnack inequality in our setting is the following:

Lemma 4.1.1. *Let $u \in C^2(B_R) \cap C(\mathbb{R}^n)$ with*

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} < +\infty \tag{4.8}$$

be a solution of

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_R, \\ u(x') = -u(x) & \text{for every } x \in \mathbb{R}^n, \\ u \geq 0 & \text{in } H^+. \end{cases}$$

There exists a constant $K > 1$ only depending on n and s such that, for every $z \in B_{R/2}^+$ and for every $x \in B_{R/4}(z) \cap B_R^+$ we have

$$\frac{1}{K} \frac{u(z)}{z_1} \leq \frac{u(x)}{x_1} \leq K \frac{u(z)}{z_1}. \tag{4.9}$$

Proof. We recall that the Poisson Kernel for the fractional Laplacian in the ball is given by (see for example [Buc16])

$$P_{n,s}(x, y) := c_{n,s} \left(\frac{R^2 - |x|^2}{|y|^2 - R^2} \right)^s \frac{1}{|x - y|^n}.$$

Hence for every $x \in B_R$, we have

$$\begin{aligned}\frac{u(x)}{c_{n,s}} &= \int_{\mathbb{R}^n\setminus B_R(0)} \left(\frac{R^2-|x|^2}{|y|^2-R^2}\right)^s \frac{1}{|x-y|^n} u(y)\,dy \\ &= \int_{H^+\setminus B_R^+} \left(\frac{R^2-|x|^2}{|y|^2-R^2}\right)^s \frac{1}{|x-y|^n} u(y)\,dy - \int_{H^+\setminus B_R^+} \left(\frac{R^2-|x|^2}{|y|^2-R^2}\right)^s \frac{1}{|x-y'|^n} u(y)\,dy \\ &= \int_{H^+\setminus B_R^+} \left(\frac{R^2-|x|^2}{|y|^2-R^2}\right)^s \left(\frac{1}{|x-y|^n} - \frac{1}{|x-y'|^n}\right) u(y)\,dy =: \int_{H^+\setminus B_R^+} \frac{T_{n,s}(x,y)}{c_{n,s}} u(y) dy.\end{aligned}$$

Our goal is to show that there exists a constant $K > 1$ depending only on n and s such that

$$\frac{1}{K}\,\frac{x_1}{z_1} \le \frac{T_{n,s}(x,y)}{T_{n,s}(z,y)} \le K\,\frac{x_1}{z_1}, \tag{4.10}$$

for every $z \in B_{R/2}^+$, $x \in B_{R/4}(z) \cap B_R^+$ and $y \in H^+ \setminus B_R^+$.

We remark that once (4.10) is established the claim in (4.9) readily follows, since

$$\frac{u(x)}{x_1} = \int_{H^+\setminus B_R^+} \frac{T_{n,s}(x,y)}{x_1} u(y) dy \le K \int_{H^+\setminus B_R^+} \frac{T_{n,s}(z,y)}{z_1} u(y) dy = K\,\frac{u(z)}{z_1},$$

which is precisely the second inequality in (4.9). The first inequality in (4.9) can be obtained similarly.

Now we prove (4.10). We notice that

$$\begin{aligned}\frac{T_{n,s}(x,y)}{T_{n,s}(z,y)} &= \left(\frac{R^2-|x|^2}{|y|^2-R^2}\right)^s \left(\frac{|y|^2-R^2}{R^2-|z|^2}\right)^s \left(\frac{1}{|x-y|^n} - \frac{1}{|x-y'|^n}\right)\left(\frac{1}{|z-y|^n} - \frac{1}{|z-y'|^n}\right)^{-1} \\ &= \left(\frac{R^2-|x|^2}{R^2-|z|^2}\right)^s \frac{|z-y|^n}{|x-y|^n}\,\frac{|z-y'|^n}{|x-y'|^n}\,\frac{|x-y'|^n-|x-y|^n}{|z-y'|^n-|z-y|^n}\,,\end{aligned} \tag{4.11}$$

and we estimate the first term as follows

$$\left(\frac{7}{16}\right)^s \le \left(\frac{R^2-(3R/4)^2}{R^2}\right)^s \le \left(\frac{R^2-|x|^2}{R^2-|z|^2}\right)^s \le \left(\frac{R^2}{R^2-(R/2)^2}\right)^s \le \left(\frac{4}{3}\right)^s. \tag{4.12}$$

Moreover, we observe that

$$\begin{aligned}\frac{|z-y|}{|x-y|} &\le \frac{|x-y|}{|x-y|} + \frac{|x-z|}{|x-y|} \le 1 + \frac{R/4}{R/4} = 2, \\ \frac{|z-y|}{|x-y|} &\ge \frac{|y|-|z|}{|y|+|x|} \ge \frac{|y|-R/2}{|y|+3R/4} \ge \frac{2}{7}.\end{aligned} \tag{4.13}$$

Now, considering the last terms in (4.11), we can write

$$\frac{|z-y'|^n}{|x-y'|^n}\,\frac{|x-y'|^n-|x-y|^n}{|z-y'|^n-|z-y|^n} =: \frac{1-\alpha^n}{1-\beta^n}\,, \tag{4.14}$$

where
$$\alpha = \frac{|x-y|}{|x-y'|} \quad \text{and} \quad \beta = \frac{|z-y|}{|z-y'|}\,.$$
We observe that
$$0 \le \alpha^2 = \frac{|x-y|^2}{|x-y'|^2} = 1 - \frac{4x_1y_1}{|x-y'|^2} \le 1 \quad \text{and} \quad 0 \le \beta^2 = \frac{|z-y|^2}{|z-y'|^2} = 1 - \frac{4z_1y_1}{|z-y'|^2} \le 1\,. \tag{4.15}$$
Going back to (4.14) we write
$$\frac{1-\alpha^n}{1-\beta^n} = \frac{(1-\alpha)(1+\alpha+\cdots+\alpha^{n-1})}{(1-\beta)(1+\beta+\cdots+\beta^{n-1})} = \left[\frac{1-\alpha^2}{1-\beta^2}\right]\frac{(1+\beta)(1+\alpha+\cdots+\alpha^{n-1})}{(1+\alpha)(1+\beta+\cdots+\beta^{n-1})}.$$
From (4.15) we easily get
$$\frac{1}{2n} \le \frac{(1+\beta)(1+\alpha+\cdots+\alpha^{n-1})}{(1+\alpha)(1+\beta+\cdots+\beta^{n-1})} \le 2n \tag{4.16}$$
and, using estimates similar to the ones in (4.13),
$$\left(\frac{2}{7}\right)^2\frac{x_1}{z_1} \le \frac{1-\alpha^2}{1-\beta^2} \le 4\,\frac{x_1}{z_1}. \tag{4.17}$$

By plugging (4.16) and (4.17) into equation (4.14) and then combining it with (4.12) and (4.13), from (4.11) we get
$$\left(\frac{7}{16}\right)^s\left(\frac{2}{7}\right)^n\frac{1}{2n}\,\frac{x_1}{z_1} \le \frac{T_{n,s}(x,y)}{T_{n,s}(z,y)} \le \left(\frac{4}{3}\right)^s 2^{n+3}\,n\,\frac{x_1}{z_1}$$
which leads to (4.10) if we set $K = K(n,s) := (4/3)^s(7/2)^{n+2}\,2n > 1$. This completes the proof. □

As a consequence of the previous result, we get the following two propositions which provide boundary Harnack's inequalities of independent interest:

Proposition 4.1.2. *Let $u \in C^2(B_R) \cap C(\mathbb{R}^n)$ be antisymmetric w.r.t. T, s-harmonic in B_R, nonnegative in H^+ and such that* (4.8) *holds. Then,*
$$\sup_{B^+_{R/2}} u \le Mu(\hat{x}),$$
where $\hat{x} = \frac{R}{2}e_1$ and $M > 0$ is a constant depending on n and s.

Proof. Let $x_\star \in \overline{B^+_{R/2}}$ be such that
$$u(x_\star) = \sup_{B^+_{R/2}} u.$$
If $u(x_\star) = 0$ the result is trivial. Therefore, we can assume $u(x_\star) > 0$ and $(x_\star)_1 > 0$.

We now point out that any point $x \in B^+_{R/2}$ can be connected to $\hat{x}$ by a Harnack chain made at most of 3 balls of radius $R/4$. Hence, by choosing x_a, $x_b \in B^+_{R/2}$ such that
$$\operatorname{dist}(x_\star, x_a) \le R/4, \quad \operatorname{dist}(x_a, x_b) \le R/4 \quad \textit{and} \quad \operatorname{dist}(x_b, x_\star) \le R/4$$

we can then apply Lemma 4.1.1 and get

$$\frac{1}{K}\,\frac{u(x_\star)}{(x_\star)_1} \le \frac{u(x_a)}{(x_a)_1} \le K\,\frac{u(x_b)}{(x_b)_1} \le K^2\,\frac{u(\hat{x})}{(\hat{x})_1}$$

which gives

$$\sup_{B^+_{R/2}} u = u(x_\star) \le K^3\,\frac{u(\hat{x})}{R/2}\,(x_\star)_1 \le K^3\,u(\hat{x}),$$

where in the last inequality we have used that $(x_\star)_1 \le R/2$. □

Proposition 4.1.3. *Let $u, v \in C^2(B_R) \cap C(\mathbb{R}^n)$ be antisymmetric w.r.t. T and satisfying* (4.8), *and assume that*

$$\begin{cases} (-\Delta)^s u = 0 = (-\Delta)^s v & \text{in } B^+_R, \\ u, v \ge 0 & \text{in } H^+. \end{cases}$$

Then

$$\sup_{B^+_{R/2}} \frac{u}{v} \le K^2 \inf_{B^+_{R/2}} \frac{u}{v},$$

where $K = K(n, s) > 1$ is the constant given in (4.9).

Proof. From Lemma 4.1.1 we have that for every $z \in B^+_{R/2}$ and every $x \in B_{R/4}(z) \cap B^+_R$

$$\frac{1}{K^2}\frac{u(z)}{v(z)} \le \frac{u(x)}{v(x)} \le K^2\frac{u(z)}{v(z)}.$$

The proof then follows by using the Harnack chain as done in the proof of Proposition 4.1.2. □

4.2 The symmetry result

The notation used for the method of moving planes can be found in Chapter 2.2. We recall here that for a given direction $e \in \mathbb{S}^{n-1}$ and an open bounded set $E \subset \mathbb{R}^n$ with boundary of class C^1 we define $\Lambda_e := \sup\{\, x \cdot e \mid x \in E \,\}$ and

$$\lambda_e = \inf\{\, \lambda \in \mathbb{R} \mid \mathcal{Q}_{\tilde{\lambda}}(E_{\tilde{\lambda}}) \subset E, \text{ for all } \tilde{\lambda} \in (\lambda, \Lambda_e) \,\}.$$

For a given direction $e \in \mathbb{S}^{n-1}$, we will refer to $T_{\lambda_e} = T$ and $E_{\lambda_e} = \widehat{E}$ as the *critical hyperplane* and the *critical cap* with respect to e, respectively, and we call λ_e the *critical value* in the direction e. As already done in Chapter 2.2, when there is no chance of ambiguity the dependence on e in the notation will be dropped. We now recall that for any given direction e at least one of the following two conditions holds:

Case 1 - The boundary of the cap reflection $\mathcal{Q}(\widehat{E})$ becomes internally tangent to the boundary of E at some point $P \notin T$;

Case 2 - the critical hyperplane T becomes orthogonal to the boundary of E at some point $Q \in T$.

Throughout this chapter, the method of moving planes will be applied to the set $E = G$, where G is the set appearing in (4.1). Hence the minimal regularity assumption that we need on G is that G is of class C^1. We also notice that, in our setting, the critical values λ_e for G are also critical values for the set Ω, even if we do not need to assume further regularity on Ω in order to apply the method of moving planes. This is the reason why in Theorem 4.0.1 we only require that G is of class C^1. We also notice that in Theorem 4.0.2 we assume that Ω is of class C^2, but this assumption is not needed for the application of the method of moving planes but it comes from using other tools in the proof.

In order to prove symmetry for the problem (4.2) with condition (4.3) we will use a fractional version of the weak and strong maximum principles and a Hopf-type Lemma for antisymmetric s-harmonic functions.

For $u, v \in H^s(\mathbb{R}^n)$, we consider the bilinear form induced by the fractional Laplacian

$$\mathcal{E}(u,v) := \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)}{|x-y|^{n+2s}} dxdy.$$

Let

$$\mathcal{D}^s(\Omega) := \{u : \mathbb{R}^n \to \mathbb{R} \quad \text{measurable} \ : \ \mathcal{E}(u,\varphi) \text{ is finite in Lebesgue sense for every } \varphi \in H_0^s(\Omega)\},$$

where

$$H_0^s(\Omega) := \{u \in H^s(\mathbb{R}^n) \ : \ u = 0 \ \text{ on } \mathbb{R}^n \setminus \Omega\}.$$

See e.g. [DNPV12, Gri11] and the references therein for further information about fractional functional spaces.

Given $g \in L^2(\Omega)$ we say that a function $u \in \mathcal{D}^s(\Omega)$ is a *solution* of

$$\begin{cases} (-\Delta)^s u = g & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.18}$$

if for all $\varphi \in H_0^s(\Omega)$ we have

$$\mathcal{E}(u,\varphi) = \int_\Omega g(x)\,\varphi(x)\,dx.$$

It will be useful to introduce the notion of entire antisymmetric supersolution. Let $H \subset \mathbb{R}^n$ be a half space and let A be an open set with $A \subset H$. Given $\tilde{g} \in L^2(A)$ we say that $v \in \mathcal{D}^s(A)$ is an *entire antisymmetric supersolution*[2] of $(-\Delta)^s v = \tilde{g}$ in A, if the following conditions hold:

- v is a supersolution of $(-\Delta)^s v = \tilde{g}$ in A, that is, for all $\varphi \in H_0^s(A)$, $\varphi \geq 0$ we have

$$\mathcal{E}(v,\varphi) \geq \int_A \tilde{g}(x)\,\varphi(x)\,dx,$$

- $v \geq 0$ in $H \setminus A$ and v is antisymmetric with respect to ∂H.

We are now ready to prove Theorem 4.0.1.

[2] Since we are going to apply the method of moving planes, the set A will typically be the intersection between the set Ω and a half space, and the function v will be the difference between the solution u of (4.18) and its reflection with respect to an hyperplane.

Proof of Theorem 4.0.1. We apply the method of moving planes to the set G. Let $e \in \mathbb{S}^{n-1}$ be a fixed direction. Without loss of generality, we can assume that $e = e_1$ and that the critical hyperplane T goes through the origin (that is, $\lambda_e = 0$). We call $H^- := \{x_1 < 0\}$ and consider the function

$$v(x) := u(x) - u(\mathcal{Q}(x)) \quad \text{for } x \in \mathbb{R}^n,$$

where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is the reflection with respect to T. We have

$$\begin{cases} (-\Delta)^s v = 0 & \mathit{in}\ \mathcal{Q}(\widehat{\Omega}), \\ v \geq 0 & \mathit{in}\ H^- \setminus \mathcal{Q}(\widehat{\Omega}), \\ v(\mathcal{Q}(x)) = -v(x) & \text{for every } x \in \mathbb{R}^n. \end{cases}$$

Thus, v is an entire antisymmetric supersolution on $\mathcal{Q}(\widehat{\Omega})$. By the weak maximum principle (see [FJ15, Proposition 3.1]) we know that $v \geq 0$ in H^-. The strong maximum principle (see [FJ15, Corollary 3.4]) then implies that either $v > 0$ in $\mathcal{Q}(\widehat{\Omega})$ or $v \equiv 0$ in $\mathbb{R}^n$. We will show that the first possibility cannot occur.

Assume by contradiction that $v > 0$ in $\mathcal{Q}(\widehat{\Omega})$. We need to distinguish between the two possible critical cases.

Case 1 - since both P and P' belong to ∂G and (4.3) holds, we immediately get that

$$v(P) = u(P) - u(P') = 0,$$

which is already a contradiction.

Case 2 - in this case the critical hyperplane $T = \{x_1 = 0\}$ is orthogonal to ∂G at some point $Q = (0, Q_2, \ldots, Q_n)$ and therefore (4.3) ensures that

$$\partial_1 v(Q) = 0. \tag{4.19}$$

On the other hand, Lemma 4.1.1 implies the following Hopf-type inequality

$$\partial_1 v(Q) < 0, \tag{4.20}$$

which contradicts (4.19) and hence (4.3). Indeed, setting $z = (-R/4, Q_2, \ldots, Q_n)$ and $x = x_t = (-t, Q_2, \ldots, Q_n) \in B_{R/4}(z)$, we have that

$$\frac{v(x_t)}{-t} \geq -\frac{4}{RK} v(z), \tag{4.21}$$

where $K > 1$ is a constant only depending on n and s. Being $z \in \mathcal{Q}(\widehat{\Omega})$, we have that $v(z) > 0$, and by letting t go to 0 (4.20) follows by (4.21).

This implies that G (and hence Ω) is symmetric with respect to the direction e. Since the direction e is arbitrary, we easily obtain that G (and hence Ω) is a ball. $\square$

An alternative approach to the Hopf-type inequality (4.20) has been developed in a very recent manuscript [DPTV24].

4.3 A quantitative maximum principle

The following lemma is a quantitative version of [FJ15, Proposition 3.3]. To state it, we adopt the notion of distance between two sets, say X and Y, defined by

$$\mathrm{dist}(X,Y) := \inf\big\{|x-y|, \quad x\in X,\ y\in Y\big\}.$$

Lemma 4.3.1. *Let $B\subset H^-$ be a ball of radius $R>0$ such that $\mathrm{dist}(B,H^+)>0$. Let $v\in C^s(B)$ be an entire antisymmetric supersolution of*

$$\begin{cases}(-\Delta)^s v = 0 & \text{in } B,\\ v\geq 0 & \text{in } H^-.\end{cases}$$

Let $K\subset H^-$ be a bounded set of positive measure such that $\overline{K}\subset(H^-\setminus\overline{B})$ and $\inf_K v>0$. Then we have that

$$v\geq C\Big[\mathrm{dist}(K,H^+)\,|K|\,\inf_K v\Big]\psi_B \quad \text{in } B, \tag{4.22}$$

where ψ_B is defined in (4.4), with

$$C := \frac{2(n+2s)\,C(n,s)\,\mathrm{dist}(B,H^+)^{n+2s+1}}{\big(\mathrm{dist}(B,H^+)^{n+2s}+C(n,s)\,|B|\,\gamma_{n,s}\,R^{2s}\big)\big(\mathrm{diam}(B)+\mathrm{diam}(K)+\mathrm{dist}(\mathcal{Q}(K),B)\big)^{n+2s+2}}.$$

Proof. We define

$$w(x) := \psi_B(x)-\psi_{\mathcal{Q}(B)}(x)+\alpha\chi_K(x)-\alpha\chi_{\mathcal{Q}(K)}(x) \quad \text{for} \quad x\in\mathbb{R}^n$$

where $\alpha>0$ is a parameter to be set later on, ψ_B is the solution of the fractional torsion problem in B and χ_A is the characteristic function of a given set A. A direct computation shows that $w\in\mathcal{D}^s(B)$.

The function w is antisymmetric and for any nonnegative test function $\varphi\in H^s_0(B)$ we have

$$\begin{aligned}\mathcal{E}(w,\varphi) &= \mathcal{E}(\psi_B,\varphi)-\mathcal{E}(\psi_{\mathcal{Q}(B)},\varphi)+\alpha\,\mathcal{E}(\chi_K,\varphi)-\alpha\,\mathcal{E}(\chi_{\mathcal{Q}(K)},\varphi)\\ &= \int_B\varphi(x)dx + C(n,s)\int_B\int_{\mathcal{Q}(B)}\frac{\psi_{\mathcal{Q}(B)}(y)\varphi(x)}{|x-y|^{n+2s}}dydx\\ &-\alpha\ C(n,s)\int_B\int_K\frac{\varphi(x)}{|x-y|^{n+2s}}dydx+\alpha\ C(n,s)\int_B\int_{\mathcal{Q}(K)}\frac{\varphi(x)}{|x-y|^{n+2s}}dydx\\ &\leq \int_B\varphi(x)dx\left[\kappa-\alpha\ C(n,s)\int_K\left(\frac{1}{|x-y|^{n+2s}}-\frac{1}{|x-y'|^{n+2s}}\right)\right],\end{aligned}$$

where

$$\kappa=\kappa(n,s,B)=1+C(n,s)\,|B|\sup_B\psi_B\sup_{x\in B,y\in H^+}\frac{1}{|x-y|^{n+2s}}<+\infty.$$

If we set

$$C_1=C_1(n,s,K,B)=C(n,s)\ |K|\inf_{x\in B,y\in K}\left(\frac{1}{|x-y|^{n+2s}}-\frac{1}{|x-y'|^{n+2s}}\right)>0, \tag{4.23}$$

then

$$\mathcal{E}(w,\varphi) \le \int_B \varphi(x)(\kappa - \alpha C_1).$$

By choosing α in such a way that $\kappa - \alpha C_1 \le 0$, we get $(-\Delta)^s w \le 0$ in B.

For concreteness, we can thus choose

$$\alpha := \frac{\kappa}{C_1}$$

to have the previous argument in place and then set

$$\tau := \inf_K \frac{v}{\alpha} > 0$$

and define

$$\tilde{v}(x) := v(x) - \tau w(x)$$

for every $x \in \mathbb{R}^n$. Recalling that w is antisymmetric and that $w \equiv 0$ on $H^- \setminus (B \cup K)$ we have

$$\begin{cases} (-\Delta)^s \tilde{v} \ge 0 & \text{in } B \\ \tilde{v} \ge 0 & \text{in } H^- \setminus B. \end{cases}$$

From the weak maximum principle we then get that $\tilde{v} \ge 0$ in B and, in particular,

$$v \ge \tau \psi_B \quad \text{in } B. \tag{4.24}$$

For every $x \in B$ and every $y \in K$ we compute

$$\begin{aligned} \frac{1}{|x-y|^{n+2s}} - \frac{1}{|x-y'|^{n+2s}} &= \frac{n+2s}{2} \int_{|x-y|^2}^{|x-y'|^2} t^{-\frac{n+2s+2}{2}} dt \\ &\ge \frac{n+2s}{2} \big(|x-y'|^2 - |x-y|^2\big) |x-y'|^{-(n+2s+2)} \\ &\ge \frac{n+2s}{2} 4x_1 y_1 |x-y'|^{-(n+2s+2)}. \end{aligned}$$

Moreover, for all $x \in B$ and $y \in K$,

$$|x-y'| \le \operatorname{diam}(B) + \operatorname{diam}(K) + \operatorname{dist}(\mathcal{Q}(K), B)$$

and consequently

$$\frac{1}{|x-y|^{n+2s}} - \frac{1}{|x-y'|^{n+2s}} \ge \frac{2(n+2s)\operatorname{dist}(B,H^+)\operatorname{dist}(K,H^+)}{\big(\operatorname{diam}(B) + \operatorname{diam}(K) + \operatorname{dist}(\mathcal{Q}(K), B)\big)^{n+2s+2}}.$$

Hence, by (4.23),

$$C_1 \ge \frac{2(n+2s)\, C(n,s)\, |K| \operatorname{dist}(B,H^+)\operatorname{dist}(K,H^+)}{\big(\operatorname{diam}(B) + \operatorname{diam}(K) + \operatorname{dist}(\mathcal{Q}(K), B)\big)^{n+2s+2}}.$$

As a result,

$$\tau = \frac{C_1}{\kappa} \inf_K v \ge \frac{2(n+2s)\, C(n,s)\, |K| \operatorname{dist}(B,H^+)\operatorname{dist}(K,H^+)}{\kappa \big(\operatorname{diam}(B) + \operatorname{diam}(K) + \operatorname{dist}(\mathcal{Q}(K), B)\big)^{n+2s+2}} \inf_K v.$$

We also observe, owing to (4.4), that

$$\sup_B \psi_B(x) = \gamma_{n,s}\, R^{2s}$$

and therefore

$$\begin{aligned}\kappa &= 1 + C(n,s)\,|B|\,\gamma_{n,s}\,R^{2s} \sup_{x\in B, y\in H^+} \frac{1}{|x-y|^{n+2s}} \\ &\le 1 + \frac{C(n,s)\,|B|\,\gamma_{n,s}\,R^{2s}}{\mathrm{dist}(B,H^+)^{n+2s}} \\ &= \frac{\mathrm{dist}(B,H^+)^{n+2s} + C(n,s)\,|B|\,\gamma_{n,s}\,R^{2s}}{\mathrm{dist}(B,H^+)^{n+2s}}.\end{aligned}$$

Accordingly,

$$\tau \ge \frac{2(n+2s)\,C(n,s)\,|K|\,\mathrm{dist}(B,H^+)^{n+2s+1}\,\mathrm{dist}(K,H^+)}{\big(\mathrm{dist}(B,H^+)^{n+2s} + C(n,s)\,|B|\,\gamma_{n,s}\,R^{2s}\big)\big(\mathrm{diam}(B)+\mathrm{diam}(K)+\mathrm{dist}(\mathcal{Q}(K),B)\big)^{n+2s+2}} \inf_K v.$$

Thus, the desired conclusion follows from (4.24). □

4.4 Approximate symmetry in one direction

As customary, we say that a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the *uniform interior ball condition* if there exists a radius $\mathfrak{r}_\Omega^i > 0$ such that for every point $x_0 \in \partial\Omega$ we can find a ball $B_i \subset \Omega$ of radius $\mathfrak{r}_\Omega^i$ with $\overline{B_i} \cap \Omega^c = \{x_0\}$.

In the next subsection, we collect some useful technical lemmas which hold true for domains satisfying such a condition.

4.4.1 Results for domains satisfying the uniform interior ball condition

As noticed in [CPY22, MP23], the following simple explicit bound for the perimeter holds true.

Lemma 4.4.2 (A general simple upper bound for the perimeter, [CPY22, MP23])**.** *Let $D \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{1,\alpha}$, with $0 < \alpha \le 1$. If D satisfies the* uniform interior ball condition *with radius $\mathfrak{r}_D^i$, the we have that*

$$|\partial D| \le \frac{n|D|}{\mathfrak{r}_D^i}. \tag{4.25}$$

Proof. By following [MP23], the desired bound can be easily obtained by considering the solution $f \in C^{1,\alpha}(\overline{D})$ to

$$\Delta f = n \text{ in } D, \quad f = 0 \text{ on } \partial D,$$

and putting together the identity

$$n|D| = \int_{\partial D} \partial_\nu f \, d\mathcal{H}^{n-1}, \text{ where } \partial_\nu \text{ denotes the outer normal derivative,}$$

with the Hopf-type inequality

$$\partial_\nu f \geq \mathfrak{r}_D^i,$$

which can be found in [MP19, Theorem 3.10].

We mention that a more general version of the bound (4.25) remains true even without assuming the uniform interior ball condition, at the cost of replacing the radius r_D of the ball condition with a parameter associated to the (weaker) pseudoball condition, which is always verified by $C^{1,\alpha}$ domains: see [CPY22, Remark 1.1] and the last displayed inequality in the proof of [CPY22, Corollary 2.1]. $\square$

The previous result is useful to prove the following.

Lemma 4.4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ of class C^2. For $\delta > 0$, we set*

$$A_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \delta\}. \tag{4.26}$$

Then, we have that

$$|A_\delta| \leq c\,\delta, \quad \text{with} \quad c := \frac{2n|\Omega|}{\mathfrak{r}_\Omega^i}, \tag{4.27}$$

where $\mathfrak{r}_\Omega^i$ is the radius of the uniform interior ball condition of Ω.

We recall that if a domain has boundary of class C^2, then it satisfies a uniform interior ball condition.

Proof of Lemma 4.4.3. We set $d_{\partial\Omega}(x) := \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. For $\delta \geq 0$, we define

$$V_\delta := \{x \in \Omega \mid d_{\partial\Omega}(x) > \delta\} \quad \text{and} \quad \Gamma_\delta := \{x \in \Omega \mid d_{\partial\Omega}(x) = \delta\}.$$

It is well-known that $d_{\partial\Omega} \in C^2(A_{r_\Omega})$ (see, e.g., [GT01, Lemma 14.16]).

We first prove the claim in the case $0 \leq \delta \leq \mathfrak{r}_\Omega^i/2$. From the coarea formula we obtain

$$|A_\delta| = \int_{A_\delta} 1\,dx = \int_{A_\delta} |\nabla d_{\partial\Omega}(x)|\,dx = \int_0^\delta \left(\int_{A_\delta \cap d_{\partial\Omega}^{-1}(t)} d\mathcal{H}^{n-1} \right) dt = \int_0^\delta |\Gamma_t|\,dt. \tag{4.28}$$

Since $t \leq \delta \leq \mathfrak{r}_\Omega^i/2$, we have that V_t is a bounded domain satisfying the uniform interior touching ball condition with radius $\mathfrak{r}_\Omega^i/2$, and with boundary Γ_t of class C^2. Thus, we can apply Lemma 4.4.2 with $D := V_t$ to get that

$$|\Gamma_t| \leq \frac{2n|V_t|}{\mathfrak{r}_\Omega^i} \leq \frac{2n|\Omega|}{\mathfrak{r}_\Omega^i}, \tag{4.29}$$

where the last inequality follows by the inclusion $V_t \subseteq \Omega$. Combining (4.28) with (4.29) immediately gives (4.27), for any $0 \leq \delta \leq \mathfrak{r}_\Omega^i/2$.

On the other hand, if $\delta \geq \mathfrak{r}_\Omega^i/2$, we easily find that

$$|A_\delta| \leq |\Omega| \leq \left[\frac{2|\Omega|}{\mathfrak{r}_\Omega^i} \right] \delta,$$

where the first inequality follows by the inclusion

$$A_\delta \subseteq \Omega, \text{ for any } \delta \geq 0.$$

Thus, (4.27) still holds true. $\square$

We now detect an optimal growth of the solution to (4.2) from the boundary, by generalizing [MP20, Lemma 3.1] to the fractional setting.

Lemma 4.4.4. *Let u satisfy* (4.2) *and let $\gamma_{n,s}$ be the constant defined in* (4.5)*. Then,*

$$u(x) \geq \gamma_{n,s} \operatorname{dist}(x, \partial\Omega)^{2s} \quad \text{for every } \ x \in \Omega. \tag{4.30}$$

Moreover, if Ω is of class C^1 and satisfies the uniform interior sphere condition with radius r_Ω, then it holds that

$$u(x) \geq \gamma_{n,s} \, (\mathfrak{r}^i_\Omega)^s \operatorname{dist}(x, \partial\Omega)^s \quad \text{for every } \ x \in \Omega. \tag{4.31}$$

Proof. Let $x \in \Omega$ and set $r := \operatorname{dist}(x, \partial\Omega)$. We consider

$$\psi(y) := \gamma_{n,s} \left(r^2 - |y - x|^2 \right)_+^s,$$

which satisfies the fractional torsion problem in $B_r(x)$, namely

$$\begin{cases} (-\Delta)^s \psi = 1 & \text{in } B_r(x), \\ \psi = 0 & \text{on } \mathbb{R}^n \setminus B_r(x). \end{cases} \tag{4.32}$$

By the comparison principle (see [FJ15, Remark 3.2]), we have that $u \geq \psi$ on $\overline{B_r(x)}$. In particular, at the center x of $B_r(x)$, we have that

$$u(x) \geq \psi(x) = \gamma_{n,s} \operatorname{dist}(x, \partial\Omega)^{2s},$$

and (4.30) follows.

Notice that (4.31) follows from (4.30) if $\operatorname{dist}(x, \partial\Omega) \geq \mathfrak{r}^i_\Omega$. Hence, from now on, we can suppose that

$$\operatorname{dist}(x, \partial\Omega) < \mathfrak{r}^i_\Omega.$$

Let $\bar{x}$ be the closest point in $\partial\Omega$ to x and call $\tilde{B} \subset \Omega$ the ball of radius $\mathfrak{r}^i_\Omega$ touching $\partial\Omega$ at $\bar{x}$ and containing x. Up to a translation, we can always suppose that

$$\text{the center of the ball } \tilde{B} \text{ is the origin.} \tag{4.33}$$

Now, we let $\tilde{\psi}$ be the solution of (4.32) in $\tilde{B}$, that is $\tilde{\psi}(y) := \gamma_{n,s} \left((\mathfrak{r}^i_\Omega)^2 - |y|^2 \right)_+^s$. By comparison ([FJ15, Remark 3.2]), we have that $u \geq \tilde{\psi}$ in $\tilde{B}$, and hence, being $x \in \tilde{B}$,

$$u(x) \geq \gamma_{n,s} \, ((\mathfrak{r}^i_\Omega)^2 - |x|^2)_+^s = \gamma_{n,s} \, (\mathfrak{r}^i_\Omega + |x|)^s (\mathfrak{r}^i_\Omega - |x|)_+^s \geq \gamma_{n,s} \, (\mathfrak{r}^i_\Omega)^s \, (\mathfrak{r}^i_\Omega - |x|)^s. \tag{4.34}$$

Moreover, from (4.33),

$$\mathfrak{r}^i_\Omega - |x| = \operatorname{dist}(x, \partial\Omega).$$

This and (4.34) give (4.31), as desired. $\square$

4.4.5 Proof of the main lemma

From now on, we let $\Omega := G + B_R(0)$, with $G \subseteq \mathbb{R}^n$ bounded, with ∂G of class C^1 and $\partial\Omega$ of class C^2.

Remark 4.4.6 (On the constants in the quantitative estimates)**.** The constants in all of our quantitative estimates can be explicitly computed and only depend on n, s, R, and $\mathrm{diam}(\Omega)$. In some of the intermediate results, the parameter $|\Omega|$ may appear. It is clear that such a parameter can be removed thanks to the bounds

$$\frac{\omega_n}{n} R^n \leq |\Omega| \leq \frac{\omega_n}{n} \mathrm{diam}(\Omega)^n, \text{ where } \frac{\omega_n}{n} \text{ is the volume of the unit ball in } \mathbb{R}^n, \tag{4.35}$$

which easily hold true in light of the monotonicity of the volume with respect to inclusion.

We remark that the estimates of the previous subsection also depend on the radius r_Ω of the uniform interior ball condition associated to Ω. Nevertheless, from now on, we have that

$$\mathfrak{r}^i_\Omega := R, \tag{4.36}$$

by the definition of $\Omega := G + B_R(0)$

We apply the method of moving planes to the set G. Hence, we fix a direction $e = e_1$ and assume the associated critical hyperplane to be $T = \{x_1 = 0\}$, with $Q : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto x'$ the reflection with respect to T. For the proofs of the next two lemmas we will use the following notation: we set for $t \geq 0$

$$G_t := G + B_t(0), \quad \widehat{G_t} := G_t \cap H^+, \quad G_t^- := G_t \cap H^- \quad U_t := \mathcal{Q}(\widehat{G_t}). \tag{4.37}$$

Note that $\Omega = G_R$.

Let $u \in C^2(\Omega) \cap C(\mathbb{R}^n)$ be a solution of (4.2). For every $x \in \mathbb{R}^n$, we set

$$v(x) := u(x) - u(x')\,.$$

Lemma 4.4.7. *Given $P \in U_R$ with $B = B_{R/8}(P)$ such that $\mathrm{dist}(B, \partial U_R) \geq R/8$, we have that*

$$|\Omega^- \setminus U_R| \leq \tilde{C}\, v(P)^{\frac{1}{2+s}}, \tag{4.38}$$

where $\tilde{C} > 0$ is an explicit constant depending only on n, s, R, and $\mathrm{diam}(\Omega)$.

Proof. For $\delta \geq 0$, we set $K_\delta := (\Omega^- \setminus U_R) \setminus (E_\delta \cup F_\delta)$, where

$$E_\delta := A_\delta \cap (\Omega^- \setminus U_R) \quad \text{ with } A_\delta \text{ as defined in (4.26)},$$

$$F_\delta := \{x \in \Omega^- \setminus U_R \,:\, \mathrm{dist}(x, T) < \delta\}.$$

With our choice of B clearly $\mathrm{dist}(B, H^+) \geq \mathrm{dist}(B, \partial U_R) \geq R/8$ and therefore, an application of Lemma 4.3.1 with $B := B_{R/8}(P)$ and $K := K_\delta$ gives that

$$v \geq \overset{\star}{C} \left[\mathrm{dist}(K_\delta, H^+)\, |K_\delta| \inf_{K_\delta} v\right] \psi_B \quad \textit{in } B, \tag{4.39}$$

holds true for a suitable explicit $\overset{\star}{C} > 0$, depending only on n, s, R, and $\operatorname{diam}(\Omega)$. Here, we used that in the present situation $K \subset \Omega$ and $B \subset U_R$.

Now looking at K_δ we have $K_\delta \subseteq (\Omega^- \setminus U_R) \subseteq H^-$ and so $\operatorname{dist}(K_\delta, B) \geq R/8$. Moreover, since $K_\delta \subseteq (G^-_{R-\delta} \setminus U_R)$ we have that $v(x) = u(x) > 0$ for every $x \in K_\delta$; hence, (4.31) and (4.36) give that

$$\inf_K v \geq [\gamma_{n,s} R^s]\ \delta^s. \tag{4.40}$$

Also, since $K_\delta \subseteq (\Omega^- \setminus U_R) \setminus F_\delta$, then

$$\operatorname{dist}(K_\delta, H^+) \geq \delta. \tag{4.41}$$

Clearly,

$$|K_\delta| = |\Omega^- \setminus U_R| - |E_\delta \cup F_\delta| \geq |\Omega^- \setminus U_R| - (|E_\delta| + |F_\delta|).$$

Since $E_\delta \subseteq A_\delta$, Lemma 4.4.3 gives that

$$|E_\delta| \leq \left[\frac{2n|\Omega|}{R}\right] \delta,$$

where we also used (4.36). Also, by definition of F_δ, it is trivial to check that

$$|F_\delta| \leq \operatorname{diam}(\Omega)^{n-1}\delta.$$

Putting together the last three displayed formulas we conclude that

$$|K_\delta| \geq |\Omega^- \setminus U_R| - \tilde{c}\,\delta, \qquad \text{with} \qquad \tilde{c} := \frac{2\omega_n \operatorname{diam}(\Omega)^n}{R} + \operatorname{diam}(\Omega)^{n-1}. \tag{4.42}$$

Here, we also used the second inequality in (4.35) to remove the dependence on $|\Omega|$ in the constant $\tilde{c}$.

Putting together (4.39), (4.40), (4.41), (4.42), and that $\psi_B(P) = \gamma_{n,s}(R/8)^{2s}$ (by (4.4) with $x_0 := P$), we get that

$$v(P) \geq \overset{\star\star}{C}\, \delta^{1+s}\big(|\Omega^- \setminus U_R| - \tilde{c}\,\delta\big) \quad \text{with} \quad \overset{\star\star}{C} := \overset{\star}{C}\ [\gamma_{n,s}R^s]\ (R/8)^{2s}\, \gamma_{n,s},$$

that is:

$$|\Omega^- \setminus U_R| \leq \frac{v(p)}{\overset{\star\star}{C}} \delta^{-(1+s)} + \tilde{c}\,\delta.$$

By minimizing in δ the right-hand side of the last inequality, we can conveniently choose

$$\delta := \left[\frac{(1+s)v(p)}{\overset{\star\star}{C}\,\tilde{c}}\right]^{\frac{1}{2+s}}$$

and obtain that (4.38) holds true with

$$\tilde{C} := \left[\frac{(1+s)\tilde{c}^{1+s}}{\overset{\star\star}{C}}\right]^{\frac{1}{2+s}}.$$

□

The next lemma uses the previous result to get a stability estimate in one specific direction.

Lemma 4.4.8 (Almost symmetry in one direction). *We have that*

$$|\Omega \setminus \mathcal{Q}(\Omega)| \ \leq \overline{C}\, [u]_{\partial G}^{\frac{1}{2+s}}, \tag{4.43}$$

where $\overline{C} > 0$ *is an explicit constant only depending on* n*,* s*,* R*, and* $\mathrm{diam}(\Omega)$*.*

Proof. We apply the method of moving planes to G in the direction e_1. We need to distinguish between some cases.
Case 1 - U_0 is internally tangent to G at a point P which is not on T. We distinguish two subcases, according to the distance of P from T.

Case 1a - We assume $\mathrm{dist}(P, T) > R/8$. Since $P \in \partial G \cap \partial U_0$ we have

$$v(P) = u(P) - u(P') \leq [u]_{\partial G}\ \mathrm{diam}(\Omega).$$

We then apply Lemma 4.4.7 to obtain that

$$|\Omega^- \setminus U_R| \leq \tilde{C}\, \mathrm{diam}(\Omega)^{\frac{1}{2+s}}\, [u]_{\partial G}^{\frac{1}{2+s}}.$$

Case 1b - $P \in \partial G \cap \partial U_0$ such that $\mathrm{dist}(P, T) \leq R/8$.

From the definitions of v and $[u]_{\partial G}$, we have that

$$\frac{v(P)}{(-P_1)} = \frac{2(u(P) - u(P'))}{\mathrm{dist}(P, P')} \leq 2[u]_{\partial G}, \tag{4.44}$$

where we adopted the notation $P = (P_1, P_2, \ldots, P_n)$.

As noticed in item (ii) of Lemma 4.8.1, we have that $B_R(P) \subset U_R \cup [\Omega \cap (H^+ \cup T)]$.

We set $\widehat{P} := (0, P_2, \ldots, P_n)$ the projection of P on the hyperplane T. We then set $\overline{P} := (-R/4, P_2, \ldots, P_n)$ so that $-\overline{P}_1 = \mathrm{dist}(\overline{P}, T) = R/4$. Using Lemma 4.1.1 with $B_R := B_{R/2}(\widehat{P})$, we see that

$$\frac{4}{R}\, v(\overline{P}) = \frac{v(\overline{P})}{(-\overline{P})_1} \leq K\ \frac{v(P)}{(-P_1)}. \tag{4.45}$$

Putting together (4.44) and (4.45) gives that

$$v(\overline{P}) \leq \frac{R}{2} K [u]_{\partial G},$$

and hence an application of Lemma 4.4.7 with $P := \overline{P}$ leads to

$$|\Omega^- \setminus U_R| \leq \tilde{C} \left(\frac{R}{2} K\right)^{\frac{1}{2+s}} [u]_{\partial G}^{\frac{1}{2+s}}.$$

Case 2 - T is orthogonal to the boundary of G at some point Q.

Again, in light of item (ii) of Lemma 4.8.1, we have that $B_R(Q) \subset U_R \cup [\Omega \cap (H^+ \cup T)]$.

We choose $\overline{P} := (-R/4, Q_2, \ldots, Q_n)$ so that $-\overline{P}_1 = \mathrm{dist}(\overline{P}, T) = R/4$.

Using Lemma 4.1.1 with $B_R := B_R(Q)$, for every $y = (y_1, Q_2, \dots, Q_n) \in B_{R/4}(\overline{P})$ we obtain that

$$\frac{v(\overline{P})}{(-\overline{P})_1} \leq K \, \frac{v(y)}{(-y_1)} \leq K \, [u]_{\partial G},$$

and hence

$$v(\overline{P}) \leq \frac{R}{4} K \, [u]_{\partial G}.$$

Again, we apply Lemma 4.4.7 with $P := \overline{P}$, and we get that

$$|\Omega^- \setminus U_R| \leq \tilde{C} \left(\frac{R}{4} \, K \right)^{\frac{1}{2+s}} \, [u]_{\partial G}^{\frac{1}{2+s}}.$$

In all cases, (4.43) holds true with

$$\overline{C} := \tilde{C} \left(\max \left\{ \mathrm{diam}(\Omega), \frac{R}{2} \, K \right\} \right)^{\frac{1}{2+s}}.$$

This completes the proof. □

4.5 The stability result

For the proof of the following lemma we closely follow [CFMN18, Lemma 4.1]. The idea is the following: for a given direction $e \in \mathcal{S}^{n-1}$ we slice the set Ω in a (finite number of) sections depending on the critical value λ_e, using the almost symmetry result in one direction of the previous section (Lemma 4.5.1). This together with a simple observation on set reflections leads to an estimate on $\lambda_e = \mathrm{dist}(0, T^e)$.

Lemma 4.5.1. *Let $\varepsilon := \min\{1/4, 1/n\} \, |\Omega| / \overline{C}$ with $\overline{C}$ as in Lemma 4.4.8. Assume that*

$$[u]_{\partial G}^{\frac{1}{s+2}} \leq \varepsilon \tag{4.46}$$

and suppose that the critical hyperplanes with respect to the coordinate directions T^{e_j} coincide with $\{x_j = 0\}$ for every $j = 1, \dots, n$. For a fixed direction $e \in \mathbb{S}^{n-1}$ we have

$$|\lambda_e| \leq \widehat{C} \, [u]_{\partial G}^{\frac{1}{s+2}} \tag{4.47}$$

where $\widehat{C} = 4 \, (n+3) \, \frac{\mathrm{diam}(\Omega)}{|\Omega|} \, \overline{C} > 0$.

Proof. We set $\Omega^0 := \{-x \mid x \in \Omega\}$. Since Ω^0 can be obtained via composition of the n reflections with respect to the hyperplanes T^{e_j} for $j = \{1, \dots, n\}$, by applying Lemma 4.4.8 n times with respect to the coordinate directions we obtain

$$|\Omega \triangle \Omega^0| \leq n \, \overline{C} \, [u]_{\partial G}^{\frac{1}{s+2}}, \tag{4.48}$$

where we define the symmetric difference between two sets A and B as $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Indeed, we first notice that

$$|\Omega \triangle \Omega^0| = 2 \, |\Omega \setminus \Omega^0|.$$

Moreover, we have that

$$|\Omega \setminus \Omega^0| \leq |\Omega \setminus Q^n(Q^{n-1}(\dots(Q^1(\Omega))\dots)| \leq |\Omega \setminus Q^n(\Omega)| + |Q^n(\Omega) \setminus Q^n(Q^{n-1}(\dots(Q^1(\Omega))\dots)|,$$

where $Q^j = Q^{e_j}$ the reflection with respect to the critical value in the coordinate direction e_j, for j from 1 to n. Now observing that

$$|Q^n(\Omega) \setminus Q^n(Q^{n-1}(\dots(Q^1(\Omega))\dots)| = |Q^n\big(\,\Omega \setminus (Q^{n-1}(\dots(Q^1(\Omega))\dots)\,\big)|,$$

using the estimate in Lemma 4.4.8 and iterating the argument we obtain (4.48).

Now, assume $\lambda_e > 0$.

We notice that $\Lambda_e \leq \operatorname{diam}(\Omega)$. In fact, if $\Lambda_e > \operatorname{diam}(\Omega)$, then $x \cdot e \geq 0$ for every $x \in \Omega$, and hence

$$|\Omega \Delta \Omega^0| = 2|\Omega|.$$

By using the last identity with (4.48), we would find

$$2|\Omega| \leq n\overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}},$$

which contradicts (4.46).

Now let $\Omega' = Q^e(\Omega)$ be the reflection of Ω about the critical hyperplane T^e. Using Lemma 4.4.8 in the direction e we get

$$|\Omega \triangle \Omega'| \leq \overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}}. \tag{4.49}$$

Recalling that $\mathcal{E}_\lambda = \{x \cdot e > \lambda\}$ and $\Omega_\lambda = \Omega \cap \mathcal{E}_\lambda$, from (4.49) we get

$$|\Omega_{\lambda_e}| \geq \frac{|\Omega|}{2} - \overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}}. \tag{4.50}$$

Moreover, if we set $\mathcal{E}_\lambda^0 := \{-x \mid x \in \mathcal{E}_\lambda\}$ we also have

$$|\Omega \cap \mathcal{E}_{\lambda_e}^0| = |\Omega^0 \cap \mathcal{E}_{\lambda_e}| \geq |\Omega_{\lambda_e}| - |\Omega \triangle \Omega^0| \geq \frac{|\Omega|}{2} - (n+1)\,\overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}},$$

which together with (4.50) gives

$$|\,\{x \in \Omega \mid \; -\lambda_e \leq x \cdot e \leq \lambda_e\}\,| \leq (n+2)\,\overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}}. \tag{4.51}$$

Since $\{\lambda_e \leq x \cdot e \leq 3\lambda_e\}$ is mapped into $\{|x \cdot e| \leq \lambda_e\}$ by the reflection with respect to T_e, using again (4.48) and (4.51) we get

$$\begin{aligned}
&|\,\{x \in \Omega \,|\, \lambda_e < x \cdot e < 3\lambda_e\}\,| \leq |\,\{x \in \Omega' \,|\, |x \cdot e| \leq \lambda_e\}\,| \leq \\
&\qquad\qquad \leq |\,\{x \in \Omega \,|\, |x \cdot e| \leq \lambda_e\}\,| + |\,\Omega \triangle \Omega'\,| \leq (n+3)\,\overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}}.
\end{aligned}$$

Now let $m_k := |\,\{x \in \Omega \mid (2k-1)\lambda_e \leq x \cdot e \leq (2k+1)\lambda_e\}\,|$ with $k \geq 1$. By the moving plane procedure the set $\Omega \cap T_\mu$ (seen as a subset in $\mathbb{R}^{n-1}$) is included in $\Omega \cap T_{\mu'}$, for every $\lambda_e \leq \mu' \leq \mu$. Therefore, m_k is a decreasing sequence and for every $k \geq 1$

$$m_k \leq m_1 \leq (n+3)\,\overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}}.$$

Now letting k_0 be the smallest natural number such that $(2k_0+1)\,\lambda_e \geq \Lambda_e$ we get

$$|\Omega_{\lambda_e}| = |\Omega \cap \{\lambda_e \leq x \cdot e \leq \Lambda_e\}| \leq \sum_{k=1}^{k_0} m_k \leq \frac{1}{2}\left(\frac{\Lambda_e}{\lambda_e}+1\right)(n+3)\,\overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}}$$

and therefore

$$|\Omega_{\lambda_e}|\,\lambda_e \leq (n+3)\,\mathrm{diam}(\Omega)\,\overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}}.$$

In light of (4.50) and (4.46), we have that $|\Omega_{\lambda_e}| \geq |\Omega|/4$, and (4.47) follows. □

We are now ready to complete the proof of the stability result in Theorem 4.0.2.

Proof of Theorem 4.0.2. Up to a translation we can assume that the critical hyperplanes T^{e_j} with respect to the n coordinate directions intersect at the origin. We choose $\varepsilon > 0$ as in the proof of Lemma 4.5.1.

Let

$$\rho_{min} := \min_{x\in\partial\Omega} |x|, \qquad \rho_{max} := \max_{x\in\partial\Omega} |x|$$

and $x, y \in \partial\Omega$ such that $|x| = \rho_{min}$ and $|y| = \rho_{max}$. Notice that, if $x = y$, then Ω is a ball, and the theorem trivially holds true. Thus, we assume $x \neq y$ and consider the unit vector

$$e = \frac{x-y}{|x-y|}$$

and the corresponding critical hyperplane T^e. The method of moving planes tells us that

$$\mathrm{dist}(x, T_e) \geq \mathrm{dist}(y, T_e). \tag{4.52}$$

Indeed, since $x = y - te$ with $t = |x-y|$, the critical position can be reached at most when y' coincides with x, which corresponds to the case in (4.52) where we have equality, while in every other case a strict inequality holds. Therefore we get

$$\rho_{max} - \rho_{min} = |y| - |x| \leq 2\,\mathrm{dist}(0, T_e) = 2|\lambda_e|. \tag{4.53}$$

Clearly, $\rho(\Omega) \leq \rho_{max} - \rho_{min}$. This, together with (4.53) and Lemma 4.5.1 gives (4.7) with $C_* = 2\widehat{C}$, if (4.46) holds true. On the other hand, if (4.46) does not hold, that is, if

$$[u]_{\partial G} > \varepsilon,$$

then it is trivial to check that

$$\rho(\Omega) \leq \mathrm{diam}(\Omega) \leq \left[\frac{\mathrm{diam}(\Omega)}{\varepsilon^{\frac{1}{s+2}}}\right][u]_{\partial G}^{\frac{1}{s+2}},$$

which is (4.7) with $C_* = \mathrm{diam}(\Omega)/\varepsilon^{1/(s+2)}$.

That is, (4.7) always holds true with

$$C_* = \max\left\{2\widehat{C}, \frac{\mathrm{diam}(\Omega)}{\varepsilon^{\frac{1}{s+2}}}\right\}.$$

As usual, the dependence on $|\Omega|$ appearing in $\widehat{C}$ and ε can be removed by using (4.35). □

4.6 On the dependence of C_* in Theorem 4.0.2 on the diameter of Ω

A natural question is whether or not the quantitative stability result in Theorem 4.0.2 holds true with a constant C_* which is independent of the diameter of Ω.

We show with an explicit example that this is not possible. The example is interesting in itself since it shows an "approximate bubbling" for remote balls. More specifically, we take $L > 10$, to be taken as large as we wish in what follows and $G := B_{1/4}(-Le_1) \cup B_{1/4}(Le_1)$. We also take $R := 3/4$ in (4.1). In this way, we have that

$$\Omega = B_1(-Le_1) \cup B_1(Le_1),$$

namely the domain is the union of two balls of unit radius located at mutual large distance.

We take u to be the corresponding torsion function as defined in (4.2). Let also v be the solution of

$$\begin{cases} (-\Delta)^s v = 1 & \text{in } B_1(-Le_1), \\ v = 0 & \text{in } \mathbb{R}^n \setminus B_1(-Le_1), \end{cases}$$

which we know to be radial.

We define $w := u - v$ and we point out that

$$\begin{cases} (-\Delta)^s w = 0 & \text{in } B_1(-Le_1), \\ w = u & \text{in } B_1(Le_1), \\ w = 0 & \text{in } \mathbb{R}^n \setminus \big(B_1(-Le_1) \cup B_1(Le_1)\big). \end{cases}$$

From this and the fractional Schauder estimates in [DSV19, Theorem 1.3], used here with $k := \ell := 0$, $f := 0$ and

$$\gamma := \begin{cases} \dfrac{11}{10} & \text{if } s \notin \left\{ \dfrac{9}{20}, \dfrac{19}{20} \right\}, \\[2ex] \dfrac{13}{10} & \text{if } s \in \left\{ \dfrac{9}{20}, \dfrac{19}{20} \right\}, \end{cases}$$

we conclude that

$$\begin{aligned} \|w\|_{C^1(B_{1/2}(-Le_1))} &\le C \int_{\mathbb{R}^n \setminus B_{1/2}(-Le_1)} \frac{|w(y)|}{|y|^{n+2s}}\,dy \\ &\le C \left[\|w\|_{L^\infty(B_1(-Le_1)\setminus B_{1/2}(-Le_1))} + \int_{B_1(Le_1)} \frac{|u(y)|}{|y|^{n+2s}}\,dy \right] \\ &\le C \left[\|w\|_{L^\infty(B_1(-Le_1)\setminus B_{1/2}(-Le_1))} + \frac{\|u\|_{L^\infty(\mathbb{R}^n)}}{L^{n+2s}} \right], \end{aligned} \tag{4.54}$$

with $C > 0$ depending only on n and s (which we feel free to rename from line to line).

Also, using the fractional Poisson Kernel P of the ball B_1 (see e.g. [Buc16, Theorem 2.10]), we have that, for all $x \in B_1$,

$$\begin{aligned}|w(x-Le_1)| &= \left|\int_{\mathbb{R}^n\setminus B_1} P(x,y)w(y-Le_1)dy\right| \le C(1-|x|^2)^s \int_{\mathbb{R}^n\setminus B_1} \frac{|w(y-Le_1)|}{(|y|^2-1)^s|x-y|^n}dy \\ &= C(1-|x|^2)^s \int_{B_1(2Le_1)} \frac{|u(y-Le_1)|}{(|y|^2-1)^s|x-y|^n}dy \le \frac{C\|u\|_{L^\infty(\mathbb{R}^n)}}{L^{n+2s}}.\end{aligned}$$

As a result,

$$\|w\|_{L^\infty(B_1(-Le_1))} \le \frac{C\|u\|_{L^\infty(\mathbb{R}^n)}}{L^{n+2s}}.$$

From this and (4.54) we arrive at

$$\|w\|_{C^1(B_{1/2}(-Le_1))} \le \frac{C\|u\|_{L^\infty(\mathbb{R}^n)}}{L^{n+2s}}. \tag{4.55}$$

Now we take $\varphi \in C^\infty(\mathbb{R}^n, [0,1])$ such that $\varphi = 1$ in $B_2(-Le_1) \cup B_2(Le_1)$ and $\varphi = 0$ outside $B_3(-Le_1) \cup B_3(Le_1)$. Thus, if $x \in B_1(-Le_1) \cup B_1(Le_1)$,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\varphi(x)-\varphi(z)}{|x-z|^{n+2s}}dz &= \int_{\mathbb{R}^n} \frac{1-\varphi(z)}{|x-z|^{n+2s}}dz \ge \int_{B_1((5-L)e_1)\cup B_1((L-5)e_1)} \frac{1-\varphi(z)}{|x-z|^{n+2s}}dz \\ &= \int_{B_1((5-L)e_1)\cup B_1((L-5)e_1)} \frac{1}{|x-z|^{n+2s}}dz \ge c,\end{aligned}$$

for some $c > 0$ depending only on n and s.

Accordingly, we can take $\psi := C\varphi$ with C large enough such that $(-\Delta)^s\psi \ge 1$. Thus, by the maximum principle, we deduce that $u \le \psi$ and accordingly $\|u\|_{L^\infty(\mathbb{R}^n)} \le C$.

Plugging this information into (4.55) we conclude that

$$\|w\|_{C^1(B_{1/2}(-Le_1))} \le \frac{C}{L^{n+2s}}.$$

Since w is antisymmetric, this gives that

$$\|w\|_{C^1(B_{1/2}(-Le_1)\cup B_{1/2}(Le_1))} \le \frac{C}{L^{n+2s}}.$$

Consequently, for all $x \neq y \in \partial B_{1/4}(-Le_1)$ (as well as for all $x \neq y \in \partial B_{1/4}(Le_1)$),

$$\frac{|w(x)-w(y)|}{|x-y|} \le \frac{C}{L^{n+2s}}.$$

Also, for all $x \in \partial B_{1/4}(-Le_1)$ and $y \in \partial B_{1/4}(Le_1)$, we have that $|x-y| \ge 1$, therefore

$$\frac{|w(x)-w(y)|}{|x-y|} \le |w(x)|+|w(y)| \le 2\|w\|_{L^\infty(B_{1/2}(-Le_1)\cup B_{1/2}(Le_1))} \le \frac{C}{L^{n+2s}}.$$

As a result,

$$\begin{aligned}[u]_{\partial G} &\le [v]_{\partial B_{1/4}(-Le_1)\cup\partial B_{1/4}(Le_1)} + [w]_{\partial B_{1/4}(-Le_1)\cup\partial B_{1/4}(Le_1)} \\ &= 0 + \sup_{x,y\in\partial B_{1/4}(-Le_1)\cup\partial B_{1/4}(Le_1),\, x\neq y} \frac{|w(x)-w(y)|}{|x-y|} \le \frac{C}{L^{n+2s}}.\end{aligned}$$

Hence, if (4.7) holded true with C_* independent of the diameter of Ω, we would have that

$$\rho(B_1(-Le_1) \cup B_1(Le_1)) \leq \frac{C}{L^{\frac{n+2s}{s+2}}}.$$

For this reason, there would exist $p \in B_1(-Le_1) \cup B_1(Le_1)$ and t, $s > 0$ such that

$$B_s(p) \subset B_1(-Le_1) \cup B_1(Le_1) \subset B_t(p)$$

and

$$|t - s| \leq \frac{C}{L^{\frac{n+2s}{s+2}}}.$$

But necessarily $s \leq 1$ and $t \geq L$, from which a contradiction plainly follows when L is sufficiently large.

4.7 Generalizations of Theorems 4.0.1 and 4.0.2

In this section we briefly describe how Theorems 4.0.1 and 4.0.2 can be slightly generalized in the case G has multiple connected components.

Let assume that $\Omega = G + B_R$, with G an open bounded set with

$$G = G_1 \cup \ldots \cup G_m\,, \tag{4.56}$$

where G_i, $i = 1, \ldots, m$, are the connected components of G and they are such that

$$(G_i + B_R) \cap (G_j + B_R) = \emptyset \quad \text{ for } i \neq j\,.$$

In this setting, the overdetermined condition (4.3) can be replaced by

$$u = c_i \quad \text{ on } \partial G_i \tag{4.57}$$

for some constants c_i, $i = 1, \ldots, m$. We have the following generalization of Theorem 4.0.1.

Theorem 4.7.1. *Let G be as in (4.56) with ∂G of class C^1 and set $\Omega := G + B_R$. There exists a solution $u \in C^s(\overline{\Omega})$ of (4.2) satisfying (4.57) if and only if G (and therefore Ω) is a ball.*

Proof. The proof is completely analogous to the one of Theorem 4.0.1. This is due to the fact that, when we apply the method of moving planes, by construction we have that the tangency point P of Case 1 and its reflected P' belong to the same connected component of G. It is clear that in Case 2 the same holds. □

We now discuss how to modify our argument for generalizing Theorem 4.0.2 in this setting. The main point is to change the definition of deficit. Indeed, in Theorem 4.0.2 we used the deficit

$$[u]_{\partial G} := \sup_{x,y \in \partial G,\, x \neq y} \frac{|u(x) - u(y)|}{|x - y|}\,.$$

It is clear that $[u]_{\partial G} \neq 0$ if $c_i \neq c_j$ for some i and j in (4.57) and then $[u]_{\partial G}$ cannot be used as a deficit in this setting. For this reason, we consider the deficit

$$[u]_* := \sup_{i=1,\ldots,m} \sup_{\substack{x,y \in \partial G_i \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|}\,. \tag{4.58}$$

By using this deficit we can argue as done for Theorem 4.0.2 and obtain the following result.

Theorem 4.7.2. *Let G be as in (4.56) with ∂G of class C^1 and let $\Omega := G + B_R$. Assume that $\partial\Omega$ is of class C^2. Let $u \in C^2(\Omega) \cap C(\mathbb{R}^n)$ be a solution of (4.2). Then, we have that*

$$\rho(\Omega) \leq C_* \, [u]_*^{\frac{1}{s+2}},$$

where $[u]_*$ is given by (4.58) and $C_* > 0$ is an explicit constant only depending on n, s, R, and the diameter $\operatorname{diam}(\Omega)$ of Ω.

Proof. By using the remark noticed in the proof of Theorem 4.7.1, the proof of the theorem is the same as the one of Theorem 4.0.2 and for this reason is omitted. □

4.8 Geometric remarks

The following technical lemma has been used in the proof of Lemma 4.4.8.

Lemma 4.8.1. *The following relations hold true.*

(i) For any two open sets A and D in $\mathbb{R}^n$, we have that

$$A + D = \overline{A} + D,$$

where $\overline{A}$ is the closure of A.

(ii) In the notation introduced in (4.37), for any point $x \in \overline{U_0} := \overline{\mathcal{Q}(G \cap H^+)}$, we have that

$$B_R(x) \subset U_R \cup \left[\Omega \cap (H^+ \cup T)\right].$$

Proof. (i) The inclusion $\subset$ is obvious. Let us prove $\supset$. For any $x \in \overline{A} + D$, we have that $x = a + d$, with $a \in \overline{A}$ and $d \in D$. Since D is open, there exists $r_d > 0$ such that $B_{r_d}(d) \subset D$. Since $a \in \overline{A}$, we can find $\underline{a} \in A$ such that $|\underline{a} - a| < r_d$. Now we notice that

$$x = a + d = \underline{a} + (a - \underline{a} + d).$$

Since the term in brackets belongs to $B_{r_d}(d) \subset D$ and $\underline{a} \in A$, we thus have proved that $x \in A + D$.

(ii) For any $x \in \overline{U_0}$, we have that

$$B_R(x) \subset \overline{U_0} + B_R(x)$$

by definition of $+$. An application of item (i) with $A := U_0$ and $D := B_R(x)$ then gives that

$$B_R(x) \subset U_0 + B_R(x).$$

The conclusion follows by noting that $U_0 \subset U_0 \cup [G \cap (H^+ \cup T)]$. □

4.9 Motivation for the overdetermined problem: fair shape of an urban settlement

A classical topic in social sciences consists in the definition and understanding of the complex transition zones (usually called “fringes”) on the periphery of urban areas, see e.g. [Pry68]. The rural-urban fringe problem aims therefore at detecting the transition in land use and demographic characteristics lying between the continuously built-up areas of a central city and the rural hinterland: this problem is of high social impact, also given the possible incomplete penetration of urban utility services in fringes.

Though the analysis of fringes is still under an intense debate and several aspects, especially the ones related to high commercial and financial pressures, are still to be considered controversial, a very simple model could be to limit our analysis to one of the features usually attributed to fringes, namely that of *low density of occupied dwellings*, and relate it to some of the characteristics that are considered inadequate for the fringe well-being such as “incomplete range and incomplete network of utility services such as reticulated water, electricity, gas and sewerage mains, fire hydrants”, etc., as well as “accessibility of schools” [Pry68].

One can also assume that *distance to urbanized areas* is a major factor to be accounted for in the analysis of the above features since “distance operates as a major constraint in shaping and facilitating urban growth, and the friction of space experienced by the rural-urban fringe is but a particular example of a principle generally accepted in human ecology and geography: the layout of a metropolis – the assignment of activities to areas – tends to be determined by a principle which may be termed the minimizing of the cost of friction” [Hai26, Pry68].

In this spirit, one can consider a model in which the environment is described by a domain Ω and the density of population (or better to say the density of occupied dwellings) is modeled by a function u. We assume that the population follows a nonlocal dispersal strategy modeled by the fractional Laplacian (see e.g. [DGV22]) and that the environment is hostile (no dwelling possible outside the domain Ω, with population “killed” if exiting the domain, corresponding to $u = 0$ outside Ω).

In this setting an equilibrium configuration for the population, subject to a growth modeled by a function $f(x, u)$, is described by the problem

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x)) & \text{for all } \ x \in \Omega, \\ u(x) = 0 & \text{for all } \ x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.59}$$

The case in which the birth and death rates of the population are negligible and the population is subject to a constant immigration factor reduces f to a constant and therefore, up to a normalization, the problem in (4.59) boils down to that in (4.2).

One could also assume that there is a small quantity, say $c > 0$, that describes the density threshold for an efficient network of utility services to develop: in this simplified model, the fringe is therefore described by the area in which the values of u belong to the interval $[0, c]$.

Clearly, the areas of major social hardship in this model would correspond to the points x of Ω in the vicinity of the boundary and with $u(x) \in [0, c]$. Assuming distance to facilities to be the leading factor towards well-being in this simplified model, the “fairest” configurations for the inhabitant of the fringe could be that in which the most remote areas are all at the same distance, say R, to the developed zone: one could therefore (at least for small c and correspondingly small R) adopt the setting in (4.1).

In this framework, the above fairest condition would translate into the requirement that the density threshold $\{u = c\}$ would coincide with ∂G, leading naturally to the overdetermined condition in (4.3).

In this spirit (and with a good degree of approximation) the overdetermined problem in (4.2) and (4.3) would correspond to that of a population in a hostile environment, with negligible birth and death rate and a constant immigration factor, that adopts a nonlocal dispersal strategy modeled by $(-\Delta)^s$, which aims at optimizing the rural-urban fringe in terms of equal maximal density to the boundary (the results presented here would give that the optimizer is given by a round city).

Chapter 5

The Parallel Surface Fractional Capacitary Problem

This chapter deals with quantitative symmetry results for overdetermined problems involving the fractional Laplacian in unbounded exterior sets or bounded annular sets. In a sense, they are the capacitary counterpart to problem (4.2)-(4.3). The proofs employ very similar techniques to the ones in Chapter 4 - that is why in this chapter you will find multiple references to the previous one.

The problems we deal with here originate from the study of capacity of a set and relative capacity which, in the classical setting, are given by

$$\operatorname{cap}(\Omega) := \inf\left\{\frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx \ : \ v \in C_c^\infty(\mathbb{R}^n),\ v|_\Omega \geq 1\right\},$$

and

$$\operatorname{cap}(\Omega; D) := \inf\left\{\frac{1}{2}\int_D |\nabla v|^2 dx : \ v \in C_c^\infty(\Omega),\ v_{|D} \geq 1\right\},$$

respectively; here D and Ω are bounded open sets, with $\overline{D} \subset \Omega \subset \mathbb{R}^n$, $n \geq 3$, and ∇v is the gradient of the function v.

Instead of the classical notion of capacity, we consider the capacity in a fractional setting. For a parameter $s \in (0,1)$, the *fractional capacity of order s* (or *s-capacity*) of the set Ω is defined as follows:

$$\operatorname{cap}_s(\Omega) := \inf\{[v]_s^2 \mid v \in C_c^\infty(\mathbb{R}^n),\ v_{|\Omega} \geq 1\}\,, \tag{5.1}$$

where $[v]_s$ is the *Gagliardo seminorm* of v which we recall from Chapter 2 to be defined as

$$[v]_s^2 := \int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dxdy\,.$$

Analogously, one can define the *relative fractional capacity of order s* of the couple of sets (Ω, D) by

$$\operatorname{cap}_s(\Omega; D) := \inf\{[v]_s^2 \mid v \in C_c^\infty(\Omega),\ v_{|D} \geq 1\}\,. \tag{5.2}$$

The Euler-Lagrange equations associated to (5.1) and (5.2) are both related to the so-called fractional Laplacian of order s. It can be proved that $\operatorname{cap}_s(\Omega)$ and $\operatorname{cap}_s(\Omega; D)$ are uniquely achieved by two functions u_Ω, $u_{\Omega,D} \in H^s(\mathbb{R}^n)$ which satisfy

$$\begin{cases} (-\Delta)^s u_\Omega = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u_\Omega = 1 & \text{in } \overline{\Omega}, \\ u_\Omega(x) \to 0 & \text{as } |x| \to +\infty\,, \end{cases} \tag{5.3}$$

and

$$\begin{cases} (-\Delta)^s u_{\Omega,D} = 0 & \text{in } A := \Omega \setminus \overline{D}, \\ u_{\Omega,D} = 1 & \text{in } \overline{D}, \\ u_{\Omega,D} = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{5.4}$$

respectively. The function u_Ω is sometimes called the *s-capacitary potential.*

Overdetermined problems for (5.3) and (5.4) have been considered in [SV19] where the overdetermined condition is given on the normal *s-derivative* at the boundary, which is assumed to be constant in the spirit of Serrin's overdetermined problem. Here we consider a somehow discrete version of Serrin's overdetermined condition, and we instead assume that the solution is constant on a surface parallel to the boundary.[1] The first result deals with solutions of problem (5.3) with the overdetermining assumption that the solution is constant on a surface parallel to $\partial\Omega$.

Theorem 5.0.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $R > 0$ and assume that $G := \Omega + B_R$ is such that ∂G of class C^1. Then, there exists a solution $u \in H^s(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ of (5.3) such that*

$$u = c \qquad \text{on } \partial G \tag{5.5}$$

for some constant c if and only if G and Ω are concentric balls and u is radially symmetric.

Once symmetry is established, one can investigate the quantitative stability result for Theorem 5.0.1. Again, just like we did for the parallel surface fractional torsion problem, we will give an estimate on $\rho(\Omega)$ in terms of the Lipschitz seminorm of the solution u on the parallel surface ∂G.

Another relevant quantity which we need to quantify the stability results is the radius of the touching ball condition. More precisely, given a set E we denote the optimal exterior and interior radii in the touching ball condition by $\mathfrak{r}_E^e$ and $\mathfrak{r}_E^i$, respectively.

The quantitative stability for the problem is given by the following theorem.

Theorem 5.0.2. *Let Ω be a bounded domain of $\mathbb{R}^n$ with $\partial\Omega$ of class C^2. Let $R > 0$ and let $G = \Omega + B_R$ be such that ∂G is of class C^2. Let $u \in C^s(\mathbb{R}^n)$ be a solution of (5.3). Then, we have that*

$$\rho(\Omega) \leq C_* \, [u]_{\partial G}^{\frac{1}{s+2}}\,, \tag{5.6}$$

with $C_* = C_*(n, s, R, \mathrm{diam}(\Omega), |\Omega|, \mathfrak{r}_\Omega^e) > 0$, where $\mathrm{diam}(\Omega)$ and $|\Omega|$ denote the diameter and the volume of Ω, respectively, and $\mathfrak{r}_\Omega^e$ is the radius of the exterior touching ball condition at Ω.

In the second part of the chapter we consider an overdetermined problem involving *annular sets.* More precisely, let $D, \Omega \subset \mathbb{R}^n$ be bounded open domains such that $\overline{D} \subset \Omega$, set

$$A := \Omega \setminus \overline{D}\,, \tag{5.7}$$

[1] Regarding problem (5.4), it is more precise to say that the solution is constant on each connected component of the parallel surface, see Theorem 5.0.3 below.

and we consider solutions to (5.4). It is clear that, since $\partial\Omega$ and ∂D do not touch, we have that

$$\overline{d} := \mathrm{dist}(\overline{D}, \mathbb{R}^n \setminus \Omega) > 0\,.$$

By choosing a positive parameter $R < \overline{d}/2$ we have that the set

$$\Gamma_R^A := \{x \in A \mid \mathrm{dist}(x, \partial A) = R\} \tag{5.8}$$

can be written as

$$\Gamma_R^A = \Gamma_R^D \cup \Gamma_R^\Omega\,,$$

with [2]

$$\begin{aligned} \Gamma_R^D &:= \{x \in A \mid \mathrm{dist}(x, \partial D) = R\}, \\ \Gamma_R^\Omega &:= \{x \in A \mid \mathrm{dist}(x, \partial\Omega) = R\}, \end{aligned}$$

with $\Gamma_R^D \cap \Gamma_R^\Omega = \emptyset$. On each of these hypersurfaces we assume that the solution satisfies the overdetermined condition

$$\begin{aligned} u &= \alpha \qquad \textit{on } \Gamma_R^D, \\ u &= \beta \qquad \textit{on } \Gamma_R^\Omega, \end{aligned} \tag{5.9}$$

where α and β are two positive constants.

We have the following symmetry result.

Theorem 5.0.3. *Let A and Γ_R^A be given by* (5.7) *and* (5.8)*, respectively, where R is such that Γ_R^A is of class C^1.*

Let $u \in H^s(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be a solution of (5.4) *satisfying the overdetermined conditions* (5.9)*. Then, D and Ω are concentric balls and u is radially symmetric.*

Now we describe the quantitative stability result that we obtain for Theorem 5.0.3. In this case, we replace the overdetermined condition (5.4) by assuming that the solution has small Lipschitz seminorm on each connected component of Γ_R^A. For this reason we define the following deficit

$$\mathrm{def}_A(u) := \max\{[u]_{\Gamma_R^D}, [u]_{\Gamma_R^\Omega}\},$$

and we have the following result.

Theorem 5.0.4. *Let A and Γ_R^A be given by* (5.7) *and* (5.8)*, respectively, and assume that ∂A and Γ_R^A are of class C^2.*

Let $u \in C^s(\mathbb{R}^n)$ be a solution of (5.4)*. Then*

$$\rho(D) + \rho(\Omega) \le C_* \mathrm{def}_A(u)^{\frac{1}{s+2}}\,, \tag{5.10}$$

with $C_* = C_*(n, s, R, \mathrm{diam}(\Omega), |\Omega|, |D|, \mathfrak{r}_D^e, \mathfrak{r}_\Omega^i) > 0$, where $\mathfrak{r}_D^e$ and $\mathfrak{r}_\Omega^i$ are the radius of the uniform exterior touching ball to D and of the interior touching ball to Ω, respectively.

[2] Notice that $\Gamma_R^A = \partial((\Omega^c + B_R) \setminus (D + B_R))$.

We notice that in Theorems 5.0.1-5.0.4 we assumed that Ω and D are connected, bounded open sets. The connectedness assumption is not necessary and it can be easily removed, and hence our results can be extended in that setting. However, this has a cost in managing the notation and it would worsen the presentation and clarity of the results. For this reason, we preferred to assume that Ω and D are connected.

The first instance in which overdetermined problems in exterior sets have been taken into account can be traced back to [Rei97]. In the paper, W. Reichel considers a semilinear equation with the standard Laplacian in a connected open set $G \subset \mathbb{R}^n$. By requiring that $\mathbb{R}^n \setminus \overline{G}$ is also connected and that $f(t)$ is a locally Lipschitz function, non-increasing for small positive values of t, the author proves that a solution $u \in C^2(\mathbb{R}^n \setminus \overline{G})$ of

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^n \setminus \overline{G}, \\ u = a > 0 & \text{on } \partial G, \\ u(x) \to 0 & \text{as } |x| \to +\infty, \\ \partial_\nu u = b \leq 0 & \text{on } \partial G, \\ 0 \leq u < a & \text{in } \mathbb{R}^n \setminus \overline{G}, \end{cases}$$

only exists if G is a ball and u is then radial and radially decreasing about the centre of G. Later in [AB98], A. Aftalion and J. Busca addressed the same problem using the method of moving planes instead; with this approach, they managed to remove the hypothesis of f being non-increasing for small positive values of its argument. B. Sirakov then in [Sir01] removed the assumption $u < a$ and allowed for possibly multi-connected sets G and for different boundary conditions on the different components of G, while taking into account a more general differential operator. For the problems in annuli, we refer to [Ale92], [Phi90], [PP89] and [Rei20].

The chapter is organized as follows. In Section 2 we present some preliminary results, including a weak maximum principle for s-harmonic functions in an unbounded domain. Section 3 is devoted to the results for exterior sets and includes the standard machinery for the method of moving planes. In Section 4 we consider the problems involving annular domains.

5.1 Fractional Maximum Principles

In this section we recall some results which will be useful in the rest of the paper. For the notation employed for the method of moving planes we refer to Chapter 2.2.

Lemma 5.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in H^s(\mathbb{R}^n)$ be a solution of*

$$\begin{cases} (-\Delta)^s u \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

Let $x_0 \in \Omega$ and $r > 0$ be such that $B_r(x_0) \subseteq \Omega$. Let $K \subset \mathbb{R}^n$ be a compact set such that

$$|K| > 0\,, \quad \operatorname{dist}(K, B_r(x_0)) > 0\,, \quad \mathrm{essinf_K u} > 0\,.$$

Then

$$u \geq C_H\, \psi_{B_r(x_0)} \qquad \text{in } B_r(x_0)\,,$$

where

$$C_H := c_{n,s}\frac{|K|\,\mathrm{essinf_K u}}{(2r+\mathrm{dist}(K,B_r(x_0))+\mathrm{diam}(K))^{n+2s}}\,, \tag{5.11}$$

with $c_{n,s}$ *and* $\psi_{B_r(x_0)}$ *given by* (2.7) *and* (2.11), *respectively.*

Lemma 5.1.1 was already proved in [GS16] and [ROS14]. Here, inspired by [FJ15] and Lemma 4.3.1, we give a proof which allows us to explicitly write the constant C_H given by (5.11), and to show its dependency on the parameters which are relevant in our problem. This will be useful when we will prove the quantitative results.

Proof of Lemma 5.1.1. We consider the barrier function

$$w(x) := \psi_B(x) + \delta\,\chi_K(x)\,,$$

where $B = B_r(x_0)$, χ_K is the characteristic function of $K \subset \mathbb{R}^n$ and $\delta > 0$ is a constant that will be chosen later.

Let $\varphi \in H^s_0(\Omega)$ be a nonnegative test function. We have

$$\mathcal{E}(w,\varphi) = \mathcal{E}(\psi_B,\varphi) + \delta\;\mathcal{E}(\chi_K,\varphi) = \int_B \varphi - \delta\, c_{n,s}\int_K\int_B \frac{\varphi(y)}{|x-y|^{n+2s}}\,dy\,dx \le$$
$$\le (1-\delta\, C)\int_B \varphi,$$

which is less or equal than zero if we choose $\delta \ge C^{-1}$ with

$$C = c_{n,s}\,|K|\inf_{x\in K, y\in B}\frac{1}{|x-y|^{n+2s}}.$$

By setting

$$\tau := \mathrm{essinf_K u}/\delta = \mathrm{C\,essinf_K u}$$

and applying the weak maximum principle for s-harmonic functions to

$$v := u - \tau\, w\,,$$

we get that

$$u \ge c_{n,s}\frac{|K|\,\mathrm{essinf_K u}}{(\mathrm{diam}(B)+\mathrm{dist}(K,B)+\mathrm{diam}(K))^{n+2s}}\,\psi_B \quad \text{in } B,$$

which is the desired result. □

As already noted in previous chapters, since our approach is based on the method of moving planes a particular attention must be given to antisymmetric s-harmonic functions. More precisely, we will have to consider functions which are antisymmetric with respect to a hyperplane which can be chosen to be $\{x_1 = 0\}$ (up to a translation and rotation).

In order to list these results, we need to introduce some notation: we set $H^+ := \{x_1 > 0\}$, $H^- := \{x_1 < 0\}$ and $T \coloneqq \{x_1 = 0\}$. Let

$$\mathcal{Q}:\mathbb{R}^n \to \mathbb{R}^n\,,\quad y \mapsto y' = (-y_1, y_2, \dots, y_n)\,,$$

be the reflection with respect to T and, for a given set E we call $E^+ := E\cap H^+$ and $E^- := E\cap H^-$.

The first result is a weak maximum principle for s-harmonic antisymmetric functions, which is stated in [FJ15, Proposition 3.1] on domains that are bounded, although for homogeneous equations this condition is not needed. We report this proposition here and we sketch a proof.

Lemma 5.1.2 (Weak maximum principle for antisymmetric functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a compact set and let $u \in H^s(\mathbb{R}^n)$ be an antisymmetric (w.r.t. $T = \{x_1 = 0\}$) solution of*

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega^c, \\ u \geq 0 & \text{in } \Omega^+. \end{cases}$$

Then, $u \geq 0$ a.e. in H^+.

Proof. Since u is s-harmonic in Ω^c then for every $\varphi \in H_0^s(\Omega^c)$ we have

$$\mathcal{E}(u, \varphi) = 0.$$

Let $\varphi = u_- \chi_{H^+} \in H_0^s(\Omega^c)$, where χ_{H^+} is the characteristic function of H^+. Following the same computations as in [FJ15, Proposition 3.1] we get

$$0 \leq \mathcal{E}(u, \varphi) \leq -\mathcal{E}(\varphi, \varphi) = -[\varphi]_s^2 \, ,$$

which immediately implies that $\varphi = 0$ a.e. and hence $u_- = 0$ a.e. in H^+. □

An analogous weak maximum principle holds for nonnegative functions in $H^s(\mathbb{R}^n)$. More precisely we have

Lemma 5.1.3 (Weak maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a compact set and let $u \in H^s(\mathbb{R}^n)$ be a solution of*

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega^c, \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

Then, $u \geq 0$ a.e. in $\mathbb{R}^n$.

Proof. The proof is analogous to the one of Lemma 5.1.2, since it is enough to consider $\varphi = u_-$. □

As an immediate consequence we have the following comparison principle for s-capacitary functions.

Corollary 5.1.4. *Let $E \subset F \subset \mathbb{R}^n$ be open bounded domains, and let u_E and u_F be the corresponding capacitary functions, i.e. the solutions to* (5.3) *for $\Omega = E$ and $\Omega = F$, respectively. Then we have*

$$u_E \leq u_F$$

in $\mathbb{R}^n$.

Proof. Since u_E is a s-capacitary function, from [War15, Lemmas 2.6 and 2.7] we have that $0 \leq u_E \leq 1$ in $\mathbb{R}^n \setminus E$. Then, by applying Lemma 5.1.3 to $v := u_F - u_E$ we obtain the result. □

From Lemma 5.1.2, we can also recover a quantitative version of the Hopf lemma for antisymmetric functions as proved in Lemma 4.3.1, which actually requires that $v \in C^s(\Omega)$, but it is straightforward to verify that the proof is still valid if one assumes $v \in H^s(\mathbb{R}^n)$. It is clear that Lemma 4.3.1 provides a quantitative version of the strong maximum principle for antisymmetric s-harmonic functions, which still holds when Ω is not bounded, as already noted in [SV19, Proposition 2.1].

Lemma 5.1.5 (Strong maximum principle for antisymmetric functions)**.** *Let Ω be an open set with $\Omega \subset H^-$ and let $v \in C(\Omega)$ be antisymmetric and a solution of*

$$\begin{cases} (-\Delta)^s v \geq 0 & \text{in } \Omega, \\ v \geq 0 & \text{in } H^-. \end{cases}$$

Then, either $v > 0$ in Ω or $v \equiv 0$ in $\mathbb{R}^n$.

Proof. From the weak maximum principle in Lemma 5.1.2 we have that $v \geq 0$ in Ω. Now assume there exists $x_0 \in \Omega$ such that $v(x_0) = 0$ and choose a ball B centered in x_0 and such that $\overline{B} \subset \Omega$. Let $K \subset \Omega$ be a compact set such that $\operatorname{dist}(B, K) > 0$ and $|K| > 0$. If we furthermore choose B and K such that $\inf_K v > 0$, by applying Lemma 4.3.1 we have

$$v \geq C\big[\operatorname{dist}(K, H^+)\,|K|\,\inf_K v\big]\psi_B \quad \text{in } B,$$

and in particular $v(x_0) > 0$, which is a contradiction. □

Another tool from Chapter 4 that we will need in our proof is the boundary Harnack inequality for s-harmonic antisymmetric functions in Lemma 4.1.1 which will be used throughout this chapter.

5.2 Exterior sets

In this section we consider the exterior overdetermined problem and prove Theorem 5.0.1 and Theorem 5.0.2.

5.2.1 The symmetry result

We start with the symmetry result given in Theorem 5.0.1.

Proof of Theorem 5.0.1. Let $e \in \mathbb{S}^{n-1}$ be a fixed direction. Withouth loss of generality we assume $e = e_1$. We recall that we are considering a solution $u \in C^s(\mathbb{R}^n)$ of (5.3) satisfying (5.5) and that $G = \Omega + B_R$, with ∂G of class C^1.

We apply the method of moving planes described in Subsection 2.1 by letting $E = G$. Without loss of generality, we can assume that $\lambda_e = 0$ (that is, the critical hyperplane T goes through the origin), and we simplify the notation by setting $H^- := \{x_1 < 0\}$, $\Omega^- := H^- \cap \Omega$ and considering

$$v(x) := u(x) - u(\mathcal{Q}(x)) \quad \text{for} \quad x \in \mathbb{R}^n,$$

where $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto x'$ is the reflection with respect to T. We have

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } H^- \setminus \Omega^- \\ v \geq 0 & \text{in } \Omega^- \\ v(\mathcal{Q}(x)) = -v(x) & \text{for every } x \in \mathbb{R}^n. \end{cases}$$

By using Lemma 5.1.2 we know that $v \geq 0$ in H^- and then Lemma 5.1.5 tells us that either $v > 0$ in $H^- \setminus \Omega^-$ or $v \equiv 0$ in $\mathbb{R}^n$. Now we show that if we assume that $v > 0$ in $H^- \setminus \Omega^-$ then we obtain a contradiction.

Case 1 - Let P be a critical point on ∂G. Since both P and its reflection P' belong to ∂G and (5.5) holds, we immediately get

$$v(P) = u(P) - u(P') = 0,$$

which is a contradiction.

Case 2 - In this case e_1 is tangent to ∂G at a point $Q \in \partial G$, and therefore we have that $\partial_1 v(Q) = 0$. On the other hand, since Q is far away from the boundary $\partial\Omega$, we can use Lemma 4.1.1 to show that $\partial_1 v(Q) < 0$, which is a contradiction.

Indeed, setting $z = (-R/4, Q_2, \ldots, Q_n)$ and $x = x_t = (-t, Q_2, \ldots, Q_n) \in B_{R/4}(z)$, with $0 < t < R/8$, we have that

$$\frac{v(x_t)}{-t} \geq -\frac{4}{RK} v(z), \tag{5.12}$$

where $K > 1$ is a constant only depending on n and s. Being $z \in H^- \setminus \Omega^-$, we have that $v(z) > 0$, and the claim follows from (5.12) by letting $t \to 0^+$. $\square$

5.2.2 Almost symmetry in one direction

Now we consider the quantitative stability result and prove Theorem 5.0.2. This will be done in two subsequent steps: we first prove the quantitative stability estimate in one direction and then, in the proof of Theorem 5.0.2 we will sketch a general idea of how to use the result in one direction to obtain the final quantitative estimate; the proof can be found in details in Section 6 of [CDP+23].

We start by proving a preliminary result which gives the behaviour of the solution to (5.3) close to the boundary.

Lemma 5.2.3. *Under the assumptions of Theorem 5.0.2, let u be a solution of* (5.3) *and let $v := 1 - u$. For any $r \leq \mathfrak{r}^e_\Omega$ we have*

$$v(x) \geq C_{cap} \,(\mathrm{dist}(x, \partial\Omega))^s \quad \text{in } (\Omega + B_r) \setminus \Omega, \tag{5.13}$$

where

$$C_{cap} := \frac{c_{n,s}\,\gamma_{n,s}\,\omega_n}{4} \, \frac{r^{n+s}}{(2r + r_0 \,\mathrm{diam}(\Omega))^{n+2s}}$$

and $r_0 > 0$ is a constant depending on n and s.

Proof. Without loss of generality, we can assume that the origin O is contained in Ω and consider the s-capacitary solution $\tilde{u}$ of the ball $B_{\mathrm{diam}(\Omega)}$ centered at the origin and of radius $\mathrm{diam}(\Omega)$:

$$\begin{cases} (-\Delta)^s \tilde{u} = 0 & \text{in } \mathbb{R}^n \setminus B_{\mathrm{diam}(\Omega)}, \\ \tilde{u} = 1 & \text{in } B_{\mathrm{diam}(\Omega)}, \\ \tilde{u}(x) \to 0 & \text{as } |x| \to +\infty. \end{cases}$$

Since $0 \leq \tilde{u} \leq 1$ and $\tilde{u}$ is radial, non-increasing and continuous (see for instance [SV19, Theorem 1.10]), there exists a radius $\tilde{R} = \tilde{R}(\mathrm{diam}(\Omega)) > 0$ such that

$$\tilde{u} < 1/2 \quad \text{in} \quad \mathbb{R}^n \setminus B_{\tilde{R}}.$$

Moreover, from Corollary 5.1.4 we have that $\tilde{u} \geq u$ in the whole space. From this we get that

$$v = 1 - u \geq 1 - \tilde{u} \geq 1/2 \quad \text{in } \mathbb{R}^n \setminus B_{\tilde{R}}.$$

We now choose $K = \overline{B_{\mathfrak{r}_\Omega^e}((\tilde{R} + \mathfrak{r}_\Omega^e)\, e_1)}$. For $x_0 \in \partial\Omega$, we now apply Lemma 5.1.1 to v with $B = B_{\mathfrak{r}_\Omega^e}(x_0)$ and K and get

$$v(x) \geq \frac{c_{n,s}\,\gamma_{n,s}\,\omega_n}{2} \frac{R^{n+s}}{(4R + \operatorname{dist}(K,B))^{n+2s}} (R - |x|)^s \quad \text{in } B.$$

We now repeat the same argument on the whole boundary $\partial\Omega$ by keeping each time the same fixed set K. We notice that in every case we have $\operatorname{dist}(K,B) \leq 2\tilde{R}$, and by using the previous inequality we obtain (5.13), where the constant $C_{cap} > 0$ can be written as

$$C_{cap} := \frac{c_{n,s}\,\gamma_{n,s}\,\omega_n}{4} \frac{R^{n+s}}{(2R + \tilde{R})^{n+2s}}.$$

In order to complete the proof, we show how $\tilde{R}$ depends on $\operatorname{diam}(\Omega)$. We consider the solution u_{B_1} to the capacitary problem (5.3) with $\Omega = B_1$, and we set

$$r_0 = \inf\{\, |x| \mid u_{B_1}(x) < 1/2\}\,.$$

By scaling properties, it is clear that $\tilde{R} = r_0 \operatorname{diam}(\Omega)$. This completes the proof. □

With this result at hand, we can prove a quantitative estimate which involves the measure of $\Omega^- \setminus \mathcal{Q}(\widehat{\Omega})$.

We fix a direction $e \in \mathbb{S}^{n-1}$. Without loss of generality, we can assume that $e = e_1$ and that the associated critical hyperplane is $T = \{x_1 = 0\}$, with $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto x'$ the reflection with respect to T. For the proof of the next lemma we will use the following notation: we set for $t \geq 0$

$$\Omega_t := \Omega + B_t(0), \quad \widehat{\Omega_t} := \Omega_t \cap H^+, \quad \Omega_t^- := \Omega_t \cap H^- \quad U_t := \mathcal{Q}(\widehat{\Omega_t}).$$

Note that $G = \Omega_R$.

Lemma 5.2.4. *Given $P \in \overline{U_R}$ with $B = B_{R/8}(P)$ such that $\operatorname{dist}(B, \partial U_0) \geq R/8$, for $\delta > 0$, we have that*

$$|\Omega^- \setminus \mathcal{Q}(\widehat{\Omega})| \;\leq \tilde{C}\,(\delta^{-(1+s)} v(P) + \delta), \tag{5.14}$$

where $\tilde{C} > 0$ is a constant depending only on n, s, R, $\mathfrak{r}_\Omega^e$ and $\operatorname{diam}(\Omega)$.

Proof. We set $K_\delta := (\Omega^- \setminus \mathcal{Q}(\widehat{\Omega})) \setminus (E_\delta \cup F_\delta)$, where

$$E_\delta := \mathcal{Q}(A_\delta) \cap (\Omega^- \setminus \mathcal{Q}(\widehat{\Omega})) \quad \text{ with } A_\delta := \{x \in \Omega^c \mid \operatorname{dist}(x, \partial\Omega) < \delta\},$$

$$F_\delta := \{x \in \Omega^- \setminus \mathcal{Q}(\widehat{\Omega}) \,|\, \operatorname{dist}(x, T) < \delta\}.$$

Using Lemma 4.3.1 with $B := B_{R/8}(P)$ and $K := K_\delta$ we obtain

$$v \geq \overset{\star}{C} \left[\operatorname{dist}(K_\delta, H^+)\, |K_\delta| \inf_{K_\delta} v\right] \psi_B \quad \text{in } B,$$

where $\overset{\star}{C} > 0$ is an explicit constant depending on n, s, R and $\mathrm{diam}(\Omega)$. Here we used that, in the present situation, we have $K \subset \Omega$ and that $\mathrm{dist}(B, U_0) \leq R$.

Since $K_\delta \subseteq (\Omega^- \setminus \mathcal{Q}(\widehat{\Omega})) \setminus F_\delta$, then

$$\mathrm{dist}(K_\delta, H^+) \geq \delta.$$

We now point out that in K_δ we have $v = u - u' = 1 - u'$; we can therefore apply Lemma 5.2.3 and get

$$v(x) \geq C_{cap}(n, s, \mathfrak{r}^e_\Omega)\, \delta^s = C_{cap}\, \delta^s \quad \text{for every} \quad x \in K_\delta.$$

Moreover, we have

$$|K_\delta| = |\Omega^- \setminus U_R| - |E_\delta \cup F_\delta| \geq |\Omega^- \setminus U_R| - (|E_\delta| + |F_\delta|). \tag{5.15}$$

By definition of F_δ, we have that

$$|F_\delta| \leq \mathrm{diam}(\Omega)^{n-1}\delta. \tag{5.16}$$

By using Lemma 5.2 in [CDP+23], since $E_\delta \subseteq A_\delta$, we have

$$|E_\delta| \leq \left[\frac{2n|\Omega|}{R}\right] \delta. \tag{5.17}$$

Putting together (5.15), (5.16) and (5.17) we get

$$|K_\delta| \geq |\Omega^- \setminus U_R| - \tilde{c}\, \delta,$$

where $\tilde{c}$ is a positive constant depending on n, $\mathrm{diam}(\Omega)$ and $\mathfrak{r}^e_\Omega$.

Hence we have proved that

$$v(P) \geq \overset{\star}{C}\, C_{cap}\, (R/8)^{2s}\, \gamma_{n,s} \delta^{1+s} \left(|\Omega^- \setminus U_R| - \tilde{c}\, \delta\right),$$

and, by choosing

$$\tilde{C} := \max\left\{ \frac{8^{2s}}{C_{cap}\, R^s\, \overset{\star}{C}}\,,\ \tilde{c} \right\},$$

we get the desired inequality (5.14). □

Once Lemma 5.2.4 is proved we can follow the same proof of Lemma 4.4.7 to get the almost symmetry in one direction and reasoning as in Section 5 of Chapter 4 we obtain the same quantitative stability estimate required for the proof of Theorem 5.0.2. This is the reason why the following proof is just a quick sum up of the main ideas.

Proof of Theorem 5.0.2. Once we have inequality (5.14), we can argue as in Lemma 4.4.7 to obtain the estimate for the almost symmetry in one direction, namely

$$|\Omega^- \setminus \mathcal{Q}(\widehat{\Omega})| \leq \overline{C}[u]_{\partial G}^{\frac{1}{s+2}}, \tag{5.18}$$

where $\overline{C} := \max\{1, \mathrm{diam}(\Omega), K\,(R/2)\}\, \tilde{C}$, with $\tilde{C}$ as in (5.14) and $K = K(n, s) \geq 1$ is the constant that appears in the boundary Harnack inequality in Lemma 4.1.1.

Now, up to a translation we can assume that the critical hyperplanes with respect to the N coordinate directions T^{e_j} coincide with $\{x_j = 0\}$ for each $j = 1, \dots, N$, that is, they all intersect at the origin.

The idea is then the following: for a given direction $e \in \mathcal{S}^{n-1}$ we slice Ω_{λ_e} in (a finite number of) sections depending on the critical value λ_e and consider their measure, namely

$$m_k := |\{x \in \Omega \mid (2k-1)\,\lambda_e \le x \cdot e \le (2k+1)\,\lambda_e\}|, \quad \text{for } k \ge 1.$$

Since Ω is bounded, $m_k > 0$ only up to an index k_0 which behaves like the inverse of λ_e. The key observation is that, by reflecting with respect to the origin and using (5.18), one has

$$m_1 = |\{x \in \Omega \mid -\lambda_e \le x \cdot e \le \lambda_e\}| \le (n+3)\,\overline{C}\,[u]_{\partial G}^{\frac{1}{s+2}}; \tag{5.19}$$

moreover, by the moving plane procedure, $m_k \le m_1$ for every k up to k_0 and therefore one can then write the expression

$$|\Omega_{\lambda_e}| \le \sum_{k=1}^{k_0} m_k \le k_0\, m_1 \le (n+3)\ \mathrm{diam}(\Omega)\,\overline{C}\,\frac{1}{\lambda_e}\,[u]_{\partial G}^{\frac{1}{s+2}}. \tag{5.20}$$

Inequalities (5.20) and some further calculations (we again refer to the proof of Lemma 4.5.1 for details) yield

$$|\lambda_e| \le 4\,(n+3)\,\frac{\mathrm{diam}(\Omega)}{|\Omega|}\,\overline{C}[u]_{\partial G}^{\frac{1}{s+2}}. \tag{5.21}$$

Now it remains to establish a relationship between $|\lambda_e|$ and $\rho(\Omega)$. We set $\rho_{min} := \min_{z \in \partial\Omega} |z|$, $\rho_{max} := \max_{z \in \partial\Omega} |z|$ and choose $x, y \in \partial\Omega$ such that $|x| = \rho_{min}$ and $|y| = \rho_{max}$. We then consider the unit vector

$$e := \frac{x-y}{|x-y|}$$

and the corresponding critical hyperplane T^e. By construction, we know that $\mathrm{dist}(x, T^e) \ge \mathrm{dist}(y, T^e)$ and therefore some simple calculations lead to

$$\rho(\Omega) \le \rho_{max} - \rho_{min} = |y| - |x| \le 2\,\mathrm{dist}(0, T^e) = 2|\lambda_e|. \tag{5.22}$$

Combining (5.21) and (5.22) leads to (5.6). It is worth pointing out that the new constants appearing in (5.19) and onward only depend on the dimension n, the diameter $\mathrm{diam}(\Omega)$ and the volume $|\Omega|$. $\square$

5.3 Annular sets

In this section we consider annular domains and prove Theorems 5.0.3 and 5.0.4. The strategy that we use is still via the method of moving planes and it is similar to the previous one; nevertheless, the method has to be carefully adapted to this situation. We recall that we are considering solutions to (5.4), where $A = \Omega \setminus \overline{D}$, with $\overline{D} \subset \Omega$ bounded open domains.

Now for a fixed direction e and a parameter $\lambda \in \mathbb{R}$ we let T_λ, H_λ, $\mathcal{Q}_\lambda$ be as in the previous section. We now consider

$$\Sigma_\lambda := (\Omega \cap H_\lambda) \setminus \mathcal{Q}_\lambda(\overline{D})$$

which is the cap of the annulus $\Omega \setminus D$. Moreover, for a given set E, we define

$$d_E := \inf\{\lambda \in \mathbb{R} \mid T_\mu \cap \overline{E} = \emptyset\}$$
$$\overline{\lambda}_E := \inf\{\lambda \leq d_E \mid \text{for every } \ \mu > \lambda, (\overline{E} \cap H_\mu)^\mu \subset (E \cap H_\mu^\mu) \ \text{ and } \ \nu(x) \cdot e > 0 \ \forall x \in T_\mu \cap \partial E\},$$

and the critical parameter $\overline{\lambda}$ is given by

$$\overline{\lambda} := \max\{\overline{\lambda}_D,\ \overline{\lambda}_\Omega\}.$$

We mention that both the function u and its reflection u' are s-harmonic in Σ_λ, as we are going to use in in the proof of Theorem 5.0.3. We also notice that, thanks to our choices, $\overline{\lambda}$ is the critical value for A with respect to the direction e, and now the critical position can occur in four possible cases (namely, Cases 1 and 2 in Subsection 2.1 for both D and Ω).

In order to avoid further technicalities we ask for the domains D and Ω to be regular (namely, with boundaries ∂G and $\partial\Omega$ of class C^2); the proof works in the same way if we instead assume that ∂A is just of class C^1, and Γ_R^G and Γ_R^Ω of class C^2.

5.3.1 The symmetry result

With this setting, we are now ready to give a proof of the symmetry result for annular sets.

Proof of Theorem 5.0.3. We fix a direction $e = e_1$ and reach the critical value $\overline{\lambda}$. Without loss of generality, we assume that $T = \{e_1 = 0\}$ and define the function $w(x) := u(x) - u(x')$ for every $x \in \mathbb{R}^n$. To simplify the notation we set $\mathcal{Q} = \mathcal{Q}_{\overline{\lambda}}$. Our aim is to show that w is actually identically zero in $\mathcal{Q}(\Sigma_{\overline{\lambda}})$. This implies that both the function w and the set A itself are symmetric with respect to direction e; since the direction e can be chosen arbitrarily, the proof is then complete.

Hence we have to show that $w \equiv 0$ in $\mathcal{Q}(\Sigma_{\overline{\lambda}})$. We notice that the function w is antisymmetric with respect to $e = e_1$ and

$$\begin{aligned}
(-\Delta)^s w(x) &= 0 && \text{for } x \in \mathcal{Q}(\Sigma_{\overline{\lambda}}),\\
w(x) = u(x) - u(x') &= 1 - u(x') \geq 0 && \text{for } x \in \overline{D} \cap \mathcal{Q}(\widehat{\Omega}),\\
w(x) = u(x) - u(x') &= u(x) \geq 0 && \text{for } x \in \Omega^- \setminus \mathcal{Q}(\widehat{\Omega}),\\
w(x) &= 0 && \text{for } x \in H^- \setminus \Omega^-.
\end{aligned}$$

In particular, the last three inequalities tell us that $w \geq 0$ in $H^- \setminus \mathcal{Q}(\Sigma_{\overline{\lambda}})$. The weak maximum principle for antisymmetric solutions in Lemma 5.1.2 implies that $w \geq 0$ in $\mathcal{Q}(\Sigma_{\overline{\lambda}})$; then, from the strong maximum principle in Lemma 5.1.5 we get that either $w > 0$ in $\mathcal{Q}(\Sigma_{\overline{\lambda}})$ or $w \equiv 0$ in $\mathcal{Q}(\Sigma_{\overline{\lambda}})$.

In order to conclude, we notice that in this case we have four possible critical cases, all of which can be treated as in the proof of Theorem 5.0.1. The conclusion then follows straightforwardly. □

5.3.2 Almost symmetry in one direction

In order to prove almost symmetry in one direction for the annular set, we need to make use again of the quantitative Hopf's type lemma (Lemma 4.3.1 in this work) and adapt it to the current problem. The first lemma in this section is about the behaviour of the solution u of (5.4) inside the annulus A with respect to the distance from the boundary. We start with a simple remark.

Remark 5.3.3. *If $u \in C^s(\mathbb{R}^n)$ solves* (5.4) *with $\partial A \in C^1$, then we have*

$$0 < u < 1 \qquad \text{in } A. \tag{5.23}$$

Indeed, applying the maximum principles for an s-harmonic function in A we get that u has to be strictly positive in A. By using the same argument for $\tilde{u} := 1 - u$ we get the latter part of (5.23).

We have the following lemma.

Lemma 5.3.4. *Under the assumptions of Theorem* (5.0.4)*, let u be a solution of* (5.4)*; then*

$$\min\{u, 1-u\}(x) \geq C^* \, (\mathrm{dist}(x, \partial A))^s \quad \text{in } A, \tag{5.24}$$

where

$$C^* := \frac{c_{n,s}\,\gamma_{n,s}}{4^{n+2s+1}} \, \frac{|D| \, \min\{\mathfrak{r}^i_\Omega, \mathfrak{r}^e_D, \overline{d}/2\}^s}{\mathrm{diam}(\Omega)^{n+2s}}.$$

Proof. We first prove (5.24) for the function u. This inequality follows by an application of Lemma 5.1.1 and therefore we fix a compact set $K_1 \subset D$ such that $|K_1| = |D|/4$. Since K_1 is compact and D is open, then $\mathrm{dist}(K_1, \partial D) > 0$.

Let $x \in A$. Assume $\mathrm{dist}(x, \partial A) = \mathrm{dist}(x, \partial\Omega)$ and let $\overline{x} \in \partial\Omega$ be such that $\mathrm{dist}(x, \partial\Omega) = \mathrm{dist}(x, \overline{x}) =: r$. We apply Lemma 5.1.1 with $K = K_1$ and $B = B_r(x)$ and get

$$u \geq c_{n,s} \, \frac{|K_1| \, \inf_{K_1} u}{(2r + \mathrm{dist}(K_1, B_r(x)) + \mathrm{diam}(K_1))^{n+2s}} \, \psi_{B_r(x)} \qquad \text{in } B_r(x). \tag{5.25}$$

Since $u = 1$ in K_1 and

$$2r + \mathrm{dist}(K_1, B_r(x)) + \mathrm{diam}(K_1) \leq 4 \, \mathrm{diam}(\Omega)\,,$$

by evaluating (5.25) at x we have

$$u(x) \geq \frac{c_{n,s}\,\gamma_{n,s}}{4^{n+2s+1}} \, \frac{|D|}{\mathrm{diam}(\Omega)^{n+2s}} \, r^{2s} = \frac{c_{n,s}\,\gamma_{n,s}}{4^{n+2s+1}} \, \frac{|D|}{\mathrm{diam}(\Omega)^{n+2s}} \, \mathrm{dist}(x, \partial\Omega)^{2s}. \tag{5.26}$$

Up until now we didn't make use of the interior radius of the touching ball condition $\mathfrak{r}^i_\Omega > 0$. If $\mathrm{dist}(x, \partial\Omega) \geq \mathfrak{r}^i_\Omega$ the equation (5.26) immediately gives (5.24). If instead $\mathrm{dist}(x, \partial\Omega) < \mathfrak{r}^i_\Omega$ we set $r_1 := \min\{\mathfrak{r}^i_\Omega, \overline{d}/2\}$, $\tilde{x} \in A$ such that $\overline{x} \in \partial\Omega \cap \partial B_{r_1}(\tilde{x})$ and apply Lemma 5.1.1 for $K = K_1$ and $B = B_{r_1}(\tilde{x})$ to get

$$u(x) \geq \frac{c_{n,s}\,\gamma_{n,s}}{4^{n+2s+1}} \, \frac{|D|}{\mathrm{diam}(\Omega)^{n+2s}} \, (r_1 + |x - \tilde{x}|)^s \, (r_1 - |x - \tilde{x}|)^s, \tag{5.27}$$

which together with the fact that $r_1 - |x - \tilde{x}| = \mathrm{dist}(x, \partial\Omega)$ gives (5.24).

Assuming now $\mathrm{dist}(x, \partial A) = \mathrm{dist}(x, \partial D)$ we can now repeat the same arguments with $r_2 := \min\{\mathfrak{r}^e_D, \overline{d}/2\}$ in place if r_1 and we obtain (5.27) with r_1 replaced by r_2.

The proof of (5.24) for $v := 1 - u$ can be carried out in the same way; we only have to fix a compact set $K_2 \subset \mathbb{R}^n \setminus \overline{\Omega}$ such that

$$\mathrm{dist}(K_1, B_r(x)) + \mathrm{diam}(K_1) \leq 2 \, \mathrm{diam}(\Omega)\,,$$

and $|K_2| = |D|/4$, and then apply Lemma 5.1.1 with $K = K_2$ as done before. $\square$

We are now ready to state a version of Lemma 5.2.4 for the annulus, under the assumptions of Theorem 5.0.4. The ball B will be chosen inside of a set where the antisymmetric function $w := u - u'$ is s-harmonic (this time, it will be $Q(\Sigma_{\overline{\lambda}})$); in this case, the compact set K consists of two components, in such a way that we can take into account the symmetric differences between the sets Ω and G and their respective reflections via the moving plane method. We define

$$
\begin{aligned}
L &:= \Omega^- \setminus \mathcal{Q}(\widehat{\Omega}) \\
M &:= D^- \setminus \mathcal{Q}(\widehat{D}) \\
\tilde{K} &:= L \cup (M \cap \mathcal{Q}(\widehat{\Omega})).
\end{aligned}
$$

We have the following lemma.

Lemma 5.3.5. *Given $P \in \mathcal{Q}(\Sigma_{\overline{\lambda}})$ with $B = B_{R/8}(P)$ such that $\operatorname{dist}(B, \mathcal{Q}(\Sigma_{\overline{\lambda}})) \geq R/8$ and given $\delta > 0$, we have that*

$$|\tilde{K}| \leq \tilde{C}\,(\delta^{-(1+s)} v(P) + \delta), \tag{5.28}$$

where $\tilde{C} > 0$ is a constant depending only on n, s, R, $\mathfrak{r}^e_D$, $\mathfrak{r}^i_\Omega$, $\operatorname{diam}(\Omega)$, $|D|$ and $|\Omega|$.

Proof. We set $K_\delta := \tilde{K} \setminus (E_\delta \cup F_\delta)$ where

$$
\begin{aligned}
E_\delta &:= [\,\{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \delta\} \cap (\Omega^- \cup \mathcal{Q}(\widehat{\Omega}))\,] \cup [G^- \setminus \mathcal{Q}(\widehat{G} + B_\delta))\,], \\
F_\delta &:= \{x \in \Omega \mid \operatorname{dist}(x, H^+) < \delta\}.
\end{aligned}
$$

We apply Lemma 4.3.1 with $B := B_{R/8}(P)$ and $K := K_\delta$ to get (4.22), i.e.

$$v \geq \overset{\star}{C}\left[\operatorname{dist}(K_\delta, H^+)\,|K_\delta| \inf_{K_\delta} v\right]\psi_B \quad \text{in } B. \tag{5.29}$$

By arguing as done for (5.15), (5.16) and (5.17) we get that

$$|\tilde{K}_\delta| \geq |\Omega^- \setminus \mathcal{Q}(\widehat{\Omega})| + |G^- \setminus \mathcal{Q}(\widehat{G})| - \tilde{c}\,\delta\,, \tag{5.30}$$

and plugging (5.30) into (5.29), together with Lemma 5.3.4 and the fact that $\operatorname{dist}(\tilde{K}_\delta, H^+) \geq \delta$ we get (5.28). $\square$

The way to use Lemma 5.3.4 to establish almost symmetry in one direction and then prove Theorem 5.0.4 is again the one sketched in the proof of Theorem 5.0.2. We just need to highlight some minor differences with the annular case, that we report below.

Proof. The first goal is to obtain the almost symmetry in one direction from (5.28). While in the proof of Theorem 5.0.2 we need to take into account the two possible critical Cases 1 and 2 for the moving plane method, with the first one being further divided into cases 1a and 1b, now the critical position can be reached for both the set D and the set Ω, resulting in a total of six possible critical cases. Nonetheless, they are tackled in the same exact way; the only thing that we need to point out is that in each of the critical cases we can write

$$v(P) \leq c_\star \max\{[u]_{\Gamma^R_\Omega}, [u]_{\Gamma^R_G}\} = c_\star \operatorname{def}_A(u)\,, \tag{5.31}$$

where $c_\star := \max\{1, \operatorname{diam}(\Omega), K\,R/2\}$. From (5.31) we can then recover the inequality

$$|\Omega^- \setminus \mathcal{Q}(\widehat{\Omega})| + |G^- \setminus \mathcal{Q}(\widehat{G})| \leq \overline{C}\operatorname{def}_A(u)^{\frac{1}{s+2}},$$

where $\overline{C} = c_\star \tilde{C}$. The slicing of the two sets can then be performed in the same way, which leads to an estimate of type

$$|\lambda_e| \leq 4\,(n+3)\frac{\operatorname{diam}(\Omega)}{|\Omega|}\,\overline{C}\,\operatorname{def}_A(u)^{\frac{1}{s+2}}, \tag{5.32}$$

where now again the bound depends on the seminorms on both of the parallel surfaces. We now only need to make sure that formula (5.22) still applies. Again, for the set Ω we define $\rho_{min} := \min_{z\in\partial\Omega}|z|$, $\rho_{max} := \max_{z\in\partial\Omega}|z|$, choose $x, y \in \partial\Omega$ such that $|x| = \rho_{min}$ and $|y| = \rho_{max}$ and consider the direction $e = y - x$ up to normalization with its critical hypeplane T^e. Since we are now in the annular case $\overline{\lambda}^e = \max\{\overline{\lambda}^e_D, \overline{\lambda}^e_\Omega\}$ and therefore the moving plane might stop before reaching the cricial position for the set Ω itself. However we can still write

$$\rho(\Omega) \leq \rho_{max} - \rho_{min} = |y| - |x| \leq 2\operatorname{dist}(0, T^e) = 2|\overline{\lambda}^e| \leq 2\,|\overline{\lambda}^e_\Omega|. \tag{5.33}$$

Combining (5.32) and (5.33) and repeating the same argument for G lead us to (5.10). $\square$

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