UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE

Corso di dottorato in matematica, ciclo XXXVI

Dipartimento di matematica Federico Enriques

Regularity results for quasilinear equations of anisotropic type and of mixed local-nonlocal type

MAT/05

Dottorando:

Carlo Alberto ANTONINI

Tutors:

Prof. Giulio CIRAOLO

Prof. Alberto FARINA

Coordinatore del dottorato:

Prof. Dario BAMBUSI

(Anno accademico 2022/2023)

Abstract

This thesis addresses interior and boundary regularity for solutions of quasilinear elliptic equations. In particular, we aim to study second order regularity of solutions to nonlinear equations driven by a local, anisotropic operator. We will also investigate first-order regularity of solutions to quasilinear equations of mixed local-nonlocal type. The thesis consists of four chapters, each one based on the original papers [8], [7], [6] and [9] respectively.

Chapter 1 deals with interior second order regularity of solutions to quasilinear equations in a possibly anisotropic setting. We deal with equations in divergence form of the kind $-\mathrm{div}\big(\mathcal{A}(\nabla u)\big) = f$, which emerge as Euler-Lagrange equations of integral functionals of the Calculus of Variations built upon possibly anisotropic norms of the gradient of trial functions. We establish interior $W^{1,2}$-Sobolev regularity for the nonlinear expression of the gradient subject to the divergence operator, the so-called *stress field* $\mathcal{A}(\nabla u)$.

Chapter 2 is about global $W^{1,2}$-Sobolev regularity of the stress field $\mathcal{A}(\nabla u)$. We study both the homogeneus Dirichlet and Neumann boundary value problems, and we provide global regularity estimates in domains enjoying minimal assumptions on the boundary. Their proofs rely on a suitable generalization of Reilly's identity, which is established for operators of Orlicz-Laplace type subject to this anisotropic regime.

Chapter 3 is devoted to a relatively transversal topic, that is the approximation of a Lipschitz domain Ω via a sequence of smooth domains. The approach here developed is different than the ones present in the literature, and it is quite flexible since our approximating sets also keep track of the (possibly) additional regularity of the boundary $\partial\Omega$. This approximation technique can be particularly useful when one considers PDEs settled in domains with minimal regularity assumptions, as in the case studied in Chapter 2.

At last, in Chapter 4 we study nonlinear equations of mixed local-nonlocal type, modeled upon the sum of a p-Laplacian operator and a fractional (s, q)-Laplacian, $-\Delta_p u + (-\Delta_q)^s u$. Under certain hypotheses on $p, q \in (1, \infty)$, $s \in (0, 1)$ and the data, we establish global Hölder continuity of the gradient of solutions to these equations, as well as a Hopf-type Lemma and a strong maximum principle.

Notation

- For $d \in \mathbb{N}$, $U \subset \mathbb{R}^d$ open, and a function $v : U \to \mathbb{R}$, we shall denote by $\nabla v = Dv$ its d-dimensional gradient, and $\nabla^2 v = D^2 v$ its hessian matrix.

  For $i, j = 1, \dots d$, we will write the partial derivatives as

$$\partial_i v = \partial_{x_i} v = \frac{\partial v}{\partial x_i}$$
$$\partial^2_{ij} v = \partial^2_{x_i x_j} v = \frac{\partial^2 v}{\partial x_i \partial x_j}\,.$$

  We will often use the short-hand notation for its level and sublevel sets

$$\{v < 0\} \coloneqq \{z \in U \,:\, v(z) < 0\}.$$
$$\{v = 0\} \coloneqq \{z \in U \,:\, v(z) = 0\}\,.$$

- For a given function $u : \Omega \to \mathbb{R}$ defined on an open set $\Omega \subset \mathbb{R}^n$, and a function $\psi : \mathbb{R}^n \to \mathbb{R}$, the notation $\nabla_\xi \psi(Du)$ means that we are differentiating the function ψ with respect to $\xi \in \mathbb{R}^n$, and evaluating it at Du.

- We denote by $W^{k,p}(\Omega)$ the usual Sobolev space of $L^p(\Omega)$ weakly differentiable functions having weak k-th order derivatives in $L^p(\Omega)$.

  For any $\alpha \in (0, 1]$, the spaces $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ will denote, respectively, the space of functions with continuous and α-Hölder continuous derivatives up to order $k \in \mathbb{N}$.

- Point of $\mathbb{R}^n$ will be written as $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We write $B_r(x)$ to denote the n-dimensional ball of radius $r > 0$ and centered at $x \in \mathbb{R}^n$. Also, $B'_r(x')$ will denote the $(n-1)$-dimensional ball of radius $r > 0$ and centered at $x' \in \mathbb{R}^{n-1}$—when the centers are omitted, the balls are assumed to be centered at the origin, i.e., $B_r := B_r(0)$ and $B'_r := B'_r(0')$.

- For $d \in \mathbb{N}$, and for a given matrix $X \in \mathbb{R}^{d\times d}$, we shall denote by $|X|$ its Frobenius Norm

$$|X| = \sqrt{\mathrm{tr}(X^t X)} = \sqrt{\sum_{i,j=1}^{d} X_{ij}^2}\,,$$

  where X^t is the transpose of X. If $X \in \mathbb{R}^{d\times d}$ is a symmetric matrix, we write $X \le c\,\mathrm{Id}$ ($X \ge c\,\mathrm{Id}$) to denote that its eigenvalues are bounded from above (below) by the constant c. From here onward, Id will denote the identity matrix.

- Given a set A, we shall write $|A|$ for its Lebesgue measure, and $\mathcal{H}^s(A)$ its s-dimensional Hausdorff measure. If A is Lebesgue measurable with $|A| < \infty$, we denote the average integral on A of a function v as

$$v_A \equiv (v)_A \coloneqq \fint_A v\,dx = \frac{1}{|A|}\int_A v\,dx\,.$$

  Also, given two open bounded sets A, B, we will denote by $\mathrm{dist}_{\mathcal{H}}(A, B)$ their Hausdorff distance.

- For a given function $\phi : U \to \mathbb{R}$ with $U \subset \mathbb{R}^{n-1}$ open, we write G_ϕ and S_ϕ to denote its graph and subgraph in $\mathbb{R}^n$, i.e.,

$$G_\phi = \{x = \big(x', \phi(x')\big) \,:\, x' \in U\} \quad \text{and} \quad S_\phi = \{x = \big(x', x_n\big) \,:\, x' \in U,\, x_n < \phi(x')\}\,.$$

Contents

Introduction

In the first part of this thesis we are going to study local second-order regularity for quasilinear equations of the form

$$-\mathrm{div}\big(\mathcal{A}(\nabla u)\big) = f \quad \text{in } \Omega \tag{0.0.1}$$

in a possibly anisotropic setting, where $\mathcal{A}(\nabla u)$ is a suitable vector-field–see (0.0.2) below, and Ω is an open subset of the Euclidean space $\mathbb{R}^n$, with $n \geq 2$.

The anisotropic term is encoded into a homogeneous, convex function H, that will be often referred to as the "anisotropy", or the "norm".
Given a norm $H = H(\xi)$ satisfying certain ellipticity assumptions, and $p \in (1, \infty)$, the term $\mathcal{A}(\nabla u)$ appearing in the divergence of (0.0.1) is given by

$$\mathcal{A}(\nabla u) := \frac{1}{p}\,\nabla_\xi H^p(\nabla u) = H^{p-1}(\nabla u)\big)\,\nabla_\xi H(\nabla u)\,, \tag{0.0.2}$$

and it is typically called *stress field*.

A classic example of equation (0.0.1) is given by the p-Laplace problem

$$-\Delta_p u := -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f\,, \tag{0.0.3}$$

which corresponds to the case of Euclidean norm $H(\xi) = |\xi|$. In particular, for $p = 2$ it reproduces the linear Poisson equation

$$-\Delta u = f\,. \tag{0.0.4}$$

Moreover, equations (0.0.1) emerge as Euler-Lagrange equations of the integral functionals

$$J_H(v) = \int_\Omega H^p(\nabla v)\,dx - \int_\Omega f\,v\,dx. \tag{0.0.5}$$

Anisotropic setting typically appears when dealing with energy functionals used to describe models of surface energy– see, e.g., [105, 186] and references therein.

Surface energy arises since the microscopic environment of the interface of a medium is different from the one in the bulk of the substance. In many concrete cases, such as for crystals or the common cooking salt, the different behavior depends significantly on the space direction, and so these energy models have now become very popular in metallurgy and crystallography, see, e.g., [11, 85, 192]. Of course, the medium may also be subject to exterior forces, and thus functional (0.0.5) results in the sum of an energy plus a potential term.

Other applications of anisotropic models related to (0.0.5) occur in noise-removal procedures in digital image processing and fluidodynamics– see, e.g., [4, 28, 67, 90, 94, 167, 60, 189] and references therein.

Equation (0.0.1)-(0.0.2) belongs to the class of quasilinear equations in divergence form of p-growth type. Namely, equations of the form

$$-\mathrm{div}\big(A(x,u,\nabla u)\big) = F(x,u,\nabla u)\,, \tag{0.0.6}$$

with appropriate conditions on $F = F(x,z,\xi)$, where the vector field $A = A(x,z,\xi)$ satisfies[1]

$$|A(x,z,\xi)| \lesssim |\xi|^{p-1} \quad \text{and} \quad A(x,z,\xi)\cdot\xi \gtrsim |\xi|^p\,. \tag{0.0.7}$$

Clearly, the prototypical example of such class of equations is given by the p-Laplace operator (0.0.3), which also features the so-called radial *Uhlenbeck type structure*, i.e., $A(x,z,\xi) \equiv \tilde{A}(|\xi|)\,\xi = |\xi|^{p-2}\xi$.

Regularity theory for quasilinear equations in divergence form has been object of study by many authors in the last century. It can be said that the seminal papers of De Giorgi, Nash and Moser [78, 163, 159, 160] opened the way on the study of this topic. Indeed, although they were treating linear problems, their proofs were based on completely nonlinear methods, i.e., the linearity of the equation was not used, and thus have been used–and improved–to treat quasilinear equations as well.

For example, concerning Hölder continuity of solutions, De Giorgi's proof was reworked and generalized to non-linear equations by Stampacchia [181, 182], Ladyzhenskaya & Uralt'seva [128] into what are now called *De Giorgi's classes*– see also [101], [108, chapter 7] and references therein. On the other hand, Moser's iteration technique was used by Serrin [179] and Trudinger [188]– see also the works of Lieberman [136, 137] for equations of Orlicz-growth and with measure data.

Further regularity of solutions is obtained by requiring A to be diffentiable in the gradient variable, and satisfying a condition of the kind[2]

$$\nabla_\xi A(x,z,\xi) \approx |\xi|^{p-2}\,\mathrm{Id}\,, \tag{0.0.8}$$

which is stronger than (0.0.7)– see, e.g., [71, Lemma 2.1]. Operators falling into this class of equations are the p-Laplacian and, as we will see, its anisotropic counterpart.

With this additional assumption in force, interior $C^{1,\alpha}$ regularity of solutions was first proven by Giaquinta & Giusti [102] in the quadratic case $p = 2$, by Evans [91] for $1 < p \leq 2$, and by Lewis [133], Tolksdorf [187] and Di Benedetto [83] for a general $1 < p < \infty$. The proof of this result is based on the so-called *perturbation method*, which can be considered as a generalization of the homonymous method used in the proof of Schauder's estimates for linear equations with Hölder continuous coefficients. Having fixed a point $x_0 \in \Omega$, the underlying idea is to "freeze" the (x,z) variables of the vector field A, i.e., to study the solution u_0 of the homogeneous problem

$$\begin{cases} -\mathrm{div}\big(A(x_0,u(x_0),\nabla u_0)\big) = 0 & \text{in } B_R(x_0)\,, \\ u_0 = u & \text{on } \partial B_R(x_0), \end{cases} \tag{0.0.9}$$

for small radius $R > 0$. Boundedness of ∇u_0 is obtained by differentiating (0.0.9) and by making use of Moser iteration. Then, via a suitable modification of De Giorgi's method, one obtains Campanato-type estimates for ∇u_0. Next, one chooses $u - u_0$ as a test function in equations (0.0.6), (0.0.9), and takes the difference between said expressions. In this way, owing to the differentiability assumption (0.0.8), the previous Campanato-type estimates, obtained for u_0, are recovered by the original solution u as well. Hence, the $C^{1,\alpha}$ regularity of u follows by Campanato's characterization of Hölder spaces [41]-[43]. We refer to Manfredi's Phd Thesis [141], and his work [142] for further details on this topic.

[1] Namely, $|A(x,z,\xi)| \leq a_0\,|\xi|^{p-1} + a_1(x)\,|z|^{p-1} + a_2(x)$ and $A(x,z,\xi)\cdot\xi \geq b_0\,|\xi|^p - b_1(x)\,|z|^p - b_2(x)$ for some positive constants a_0, b_0, and suitable functions a_1, a_2, b_1, b_1.

[2] Specifically, there exist two constants $c, C > 0$ such that $c\,|\xi|^{p-2}\,\mathrm{Id} \leq \nabla_\xi A(x,z,\xi) \leq C\,|\xi|^{p-2}\,\mathrm{Id}$.

See also [136] for the study of interior $C^{1,\alpha}$ regularity of solutions to Orlicz-growth type equations, which we will discuss in Chapter 2. Let us point out that, in general, solutions to (0.0.3) do not have any better classical regularity than $C^{1,\alpha}$– see [12, 119].

Other results concerning interior Hölder continuity and gradient regularity of p-Laplace type equations can be found in [15, 17, 34, 38, 55, 56, 76, 87, 88, 89, 125, 126, 127, 143, 144, 154, 155].

So far, we have given a brief overview on first-order regularity theory for solutions to quasilinear equations. As already mentioned, the main results of the first two chapters of this thesis are instead focused on second order regularity for solutions to equation (0.0.1).

Early contributions on this topic date back to the works of Bernstein [21] and Schauder [175] for the Poisson equation (0.0.4). Their generalization to linear equations in divergence form was proved by various authors including Friedrichs [96], Browder [37], Lax [129] and Nirenberg [165, 166]. The result can be stated as follows: if $u \in W^{1,2}_{loc}(\Omega)$ is a local weak solution of (0.0.4), then

$$u \in W^{2,2}_{loc}(\Omega) \iff f \in L^2_{loc}(\Omega)\,,$$

or equivalently

$$\nabla u \in W^{1,2}_{loc}(\Omega) \iff f \in L^2_{loc}(\Omega). \tag{0.0.10}$$

The proof is nowadays well known, and is based on the so-called *difference quotients* method–see for instance [106, Sections 8.3-8.4] and [36, Section 9.6].

Extensions of these results, still based on the difference quotients technique, were first obtained for quasilinear equations of the form

$$-\mathrm{div}\Big([\varepsilon^2 + |\nabla u|^2]^{\frac{p-2}{2}} \nabla u\Big) = f\,, \quad \text{for } \varepsilon > 0\,, \tag{0.0.11}$$

by requiring further integrability on f– see, e.g., [128, pp. 277], [187, Proposition 1], [108, Chapter 8].

However, for the p-Laplace equation (0.0.3)– that is in the case $\varepsilon = 0$ in (0.0.11)– the equivalence relation (0.0.10) is in general false. Indeed, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\beta \geq 1$, the function $u_0(x) = |x_1|^\beta$ is a local weak solution of (0.0.3) in the unit ball B_1, with right hand-side $f_0 \in L^\infty(B_1)$ if $p > 2$ is large enough, but $u_0 \notin W^{2,2}_{loc}(B_1)$ for $\beta \leq \frac{3}{2}$. In fact, $u_0 \notin W^{2,q}_{loc}(B_1)$ for all $q > 1$ provided β is sufficiently close to 1.

Therefore, in order to study L^2-second order regularity for quasilinear equations of p-Laplace type, it is more appropriate to look at the regularity of other suitable quantities than the solution u itself, such as the stress field $\mathcal{A}(\nabla u)$. In this regard, *sharp* L^2 second-order regularity of solutions, i.e., the extension of (0.0.10) to the p-Poisson problem (0.0.3), was obtained by Cianchi & Maz'ya [58] and reads as follows: if u is a local solution of equation (0.0.3), then

$$\mathcal{A}(\nabla u) \in W^{1,2}_{loc}(\Omega) \iff f \in L^2_{loc}(\Omega)\,, \tag{0.0.12}$$

and the following local quantitative estimate holds true

$$\begin{aligned} &R^{-1}\|\mathcal{A}(\nabla u)\|_{L^2(B_R)} + \|\nabla \mathcal{A}(\nabla u)\|_{L^2(B_R)} \\ &\qquad \leq c\Big(\|f\|_{L^2(B_{2R})} + R^{-\frac{n}{2}-1}\,\|\nabla u\|^{p-1}_{L^{p-1}(B_{2R})}\Big), \end{aligned} \tag{0.0.13}$$

for every open ball $B_{2R} \subset\subset \Omega$ with radius $R \leq 1$, where $c = c(n,p) > 0$.

Related second-order regularity results, with additional regularity assumptions on f can also be found in other works. For instance, Lou [139] showed a similar, yet weaker result, that is $|\mathcal{A}(\nabla u)| = |\nabla u|^{p-1} \in W^{1,2}_{loc}(\Omega)$ if u is a local weak solution of (0.0.3) and assuming $f \in L^q_{loc}(\Omega)$ with $q > n/p$, but without providing any quantitative estimates.

Local fractional differentiability of the stress field has been recently studied by Avelin, Kuusi & Mingione [13], and Balci, Diening & Weimar [16]. BMO-type estimates on $\mathcal{A}(\nabla u)$ have been obtained by Breit, Cianchi, Diening, Kaplický & Schwarzacher [84, 34].

Similar regularity results have also been obtained for vector fields of the form $V_\alpha = |\nabla u|^{\alpha-1}\nabla u$ under suitable assumptions on $\alpha \in (0,\infty)$, $p \in (1,\infty)$ and the source term f. For example, a classical result used in the proof of the $C^{1,\alpha}$ regularity [128, 83] tells us that if u solves (0.0.3) with *bounded* f, then

$$|\nabla u|^{\frac{p-2}{2}}\nabla u \in W^{1,2}_{loc}(\Omega),$$

for any $p \in (1,\infty)$.

Other contributions due to Simon [180] and De Téhlin [79] show that $\nabla u \in W^{1,p}_{loc}$ provided $f \in L^{p'}_{loc}$ and $1 < p \le 2$. Similar results for the Orlicz-Laplace equation can be found in [46].

When $p > 2$, Cellina [44] showed that $\nabla u \in W^{1,2}_{loc}$ if $f \in W^{1,2}_{loc}$, for $p \in [2,3)$– see also [156] for a generalization– whereas in [45] the author proves that $\nabla u \in W^{s,2}_{loc}$ for $p \in [3,4)$ and for all $0 < s < 4-p$. Similar, interior regularity results have been obtained in [39] and [109], the latter in a very general setting. Let us also mention that Damascelli & Sciunzi [72] obtained interior weighted L^2-Hessian regularity results for all $p \in (1,\infty)$, a suitable source term f, but without providing any quantitative estimates.

For what concerns boundary regularity of solutions to quasilinear equations, the first results were obtained almost simultaneously together with the interior ones. For instance, global boundedness and Hölder continuity for equations of p-growth type (0.0.7) can be found in the book of Ladyzhenskaya & Uralt'seva [128]– see also the work of Trudinger [188] and references therein. Boundary $C^{1,\alpha}$-regularity was proven in [103] in the quadratic case $p = 2$ and in [135] for $p \in (1,\infty)$. As in the local case, the vector field $A(x,z,\xi)$ is required to be differentiable in the ξ variable, with (0.0.8) in force.

The proof still makes use of the perturbation method previously described. The only difference when dealing with the Dirichlet problem is the study of the frozen equation of u_0, which takes place on the half ball $B^+_R(x_0)$, i.e.,

$$\begin{cases} -\mathrm{div}\big(A(x_0,u(x_0),\nabla u_0) = 0 \quad \text{in } B^+_R(x_0)\,, \\ u_0 = u \quad \text{on } \partial B^+_R(x_0) \end{cases} \tag{0.0.14}$$

for which boundedness and Campanato estimates of the gradient ∇u_0 follow from a delicate barrier argument and weak Harnack inequalities. A related contribution is due to Fan [92] for variable exponent quasilinear operators, i.e., of $p(x)$-Laplace type, and the Orlicz case is discussed in [136].

More recently, under minimal regularity assumptions on the boundary and the source term f, global Lipschitz regularity was obtained by Cianchi & Maz'ya [53, 54] for equations featuring radial Uhlenbeck structure. Their proof is based on integration of (0.0.3) (multiplied by Δu) over the level sets of $|\nabla u|$, followed by a careful analysis of each integral term via (pseudo-)rearrangements and distribution functions. Very recently, a similar result has been obtained (on convex sets) by De Filippis & Piccinini [77] for equations of (p,q)-growth type, by using a global De Giorgi type iteration.

Regarding second-order regularity, global $W^{2,2}$-estimates of solutions u to linear Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.0.15}$$

can still be established by using the difference quotients method, as long as the boundary $\partial\Omega$ is sufficiently regular, e.g., $\partial\Omega \in C^2$. In particular, the result states that if u is a weak solution to (0.0.15), then

$$\nabla u \in W^{1,2}(\Omega) \iff f \in L^2(\Omega)\,. \tag{0.0.16}$$

The same global results for linear problems in non-smooth domains satisfying minimal regularity assumptions were established in [148, 149]. The approach of the proof is different, since it is based on a Reilly's type identity

$$\int_\Omega (\Delta u)^2 dx = \int_\Omega |\nabla^2 u|^2 dx + \int_{\partial\Omega} (\partial_\nu u)^2 \,\mathrm{tr}\,\mathcal{B}\, d\mathcal{H}^{n-1}\,, \quad u = 0 \text{ on } \partial\Omega, \tag{0.0.17}$$

where $\mathrm{tr}\,\mathcal{B}$ stands for the *mean curvature* of $\partial\Omega$.

The extension of this result to the p-Laplace operator can still be found in the work of Cianchi & Maz'ya [58]–see also [59, 14] for the case of p-Laplace systems. Namely, if u is a solution to

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.0.18}$$

then we have the global counterpart to (0.0.12), i.e.,

$$\mathcal{A}(\nabla u) \in W^{1,2}(\Omega) \iff f \in L^2(\Omega)\,, \tag{0.0.19}$$

provided that Ω is convex, or its boundary satisfies minimal regularity assumptions. Their proof of this result essentially relies on an integral inequality, which extends (0.0.17) to the p-Poisson problem:

$$\int_\Omega \big|\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)\big|^2 dx \geq c \int_\Omega |\nabla u|^{2(p-2)} |\nabla^2 u|^2\, dx + \int_{\partial\Omega} |\nabla u|^{2(p-2)}\, (\partial_\nu u)^2 \,\mathrm{tr}\,\mathcal{B}\, d\mathcal{H}^{n-1}\,, \tag{0.0.20}$$

if $u = 0$ on $\partial\Omega$. Moreover, they enstablish *sharp* two-sided estimates

$$\|f\|_{L^2(\Omega)} \leq \|\nabla \mathcal{A}(\nabla u)\|_{L^2(\Omega)} \leq c\, \|f\|_{L^2(\Omega)}\,, \tag{0.0.21}$$

for some constant $c = c(n, p, \Omega) > 0$.

We also point out that boundary $W^{1,2}$-regularity for vector fields $V_\alpha = |\nabla u|^{\alpha-1}\nabla u$ has very recently been established in [158], for $\alpha > \frac{p-1}{2}$ on assuming $f \in W^{1,1}(\Omega) \cap L^q(\Omega)$ with $q > n$; related boundary regularity of p-harmonic functions is studied in [114].

In the final part of this thesis we will focus our attention to first order regularity of solutions to a different class of quasilinear equations. Indeed, we will study operators of mixed local-nonlocal type given by the sum of a local second-order elliptic operator and a nonlocal integrodifferential operator of fractional order. The model example is given by the sum of a p-Laplacian, and an (s, q)-fractional Laplacian:

$$-\Delta_p u + (-\Delta_q)^s u\,, \tag{0.0.22}$$

with constants $p, q \in (1, \infty)$, $s \in (0, 1)$, and the nonlocal term

$$(-\Delta_q)^s u(x) \coloneqq 2\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{q-2}\big(u(x) - u(y)\big)}{|x - y|^{n+sq}} dy\,,$$

where P.V. stands for the principal value of an integral.

These types of operators are connected with diffusion processes, as, by the Lévy-Khintchine formula, they are the infinitesimal generator of a general Lévy process, where the local operator is associated with a Brownian motion, while the nonlocal operator models a Lévy flight– see [73]. Mixed operators also appear in the description of a biological population in an ecological niche– see [74, 157, 168].

In the last twenty years a great deal of research has focused on the study of purely nonlocal operators. Comparatively, much fewer articles investigated the effect of coupling local and nonlocal terms. For instance, we point out the papers [47, 48] by Chen, Kim, Song & Vondraček, which established Green function estimates and a boundary Harnack inequality for the Dirichlet problem, and the series of contributions [22, 23, 25, 26] by Biagi, Dipierro, Valdinoci & Vecchi, where a number of properties enjoyed by the solutions of linear and semilinear equations are studied. We note in passing that the scope of these papers was confined to the linear case $p = q = 2$ in (0.0.22), i.e., the model operator is determined by the superposition of the Laplacian and of one of its fractional powers. More recently, attention has been given to quasilinear generalizations of these models such as (0.0.22) like for instance [24, 27, 30, 69, 75, 97, 98]. The article [75] by De Filippis & Mingione, in particular, obtained several regularity, such as the interior Hölder continuity of the gradient of the solutions and their global almost Lipschitz character, under the assumption that $p > sq$ in (0.0.22). These results will be the starting point of our investigations in the last chapter.

Goal of Chapter 1. In this chapter we study local second-order regularity of solutions to the *anisotropic p-Laplace operator* (also called *Finsler p-Laplace operator*)

$$-\Delta_p^H u := -\mathrm{div}\Big(\tfrac{1}{p}\nabla_\xi H^p(\nabla u)\Big) = f\,, \tag{0.0.23}$$

where H is a norm on $\mathbb{R}^n$ satisfying suitable *ellipticity assumptions*– see Section 1.2.

We will establish local $W^{1,2}$-Sobolev regularity for $\mathcal{A}(\nabla u)$, and we will also provide local quantitative estimates analogous to (0.0.13)– see equations (1.1.8)-(1.1.10) below. Moreover, on assuming f to enjoy higher integrability, we will also obtain L^2-weighted regularity estimates for the Hessian of u, the weight given by $H^{2(p-2)}(\nabla u)$– see estimate (1.1.11).

We highlight that these results are not just a trivial generalization of the previous ones concerning the p-Laplace operator, i.e., when $H(\xi) = |\xi|$. Indeed, when dealing with anisotropic equations (0.0.23), two main difficulties arise compared to the usual Euclidean setting.

- The general lack of regularity of the norm squared at the origin, that is $H^2 \in C^2(\mathbb{R}^n \setminus \{0\})$ – see Remark 1.2.1 below. To fix the ideas, we recall that a classical approximation procedure for the p-Laplace operator (0.0.3) consists in studying solutions u_ε to (0.0.11), i.e.,

$$-\mathrm{div}\Big(\big(\varepsilon^2 + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}}\nabla u_\varepsilon\Big) = f \quad \text{in } \Omega.$$

For smooth f and $\varepsilon > 0$, standard elliptic regularity theory [108, 106] ensures that $u_\varepsilon \in C^\infty(\Omega)$, and u_ε converge to u in the energy space $W^{1,p}_{loc}(\Omega)$. Similarly, the approximation technique used in Section 1.3 to deal with the anisotropic operator (0.0.23) is to consider u_ε solution to

$$-\mathrm{div}\Big(\big(\varepsilon^2 + H^2(\nabla u_\varepsilon)\big)^{\frac{p-2}{2}}\tfrac{1}{2}\nabla_\xi H^2(\nabla u_\varepsilon)\Big) = f \quad \text{in } \Omega. \tag{0.0.24}$$

In this case, no matter how regular the source f is, the solutions u_ε are not a priori smooth due to the lack of regularity of H^2 in the origin, but only belong to $W^{2,2}_{loc}(\Omega) \cap C^{1,\theta}_{loc}(\Omega)$ at most– see Lemma 1.3.2 below. Hence, the computations required to establish local quantitative estimates (independent on ε) are to be performed with due care keeping in mind this a priori regularity properties.

- The loss of rotational invariance whenever H is not the Euclidean norm (or more generally an Hilbert norm– see identity (1.2.4)). Indeed, the authors in [58] establish (0.0.12) by exploiting an intermediate inequality for the square of the differential operator which crucially relies on the radial Uhlenbeck type structure of the standard p-Laplace operator (0.0.3). Therefore, even in the quadratic case $p = 2$, the loss of this structure for general norms H and its consequent lack of rotational invariance call for a different approach, suitable adapted to the anisotropic setting in consideration.

Goal of Chapter 2. The main objective of this chapter is to establish global $W^{1,2}$-regularity of the stress field. Thus, we prove that the regularity results of the previous chapter hold true up to the boundary, and for a wider class of operators, the so-called *anisotropic Orlicz-Laplace operators.*

Specifically, we are going to deal with solutions to Dirichlet problems of the form

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}(\nabla u)\big) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{0.0.25}$$

and co-normal Neumann problems of the form

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}(\nabla u)\big) = f & \text{in } \Omega \\ \mathcal{A}(\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{0.0.26}$$

Here, ν denotes the outward normal to $\partial\Omega$, and the vector field $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\mathcal{A}(\xi) = \nabla_\xi B\big(H(\xi)\big) = \begin{cases} B'\big(H(\xi)\big)\nabla_\xi H(\xi) & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0, \end{cases} \tag{0.0.27}$$

where B is a suitable convex function, also called *Young function.* When $B(t) = \frac{t^p}{p}$, we recover the anisotropic p-Laplace operator (0.0.23), whereas in the case of general Young functions B equations (0.0.25)-(0.0.26) fall into the class of quasilinear operators with Orlicz-type growth, i.e., for which (0.0.8) is valid with $\frac{B'(|\xi|)}{|\xi|}$ in place of $|\xi|^{p-2}$– see Section 2.2 for details.

Under the same ellipticity assumptions on the norm H as in Chapter 1, and assuming minimal regularity hypothesis on the source term $f \in L^2(\Omega)$ and the domain Ω, we will show that $\mathcal{A}(\nabla u) \in W^{1,2}(\Omega)$, hence the equivalence

$$\mathcal{A}(\nabla u) \in W^{1,2}(\Omega) \iff f \in L^2(\Omega)\,, \tag{0.0.28}$$

in the anisotropic setting and under Orlicz-type growth conditions. Sharp quantitative estimates

$$\|f\|_{L^2(\Omega)} \leq \|\nabla\mathcal{A}(\nabla u)\|_{L^2(\Omega)} \leq c\,\|f\|_{L^2(\Omega)}\,, \tag{0.0.29}$$

are provided together with explicit information on the dependence of the constants.

As much as the local case, the loss of rotational invariance of the anisotropic operator, and the lack of regularity of H^2 at the origin call for a different approach compared to isotropic problems. Indeed, inequalities (0.0.17) and (0.0.20) cannot be extended to the anisotropic equation (0.0.25), since they crucially rely on the rotational invariance of the Laplace and p-Laplace operators respectively.

To overcome this issue, we will introduce new differential and integral identities and inequalities for vector fields. In particular, we establish an *anisotropic Reilly's type identity*, which roughly reads as

$$\int_\Omega \big|\mathrm{div}\big(\mathcal{A}(\nabla u)\big)\big|^2 dx = \int_\Omega \mathrm{tr}\Big(\big(\nabla\mathcal{A}(\nabla u)\big)^2\Big)\,dx + \int_{\partial\Omega} H^{2(p-1)}(\nabla u)\,H(\nu)\,\mathrm{tr}\,\mathcal{B}^H\,d\mathcal{H}^{n-1}\,, \tag{0.0.30}$$

where $\operatorname{tr}\mathcal{B}^H$ is the *anisotropic mean curvature* of $\partial\Omega$ associated with the norm H– see Definition 2.3.18 below.

Once this is established, our proof will consist, loosely speaking, in squaring both sides of equation (2.1.1), integrating the resultant equation over Ω, and exploiting the anisotropic Reilly's identity (0.0.30) and some integral inequalities which eventually yield the desired Sobolev regularity of $\mathcal{A}(\nabla u)$, via estimates for the corresponding norm.

Nevertheless, the overall argument just described requires a degree of smoothness of the function u and of the domain Ω which are not guaranteed for the solutions to problems (0.0.25) and (0.0.26) under the assumptions to be imposed on B, H, f and Ω. Henceforth, this approach entails approximations at various levels, involving two smoothing procedures of the differential operator–one of which due to the degeneracy of H^2 at the origin– the regularization of right-hand side f of the equation, as well as an approximation procedure of the domain Ω which is the main content of Chapter 3.

Goal of Chapter 3. As already mentioned, Chapter 3 yields a novel approximation technique concerning domains with sufficiently regular boundary. More precisely, we assume that Ω is a Lipschitz domain of class $W^{2,q}$– i.e., its boundary can be locally described as the graph of a function of $(n-1)$-variables which is Lipschitz continuous, and belongs to the Sobolev space $W^{2,q}$, $q \in [1,\infty)$. In a certain sense, the latter assumption involves weak second-order derivatives of the boundary, hence it permits us to define the notion of weak curvature $\mathcal{B}$ of $\partial\Omega$ such that $|\mathcal{B}| \in L^q(\partial\Omega)$–see Definition 3.1.2 below.

Here, we shall construct a sequence of smooth domains $\{\Omega_m\}_{m\in\mathbb{N}}$ strictly containing Ω such that $\partial\Omega_m \xrightarrow{m\to\infty} \partial\Omega$ in the Lebesgue sense, and in the Hausdorff sense.

The latter convergence will be also quantified in terms of the Lipschitz constant of Ω– see estimate (3.2.3). This means that the boundaries $\partial\Omega_m$ uniformly converge to $\partial\Omega$ in a quantified way as $m \to \infty$.

Furthermore, our approximation procedure keeps track of the regularity properties of $\partial\Omega$, and provides "curvature convergence" as well. Loosely speaking, we have that $\mathcal{B}_{\Omega_m}$ converge in L^q to $\mathcal{B}_\Omega$ as $m \to \infty$– see, e.g., equation (3.2.8). This is analogous to the case of Lipschitz functions $f \in W^{2,q}$; its regularizations f_m, obtained via convolution, converge uniformly to f, and their Hessian $\nabla^2 f_m$ (and so the curvature $\mathcal{B}_{f_m}$ of their graphs) converge to $\nabla^2 f$ (the curvature $\mathcal{B}_f$) in L^q.

This analogy is not surprising, since the very first step of our proof consists in regularizing (via convolution) the functions which locally describe the boundary $\partial\Omega$. We refer to Section 3.2 for further details in the construction of the sets Ω_m.

Finally, thanks to our construction and its convergence properties, some of the geometric quantities characterizing the set Ω such as its diameter, the Lipschitz characteristics, and certain capacity estimates of $\partial\Omega$ are comparable to the corresponding ones of the domains Ω_m– see Sections 3.1.1-3.2. All of this information will be crucial in order to quantitatively keep track of the constants in the proof of estimate (0.0.29) in Chapter 2.

Goal of Chapter 4. In the last chapter of this thesis we move our attention to quasilinear operators of mixed local-nonlocal type such as (0.0.22). We will focus on boundary properties of solutions to such equations. First, we establish global $C^{1,\theta}$-regularity of solutions under suitable assumptions on $p, q \in (1,\infty)$, $s \in (0,1)$, the boundary $\partial\Omega$ and the source term f.

This result is proven via the perturbation method, in a similar manner as described for purely local operators above. Clearly, the major differences and difficulties in this approach lie in the presence of the nonlocal integrodifferential term, which is handled by imposing the condition

$$p > sq\,. \tag{0.0.31}$$

This roughly says that the $W^{s,q}$-capacity generated by the nonlocal term is controlled by the $W^{1,p}$-capacity of the local term in (0.0.22), so that the latter term has a greater regularizing effect than the nonlocal one, and Hölder continuity of the gradient follows. We also expect assumption (0.0.31) to be somewhat necessary for, in the case $p < sq$, the nonlocal term in (0.0.22) becomes the leading one, and one should not be able to extract more than the Hölder continuity of solutions in view of the known regularity results for purely nonlocal equations–see, e.g., [98, 32].

In the last part of this chapter, we show the validity of a Hopf-type lemma. Here we impose no restriction on the parameters p, q and s, since in the proof we treat both operators of (0.0.22) separately.

In its general spirit, the proof proceeds similarly to those usually employed to establish Hopf lemmas, i.e., via a barrier type argument. Specifically, we construct a suitable positive subsolution to both local and nonlocal operators– the *barrier function*– and the conclusion then follows from the weak comparison principle for such operators. As a byproduct of this Hopf-type lemma, we immediately infer a strong maximum principle, too.

Chapter 1

Interior regularity for anisotropic quasilinear equations

1.1 Main results

Throughout this chapter, Ω is an open subset of $\mathbb{R}^n$ with $n \geq 2$ and, for $1 < p < +\infty$, we will consider a local weak solution $u \in W^{1,p}_{loc}(\Omega)$ to the anisotropic p-Laplace equation

$$-\mathrm{div}\,(\mathcal{A}(\nabla u)) = f \quad \text{in } \Omega. \tag{1.1.1}$$

This means that

$$\int_\Omega \mathcal{A}(\nabla u) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \qquad \forall\, \varphi \in W^{1,p}_c(\Omega), \tag{1.1.2}$$

where $W^{1,p}_c(\Omega)$ denotes the the set of compactly supported members of $W^{1,p}(\Omega)$, and $\mathcal{A} = \mathcal{A}(\xi)$ is the vector field given by

$$\mathcal{A}(\xi) = \nabla_\xi B\big(H(\xi)\big) = \begin{cases} H^{p-1}(\xi)\, \nabla_\xi H(\xi) & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0\,, \end{cases} \tag{1.1.3}$$

where $B(t)$ is the polynomial function

$$B(t) = \frac{t^p}{p}\,, \tag{1.1.4}$$

and $H \in C^2(\mathbb{R}^n \setminus \{0\})$ is a uniformly convex norm, that is it satisfies

$$\lambda\, |\eta|^2 \leq \tfrac{1}{2} \nabla^2_\xi H^2(\xi)\, \eta \cdot \eta \leq \Lambda\, |\eta|^2 \quad \text{for } \xi \neq 0, \text{ and } \eta \in \mathbb{R}^n. \tag{1.1.5}$$

for some constants $\lambda, \Lambda > 0$, which we will refer as *ellipticity constants* of H.

Observe also that, in view of the smoothness assumptions on the norm H, we have

$$B\big(H(\xi)\big) = \frac{H^p(\xi)}{p} \in C^1(\mathbb{R}^n) \cap C^2\big(\mathbb{R}^n \setminus \{0\}\big),$$

Concerning the source term, we ask for integrability $f \in L^q_{loc}(\Omega)$, with

$$q = \begin{cases} 2 & \text{if } p \geq \frac{2n}{n+2} \\ (p^*)' & \text{if } 1 < p < \frac{2n}{n+2}, \end{cases} \tag{1.1.6}$$

where $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent.

Let us remark that the assumption $q = (p^*)'$, when $1 < p < \frac{2n}{n+2}$, is the least one on the source term f in order to have the right-hand side of equation (1.1.2) well defined. It also follows that equation (1.1.1) has a variational structure, since it is the Euler-Lagrange equation of the functional

$$\mathcal{J}(v) = \frac{1}{p}\int_\Omega H^p(\nabla v)\,dx - \int_\Omega f v\,dx\,.$$

In particular, if H is the standard Euclidean norm, this is the integral functional associated to the standard p-Laplace operator $-\Delta_p u = -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)$.

Notice also that $2 = \left(\frac{2n}{n+2}\right)^* = \left(\left(\frac{2n}{n+2}\right)^*\right)' < (p^*)' = \frac{np}{np-n+p} < \frac{n}{p}$. Therefore, the integrability assumption (1.1.6) on f is weaker than the one in [139] and [179].

In the case $p \geq 2n/(n+2)$, by Sobolev and Hölder inequalities we have

$$\begin{aligned} \|v\|_{L^{q'}(\omega)} = \|v\|_{L^2(\omega)} &\leq C_s\left(\frac{2n}{n+2}, n\right)\|\nabla v\|_{L^{2n/(n+2)}(\omega')} \\ &\leq C_s\left(\frac{2n}{n+2}, n\right)|\omega|^{\frac{1}{2}+\frac{1}{n}-\frac{1}{p}}\|\nabla v\|_{L^p(\omega)} \qquad \forall\, v \in W_0^{1,p}(\omega) \end{aligned} \tag{1.1.7}$$

where ω is any open bounded subset of $\mathbb{R}^N$. Here, we denoted by $C_s(r,n)$ the Sobolev constant of the embedding $W^{1,r} \hookrightarrow L^{r^*}$ in $\mathbb{R}^n$, for $1 < r < n$.

We now state our main results. The first one concerns the local $W^{1,2}$-Sobolev regularity of the stress field $\mathcal{A}(\nabla u) = \frac{1}{p}\nabla_\xi H^p(\nabla u)$ for solutions of equation (1.1.1), together with the corresponding quantitative estimates; precisely we have

Theorem 1.1.1. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of (1.1.1), with $f \in L^q_{loc}(\Omega)$ and where $H \in C^2(\mathbb{R}^n \setminus \{0\})$ and q satisfy (1.1.5) and (1.1.6), respectively. Then*

$$\mathcal{A}(\nabla u) \in W^{1,2}_{loc}(\Omega)$$

and there exists a constant $c > 0$, depending only on n, p, λ, Λ, such that

$$\|\nabla\mathcal{A}(\nabla u)\|_{L^2(B_{R/2})} \leq c\Big[(R^{-\frac{n}{2}-1})\|\mathcal{A}(\nabla u)\|_{L^1(B_{2R}\setminus B_R)} + \|f\|_{L^2(B_{2R})}\Big], \tag{1.1.8}$$

$$\|\mathcal{A}(\nabla u)\|_{L^2(B_R)} \leq c\Big[R^{-\frac{n}{2}}\|\mathcal{A}(\nabla u)\|_{L^1(B_{2R}\setminus B_R)} + R\|f\|_{L^2(B_{2R})}\Big], \tag{1.1.9}$$

$$\|\mathcal{A}(\nabla u)\|_{L^1(B_{2R}\setminus B_R)} \leq c\|\nabla u\|^{p-1}_{L^{p-1}(B_{2R}\setminus B_R)}\,, \tag{1.1.10}$$

for any open ball $B_{2R} \subset\subset \Omega$.

A few comments are in order; estimates (1.1.8)-(1.1.9) are the conterpart of (0.0.13) in the anisotropic setting. Their right hand sides only contain the L^2-norm of the source term f and, being a local estimate, the L^1-norm of $\mathcal{A}(\nabla u)$, which is quantified in (1.1.10) in terms of the L^{p-1}-norm of ∇u. This is always finite since $u \in W^{1,p}_{loc}(\Omega)$.

Also, the constant c appearing in these inequalities does not depend on the norm H itself, but only on its ellipticity constants λ, Λ. Hence different norms having the same lower and upper bounds on the curvatures of their anisotropic unit balls, provide the same quantitative estimates on the solution to the corresponding anisotropic equations (1.1.1). Hence this fact, well known in the case of linear equations (1.2.5), i.e., for Hilbert norms–see [106, Chapter 8]– turns out to be true for any uniformly elliptic norm H. Finally we point out that the exponents on the radius R appearing in (1.1.8)-(1.1.9) are sharp due to a scaling argument.

We also remark that recently Guarnotta & Mosconi [112] have obtained a similar regularity result for a wide class of operators. Namely, they consider stress fields $\mathcal{A}(\xi) = \nabla_\xi F(\xi)$, where F is a *quasi-uniformly convex function*, i.e., $F \in C^1(\mathbb{R}^n) \cap W^{2,1}_{loc}(\mathbb{R}^n)$ and the ratio of the eigenvalues of $\nabla^2_\xi F(\xi)$ is bounded for almost every $\xi \in \mathbb{R}^n$. In our case, we have $F(\xi) = H^p(\xi)$.

In our next result, on assuming that the source term f enjoys better integrability properties and $p \le 2$, we prove some regularity results regarding the Hessian of the solutions to (1.1.1).

Theorem 1.1.2. *Assume $1 < p \le 2$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of* (1.1.1) *where $H \in C^2(\mathbb{R}^n \setminus \{0\})$ satisfies* (1.1.5) *and $f \in L^r_{loc}(\Omega)$, $r > n$. Then*

$$u \in W^{2,2}_{loc}(\Omega) \cap C^{1,\beta}_{loc}(\Omega)$$

for some $\beta \in (0,1)$ depending only on $n, p, r, \lambda, \Lambda$.

Moreover, for any open ball $B_{2R} \subset\subset \Omega$ we have

$$\int_{B_{R/2}} \left|D^2 u\right|^2 dx \le c\Big[R^{-n-2}\|\mathcal{A}(\nabla u)\|^2_{L^1(B_{2R}\setminus B_R)} + \|f\|^2_{L^2(B_{2R})}\Big],$$

where c is a constant depending only on $p, n, \lambda, \Lambda, r, B_R, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^r(B_{2R})}$.

In particular, when $p = 2$ we have

$$\int_{B_{R/2}} \left|D^2 u\right|^2 dx \le c\Big[R^{-n-2}\|\mathcal{A}(\nabla u)\|^2_{L^1(B_{2R}\setminus B_R)} + \|f\|^2_{L^2(B_{2R})}\Big],$$

where c is a constant depending only on n, λ, Λ.

Remark 1.1.3. *Theorem 1.1.2 is a special case of a more general result involving a source term f satisfying some weaker integrability conditions. See Theorem 1.5.2 and Remark 1.5.1 in Section 1.5.*

As mentioned in the Introduction, for $p > 2$ Theorem 1.1.2 is in general false. Nevertheless, for any $p > 1$ we have the following weighted integral estimate for the Hessian of the solution u.

Theorem 1.1.4. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a local solution of* (1.1.1)*, where H satisfies* (1.1.5) *and $f \in L^r_{loc}(\Omega)$ with $r > n$. Then*

$$u \in W^{2,2}_{loc}(\Omega \setminus Z) \cap C^{1,\beta}_{loc}(\Omega)$$

where Z denotes the set of critical points of u and $\beta \in (0,1)$ depends only on $n, p, r, \lambda, \Lambda$.

Moreover, for any open ball $B_{2R} \subset\subset \Omega$ we have

$$\int_{B_{R/2}\setminus Z} \left[H^2(\nabla u)\right]^{p-2} \left|D^2 u\right|^2 dx \le c, \tag{1.1.11}$$

where c is a constant depending only on $p, n, \lambda, \Lambda, r, B_R, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^r(B_{2R})}$.

Remark 1.1.5. *Theorem 1.1.4 is a special case of a more general result involving a source term f satisfying some weaker integrability conditions. See Theorem 1.5.3 and Remark 1.5.1 in Section 1.5.*

Next, as a consequence of Theorem 1.1.1, we prove two interesting results. The first application is related to the measure of critical points, and it was firstly proved in [139] in the Euclidean case and under more restrictive assumptions on f (see also [62]).

Proposition 1.1.6. *Let $u \in W^{1,p}(\Omega)$ be a weak solution of* (1.1.1) *and assume that the assumptions of Theorem 1.1.1 are fulfilled. Then*

$$f(x) = 0 \quad a.e. \ x \in \{\nabla u = 0\}.$$

An immediate consequence of Proposition 1.1.6 is the following corollary.

Corollary 1.1.7. *Under the assumptions of Proposition 1.1.6, if $f(x) \neq 0$ for almost all $x \in \Omega$, then the Lebesgue measure of the singular set $\{\nabla u = 0\}$ is zero.*

In particular, for any $C \in \mathbb{R}$, the level set $\{u = C\}$ has zero measure.

Outline of the proofs. Here we describe the main steps of our proofs. In order to prove Theorem 1.1.1 we first perform a suitable approximation procedure as described in (0.0.24), and then we take the square of both sides of the equation of approximate solutions u_ε; by making use of the ellipticity assumption (1.1.5) coupled with elementary inequalities such as (1.2.23), we manage to obtain weighted L^2-estimates on $D^2 u_\varepsilon$ and Caccioppoli-type inequality on the approximate stress fields $\mathcal{A}_\varepsilon(\nabla u_\varepsilon)$– see estimates (1.4.2), (1.4.10) and (1.4.11) below. As a next step we exploit an iterative argument coupled with uniform a priori energy estimates on u_ε, and obtain local $W^{1,2}$-quantitative estimates on $\mathcal{A}_\varepsilon(\nabla u_\varepsilon)$ independent on $\varepsilon > 0$, and thus Theorem 1.1.1 will follow by letting $\varepsilon \to 0$.

Next, in order to prove Theorems 1.1.2-1.1.4, we just need to let $\varepsilon \to 0$ in the previously obtained weighted estimates on $D^2 u_\varepsilon$ in conjunction with uniform a priori $C^{1,\beta}_{loc}$- estimates on u_ε, which follow from the additional integrability assumptions on f.

We conclude with the proof of Proposition 1.1.6 and Corollary 1.1.7: owing to the regularity result $\mathcal{A}(\nabla u) \in W^{1,2}_{loc}(\Omega)$, we may consider

$$\frac{|\mathcal{A}(\nabla u)|}{\varepsilon + |\mathcal{A}(\nabla u)|}\,\varphi\,, \quad \varphi \in C^\infty_c(\Omega)\,, \varepsilon > 0$$

as an admissible test function in equation (1.1.1), and then let $\varepsilon \to 0$ by Lebesgue dominated convergence theorem to conclude.

1.2 The norm H

Here we introduce the relevant definition and some properties concerning the norm H, together with a few examples.

We recall that a function $H : \mathbb{R}^n \to [0, \infty)$ is a *norm* on $\mathbb{R}^n$, if it satisfies

1. $H(0) = 0$ and $H(\xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$;
2. H is positively 1-homogeneous, i.e., $H(t\,\xi) = t\,H(\xi)$ for all $t \geq 0$ and $\xi \in \mathbb{R}^n$;
3. H is a convex, symmetric function, i.e., $H(\xi) = H(-\xi)$ for all $\xi \in \mathbb{R}^n$.

In order to obtain the regularity results, we will always assume that

$$H \in C^2(\mathbb{R}^n \setminus \{0\})\,. \tag{1.2.1}$$

Under this assumption, since H^2 is homogeneous of degree 2, it follows that the largest eigenvalue of the matrix $\frac{1}{2}\nabla^2_\xi H^2(\xi)$ is uniformly bounded from above for $\xi \neq 0$ by some positive constant $\Lambda > 0$. As an ellipticity condition, we also assume that its smallest eigenvalue is uniformly bounded from below by some positive constant $\lambda > 0$. Therefore

$$\lambda\,|\eta|^2 \leq \tfrac{1}{2}\nabla^2_\xi H^2(\xi)\,\eta\cdot\eta \leq \Lambda\,|\eta|^2 \quad \text{for } \xi \neq 0, \text{ and } \eta \in \mathbb{R}^n. \tag{1.2.2}$$

Owing to the results of [68, Proposition 3.1], the first inequality in (1.2.2) is equivalent to the geometric condition that

$$\text{the unit ball } B_1^H = \{\xi \in \mathbb{R}^n \,:\, H(\xi) < 1\} \text{ is uniformly convex}\,, \tag{1.2.3}$$

that is all the principal curvatures of its boundary are bounded away from zero. For this very reason, we say that H a *uniformly convex norm*–or *uniformly elliptic norm*– if it satisfies (1.2.1) and (1.2.2).

Examples of norms

(i) The simplest example is given by the so-called **Hilbert norms**, that is norms H of the form

$$H(\xi) = \sqrt{A\xi \cdot \xi}\,, \quad \xi \in \mathbb{R}^n\,, \tag{1.2.4}$$

where A is a constant, symmetric and positive definite matrix of $\mathbb{R}^n$. With such a choice of norm, the anisotropic Laplace equation (0.0.23) (i.e for $p = 2$), becomes

$$-\mathrm{div}\big(A\nabla u\big) = f\,. \tag{1.2.5}$$

Observe that, since in this case $A = \frac{1}{2}\nabla_\xi^2 H^2(\xi)$ for all $\xi \in \mathbb{R}^n$, the ellipticity assumption (1.2.2) translates into the classical ellipticity condition

$$\lambda\,\mathrm{Id} \leq A \leq \Lambda\,\mathrm{Id}\,.$$

Remark 1.2.1. Let us emphasize that if $H^2 \in C^2(\mathbb{R}^n)$, i.e., H^2 is smooth at the origin, then necessarily H is a Hilbert norm. Indeed $\nabla_\xi^2 H^2(\xi)$ is a zero-homogeneous function on $\mathbb{R}^n \setminus \{0\}$, and thus its continuity at the origin would imply that it is constant on $\mathbb{R}^n$, i.e., there exists a matrix $A \equiv \frac{1}{2}\nabla_\xi^2 H^2(\xi)$ for all $\xi \in \mathbb{R}^n$. It follows that H is a Hilbert norm, since $H^2(\xi) = \frac{1}{2}\nabla_\xi^2 H^2(\xi)\xi\cdot\xi$ owing to the homogeneity properties of H^2.

(ii) Given two norm $H_\#$ and H_* of class $C^2(\mathbb{R}^n \setminus \{0\})$, such that *at least one* of them satisfies condition (1.2.2), then for all $a, b > 0$ and $q \in (1,\infty)$, the function

$$H(\xi) := \Big(a\, H_\#^q(\xi) + b\, H_*^q(\xi)\Big)^{1/q}. \tag{1.2.6}$$

is a norm of class $C^2(\mathbb{R}^n \setminus \{0\})$, and it satisfies the ellipticity condition (1.2.2) as well. Clearly, H such is a norm and

$$\frac{H^q(\xi)}{q} = \frac{a}{q}H_\sharp^q(\xi) + \frac{b}{q}H_*^q(\xi)\,.$$

To prove (1.2.2) we observe that, owing to [68, Proposition 3.1], this is equivalent to show the existence of two constants $c, C > 0$ such that

$$c\,|\xi|^{q-2}\,\mathrm{Id} \leq \tfrac{1}{q}\nabla_\xi^2 H^q(\xi) \leq C\,|\xi|^{q-2}\,\mathrm{Id}, \quad \text{for all } \xi \neq 0 \tag{1.2.7}$$

Then, since H_* and $H_\sharp$ are one-homogeneous and of class C^2 outside the origin, we have that $\nabla_\xi^2 H_*^q$ and $\nabla_\xi^2 H_\sharp^q$ are homogeneous of degree $q-2$, and so the right inequality in (1.2.7) is fulfilled. The first condition in (1.2.7) follows from the fact that $H_\sharp$ satisfies (1.2.7) being uniformly convex, and since H_*^q is convex so its Hessian is nonnegative definite outside the origin.

As a particular case, we have that

$$H(\xi) = \Big(a\,|\xi|^2 + H_*^2(\xi)\Big)^{1/2}$$

is a uniformly convex norm for all $a > 0$ and norms $H_* \in C^2(\mathbb{R}^n \setminus \{0\})$.

(iii) Let K be a bounded, symmetric, C^2 uniformly convex domain, such that $0 \in K$; then its Minkowski Gauge

$$p_K(x) := \big\{ r > 0 \,:\, x \in r\,K \big\}\,,$$

is a norm on $\mathbb{R}^n$, and it satisfies (1.2.2) since its anisotropic unit ball is K– see for instance [174, Theorem 1.36] or [177, Lemma 1.7.13] .

1.2.1 Properties of uniformly convex norms

Here we collect some analytic properties of norms $H \in C^2(\mathbb{R}^n \setminus \{0\})$ satisfying the ellipticity condition (1.2.2), which are essential in our proofs.

To begin with, observe that

$$H^2 \in C^1(\mathbb{R}^n),$$

inasmuch as $H \in C^2(\mathbb{R}^n \setminus \{0\})$ and H^2 is 2-homogeneous. The homogeneity of H^2 also implies

$$\tfrac{1}{2}\nabla^2_\xi H^2(\xi)\,\xi \cdot \xi = H^2(\xi) \quad \text{for } \xi \neq 0. \tag{1.2.8}$$

Hence, owing to assumption (1.2.2), we infer

$$\lambda\,|\xi|^2 \leq H^2(\xi) \leq \Lambda\,|\xi|^2 \quad \text{for } \xi \in \mathbb{R}^n. \tag{1.2.9}$$

We denote by H_0 the dual norm of H, defined as

$$H_0(x) = \sup_{\xi \neq 0} \frac{\xi \cdot x}{H(\xi)} \quad \text{for } x \in \mathbb{R}^n. \tag{1.2.10}$$

As a consequence of (1.2.9) and (1.2.10), one has that

$$\frac{1}{\Lambda}|x|^2 \leq H_0^2(x) \leq \frac{1}{\lambda}|x|^2 \quad \text{for } x \in \mathbb{R}^n. \tag{1.2.11}$$

Inequalities (1.2.9) and (1.2.11) tell us that the Euclidean norm $|\cdot|$ and the norms H, H_0 are equivalent up to constants which only depend on λ, Λ, and not on the norm H itself.

Next, by the results of [60, Lemma 3.1], we have

$$H_0(\nabla_\xi H(\xi)) = 1 \quad \text{for } \xi \neq 0. \tag{1.2.12}$$

Thereby, from (1.2.11) we infer that

$$\sqrt{\lambda} \leq |\nabla_\xi H(\xi)| \leq \sqrt{\Lambda} \quad \text{for } \xi \neq 0. \tag{1.2.13}$$

The homogeneity of the function H ensures that

$$\xi \cdot \nabla_\xi H(\xi) = H(\xi) \quad \text{for } \xi \neq 0. \tag{1.2.14}$$

and

$$\nabla^2_\xi H(\xi)\,\xi = 0 \quad \text{for } \xi \neq 0. \tag{1.2.15}$$

We can also describe the behavior of the matrix $\nabla^2_\xi H(\xi)$ when acting on $\xi^\perp$, the subspace of $\mathbb{R}^n$ orthogonal to the vector $\xi \neq 0$. This is the content of the following lemma.

Lemma 1.2.2. *Let $\xi \in \mathbb{S}^{n-1}$. Then,*

$$\nabla_\xi^2 H(\xi) \, : \xi^\perp \to \xi^\perp. \tag{1.2.16}$$

Moreover, tha map (1.2.16) *is an isomorphism and*

$$\frac{\lambda}{\sqrt{\Lambda}}\mathrm{Id} \le \nabla_\xi^2 H(\xi) \le \frac{\Lambda}{\sqrt{\lambda}}\left(1+\frac{\Lambda}{\lambda}\right)\mathrm{Id} \qquad \text{on } \xi^\perp. \tag{1.2.17}$$

Proof. We recall that

$$\tfrac{1}{2}\nabla_\xi^2 H^2(\xi) = H(\xi)\,\nabla_\xi^2 H(\xi) + \nabla_\xi H(\xi) \otimes \nabla_\xi H(\xi) \qquad \text{for } \xi \neq 0. \tag{1.2.18}$$

Let $\xi \in \mathbb{S}^{n-1}$. Since (1.2.14) implies $\xi \cdot \nabla_\xi H(\xi) \neq 0$, we have that ξ and $\nabla_\xi H(\xi)^\perp$ span the whole $\mathbb{R}^n$. Thus, for every $\eta \in \xi^\perp$ such that $|\eta| = 1$, there exist $\alpha \in \mathbb{R}$ and $\zeta \in \nabla_\xi H(\xi)^\perp$ such that

$$\eta = \alpha\,\xi + \zeta.$$

In particular

$$\eta \cdot \nabla_\xi H(\xi) = \alpha\,\xi \cdot \nabla_\xi H(\xi) = \alpha\, H(\xi).$$

Thus, from estimates (1.2.9), (1.2.13) and $|\eta| = |\xi| = 1$, we infer

$$|\alpha| = \left|\frac{\eta \cdot \nabla_\xi H(\xi)}{H(\xi)}\right| \le \sqrt{\frac{\Lambda}{\lambda}}. \tag{1.2.19}$$

Since $\eta \perp \xi$,

$$|\zeta|^2 = |\eta - \alpha\,\xi|^2 = 1 + \alpha^2.$$

An application of inequalities (1.2.2) with $\eta = \zeta$, equation (1.2.18), and the fact that $\zeta \in \nabla_\xi H(\xi)^\perp$ imply

$$\lambda\,(1+\alpha^2) \le H(\xi)\,\nabla_\xi^2 H(\xi)\zeta \cdot \zeta \le \Lambda\,(1+\alpha^2), \tag{1.2.20}$$

Inasmuch as $\nabla_\xi^2 H(\xi)\,\xi = 0$ and $\zeta = \eta - \alpha\xi$,

$$\nabla_\xi^2 H(\xi)\zeta \cdot \zeta = \nabla_\xi^2 H(\xi)\,\eta \cdot \eta.$$

Coupling the latter equality with inequalities (1.2.20) implies that

$$\lambda \le \lambda\,(1+\alpha^2) \le H(\xi)\,\nabla_\xi^2 H(\xi)\,\eta \cdot \eta \le \Lambda\,(1+\alpha^2). \tag{1.2.21}$$

Hence, (1.2.17) follows via (1.2.19).
From the symmetry of the matrix $\nabla_\xi^2 H(\xi)$ one can deduce that it maps $\mathbb{R}^n$ into $\xi^\perp$. Furthermore, thanks to property (1.2.17), $\nabla_\xi^2 H(\xi)\,\eta \neq 0$ if $\eta \in \xi^\perp \setminus \{0\}$. Hence, the map (1.2.16) is actually an isomorphism. □

We conclude this section with a simple, yet very useful algebraic inequality.
We recall that $|M|^2 = \sum_{i,j=1}^{n} M_{ij}^2$ stands for the Frobenius norm of the matrix matrix $M = (M_{ij}) \in \mathbb{R}^{n\times n}$.

Lemma 1.2.3. *Let $X, Y \in \mathbb{R}^{n\times n}$. Assume that Y is symmetric and X is symmetric and positive definite. Denote by $\lambda_{\min}$ and $\lambda_{\max}$ the smallest and the largest eigenvalue of X, respectively. Then,*

$$\mathrm{tr}\big((X\,Y)^2\big) \ge \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^2 |X\,Y|^2. \tag{1.2.22}$$

Proof. The elementary inequality

$$\lambda_{\min} \operatorname{tr}(M) \leq \operatorname{tr}(XM) = \operatorname{tr}(MX) \leq \lambda_{\max} \operatorname{tr}(M)\,, \tag{1.2.23}$$

holds for any positive semi-definite matrix M. To verify this fact, recall that there exist unitary matrix U and a diagonal matrix whose entries are the eigenvalues of X such that $X = U^T \Lambda U$. Hence,

$$\operatorname{tr}(XM) = \operatorname{tr}(U^T \Lambda U M) = \operatorname{tr}(\Lambda U M U^T) \geq \lambda_{\min} \operatorname{tr}(UMU^T) = \lambda_{\min} \operatorname{tr}(M).$$

Note that the inequality holds since the matrix UMU^T is positive semi-definite, and hence all the entries in its diagonal are nonnegative. This establishes the first inequality in (1.2.23). The second one follows analogously. Thanks to the first inequality in (1.2.23) we have that

$$\operatorname{tr}\big((XY)^2\big) = \operatorname{tr}(XYXY) \geq \lambda_{\min} \operatorname{tr}(YXY) = \lambda_{\min} \operatorname{tr}(XY^2) \geq \lambda_{\min}^2 \operatorname{tr}(Y^2) = \lambda_{\min}^2 |Y|^2, \tag{1.2.24}$$

where we have made use of the fact that, by the very definition, the matrix YXY is symmetric and positive definite since Y is symmetric and X is symmetric and positive definite. Analogously, from the second inequality in (1.2.23) we obtain

$$\operatorname{tr}\big((XY)^2\big) = \operatorname{tr}(XYXY) \leq \lambda_{\max} \operatorname{tr}(YXY) = \lambda_{\max} \operatorname{tr}(XY^2) \leq \lambda_{\max}^2 \operatorname{tr}(Y^2) = \lambda_{\max}^2 |Y|^2\,. \tag{1.2.25}$$

Furthermore, still from the second inequality in (1.2.23) we infer that

$$|XY|^2 = \operatorname{tr}\big(XY(XY)^t\big) = \operatorname{tr}(XYYX) = \operatorname{tr}(X^2Y^2) \leq \lambda_{\max}^2 \operatorname{tr}(Y^2) = \lambda_{\max}^2 |Y|^2, \tag{1.2.26}$$

thanks to the fact that X^2 is symmetric and its eigenvalues agree with the eigenvalues of X squared. Inequality (1.2.22) is then a consequence of inequalities (1.2.24) and (1.2.26). □

1.3 The approximation argument

As usual in regularity theory, the starting point of our argument is the choice of an approximating procedure. In this section we set the approximation argument and obtain a preliminary uniform bound which will be useful later.

For all $\varepsilon \in [0,1)$, consider the function $B_\varepsilon(t) = B(\sqrt{\varepsilon^2 + t^2}) - B(\varepsilon)$ with B given by (1.1.4), i.e.,

$$B_\varepsilon(t) = \frac{1}{p}\left(\varepsilon^2 + t^2\right)^{\frac{p}{2}} - \frac{\varepsilon^p}{p} \quad t \geq 0, \tag{1.3.1}$$

and

$$\mathcal{A}_\varepsilon(\xi) := \nabla_\xi \big(B_\varepsilon \circ H\big)(\xi) = \begin{cases} \left[\varepsilon^2 + H^2(\xi)\right]^{\frac{p-2}{2}} \frac{1}{2} \nabla_\xi H^2(\xi) & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}. \tag{1.3.2}$$

Notice also the alternative formula

$$\mathcal{A}_\varepsilon(\xi) = \begin{cases} \left[\varepsilon^2 + H^2(\xi)\right]^{\frac{p-1}{2}} H(\xi)\, \nabla_\xi H(\xi) & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0\,. \end{cases} \tag{1.3.3}$$

In particular, (1.2.13) and (1.3.3) imply

$$\sqrt{\lambda}\left[\varepsilon^2 + H^2(\xi)\right]^{\frac{p-1}{2}} H(\xi) \leq |\mathcal{A}_\varepsilon(\xi)| \leq \sqrt{\Lambda}\left[\varepsilon^2 + H^2(\xi)\right]^{\frac{p-1}{2}} H(\xi)\,, \tag{1.3.4}$$

for all $\xi \in \mathbb{R}^n$.

Furthermore, owing to equation (1.3.2), the ellipticity assumption (1.2.2) on H, and the results of [68]– see also Lemma 2.2.1 below– we have that the symmetric matrix

$$\nabla_\xi \mathcal{A}_\varepsilon(\xi) = \left(\frac{\partial \mathcal{A}^i_\varepsilon(\xi)}{\partial \xi_j} \right)_{i,j=1,\dots,n} \qquad \text{for } \xi \neq 0$$

satisfies

$$\lambda \min\{(p-1),1\} \left(\varepsilon^2 + H^2(\xi)\right)^{\frac{p-2}{2}} \mathrm{Id} \leq \nabla_\xi \mathcal{A}_\varepsilon(\xi) \leq \Lambda \max\{(p-1),1\} \left(\varepsilon^2 + H^2(\xi)\right)^{\frac{p-2}{2}} \mathrm{Id} \tag{1.3.5}$$

for all $\xi \neq 0$.

Now we set $f_0 := f$ and

$$f_\varepsilon := \min\left\{ \max\{f, -\varepsilon^{-1}\}, \varepsilon^{-1} \right\} \qquad \forall\, \varepsilon \in (0,1)\,; \tag{1.3.6}$$

then

$$\begin{cases} f_\varepsilon \in L^\infty(\Omega), \quad |f_\varepsilon| \leq |f| \quad \text{a.e. in } \Omega, \\ f_\varepsilon \to f \quad \text{in } L^q_{loc}(\Omega). \end{cases} \tag{1.3.7}$$

Let us fix a subdomain $\Omega' \subset\subset \Omega$ (i.e., compactly contained in Ω) and let u_ε be the unique weak solution of

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big) = f_\varepsilon \quad \text{in } \Omega' \\ u_\varepsilon = u \quad \text{on } \partial\Omega', \end{cases} \tag{1.3.8}$$

where the boundary condition is to be intended as

$$u_\varepsilon - u \in W^{1,p}_0(\Omega').$$

It is classical that, for every $\varepsilon \in [0,1)$, u_ε is the unique minimizer of the strictly convex, coercive and weakly lower semicontinuous functional

$$\mathcal{J}_\varepsilon(v) = \frac{1}{p} \int_{\Omega'} \left(\varepsilon^2 + H^2(\nabla v)\right)^{\frac{p}{2}} \, dx - \int_{\Omega'} f_\varepsilon v \, dx, \tag{1.3.9}$$

in the closed and convex set

$$W^{1,p}_u(\Omega') = u + W^{1,p}_0(\Omega').$$

The following lemma provides a first useful bound on the approximating functions u_ε.

Lemma 1.3.1. *Let $u_\varepsilon \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of (1.3.8). Then, for any $\Omega' \subset\subset \Omega$ and for any $\varepsilon \in (0,1)$,*

$$\int_{\Omega'} \left(\varepsilon^2 + H^2(\nabla u_\varepsilon)\right)^{\frac{p}{2}} dx \leq K_{\Omega'} + 2^p \varepsilon^p |\Omega'| \tag{1.3.10}$$

with

$$K_{\Omega'} = (2^p + 1) \int_{\Omega'} H^p(\nabla u) \, dx + \underline{C} \, \|f\|^{p'}_{L^q(\Omega')}. \tag{1.3.11}$$

Here $|\Omega'|$ denotes the Lebesgue measure of Ω' and $\underline{C} = \underline{C}(n,p,\lambda,|\Omega'|)$ is a non-negative constant, independent of ε, that can be explicitly determined. [1]

Furthermore, we have that

$$u_\varepsilon \longrightarrow u \quad \textit{strongly in } W^{1,p}(\Omega').$$

[1] $\underline{C} = 2^{p'+1}(p-1)\lambda^{-p'} C_0^{p'}$, where C_0 is given by (1.3.15).

Proof. Since u_ε minimizes the functional (1.3.9) over $W^{1,p}_u(\Omega') = u + W^{1,p}_0(\Omega')$, we can take u as a competitor. This choice leads to

$$\begin{aligned}\frac{1}{p}\int_{\Omega'}\left(\varepsilon^2+H^2(\nabla u_\varepsilon)\right)^{\frac{p}{2}}\,dx &\le \frac{1}{p}\int_{\Omega'}\left(\varepsilon^2+H^2(\nabla u)\right)^{\frac{p}{2}}\,dx + \int_{\Omega'} f_\varepsilon(u_\varepsilon-u)\,dx\\ &\le \frac{1}{p}\int_{\Omega'}\left(\varepsilon^2+H^2(\nabla u)\right)^{\frac{p}{2}}\,dx + \|f_\varepsilon\|_{L^q(\Omega')}\|u_\varepsilon-u\|_{L^{q'}(\Omega')}.\end{aligned}\tag{1.3.12}$$

Then,

$$\begin{aligned}\|u_\varepsilon-u\|_{L^{q'}(\Omega')} &= \begin{cases}\|u_\varepsilon-u\|_{L^2} & \text{if } p\ge 2n/(n+2)\\[2mm] \|u_\varepsilon-u\|_{L^{p^*}} & \text{if } p< 2n/(n+2)\end{cases}\\ &\le \begin{cases} C_s\left(\frac{2n}{n+2},n\right)\|\nabla u_\varepsilon-\nabla u\|_p\,|\Omega'|^{\frac12+\frac1n-\frac1p} & \text{if } p\ge 2n/(n+2)\\[2mm] C_s(p,n)\|\nabla u_\varepsilon-\nabla u\|_p & \text{if } p<2n/(n+2)\end{cases}\end{aligned}\tag{1.3.13}$$

where in the latter we have used (1.1.7). Hence,

$$\|u_\varepsilon-u\|_{L^{q'}(\Omega')} \le C_0\Big(\|\nabla u_\varepsilon\|_{L^p(\Omega')}+\|\nabla u\|_{L^p(\Omega')}\Big)\tag{1.3.14}$$

where

$$C_0 = \begin{cases} C_s\left(\frac{2n}{n+2},n\right)\,|\Omega'|^{\frac12+\frac1n-\frac1p} & \text{if } p\ge 2n/(n+2)\\[2mm] C_s(p,n) & \text{if } p<2n/(n+2)\,.\end{cases}\tag{1.3.15}$$

Therefore, for any $\delta>0$, by weighted Young's inequality we obtain

$$\begin{aligned}\|f_\varepsilon\|_{L^q(\Omega')}\,\|u_\varepsilon-u\|_{L^{q'}(\Omega')} &\le \Big(\|\nabla u_\varepsilon\|_{L^p(\Omega')}+\|\nabla u\|_{L^p(\Omega')}\Big)C_0\|f_\varepsilon\|_{L^q(\Omega')}\\ &\le \frac{\delta^p}{p}\Big(\|\nabla u_\varepsilon\|_{L^p(\Omega')}+\|\nabla u\|_{L^p(\Omega')}\Big)^p + \frac{\left(C_0\|f_\varepsilon\|_{L^q(\Omega')}\right)^{p'}}{\delta^{p'}\,p'}\,.\end{aligned}$$

By plugging the above inequality in (1.3.12), we infer

$$\begin{aligned}\frac{1}{p}\int_{\Omega'}\left(\varepsilon^2+H^2(\nabla u_\varepsilon)\right)^{\frac{p}{2}}\,dx &\le \frac{1}{p}\int_{\Omega'}\left(\varepsilon^2+H^2(\nabla u)\right)^{\frac{p}{2}}\,dx +\\ &+\frac{2^{p-1}\delta^p}{p\,\lambda^p}\left(\int_{\Omega'}H^p(\nabla u_\varepsilon)dx+\int_{\Omega'}H^p(\nabla u)dx\right)+\frac{C_0^{p'}\|f_\varepsilon\|^{p'}_{L^q(\Omega')}}{\delta^{p'}p'},\end{aligned}\tag{1.3.16}$$

where we also used inequality (1.2.9). By choosing $\delta=\lambda/2$ we find

$$\begin{aligned}\int_{\Omega'}\left(\varepsilon^2+H^2(\nabla u_\varepsilon)\right)^{\frac{p}{2}}dx &\le 2\int_{\Omega'}\left(\varepsilon^2+H^2(\nabla u)\right)^{\frac{p}{2}}\,dx+\\ &+\int_{\Omega'}H^p(\nabla u)dx + 2^{p'+1}(p-1)\lambda^{-p'}C_0^{p'}\|f_\varepsilon\|^{p'}_{L^q(\Omega')}\\ &\le (2^p+1)\int_{\Omega'}H^p(\nabla u)dx + 2^{p'+1}(p-1)\lambda^{-p'}C_0^{p'}\|f_\varepsilon\|^{p'}_{L^q(\Omega')} + 2^p\varepsilon^p|\Omega'|\end{aligned}\tag{1.3.17}$$

and the desired inequality (1.3.10) follows by recalling (1.3.7).

Now we show that

$$u_\varepsilon \to u \quad \text{in } W^{1,p}(\Omega').$$

We first notice that $\|u_\varepsilon\|_{W^{1,p}(\Omega')}$ is uniformly bounded in ε thanks to Poincaré inequality on Ω' and (1.3.10). We can therefore extract a subsequence, relabeled as u_ε, such that

$$u_\varepsilon \rightharpoonup w \quad \text{weakly in } W^{1,p}(\Omega'),$$

for some function $w \in W^{1,p}_u(\Omega')$, since this set is weakly closed (being closed and convex). We want to show that $w = u$ on Ω'.

We recall that u is the unique minimizer of the functional

$$\mathcal{J}[v] := \frac{1}{p}\int_{\Omega'} H^p(\nabla v)\,dx - \int_{\Omega'} f\,v\,dx \quad \text{in } W^{1,p}_u(\Omega').$$

Again, since $\mathcal{J}_\varepsilon[u_\varepsilon] \le \mathcal{J}_\varepsilon[u]$, we obtain

$$\int_{\Omega'} \frac{H^p(\nabla u_\varepsilon)}{p}\,dx \le \frac{1}{p}\int_{\Omega'} \left(\varepsilon^2 + H^2(\nabla u_\varepsilon)\right)^{\frac{p}{2}} dx \le \frac{1}{p}\int_{\Omega'} \left(\varepsilon^2 + H^2(\nabla u)\right)^{\frac{p}{2}} + \int_{\Omega'} f_\varepsilon (u_\varepsilon - u)\,dx. \tag{1.3.18}$$

Therefore

$$\begin{aligned} \mathcal{J}[u_\varepsilon] &= \frac{1}{p}\int_{\Omega'} H^p(\nabla u_\varepsilon)\,dx - \int_{\Omega'} f\,u_\varepsilon\,dx \\ &\le \frac{1}{p}\int_{\Omega'} \left(\varepsilon^2 + H^2(\nabla u)\right)^{\frac{p}{2}} - \int_{\Omega} f_\varepsilon\,u\,dx + \int_{\Omega'} (f_\varepsilon - f)\,u_\varepsilon\,dx \\ &= \mathcal{J}_\varepsilon[u] + \int_{\Omega'} (f_\varepsilon - f)\,u_\varepsilon\,dx. \end{aligned} \tag{1.3.19}$$

We know that $f_\varepsilon \to f$ in $L^q(\Omega)$ and u_ε is uniformly bounded in $L^{q'}(\Omega')$ by Sobolev inequality; hence

$$\int_{\Omega'} (f_\varepsilon - f)\,u_\varepsilon\,dx \to 0 \quad \text{as } \varepsilon \to 0.$$

By the weak lower semicontinuity of the functional $\mathcal{J}$ and (1.3.19), we then infer

$$\mathcal{J}[w] \le \liminf_{\varepsilon\to 0} \mathcal{J}[u_\varepsilon] \le \liminf_{\varepsilon\to 0} \left(\mathcal{J}_\varepsilon[u] + \int_{\Omega'} (f_\varepsilon - f)\,u_\varepsilon\,dx\right) = \mathcal{J}[u], \tag{1.3.20}$$

which implies that $w = u$ on Ω' by the uniqueness of minimizers of $\mathcal{J}$. By repeating the above argument for any subsequence $\{u_{\varepsilon_k}\} \subset \{u_\varepsilon\}$, we infer that the whole sequence $u_\varepsilon \to u$ weakly in $W^{1,p}(\Omega')$.

We now show that $u_\varepsilon \to u$ strongly in $W^{1,p}(\Omega)$. By [187, Lemma 1], we have

$$[\mathcal{A}_\varepsilon(\nabla u) - \mathcal{A}_\varepsilon(\nabla u_\varepsilon)]\cdot[\nabla u - \nabla u_\varepsilon] \ge G_\varepsilon := \gamma_0 \begin{cases} (1 + |\nabla u| + |\nabla u_\varepsilon|)^{p-2}|\nabla u - \nabla u_\varepsilon|^2 & p < 2 \\ |\nabla u - \nabla u_\varepsilon|^p & p \ge 2, \end{cases} \tag{1.3.21}$$

where $\mathcal{A}_\varepsilon$ is the vector field given by (1.3.2), and γ_0 is a positive constant depending on n, p, λ, Λ.

Therefore, recalling that u_ε is a weak solution to (1.3.8), we get

$$\begin{aligned}0 \le \int_{\Omega'} G_\varepsilon \, dx &\le \int_{\Omega'} [\mathcal{A}_\varepsilon(\nabla u) - \mathcal{A}_\varepsilon(\nabla u_\varepsilon)] \cdot [\nabla u - \nabla u_\varepsilon] \, dx \\ &= \int_{\Omega'} \mathcal{A}_\varepsilon(\nabla u) \cdot [\nabla u - \nabla u_\varepsilon] \, dx - \int_{\Omega'} \mathcal{A}_\varepsilon(\nabla u_\varepsilon) \cdot [\nabla u - \nabla u_\varepsilon] \, dx \\ &= I_1(\varepsilon) + I_2(\varepsilon).\end{aligned} \tag{1.3.22}$$

Now we show that $I_1(\varepsilon)$ and $I_2(\varepsilon)$ vanish at the limit $\varepsilon \to 0$. To this end, we observe that (1.3.4) and (1.2.9) imply

$$|\mathcal{A}_\varepsilon(\nabla u)| \le C(\lambda, \Lambda, p)\,(1 + |\nabla u|)^{p-1} \quad \text{a.e. in } \Omega', \quad \forall \varepsilon \in (0,1),$$

and so $\mathcal{A}_\varepsilon(\nabla u) \to \mathcal{A}(\nabla u)$ in $L^{p'}(\Omega')$, by dominated convergence. Since $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^p(\Omega')$, we immediately obtain that

$$I_1(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

Regarding $I_2(\varepsilon)$, we notice that by testing equation (1.3.8) with the test function $u - u_\varepsilon$, we have

$$I_2(\varepsilon) = -\int_{\Omega'} f_\varepsilon\,(u - u_\varepsilon)\, dx. \tag{1.3.23}$$

We first recall that $u_\varepsilon \to u$ weakly in $W^{1,p}(\Omega')$. Moreover, as seen before, u_ε is uniformly bounded in $L^{q'}(\Omega')$ w.r.t. ε, then, up to a subsequence, $u_{\varepsilon_k} \to u$ weakly in $L^{q'}(\Omega')$. Again, by repeating the argument for any subsequence, we find

$$u_\varepsilon \to u \quad \text{weakly in } L^{q'}(\Omega') \quad \text{and} \quad f_\varepsilon \to f \quad \text{strongly in } L^q(\Omega)\,,$$

which imply $I_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus we have obtained that

$$\int_{\Omega'} G_\varepsilon \, dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.3.24}$$

If $p \ge 2$ then this is exactly the strong convergence of u_ε to u in $W^{1,p}(\Omega')$.

When $p < 2$, by Holder's inequality we have

$$\begin{aligned}\int_{\Omega'} |\nabla(u_\varepsilon - u)|^p dx \le &\left(\int_{\Omega'} (1 + |\nabla u| + |\nabla u_\varepsilon|)^{p-2} |\nabla(u_\varepsilon - u)|^2 dx \right)^{\frac{p}{2}} \\ &\times \left(\int_{\Omega'} (1 + |\nabla u| + |\nabla u_\varepsilon|)^p dx \right)^{\frac{2-p}{2}},\end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. The latter implies the desired conclusion also for $p < 2$, which concludes the proof. □

The following lemma collects some properties for u_ε which will be useful later.

Lemma 1.3.2. *Let u_ε be a solution of* (1.3.8). *Then,*

$$u_\varepsilon \in W^{2,2}_{loc}(\Omega) \cap C^1(\Omega)$$

and

$$\mathcal{A}_\varepsilon(\nabla u_\varepsilon) = \Big(\mathcal{A}^1_\varepsilon(\nabla u_\varepsilon), ..., \mathcal{A}^n_\varepsilon(\nabla u_\varepsilon) \Big) \in W^{1,2}_{loc}(\Omega; \mathbb{R}^n).$$

Furthermore, for any $j, k = 1, ..., n$,

$$\partial_{x_k}\mathcal{A}^j_\varepsilon(\nabla u_\varepsilon) = \sum_{m=1}^{n} \frac{\partial \mathcal{A}^j_\varepsilon}{\partial \xi_m}(\nabla u_\varepsilon)\frac{\partial}{\partial x_k}\Big(\frac{\partial u_\varepsilon}{\partial x_m}\Big) \qquad a.e. \quad in \quad \Omega, \tag{1.3.25}$$

where the products on the r-h-s are to be interpreted as zero whenever their second factor is zero, irrespective of whether $\frac{\partial \mathcal{A}^j_\varepsilon}{\partial \xi_m}$ is defined.

Proof. Since $f_\varepsilon \in L^\infty_{loc}(\Omega)$, thanks to [179] we have that $u_\varepsilon \in C^0(\Omega)$. Then, thanks to the ellipticity condition (1.3.5) and (1.2.9), we may apply [187, Theorem 1, Proposition 1] and obtain

$$\begin{aligned} &u_\varepsilon \in W^{2,2}_{loc}(\Omega) \cap C^1(\Omega) \quad \text{if } p \geq 2 \\ &u_\varepsilon \in W^{2,p}_{loc}(\Omega) \cap C^1(\Omega) \quad \text{if } p \leq 2. \end{aligned} \tag{1.3.26}$$

Since $\nabla u_\varepsilon \in C^0(\Omega) \subset L^\infty_{loc}(\Omega)$, we infer

$$u_\varepsilon \in W^{2,2}_{loc}(\Omega)$$

also in the case $p \leq 2$ by applying [68, Proposition 4.3].

Now we notice that [68, Lemma 4.1] implies

$$\mathcal{A}_\varepsilon(\xi) \in C^1(\mathbb{R}^n \setminus \{0\}) \cap Lip_{loc}(\mathbb{R}^n)$$

and, from the chain rule of [145, Theorem 2.1] (see also [132, section 11]), we obtain that

$$\mathcal{A}_\varepsilon(\nabla u_\varepsilon) \in W^{1,2}_{loc}(\Omega; \mathbb{R}^n)$$

and (1.3.25), which completes the proof. □

1.4 Preliminary uniform bounds

In this section we obtain some crucial integral inequalities for the solutions u_ε of the approximating problems, which allow us to bound some relevant integral quantities uniformly in ε.

Let

$$Z_\varepsilon = \{x \in \Omega \, : \nabla u_\varepsilon = 0\}$$

be the set of critical points of u_ε. Therefore, in view of Lemma 1.3.2, we have

$$D^2 u_\varepsilon = 0 \quad \text{a.e. in } Z_\varepsilon,$$

and so

$$\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon) = \begin{cases} \nabla_\xi A_\varepsilon(\nabla u_\varepsilon)\, D^2 u_\varepsilon & \text{a.e. on } Z^c_\varepsilon, \\ 0 & \text{a.e. on } Z_\varepsilon. \end{cases} \tag{1.4.1}$$

Proposition 1.4.1. *Let u_ε be a solution of (1.3.8). Then there exists a constant $C_1 = C_1(n, p, \lambda, \Lambda)$ such that, for any function $\eta \in C^{0,1}_c(\Omega)$ and for any $\varepsilon \in (0, 1)$, we have*

$$\begin{aligned} \int_\Omega \eta^2[\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{p-2}\big|D^2 u_\varepsilon\big|^2 dx \leq & C_1 \int_\Omega [\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{p-2} H^2(\nabla u_\varepsilon)|\nabla\eta|^2\, dx \\ & + C_1 \int_\Omega \eta^2 f^2_\varepsilon dx. \end{aligned} \tag{1.4.2}$$

Proof. Since from Lemma 1.3.2 we have that $\mathcal{A}_\varepsilon(\nabla u_\varepsilon) \in W^{1,2}_{loc}(\Omega)$, we can differentiate the equation (1.3.8) to obtain

$$-\mathrm{div}\,(\partial_{x_k}\mathcal{A}_\varepsilon(\nabla u_\varepsilon)) = \partial_{x_k} f_\varepsilon \quad \text{in} \quad \mathcal{D}'(\Omega), \qquad k = 1, ..., n, \tag{1.4.3}$$

and so

$$\sum_{j=1}^{n} \int_\Omega \partial_{x_k}\mathcal{A}^j_\varepsilon(\nabla u_\varepsilon)\partial_{x_j}\varphi = -\int_\Omega f_\varepsilon \partial_{x_k}\varphi \qquad k = 1, ..., n, \tag{1.4.4}$$

holds true for any $\varphi \in W^{1,2}_c(\Omega)$, the set of compactly supported members of $W^{1,2}(\Omega)$.

For any $\eta \in C^{0,1}_c(\Omega)$ and any $k = 1, ..., n$ we first choose $\varphi = \eta^2\mathcal{A}^k_\varepsilon(\nabla u_\varepsilon) \in W^{1,2}_c(\Omega)$ as test function in (1.4.4) and then we sum the obtained identities from $k = 1$ to n as to obtain

$$\begin{aligned} 0 &= \int_\Omega \eta^2 \mathrm{tr}\left[(\nabla\mathcal{A}_\varepsilon(\nabla u_\varepsilon))^2\right]\,dx + 2\int_\Omega \eta\langle\nabla\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,\mathcal{A}_\varepsilon(\nabla u_\varepsilon), \nabla\eta\rangle dx \\ &+ \sum_{k=1}^{n}\int_\Omega \eta^2\partial_{x_k}\mathcal{A}^k_\varepsilon(\nabla u_\varepsilon) f_\varepsilon\,dx + 2\int_\Omega \eta\, f_\varepsilon\langle\mathcal{A}_\varepsilon(\nabla u_\varepsilon), \nabla\eta\rangle\,dx = \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{1.4.5}$$

Therefore, from (1.4.1), (1.2.23) and (1.3.5), we infer

$$\begin{aligned} I_1 &= \int_{\Omega\setminus Z_\varepsilon} \eta^2\mathrm{tr}\left[\nabla_\xi\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,D^2u_\varepsilon\,\nabla_\xi\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,D^2u_\varepsilon\right]\,dx \\ &\geq \lambda\,\min\{(p-1),1\}\int_{\Omega\setminus Z_\varepsilon} \eta^2[\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{\frac{p-2}{2}}\,\mathrm{tr}\left(D^2u_\varepsilon\,\nabla_\xi\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,D^2u_\varepsilon\right)\,dx \\ &= \lambda\,\min\{(p-1),1\}\int_{\Omega\setminus Z_\varepsilon} \eta^2[\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{\frac{p-2}{2}}\,\mathrm{tr}\left(\nabla_\xi\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,D^2u_\varepsilon\,D^2u_\varepsilon\right)\,dx \\ &\geq \lambda^2\left(\min\{(p-1),1\}\right)^2\int_{\Omega\setminus Z_\varepsilon} \eta^2[\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{p-2}\mathrm{tr}\left(D^2u_\varepsilon\,D^2u_\varepsilon\right)\,dx \\ &= \lambda^2\,\left(\min\{(p-1),1\}\right)^2\int_\Omega \eta^2[\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{p-2}\left|D^2u_\varepsilon\right|^2 dx, \end{aligned} \tag{1.4.6}$$

where we used the symmetry of D^2u_ε and of $\nabla_\xi\mathcal{A}_\varepsilon(\nabla u_\varepsilon)$.

From (1.3.4), (1.4.1) and (1.3.5), we find that

$$\begin{aligned} |I_2| &= \left|2\int_{\Omega\setminus Z_\varepsilon} \eta\langle\nabla_\xi\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,D^2u_\varepsilon\,\mathcal{A}_\varepsilon(\nabla u_\varepsilon), \nabla\eta\rangle dx\right| \\ &\leq 2\Lambda^{3/2}\,\max\{(p-1),1\}\int_\Omega \eta\,[\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{p-2}H(\nabla u_\varepsilon)|\nabla\eta|\,\left|D^2u_\varepsilon\right|dx \\ &\leq \delta\int_\Omega \eta^2\,[\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{p-2}\left|D^2u_\varepsilon\right|^2 dx \\ &\quad + \frac{\Lambda^3\,(\max\{(p-1),1\})^2}{\delta}\int_\Omega [\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{p-2}H^2(\nabla u_\varepsilon)|\nabla\eta|^2\,dx, \end{aligned} \tag{1.4.7}$$

where in the last inequality we applied weighted Young's inequality with a weight $\delta > 0$ to be chosen later. From (1.4.1), (1.3.5), Hölder and Young inequalities, we obtain

$$
\begin{aligned}
|I_3| &= \Big| \int_{\Omega\setminus Z_\varepsilon} \eta^2 \operatorname{tr}\Big(\nabla_\xi \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\, D^2 u_\varepsilon\Big) f_\varepsilon \, dx \Big| \le \int_{\Omega\setminus Z_\varepsilon} \eta^2 \left|\nabla_\xi \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\right| \left|D^2 u_\varepsilon\right| |f_\varepsilon| \, dx \\
&\le \sqrt{n}\,\Lambda \max\{(p-1),1\} \int_\Omega \eta^2 \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{\frac{p-2}{2}} \left|D^2 u_\varepsilon\right| |f_\varepsilon| \, dx \\
&\le \delta \int_\Omega \eta^2 \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} \left|D^2 u_\varepsilon\right|^2 dx + \frac{n\Lambda^2 \left(\max\{(p-1),1\}\right)^2}{4\delta} \int_\Omega \eta^2 f_\varepsilon^2 dx.
\end{aligned} \tag{1.4.8}
$$

Finally, via Young's inequality,

$$
\begin{aligned}
|I_4| &\le 2\sqrt{\Lambda} \int_\Omega |\eta| \left(\varepsilon^2 + H^2(\nabla u_\varepsilon)\right)^{\frac{p-2}{2}} H(\nabla u_\varepsilon) |f_\varepsilon| \, |\nabla \eta| \, dx \\
&\le \Lambda \int_\Omega \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} H^2(\nabla u_\varepsilon) |\nabla \eta|^2 \, dx + \int_\Omega \eta^2 f_\varepsilon^2 dx.
\end{aligned} \tag{1.4.9}
$$

By combining (1.4.6)-(1.4.9) we get

$$
\begin{aligned}
&\Big(\lambda^2 \big(\min\{(p-1),1\}\big)^2 - 2\delta\Big) \int_\Omega \eta^2 \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} \left|D^2 u_\varepsilon\right|^2 dx \le \\
&\le \Lambda \left(1 + \frac{\Lambda^2 \left(\max\{(p-1),1\}\right)^2}{\delta}\right) \int_\Omega \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} H^2(\nabla u_\varepsilon) |\nabla \eta|^2 \, dx \\
&+ \left(1 + \frac{n\Lambda^2 \left(\max\{(p-1),1\}\right)^2}{4\delta}\right) \int_\Omega \eta^2 f_\varepsilon^2 dx
\end{aligned}
$$

and, by choosing $\delta = \frac{\lambda^2 (\min\{(p-1),1\})^2}{4}$ in the latter, we find

$$
\begin{aligned}
&\int_\Omega \eta^2 \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} \left|D^2 u_\varepsilon\right|^2 dx \le \\
&\le \frac{2\Lambda}{\lambda^2 (\min\{(p-1),1\})^2} \left(1 + \frac{4\Lambda^2 (\max\{(p-1),1\})^2}{\lambda^2 (\min\{(p-1),1\})^2}\right) \int_\Omega \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} H^2(\nabla u_\varepsilon)\, |\nabla \eta|^2 \, dx \\
&+ \frac{2}{\lambda^2 (\min\{(p-1),1\})^2} \left(1 + \frac{n\Lambda^2 (\max\{(p-1),1\})^2}{\lambda^2 (\min\{(p-1),1\})^2}\right) \int_\Omega \eta^2 f_\varepsilon^2 dx
\end{aligned}
$$

which completes the proof. □

The following corollary is a consequence of Proposition 1.4.1. It will be crucial in the proof of Theorem 1.1.1.

Corollary 1.4.2. *Let u_ε be a solution of* (1.3.8). *Then for any function $\eta \in C_c^{0,1}(\Omega)$ and for any $\varepsilon \in (0,1)$, we have*

$$
\int_\Omega \eta^2 \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} \left|D^2 u_\varepsilon\right|^2 dx \le C_2 \int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 |\nabla \eta|^2 dx + C_2 \int_\Omega \eta^2 f_\varepsilon^2 dx \tag{1.4.10}
$$

and

$$
\int_\Omega \eta^2 \left|\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\right|^2 dx \le C_2 \int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 |\nabla \eta|^2 dx + C_2 \int_\Omega \eta^2 f_\varepsilon^2 \, dx\,, \tag{1.4.11}
$$

where C_2 is a constant depending only on n, p, λ and Λ.

Proof. First, we notice that (1.4.10) readily follows from (1.3.4) and (1.4.2).

Also, by (1.4.1) and (1.3.5), we have that

$$\left|\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\right| = \left|\nabla_\xi \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\, D^2 u_\varepsilon\right| \le C(n,p,\lambda,\Lambda)\,[\varepsilon^2 + H^2(\nabla u_\varepsilon)]^{\frac{p-2}{2}} \left|D^2 u_\varepsilon\right| \quad \text{a.e. on } \Omega, \tag{1.4.12}$$

therefore (1.4.11) follows immediately from (1.4.10). □

Now we proceed estimate the term $\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx$, where B_R is any open ball such that $\overline{B_{2R}} \subset \Omega$. More precisely we have the following result.

Proposition 1.4.3. *Let u_ε be a solution of (1.3.8). Then, for any $\varepsilon \in (0,1)$ and for any open ball $B_{2R} \subset\subset \Omega$ we have*

$$\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx \le C_3 \Big[R^{-n} \left(\int_{B_{2R}\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)| dx \right)^2 + R^2 \int_{B_{2R}} f_\varepsilon^2 dx \Big] \tag{1.4.13}$$

$$\int_{B_{\frac{R}{2}}} \|\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\|^2 dx \le C_4 \Big[R^{-n-2} \left(\int_{B_{2R}\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)| dx \right)^2 + \int_{B_{2R}} f_\varepsilon^2 dx \Big] \tag{1.4.14}$$

where C_3, C_4 are constants depending only on n, p, λ and Λ.

Proof. Thanks to Lemma 1.3.2 we have that $\eta \mathcal{A}_\varepsilon^k(\nabla u_\varepsilon) \in W_c^{1,2}(\Omega)$ for any $k = 1, ..., n$ and for $\eta \in C_c^{0,1}(\Omega)$ whose support is contained in $\overline{B_{2R}} \subset \Omega$.
We first consider the case $n \ge 3$.
Case $n \ge 3$. Since

$$\begin{aligned} \int_\Omega |\eta\, \mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx &= \int_\Omega \left(|\eta\, \mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\right)^{\frac{2^*}{2}} dx \\ &= \int_\Omega \left(\sum_{k=1}^n |\eta\, \mathcal{A}_\varepsilon^k(\nabla u_\varepsilon)|^2 \right)^{\frac{2^*}{2}} dx \le C(n) \int_\Omega \sum_{k=1}^n |\eta\, \mathcal{A}_\varepsilon^k(\nabla u_\varepsilon)|^{2^*} dx\,, \end{aligned} \tag{1.4.15}$$

then the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ yields

$$\begin{aligned} \int_\Omega |\eta\, \mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx &\le C'(n) \left[\sum_{k=1}^n \left(\int_\Omega |\nabla(\eta\, \mathcal{A}_\varepsilon^k(\nabla u_\varepsilon))|^2 dx \right)^{\frac{2^*}{2}} \right] \\ &\le C''(n) \sum_{k=1}^n \left[\int_\Omega \left(\eta^2 |\nabla \mathcal{A}_\varepsilon^k(\nabla u_\varepsilon))|^2 + |\mathcal{A}_\varepsilon^k(\nabla u_\varepsilon))|^2 |\nabla \eta|^2 \right) dx \right]^{\frac{2^*}{2}} \\ &\le nC''(n) \left[\int_\Omega \left(\eta^2 \|\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon))\|^2 + |\mathcal{A}_\varepsilon(\nabla u_\varepsilon))|^2 |\nabla \eta|^2 \right) dx \right]^{\frac{2^*}{2}} . \end{aligned} \tag{1.4.16}$$

Now we use (1.4.11) in the latter to infer

$$\int_\Omega |\eta\, \mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx \le nC''(n) \left[(C_2+1)\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 |\nabla \eta|^2 dx + C_2 \int_\Omega \eta^2 f_\varepsilon^2 dx\right]^{\frac{2^*}{2}}$$

(1.4.17)
$$\le C(n,p,\lambda,\Lambda)\left[\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2|\nabla\eta|^2 dx + \int_\Omega \eta^2 f_\varepsilon^2 dx\right]^{\frac{2^*}{2}}$$

$$\le C'(n,p,\lambda,\Lambda)\left[\left(\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2|\nabla\eta|^2\right)^{\frac{2^*}{2}} dx + \left(\int_\Omega \eta^2 f_\varepsilon^2 dx\right)^{\frac{2^*}{2}}\right].$$

Let $R \le t < s \le 2R$ and let $\eta = \eta_{t,s} \in C_c^{0,1}(\Omega)$ be a cut off function with $0 \le \eta \le 1$ and such that

(1.4.18)
$$\eta \equiv 1 \quad \text{on } B_t, \quad \eta = 0 \quad \text{on } \Omega \setminus B_s, \quad |\nabla \eta| \le \frac{1}{s-t} \quad \text{on} \quad \Omega.$$

Then from (1.4.17) we have
(1.4.19)

$$\int_{B_t} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx \le C''(n,p,\lambda,\Lambda)\left[\frac{1}{(s-t)^{2^*}}\left(\int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx\right)^{\frac{2^*}{2}} dx + \left(\int_{B_{2R}} f_\varepsilon^2 dx\right)^{\frac{2^*}{2}}\right].$$

Following [108, Remark 6.12], let $r = 2^*/2 > 1$ and consider $\sigma \in (0,1)$. Let

$$\alpha = \left(\frac{1-\sigma}{r-\sigma}\right) r \in (0,1)$$

so that

$$\frac{r}{\alpha} = \frac{r-\sigma}{1-\sigma} > 1, \quad \left(\frac{r}{\alpha}\right)' = \frac{r-\sigma}{r-1} \quad \text{and} \quad (1-\alpha)\left(\frac{r}{\alpha}\right)' = \sigma.$$

By Holder's inequality we have

$$\int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx = \int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2\alpha}\, |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2(1-\alpha)} dx$$

(1.4.20)
$$\le \left(\int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2r} dx\right)^{\frac{1-\sigma}{r-\sigma}} \left(\int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2\sigma} dx\right)^{\frac{r-1}{r-\sigma}}.$$

Thus, since $-2^* = -2n/(n-2) = n(1-r)$, from (1.4.19) and the latter we obtain

(1.4.21)

$$\int_{B_t} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx$$

$$\le C''(n,p,\lambda,\Lambda)(s-t)^{n(1-r)}\left(\int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2r} dx\right)^{r\left(\frac{1-\sigma}{r-\sigma}\right)} \left(\int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2\sigma} dx\right)^{r\left(\frac{r-1}{r-\sigma}\right)}$$

$$+ C''(n,p,\lambda,\Lambda)\left(\int_{B_{2R}} f_\varepsilon^2 dx\right)^r$$

$$\le \left(\int_{B_s} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2r} dx\right)^{r\left(\frac{1-\sigma}{r-\sigma}\right)} \left[C''(n,p,\lambda,\Lambda)(s-t)^{n(1-r)}\left(\int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2\sigma} dx\right)^{r\left(\frac{r-1}{r-\sigma}\right)}\right]$$

$$+ C''(n,p,\lambda,\Lambda)\left(\int_{B_{2R}} f_\varepsilon^2 dx\right)^r$$

and therefore, via weighted Young's inequality with conjugate exponents $\frac{r-\sigma}{r(1-\sigma)}$ and $\frac{r-\sigma}{\sigma(r-1)}$, we obtain

$$\begin{aligned}\int_{B_t} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx &\le \frac{1}{2}\int_{B_s} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2r} dx + \tilde{C}(s-t)^{-(r-\sigma)\frac{n}{\sigma}} \left(\int_{B_s\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2\sigma} dx\right)^{\frac{r}{\sigma}} \\ &\quad + C''(n,p,\lambda,\Lambda)\left(\int_{B_{2R}} f_\varepsilon^2 dx\right)^r \\ &\le \tilde{C}(s-t)^{-(r-\sigma)\frac{n}{\sigma}} \left(\int_{B_{2R}\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2\sigma} dx\right)^{\frac{r}{\sigma}} + C''(n,p,\lambda,\Lambda)\left(\int_{B_{2R}} f_\varepsilon^2 dx\right)^r \\ &\quad + \frac{1}{2}\int_{B_s} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx\end{aligned}$$

where $\tilde{C}$ is a constant depending only on $n, p, \sigma, \lambda, \Lambda$.

By applying [108, Lemma 6.1] with

$$Z(t) = \int_{B_t} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx,$$

and by choosing $\sigma = \frac{1}{2}$, from the above inequality we obtain

$$\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2^*} dx \le C'''\, R^{-(r-\sigma)\frac{n}{\sigma}} \left(\int_{B_{2R}\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)| dx\right)^{2r} + C''' \left(\int_{B_{2R}} f_\varepsilon^2 dx\right)^r \tag{1.4.22}$$

where C''' is a constant depending only on n, p, λ, Λ.

Then Hölder's inequality and (1.4.22) imply

$$\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx \le C_1'''|B_R|^{2/n}\Big[R^{-(r-\sigma)\frac{n-2}{\sigma}} \left(\int_{B_{2R}\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)| dx\right)^2 + \int_{B_{2R}} f_\varepsilon^2 dx\Big] \tag{1.4.23}$$

where C_1''' is a constant depending only on n, p, λ, Λ.

A short computation yields $(r-\sigma)\frac{n-2}{\sigma} = n+2$, therefore the latter gives

$$\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx \le C_2''' R^2\Big[R^{-(n+2)} \left(\int_{B_{2R}\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)| dx\right)^2 + \int_{B_{2R}} f_\varepsilon^2 dx\Big] \tag{1.4.24}$$

where C_2''' is a constant depending only on n, p, λ, Λ. This proves (1.4.13).

To prove (1.4.14) we make use of (1.4.11) by letting $\eta \in C_c^{0,1}(\Omega)$ be a cut-off function with $0 \le \eta \le 1$ and such that

$$\eta \equiv 1 \quad \text{in } B_{R/2}, \quad \eta = 0 \quad \text{on } \Omega \setminus B_R, \quad |\nabla \eta| \le 2/R \quad \text{on } \Omega,$$

which leads to

$$\int_{B_{\frac{R}{2}}} \left|\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\right|^2 dx \le 4C_2 R^{-2} \int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx + C_2 \int_{B_R} f_\varepsilon^2 dx \tag{1.4.25}$$

Inserting (1.4.24) into the latter yields (1.4.14).

Case $n = 2$. In this case we observe that, for any $\theta > 2$, it holds

(1.4.26)
$$\int_\Omega |\eta \mathcal{A}_\varepsilon^k(\nabla u_\varepsilon)|^\theta dx \le C(\theta) R^2 \left(\int_\Omega \left| \nabla (\eta \mathcal{A}_\varepsilon^k(\nabla u_\varepsilon) \right|^2 dx \right)^{\frac{\theta}{2}} .$$

Here we have used that $\eta \mathcal{A}_\varepsilon^k(\nabla u_\varepsilon) \in W_c^{1,2}(\Omega)$ and its support is contained in $\overline{B_{2R}} \subset \Omega$ (see for instance [132, Theorem 12.33]). Now we repeat the previous computations with any $\theta > 2$ fixed. This leads to (1.4.19) with 2^* replaced by θ and $C''(n,p,\lambda,\Lambda)$ replaced by $C''(n,p,\lambda,\Lambda,\theta)R^2$, i.e.,

(1.4.27)
$$\int_{B_t} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^\theta dx \le C''(n,p,\lambda,\Lambda,\theta) R^2 \left[\frac{1}{(s-t)^\theta} \left(\int_{B_s} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx \right)^{\frac{\theta}{2}} + \left(\int_{B_{2R}} f_\varepsilon^2 dx \right)^{\frac{\theta}{2}} \right].$$

Now we choose $r = \frac{\theta}{2} > 1$ and we repeat the computations after formula (1.4.19). This leads to

(1.4.28)
$$\begin{aligned} &\int_{B_t} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^\theta dx \\ &\le C''(n,p,\lambda,\Lambda,\theta) R^2 (s-t)^{-2r} \left(\int_{B_s \setminus B_R} \left| \mathcal{A}_\varepsilon(\nabla u_\varepsilon) \right|^{2r} dx \right)^{r\left(\frac{1-\sigma}{r-\sigma}\right)} \left(\int_{B_s \setminus B_R} \left| \mathcal{A}_\varepsilon(\nabla u_\varepsilon) \right|^{2\sigma} dx \right)^{r\left(\frac{r-1}{r-\sigma}\right)} \\ &+ C''(n,p,\lambda,\Lambda,\theta) R^2 \left(\int_{B_{2R}} f_\varepsilon^2 dx \right)^r \le \tilde{C} R^{\frac{2(r-\sigma)}{\sigma(r-1)}} (s-t)^{-\frac{2r(r-\sigma)}{\sigma(r-1)}} \left(\int_{B_{2R} \setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2\sigma} dx \right)^{\frac{r}{\sigma}} \\ &+ C''(n,p,\lambda,\Lambda,\theta) R^2 \left(\int_{B_{2R}} f_\varepsilon^2 dx \right)^r + \frac{1}{2} \int_{B_s} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^\theta dx \end{aligned}$$

where $\tilde{C}$ is a constant depending only on $n, p, \sigma, \lambda, \Lambda$ and θ.

By choosing $\sigma = \frac{1}{2}$ and applying [108, Lemma 6.1] we obtain

(1.4.29)
$$\begin{aligned} &\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^\theta dx \le C'''\, R^{\frac{2(r-\sigma)}{\sigma(r-1)}} R^{-\frac{2r(r-\sigma)}{\sigma(r-1)}} \left(\int_{B_{2R} \setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^{2\sigma} dx \right)^{\frac{r}{\sigma}} + C''' R^2 \left(\int_{B_{2R}} f_\varepsilon^2 dx \right)^r \\ &= C'''\, R^{-2(\theta-1)} \left(\int_{B_{2R} \setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)| dx \right)^\theta + C''' R^2 \left(\int_{B_{2R}} f_\varepsilon^2 dx \right)^{\frac{\theta}{2}} \end{aligned}$$

where C''' is a constant depending only on n, p, λ, Λ and θ.

Then Hölder's inequality and (1.4.29) imply

$$\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx \le C_1''' \Big[R^{-2} \left(\int_{B_{2R} \setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)| dx \right)^2 + R^2 \int_{B_{2R}} f_\varepsilon^2 dx \Big]$$

where C_1''' is a constant depending only on n, p, λ, Λ and θ. Then (1.4.13) follows by fixing a value of $\theta > 2$. From the latter it is immediate to infer (1.4.14). □

1.5 Proof of the main results

The following section is devoted to the proof of the main results of this chapter.

Proof of Theorem 1.1.1. It suffices to apply the estimates we have found in the previous sections for the approximating sequence u_ε, and then pass to the limit as $\varepsilon \to 0$.

Let us fix $\Omega' \subset\subset \Omega$ and consider u_ε solutions to (1.3.8). From (1.3.4), (1.3.10) and Hölder's inequality, we have

$$\|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\|_{L^1(\Omega')} \leq C\,,$$

where C does not depend on ε. Then from Proposition (1.4.3) and a standard covering argument we infer that

$$\|\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\|_{W^{1,2}(\Omega')} \leq C, \tag{1.5.1}$$

where C does not depend on ε.

Since those estimates are uniform in ε, we can extract a subsequence, relabelled as u_ε, such that

$$\mathcal{A}_\varepsilon(\nabla u_\varepsilon) \to h \quad \text{weakly in } W^{1,2}_{loc}(\Omega), \quad \text{strongly in } L^2_{loc}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega, \tag{1.5.2}$$

for some $h \in W^{1,2}_{loc}(\Omega;\mathbb{R}^n)$.

From the L^p convergence $\nabla u_\varepsilon \to \nabla u$, we have (up to a subsequence, still denoted by u_ε)

$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in} \quad \Omega.$$

Hence

$$\mathcal{A}_\varepsilon(\nabla u_\varepsilon) \to \mathcal{A}(\nabla u) \quad \text{a.e. in} \quad \Omega,$$

and so $h = \mathcal{A}(\nabla u)$ thanks to (1.5.2).

Estimates (1.1.8) and (1.1.9) then follows by letting $\varepsilon \to 0$ in Proposition 1.4.3. Finally, the estimate (1.1.10) follows immediately from (1.1.3). □

As already observed in Remark 1.1.3 and Remark 1.1.5, Theorem 1.1.2 and Theorem 1.1.4 are special cases of two more general results that we state and prove hereafter. To this end we first introduce the assumptions on the source term f :

$$\begin{cases} \text{if} \quad p > \frac{n}{2} & \exists\, \gamma \in (n-2,n) \qquad\qquad : \quad f \in \mathcal{M}^{2,\gamma}_{loc}(\Omega), \\[2ex] \text{if} \quad p \leq \frac{n}{2} & \exists\, \gamma \in (n-2,n),\ \exists\, s > \frac{n}{p} \quad : \quad f \in L^s_{loc}(\Omega) \cap \mathcal{M}^{2,\gamma}_{loc}(\Omega), \end{cases} \tag{1.5.3}$$

where we have denoted by $\mathcal{M}^{2,\gamma}$ the classical Morrey space. Then we have

Remark 1.5.1.

i) If f satisfies (1.5.3)*, then $f \in L^q_{loc}(\Omega)$ where q fulfills* (1.1.6)*, and therefore Theorem 1.1.1 applies.*

ii) If $f \in L^r_{loc}(\Omega)$, $r > n$, then f satisfies (1.5.3)*. Indeed, by Holder inequality, we have that $f \in \mathcal{M}^{2,n-\frac{2n}{r}}_{loc}(\Omega)$ (and $n - \frac{2n}{r} \in (n-2,n)$, since $r > n$). Moreover, $\|f\|_{\mathcal{M}^{2,n-\frac{2n}{r}}(\Omega')} \leq C(n,r)\|f\|_{L^r(\Omega')}$ for any open subset $\Omega' \subset\subset \Omega$. Therefore, Theorem 1.1.2 and Theorem 1.1.4 are special cases of the two following general results.*

Theorem 1.5.2. *Assume* $1 < p \leq 2$ *and let* $u \in W^{1,p}_{loc}(\Omega)$ *be a local weak solution of* (1.1.1) *where* H *satisfies* (1.2.2) *and* f *satisfies* (1.5.3)*. Then*

$$u \in W^{2,2}_{loc}(\Omega) \cap C^{1,\beta}_{loc}(\Omega)$$

for some $\beta \in (0,1)$ *depending only on* n, p, λ, Λ *and* γ.

Moreover, for any open ball $B_{2R} \subset\subset \Omega$ *we have*

$$\int_{B_{R/2}} \left|D^2 u\right|^2 dx \leq C\Big[R^{-n-2}\|\mathcal{A}(\nabla u)\|^2_{L^1(B_{2R}\setminus B_R)} + \|f\|^2_{L^2(B_{2R})}\Big],$$

where C *is a constant depending on* $p, n, \lambda, \Lambda, \gamma, B_R, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^{\max\{2,s\}}(B_{2R})}$ *and* $\|f\|_{\mathcal{M}^{2,\gamma}(B_{2R})}$.

In particular, when $p = 2$ *we have*

$$\int_{B_{R/2}} \left|D^2 u\right|^2 dx \leq C\Big[R^{-n-2}\|\mathcal{A}(\nabla u)\|^2_{L^1(B_{2R}\setminus B_R)} + \|f\|^2_{L^2(B_{2R})}\Big],$$

where C *is a constant depending only on* n, λ, Λ.

Theorem 1.5.3. *Let* $u \in W^{1,p}_{loc}(\Omega)$ *be a local solution of* (1.1.1)*, where* H *satisfies* (1.2.2) *and* f *satisfies* (1.5.3)*. Then*

$$u \in C^{1,\beta}_{loc}(\Omega)$$

for some $\beta \in (0,1)$ *depending only on* n, p, λ, Λ *and* γ.

Moreover, for any open ball $B_{2R} \subset\subset \Omega$ *we have*

$$\int_{B_{R/2}\setminus Z} \left[H^2(\nabla u)\right]^{p-2} \left|D^2 u\right|^2 dx \leq C, \tag{1.5.4}$$

where Z *denotes the set of critical points of* u *and* C *is a constant depending on* $p, n, \lambda, \Lambda\ \gamma, B_R, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f$ *and* $\|f\|_{\mathcal{M}^{2,\gamma}(B_{2R})}$.

To prove Theorem 1.5.2 and Theorem 1.5.3 we need the following useful auxiliary result (inspired by the reading of Section 5 of [137]).

Lemma 1.5.4. *Assume* $n \geq 2$ *and let* U *be an open bounded set of* $\mathbb{R}^n$ *of class* C^2*. Let* f *be a function belonging to the Morrey space* $\mathcal{M}^{2,\gamma}(U)$ *with* $n-2 < \gamma < n$ *and set* $\alpha = \frac{\gamma-n+2}{2} \in (0,1)$*. Then there exists* $F \in W^{1,2}(U;\mathbb{R}^n) \cap C^{0,\alpha}_{loc}(U;\mathbb{R}^n)$ *such that*

$$-\mathrm{div} F = f \quad in \quad U \tag{1.5.5}$$

and, for any open Lipschitz set $U' \subset\subset U$,

$$\|F\|_{C^{0,\alpha}(U')} \leq C\|f\|_{\mathcal{M}^{2,\gamma}(U)}, \tag{1.5.6}$$

where C *is a constant depending only on* n, γ, U' *and* U.

Proof. The proof relies on some results of Campanato and Morrey. [2] and [104][chapter 5].

[2]Recall that the Morrey space $\mathcal{M}^{2,\gamma}(A)$ is isomorphic (as Banach space) to the Campanato space $\mathcal{L}^{2,\gamma}(A)$ whenever A is an open bounded Lipschitz set of $\mathbb{R}^n$ and $0 \leq \gamma < n$. We shall freely use this result in the course of the proof. More details on this property as well as other useful results used in this paper on Morrey's and Campanato's spaces can be found in [108][Section 2.3]

Let $u \in W_0^{1,2}(U) \cap W^{2,2}(U)$ be the unique weak solution to $-\Delta u = f$ in U and recall that $\|u\|^2_{W^{2,2}(U)} \leq C_1 \|f\|^2_{L^2(U)}$, for some constant C_1 depending only on n and U. Also, by a result of Campanato [43, Teorema 10.I] (see also [104, Chapter 5]) we know that

$$\|\partial^2_{x_j x_k} u\|^2_{\mathcal{M}^{2,\gamma}(U')} \leq C_2 \left[\|u\|^2_{W^{2,2}(U)} + \|f\|^2_{\mathcal{M}^{2,\gamma}(U)} \right] \qquad \forall\, j,k = 1,...,n \tag{1.5.7}$$

where the constant C_2 depends only on γ, n and U'. Hence,

$$\|\nabla \partial_{x_k} u\|^2_{\mathcal{M}^{2,\gamma}(U')} \leq C_3 \|f\|^2_{\mathcal{M}^{2,\gamma}(U)} \qquad \forall\, j,k = 1,...,n \tag{1.5.8}$$

where C_3 is a constant that depends only on γ, n, U' and U. Set $w = \partial_{x_k} u$, then Poincaré inequality and (1.5.8) imply that, for any $x_0 \in U'$ and any $0 < \rho < \frac{dist(U', \partial U)}{2}$,

$$\int_{B_\rho(x_0)} |w - w_{B_\rho(x_0)}|^2 dx \leq c\rho^2 \int_{B_\rho(x_0)} |\nabla \partial_{x_k} u|^2 \leq c\rho^2 C_3 \|f\|^2_{\mathcal{M}^{2,\gamma}(U)} \rho^\gamma = cC_3 \|f\|^2_{\mathcal{M}^{2,\gamma}(U)} \rho^{\gamma+2} \tag{1.5.9}$$

where $w_\omega := \frac{1}{|\omega|} \int_\omega w\, dx$ and $c = c(n)$.

Moreover, when $\rho \geq \frac{dist(U', \partial U)}{2}$, we have

$$\begin{aligned} &\int_{U \cap B_\rho(x_0)} |w - w_{U \cap B_\rho(x_0)}|^2 dx \leq 2\|w\|^2_{L^2(U)} \leq 2\|w\|^2_{L^2(U)} \left[\frac{2\rho}{dist(U', \partial U)} \right]^{\gamma+2} \\ &= 2 \left[\frac{2}{dist(U', \partial U)} \right]^{\gamma+2} \|\partial_{x_k} u\|^2_{L^2(U)} \rho^{\gamma+2} \leq 2 \left[\frac{2}{dist(U', \partial U)} \right]^{\gamma+2} C_1^2 \|f\|^2_{L^2(U)} \rho^{\gamma+2} \end{aligned} \tag{1.5.10}$$

Combining (1.5.9) and (1.5.10) we immediately get that $\partial_{x_k} u$ belongs to the Campanato space $\mathcal{L}^{2,\gamma+2}(U')$ and

$$\|\partial_{x_k} u\|^2_{\mathcal{L}^{2,\gamma+2}(U')} \leq C_4 \|f\|^2_{\mathcal{M}^{2,\gamma}(U)} \quad \text{for all } k = 1, \ldots, n, \tag{1.5.11}$$

where C_4 is a constant depending only on n, γ, U' and U.

Now, since $n < \gamma + 2 < n + 2$, the well-known integral characterisation of Holder spaces by Campanato [41, 42, 108] tell us that

$$\partial_{x_k} u \in C^{0, \frac{\gamma - n + 2}{2}}(\overline{U'}), \quad \|\partial_{x_k} u\|_{C^{0, \frac{\gamma - n + 2}{2}}(\overline{U'})} \leq C_5 \|f\|_{\mathcal{M}^{2,\gamma}(U)} \qquad \forall\, k = 1,...,n \tag{1.5.12}$$

where C_5 is a constant depending only on n, γ, U' and U.

The desired conclusion then follows by taking $F = \nabla u$. □

We are now ready to prove Theorem 1.5.2.

Proof of Theorem 1.5.2. Set

$$\tilde{s} := \begin{cases} 2 & \text{if } \ p > \frac{n}{2}, \\ s & \text{if } \ p \leq \frac{n}{2}. \end{cases} \tag{1.5.13}$$

Let us consider an open ball $B_{2R} \subset\subset \Omega$ and let f_ε and u_ε be as in Section 1.3. Recall that, in the course of the proof of Theorem 1.1.1, we proved that

$$\|u_\varepsilon\|_{W^{1,p}(B_{2R})} \leq C_1' := C_1'(p, n, \lambda, \Lambda, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^{\tilde{s}}(B_{2R})}) \tag{1.5.14}$$

$$\|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\|_{L^1(B_{2R})} \le C_1' \tag{1.5.15}$$

and that, up to a subsequence,

$$\nabla u_\varepsilon \to \nabla u \quad \text{strongly in } W^{1,p}_{loc}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega, \tag{1.5.16}$$

$$\mathcal{A}_\varepsilon(\nabla u_\varepsilon) \to \mathcal{A}(\nabla u) \quad \text{weakly in } W^{1,2}_{loc}(\Omega), \quad \text{strongly in } L^2_{loc}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega. \tag{1.5.17}$$

By making use of (1.5.14) we have $u_\varepsilon \in C^0(\Omega)$ and the following bound

$$\|u_\varepsilon\|_{L^\infty(B_R)} \le C_2' := C_2'(p, n, \lambda, \Lambda, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^{\bar{s}}(B_{2R})}). \tag{1.5.18}$$

Indeed, if $p > n$ we have $\|u_\varepsilon\|_{L^\infty(B_R)} \le C(B_R, p)\|u_\varepsilon\|_{W^{1,p}(B_R)}$ by Sobolev embedding, and so (1.5.18) follows from (1.5.14). When $p \le n$ we have $\|u_\varepsilon\|_{L^\infty(B_R)} \le C'(p, n, \lambda, \Lambda, B_{2R}, \|u_\varepsilon\|_{L^p(B_{2R})}, \|f\|_{L^{\bar{s}}(B_{2R})})$, by the celebrated results in [179], and once again (1.5.18) follows from (1.5.14).

Now we observe that also $f_\varepsilon \in \mathcal{M}^{2,\gamma}(B_{2R})$, and $\|f_\varepsilon\|_{\mathcal{M}^{2,\gamma}(B_{2R})} \le \|f\|_{\mathcal{M}^{2,\gamma}(B_{2R})}$. We can therefore use Lemma 1.5.4 to obtain vector fields $F_\varepsilon \in C^{0,\alpha}(\overline{B_R})$ such that

$$\|F_\varepsilon\|_{C^{0,\alpha}(B_R)} \le C\|f_\varepsilon\|_{\mathcal{M}^{2,\gamma}(B_{2R})} \le C\|f\|_{\mathcal{M}^{2,\gamma}(B_{2R})} \tag{1.5.19}$$

where $\alpha = \frac{\gamma-n+2}{2} \in (0,1)$ and C is a constant depending only on n, γ, B_R and B_{2R}.

Now we set $A_\varepsilon(x,\xi) := \mathcal{A}_\varepsilon(\xi) - F_\varepsilon(x)$, $(x,\xi) \in B_R \times (\mathbb{R}^n \setminus \{0\})$ and observe that

$$-\operatorname{div}(A_\varepsilon(x, \nabla u_\varepsilon)) = 0 \quad \text{in } B_R. \tag{1.5.20}$$

We can therefore apply [136, Theorem 1.7] to obtain $\beta = \beta(n, p, \lambda, \Lambda, \gamma) \in (0,1)$ such that

$$\|u_\varepsilon\|_{C^{1,\beta}(B_{\frac{R}{2}})} \le C_3' = C_3'(p, n, \lambda, \Lambda, \gamma, B_R, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^{\bar{s}}(B_{2R})}, \|f\|_{\mathcal{M}^{2,\gamma}(B_{2R})}). \tag{1.5.21}$$

Hence, up to a subsequence, $u_\varepsilon \to u$ in $C^1_{loc}(\Omega)$, $u \in C^{1,\beta}_{loc}(\Omega)$.

By (1.4.10) and $p \le 2$ we get

$$\begin{aligned}\int_{B_{\frac{R}{2}}} \left|D^2 u_\varepsilon\right|^2 dx &\le C_4' \int_{B_{\frac{R}{2}}} \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} \left|D^2 u_\varepsilon\right|^2 dx \le \\ C_4' C_2 &\left[\frac{4}{R^2}\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx + \int_{B_R} f_\varepsilon^2 dx\right]\end{aligned} \tag{1.5.22}$$

where C_2 is a constant depending only on n, p, λ, Λ and C_4' is a positive constant depending only on C_3' (note that one can take $C_4' = 1$ when $p = 2$). Then, inserting (1.4.13) into the latter yields

$$\begin{aligned}\int_{B_{\frac{R}{2}}} \left|D^2 u_\varepsilon\right|^2 dx &\le C_4' C_2 \left[\frac{4}{R^2}\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx + \int_{B_R} f_\varepsilon^2 dx\right] \\ &\le C_4' C(n,p,\lambda,\Lambda)\Big[R^{-n-2}\left(\int_{B_{2R}\setminus B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)| dx\right)^2 + \int_{B_{2R}} f^2 dx\Big] \\ &\le C_5' = C_5'(p, n, \lambda, \Lambda, \gamma, B_R, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^{\bar{s}}(B_{2R})}, \|f\|_{\mathcal{M}^{2,\gamma}(B_{2R})})\end{aligned} \tag{1.5.23}$$

where in the last inequality we have used (1.5.15). Therefore, up to a subsequence, $u_\varepsilon \to u$ weakly in $W^{2,2}_{loc}(\Omega)$ and the thesis follows by letting $\varepsilon \to 0$ in (1.5.22) and then recalling (1.1.9). □

Proof of Theorem 1.5.3. We repeat the proof of Theorem 1.1.2 until the estimate (1.5.21). Hence, up to a subsequence,

$$u_\varepsilon \to u \quad \text{in} \quad C^1_{loc}(\Omega), \qquad u \in C^{1,\beta}_{loc}(\Omega). \tag{1.5.24}$$

By (1.4.10), (1.4.13) and (1.5.15) we have that

$$\begin{aligned}\int_{B_{\frac{R}{2}}} \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} \left|D^2 u_\varepsilon\right|^2 dx \le C_2 \left[\frac{4}{R^2}\int_{B_R} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx + \int_{B_R} f_\varepsilon^2 dx\right] \\ \le C_5' = C_5'(p, n, \lambda, \Lambda, \gamma, B_R, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^{\tilde{s}}(B_{2R})}, \|f\|_{\mathcal{M}^{2,\gamma}(B_{2R})});\end{aligned} \tag{1.5.25}$$

therefore, for every $i, j \in \{1, \dots, n\}$,

$$\phi_\varepsilon^{i,j} := \left(\varepsilon^2 + |\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}} \partial^2_{x_i x_j} u_\varepsilon \tag{1.5.26}$$

is uniformly bounded in $L^2_{loc}(\Omega)$ w.r.t. $\varepsilon > 0$. Hence, up to a subsequence,

$$\phi_\varepsilon^{i,j} \to \phi^{i,j} \quad \text{weakly in } L^2_{loc}(\Omega) \quad \text{as } \varepsilon \to 0. \tag{1.5.27}$$

In view of (1.5.25), (1.5.27) and the weak lower semicontinuity of the L^2 norm, to get our thesis it is enough to prove that

$$\phi^{i,j} = |\nabla u|^{p-2} \partial^2_{x_i x_j} u \quad \text{a.e. in } \Omega \setminus Z. \tag{1.5.28}$$

To this end, we fix an arbitrary open ball $\mathcal{B}_{2R} \subset\subset \Omega \setminus Z$, then $|\nabla u| \ge 2c > 0$ in $\mathcal{B}_{2R}$ by definition of Z. Hence, by (1.5.24), we have

$$|\nabla u_\varepsilon| \ge c \quad \text{in } \mathcal{B}_{2R}, \quad \text{for all small enough } \varepsilon. \tag{1.5.29}$$

By using (1.5.25), (1.5.29) and (1.5.21) we find

$$\begin{aligned}\int_{\mathcal{B}_{\frac{R}{2}}} \left|D^2 u_\varepsilon\right|^2 dx \le C(c, p, \lambda, \Lambda, C_3') \int_{\mathcal{B}_{\frac{R}{2}}} \left[\varepsilon^2 + H^2(\nabla u_\varepsilon)\right]^{p-2} \left|D^2 u_\varepsilon\right|^2 dx \\ \le C_6' = C_6'(c, p, n, \lambda, \Lambda, \gamma, B_R, B_{2R}, \|u\|_{W^{1,p}(B_{2R})}, \|f\|_{L^{\tilde{s}}(B_{2R})}, \|f\|_{\mathcal{M}^{2,\gamma}(B_{2R})}),\end{aligned}$$

which implies that u_ε is uniformly bounded in $W^{2,2}_{loc}(\Omega \setminus Z)$ and then, up to a subsequence, $u_\varepsilon \to u$ weakly in $W^{2,2}_{loc}(\Omega \setminus Z)$. The latter and (1.5.24) yield

$$\phi_\varepsilon^{i,j} = \left(\varepsilon^2 + |\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}} \partial^2_{x_i x_j} u_\varepsilon \to |\nabla u|^{p-2} \partial^2_{x_i x_j} u,$$

weakly in $L^2_{loc}(\Omega \setminus Z)$, which proves (1.5.28) and concludes the proof. □

Proof of Proposition 1.1.6. From Theorem 1.1.1 we know that

$$|\mathcal{A}(\nabla u)| \in W^{1,2}_{loc}(\Omega).$$

Thanks to a well-known result due to Stampacchia [181] we infer that

$$\frac{|\mathcal{A}(\nabla u)|}{\varepsilon + |\mathcal{A}(\nabla u)|} \in W^{1,2}_{loc}(\Omega)$$

for any $\varepsilon > 0$. Therefore, for any $\varphi \in C_c^\infty(\Omega)$, we can use

$$\frac{|\mathcal{A}(\nabla u)|}{\varepsilon + |\mathcal{A}(\nabla u)|}\varphi$$

as a test function in (1.1.2) and we have

$$\int_\Omega \frac{|\mathcal{A}(\nabla u)|}{\varepsilon + |\mathcal{A}(\nabla u)|}\varphi\, f\, dx = \int_\Omega \frac{|\mathcal{A}(\nabla u)|}{\varepsilon + |\mathcal{A}(\nabla u)|}\mathcal{A}(\nabla u)\cdot\nabla\varphi\, dx + \varepsilon\int_\Omega \frac{\mathcal{A}(\nabla u)\cdot\nabla(|\mathcal{A}(\nabla u)|)}{(\varepsilon + |\mathcal{A}(\nabla u)|)^2}\varphi\, dx. \tag{1.5.30}$$

We first notice that

$$\int_\Omega \frac{|\mathcal{A}(\nabla u)|}{\varepsilon + |\mathcal{A}(\nabla u)|}\varphi\, f\, dx = \int_{\Omega\setminus\{\nabla u = 0\}} \frac{|\mathcal{A}(\nabla u)|}{\varepsilon + |\mathcal{A}(\nabla u)|}\varphi\, f\, dx. \tag{1.5.31}$$

Moreover we have

$$\left|\varepsilon\frac{\mathcal{A}(\nabla u)\cdot\nabla(|\mathcal{A}(\nabla u)|)}{(\varepsilon + |\mathcal{A}(\nabla u)|)^2}\varphi\right| \le \nabla(|\mathcal{A}(\nabla u)|)|\varphi|$$

where the latter function belongs to $L^1(\Omega)$, independently on ε. This implies that we can use the dominated convergence theorem in (1.5.30) as $\varepsilon \to 0^+$ and, from (1.5.31), we obtain

$$\int_{\Omega\setminus\{\nabla u = 0\}} \varphi\, f\, dx = \int_\Omega \mathcal{A}(\nabla u)\cdot\nabla\varphi\, dx = \int_\Omega \varphi\, f\, dx\,,$$

where in the last equality we used again the equation of u. Since φ is any function in $C_c^\infty(\Omega)$, we get the desired conclusion. □

Proof of Corollary 1.1.7. This corollary is a straightforward consequence of Proposition 1.1.6. Indeed, the singular set $Z = \{\nabla u = 0\}$ is contained into the set $\{f = 0\}$ up to a set of measure zero. Since $|\{f = 0\}| = 0$ then $|\{\nabla u = 0\}|$. □

Chapter 2

Global Regularity for anisotropic elliptic problems

2.1 Main results

As mentioned in the introduction, in the following chapter we will be studying solutions to boundary value problems for equation

$$-\operatorname{div}\big(\mathcal{A}(\nabla u)\big) = f \quad \text{in } \Omega\,, \tag{2.1.1}$$

under homogeneous Dirichlet conditions

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(\nabla u)\big) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.1.2}$$

or Neumann condition

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(\nabla u)\big) = f & \text{in } \Omega \\ \mathcal{A}(\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.1.3}$$

the latter with compatibility condition $\int_\Omega f\,dx = 0$. The vector field $\mathcal{A}:\mathbb{R}^n\to\mathbb{R}^n$ is now given by

$$\mathcal{A}(\xi) = \nabla_\xi B\big(H(\xi)\big) = \begin{cases} b\big(H(\xi)\big)\nabla_\xi H(\xi) & \text{if } \xi\neq 0 \\ 0 & \text{if } \xi = 0, \end{cases} \tag{2.1.4}$$

where H is a uniformly convex norm as described in Section 1.2, and

$$B(t) = \int_0^t b(s)\,ds$$

is a C^2-convex function satisfying certain nonlinear growth assumptions. A particular case is the polynomial growth studied in the previous chapter where $B(t) = \frac{t^p}{p}$, though here we allow for nonlinearities of non-polynomial type as well. Further details and properties of B are given in Section 2.2.

The function B is typically called Young function and, as we will show in Section 2.2, equation (2.1.1) belongs to the class of quasilinear equation of *Orlicz*-growth type, i.e., equations of the form

$$-\operatorname{div}\big(A(x,u,\nabla u)\big) = F(x,u,\nabla u)\,, \tag{2.1.5}$$

where the vector field $A = A(x, z, \xi)$ satisfies

$$|A(x,z,\xi)| \lesssim B'\big(|\xi|\big) \quad \text{and} \quad A(x,z,\xi)\cdot\xi \gtrsim B\big(|\xi|\big)\,. \tag{2.1.6}$$

The function f in equations (2.1.1)-(2.1.3) is supposed to belong to $L^2(\Omega)$. As already observed in the previous chapter, weak solutions to problems (2.1.2) and (2.1.3) are not well defined and need not exist under this assumption on f, since the space $L^2(\Omega)$ is not included in the dual of the natural Orlicz-Sobolev space associated with these problems unless B grows fast enough near infinity. For instance, as mentioned in Chapter 1, if $B(t)$ behaves like t^p near infinity, then p has to exceed $\frac{2n}{n+2}$. Thus, an even weaker notion of generalized solution has to be employed. Various definitions of solutions to nonlinear elliptic equations in divergence form, with a right-hand side affected by a low integrability degree, have been introduced in the literature, see, e.g., [31, 19, 70, 138].

A posteriori, they turn out to be equivalent. Precise formulations of the definitions adopted here are given at the beginning of Sections 2.5, 2.6 and 2.7, that deal with local solutions, solutions to Dirichlet problems and solutions to Neumann problems respectively. Let us just disclose here that the solutions in question are not weakly differentiable in general. Hence, the expression ∇u appearing in our statements is an abuse of notation for a surrogate gradient which has to be properly interpreted. In this connection, we point out one trait of our results, which, in particular, reveals the regularizing effect of the nonlinear function $\mathcal{A}$, that turns $\mathcal{A}(\nabla u)$ into a true Sobolev map.

The first result we provide is a local estimate which complements Theorem 1.1.1 in the Orlicz setting. Being a local result, no additional assumption on Ω is required.

Theorem 2.1.1 (Local estimate)**.** *Assume that $B \in C^2(0,\infty)$ is a Young function fulfilling conditions* (2.2.5) *and* (2.2.6)*, and that $H \in C^2(\mathbb{R}^n \setminus \{0\})$ is a norm satisfying property* (1.2.2)*. Let Ω be any open set in $\mathbb{R}^n$. Assume that $f \in L^2_{loc}(\Omega)$, and let u be a generalized local solution to equation* (2.1.1)*. Then,*

$$\mathcal{A}(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\Omega), \tag{2.1.7}$$

and there exists a constant $c = c(n, i_b, s_b, \lambda, \Lambda)$ such that

$$\begin{aligned} &\|\mathcal{A}(\nabla u)\|_{L^2(B_R)} \le c\,R\|f\|_{L^2(B_{2R})} + c\,R^{-\frac{n}{2}}\|\mathcal{A}(\nabla u)\|_{L^1(B_{2R})} \\ &\|\nabla(\mathcal{A}(\nabla u))\|_{L^2(B_R)} \le c\,\|f\|_{L^2(B_{2R})} + c\,R^{-\frac{n}{2}-1}\|\mathcal{A}(\nabla u)\|_{L^1(B_{2R}\setminus B_R)} \end{aligned} \tag{2.1.8}$$

for every ball $B_{2R} \subset\subset \Omega$.

When it comes to global estimates – the core of the investigations of this chapter – the geometry of Ω plays a crucial role. The first result with this regard deals with plainly bounded convex domains Ω. In this case, no additional regularity of Ω is needed. As will be clear from the proof, this is possible thanks to the fact that a priori bounds for $\|\mathcal{A}(\nabla u)\|_{W^{1,2}(\Omega)}$ involve certain integrals over $\partial\Omega$, which depend on its curvatures. If Ω is convex, these integrals have a definite sign, which makes the integrals in question negligible in the relevant bounds. Importantly, the constants in these bounds depend on the convex domain Ω only through its diameter d_Ω.

Theorem 2.1.2 (Convex domains)**.** *Let B and H be as in Theorem 2.1.1. Let Ω be a bounded convex set in $\mathbb{R}^n$. Assume that $f \in L^2(\Omega)$ and let u be a generalized solution to either the Dirichlet problem* (2.1.2) *or the Neumann problem* (2.1.3)*. Then,*

$$\mathcal{A}(\nabla u) \in W^{1,2}(\Omega).$$

Moreover,

$$\|\mathcal{A}(\nabla u)\|_{L^2(\Omega)} \le c_1 \|f\|_{L^2(\Omega)} \quad \textit{and} \quad \|\nabla(\mathcal{A}(\nabla u))\|_{L^2(\Omega)} \le c_2 \|f\|_{L^2(\Omega)}, \tag{2.1.9}$$

where

$$c_1 = c(n, i_b, s_b, \lambda, \Lambda)\, d_\Omega \quad \textit{and} \quad c_2 = \frac{\Lambda \max\{1, s_b\}}{\lambda \min\{1, i_b\}}.$$

In particular, the constant c_2 is independent of Ω.

As soon as the realm of convex domains is abandoned, the conclusions of Theorem (2.1.2) can fail, in the absence of additional assumptions on the curvatures of $\partial\Omega$. Indeed, counterexamples in this connection can be exhibited, even for the plain Laplace operator, for slight perturbations of convex domains. Consider, for instance, a bounded open set Ω whose boundary is smooth outside a small portion, where it agrees with the graph of a function Θ of the variables $(x_1, \dots, x_n)$, given by

$$\Theta(x_1, \dots, x_n) = \frac{c|x_1|}{\log|x_1|} \tag{2.1.10}$$

for some constant c and for small x_1. As shown in [148, 149], if the constant c is not small enough, then one can exhibit Dirichlet problems for the Laplacian, with smooth right-hand sides, whose solutions do not belong to $W^{2,2}(\Omega)$.

A suitable assumption on Ω that restores the result involves integrability properties of the weak curvatures of $\partial\Omega$. One can request that Ω is a bounded Lipschitz domain such that the functions of $(n-1)$ variables, that locally describe Ω around boundary points, are endowed with second-order weak derivatives which belong to a specific Marcinkiewicz space depending on the dimension n. Also, the norm of the curvatures in this space, evaluated on balls centered on $\partial\Omega$, has to be sufficiently small for small radii of the balls. Specifically, denote by $\mathcal{B}$ the weak second fundamental form on $\partial\Omega$, and define the function $\Psi_\Omega : (0,\infty) \to [0,\infty]$ as

$$\Psi_\Omega(r) = \begin{cases} \sup\limits_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{n-1,\infty}(\partial\Omega \cap B_r(x))} & \text{if } n \ge 3, \\[2ex] \sup\limits_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{1,\infty}\log L(\partial\Omega \cap B_r(x))} & \text{if } n = 2 \end{cases} \tag{2.1.11}$$

for $r > 0$. Here, $L^{n-1,\infty}$ and $L^{1,\infty}\log L$ denote Marcinkiewicz type spaces, with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on $\partial\Omega$. Recall that

$$\|g\|_{L^{n-1,\infty}(\partial\Omega \cap B_r(x))} = \sup_{s \in (0, \mathcal{H}^{n-1}(\partial\Omega \cap B_r(x)))} s^{\frac{1}{n-1}} g^{**}(s)$$

for $n \ge 3$, and

$$\|g\|_{L^{1,\infty}\log L(\partial\Omega \cap B_r(x))} = \sup_{s \in (0, \mathcal{H}^{n-1}(\partial\Omega \cap B_r(x)))} s \log\left(1 + \tfrac{1}{s}\right) g^{**}(s),$$

for a measurable function $g : \partial\Omega \to \mathbb{R}$. Here g^* denotes the decreasing rearrangement of g with respect to the measure $\mathcal{H}^{n-1}$, and $g^{**}(s) = s^{-1}\int_0^s g^*(r)\,dr$ for $s > 0$. In what follows, the notation $\partial\Omega \in W^2L^{n-1,\infty}$ or $\partial\Omega \in W^2L^{1,\infty}\log L$ means that the weak curvatures of $\partial\Omega$ belong to the respective Marcinkiewicz spaces. An analogous notation will be adopted to denote that the weak curvatures in question belong to some other space.

Furthermore, by $\mathfrak{L}_\Omega$ we indicate a Lipschitz characteristic of Ω, which is constituted by the Lipschitz constant L_Ω of the functions which locally describe $\partial\Omega$, and by the radius R_Ω of their ball domains. We refer to Chapter 3 below for the precise definition of $\mathcal{L}_\Omega$-Lipschitz domains, of domains with boundary $\partial\Omega \in W^2\mathcal{M}$, for any given Marcinkiewicz space $\mathcal{M}$, and the definition of weak curvature $\mathcal{B}$.

Theorem 2.1.3 (Domains with minimally integrable curvatures)**.** *Let B and H be as in Theorem 2.1.1. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz characteristic $\mathfrak{L}_\Omega$, such that $\partial\Omega \in W^2L^{n-1,\infty}$ if $n \geq 3$ or $\partial\Omega \in W^2L^{1,\infty}\log L$ if $n = 2$. Assume that $f \in L^2(\Omega)$ and let u be a generalized solution to either the Dirichlet problem* (2.1.2) *or the Neumann problem* (2.1.3)*. There exists a constant $\kappa_0 = \kappa_0(n, i_b, s_b, \lambda, \Lambda, \mathfrak{L}_\Omega, d_\Omega)$ such that, if*

$$\lim_{r\to 0^+} \Psi_\Omega(r) < \kappa_0, \tag{2.1.12}$$

then,

$$\mathcal{A}(\nabla u) \in W^{1,2}(\Omega).$$

Moreover,

$$\|\mathcal{A}(\nabla u)\|_{W^{1,2}(\Omega)} \leq c\,\|f\|_{L^2(\Omega)} \tag{2.1.13}$$

for a suitable constant $c = c(i_b, s_b, \lambda, \Lambda, \Omega)$.

We stress that the use of Marcinkiewicz norms and, in particular, the smallness condition (2.1.12) are not just due to technical reasons. They are in fact minimal assumptions in terms of integrability properties of the curvatures of $\partial\Omega$, for $\mathcal{A}(\nabla u)$ to belong to $W^{1,2}(\Omega)$. This can be shown, for instance, via an example from [123], for $n = 3$ and $p \in (\frac{2}{3}, 2]$, in the standard isotropic case. In that paper, open sets $\Omega \subset \mathbb{R}^3$ are displayed such that $\partial\Omega \in W^2L^{2,\infty}$, for which the limit in (2.1.12) is yet too large, and the solution u to the Dirichlet problem for the p-Laplace equation, with a smooth right-hand side, is such that $|\nabla u|^{p-2}\nabla u \notin W^{1,2}(\Omega)$. In [148] two-dimensional Dirichlet problems for the Laplace operators are considered. In particular, open sets Ω with $\partial\Omega \in W^2L^{1,\infty}\log L$ are exhibited where the solution to the Poisson equation with a smooth right-hand side does not belong in $W^{2,2}(\Omega)$. This is again due to a large value of the limit in (2.1.12). Related Neumann problems are considered in [151, Section 14.6.1].

The result of Theorem 2.1.3 can still be sharpened, if assumptions of a somewhat different nature are allowed. They entail the use of a weighted isocapacitary function for subsets of $\partial\Omega$, the weight being the norm of the second fundamental form on $\partial\Omega$. This function is denoted by $\mathcal{K}_\Omega : (0,\infty) \to [0,\infty)$ and defined by

$$\mathcal{K}_\Omega(r) = \sup_{\substack{E \subset B_r(x)\\ x \in \partial\Omega}} \frac{\int_{\partial\Omega\cap E} |\mathcal{B}|\, d\mathcal{H}^{n-1}}{\operatorname{cap}(E, B_r(x))} \qquad \text{for } r > 0\,. \tag{2.1.14}$$

Here, $B_r(x)$ stands for the ball centered at x, with radius r, and $\operatorname{cap}(E, B_r(x))$ for the classical capacity of a compact set E relative to $B_r(x)$, i.e.,

$$\operatorname{cap}\big(E, B_r(x)\big) = \inf\left\{ \int_{B_r(x)} |\nabla v|^2\, dy \,:\, v \in C^{0,1}_c\big(B_r(x)\big)\,,\, v \geq 1 \text{ on } E\right\}.$$

This is the content of the next result.

Theorem 2.1.4 (Domains satisfying a boundary isocapacitary inequality)**.** *Let B and H be as in Theorem 2.1.1. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz characteristic $\mathfrak{L}_\Omega$, such that $\partial\Omega \in W^{2,1}$. Assume that $f \in L^2(\Omega)$ and let u be a generalized solution to either the Dirichlet problem* (2.1.2) *or the Neumann problem* (2.1.3)*. There exists a constant $\kappa_1 = \kappa_1(n, i_b, s_b, \lambda, \Lambda, \mathfrak{L}_\Omega, d_\Omega)$ such that, if*

$$\lim_{r\to 0^+} \mathcal{K}_\Omega(r) < \kappa_1, \tag{2.1.15}$$

then,

$$\mathcal{A}(\nabla u) \in W^{1,2}(\Omega).$$

Moreover,

$$\|\mathcal{A}(\nabla u)\|_{W^{1,2}(\Omega)} \le c\,\|f\|_{L^2(\Omega)} \tag{2.1.16}$$

for a suitable constant $c = c(i_b, s_b, \lambda, \Lambda, \Omega)$.

Theorem 2.1.4 is stronger than 2.1.3. Indeed, the former not only implies the latter, but also applies to less regular domains. This is the case, for instance, of the sets described above, whose boundary locally agrees with the graph of the function Θ given by (2.1.10). Actually, condition (2.1.15) is fulfilled by these domains, provided that the constant c appearing in (2.1.10) is small enough, whereas $\partial\Omega \notin W^2L^{n-1,\infty}$. The same domains also demonstrate the necessity of the smallness condition (2.1.15), since, as mentioned above, the conclusions of Theorem 2.1.4 fail if the constant c, and hence the limit in (2.1.15), exceeds some threshold.

Let us quickly point out that Theorem 2.1.3 has also been established by Miao, Fa Peng & Zhou [153] for non-autonomous Hilbert norms, i.e., norms of the form $H(x,\xi) = \sqrt{A(x)\xi \cdot \xi}$.

We conclude this section with a statement concerning sufficiently regular domains – specifically, domains Ω such that $\partial\Omega \in C^{2,\alpha}$. Under this assumption, the constants appearing in the $W^{1,2}(\Omega)$ estimate of the stress field $\mathcal{A}(\nabla u)$ admit bounds with an explicit dependence on d_Ω, L_Ω, R_Ω, and $\|\mathcal{B}\|_{L^\infty(\partial\Omega)}$. Thanks to the monotonicity of this dependence, the bounds in question are uniform in classes of domains Ω where d_Ω, L_Ω and $\|\mathcal{B}\|_{L^\infty(\partial\Omega)}$ are uniformly bounded from above, and R_Ω from below.

Theorem 2.1.5 (Domains with bounded curvatures)**.** *Let B and H be as in Theorem 2.1.1. Let Ω be a bounded open set in $\mathbb{R}^n$ such that $\partial\Omega \in C^{2,\alpha}$ for some $\alpha \in (0,1)$, and let $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$ be a Lipschitz characteristic of Ω. Assume that $f \in L^2(\Omega)$ and let u be a generalized solution to either the Dirichlet problem* (2.1.2) *or the Neumann problem* (2.1.3)*. Then,*

$$\mathcal{A}(\nabla u) \in W^{1,2}(\Omega)\,.$$

Moreover,

$$\|\mathcal{A}(\nabla u)\|_{L^2(\Omega)} \le c_1\,\|f\|_{L^2(\Omega)} \quad \text{and} \quad \|\nabla\big(\mathcal{A}(\nabla u)\big)\|_{L^2(\Omega)} \le c_2\,\|f\|_{L^2(\Omega)}\,, \tag{2.1.17}$$

where:
if $n \ge 3$, then

$$\begin{cases} c_1 = c\, d_\Omega^{p(n)}\,(1+L_\Omega)^{n+2}\,\max\Big\{(1+L_\Omega)^{t(n)}\,\|\mathcal{B}\|_{L^\infty(\partial\Omega)}^{(2n+2)(n+2)}, R_\Omega^{-(2n+2)(n+2)}\Big\} \\ \\ c_2 = c\, d_\Omega^{p(n)+n}\,(1+L_\Omega)^{n+2}\,\max\Big\{(1+L_\Omega)^{t(n)+9(n+2)}\,\|\mathcal{B}\|_{L^\infty(\partial\Omega)}^{(2n+3)(n+2)}, R_\Omega^{-(2n+3)(n+2)}\Big\} \end{cases}$$

and $c = c(n, \lambda, \Lambda, i_b, s_b)$, $p(n) = (2n+1)(n+2)+n$, and $t(n) = 9(n+2)(2n+2)$.
If $n = 2$, then

$$\begin{cases} c_1 \;\; = c'\, d_\Omega^{22}\,(1+L_\Omega)^4\,\max\left\{\dfrac{(1+L_\Omega)^{288}\,\big(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)}\big)^{24}}{\log^{24}\big(1+c(1+L_\Omega)\big(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)}\big)\big)}, R_\Omega^{-24}\right\} \\ \\ c_2 \;\; = c'\, d_\Omega^{24}\,(1+L_\Omega)^4\,\max\left\{\dfrac{(1+L_\Omega)^{336}\,\big(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)}\big)^{28}}{\log^{28}\big(1+c(1+L_\Omega)\big(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)}\big)\big)}, R_\Omega^{-28}\right\}, \end{cases}$$

where $c = c(\lambda, \Lambda, i_b, s_b)$ and $c' = c'(\lambda, \Lambda, i_b, s_b)$.

Let us point out that, in the case of solutions to Neumann problems, our proof also applies as soon as that $\partial\Omega \in C^2$. In fact, a refinement of the arguments in the proof of Theorem 2.1.5 would lead to the same conclusions, for both Dirichlet and Neumann problems, under the weaker assumption that $\partial\Omega \in C^{1,1}$. This generalization is skipped in order to avoid additional technicalities.

Outline of the proofs. Here we sketch the main ideas of the proofs, which were already hinted in the Introduction. Concerning Dirichlet problems (2.1.2), we first establish pointwise and integral identities for smooth vector fields– see Lemmas 2.3.1-2.3.2. Then, by exploiting the homogeneity properties of H, these identities will allow us to obtain the anisotropic Reilly's identity (2.3.33). Since this formula is valid for sufficiently smooth source terms f, smooth domains Ω and regular stress fields, we need to resort to a cascade of smoothing procedures on these objects.

Specifically, we will consider u_ε solutions of

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \Omega \end{cases} \tag{2.1.18}$$

for smooth f and Ω, where $\mathcal{A}_\varepsilon$ is a proper approximate stress field– see Section 2.4. Owing to the degeneracy of H^2 at the origin, we apply an additional approximation procedure on the stress field $\mathcal{A}_\varepsilon$, as to obtain smooth functions $u_{\varepsilon,m}$ solving a similar equation to (2.1.18). By taking the square of such equation, applying Reilly's formula (2.3.33) and letting $m \to \infty$, we manage to show that Reilly's identity (2.3.33) holds true for u_ε as well– see formula (2.6.40).

For convex domains Ω, the boundary term in this formula has a definite sign, hence via estimates on the norm H and elementary algebraic we easily get $W^{1,2}$-global estimates on $\mathcal{A}_\varepsilon(\nabla u_\varepsilon)$ independent on ε, and the theorem will follow by letting $\varepsilon \to 0$.

For nonconvex domains, either the isocapacitary assumption (2.1.15) or hypothesis (2.1.12) will help us control the boundary integral in Reilly's formula, so that we can once again obtain uniform $W^{1,2}$-global estimates on the stress field.

Once this is proven, we remove the smoothness assumption on Ω by considering a suitable approximation of this set which will help us keep track of the quantitative constants in the regularity estimates. This will be the main content of Chapter 3 for nonconvex domains. Finally, we get rid of the regularity assumption on f by taking a smooth sequence f_k such that $f_k \to f$ in $L^2(\Omega)$, thus completing the proof for Dirichlet problems (2.1.2).

The proof for Neumann problems (2.1.3) is pretty much identical, save that we make use of identity (2.7.14) in place of (2.6.40).

2.2 The Young function B and the stress field $\mathcal{A}$

In this section we recollect some properties of Young functions B and the associated stress field $\mathcal{A}(\nabla u)$ defined in (2.1.4). We also recall the definition of Orlicz-Sobolev spaces and we specify the required assumptions on B which will ensure the validity of the desired regularity results. We refer to [1, Chapter 8], [49], [115], [171] and [169, Chapter 4] for comprehensive treatments of Young functions and Orlicz-Sobolev spaces.

Let $B : [0,\infty) \to [0,\infty)$ be a convex function such that $B(0) = 0$, and $B(t) > 0$ for $t > 0$. Any such function B is called *Young function* and takes the form

$$B(t) = \int_0^t b(s)\,ds \qquad \text{for } t \geq 0, \tag{2.2.1}$$

for some non-decreasing function $b : [0,\infty) \to [0,\infty)$.

We define $\widetilde{B}$ the conjugate function to B

$$\widetilde{B}(t) = \sup\{s\,t - B(s) \,:\, s \geq 0\}\,. \tag{2.2.2}$$

This is also a Young function, which can be written as

$$\widetilde{B}(t) = \int_0^t b^{-1}(s)\,ds\,, \tag{2.2.3}$$

where $b^{-1}(s)$ is the generalized inverse of b.

The function B is supposed to be twice continuously differentiable $B \in C^2(0,\infty)$ and to have a nonlinear growth. Precisely, if $b(t) = B'(t)$ is the function appearing in equation (2.2.1), on setting

$$i_b = \inf_{t>0} \frac{t\,b'(t)}{b(t)} \quad \text{and} \quad s_b = \sup_{t>0} \frac{t\,b'(t)}{b(t)}, \tag{2.2.4}$$

the nonlinear growth condition on the function B is imposed by requiring that

$$i_b > 0. \tag{2.2.5}$$

Property (2.2.5) is equivalent to the so-called ∇_2-condition in the theory of Young functions. A doubling condition, known as Δ_2-condition in this theory, is also demanded on B. The latter is equivalent to

$$s_b < \infty. \tag{2.2.6}$$

The standard choice

$$B(t) = \tfrac{1}{p} t^p, \tag{2.2.7}$$

corresponds to operators with plain p-growth, with $i_b = s_b = p > 1$. Multiplying the function in (2.2.7) by powers of logarithms results in functions, that are still admissible, or the form:

$$B(t) = t^p \log^q(c+t),$$

where $p > 1$, $q \in \mathbb{R}$, and c is a positive, sufficiently large constant for B to be convex.
More elaborated instances, borrowed from [185], are:

$$B(t) = t^3(1+(\ln t)^2)^{-\frac{1}{2}} \exp(\ln t \arctan(\ln t));$$
$$B(t) = t^{4+\sin\sqrt{1+(\ln t)^2}}.$$

Now, owing to assumption (2.2.5), we have $b'(t) > 0$ for $t > 0$, and

$$\lim_{t\to 0^+} b(t) = 0, \tag{2.2.8}$$

so that $b \in C^1([0,\infty))$, thus $B \in C^1([0,\infty)) \cap C^2(0,\infty)$.

The monotonicity of the function b ensures that

$$\tfrac{t}{2} b(\tfrac{t}{2}) \leq B(t) \leq b(t)t \quad \text{for } t \geq 0. \tag{2.2.9}$$

Also, as a consequence of assumption (2.2.4), the functions $\frac{b(t)}{t^{i_b}}$ and $\frac{b(t)}{t^{s_b}}$ are non-decreasing and non-increasing, respectively. Hence,

$$b(1)\,\min\{t^{i_b}, t^{s_b}\} \leq b(t) \leq b(1)\,\max\{t^{i_b}, t^{s_b}\} \quad \text{for } t \geq 0, \tag{2.2.10}$$

and there exist positive constants c and C, depending only on i_b and s_b, such that

$$c\, b(s) \leq b(t) \leq C\, b(s) \quad \text{if } 0 \leq s \leq t \leq 2s. \tag{2.2.11}$$

Now, define the function $a : (0, \infty) \to [0, \infty)$ as

$$a(t) = \frac{b(t)}{t} \qquad \text{for } t > 0. \tag{2.2.12}$$

Thereby, $a \in C^1(0, \infty)$, and on setting

$$i_a = \inf_{t>0} \frac{t\, a'(t)}{a(t)} \quad \text{and} \quad s_a = \sup_{t>0} \frac{t\, a'(t)}{a(t)}, \tag{2.2.13}$$

one has that

$$i_b = i_a + 1 \quad \text{and} \quad s_b = s_a + 1. \tag{2.2.14}$$

Hence, assumptions (2.2.5) and (2.2.6) are equivalent to

$$-1 < i_a \leq s_a < \infty\,, \tag{2.2.15}$$

and a counterpart of (2.2.11) holds; namely

$$c\, a(s) \leq a(t) \leq C\, a(s) \quad \text{if } 0 < s \leq t \leq 2s. \tag{2.2.16}$$

Also

$$i_B \geq i_b + 1 \quad \text{and} \quad s_B \leq s_b + 1. \tag{2.2.17}$$

Thus, assumptions (2.2.5), (2.2.6) imply that $i_B \geq 1$ and $s_B < \infty$ as well.
Assumption (2.2.6) also ensures that for every $k > 0$ there exists a constant c, depending only on k and s_b, such that

$$b(kt) \leq cb(t) \qquad \text{for } t \geq 0. \tag{2.2.18}$$

Similarly, the fact that $s_B < \infty$ ensures that for every $k > 0$ there exists a constant c, depending only on k and s_B, such that

$$B(kt) \leq cB(t) \qquad \text{for } t \geq 0. \tag{2.2.19}$$

Moreover, since $i_B > 1$, a parallel property holds for the Young conjugate $\widetilde{B}$. Namely, for every $k > 0$ there exists a constant c, depending only on k and i_B, such that

$$\widetilde{B}(kt) \leq c\widetilde{B}(t) \qquad \text{for } t \geq 0. \tag{2.2.20}$$

The property $s_B < \infty$ also implies that there exists a constant c such that

$$\widetilde{B}(b(t)) \leq cB(t) \qquad \text{for } t \geq 0. \tag{2.2.21}$$

We close this section recalling the definitions of Orlicz spaces.
The Orlicz-Lebesgue space $L^B(\Omega)$ is defined as the set of measureable functions u on Ω whose Luxemburg norm

$$\|u\|_{L^B(\Omega)} = \inf\left\{\lambda > 0\ :\ \int_\Omega B\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}$$

is finite. The Orlicz-Sobolev spaces $W^{1,B}(\Omega)$ is the set consisting of functions $u \in L^B(\Omega)$ whose distributional gradient $\nabla u \in L^B(\Omega)$.

For instance, the standard choice $B(t) = t^p$ with $p > 1$ yields $L^B(\Omega) = L^p(\Omega)$ and $W^{1,B}(\Omega) = W^{1,p}(\Omega)$.

The stress field $\mathcal{A}$

Given a uniformly elliptic norm H, and a Young function B as above, we now want to obtain some information concerning the stress field

$$\mathcal{A}(\xi) = \nabla_\xi B\big(H(\xi)\big) = \begin{cases} b\big(H(\xi)\big)\,\nabla_\xi H(\xi) & \xi \neq 0 \\ 0 & \xi = 0 \end{cases}.$$

First observe that, thanks to (2.2.19), inequalities (1.2.9) imply that

$$c\,B(|\xi|) \leq B(H(\xi)) \leq C\,B(|\xi|) \quad \text{for } \xi \in \mathbb{R}^n, \tag{2.2.22}$$

for suitable constants $c = c(s_B, \lambda, \Lambda)$ and $C = C(s_B, \lambda, \Lambda)$, and

$$cb(|\xi|) \leq b(H(\xi)) \leq Cb(|\xi|) \quad \text{for } \xi \in \mathbb{R}^n, \tag{2.2.23}$$

for suitable constants $c = c(s_b, \lambda, \Lambda)$ and $C = C(s_b, \lambda, \Lambda)$. Moreover, inequality (1.2.13) yields

$$\sqrt{\lambda}\,b(H(\xi)) \leq |\mathcal{A}(\xi)| \leq \sqrt{\Lambda}\,b(H(\xi)) \quad \text{for } \xi \in \mathbb{R}^n. \tag{2.2.24}$$

Owing to (2.2.12), the function $\mathcal{A}$ admits the alternate expression

$$\mathcal{A}(\xi) = a\big(H(\xi)\big)\tfrac{1}{2}\nabla_\xi H^2(\xi) \quad \text{for } \xi \neq 0. \tag{2.2.25}$$

Hence, via equation (1.2.8) and the homogeneity of H^2, we deduce that

$$\mathcal{A}(\xi)\cdot\xi = a(H(\xi))H^2(\xi) = b(H(\xi))H(\xi) \quad \text{for } \xi \in \mathbb{R}^n.$$

Coupling the latter equation with inequality (2.2.9) yields

$$\mathcal{A}(\xi)\cdot\xi \geq B\big(H(\xi)\big) \quad \text{for } \xi \in \mathbb{R}^n. \tag{2.2.26}$$

Observe that inequalities (2.2.24) and (2.2.26) tell us that the vector field $\mathcal{A}(\xi)$ and its associated quasilinear equation (2.1.1) satisfy the Orlicz-type growth condition (2.1.6). When $B(t) = \frac{1}{p}t^p$, this corresponds to the p-growth hypothesis (0.0.7).

Next, we state and prove a lemma which provides us with some additional properties of the function $\mathcal{A}$, and can be seen as an extension of [68, Theorem 1.5] to the Orlicz setting.

Furthermore, the following lemma, and in particular inequalities (2.2.28) below show that the stress field $\mathcal{A}(\xi)$ is differentiable, and its gradient satisfies natural growth condition (2.1.6).

In the statement, $\lambda_{\min}(\xi)$ and $\lambda_{\max}(\xi)$ denote the smallest and largest eigenvalue, respectively, of the symmetric matrix $\nabla_\xi\mathcal{A}(\xi)$ given by

$$\nabla_\xi\mathcal{A}(\xi) = \left(\frac{\partial\mathcal{A}^i(\xi)}{\partial\xi_j}\right)_{i,j=1,\dots,n} \quad \text{for } \xi \neq 0.$$

Lemma 2.2.1. *We have that $\mathcal{A}(\xi)$ is coercive, i.e.,*

$$\big(\mathcal{A}(\xi) - \mathcal{A}(\eta)\big)\cdot(\xi - \eta) > 0 \quad \textit{for } \xi, \eta \in \mathbb{R}^n, \textit{ with } \xi \neq \eta. \tag{2.2.27}$$

Moreover,

$$\lambda\,\min\{1, i_b\}\,a\big(H(\xi)\big)\,|\eta|^2 \leq \nabla_\xi\mathcal{A}(\xi)\,\eta\cdot\eta \leq \Lambda\,\max\{1, s_b\}\,a\big(H(\xi)\big)\,|\eta|^2, \tag{2.2.28}$$

for $\xi \neq 0$ and $\eta \in \mathbb{R}^n$. In particular

$$\frac{\lambda_{\max}(\xi)}{\lambda_{\min}(\xi)} \leq \frac{\Lambda\,\max\{1, s_b\}}{\lambda\,\min\{1, i_b\}} \quad \textit{for } \xi \neq 0. \tag{2.2.29}$$

Proof. Let $\xi \neq 0$ and $i, j \in \{1, \dots, n\}$. Computations show that

$$\frac{\partial \mathcal{A}^i(\xi)}{\partial \xi_j} = a\big(H(\xi)\big)\Big\{\Big[1 + \frac{H(\xi)\, a'\big(H(\xi)\big)}{a\big(H(\xi)\big)}\Big]\, \partial_{\xi_i} H(\xi)\, \partial_{\xi_j} H(\xi) + H(\xi)\, \partial_{\xi_i \xi_j} H(\xi)\Big\}. \tag{2.2.30}$$

From (1.2.2), (1.2.18) and (2.2.15) we deduce that

$$\begin{aligned}\sum_{i,j=1}^{n} \frac{\partial \mathcal{A}^i(\xi)}{\partial \xi_j}\, \eta_i \eta_j &\geq a\big(H(\xi)\big) \min\{1, 1 + i_a\}\Big\{\partial_{\xi_i} H(\xi)\, \partial_{\xi_j} H(\xi)\, \eta_i\, \eta_j + H(\xi)\, \partial_{\xi_i \xi_j} H(\xi)\, \eta_i\, \eta_j\Big\} \\ &= a\big(H(\xi)\big) \min\{1, 1 + i_a\}\, \tfrac{1}{2} \nabla^2_\xi H^2(\xi)\, \eta \cdot \eta \geq \lambda \min\{1, 1 + i_a\}\, a\big(H(\xi)\big)\, |\eta|^2 ,\end{aligned} \tag{2.2.31}$$

for $\eta \in \mathbb{R}^n$ and $\xi \neq 0$. Hence, the first inequality in (2.2.28) follows, thanks to equation (2.2.14).
By (2.2.14), the second inequality in (2.2.28) can be deduced from equations (1.2.18), (2.2.30) and (2.2.15), which imply that

$$\sum_{i,j=1}^{n} \frac{\partial \mathcal{A}^i(\xi)}{\partial \xi_j} \eta_i\, \eta_j \leq a\big(H(\xi)\big)\, \max\{1, 1 + s_a\}\, \tfrac{1}{2} \nabla^2_\xi H^2(\xi)\, \eta \cdot \eta \qquad \text{for } \eta \in \mathbb{R}^n.$$

Equation (2.2.8) ensures that the function $\mathcal{A}$ is continuous also at 0. Therefore,

$$\begin{aligned}\big(\mathcal{A}(\xi) - \mathcal{A}(\eta)\big) \cdot (\xi - \eta) &= \int_0^1 \frac{d}{dt} \mathcal{A}\big(t\, \xi + (1-t)\eta\big) \cdot (\xi - \eta)\, dt = \\ &= \int_0^1 \frac{\partial \mathcal{A}^i}{\partial \xi_j}\big(t\, \xi + (1-t)\eta\big)\, (\xi - \eta)_i\, (\xi - \eta)_j\, dt \quad \text{for } \xi, \eta \in \mathbb{R}^n.\end{aligned}$$

Hence, by inequality (2.2.31),

$$\big(\mathcal{A}(\xi) - \mathcal{A}(\eta)\big) \cdot (\xi - \eta) \geq \lambda \min\{1, 1 + i_a\}\Big(\int_0^1 a\big(H(t\, \xi + (1-t)\eta)\big)\, dt\Big)\, |\xi - \eta|^2 > 0$$

if $\xi \neq \eta$. This establishes inequality (2.2.27).
Finally, (2.2.29) is a straightforward consequence of (2.2.28). □

2.3 Fundamental lemmas for vector fields

Several pointwise identities and inequalities involving functions and vector fields are offered in this section. They are critical in the proofs of our regularity estimates.

We begin with an identity for vector fields $V : \Omega \to \mathbb{R}^n$. If $V = (V^1, \dots, V^n)$, then we set

$$\nabla V = (\partial_j V^i)_{ij}.$$

Hence, $\nabla V\, V$ is the vector whose i-th component agrees with $V^j \partial_j V^i$. Here, and in what follows, we adopt the convention about summation over repeated indices.

Lemma 2.3.1. *Assume that $V : \Omega \to \mathbb{R}^n$ and $V \in C^2(\Omega)$. Then*

$$\big(\operatorname{div} V\big)^2 = \operatorname{tr}\big((\nabla V)^2\big) + \operatorname{div}\Big(V \operatorname{div} V - \nabla V\, V\Big)\,. \tag{2.3.1}$$

Proof. Schwarz's theorem on second mixed derivatives and an exchange of the indices i and j ensure that

$$(\partial_j\partial_i V^i)\, V^j = (\partial_j\partial_i V^j)\, V^i. \tag{2.3.2}$$

Notice that

$$\operatorname{div}(\nabla V\, V) = \partial_j\Big(V^i\,\partial_i V^j\Big) = \partial_j V^i\,\partial_i V^j + V^i\,\partial_j\partial_i V^j = \operatorname{tr}((\nabla V)^2) + V^i\,\partial_j\partial_i V^j\,, \tag{2.3.3}$$

and

$$\operatorname{div}\Big(V\operatorname{div}V\Big) = \big(\operatorname{div}V\big)^2 + V\cdot\nabla(\operatorname{div}V) = \big(\operatorname{div}V\big)^2 + V^i\,\partial_i\partial_j V^j. \tag{2.3.4}$$

Subtracting equations (2.3.4) and (2.3.3) and the use (2.3.2) yield

$$\operatorname{div}\Big(V\operatorname{div}V - \nabla V\, V\Big) = \big(\operatorname{div}V\big)^2 - \operatorname{tr}\big((\nabla V)^2\big),$$

namely equation (2.3.1). □

Let Ω be an open set in $\mathbb{R}^n$ such that $\partial\Omega \in C^1$. We denote by $\nu = \nu(x)$ the outward unit normal to Ω at a point $x \in \partial\Omega$. Given a vector field $V : \partial\Omega \to \mathbb{R}^n$, its tangential component V_T is defined by

$$V_T = V - (V\cdot\nu)\,\nu.$$

The notations ∇_T and div_T are adopted for the tangential gradient and divergence operators on $\partial\Omega$. Therefore, if $u \in C^1(\overline{\Omega})$, then

$$\nabla_T u = \nabla u - (\partial_\nu u)\,\nu \quad \text{on } \partial\Omega,$$

where $\partial_\nu u$ is the normal derivative of u. Moreover, if $V \in C^1(\overline{\Omega})$, then

$$\operatorname{div}_T V = \operatorname{div}V - (\partial_\nu V\cdot\nu) \quad \text{on } \partial\Omega.$$

As a consequence, we obtain the following lemma.

Lemma 2.3.2. *Let Ω be a bounded open set in $\mathbb{R}^n$ such that $\partial\Omega \in C^1$. Assume that $V : \mathbb{R}^n \to \mathbb{R}^n$, $V \in C^{0,1}(\mathbb{R}^n)$, and there exists a closed set Z such that*

$$V(x) = 0 \quad \text{if } x \in Z, \tag{2.3.5}$$

and

$$V \in C^1(\mathbb{R}^n \setminus Z). \tag{2.3.6}$$

Then,

$$\int_\Omega \big(\operatorname{div}V\big)^2\,\phi\,dx = \int_\Omega \operatorname{tr}\big((\nabla V)^2\big)\phi\,dx + \int_{\partial\Omega}\Big((\operatorname{div}_T V)\,V\cdot\nu - \nabla_T V\,V_T\cdot\nu\Big)\,\phi\,d\mathcal{H}^{n-1} - \int_\Omega\Big((\operatorname{div}V)\,V\cdot\nabla\phi - \nabla V\,V\cdot\nabla\phi\Big)dx. \tag{2.3.7}$$

for every $\phi \in C^\infty(\mathbb{R}^n)$. In particular

$$\int_\Omega \big(\operatorname{div}V\big)^2\,dx = \int_\Omega \operatorname{tr}\big((\nabla V)^2\big)dx + \int_{\partial\Omega}\Big((\operatorname{div}_T V)\,V\cdot\nu - \nabla_T V\,V_T\cdot\nu\Big)d\mathcal{H}^{n-1}. \tag{2.3.8}$$

In equations (2.3.7) and (2.3.8), the functions $(\operatorname{div}_T V)\,V$ and $\nabla_T V\,V_T$ are defined as 0 in the set Z.

Proof. By multiplying the vector field V by a smooth compactly supported function, whose support contains $\overline{\Omega}$, we may assume, without loss of generality, that V is compactly supported in $\mathbb{R}^n$. Since $V \in \mathrm{Lip}(\mathbb{R}^n)$ and assumption (2.3.6) is in force, the vector field V can be approximated, via standard convolutions, by a sequence $\{V_k\}$ of smooth, compactly supported functions in $\mathbb{R}^n$, such that

$$V_k(x) \to V(x) \quad \text{for every } x \in \mathbb{R}^n, \tag{2.3.9}$$

$$\nabla V_k(x) \to \nabla V(x) \quad \text{for every } x \in \mathbb{R}^n \setminus Z, \tag{2.3.10}$$

and

$$|V_k(x)| \le c, \quad |\nabla V_k(x)| \le c \quad \text{for every } x \in \mathbb{R}^n, \tag{2.3.11}$$

for some constant c. Assumption (2.3.5) and the second inequality in (2.3.11) also imply that

$$\mathrm{div}_T V_k(x)\, V_k(x) \to 0 \quad \text{and} \quad \nabla_T V_k(x)\, (V_k)_T(x) \to 0 \quad \text{for every } x \in \partial\Omega \cap Z. \tag{2.3.12}$$

Fix $k \in \mathbb{N}$. Given $\phi \in C^\infty(\mathbb{R}^n)$, by (2.3.1) and the divergence theorem we have that

$$\begin{aligned}\int_\Omega \big(\mathrm{div}\, V_k\big)^2 \phi\, dx = \int_\Omega \mathrm{tr}\big((\nabla V_k)^2\big)\phi\, dx + \int_{\partial\Omega} \Big((\mathrm{div} V_k)\, V_k - \nabla V_k\, V_k\Big) \cdot \nu\, \phi\, d\mathcal{H}^{n-1} \\ - \int_\Omega \Big((\mathrm{div}\, V_k)\, V_k \cdot \nabla\phi - \nabla V_k\, V_k \cdot \nabla\phi\Big)\, dx.\end{aligned} \tag{2.3.13}$$

Subtracting the equations

$$(\mathrm{div}\, V_k)\, V_k \cdot \nu = (\mathrm{div}_T\, V_k)\, V_k \cdot \nu + (\partial_\nu V_k \cdot \nu)\, V_k \cdot \nu\,,$$

and

$$\nabla V_k\, V_k \cdot \nu = \nabla_T V_k\, (V_k)_T \cdot \nu + (V_k \cdot \nu)\, \partial_\nu V_k \cdot \nu\,,$$

results in

$$(\mathrm{div}\, V_k)\, V_k \cdot \nu - \nabla V_k\, V_k \cdot \nu = (\mathrm{div}_T V_k)\, V_k \cdot \nu - \nabla_T V_k\, (V_k)_T \cdot \nu \quad \text{on } \partial\Omega\,. \tag{2.3.14}$$

From equations (2.3.13) and (2.3.14) one deduces that

$$\begin{aligned}\int_\Omega \big(\mathrm{div}\, V_k\big)^2 \phi\, dx = \int_\Omega \mathrm{tr}\big((\nabla V_k)^2\big)\phi\, dx + \int_{\partial\Omega} \Big((\mathrm{div}_T V_k)\, V_k \cdot \nu - \nabla_T V_k\, (V_k)_T \cdot \nu\Big)\, \phi\, d\mathcal{H}^{n-1} \\ - \int_\Omega \Big((\mathrm{div}\, V_k)\, V_k \cdot \nabla\phi - \nabla V_k\, V_k \cdot \nabla\phi\Big)\, dx.\end{aligned} \tag{2.3.15}$$

Owing to properties (2.3.9)–(2.3.12), passing to the limit as $k \to \infty$ in the latter equation yields (2.3.7), via the dominated convergence theorem. □

Our next task is a proof of a generalization to the anisotropic setting of the classical Reilly's identity [172]. It involves the notion of anisotropic second fundamental form of the boundary of a set Ω, and its anisotropic mean curvature – see, e.g., [64, 65, 66, 189, 190].

Recall that the shape operator (also called Weingarten operator) on $\partial\Omega$ agrees with $\nabla_T \nu$. Since $(\nabla_T \nu) : \nu^\perp \to \nu^\perp$, owing to (1.2.16) one also has that

$$\nabla_\xi^2 H(\nu)\, (\nabla_T \nu) : \nu^\perp \to \nu^\perp. \tag{2.3.16}$$

The *anisotropic second fundamental form* $\mathcal{B}^H$ of $\partial\Omega$ is defined by

$$\nabla^2_\xi H(\nu)\,(\nabla_T \nu)\,\eta\cdot\zeta \qquad \text{for } \eta\,,\zeta \in \nu^\perp.$$

Namely,

$$\mathcal{B}^H = \nabla_T\big(\nabla_\xi H(\nu)\big) = \nabla^2_\xi H(\nu)\,\nabla_T \nu. \tag{2.3.17}$$

Furthermore, the *anisotropic mean curvature* is given by

$$\operatorname{tr}\mathcal{B}^H = \operatorname{div}_T\big(\nabla_\xi H(\nu)\big)\,. \tag{2.3.18}$$

Clearly, when H is the Euclidean norm, $\nabla^2_\xi H(\nu) = \mathrm{Id}_{\nu^\perp}$, and hence $\mathcal{B}^H = \mathcal{B}$, the standard second fundamental form on $\partial\Omega$.
The functions $\Psi^H_\Omega : (0,\infty) \to [0,\infty)$ and $\mathcal{K}^H_\Omega : (0,\infty)\to[0,\infty)$ are defined as in (2.1.11) and (2.1.14), with $\mathcal{B}$ replaced by $\mathcal{B}^H$. Namely,

$$\Psi^H_\Omega(r) = \begin{cases} \sup\limits_{x\in\partial\Omega} ||\mathcal{B}^H||_{L^{n-1,\infty}(\partial\Omega\cap B_r(x))} & \text{if } n\geq 3,\\[2ex] \sup\limits_{x\in\partial\Omega} ||\mathcal{B}^H||_{L^{1,\infty}\log L(\partial\Omega\cap B_r(x))} & \text{if } n = 2\end{cases} \tag{2.3.19}$$

for $r>0$, and

$$\mathcal{K}^H_\Omega(r) = \sup_{\substack{E\,\subset\, B_r(x)\\ x\,\in\,\partial\Omega}} \frac{\int_{\partial\Omega\cap E}|\mathcal{B}^H|d\mathcal{H}^{n-1}}{\operatorname{cap}(E, B_r(x))} \qquad \text{for } r>0\,. \tag{2.3.20}$$

As a consequence of equation (2.3.17), Lemma 1.2.2, and that the curvature $\mathcal{B} = \nabla_T\nu$, there exist positive constants $c = c(n,\lambda,\Lambda)$ and $C = C(n,\lambda,\Lambda)$ such that

$$c\,|\mathcal{B}| \leq |\mathcal{B}^H| \leq C\,|\mathcal{B}| \qquad \text{on } \partial\Omega. \tag{2.3.21}$$

Hence,

$$c\,\Psi_\Omega(r) \leq \Psi^H_\Omega(r) \leq C\,\Psi_\Omega(r) \quad \text{for } r>0, \tag{2.3.22}$$

and

$$c\,\mathcal{K}_\Omega(r) \leq \mathcal{K}^H_\Omega(r) \leq C\,\mathcal{K}_\Omega(r) \quad \text{for } r>0. \tag{2.3.23}$$

Also, equation (2.3.17) and the first inequality in (1.2.23) imply that

$$\text{if} \quad \mathcal{B}\geq 0\ [>0], \quad \text{then} \quad \operatorname{tr}(\mathcal{B}^H)\geq 0\ [>0]. \tag{2.3.24}$$

Lemma 2.3.3. *Let Ω be a bounded open set in $\mathbb{R}^n$ such that $\partial\Omega\in C^2$. Assume that $h\in C^1(\overline{\Omega})$, $v\in C^2(\overline{\Omega})$, and $v=0$ on $\partial\Omega$. Then,*

$$\begin{aligned}\operatorname{div}_T\Big(h\,\tfrac12\nabla_\xi H^2(\nabla v)\Big)\,h\,\tfrac12\nabla_\xi H^2(\nabla v)\cdot\nu - h\,\nabla_T\Big[h\,\tfrac12\nabla_\xi H^2(\nabla v)\Big]\,\Big[\tfrac12\nabla_\xi H^2(\nabla v)\Big]_T\cdot\nu\\ = h^2\,H(\nu)\,H^2(\nabla v)\operatorname{tr}\mathcal{B}^H\end{aligned} \tag{2.3.25}$$

on $\partial\Omega\cap\{\nabla v\neq 0\}$.

Proof. Throughout this proof, all formulas are understood to hold, without further mentioning, in the set $\partial\Omega \cap \{\nabla v \neq 0\}$. Computations show that

$$\begin{aligned}(2.3.26)\qquad &\mathrm{div}_T\Big(h\,\tfrac12\nabla_\xi H^2(\nabla v)\Big)\, h\,\tfrac12\nabla_\xi H^2(\nabla v)\cdot\nu - h\,\nabla_T\Big[h\,\tfrac12\nabla_\xi H^2(\nabla v)\Big]\,\Big[\tfrac12\nabla_\xi H^2(\nabla v)\Big]_T\cdot\nu\\ &= h^2\Big\{\mathrm{div}_T\Big(\tfrac12\nabla_\xi H^2(\nabla v)\Big)\,\tfrac12\nabla_\xi H^2(\nabla v)\cdot\nu - \nabla_T\big[\tfrac12\nabla_\xi H^2(\nabla v)\big]\,\big[\tfrac12\nabla_\xi H^2(\nabla v)\big]_T\cdot\nu\Big\}.\end{aligned}$$

Hence, equation (2.3.25) will follow if we show that

$$\begin{aligned}&\mathrm{div}_T\Big(\tfrac12\nabla_\xi H^2(\nabla v))\Big)\,\tfrac12\nabla_\xi H^2(\nabla v)\cdot\nu - \nabla_T\big[\tfrac12\nabla_\xi H^2(\nabla v)\big]\,\big[\tfrac12\nabla_\xi H^2(\nabla v)\big]_T\cdot\nu\\ &\qquad = H^2(\nabla v)\,H(\nu)\,\mathrm{tr}\,\mathcal{B}^H\,.\end{aligned}$$

Inasmuch as v vanishes on $\partial\Omega$, one has that

$$(2.3.27)\qquad \nabla v = (\partial_\nu v)\,\nu \quad \text{on } \partial\Omega,$$

and, by the homogeneity of H^2,

$$(2.3.28)\qquad \tfrac12\nabla_\xi H^2(\nabla v)\cdot(\partial_\nu v)\nu = H^2(\nabla v)\,.$$

Since $\nabla_\xi H$ is homogeneous of degree zero, equation (2.3.27) ensures that

$$\begin{aligned}(2.3.29)\qquad \mathrm{div}_T\Big(\tfrac12\nabla_\xi H^2(\nabla v)\Big) &= \mathrm{div}_T\Big(H(\nabla v)\,\nabla_\xi H(\nabla v)\Big)\\ &= H(\nabla v)\,\mathrm{sign}(\partial_\nu v)\,\mathrm{div}_T\big(\nabla_\xi H(\nu)\big) + \nabla_T(H(\nabla v))\cdot\nabla_\xi H(\nabla v)\\ &= H(\nabla v)\,\mathrm{sign}(\partial_\nu v)\,\mathrm{tr}\mathcal{B}^H + \nabla_T(H(\nabla v))\cdot\big[\nabla_\xi H(\nabla v)\big]_T.\end{aligned}$$

Notice that in the second equality we have made use of the fact that sign $\partial_\nu v$ is constant in the sets $\{\partial_\nu v > 0\}$ and $\{\partial_\nu v < 0\}$, which are open on $\partial\Omega$ in the topology induced by $\mathbb{R}^n$. Combining equations (2.3.28) and (2.3.29) tells us that

$$\begin{aligned}(2.3.30)\qquad &\mathrm{div}_T\Big(\tfrac12\nabla_\xi H^2(\nabla v))\Big)\,\tfrac12\nabla_\xi H^2(\nabla v)\cdot\nu\\ &= \frac{H(\nabla v)}{|\partial_\nu v|}\,H^2(\nabla v)\,\mathrm{tr}\mathcal{B}^H + \frac{H^2(\nabla v)}{\partial_\nu v}\nabla_T(H(\nabla v))\cdot\big[\nabla_\xi H(\nabla v)\big]_T\\ &= H(\nu)\,H^2(\nabla v)\,\mathrm{tr}\mathcal{B}^H + \frac{H^2(\nabla v)}{\partial_\nu v}\nabla_T(H(\nabla v))\cdot\big[\nabla_\xi H(\nabla v)\big]_T.\end{aligned}$$

Next, we have that

$$\begin{aligned}(2.3.31)\qquad &\nabla_T\big[\tfrac12\nabla_\xi H^2(\nabla v)\big]\,\big[\tfrac12\nabla_\xi H^2(\nabla v)\big]_T\cdot\nu = H(\nabla v)\,\nabla_T\big(H(\nabla v)\,\nabla_\xi H(\nabla v)\big)\,\big[\nabla_\xi H(\nabla v)\big]_T\cdot\frac{\nabla v}{\partial_\nu v}\\ &= \frac{H^2(\nabla v)}{\partial_\nu v}\Big\{\Big(\nabla_\xi^2 H(\nabla v)\,\nabla v\Big)\cdot\Big(\nabla_T(\nabla v)\,\big[\nabla_\xi H(\nabla v)\big]_T\Big) + \big[\nabla_\xi H(\nabla v)\big]_T\cdot\nabla_T H(\nabla v)\Big\}\\ &= \frac{H^2(\nabla v)}{\partial_\nu v}\big[\nabla_\xi H(\nabla v)\big]_T\cdot\nabla_T H(\nabla v),\end{aligned}$$

where the first equality holds owing to equation (2.3.27), the second one to the chain rule and equality (1.2.14), and the last one to equation (1.2.15). Equation (2.3.27) follows from (2.3.30) and (2.3.31). □

Remark 2.3.4. Notice that identity (2.3.25) can be extended by continuity also to those points $\overline{x} \in \partial\Omega$ such that $\nabla v(\overline{x}) = 0$. Indeed, since the tangential derivatives appearing in (2.3.25) are bounded in $\overline{\Omega} \setminus \{\nabla v = 0\}$, the two sides of identity (2.3.25) approach zero when x tends to $\overline{x}$, inasmuch as $\nabla_\xi H^2(\nabla v)$ is Lipschitz continuous and vanishes at such points.

Theorem 2.3.5 (Anisotropic Reilly's identity)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$ such that $\partial\Omega \in C^2$. Assume that $h \in C^1(\overline{\Omega})$, $v \in C^2(\overline{\Omega})$, and $v = 0$ on $\partial\Omega$. Set*

$$W = h\,\tfrac{1}{2}\nabla_\xi H^2(\nabla v) \quad \text{in } \overline{\Omega}. \tag{2.3.32}$$

Then,

$$\int_\Omega \big(\operatorname{div} W\big)^2 \phi\,dx = \int_\Omega \operatorname{tr}\big((\nabla W)^2\big)\,\phi\,dx + \int_{\partial\Omega} h^2\,H(\nu)\,H^2(\nabla v)\operatorname{tr}\mathcal{B}^H\,\phi\,d\mathcal{H}^{n-1} - \int_\Omega \big\{(\operatorname{div} W)\,W\cdot\nabla\phi - \nabla W\,W\cdot\nabla\phi\big\}\,dx \tag{2.3.33}$$

for every $\phi \in C^\infty(\mathbb{R}^n)$.

Let us mention that formula (2.3.33) was established in [82, formula 3.11] in the special case when both h and $\operatorname{div} W$ are constant.

Proof of Theorem 2.3.5. Our assumptions on the domain Ω, and on the functions h and v ensure that they are, in fact, restrictions to $\overline{\Omega}$ of functions defined in the entire $\mathbb{R}^n$ and enjoying the same regularity properties and compactly supported in $\mathbb{R}^n$. Therefore, the function W is also defined on all $\mathbb{R}^n$ via formula (2.3.32). The fact that H is a norm and $H \in C^2(\mathbb{R}^n \setminus \{0\})$ ensures that the hypotheses of Lemma 2.3.2 are fulfilled with $V = W$, and $Z = \{x \in \mathbb{R}^n : \nabla v(x) = 0\}$, hence the thesis follows by Lemmas 2.3.2-2.3.3 . □

2.4 The approximation argument

The key tool in the proof of the global regularity results is Theorem 2.3.5 coupled with the choice of a suitable approximation procedure. In order to deal with the (possibly) non-polynomial growth of the Young function B, here we follow a different approach than the one used in Section 1.3.

Specifically we shall approximate the function $\mathcal{A}$ via a family of functions with quadratic growth. To this purpose, for $\varepsilon \in (0,1)$, we define the function $a_\varepsilon(t) : [0,\infty) \to [0,\infty)$ as

$$a_\varepsilon(t) = \frac{a\big(\sqrt{\varepsilon + t^2}\big) + \varepsilon}{1 + \varepsilon\, a\big(\sqrt{\varepsilon + t^2}\big)} \quad \text{for } t \geq 0, \tag{2.4.1}$$

the function $b_\varepsilon(t) : [0,\infty) \to [0,\infty)$ as

$$b_\varepsilon(t) = a_\varepsilon(t)t \quad \text{for } t \geq 0, \tag{2.4.2}$$

and the function $\mathcal{A}_\varepsilon(\xi) : \mathbb{R}^n \to [0,\infty)$ as

$$\mathcal{A}_\varepsilon(\xi) = \begin{cases} b_\varepsilon\big(H(\xi)\big)\nabla_\xi H(\xi) & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases} \tag{2.4.3}$$

Notice the alternative formula

$$\mathcal{A}_\varepsilon(\xi) = a_\varepsilon\big(H(\xi)\big)\tfrac{1}{2}\nabla_\xi H^2(\xi) \quad \text{if } \xi \neq 0. \tag{2.4.4}$$

Some basic properties of these functions as $\varepsilon \to 0^+$ are provided by the following lemma.

Lemma 2.4.1. *Let* $\varepsilon \in (0,1)$. *Then* $a_\varepsilon \in C^1([0,\infty))$,

$$\varepsilon \le a_\varepsilon(t) \le \varepsilon^{-1} \quad \text{for } t \ge 0, \tag{2.4.5}$$

and

$$\min\{i_a, 0\} \le i_{a_\varepsilon} \le s_{a_\varepsilon} \le \max\{s_a, 0\}. \tag{2.4.6}$$

Moreover, $\mathcal{A}_\varepsilon(\xi) \in C^1(\mathbb{R}^n \setminus \{0\})$ *and, given any* $M > 0$,

$$\lim_{\varepsilon\to 0^+} b_\varepsilon(t) = b(t) \tag{2.4.7}$$

uniformly in $[0, M]$, *and*

$$\lim_{\varepsilon\to 0^+} \mathcal{A}_\varepsilon(\xi) = \mathcal{A}(\xi) \tag{2.4.8}$$

uniformly in $\{\xi \in \mathbb{R}^n : |\xi| \le M\}$.

The proof of this lemma is analogous to that of [54, Proof of Lemma 4.5] and will be omitted.

The next result provides us with information about the symmetric matrix $\nabla_\xi A_\varepsilon(\xi)$ given by

$$\nabla_\xi A_\varepsilon(\xi) = \left(\frac{\partial \mathcal{A}^i_\varepsilon(\xi)}{\partial \xi_j}\right)_{i,j=1,\dots,n} \qquad \text{for } \xi \ne 0,$$

for $\varepsilon \in (0,1)$, and about its smallest and largest eigenvalues $\lambda^\varepsilon_{\min}(\xi)$ and $\lambda^\varepsilon_{\max}(\xi)$.

Lemma 2.4.2. *Let* $\varepsilon \in (0,1)$. *Then,*

$$\lambda \min\{1, i_b\}\, a_\varepsilon\big(H(\xi)\big)\, |\eta|^2 \le \nabla_\xi \mathcal{A}_\varepsilon(\xi)\, \eta \cdot \eta \le \Lambda \max\{1, s_b\}\, a_\varepsilon\big(H(\xi)\big)\, |\eta|^2 \tag{2.4.9}$$

for $\xi \ne 0$ *and* $\eta \in \mathbb{R}^n$. *In particular,*

$$\frac{\lambda^\varepsilon_{\max}(\xi)}{\lambda^\varepsilon_{\min}(\xi)} \le \frac{\Lambda \max\{1, s_b\}}{\lambda \min\{1, i_b\}} \quad \text{for } \xi \ne 0, \tag{2.4.10}$$

and

$$\varepsilon\, \lambda \min\{1, i_b\}\, \mathrm{Id} \le \nabla_\xi \mathcal{A}_\varepsilon(\xi) \le \varepsilon^{-1}\, \Lambda \max\{1, s_b\}\, \mathrm{Id} \quad \text{for } \xi \ne 0. \tag{2.4.11}$$

Hence, the function $\mathcal{A}_\varepsilon : \mathbb{R}^n \to [0,\infty)$ *is Lipschitz continuous.*

Lemma 2.4.2 follows via Lemma 2.2.1, applied with the function a replaced with a_ε, and Lemma 2.4.1.

Finally, by exploiting (2.4.10), the symmetry of the matrix $\nabla_\xi \mathcal{A}(\xi)$, and the algebraic Lemma 1.2.3, we infer the following result.

Lemma 2.4.3. *Let* $\varepsilon \in (0,1)$ *and let* $M \in \mathbb{R}^{n\times n}$ *be a symmetric matrix. Then,*

$$\mathrm{tr}\big((\nabla_\xi \mathcal{A}_\varepsilon(\xi) M)^2\big) \ge \left(\frac{\lambda \min\{1, i_b\}}{\Lambda \max\{1, s_b\}}\right)^2 \big|\nabla_\xi \mathcal{A}_\varepsilon(\xi) M\big|^2 \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.4.12}$$

2.5 Local regularity

This section is devoted to the proof of Theorem 2.1.1. The definition of generalized local solutions to the equations considered in this theorem involves the use of spaces of functions whose truncations are weakly differentiable. For $t > 0$, denote by $T_t : \mathbb{R} \to \mathbb{R}$ the function defined as

$$T_t(s) = \begin{cases} s & \text{if } |s| \le t \\ t\,\mathrm{sign}(s) & \text{if } |s| > t\,. \end{cases}$$

Given an open set Ω in $\mathbb{R}^n$, define the space

$$\mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega) = \Big\{ u \,\text{is measurable in}\ \Omega\ :\ T_t(u) \in W^{1,1}_{\mathrm{loc}}(\Omega) \text{ for every } t > 0 \Big\}\,. \tag{2.5.1}$$

When Ω is bounded, the spaces $\mathcal{T}^{1,1}(\Omega)$ and $\mathcal{T}^{1,1}_0(\Omega)$ are defined accordingly, on replacing $W^{1,1}_{\mathrm{loc}}(\Omega)$ with $W^{1,1}(\Omega)$ and $W^{1,1}_0(\Omega)$ in (2.5.1).
As shown in [19, Lemma 2.1], to each function $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ one can associate a (unique) measurable function $Z_u : \Omega \to \mathbb{R}^n$ such that

$$\nabla\big(T_t(u)\big) = \chi_{\{|u|<t\}} Z_u \qquad \text{a.e. in } \Omega \tag{2.5.2}$$

for every $t > 0$. Here χ_E denotes the characteristic function of the set E. With abuse of notation, the function Z_u will be simply denoted by ∇u in what follows.

Assume that $f \in L^2_{\mathrm{loc}}(\Omega)$. A function $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ is called a generalized local solution to equation (2.1.1) if $\mathcal{A}(\nabla u) \in L^1_{\mathrm{loc}}(\Omega)$, the equation

$$\int_\Omega \mathcal{A}(\nabla u) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \tag{2.5.3}$$

holds for every $\varphi \in C^\infty_0(\Omega)$, and there exists a sequence $\{f_k\} \subset C^\infty_0(\Omega)$ and a corresponding sequence of local weak solutions $\{u_k\}$ to equation (2.1.1), with f replaced by f_k, such that $f_k \to f$ in $L^2(\Omega')$ for every open set $\Omega' \subset\subset \Omega$,

$$u_k \to u \quad \text{and} \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega, \tag{2.5.4}$$

and

$$\lim_{k\to\infty} \int_{\Omega'} |\mathcal{A}(\nabla u_k)| \, dx = \int_{\Omega'} |\mathcal{A}(\nabla u)| \, dx. \tag{2.5.5}$$

Here, ∇u stands for the function Z_u satisfying property (2.5.2).

Proof of Theorem 2.1.1. For simplicity of notation, we shall prove the result with balls B_{2R} replaced by B_{3R}.
Assume, for the time being, that $f \in L^\infty(\Omega)$. Under this assumption, thanks to [121, Theorem 5.1], the function u belongs to $L^\infty_{loc}(\Omega)$ and, in particular, for any $\Omega' \subset\subset \Omega$ there exists a constant $c = c(n, i_b, s_b, \lambda, \Lambda, \Omega', \Omega, \|f\|_{L^\infty(\Omega)})$ such that

$$\|u\|_{L^\infty(\Omega')} \le c\,. \tag{2.5.6}$$

Thus we may apply [136, Theorem 1.7] and infer that

$$u \in C^{1,\theta}(\Omega')\,, \tag{2.5.7}$$

for some $\theta \in (0,1)$ depending on $n, \lambda, \Lambda, s_b, i_b, \Omega', \Omega, \|f\|_{L^\infty(\Omega)}$.
Next, fix $\varepsilon \in (0,1)$, and consider the weak solution u_ε to the Dirichlet problem

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big) = f & \text{in } B_{3R} \\ u_\varepsilon = u & \text{on } \partial B_{3R}, \end{cases} \tag{2.5.8}$$

where $\mathcal{A}_\varepsilon$ is the function defined by (2.4.3). The function u_ε agrees with the unique minimizer of the strictly convex functional J^H_ε defined as

$$J^H_\varepsilon(w) = \int_{B_{3R}} B_\varepsilon\big(H(\nabla w)\big)\,dx - \int_{B_{3R}} f\,w\,dx\,, \tag{2.5.9}$$

among all functions $w \in W^{1,2}(B_{3R})$ such that $u = w$ on ∂B_{3R}. Here, B_ε is the Young function given by

$$B_\varepsilon(t) = \int_0^t b_\varepsilon(s)\,ds \quad \text{for } t \geq 0.$$

Owing to (2.4.5), (2.4.9) and Lemma 1.3.2– see also [20, Theorem 8.1]– one has that $u_\varepsilon \in W^{2,2}_{\mathrm{loc}}(B_{3R})$. Since $\mathcal{A}_\varepsilon(\xi) \in C^{0,1}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$, the chain rule for vector-valued functions [145] ensures that $\mathcal{A}_\varepsilon(\nabla u_\varepsilon) \in W^{1,2}_{loc}(B_{3R})$ and

$$\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)) = \nabla_\xi \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\nabla^2 u_\varepsilon \quad \text{a.e. in } B_{3R}.$$

Now, fix $R \leq \sigma < \tau \leq 2R$ and let φ be a cut-off function such that $\varphi \in C^\infty_0(B_\tau)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in B_σ, and

$$\varphi = 1 \quad \text{on } B_\sigma \quad \text{and } |\nabla\varphi| \leq c(n)/(\tau - \sigma). \tag{2.5.10}$$

Extend the vector field $\mathcal{A}_\varepsilon(\nabla u_\varepsilon)$ to the whole $\mathbb{R}^n$ and, via convolution, consider its regularization $V_{\varepsilon,\delta} = \mathcal{A}_\varepsilon(\nabla u_\varepsilon) * \varrho_\delta$. Here $\{\varrho_\delta\}_{\delta>0}$ denotes a family of standard, radially symmetric mollifiers.
Standard properties of convolution imply that $V_{\varepsilon,\delta} \xrightarrow{\delta\to 0} \mathcal{A}_\varepsilon(\nabla u_\varepsilon)$ in $W^{1,2}_{loc}(B_{3R})$. Thus, an application of equation (2.3.7) with $V = V_{\varepsilon,\delta}$, $Z = \emptyset$ and $\phi = \varphi^2$ yields, after letting $\delta \to 0$,

$$\begin{aligned} \int_\Omega f^2\,\varphi^2\,dx = {} & \int_\Omega \mathrm{tr}\big((\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)))^2\big)\varphi^2\,dx \\ & - 2\int_\Omega \Big\{ \mathrm{div}\,\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\varphi - \nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\varphi \Big\}\,\varphi\,dx. \end{aligned} \tag{2.5.11}$$

From Young's inequality, we deduce that

$$\begin{aligned} \Big| \int_\Omega 2 \Big\{ \big(\mathrm{div}\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)\,\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\varphi - \nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\varphi\big) \Big\}\,\varphi\,dx \Big| \leq \\ \leq \gamma \int_\Omega |\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))|^2\,\varphi^2\,dx + \frac{c}{\gamma}\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,|\nabla\varphi|^2\,dx \end{aligned} \tag{2.5.12}$$

for some constant $c = c(n)$ and for every $\gamma > 0$. Combining (2.5.11) and (2.5.12), making use of inequality (2.4.12), and recalling property (2.5.10) enable us to deduce that

$$\int_{B_\sigma} \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2\,dx \leq c\int_{B_{2R}} f^2\,dx + \frac{c}{(\tau-\sigma)^2}\int_{B_\tau\setminus B_\sigma} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,dx, \tag{2.5.13}$$

where $c = c(n, i_a, s_a, \lambda, \Lambda)$. Thanks to a Sobolev type inequality on annuli (see, e.g., [58, formula (5.4)]), one has that

$$\frac{1}{(\tau-\sigma)^2}\int_{B_\tau\setminus B_\sigma} |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,dx \le c\int_{B_\tau\setminus B_\sigma}\big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2\,dx + \frac{c\,R}{(\tau-\sigma)^{n+3}}\Big(\int_{B_\tau\setminus B_\sigma}|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|\,dx\Big)^2 \tag{2.5.14}$$

for some constant $c = c(n)$. Coupling inequality (2.5.13) with (2.5.14) tells us that

$$\int_{B_\sigma}\big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2\,dx \le c\int_{B_\tau\setminus B_\sigma}\big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2\,dx + c\int_{B_{2R}} f^2\,dx + \frac{c\,R}{(\tau-\sigma)^{n+3}}\Big(\int_{B_\tau\setminus B_\sigma}|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|\,dx\Big)^2$$

for some constant $c = c(n, i_a, s_a, \lambda, \Lambda)$. After adding the quantity $c\int_{B_\sigma}\big|\nabla\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big|^2\,dx$ to both sides of this inequality one infers that

$$\int_{B_\sigma}\big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2\,dx \le \frac{c}{1+c}\int_{B_\tau}\big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2\,dx + c'\int_{B_{2R}} f^2\,dx + \frac{c'\,R}{(\tau-\sigma)^{n+3}}\Big(\int_{B_{2R}\setminus B_R}|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|\,dx\Big)^2. \tag{2.5.15}$$

for some constant $c' = c'(n, i_a, s_a, \lambda, \Lambda)$. A standard iteration argument (see, e.g., [100, Lemma 3.1, Chapter 5]) enables us to deduce from inequality (2.5.15) that

$$\int_{B_R}\big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2\,dx \le c\int_{B_{2R}} f^2\,dx + \frac{c}{R^{n+2}}\Big(\int_{B_{2R}\setminus B_R}|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|\,dx\Big)^2, \tag{2.5.16}$$

for some constant $c = c(n, i_a, s_a, \lambda, \Lambda)$. Moreover, a Poincaré type inequality implies that

$$\int_{B_R}|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,dx \le cR^2\int_{B_R}\big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2\,dx + \frac{c}{R^n}\Big(\int_{B_R}|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|\,dx\Big)^2$$

for some constant $c = c(n)$. Hence, via inequality (2.5.16), we obtain that

$$\int_{B_R}|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,dx \le c\,R^2\int_{B_{2R}} f^2\,dx + \frac{c}{R^n}\Big(\int_{B_{2R}}|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|\,dx\Big)^2 \tag{2.5.17}$$

for some constant $c = c(n, i_a, s_a, \lambda, \Lambda)$.
From [184, Theorem 2] one can deduce that

$$\|u_\varepsilon\|_{L^\infty(B_{3R})} \le \|u\|_{L^\infty(B_{3R})} + c\,R\,\hat{b}_\varepsilon^{-1}\Big(c\|f\|_{L^n(B_{3R})}\Big), \tag{2.5.18}$$

where $\hat{b}_\varepsilon$ is the function defined by $\hat{b}_\varepsilon(t) = B_\varepsilon(t)/t$. Hence, thanks to equations (2.2.9) and (2.2.14), applied with b replaced by b_ε, and formulas (2.4.6) and (2.5.6), one can deduce that

$$\|u_\varepsilon\|_{L^\infty(B_{3R})} \le c \tag{2.5.19}$$

for some constant c independent of ε.
This enables us to apply [136, Theorem 1.7] and obtain that

$$\|u_\varepsilon\|_{C^{1,\theta}(B')} \le c, \tag{2.5.20}$$

for some constant c independent of $\varepsilon \in (0,1)$ and for every ball $B' \subset\subset B_{3R}$. Hence, there exist a function $v \in C^1(B_{3R})$ and a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0^+$ and

$$u_{\varepsilon_k} \to v \quad \text{in } C^{1,\theta'}_{\mathrm{loc}}(B_{3R}) \tag{2.5.21}$$

for every $\theta' < \theta$, In particular, this convergence and inequality (2.5.19) imply that $v \in L^\infty(B_{3R})$.
Moreover, by equation (2.4.8) and (2.5.20), the norms $\|\mathcal{A}_{\varepsilon_k}(\nabla u_{\varepsilon_k})\|_{L^\infty(B_{2R})}$ are uniformly bounded for $k \in \mathbb{N}$. This piece of information, coupled with inequalities (2.5.16) and (2.5.17), entails that the sequence $\{\mathcal{A}_{\varepsilon_k}(\nabla u_{\varepsilon_k})\}$ is uniformly bounded in $W^{1,2}(B_{2R})$. As a consequence, there exists a subsequence, still denoted by $\{u_{\varepsilon_k}\}$, such that

$$\mathcal{A}_{\varepsilon_k}(\nabla u_{\varepsilon_k}) \rightharpoonup \mathcal{A}(\nabla v) \quad \text{weakly in } W^{1,2}(B_{2R})\,. \tag{2.5.22}$$

Hence, from inequalities (2.5.16) and (2.5.17) we infer that

$$\int_{B_R} \big|\nabla(\mathcal{A}(\nabla v))\big|^2\,dx \le c \int_{B_{2R}} f^2\,dx + \frac{c}{R^{n+2}}\Big(\int_{B_{2R}\setminus B_R} |\mathcal{A}(\nabla v)|\,dx\Big)^2, \tag{2.5.23}$$

and

$$\int_{B_R} |\mathcal{A}(\nabla v)|^2\,dx \le c\,R^2 \int_{B_{2R}} f^2\,dx + \frac{c}{R^n}\Big(\int_{B_{2R}} |\mathcal{A}(\nabla v)|\,dx\Big)^2. \tag{2.5.24}$$

Also, passing to the limit in the weak formulation of problem (2.5.8) tells us that

$$\int_{B_{3R}} \mathcal{A}(\nabla v)\cdot\nabla\varphi\,dx = \int_{B_{3R}} f\,\varphi\,dx \tag{2.5.25}$$

for every function $\varphi \in C^\infty_0(B_{3R})$.
We claim that

$$v - u \in W^{1,B}_0(B_{3R}). \tag{2.5.26}$$

To verify this claim notice that, thanks to (2.5.7), we can exploit the minimizing property of the function u_ε, which tells us that is $J^H_\varepsilon(u_\varepsilon) \le J^H_\varepsilon(u)$. Coupling this piece of information with (1.2.9), and properties (2.2.19) and (2.2.20) for B and B_ε ensures that

$$\int_{B_{3R}} B_\varepsilon\big(|\nabla u_\varepsilon|\big)\,dx \le c \int_{B_{3R}} B_\varepsilon\big(|\nabla u|\big)\,dx + c\int_{B_{3R}} f\,(u_\varepsilon - u)\,dx\,, \tag{2.5.27}$$

for some positive constant $c = c(n, i_b, s_b, \lambda, \Lambda)$. In particular, such a constant is independent of ε, thanks to inequalities (2.4.6), inasmuch as it depends on ε only through a lower bound on i_{B_ε} and an upper bound on s_{B_ε}.

Owing to bounds (2.5.6) and (2.5.19), the inequality (2.5.27) yields:

$$\int_{B_{3R}} B_\varepsilon(|\nabla u_\varepsilon|)\,dx \le c \int_{B_{3R}} B_\varepsilon(|\nabla u|)\,dx + c \tag{2.5.28}$$

for some constant c independent of ε. From (2.2.9), (2.2.10) for the function b_ε, and (2.4.6), there exist positive constants c, c' such that

$$c\,t^{\min\{i_b+1,2\}} \le B_\varepsilon(t) \le c'\,t^{\max\{s_b+1,2\}} \quad \text{for } t \ge 1. \tag{2.5.29}$$

Since $B_\varepsilon \to B$ locally uniformly in $[0,\infty)$, property (2.5.21) implies that $B_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|) \to B(|\nabla v|)$ everywhere in B_{3R}. Thus, from (2.5.7), (2.5.28) and (2.5.29) we deduce, via Fatou's Lemma, that

$$
\begin{aligned}
\int_{B_{3R}} B(|\nabla v|)\, dx &\le c \liminf_{k\to\infty} \int_{B_{3R}} B_{\varepsilon_k}(|\nabla u|)\, dx + c \\
&\le c|B_{3R}|\Big(\|\nabla u\|_{L^\infty(B_{3R})} + 1\Big)^{\max\{s_b+1,2\}} + c\,.
\end{aligned}
\tag{2.5.30}
$$

Hence, $v \in W^{1,B}(B_{3R})$. On the other hand, from (2.5.28), (2.5.7) and (2.5.29) we infer that

$$
\begin{aligned}
\int_{B_{3R}} |\nabla u_\varepsilon|^{\min\{i_b+1,2\}}\, dx &\le c \int_{B_{3R}} B_\varepsilon(|\nabla u|)\, dx + c \\
&\le c|B_{3R}|\Big(\|\nabla u\|_{L^\infty(B_{3R})} + 1\Big)^{\max\{s_b+1,2\}} + c.
\end{aligned}
$$

The latter bound implies that the family of functions $\{u_\varepsilon - u\}$ is uniformly bounded in the Sobolev space $W^{1,\min\{i_b+1,2\}}_0(B_{3R})$ for $\varepsilon \in (0,1)$. The reflexivity of this space implies that $v-u \in W^{1,\min\{i_b+1,2\}}_0(B_{3R})$, Combining this membership with inequality (2.5.30) yields (2.5.26). Our claim is thus proved. Thanks to (2.5.26), (1.2.9), (2.2.21) and (2.2.24), we have that the vector field $\mathcal{A}(\nabla v) \in L^{\widetilde{B}}(B_{3R})$. The density of the space $C^\infty_0(B_{3R})$ in the space $W^{1,B}_0(B_{3R})$ and Hölder's inequality in Orlicz spaces [169, Theorem 4.7.5] ensure that equation (2.5.25) holds, in fact, for every function $\varphi \in W^{1,B}_0(B_{3R})$. Hence, v is a weak solution to the problem

$$
\begin{cases}
-\mathrm{div}\big(\mathcal{A}(\nabla v)\big) = f & \text{in } B_{3R} \\
v = u & \text{on } \partial B_{3R}.
\end{cases}
\tag{2.5.31}
$$

The uniqueness of this solution implies that $v = u$. Therefore, inequalities (2.5.23) and (2.5.24) read

$$
\int_{B_R} |\nabla(\mathcal{A}(\nabla u))|^2\, dx \le c \int_{B_{2R}} f^2\, dx + \frac{c}{R^{n+2}}\Big(\int_{B_{2R}\setminus B_R} |\mathcal{A}(\nabla u)|\, dx\Big)^2,
\tag{2.5.32}
$$

and

$$
\int_{B_R} |\mathcal{A}(\nabla u)|^2\, dx \le c\, R^2 \int_{B_{2R}} f^2\, dx + \frac{c}{R^{n}}\Big(\int_{B_{2R}} |\mathcal{A}(\nabla u)|\, dx\Big)^2.
\tag{2.5.33}
$$

We conclude the proof by removing the assumption that $f \in L^\infty(\Omega)$. Let $f \in L^2_{\mathrm{loc}}(\Omega)$, and let $\{f_k\}$ and $\{u_k\}$ be sequences as in the definition of local approximable solution u to equation (2.1.1). Hence, (2.5.4) and (2.5.5) hold.

Inequalities (2.5.32) and (2.5.33), applied with u replaced by u_k, and equation (2.5.5) tell us that

$$
\int_{B_R} |\nabla(\mathcal{A}(\nabla u_k))|^2\, dx \le c \int_{B_{2R}} f_k^2\, dx + \frac{c}{R^{n+2}}\Big(\int_{B_{2R}\setminus B_R} |\mathcal{A}(\nabla u_k)|\, dx\Big)^2,
\tag{2.5.34}
$$

and

$$
\int_{B_R} |\mathcal{A}(\nabla u_k)|^2\, dx \le c\, R^2 \int_{B_{2R}} f_k^2\, dx + \frac{c}{R^{n}}\Big(\int_{B_{2R}} |\mathcal{A}(\nabla u_k)|\, dx\Big)^2
\tag{2.5.35}
$$

for $k \in \mathbb{N}$. Therefore, the sequence $\{\mathcal{A}(\nabla u_k)\}$ is bounded in $W^{1,2}(B_{2R})$. As a consequence, there exist a function $U : B_{2R} \to \mathbb{R}^n$, such that $U \in W^{1,2}(B_{2R})$, and a subsequence of $\{\mathcal{A}(\nabla u_k)\}$, still indexed by k, such that

$$
\mathcal{A}(\nabla u_k) \to U \quad \text{in } L^2(B_{2R}) \quad \text{and } \mathcal{A}(\nabla u_k) \rightharpoonup U \quad \text{weakly in } W^{1,2}(B_{2R}).
\tag{2.5.36}
$$

By (2.5.4), we thus deduce that $U = \mathcal{A}(\nabla u)$ a.e. in B_{2R}. Hence, property (2.1.7) holds, and inequalities (2.1.8) follow on passing to the limit in (2.5.34) and (2.5.35). □

2.6 Global estimates: Dirichlet problems

Here, we are concerned with proofs of our global results for solutions to Dirichlet problems. Assume that $f \in L^2(\Omega)$. A function $u \in \mathcal{T}_0^{1,1}(\Omega)$ will be called a generalized solution to the Dirichlet problem (2.1.2) if $\mathcal{A}(\nabla u) \in L^1(\Omega)$,

$$\int_\Omega \mathcal{A}(\nabla u) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \tag{2.6.1}$$

for every $\varphi \in C_0^\infty(\Omega)$, and there exists a sequence $\{f_k\} \subset C_0^\infty(\Omega)$ such that $f_k \to f$ in $L^2(\Omega)$ and the sequence of weak solutions $\{u_k\}$ to problem (2.1.2), with f replaced by f_k, satisfies

$$u_k \to u \quad \text{a.e. in } \Omega.$$

In (2.6.1), ∇u stands for the function Z_u fulfilling (2.5.2).
By [57, Theorem 3.2], there exists a unique generalized solution u to problem (2.1.2), and

$$\int_\Omega |\mathcal{A}(\nabla u)| \, dx \le c \, |\Omega|^{1/n} \int_\Omega |f| \, dx \tag{2.6.2}$$

for some constant $c = c(n, i_b, s_b, \lambda, \Lambda)$. Moreover, if $\{f_k\}$ is any sequence as above, and $\{u_k\}$ is the associated sequence of weak solutions, then

$$u_k \to u \quad \text{and} \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega, \tag{2.6.3}$$

up to subsequences.

The generalized solutions introduced above agree with the classical weak solutions, provided that the function f has a sufficiently high degree of integrability. Recall that a function u is called a weak solution to the Dirichlet problem (2.1.2) if $u \in W_0^{1,B}(\Omega)$ and equation (2.6.1) holds for every function $\varphi \in W_0^{1,B}(\Omega)$. Here, $W_0^{1,B}(\Omega)$ denotes the Orlicz-Sobolev space, built upon B, of those functions vanishing in the usual appropriate sense on $\partial\Omega$. Minimal conditions on f for a weak solution to be well-defined and to exist can be exhibited – see [2]. They rely upon a sharp embedding theorem for Orlicz-Sobolev spaces [51, 52]. In view of our purposes, we shall only need to deal with weak solutions under the assumption that $f \in L^\infty(\Omega)$, in which case they certainly exist whatever B is.

Having dispensed with the necessary definitions, we begin preparing for our proofs with a few lemmas concerning Sobolev functions. The first one deals with the continuity in Sobolev spaces of the composition operator for vector-valued functions, see [162, Proposition 2.6].

Lemma A. *Let Ω be a bounded open set in $\mathbb{R}^n$ and let $F \in C^{0,1}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$. Assume that $V : \Omega \to \mathbb{R}^n$ is such that $V \in W^{1,2}(\Omega)$, and let $\{V_m\}$ be a sequence of functions $V_m : \Omega \to \mathbb{R}^n$ such that $V_m \in W^{1,2}(\Omega)$ for $m \in \mathbb{N}$. If*

$$V_m \to V \quad \text{in } W^{1,2}(\Omega), \tag{2.6.4}$$

then,

$$F(V_m) \to F(V) \quad \text{in } W^{1,2}(\Omega). \tag{2.6.5}$$

The following lemma is a straightforward consequence of Propositions 3.5.1 and 3.5.2, applied with $\varrho = |\mathcal{B}^H|$, and formulas (2.3.22)–(2.3.23).

Lemma 2.6.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$. Assume that $\partial\Omega \in W^{2,1}$.*
(i) If $\mathcal{K}_\Omega(r) < \infty$ for $r \in \big(0, R_\Omega]$, then,

$$\int_{\partial\Omega\cap B_r(x)} v^2\,|\mathcal{B}^H|\,d\mathcal{H}^{n-1} \le c_0(n,\lambda,\Lambda)\,(1+L_\Omega)^4\,\mathcal{K}_\Omega(r)\int_{\Omega\cap B_r(x)} |\nabla v|^2\,dy, \tag{2.6.6}$$

for every $x \in \partial\Omega$, $r \in \big(0, R_\Omega]$ and $v \in W^{1,2}_0(B_r(x))$.
(ii) If $\Psi_\Omega(r) < \infty$ for $r \in \big(0, R_\Omega]$, then,

$$\int_{\partial\Omega\cap B_r(x)} v^2\,|\mathcal{B}^H|\,d\mathcal{H}^{n-1} \le c_0(n,\lambda,\Lambda)\,\big(1+L_\Omega\big)^{11}\,\Psi_\Omega(r)\int_{\Omega\cap B_r(x)} |\nabla v|^2\,dy, \tag{2.6.7}$$

for every $x \in \partial\Omega$, $r \in \big(0, R_\Omega]$ and $v \in W^{1,2}_0(B_r(x))$.

The inequality provided by the next lemma is well known. The point here is the dependence of the constants on Lipschitz characteristics of domains.

Lemma 2.6.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$. Then,*

$$\|v\|^2_{L^2(\Omega)} \le \sigma\,\|\nabla v\|^2_{L^2(\Omega)} + c\,\frac{(1+\sigma)^2}{\sigma}\,\frac{d_\Omega^{2n(n+2)}}{r^{(2n+1)(n+2)}}\,(1+L_\Omega)^{n+2}\,\|v\|^2_{L^1(\Omega)} \tag{2.6.8}$$

for some constant $c = c(n)$ and for every $\sigma > 0$, $r \in (0, R_\Omega]$ and $v \in W^{1,2}(\Omega)$.

Proof. An application of an extension theorem by Stein, in the form of [132, Theorem 13.17], ensures that there exists a bounded linear operator $\mathcal{E} : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^n)$ such that,

$$\begin{aligned}
\|\mathcal{E}(v)\|_{L^2(\mathbb{R}^n)} &\le c(n)\left(\frac{d_\Omega}{r}\right)^n \|v\|_{L^2(\Omega)}\\
\|\nabla\mathcal{E}(v)\|_{L^2(\mathbb{R}^n)} &\le c(n)\,\frac{d_\Omega^{2n}}{r^{2n+1}}(1+L_\Omega)\left(\|v\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\right)
\end{aligned} \tag{2.6.9}$$

for $r \in (0, R_\Omega)$ and for $v \in W^{1,2}(\Omega)$. Set

$$s = \begin{cases} \frac{2n}{n-2} & \text{if } n \ge 3\\ 4 & \text{if } n = 2,\end{cases}$$

and

$$\alpha = \begin{cases} \frac{2}{n+2} & \text{if } n \ge 3\\ \frac{1}{3} & \text{if } n = 2,\end{cases}$$

whence $\frac{1}{2} = \alpha + \frac{1-\alpha}{s}$. From Hölder's inequality, the Sobolev inequality, and the inequalities in (2.6.9) one deduces that

$$\begin{aligned}
\|v\|_{L^2(\Omega)} &\le \|v\|^\alpha_{L^1(\Omega)}\,\|v\|^{1-\alpha}_{L^s(\Omega)} = \|v\|^\alpha_{L^1(\Omega)}\,\|\mathcal{E}(v)\|^{1-\alpha}_{L^s(\Omega)}\\
&\le \|v\|^\alpha_{L^1(\Omega)}\,\|\mathcal{E}(v)\|^{1-\alpha}_{L^s(\mathbb{R}^n)} \le c(n)\,\|v\|^\alpha_{L^1(\Omega)}\,\|E(v)\|^{1-\alpha}_{W^{1,2}(\mathbb{R}^n)}\\
&\le c(n)\,C^{1-\alpha}\,\|v\|^\alpha_{L^1(\Omega)}\left(\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\right)^{1-\alpha},
\end{aligned}$$

where we have set

$$C = \frac{d_\Omega^{2n}}{r^{2n+1}}\,(1+L_\Omega)\,.$$

An application of Young's inequality with exponents $\frac{1}{\alpha}$ and $\frac{1}{1-\alpha}$ yields

$$\begin{aligned}\|v\|^2_{L^2(\Omega)} &\le c(n)\,C^{2(1-\alpha)}\,\|v\|^{2\alpha}_{L^1(\Omega)}\Big(\|v\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\Big)^{1-\alpha}\\ &\le \varepsilon\Big(\|v\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\Big)+c'(n)\,\varepsilon^{-\frac{\alpha}{1-\alpha}}\,C^{2\frac{1-\alpha}{\alpha}}\,\|v\|^2_{L^1(\Omega)}\,,\end{aligned}$$

for $\varepsilon\in(0,1)$. Hence, by setting $\sigma=\frac{\varepsilon}{1-\varepsilon}$, one obtains that

$$\|v\|^2_{L^2(\Omega)} \le \sigma\,\|\nabla v\|^2_{L^2(\Omega)} + c'(n)\,\Big(1+\frac{1}{\sigma}\Big)^{\frac{\alpha}{1-\alpha}}\,(1+\sigma)\,C^{2\frac{1-\alpha}{\alpha}}\,\|v\|^2_{L^1(\Omega)} \tag{2.6.10}$$

Notice that

$$\Big(1+\frac{1}{\sigma}\Big)^{\frac{\alpha}{1-\alpha}}\,(1+\sigma) \le \frac{(1+\sigma)^2}{\sigma}.$$

Moreover, since $r\le R_\Omega\le d_\Omega$, $\frac{2(1-\alpha)}{\alpha}\le n+2$ and $C\ge 1$, we have that

$$C^{2\frac{1-\alpha}{\alpha}} \le C^{n+2} = \Big(\frac{d_\Omega^{2n}}{r^{2n+1}}\Big)^{n+2}\,(1+L_\Omega)^{n+2}.$$

Inequality (2.6.8) thus follows from (2.6.10). □

Proof of Theorem 2.1.4, Dirichlet problems. We split the proof into several steps.
Step 1. Here we assume that

$$f\in C^{0,\alpha}_0(\Omega) \tag{2.6.11}$$

and

$$\partial\Omega\in C^{2,\alpha}. \tag{2.6.12}$$

For every $\varepsilon\in(0,1)$, let $\mathcal{A}_\varepsilon$ be the function defined as in (2.4.3), and denote by u_ε the weak solution to the Dirichlet problem

$$\begin{cases}-\mathrm{div}\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)=f & \text{in } \Omega\\ u_\varepsilon=0 & \text{on } \partial\Omega\,.\end{cases} \tag{2.6.13}$$

The same argument exploited in connection with problem (2.5.8) ensures that there exists a unique solution $u_\varepsilon\in W^{1,2}_0(\Omega)$ to problem (2.6.13).
We claim that there exists $\theta=\theta(n,i_b,s_b,\lambda,\Lambda,\|f\|_\infty)\in(0,1)$ such that

$$u_\varepsilon\in W^{2,2}(\Omega)\cap C^{1,\theta}(\overline{\Omega})\,, \tag{2.6.14}$$

and

$$u_\varepsilon\to u \quad \text{in } C^{1,\theta'}_{\mathrm{loc}}(\Omega)\,, \tag{2.6.15}$$

for every $0<\theta'<\theta$.
Furthermore, fixing any $\varepsilon>0$, there exists a sequence $\{u_{\varepsilon,m}\}$ such that

$$u_{\varepsilon,m}\in C^{2,\alpha}(\overline{\Omega}) \quad\text{and}\quad u_{\varepsilon,m}=0 \quad \text{on } \partial\Omega\,, \tag{2.6.16}$$

and

$$u_{\varepsilon,m} \xrightarrow{m\to\infty} u_\varepsilon \quad \text{in } W^{2,2}(\Omega) \text{ and } C^{1,\theta'}(\overline{\Omega}). \tag{2.6.17}$$

To prove our claims, we make use of an argument from [61, Section 3], and define, for $\varepsilon \in (0,1)$ and $\delta > 0$ the regularized vector field $\mathcal{A}_{\varepsilon,\delta} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathcal{A}_{\varepsilon,\delta} = \mathcal{A}_\varepsilon * \rho_\delta \qquad \text{in } \mathbb{R}^n. \tag{2.6.18}$$

Here, $\{\rho_\delta\}$ denotes a family of standard, radially symmetric mollifiers. By properties of convolutions, $\mathcal{A}_{\varepsilon,\delta} \in C^\infty(\mathbb{R}^n)$ and $\lim_{\delta\to 0^+} \mathcal{A}_{\varepsilon,\delta} = \mathcal{A}_\varepsilon$ locally uniformly in $\mathbb{R}^n$. Also, thanks to inequalities (2.4.11), one can readily verify that

$$\varepsilon\,\lambda\,\min\{1, i_b\}\,\mathrm{Id} \le \nabla_\xi \mathcal{A}_{\varepsilon,\delta}(\xi) \le \varepsilon^{-1}\,\Lambda\,\max\{1, s_b\}\,\mathrm{Id} \quad \text{for } \xi \in \mathbb{R}^n. \tag{2.6.19}$$

Next consider the family $\{w_{\varepsilon,\delta}\}$ of the unique solutions to the problems

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\big) = f & \text{in } \Omega \\ w_{\varepsilon,\delta} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.6.20}$$

Thanks to (2.6.19), classical results tell us that $w_{\varepsilon,\delta} \in W^{2,2}(\Omega)$, and

$$\|w_{\varepsilon,\delta}\|_{W^{2,2}(\Omega)} \le c_0 \tag{2.6.21}$$

for some constant $c_0 = c_0(n, \lambda, \Lambda, \varepsilon, \Omega, \|f\|_{L^\infty(\Omega)})$, see, e.g., [128, pp. 270-277], or [20, Theorem 8.2], or [108, Chapter 8.4]. Notice that, by standard elliptic regularity theory ([106, Theorem 9.19] or [128, Theorem 6.3,pag. 283]), we have

$$w_{\varepsilon,\delta} \in C^{2,\alpha}(\overline{\Omega})\,. \tag{2.6.22}$$

Next, by [181, Corollary 6.1], there exists a constant c_1, depending on the same quantities as c_0, such that

$$\|w_{\varepsilon,\delta}\|_{L^\infty(\Omega)} \le c_1. \tag{2.6.23}$$

Coupling this piece of information with [135, Theorem 1] entails that

$$\|w_{\varepsilon,\delta}\|_{C^{1,\theta}(\overline{\Omega})} \le c_2\,, \tag{2.6.24}$$

for some constant c_2, with the same dependence as c_0 and c_1. In particular, these constants are independent of δ.

Thanks to inequalities (2.6.21) and (2.6.24), there exist a function $w_\varepsilon \in W^{2,2}(\Omega) \cap C^{1,\theta}(\overline{\Omega})$ and a sequence $\{\delta_k\}$ such that $\delta_k \to 0^+$,

$$w_{\varepsilon,\delta_k} \xrightarrow{k\to\infty} w_\varepsilon \quad \text{in } C^{1,\theta'}(\overline{\Omega}) \quad \text{and} \quad w_{\varepsilon,\delta_k} \xrightarrow{k\to\infty} w_\varepsilon \quad \text{weakly in } W^{2,2}(\Omega) \tag{2.6.25}$$

for every $\theta' \in (0,\theta)$.

Passing to the limit as $k \to \infty$ in the weak formulation of problem (2.6.20) shows that w_ε is solution to problem (2.6.13), whence $w_\varepsilon = u_\varepsilon$ by the uniqueness of the solution. Property (2.6.14) is thus established.

We next prove properties (2.6.16) and (2.6.17). Thanks to the Banach-Saks theorem, the weak convergence (2.6.25) in the Hilbert space $W^{2,2}(\Omega)$ ensures that there exists a subsequence $\{\delta_{k_l}\}_{l\in\mathbb{N}}$ such that, on setting

$$u_{\varepsilon,m} = \frac{1}{m}\sum_{l=1}^{m} w_{\varepsilon,\delta_{k_l}},$$

one has that

$$u_{\varepsilon,m} \xrightarrow{m\to\infty} u_\varepsilon \quad \text{in } W^{2,2}(\Omega). \tag{2.6.26}$$

Moreover, by the convergence of w_{ε,δ_k} to u_ε in $C^{1,\theta'}(\overline{\Omega})$,

$$u_{\varepsilon,m} \xrightarrow{m\to\infty} u_\varepsilon \quad \text{in } C^{1,\theta'}(\overline{\Omega})\,. \tag{2.6.27}$$

Thanks to (2.6.22),(2.6.26) and (2.6.27), the sequence $\{u_{\varepsilon,m}\}$ satisfies properties (2.6.16) and (2.6.17). To complete the proof of this step, we establish the convergence in (2.6.15). By the minimizing property of the function u_ε for the functional J^H_ε, defined as in (2.5.9) with B_{3R} replaced by Ω, we have that $J^H_\varepsilon(u_\varepsilon) \le J^H_\varepsilon(0)$, hence

$$\int_\Omega B_\varepsilon\big(H(\nabla u_\varepsilon)\big)\,dx \le \int_\Omega f\,u_\varepsilon\,dx. \tag{2.6.28}$$

Thanks to [184, Theorem 2], inequalities (2.4.6), and property (2.4.7), there exists a constant $c = c(n, i_b, s_b, \lambda, \Lambda, |\Omega|, \|f\|_{L^\infty(\Omega)})$ such that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le c. \tag{2.6.29}$$

Owing to inequalities (1.2.9), (2.4.6), and (2.6.29), one can deduce from (2.6.28) that

$$\int_\Omega B_\varepsilon(|\nabla u_\varepsilon|)\,dx \le c \tag{2.6.30}$$

for some constant $c = c(n, i_b, s_b, \lambda, \Lambda, |\Omega|, \|f\|_{L^\infty(\Omega)})$. Moreover, inequality (2.6.29) allows one to apply [136, Theorem 1.7] and obtain

$$\|u_\varepsilon\|_{C^{1,\theta}(\Omega')} \le c \quad \text{for every open set } \Omega' \subset\subset \Omega,$$

and for some constant $c = c(n, \lambda, \Lambda, i_b, s_b, \lambda, \Lambda, \Omega', \Omega, \|f\|_{L^\infty(\Omega)})$.
Thus, there exists a sequence of $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0^+$ and

$$u_{\varepsilon_k} \to v \quad \text{in } C^1_{\text{loc}}(\Omega)\,, \tag{2.6.31}$$

for some function $v \in C^1(\Omega)$.
Now, we want to show that

$$v = u. \tag{2.6.32}$$

To this purpose, one can use an analogous argument as at the end of Step 1 of the proof Theorem 2.1.1. Specifically, since $B_{\varepsilon_k}(t) \to B(t)$ locally uniformly in $[0,\infty)$, from (2.6.31) we have that $B_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|) \to B(|\nabla v|)$ everywhere in Ω. From inequalities (2.6.30) and (2.5.29), and Fatou's Lemma we obtain that

$$\int_\Omega B(|\nabla v|)\,dx \le c \quad \text{and} \quad \int_\Omega |\nabla u_\varepsilon|^{\min\{i_b+1,2\}}\,dx \le c\,, \tag{2.6.33}$$

for some constant c independent on ε. Thanks to the reflexivity of the space $W_0^{1,\min\{i_b+1,2\}}(\Omega)$, inequalities (2.6.33) imply that $v \in W_0^{1,B}(\Omega)$.
Owing to (2.6.31), passing to the limit as $\varepsilon \to 0^+$ in (2.6.13) yields:

$$\int_\Omega \mathcal{A}(\nabla v) \cdot \nabla \varphi \, dx = \int_\Omega f \, \varphi \, dx \tag{2.6.34}$$

for every $\varphi \in C_0^\infty(\Omega)$. A density argument as at the end of the proof of Step 1 of Theorem 2.1.1 implies that equation (2.6.34) holds, in fact, for every function $\varphi \in W_0^{1,B}(\Omega)$. Thus, v is the weak solution to problem (2.1.2), whence, by its uniqueness, equality (2.6.32) follows. Thereby, property (2.6.15) is a consequence of (2.6.31) and of the fact that the preceding argument applies to any sequence extracted from the family $\{u_\varepsilon\}$.

Step 2. We show that, given any $L, d, M > 0$ and $\overline{r} \in (0,1)$, there exists a positive constant $\overline{c} = \overline{c}(n, \lambda, \Lambda, i_b, s_b, L)$ such that, if Ω is a bounded domain of class $C^{2,\alpha}$ and Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$ satisfying $L_\Omega \leq L$, $d_\Omega \leq d$, $R_\Omega \geq \overline{r}$ and

$$\mathcal{K}_\Omega(r) \leq \overline{c} \qquad \text{for } r \in (0, \overline{r}]\,, \tag{2.6.35}$$

and $u \in W_0^{1,B}(\Omega)$ is a weak solution to problem (2.6.1) with $\|f\|_{L^2(\Omega)} \leq M$, then

$$\|\mathcal{A}(\nabla u)\|^2_{L^2(\Omega)} \leq c_1 \, \|f\|^2_{L^2(\Omega)} \quad \text{and} \quad \|\nabla \mathcal{A}(\nabla u)\|^2_{L^2(\Omega)} \leq c_2 \, \|f\|^2_{L^2(\Omega)}\,. \tag{2.6.36}$$

Here, $\overline{c}$, c_1 and c_2 are constants of the form:

$$\begin{aligned} \overline{c} &= c(n, \lambda, \Lambda, i_b, s_b) \, \frac{1}{(1+L)^4} \\ c_1 &= c(n, \lambda, \Lambda, i_b, s_b) \, \frac{d_\Omega^{p(n)} \, (1+L_\Omega)^{n+2}}{\overline{r}^{(2n+2)(n+2)}} \\ c_2 &= c(n, \lambda, \Lambda, i_b, s_b) \, \frac{d_\Omega^{p(n)+n} \, (1+L_\Omega)^{n+2}}{\overline{r}^{(2n+3)(n+2)}}\,, \end{aligned} \tag{2.6.37}$$

where $p(n) = (2n+1)(n+2) + n$.

In order to prove this assertion, let us first consider the families of functions $\{u_\varepsilon\}$ and $\{u_{\varepsilon,m}\}$ defined in Step 1, and let $\varphi \in C_0^\infty(\mathbb{R}^n)$. An application of formula (2.3.33), with $v = u_{\varepsilon,m}$, $h = a_\varepsilon\big(H(\nabla u_{\varepsilon,m}(x))\big)$, and $\phi = \varphi^2$, yields

$$\begin{aligned} &\int_\Omega \operatorname{div}\big(\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m})\big)^2 \varphi^2 \, dx \\ &\quad = \int_\Omega \operatorname{tr}\big((\nabla(\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m})))^2\big) \, dx + \int_{\partial\Omega} a_\varepsilon\big(H(\nabla u_{\varepsilon,m})\big)^2 \, H(\nu) \, H^2(\nabla u_{\varepsilon,m}) \operatorname{tr} \mathcal{B}^H \, \varphi^2 \, d\mathcal{H}^{n-1} \\ &\quad - 2 \int_\Omega \Big\{ \operatorname{div}(\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m})) \, \mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m}) \cdot \nabla \varphi - \nabla(\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m}))\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m}) \cdot \nabla \varphi \Big\} \, \varphi \, dx. \end{aligned} \tag{2.6.38}$$

From (2.6.17) we deduce, via Lemma A, that

$$\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m}) \xrightarrow{m \to \infty} \mathcal{A}_\varepsilon(\nabla u_\varepsilon) \quad \text{in } W^{1,2}(\Omega) \text{ and } C^{0,\theta'}(\overline{\Omega}). \tag{2.6.39}$$

In particular, $\operatorname{div}\big(\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m})\big) \xrightarrow{m \to \infty} f$ in $L^2(\Omega)$. Therefore, passing to the limit as $m \to \infty$ in equation (2.6.38) yields:

$$\begin{aligned} \int_\Omega f^2 \, \varphi^2 \, dx &= \int_\Omega \operatorname{tr}\big((\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon))^2\big) \, dx + \int_{\partial\Omega} a_\varepsilon(H(\nabla u_\varepsilon))^2 \, H(\nu) \, H^2(\nabla u_\varepsilon) \operatorname{tr} \mathcal{B}^H \, \varphi^2 \, d\mathcal{H}^{n-1} + \\ &\quad - 2 \int_\Omega \Big\{ \operatorname{div}\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big) \, \mathcal{A}_\varepsilon(\nabla u_\varepsilon) \cdot \nabla \varphi - \nabla\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)\mathcal{A}_\varepsilon(\nabla u_\varepsilon) \cdot \nabla \varphi \Big\} \, \varphi \, dx. \end{aligned} \tag{2.6.40}$$

We begin by estimating the boundary integral in equation (2.6.40). Let $x \in \partial\Omega$, $r \in (0, \overline{r}]$, and $\varphi \in C_0^\infty(B_r(x))$. Choosing $v = \mathcal{A}^i_\varepsilon(\nabla u_\varepsilon)\,\varphi$ in inequality (2.6.6) and summing over $i = 1, \dots, n$ imply that

(2.6.41)

$$\int_{\partial\Omega\cap B_r(x)} \Big|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,\varphi\Big|^2 \big|\mathrm{tr}\,\mathcal{B}^H\big|\,d\mathcal{H}^{n-1} \le c_0\,(1+L_\Omega)^4\,\mathcal{K}_\Omega(r)\int_{\Omega\cap B_r(x)} \Big|\nabla\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,\varphi\big)\Big|^2\,dx$$
$$\le 2\,c_0\,(1+L_\Omega)^4\,\mathcal{K}_\Omega(r)\bigg\{\int_{\Omega\cap B_r(x)} \Big|\nabla\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)\Big|^2\varphi^2\,dx + \int_{\Omega\cap B_r(x)} \big|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big|^2\,|\nabla\varphi|^2\,dx\bigg\}.$$

Observe that, by equation (1.2.12) and the definition of $\mathcal{A}_\varepsilon$, we have $H_0(\mathcal{A}_\varepsilon(\xi)) = a_\varepsilon(H(\xi))\,H(\xi)$ for $\xi \neq 0$. Also, owing to the second inequality in (1.2.9), $H(\nu) \le \sqrt{\Lambda}$. Thus, from inequality (2.6.41), we deduce, via (1.2.11), that

(2.6.42)

$$\int_{\partial\Omega} a_\varepsilon(H(\nabla u_\varepsilon))^2\,H(\nu)\,H^2(\nabla u_\varepsilon)\,\mathrm{tr}\,\mathcal{B}^H\,\varphi^2\,d\mathcal{H}^{n-1}$$
$$= \int_{\partial\Omega} H_0\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)^2\,H(\nu)\,\mathrm{tr}\,\mathcal{B}^H\,\varphi^2\,d\mathcal{H}^{n-1} \le \frac{\sqrt{\Lambda}}{\lambda}\int_{\partial\Omega}\big|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big|^2\,\big|\mathrm{tr}\,\mathcal{B}^H\big|\,\varphi^2\,d\mathcal{H}^{n-1}$$
$$\le \frac{2\,c_0\,(1+L_\Omega)^4\sqrt{\Lambda}}{\lambda}\,\mathcal{K}_\Omega(r)\bigg\{\int_{\Omega\cap B_r(x)}\Big|\nabla\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)\Big|^2\varphi^2\,dx + \int_{\Omega\cap B_r(x)}\big|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big|^2\,|\nabla\varphi|^2\,dx\bigg\}.$$

Next, we use Young's inequality to bound the last integral on the right-hand side of inequality (2.6.40) and obtain

$$\bigg|2\int_\Omega \Big\{\mathrm{div}\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)\,\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\varphi - \nabla\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\varphi\Big\}\,\varphi\,dx\bigg|$$
$$\le \gamma\int_\Omega |\nabla\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)|^2\,\varphi^2\,dx + \frac{c_1}{\gamma}\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,|\nabla\varphi|^2\,dx, \tag{2.6.43}$$

for some constant $c_1 = c_1(n)$ and every $\gamma > 0$. A combination of (2.6.40), (2.6.42), and (2.6.43) enables us to deduce, via inequality (2.4.12), that

(2.6.44)

$$\bigg(\Big(\frac{\lambda\,\min\{1,i_b\}}{\Lambda\,\max\{1,s_b\}}\Big)^2 - \frac{2\,c_0\,(1+L_\Omega)^4\sqrt{\Lambda}}{\lambda}\,\mathcal{K}_\Omega(r) - \gamma\bigg)\int_\Omega |\nabla\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big)|^2\,\varphi^2\,dx \le$$
$$\le \int_\Omega f^2\,\varphi^2\,dx + \Big(\frac{2\,c_0\,(1+L_\Omega)^4\sqrt{\Lambda}}{\lambda}\,\mathcal{K}_\Omega(r) + \frac{c}{\gamma}\Big)\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,|\nabla\varphi|^2\,dx.$$

Now we choose

$$\gamma = \frac{1}{2}\Big(\frac{\lambda\,\min\{1,i_b\}}{\Lambda\,\max\{1,s_b\}}\Big)^2,$$

and assume that (2.6.35) is in force with

$$\overline{c} = \frac{1}{8}\left(\frac{\lambda\,\min\{1,i_b\}}{\Lambda\,\max\{1,s_b\}}\right)^2\frac{\lambda}{\sqrt{\Lambda}}\,\frac{1}{c_0\,\big(1+L\big)^4}, \tag{2.6.45}$$

where $c_0 = c_0(n, \lambda, \Lambda)$ is the constant appearing in (2.6.6). With this choice of the constants, from (2.6.44) we obtain that

$$\int_\Omega |\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))|^2 \varphi^2\, dx \le c_2 \int_\Omega f^2 \varphi^2\, dx + c_2 \int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 |\nabla\varphi|^2\, dx \tag{2.6.46}$$

for some constant $c_2 = c_2(n, \lambda, \Lambda, i_b, s_b)$ and for every $r \in (0, \overline{r}]$, $x \in \partial\Omega$ and $\varphi \in C_0^\infty(B_r(x))$.
On the other hand, inequality (2.6.46) continues to hold if $B_r(x) \subset\subset \Omega$, since the boundary integral in (2.6.40) simply vanishes in this case.
Let us now chose a finite covering of Ω by balls

$$B_{\overline{r}/4}(x_j), \text{ with } x_j \in \partial\Omega,\ j = 1, \dots, N_B$$

and

$$B_{\overline{r}/40}(z_i), \text{ with } z_i \in \Omega \text{ and } B_{\overline{r}/10}(z_i) \subset\subset \Omega,\ i = 1, \dots, N_I,$$

where $N_B \in \mathbb{N}$ and $N_I \in \mathbb{N}$. Notice that such a covering can be chosen in such a way its cardinality $N = N_B + N_I$ admits the bound

$$N \le c(n) \left(\frac{d_\Omega}{\overline{r}}\right)^n. \tag{2.6.47}$$

Denote by B_k, with $k = 1, \dots, N$, a generic ball from this covering, and let $\{\varphi_k\}_{k=1,\dots,N}$ be a family of functions $\varphi_k \in C_0^\infty(4B_k)$, such that $0 \le \varphi_k \le 1$ on $4B_k$,

$$\varphi_k = 1 \quad \text{on } B_k \quad \text{and} \quad |\nabla\varphi_k| \le \frac{80}{\overline{r}} \quad \text{on } 4B_k,$$

where $4B_k$ denotes the ball having the same center as B_k and whose radius is four times the radius of B_k.
Applying inequality (2.6.46) with $\varphi = \varphi_k$, for $k = 1, \dots, N$, and adding the resultant inequalities yields:

$$\begin{aligned}\int_\Omega \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2 dx &\le \sum_{k=1}^N \int_{\Omega\cap B_k} \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2 dx \le \sum_{k=1}^N \int_{\Omega\cap 4B_k} \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2 \varphi_k^2\, dx \\ &= \sum_{k=1}^N \int_\Omega \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2 \varphi_k^2\, dx \le \sum_{k=1}^N c_2 \int_\Omega f^2 \varphi_k^2\, dx + \sum_{k=1}^N c_2 \int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 |\nabla\varphi_k|^2\, dx \\ &\le N c_2 \int_\Omega f^2\, dx + \frac{6400\, N\, c_2}{\overline{r}^2} \int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\, dx.\end{aligned} \tag{2.6.48}$$

Lemma 2.6.2 ensures that

$$\begin{aligned}&\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\, dx \\ &\le \sigma \int_\Omega \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2 dx + c(n) \frac{(1+\sigma)^2}{\sigma} \frac{d_\Omega^{2n(n+2)}}{r^{(2n+1)(n+2)}} (1 + L_\Omega)^{n+2} \left(\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|\, dx\right)^2\end{aligned} \tag{2.6.49}$$

for every $\sigma > 0$ and $r \in (0, R_\Omega)$. Owing to [57, Proposition 5.1], the last integral on the right-hand side of inequality (2.6.49) can be bounded by a constant $c = c(n, \lambda, \Lambda, i_b, s_b)$ times $|\Omega|^{1/n} \int_\Omega |f|\, dx$– see inequality (2.6.2) above. Hence, from Hölder's inequality we deduce that

$$\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\, dx \le \sigma \int_\Omega \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2 dx + c_3\, \vartheta(\sigma, n, r, L_\Omega, d_\Omega) \int_\Omega f^2\, dx\,, \tag{2.6.50}$$

for some constant $c_3 = c_3(n, \lambda, \Lambda, i_b, s_b)$, where

$$\vartheta(\sigma, n, r, L_\Omega, d_\Omega) = \frac{(1+\sigma)^2}{\sigma} \frac{d_\Omega^{(2n+1)(n+2)}}{r^{(2n+1)(n+2)}} (1+L_\Omega)^{n+2}. \tag{2.6.51}$$

Coupling inequalities (2.6.48) and (2.6.50), and making use of the bound from (2.6.47) entail that

$$\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\, dx \le c_4 \Big[\sigma \Big(\frac{d_\Omega}{\overline{r}}\Big)^n + \vartheta\Big] \int_\Omega f^2 dx + \sigma\, c_4 \frac{d_\Omega^n}{\overline{r}^{n+2}} \int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\, dx\,,$$

for some constant $c_4 = c_4(n, \lambda, \Lambda, i_b, s_b)$, which can be assumed to be larger than 1. Now choose $\sigma > 0$ in the above expression in such a way that

$$\sigma\, c_4 \frac{d_\Omega^n}{\overline{r}^{n+2}} = \frac{1}{2}$$

and observe that $\sigma < 1$ and $\vartheta(\sigma, n, r, L_\Omega, d_\Omega) > 1$ since $r < \overline{r} \le R_\Omega \le d_\Omega$ and $c_4 > 1$. Then, from definition (2.6.51) to deduce that

$$\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\, dx \le (1 + 2c_4\vartheta) \int_\Omega f^2\, dx \le 32 c_4^2 \frac{d_\Omega^{(2n+1)(n+2)+n}}{\overline{r}^{n+2}} \frac{(1+L_\Omega)^{n+2}}{r^{(2n+1)(n+2)}} \int_\Omega f^2\, dx \tag{2.6.52}$$

for every $r \in (0, \overline{r})$. On the other hand, from inequalities (2.6.47), (2.6.48) and (2.6.52) one can deduce that

$$\int_\Omega \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\big|^2 dx \le c_5 \frac{d_\Omega^{(2n+1)(n+2)+2n}}{\overline{r}^{2(n+2)}} \frac{(1+L_\Omega)^{n+2}}{r^{(2n+1)(n+2)}} \int_\Omega f^2\, dx\,, \tag{2.6.53}$$

for some constant $c_5 = c_5(n, \lambda, \Lambda, i_b, s_b)$ and for every $r \in (0, \overline{r})$.
The choice $r = \frac{\overline{r}}{2}$ in (2.6.52) and (2.6.53) implies that

$$\|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\|_{W^{1,2}(\Omega)} \le c(n, \lambda, \Lambda, i_b, s_b, L, d, \overline{r}, M).$$

Combining the latter inequality with (2.4.8) and (2.6.15) entails that there exists a sequence ε_k such that

$$\mathcal{A}_{\varepsilon_k}(\nabla u_{\varepsilon_k}) \rightharpoonup \mathcal{A}(\nabla u) \quad \text{weakly in } W^{1,2}(\Omega).$$

Estimate (2.6.36) thus follows by choosing $\varepsilon = \varepsilon_k$ and $r = \frac{\overline{r}}{2}$ in inequalities (2.6.52), (2.6.53) and passing to the limit as $k \to \infty$.

Step 3. Our task in this step is to remove assumption (2.6.12), while maintaining (2.6.11). To this purpose, let us extend f to the whole of $\mathbb{R}^n$ by setting $f = 0$ outside Ω.

Next, by using the results of Theorem 3.2.1, we may find positive constants $\widehat{c} = \widehat{c}(n, \mathfrak{L}_\Omega, d_\Omega)$, $\widehat{r} = \widehat{r}(n, \mathfrak{L}_\Omega, d_\Omega) < 1$, and a sequence $\{\Omega_m\}$ of open sets of $\mathbb{R}^n$ such that:
$\partial\Omega_m \in C^\infty$, $\Omega \Subset \Omega_m$, $\lim_{m\to\infty} |\Omega_m \setminus \Omega| = 0$, the Hausdorff distance between Ω_m and Ω tends to 0 as $m \to \infty$,

$$L_{\Omega_m} \le \widehat{c}, \quad R_{\Omega_m} \ge 1/\widehat{c}, \quad d_{\Omega_m} \le c(n)\, d_\Omega, \tag{2.6.54}$$

and

$$\mathcal{K}_{\Omega_m}(r) \le \begin{cases} \widehat{c}\Big(\mathcal{K}_\Omega\big(\widehat{c}\,(r + \frac{1}{m})\big) + r\Big) & \text{if } n \ge 3 \\[2ex] \widehat{c}\Big(\mathcal{K}_\Omega\big(\widehat{c}\,(r + \frac{1}{m})\big) + r\, \log(1 + \frac{1}{r})\Big) & \text{if } n = 2 \end{cases} \tag{2.6.55}$$

for $m \in \mathbb{N}$ and $r \in (0, \widehat{r})$.
Now let u_m be the weak solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}(\nabla u_m)\big) = f & \text{in } \Omega_m \\ u_m = 0 & \text{on } \partial\Omega_m. \end{cases} \tag{2.6.56}$$

Set $L = \widehat{c}$, $d = c(n)\, d_\Omega$ and $M = \|f\|_{L^2(\Omega)}$ in Step 2, and assume that condition (2.1.15) is fulfilled with

$$\kappa_1 = \kappa_1(n, \lambda, \Lambda, i_b, s_b, \mathfrak{L}_\Omega, d_\Omega) = \overline{c}/(2\widehat{c}),$$

where $\overline{c}$ is the constant defined by (2.6.37) in Step 2.
This piece of information, combined with (2.6.55), implies that there exist a positive real number $\overline{r} = \overline{r}(\Omega) < \min\left\{1/\widehat{c}, \widehat{r}\right\} < 1$ and a positive integer $\overline{m} = \overline{m}(\Omega)$ such that

$$\mathcal{K}_{\Omega_m}(r) \le \overline{c}$$

for $r \in (0, \overline{r})$ and $m > \overline{m}$.
Hence, we may apply the result of Step 2 to problem (2.6.56), and obtain

$$\|\mathcal{A}(\nabla u_m)\|_{W^{1,2}(\Omega)} \le \|\mathcal{A}(\nabla u_m)\|_{W^{1,2}(\Omega_m)} \le c\, \|f\|_{L^2(\Omega_m)} = c\, \|f\|_{L^2(\Omega)} \tag{2.6.57}$$

for some constant $c = c(i_b, s_b, \lambda, \Lambda, \Omega)$. Consequently, there exist a subsequence of $\{u_m\}$, still indexed by m, and a vector-valued function $U : \Omega \to \mathbb{R}^n$ such that $U \in W^{1,2}(\Omega)$ and

$$\mathcal{A}(\nabla u_m) \rightharpoonup U \quad \text{weakly in } W^{1,2}(\Omega). \tag{2.6.58}$$

Via an analogous argument as in the proof of inequality (2.5.19), one infers from [184, Theorem 2] that there exists a constant c, independent on m, such that

$$\|u_m\|_{L^\infty(\Omega_m)} \le c. \tag{2.6.59}$$

Thereby, thanks to [136, Theorem 1.7], given any $\Omega' \subset\subset \Omega$, there exist $\theta \in (0,1)$ and a constant c independent of m, such that $\|u_m\|_{C^{1,\theta}(\Omega')} \le c$. Hence, there exist a further subsequence, still denoted by $\{u_m\}$, and a function $v \in C^{1,\theta}_{\mathrm{loc}}(\Omega)$ such that

$$u_m \to v \quad \text{in } C^{1,\theta'}_{\mathrm{loc}}(\Omega) \tag{2.6.60}$$

for every $0 < \theta' < \theta$. Owing to (2.6.58), this implies that $A(\nabla v) = U$, whence

$$\mathcal{A}(\nabla u_m) \rightharpoonup \mathcal{A}(\nabla v) \quad \text{weakly in } W^{1,2}(\Omega). \tag{2.6.61}$$

On passing to the limit as $m \to \infty$ in the weak formulation of problem (2.6.56), from (2.6.61) we infer that

$$\int_\Omega \mathcal{A}(\nabla v) \cdot \nabla\varphi\, dx = \int_\Omega f\, \varphi\, dx \tag{2.6.62}$$

for every $\varphi \in C^\infty_0(\Omega)$.
Now, consider a ball B_R such that $\Omega \subset\subset B_R$ and extend u_m to B_R by setting $u_m = 0$ in $B_R \setminus \Omega_m$. Since $u_m \in W^{1,B}_0(\Omega_m)$, such an extension belongs to $W^{1,B}_0(B_R)$. By the minimality property of the

function u_m for the functional associated with problem (2.6.56), the fact that $f = 0$ in $B_R \setminus \Omega$, and inequalities (2.2.22) and (2.6.59), we have that

$$c_1 \int_{B_R} B(|\nabla u_m|)\, dx \le \int_{B_R} B\big(H(\nabla u_m)\big)\, dx \le \int_{\Omega_m} f\, u_m\, dx \le c_2$$

for suitable positive constants c_1 and c_2 independent of m. Hence, via a Poincaré type inequality for functions in the space $W^{1,B}_0(B_R)$, the sequence $\{u_m\}$ is bounded in $W^{1,B}_0(B_R)$. The reflexivity of this space and the compactness of the embedding of this space into $L^1(\Omega)$ entail that there exists a subsequence, again still denoted by $\{u_m\}$, and a function $w \in W^{1,B}_0(B_R)$ such that

$$u_m \rightharpoonup w \quad \text{weakly in } W^{1,B}_0(B_R) \quad and \quad u_m \to w \quad \text{a.e. in } B_R.$$

Since the Hausdorff distance between Ω_m and Ω tends to zero, and $u_m = 0$ in $B_R \setminus \Omega_m$, we have that $w = 0$ almost everywhere in $B_R \setminus \Omega$. Inasmuch as $w = v$ in Ω, we can conclude that $v \in W^{1,B}_0(\Omega)$. As in the previous steps, a density argument now ensures that (2.6.62) holds for any function $\varphi \in W^{1,B}_0(\Omega)$, and hence v is a weak solution to problem (2.1.2). The uniqueness of such a solution implies that $u = v$. Passing to the limit as $m \to \infty$ in (2.6.57), and recalling (2.6.61) yield (2.1.16).

<u>Step 4.</u> We conclude the proof by removing the remaining additional assumption (2.6.11). Suppose that $f \in L^2(\Omega)$ and let $\{f_k\} \subset C^\infty_0(\Omega)$ be any sequence such that $f_k \to f$ in $L^2(\Omega)$. Let $\{u_k\}$ be the sequence of weak solutions to the Dirichlet problems

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}(\nabla u_k)\big) = f_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6.63}$$

Thanks to property (2.6.3), one has that

$$u_k \to u \quad \text{and} \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega. \tag{2.6.64}$$

By Step 3, there exists a constant $c = c(i_b, s_b, \lambda, \Lambda, \Omega)$, such that for any $k \ge 1$,

$$\|\mathcal{A}(\nabla u_k)\|_{W^{1,2}(\Omega)} \le c\, \|f_k\|_{L^2(\Omega)} \tag{2.6.65}$$

Since $f_k \to f$ in $L^2(\Omega)$, there exists a subsequence, still indexed by k, satisfying

$$\mathcal{A}(\nabla u_k) \to U \quad \text{in } L^2(\Omega) \quad \text{and} \quad \mathcal{A}(\nabla u_k) \rightharpoonup U \quad \text{weakly in } W^{1,2}(\Omega), \tag{2.6.66}$$

for some function $U : \Omega \to \mathbb{R}^n$ such that $U \in W^{1,2}(\Omega)$. From properties (2.6.64), we infer that $\mathcal{A}(\nabla u) = U$. Hence, $\mathcal{A}(\nabla u) \in W^{1,2}(\Omega)$ and inequality (2.1.16) follows by passing to the limit as $k \to \infty$ in estimate (2.6.65). □

Proof of Theorem 2.1.3, Dirichlet problems. The proof proceeds through the same steps as that of Theorem 2.1.4. We limit ourselves to sketching the necessary changes.

<u>Step 1</u> is unchanged.

<u>Step 2.</u> One has to replace condition (2.6.35) with

$$\Psi_\Omega(r) \le \overline{c}_1 \quad \text{for } r \in (0, \overline{r}], \tag{2.6.67}$$

where the constant $\overline{c}_1$ is given by

$$\overline{c}_1 = \frac{1}{4}\left(\frac{\lambda \min\{1, i_b\}}{\Lambda \max\{1, s_b\}}\right)^2 \frac{\lambda}{\sqrt{\Lambda}}\, \frac{1}{c_0\,(1+L)^{11}}.$$

One then makes use of Part (ii) of Lemma 2.6.1, instead of Part (i), in order to estimate the boundary term in (2.6.41). Inequality (2.6.36) hence follows.

Step 3. Coupling inequality (2.6.55) with (3.5.26) below tells us that there exist constants $\widehat{c} = \widehat{c}(n, \mathfrak{L}_\Omega, d_\Omega)$ and $\widehat{r} = \widehat{r}(n, \mathfrak{L}_\Omega, d_\Omega)$ such that

$$\mathcal{K}_{\Omega_m}(r) \leq \begin{cases} \widehat{c}\Big(\Psi_\Omega\big(\widehat{c}\,(r + \tfrac{1}{m})\big) + r\Big) & \text{if } n \geq 3 \\ \widehat{c}\Big(\Psi_\Omega\big(\widehat{c}\,(r + \tfrac{1}{m})\big) + r\,\log(1 + \tfrac{1}{r})\Big) & \text{if } n = 2 \end{cases} \tag{2.6.68}$$

for $r \in (0, \widehat{r})$. Assume that condition (2.1.12) is in force with constant

$$\kappa_0 = \kappa_0(n, \lambda, \Lambda, i_b, s_b, \mathfrak{L}_\Omega, d_\Omega) = \overline{c}/(2\widehat{c}),$$

where $\overline{c}$ is defined in (2.6.45). From (2.6.68) we infer that there exist constants $\overline{r} = \overline{r}(\Omega)$ and $\overline{m} = \overline{m}(\Omega)$ such that

$$\mathcal{K}_{\Omega_m}(r) \leq \overline{c}$$

for $r \in \big(0, \overline{r}(\Omega)\big)$ and $m > \overline{m}(\Omega)$. Therefore, starting from estimate (2.6.57), one can now conclude as in the proof Step 3 of Theorem 2.1.4.

Step 4 is unchanged. □

Proof of Theorem 2.1.2, Dirichlet problems. The proof parallels that of Theorems 2.1.4 and 2.1.3. It is indeed simpler, since the boundary terms in the a priori estimates can just be disregarded, thanks to their sign. In what follows, we just point out the necessary variants and simplifications.

Step 1. This step agrees with that of Theorem 2.1.4.

Step 2. The convexity of the set Ω plays a major role in this step. Owing to property (2.3.24), it ensures that $\operatorname{tr}\mathcal{B}^H \geq 0$ on $\partial\Omega$. Therefore, an application of equation (2.3.33), with $v = u_{\varepsilon,m}$, $h = a_\varepsilon\big(H(\nabla u_{\varepsilon,m}(x))\big)$, and $\phi = 1$ tells us that

$$\int_\Omega \operatorname{div}\big(\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m})\big)^2\,dx \geq \int_\Omega \operatorname{tr}\Big(\nabla\big(\mathcal{A}_\varepsilon(\nabla u_{\varepsilon,m})\big)^2\Big)\,dx. \tag{2.6.69}$$

Thanks to (2.6.39), passing to the limit in inequality (2.6.69) as $m \to \infty$ and using inequality (2.4.12) yield:

$$\int_\Omega f^2\,dx \geq \int_\Omega \operatorname{tr}\big((\nabla\mathcal{A}_\varepsilon(\nabla u_\varepsilon))^2\big)\,dx \geq \left(\frac{\lambda\,\min\{1, i_b\}}{\Lambda\,\max\{1, s_b\}}\right)^2 \int_\Omega |\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))|^2\,dx. \tag{2.6.70}$$

In order to estimate the L^2-norm of $\mathcal{A}_\varepsilon(\nabla u_\varepsilon)$, we exploit the fact that, since Ω is a bounded convex domain in $\mathbb{R}^n$, the constant in the Poincaré inequality on Ω depends only on d_Ω and n. Thus, on denoting by $\mathcal{A}_\varepsilon(\nabla u_\varepsilon)_\Omega$ the vector-valued mean value of $\mathcal{A}_\varepsilon(\nabla u_\varepsilon)$ over Ω, we have that

$$\begin{aligned} \int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,dx &\leq 2\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon) - \mathcal{A}_\varepsilon(\nabla u_\varepsilon)_\Omega|^2\,dx + 2\,|\Omega|\,|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)_\Omega|^2 \\ &\leq 2\,c\,d_\Omega^2 \int_\Omega |\nabla\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx + 2\,|\Omega|^{-1}\Big(\int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|\,dx\Big)^2 \end{aligned} \tag{2.6.71}$$

for some constant $c = c(n)$. The following chain holds:

$$\begin{aligned}\text{(2.6.72)}\quad \int_\Omega |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2\,dx &\le 2\,c(n)\,d_\Omega^2 \int_\Omega |\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx + 2\,|\Omega|^{-1} c(n,\lambda,\Lambda,i_b,s_b)\left(|\Omega|^{1/n}\int_\Omega |f|\,dx\right)^2\\ &\le 2\,c(n)\,d_\Omega^2 \int_\Omega |\nabla \mathcal{A}_\varepsilon(\nabla u_\varepsilon)|^2 dx + c(n,\lambda,\Lambda,i_b,s_b)\,d_\Omega^2 \int_\Omega f^2\,dx\\ &\le c'(n,\lambda,\Lambda,i_b,s_b)\,d_\Omega^2 \int_\Omega f^2\,dx\,,\end{aligned}$$

where the first inequality is a consequence of inequality (2.6.71) and of inequality (2.6.2), with $\mathcal{A}$ and u replaced by $\mathcal{A}_\varepsilon$ and u_ε, the second inequality follows via Hölder's inequality and the fact that $|\Omega| \le c(n)\,d_\Omega^n$, and the last one is due to (2.6.70).
Starting from inequalities (2.6.70) and (2.6.72), instead of (2.6.46), estimate (2.1.9) follows via an analogous argument as in the proof of Theorem 2.1.4 .

Step 3. The proof is the same as that of Theorem 2.1.4, save that the approximating domains Ω_m have to be taken convex, and the bounds in (2.1.9), with Ω replaced by Ω_m, have to be used. In order to construct the convex approximating domains Ω_m, one can employ the regularized signed distance ρ of [134, Theorem 1.4]. Since the latter is a concave function, which is smooth outside $\partial\Omega$, the open sets

$$\text{(2.6.73)}\qquad \Omega_m = \{x \in \mathbb{R}^n \,:\, -\rho(x) < 1/m\}$$

satisfy the desired properties.

Step 4. This step is the same as that of Theorem 2.1.4. □

Proof of Theorem 2.1.5, Dirichlet problems. We start by recalling estimate (3.5.28), which will be proven in Chapter 3:

$$\text{(2.6.74)}\qquad \mathcal{K}_\Omega(r) \le \begin{cases} c(n)\,(1+L_\Omega)^5\, r\,\|\mathcal{B}\|_{L^\infty(\partial\Omega)} & \text{if } n \ge 3\\[2ex] c\,(1+L_\Omega)^8\,\omega(r)\big(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)}\big) & \text{if } n = 2,\end{cases}$$

for $r \in (0, R_\Omega)$, where $\omega : (0,\infty) \to [0,\infty)$ denotes the function given by

$$\omega(r) = r\,\log\left(1+\frac{1}{r}\right) \quad \text{for } r \in (0,\infty).$$

Now we apply the result of Step 2 of Theorem 2.1.4 with

$$L = L_\Omega\,, \quad d = d_\Omega\,, \quad M = \|f\|_{L^2(\Omega)}$$

and a suitable $\overline{r} = \overline{r}(n, i_b, s_b, \lambda, \Lambda, L, R_\Omega, \|\mathcal{B}\|_{L^\infty(\partial\Omega)})$ such that inequality (2.6.35) is satisfied. Hence, inequalities (2.6.36) will hold with constants c_1, c_2 now depending only on $n, i_b, s_b, \lambda, \Lambda, L, R_\Omega, d, \|\mathcal{B}\|_{L^\infty(\partial\Omega)}$. Specifically, when $n \ge 3$, then inequality (2.6.35) is fulfilled provided that

$$\overline{r} \le \min\left\{\frac{c}{(1+L)^9\,\|\mathcal{B}\|_{L^\infty(\partial\Omega)}}\,, R_\Omega\right\}$$

for a suitable constant $c = c(n,\lambda,\Lambda,i_b,s_b)$. Thus, the inequalities in (2.6.36) follow, with

$$c_1 = c\,d^{p(n)}\,(1+L)^{(n+2)}\,\max\left\{(1+L)^{t(n)}\,\|\mathcal{B}\|_{L^\infty(\partial\Omega)}^{(2n+2)(n+2)},\ R_\Omega^{-(2n+2)(n+2)}\right\}$$

$$c_2 = c\, d^{p(n)+n}\, (1+L)^{(n+2)}\, \max\left\{(1+L)^{t(n)+9(n+2)}\, \|\mathcal{B}\|_{L^\infty(\partial\Omega)}^{(2n+3)(n+2)},\, R_\Omega^{-(2n+3)(n+2)}\right\},$$

where $p(n) = (2n+1)(n+2)+n$, $t(n) = 9\,(n+2)(2n+2)$ and $c = c(n, i_b, s_b, \lambda, \Lambda)$.
When $n = 2$, observe that the function ω is increasing and, for every $s_0 \in (0,1)$, there exist constants c_1 and c_2 such that

$$c_1 \frac{s}{\log(1+\frac{1}{s})} \le \omega^{-1}(s) \le c_2 \frac{s}{\log(1+\frac{1}{s})} \quad \text{for } s \in (0, s_0). \tag{2.6.75}$$

Thereby, inequality (2.6.35) holds if

$$\bar{r} \le \min\left\{\frac{c\, \log\big(1 + c(1+L)(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)})\big)}{(1+L)^{12}\,(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)})}\,, R_\Omega\right\}$$

for a suitable constant $c = c(n, \lambda, \Lambda, i_b, s_b)$.

As a consequence, the inequalities in (2.6.36) are fulfilled with

$$c_1 = c'\, d^{22}\, (1+L)^4\, \max\left\{\frac{(1+L)^{288}\,(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)})^{24}}{\log^{24}\big(1+c(1+L)(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)})\big)}\,, R_\Omega^{-24}\right\}$$

$$c_2 = c'\, d^{24}(1+L)^4\, \max\left\{\frac{(1+L)^{336}\,(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)})^{28}}{\log^{28}\big(1+c(1+L)(1+\|\mathcal{B}\|_{L^\infty(\partial\Omega)})\big)}\,, R_\Omega^{-28}\right\}$$

for some constant $c = c(n, i_b, s_b, \lambda, \Lambda)$ and $c' = c'(n, i_b, s_b, \lambda, \Lambda)$. □

2.7 Global estimates: Neumann problems

We conclude with proofs of our global regularity results to Neumann problems of the form (2.1.3). The definition of generalized solutions to these problems can be given in a spirit analogous to that presented for Dirichlet problems in Section 2.6. Assume that $f \in L^2(\Omega)$ and

$$\int_\Omega f\, dx = 0. \tag{2.7.1}$$

A function $u \in \mathcal{T}^{1,1}(\Omega)$ will be called a generalized solution to problem (2.1.3) if $\mathcal{A}(\nabla u) \in L^1(\Omega)$,

$$\int_\Omega \mathcal{A}(\nabla u)\cdot\nabla\varphi\, dx = \int_\Omega f\varphi\, dx \tag{2.7.2}$$

for every $\varphi \in C^\infty(\Omega) \cap W^{1,\infty}(\Omega)$, and there exists a sequence $\{f_k\} \subset C_0^\infty(\Omega)$, with $\int_\Omega f_k(x)\, dx = 0$ for $k \in \mathbb{N}$, such that $f_k \to f$ in $L^2(\Omega)$ and the sequence of (suitably normalized by additive constants) weak solutions $\{u_k\}$ to the problem (2.1.3), with f replaced by f_k, satisfies

$$u_k \to u \quad \text{a.e. in } \Omega.$$

Owing to [57, Theorem 3.8], if Ω is a bounded Lipschitz domain, then there exists a unique (up to additive constants) generalized solution u to problem (2.1.3), and

$$\int_\Omega |\mathcal{A}(\nabla u)|\, dx \le c\, |\Omega|^{1/n} \int_\Omega |f|\, dx \tag{2.7.3}$$

for some constant $c = c(n, \lambda, \Lambda, i_b, s_b)$. Moreover, if $\{f_k\}$ is any sequence as above, and $\{u_k\}$ is the associated sequence of (normalized) weak solutions, then

$$u_k \to u \quad \text{and} \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega, \tag{2.7.4}$$

up to subsequences.

Recall that a function $u \in W^{1,B}(\Omega)$ is called a weak solution to the Neumann problem (2.1.3) if equation (2.7.2) holds for every function $\varphi \in W^{1,B}(\Omega)$. If Ω is a bounded Lipschitz domain and $f \in L^\infty(\Omega)$ and fufills condition (2.7.1), then one can conclude as in [53, Theorems 2.13 and 2.14] that there exists a unique (up to additive constants) weak solution u to the Neumann problem (2.1.3). Moreover,

$$\int_\Omega B(|\nabla u|)\, dx \le c\|f\|_{L^\infty(\Omega)} b^{-1}(\|f\|_{L^\infty(\Omega)})$$

for some constant $c = c(|\Omega|, i_b, s_b, \lambda, \Lambda)$.

Proof of Theorem 2.1.4, Neumann problems. We split the proof into steps.
Step 1. We begin by imposing the additional assumptions that

$$f \in C_0^\infty(\Omega) \tag{2.7.5}$$

and

$$\partial\Omega \in C^2. \tag{2.7.6}$$

For every $\varepsilon \in (0, 1)$, let $\mathcal{A}_\varepsilon$ be the function defined by in (2.4.3). Let u_ε the (unique up to additive constants) weak solution to the Neumann problem

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big) = f & \text{in } \Omega \\ \mathcal{A}_\varepsilon(\nabla u_\varepsilon) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7.7}$$

We claim that

$$u_\varepsilon \in W^{2,2}(\Omega) \cap C^{1,\theta}(\overline{\Omega}), \tag{2.7.8}$$

and the solutions u_ε can be defined with suitable additive constants in such a way that there exists a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0^+$ and

$$u_{\varepsilon_k} \to u \quad \text{in } C^{1,\theta'}_{\mathrm{loc}}(\Omega). \tag{2.7.9}$$

To this purpose, for $\delta > 0$ consider the (unique up to additive constants) solution $w_{\varepsilon,\delta} \in W^{1,2}(\Omega)$ to the problem

$$\begin{cases} -\mathrm{div}\big(\mathcal{A}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\big) = f & \text{in } \Omega \\ \mathcal{A}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7.10}$$

where the function $\mathcal{A}_{\varepsilon,\delta}$ is defined as in (2.6.18).
An application of [50, Theorem 3.1 (a)] ensures that the solutions u_ε and $w_{\varepsilon,\delta}$ can be chosen with proper additive constants so that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le c \quad \text{and} \quad \|w_{\varepsilon,\delta}\|_{L^\infty(\Omega)} \le c \tag{2.7.11}$$

for some contant c independent of δ and ε.
Thanks to the latter inequality, an application of [135, Theorem 2] tells us that there exists a constant c, independent of δ, such that

$$\|w_{\varepsilon,\delta}\|_{C^{1,\theta}(\overline{\Omega})} \le c\,. \tag{2.7.12}$$

On the other hand, via an analogous argument as in the proof of [20, Theorem 8.2], adapted to (homogeneous) Neumann boundary condition, one can show that

$$\|w_{\varepsilon,\delta}\|_{W^{2,2}(\Omega)} \le c \tag{2.7.13}$$

for some constant c independent of δ. Bounds (2.7.12) and (2.7.13) imply that there exist a sequence $\{\delta_k\}$ and a function $w_\varepsilon \in C^{1,\theta'}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ such that $\delta_k \to 0^+$,

$$w_{\varepsilon,\delta_k} \to w_\varepsilon \quad \text{in } C^{1,\theta'}(\overline{\Omega}), \text{ and } \quad w_{\varepsilon,\delta_k} \rightharpoonup w_\varepsilon \quad \text{weakly in } W^{2,2}(\Omega).$$

Passing to the limit in the weak formulation of problem (2.7.10) as $k \to \infty$ shows that w_ε is a solution to problem (2.7.7). Hence, $w_\varepsilon = u_\varepsilon$, up to additive constants. Property (2.7.8) is thus established.
Next, the first inequality in (2.7.11) enables one to apply [136, Theorem 7] and infer that, for every open set $\Omega' \subset\subset \Omega$, there exists a constant c independent of ε such that

$$\|u_\varepsilon\|_{C^{1,\theta}(\Omega')} \le c.$$

Hence, there exist a function $v \in C^{1,\theta'}_{\mathrm{loc}}(\Omega)$ and a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ and

$$u_{\varepsilon_k} \to v \quad \text{in } C^{1,\theta'}(\Omega')\,,$$

for every $\theta' \in (0,\theta)$ and every open set $\Omega' \subset\subset \Omega$.
Via a similar argument as in the proof of equation (2.6.32) one can show that $v \in W^{1,B}(\Omega)$. Hence, passing to the limit as $k \to \infty$ in the weak formulation of problem (2.7.7) with $\varepsilon = \varepsilon_k$, implies that v is a solution to the Neumann problem (2.1.3). Thus, $v = u + c$ for some constant c, and (2.7.9) follows.

Step 2. Assume that hypotheses (2.7.5) and (2.7.6) are still satisfied. The following identity holds for $\varepsilon \in (0,1)$, and is a consequence of [111, Theorem 3.1.1.1]:

$$\begin{aligned} \int_\Omega f^2\, \phi\, dx = \int_\Omega \mathrm{tr}\big((\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)))^2\big)\phi\, dx + \int_{\partial\Omega} \mathcal{B}\Big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)_T, \,\mathcal{A}_\varepsilon(\nabla u_\varepsilon)_T\Big)\, \phi\, d\mathcal{H}^{n-1} \\ - \int_\Omega \Big\{ \mathrm{div}\,(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\, \mathcal{A}_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\phi - \nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon))\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\phi \Big\}\, dx. \end{aligned} \tag{2.7.14}$$

Let us incidentally note the latter identity could also be deduced from Lemma 2.3.2, via an approximation argument. This identity plays a role in the Neumann problem parallel to that of (2.6.40) in the Dirichlet problem. Since $\mathcal{A}_\varepsilon(\xi) \in C^{0,1}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$, and $u_\varepsilon \in W^{2,2}(\Omega)$, by the chain rule for vector-valued functions [145], we have that

$$\mathcal{A}_\varepsilon(\nabla u_\varepsilon) \in W^{1,2}(\Omega).$$

Now, let $x \in \partial\Omega$ and $r \in (0, R_\Omega)$, and choose $\phi = \varphi^2$, with $\varphi \in C^\infty_0(B_r(x))$ in identity (2.7.14). From inequality (2.6.6), appplied with $v = \mathcal{A}^i_\varepsilon(\nabla u_\varepsilon)\,\varphi$, and Young's inequality one deduces that

$$\Big| \int_{\partial\Omega} \mathcal{B}\Big(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)_T, \,\mathcal{A}_\varepsilon(\nabla u_\varepsilon)_T\Big)\, \varphi^2\, d\mathcal{H}^{n-1} \Big| \le \int_{\partial\Omega} |\mathcal{B}|\, |\mathcal{A}_\varepsilon(\nabla u_\varepsilon)_T|^2\, \varphi^2\, d\mathcal{H}^{n-1} \tag{2.7.15}$$

$$\leq c_0\,(1+L_\Omega)^4\,\mathcal{K}_\Omega(r)\int_\Omega \big|\nabla(\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\,\varphi)\big|^2\,dx$$
$$\leq 2\,c_0\,(1+L_\Omega)^4\,\mathcal{K}_\Omega(r)\bigg\{\int_{\Omega\cap B_r(x)}\Big|\nabla\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\Big|^2\varphi^2\,dx+\int_{\Omega\cap B_r(x)}\big|\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big|^2\,|\nabla\varphi|^2\,dx\bigg\}.$$

Thus, from (2.4.12), (2.6.43), (2.7.14) and (2.7.15) we obtain inequality (2.6.44) again. Starting from that inequality, one can proceed exactly as in the proof for Dirichlet problems and derive inequality (2.1.16) under the current assumptions (2.7.5) and (2.7.6). Just notice that properties (2.7.8) and (2.7.9) have to used in this derivation instead of (2.6.14) and (2.6.15).

Step 3. Here we still assume (2.7.5), but remove the restriction (2.7.6).
To this purpose, extend the function f to $\mathbb{R}^n$ by 0 in $\mathbb{R}^n\setminus\Omega$, and consider a sequence of sets $\{\Omega_m\}$ as in Step 3 of the proof for Dirichlet problems. For each $m\in\mathbb{N}$, let u_m be the unique (up to additive constants) solution to the problem

$$\begin{cases}-\mathrm{div}\big(\mathcal{A}(\nabla u_m)\big)=f & \text{in } \Omega_m\\ \mathcal{A}(\nabla u_m)\cdot\nu=0 & \text{on } \partial\Omega_m.\end{cases}\tag{2.7.16}$$

By [50, Theorem 3.1 (a)] and [136, Theorem 1.7], there exists a sequence of functions $\{u_m\}$, suitably normalized by additive constants and still indexed by m, such that

$$u_m\to v \quad \text{in } C^1_{\rm loc}(\Omega)\,,$$

for some function $v\in C^1(\Omega)$.
An analogous argument as in the proof of Step 3 for Dirichlet problems, which relies upon [53, Theorem 2.14] and [50, Theorem 3.1 (b)], enables one to infer that $v\in W^{1,B}(\Omega)$.
In order to show that v agrees with u, up to an additive constant, it suffices to prove that v solves the Neumann problem (2.1.3). Thanks to properties (2.6.54) and (2.6.55), by Step 2 the sequence $\{\mathcal{A}(\nabla u_m)\}$ is uniformly bounded in $W^{1,2}(\Omega)$, inasmuch as

$$\|\mathcal{A}(\nabla u_m)\|_{W^{1,2}(\Omega)}\leq\|\mathcal{A}(\nabla u_m)\|_{W^{1,2}(\Omega_m)}\leq c\,\|f\|_{L^2(\Omega_m)}=c\,\|f\|_{L^2(\Omega)}\tag{2.7.17}$$

for some constant c independent of m.
Hence, there exists a subsequence of $\{u_m\}$, still indexed by m, such that

$$A(\nabla u_m)\rightharpoonup\mathcal{A}(\nabla v)\quad\text{weakly in } W^{1,2}(\Omega)\,.\tag{2.7.18}$$

Let us extend any test function $\varphi\in W^{1,\infty}(\Omega)$ to a function in $W^{1,\infty}(\mathbb{R}^n)$, and still denote this extension by φ. The definition of weak solution to problem (2.7.16) implies that

$$\int_\Omega f\,\varphi\,dx=\int_{\Omega_m}\mathcal{A}(\nabla u_m)\cdot\nabla\varphi\,dx=\int_\Omega\mathcal{A}(\nabla u_m)\cdot\nabla\varphi\,dx+\int_{\Omega_m\setminus\Omega}\mathcal{A}(\nabla u_m)\cdot\nabla\varphi\,dx\tag{2.7.19}$$

for $m\in\mathbb{N}$. Property (2.7.18) ensures that

$$\lim_{m\to\infty}\int_\Omega\mathcal{A}(\nabla u_m)\cdot\nabla\varphi\,dx=\int_\Omega\mathcal{A}(\nabla v)\cdot\nabla\varphi\,dx.\tag{2.7.20}$$

On the other hand, the fact that $|\Omega_m\setminus\Omega|\to 0$ and the dominated convergence theorem yield

$$\lim_{m\to\infty}\int_{\Omega_m\setminus\Omega}\mathcal{A}(\nabla u_m)\cdot\nabla\varphi\,dx=0.\tag{2.7.21}$$

Combining equations (2.7.19)–(2.7.21) tell us that the function v satifies the equality

$$\int_\Omega \mathcal{A}(\nabla v)\cdot\nabla\varphi\,dx = \int_\Omega f\,\varphi\,dx \tag{2.7.22}$$

for every $\varphi \in W^{1,B}(\Omega)$. Thus, v is a weak solution to the Neumann problem (2.1.3), whence $v = u$, up to additive constants. Inequality (2.1.16) follows via (2.7.17).

Step 4. The remaining additional assumption (2.7.5) can be removed by approximating f by a sequence of smooth functions $\{f_k\}$, via the same argument as in Step 4 of the proof for Dirichlet problems. One has just to choose the sequence in such a way the compatibility condition $\int_\Omega f_k(x)\,dx = 0$ is fulfilled for $k \in \mathbb{N}$. □

Proof of Theorem 2.1.3, Neumann problems. The proof is the same as that of Theorem 2.1.4, save that Part (i) has to be replaced with Part (ii) in the application of Lemma 2.6.1 in Step 2, and equation (2.6.68) has to be used in Step 3 as in the proof of the corresponding Dirichlet problem. □

Proof of Theorem 2.1.2, Neumann problems. The only variants with respect to the proof of Theorem 2.1.4 concern Steps 2 and 3.

Step 2. The convexity assumption on Ω ensures that the quadratic form $\mathcal{B}$ is nonnegative on $\partial\Omega$. Thus, choosing $\phi = 1$ in equation (2.7.14) and exploiting inequality (2.4.12) yield

$$\int_\Omega f^2\,dx \geq \int_\Omega \operatorname{tr}\big((\nabla\,\mathcal{A}_\varepsilon(\nabla u_\varepsilon))^2\big)\,dx \geq \left(\frac{\lambda\,\min\{1,i_b\}}{\Lambda\,\max\{1,s_b\}}\right)^2 \int_\Omega \big|\nabla\mathcal{A}_\varepsilon(\nabla u_\varepsilon)\big|^2\,dx. \tag{2.7.23}$$

This inequality replaces (and simplifies) the use of inequality (2.6.44) in the derivation of (2.1.9) in the case when $f \in C_0^\infty(\Omega)$ and $\partial\Omega \in C^2$.

Step 3. The sole variant here is in that the approximating smooth domains Ω_m have to be chosen convex, as defined as in equation (2.6.73), for instance. □

Proof of Theorem 2.1.5, Neumann problems. Thanks to estimate (2.6.74), the conclusions can be deduced via a slight variant of the proof of Theorem 2.1.4 for Neumann problems. The necessary modifications parallel those mentioned in the proof of the present theorem for Dirichlet problems. The details are omitted for brevity. □

Chapter 3

Smooth approximation of Lipschitz domains, weak curvatures and isocapacitary estimates

3.1 Introduction and definitions

As already mentioned, in the following chapter we carry out an approximation procedure on bounded Lipschitz domains Ω of $\mathbb{R}^n$. We construct two sequences of C^∞-smooth bounded domains $\{\omega_m\}, \{\Omega_m\}$ such that $\omega_m \Subset \Omega \Subset \Omega_m$ for all $m \in \mathbb{N}$, which also satisfy natural covergence properties like, for instance, in the sense of the Lebesgue measure and in the sense of Hausdorff to Ω.

Our construction will allow us to keep track of some geometric quantities of Ω like a Lipschitz characteristic $\mathcal{L}_\Omega = (L_\Omega, R_\Omega)$ and its diameter d_Ω, so that these will comparable to the corresponding ones of its approximating sets ω_m, Ω_m as we have stated in estimates (2.6.54). We recall that the constant R_Ω stands for the radius of the ball domains on which the boundary $\partial\Omega$ can be described as a function of $(n-1)$-variables– the so-called *local boundary chart*– and L_Ω is their Lipschitz constant– we refer to Section 3.1.1 for the precise definition of a Lipschitz characteristic of Ω and its properties.

The smooth charts locally describing the boundaries $\partial\omega_m, \partial\Omega_m$ will be defined on the same reference systems as the local charts describing $\partial\Omega$, and owing to the Lipschitz continuity of the latter boundary, we will also have strong convergence of the local charts in the Sobolev space $W^{1,p}$ for all $p \in [1, \infty)$.

Furthermore, if the boundary Ω enjoys additional regularity properties as, for instance, its local boundary charts belong to the Sobolev space $W^{2,q}$ (or they are of class $C^{1,\alpha}$) for some $q \in [1, \infty)$, then the corresponding boundary charts of the approximating sets will also convergence in the $W^{2,q}$-sense (in $C^{1,\alpha'}$ fo all $0 < \alpha' < \alpha$). In a certain way, this means that the second fundamental forms $\mathcal{B}_{\omega_m}$ and $\mathcal{B}_{\Omega_m}$ of the regularized sets converge in L^q to the "weak" curvature $\mathcal{B}_\Omega$ of the initial domain Ω. We refer to Theorems 3.2.1,3.2.2 below.

Concerning its applications, in the previous chapters we have observed that approximating rough domains via a sequence of smooth bounded domains is somewhat necessary when dealing with boundary value problems in Partial Differential Equations. By tackling the same boundary value problem (or its suitable regularization) on smoother domains, accordingly one obtains smoother solutions, hence it is possible to perform all the desired computations and infer a priori estimates which do not depend on the full regularity of the approximating sets Ω_m, but only on some geometric constants of theirs like a Lipschitz characteristics, or other suitable quantities possibly depending on the second fundamental form $\mathcal{B}_{\Omega_m}$.

The notable example was already provided by the weighted isocapacitary function (2.1.14)

$$\mathcal{K}_\Omega(r) = \sup_{\substack{E \subset B_r(x) \\ x \in \partial\Omega}} \frac{\int_{\partial\Omega\cap E} |\mathcal{B}_\Omega| d\mathcal{H}^{n-1}}{\mathrm{cap}(E, B_r(x))} \qquad \text{for } r > 0\,, \tag{3.1.1}$$

used in the characterization of global Sobolev regularity for the Stress field $\mathcal{A}(\nabla u)$ in Chapter 2.

We remark that in order for $\mathcal{K}_\Omega(r)$ to be well defined, it suffices that $\partial\Omega$ is Lipschitz continuous and belongs to $W^{2,1}$, as it can be inferred from inequalities (3.1.11) below.

Before introducing the necessary definitions and stating the main theorems, we briefly review the history of results related to ours, and highlight the differences and the novelties of our methods. Smooth approximation of open sets, not necessarily having Lipschitzian boundary, has been object of study by many authors. To the best of our knowledge, the first author who provided an approximation of this kind is Nečas [164], followed by Massari & Pepe [146] and Doktor [86]. The underlying idea behind their proof is nowadays standard, and it is typically used to approximate sets of finite perimeter. This consists in regularizing the characteristic function of Ω via mollification and convolution, and then define the approximating set Ω_m as a suitable superlevel set of the mollified characteristic functions– see for instance [5, Theorem 3.42] or [140, Section 13.2]. We point out that Schmidt [176] and Gui, Hu & Li [113] constructed smooth approximating domains *strictly contained* in Ω under additional assumptions on the finite perimeter domain Ω, whereas an outer approximation via smooth sets is given by Doktor [86] when the domain Ω is endowed with a Lipschitz continuous boundary.

A different kind of approach, which makes use of Stein's regularized distance, has been recently developed by Ball & Zarnescu [18]. Here, the authors deal with C^0 domains, i.e., domains whose boundary can be locally described by merely continuous charts, and hence need not have finite perimeter. We mention that their regularized domains Ω_ε are defined as the ε-superlevel set of the regularized distance function, which in turn is obtained via mollification of the usual signed distance function. Here, the parameter ε can be taken either positive or negative, according to the preferred method of approximation, whether from the inside or outside of Ω.

The aforementioned techniques have thus been used to treat domains with "rough" boundaries; however, they do not seem suitable to approximate domains which possess weakly defined curvatures, even in the case of domains having bounded curvatures, e.g., $\partial\Omega \in C^{1,1}$. Namely, we do not recover any quantitative information or convergence property regarding the second fundamental forms $\mathcal{B}_{\Omega_m}$ from the original one $\mathcal{B}_\Omega$. This is because first-order estimates regarding Ω_m are proven by a careful pointwise analysis of the gradient of the local charts describing their boundaries. In order to obtain estimates about their second fundamental form $\mathcal{B}_{\Omega_m}$, such pointwise analysis needs to be extended to second-order derivatives, and this calls for the application of the implicit function theorem, for which Ω is required to be at least of class C^2.

This drawback is probably due to the fact that the above regularization procedures are global in nature, i.e., they are obtained via mollification of functions "globally" describing Ω, like its characteristic function or signed distance, whereas the second fundamental form of hypersurfaces of $\mathbb{R}^n$ is defined via local parametrizations.

Comparatively, our proof relies on techniques which, in a sense, can be deemed as local in nature, since the starting point of our method is the regularization of the functions of $(n-1)$-variables which locally describe $\partial\Omega$. Thus, the approach here propose seems more suitable when dealing with weak curvatures, though at the cost of requiring Ω to have a Lipschitz continuous boundary.

3.1.1 Basic definitions

Here we provide the relevant definitions of use throughout the rest of the chapter.

From here onward, we will denote by $\rho = \rho(x')$ the standard, radially symmetric convolution Kernel in $\mathbb{R}^{n-1}$, i.e.,

$$\rho(x') = \begin{cases} \exp\left\{ -\dfrac{1}{1-|x'|^2} \right\} & \text{if } |x'| < 1 \\ 0 & \text{if } |x'| \geq 1\,, \end{cases}$$

and we shall write

$$\rho_m(x') := m^{n-1}\rho\big(m\,x'\big)$$

for $m \in \mathbb{N}$. Also, given a function $h \in L^1_{loc}(\mathbb{R}^{n-1})$, the convolution operator $M_m(h)$ will be defined as

$$M_m(h)(x') = h * \rho_m(x') = \int_{\mathbb{R}^{n-1}} h(y')\,\rho_m(x'-y')\,dy'\,.$$

We now give the precise definitions of Lipschitz domain and of Lipschitz characteristic.

Definition 3.1.1 (Lipschitz characteristic of a domain)**.** An open set Ω in $\mathbb{R}^n$ is called a Lipschitz domain if there exist constants $L_\Omega > 0$ and $R_\Omega \in (0,1)$ such that, for every $x_0 \in \partial\Omega$ and $R \in (0, R_\Omega]$ there exist an orthogonal coordinate system centered at $0 \in \mathbb{R}^n$ and an L_Ω-Lipschitz continuous function $\phi : B'_R \to (-\ell, \ell)$, where B'_R denotes the ball in $\mathbb{R}^{n-1}$, centered at $0' \in \mathbb{R}^{n-1}$ and with radius R, and

$$\ell = R(1 + L_\Omega), \tag{3.1.2}$$

satisfying

$$\begin{aligned} &\partial\Omega \cap \big(B'_R \times (-\ell,\ell)\big) = \{(x', \phi(x'))\,:\,x' \in B'_R\}, \\ &\Omega \cap \big(B'_R \times (-\ell,\ell)\big) = \{(x', x_n)\,:\,x' \in B'_R\,,\,-\ell < x_n < \phi(x')\}. \end{aligned} \tag{3.1.3}$$

Moreover, we set

$$\mathfrak{L}_\Omega = (L_\Omega, R_\Omega), \tag{3.1.4}$$

and call $\mathfrak{L}_\Omega$ a Lipschitz characteristic of Ω.

Definition 3.1.1 and identities (3.1.3) tell us that in the coordinate cylinder $B'_{R_\Omega} \times (-\ell, \ell)$ centered at a point $x_0 \in \partial\Omega$, we can represent $\partial\Omega$ and Ω as the graph and subgraph of an L_Ω-Lipschitz function ϕ of $(n-1)$-variables, respectively.

It is easily seen that this definition coincides with the standard one for uniformly Lipschitz domains–see, e.g., [116, Section 2.4]. Our definition has the advantage of pointing out $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$, which appears in the characterization of our approximation sets, and was seen in the quantitative estimates of the global regularity results.

Remark. *Generally speaking, a Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$ is not uniquely determined. For instance, if $\partial\Omega \in C^1$, then L_Ω may be taken arbitrarily small, provided that R_Ω is chosen sufficiently small.*

The function ϕ in definition 3.1.1 is typically called *local (boundary) chart.* By Rademacher's theorem, this function is differentiable for $\mathcal{H}^{n-1}$-almost every x', with gradient $\nabla\phi$ bounded by L_Ω. In particular, this implies that any Lipschitz domain Ω admits a tangent plane on $\mathcal{H}^{n-1}$-almost every point of its boundary.

The local chart ϕ naturally endows $\partial\Omega$ of a local parametrization

$$\iota_\phi(x') = \big(x', \phi(x')\big) \tag{3.1.5}$$

under which, whenever ϕ is differentiable at x', a basis of the tangent space at the point $(x', \phi(x'))$ is given by

$$\mathcal{E}_\phi = \left\{ e_i + \frac{\partial \phi(x')}{\partial x'_i} \right\}_{i=1,\dots,n-1} \tag{3.1.6}$$

where $e_i = (0, \dots, 1, \dots, 0)$ is the i-th canonical unit vector of $\mathbb{R}^n$.

Moreover, via such parametrization $\iota_\phi(x')$, the first fundamental form $g = \{g_{ij}\}_{i,j=1}^{n-1}$ can be computed as

$$g_{ij}(x') = \delta_{ij} + \frac{\partial \phi(x')}{\partial x'_i} \frac{\partial \phi(x')}{\partial x'_j}\,, \tag{3.1.7}$$

where δ_{ij} denotes the Kronecker's delta, and x' is a point of differentiability of ϕ. Then, the inverse matrix $g^{-1} = \{g^{ij}\}_{i,j=1}^{n-1}$ can be explictly computed:

$$g^{ij}(x') = \delta_{ij} - \frac{1}{1 + |\nabla \phi(x')|^2} \frac{\partial \phi(x')}{\partial x'_i} \frac{\partial \phi(x')}{\partial x'_j}\,. \tag{3.1.8}$$

For such points $x_0 = \big(x', \phi(x')\big) \in \partial\Omega$, we shall denote by $T_{x_0}\partial\Omega = T_{x'}\partial\Omega$ the tangent space at x^0. From the discussion above, $\partial\Omega$ admits a tangent plane $\mathcal{H}^{n-1}$-almost every point $x_0 \in \partial\Omega$, hence we may want to define a notion of weak second fundamental form which extends the classical one for C^∞-smooth domains of $\mathbb{R}^n$.

For this purpose, we need some additional regularity assumptions on ϕ, and in particular on its second-order derivatives.

Definition 3.1.2 ($W^{2,q}$ domains and weak curvature)**.** Let $q \in [1, \infty)$. We say that a bounded Lipschitz domain Ω is of class $W^{2,q}$ if the local boundary chart ϕ satisfying (3.1.3) belongs to the Sobolev space $W^{2,q}(B'_R)$. If $\phi \in W^{2,\infty}(B'_R)$, we say that $\partial\Omega \in C^{1,1}$ (or $\partial\Omega \in W^{2,\infty}$).
More generally, if $\mathcal{M}$ is a function space containing $L^1(B'_R)$, we say that $\partial\Omega \in W^2\mathcal{M}$ if the local boundary chart $\phi \in W^2\mathcal{M}$.
When $\partial\Omega \in W^{2,1}$, the weak curvature $\mathcal{B}_\Omega$ of $\partial\Omega$ is a bilinear operator $\mathcal{B}_\Omega(x_0) : T_{x_0}\partial\Omega \times T_{x_0}\partial\Omega \to \mathbb{R}$ defined for $\mathcal{H}^{n-1}$-almost every point $x_0 \in \partial\Omega$ such that, under the choice of local parametrization ι_ϕ in (3.1.5), its components $\big\{\mathcal{B}_{ij}\big\}_{i,j=1}^{n-1}$ with respect to the basis $\mathcal{E}_\phi$ in (3.1.6) of $T_{x'}\partial\Omega$ are locally defined as

$$\mathcal{B}_{ij}(x') = -\frac{1}{\sqrt{1 + |\nabla \phi(x')|^2}} \frac{\partial^2 \phi(x')}{\partial x'_i \partial x'_j}\,, \tag{3.1.9}$$

for $\mathcal{H}^{n-1}$-almost every points $x' \in B'_R$ of differentiability of ϕ. Its norm is then given by

$$|\mathcal{B}_\Omega(x')| = \frac{\sqrt{\operatorname{trace}\big((g^{-1}\,\nabla^2\phi)^2\big)}}{\sqrt{1 + |\nabla \phi(x')|^2}}, \tag{3.1.10}$$

where g^{-1} is the inverse matrix of g given by (3.1.8).

The reader may verify that identities (3.1.5)-(3.1.10) concur with the usual ones when $\partial\Omega$ is a smooth hypersurface of $\mathbb{R}^n$–see, e.g., [131, pp. 246-249]. However, these definitions also make sense when ϕ is merely Lipschitz continuous and belongs to the Sobolev space $W^{2,1}$. Indeed, the following inequalities hold true:

$$\frac{|\nabla^2\phi(x')|}{(1+L_\Omega^2)^{3/2}} \le |\mathcal{B}_\Omega(x')| \le |\nabla^2\phi(x')| \,. \tag{3.1.11}$$

In order to prove (3.1.11), recalling (1.2.24)-(1.2.25) above, we have the elementary linear algebra inequalities

$$\lambda_{\min}^2|Y|^2 \le \mathrm{tr}\big((XY)^2\big) \le \lambda_{\max}^2\,|Y|^2\,,$$

for all symmetric matrices X, Y, with X definite positive, where $\lambda_{\min}, \lambda_{\max}$ denote the smallest and largest eigenvalues of X. Then, owing to (3.1.8), we observe that the largest and smallest eigenvalues of the matrix g^{-1} are respectively 1 and $(1+|\nabla\phi|^2)^{-1}$, and since $|\nabla\phi| \le L_\Omega$ we immediately infer (3.1.11). Inequalities (3.1.11) also show that (locally) second fundamental form $\mathcal{B}_\Omega$ is equivalent to the second-order derivatives of the local charts. In particular, we have that $|\mathcal{B}_\Omega| \in L^1(\partial\Omega)$ if $\partial\Omega \in W^{2,1}$.

We close this section by pointing out that the above definitions can be easily extended to more general domains. For instance, for $k \ge 2$ and given a Marcinkiewicz space $\mathcal{M}$ we say that $\partial\Omega \in W^k\mathcal{M}$ if the boundary chart $\phi \in W^k\mathcal{M}(B'_R)$, that is all of its derivatives up to order k belong to the function space $\mathcal{M}$ in B'_R. Similarly, $\partial\Omega \in C^k$ ($\partial\Omega \in C^{k,\alpha}$) if $\phi \in C^k(B'_R)$ ($\phi \in C^{k,\alpha}(B'_R)$).

3.2 Main results

Having dispensed of the necessary definitions and notations, we can now give a precise statement of our main results. This is the content of this section, coupled with a few comments and an outline of the proofs. Our first main result reads as follows.

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, Lipschitz domain, with Lipschitz characteristic $\mathcal{L}_\Omega = (L_\Omega, R_\Omega)$.*
(i) There exist sequences of bounded domains $\{\omega_m\}, \{\Omega_m\}$, such that $\partial\omega_m \in C^\infty$, $\partial\Omega_m \in C^\infty$, and

$$\omega_m \Subset \Omega \Subset \Omega_m \quad \text{for all } m \in \mathbb{N}.$$

Their diameters satisfy

$$d_{\Omega_m} \le c(n)\, d_\Omega\,, \quad d_{\omega_m} \le c(n)\, d_\Omega\,, \tag{3.2.1}$$

the following convergence property hold true

$$\lim_{m\to\infty} |\Omega_m \setminus \Omega| = 0\,, \quad \lim_{m\to\infty} |\Omega \setminus \omega_m| = 0\,, \tag{3.2.2}$$

the Hausdorff distances safisfy

$$\mathrm{dist}_{\mathcal{H}}(\omega_m, \Omega) + \mathrm{dist}_{\mathcal{H}}(\Omega_m, \Omega) \le \frac{12\, L_\Omega\sqrt{1+L_\Omega^2}}{m} \quad \text{for all } m \in \mathbb{N}, \tag{3.2.3}$$

and we may choose their Lipschitz characteristics $\mathcal{L}_{\Omega_m} = (L_{\Omega_m}, R_{\Omega_m})$ and $\mathcal{L}_{\omega_m} = (L_{\omega_m}, R_{\omega_m})$ such that

$$\begin{aligned} &L_{\Omega_m} \le c(n)(1+L_\Omega^2)\,, \quad R_{\Omega_m} \ge R_\Omega/\big(c(n)(1+L_\Omega^2)\big) \\ &L_{\omega_m} \le c(n)(1+L_\Omega^2)\,, \quad R_{\omega_m} \ge R_\Omega/\big(c(n)(1+L_\Omega^2)\big)\,, \quad \text{for all } m \in \mathbb{N}. \end{aligned} \tag{3.2.4}$$

Moreover, the smooth boundaries $\partial\omega_m, \partial\Omega_m$ are described with the help of the same co-ordinate systems as $\partial\Omega$, i.e., there exist finite number of local boundary charts $\{\phi^i\}_{i=1}^N, \{\psi_m^i\}_{i=1}^N$ and $\{\varphi_m^i\}_{i=1}^N$ which describe $\partial\Omega$, $\partial\Omega_m$ and $\partial\omega_m$ respectively, such that for each $i = 1, \dots, N$ the functions $\psi_m^i, \varphi_m^i \in C^\infty$ are defined on the same reference system as ϕ^i, and

$$\psi_m^i \xrightarrow{m\to\infty} \phi^i \quad \text{and} \quad \varphi_m^i \xrightarrow{m\to\infty} \phi^i \quad \text{in } W^{1,p}(B'_{R_\Omega-\varepsilon_0}), \tag{3.2.5}$$

for all $p \in [1, \infty)$, for all $i = 1, \dots, N$, and any fixed constant $\varepsilon_0 \in (0, R_\Omega/2)$.
(ii) If in addition $\partial\Omega \in W^{2,q}$ for some $q \in [1, \infty)$, then

$$\psi_m^i \xrightarrow{m\to\infty} \phi^i \quad \text{and} \quad \varphi_m^i \xrightarrow{m\to\infty} \phi^i \quad \text{in } W^{2,q}(B'_{R_\Omega-\varepsilon_0}), \tag{3.2.6}$$

and there exists a constant $\widehat{c} = \widehat{c}(n, \mathcal{L}_\Omega, d_\Omega)$ such that

$$\mathcal{K}_{\Omega_m}(r) + \mathcal{K}_{\omega_m}(r) \le \begin{cases} \widehat{c}\left\{\mathcal{K}_\Omega\big(\widehat{c}\,(r + \frac{1}{m})\big) + r\right\} & \text{if } n \ge 3 \\ \\ \widehat{c}\left\{\mathcal{K}_\Omega\big(\widehat{c}\,(r + \frac{1}{m})\big) + r\,\log(1 + \frac{1}{r})\right\} & \text{if } n = 2 \end{cases} \tag{3.2.7}$$

for all $m \in \mathbb{N}$ and $r \le r_0(n, \mathcal{L}_\Omega)$.

Let us briefly comment on our result. Part (i) of Theorem 3.2.1 is mostly analogous to [86, Theorem 5.1]; as expected from domains Ω with Lipschitz continuous boundary, the local charts of $\partial\Omega_m, \partial\omega_m$ converge to the corresponding local charts of $\partial\Omega$ in $W^{1,p}$ for all $p \in [1, \infty)$. In particular, by the classical Morrey-Sobolev's embedding Theorems, this entails an "almost Lipschitz convergence", i.e., the local charts ψ_m^i and φ_m^i converge to ϕ^i in every Hölder space $C^{0,\alpha}$ with $\alpha \in (0, 1)$.

The main novelty of our result is given in Part (ii), where information about the second fundamental forms $\mathcal{B}_{\omega_m}$ and $\mathcal{B}_{\Omega_m}$ (or equivalently $\nabla^2\varphi_m^i$ and $\nabla^2\psi_m^i$) is retrieved when $\partial\Omega$ is endowed with a weak curvature. For instance, by definition (3.1.9) and from the results of Theorem 3.2.1, via a standard covering argument it is easy to show that

$$\int_{\partial\Omega_m} |\mathcal{B}_{\Omega_m}|^q d\mathcal{H}^{n-1} \to \int_{\partial\Omega} |\mathcal{B}_\Omega|^q d\mathcal{H}^{n-1} \quad \text{and} \quad \int_{\partial\omega_m} |\mathcal{B}_{\omega_m}|^q d\mathcal{H}^{n-1} \to \int_{\partial\Omega} |\mathcal{B}_\Omega|^q d\mathcal{H}^{n-1}, \tag{3.2.8}$$

for all $q \in [1, \infty)$ such that $\partial\Omega \in W^{2,q}$.

Other than this convergence property, we obtain the isocapacitary estimate (3.2.7), where $\mathcal{K}_\Omega$ and $\mathcal{K}_{\Omega_m}, \mathcal{K}_{\omega_m}$ are the functions defined in (2.1.14) relative to Ω, Ω_m and ω_m, respectively. In the proof of (3.2.7), we will also explicitly write the constant $\widehat{c}$ appearing therein.

Finally, the fixed parameter $\varepsilon_0 \in (0, R_\Omega/2)$ appearing in (3.2.5) and (3.2.6) is purely technical, and does not affect the validity of the convergence results since the boundaries $\partial\Omega$, $\partial\Omega_m$ and $\partial\omega_m$ all share the same coordinate cylinders of the kind $B'_{R_\Omega/2} \times (-\ell, \ell)$, where $\ell = (1 + L_\Omega)\,R_\Omega$.

Outline of the proof. We fix a covering of $\partial\Omega$, with corresponding partition of unity $\{\xi_i\}_i$ and local boundary charts $\{\phi^i\}_i$, which are L_Ω-Lipschitz continuous.

Then we regularize each function ϕ^i via convolution, and add (or subtract) a suitable constant, so that we obtain C^∞-smooth functions $\{\phi_m^i\}_i$ such that $\phi_m^i > \phi^i$ (or $\phi_m^i < \phi^i$).

However, in the original reference system, the graphs of these smooth functions $G_{\phi_m^i}$ are not "glued" together, and thus their union is not the boundary of a domain, unlike the graphs G_{ϕ^i} whose union describes $\partial\Omega$– see Figure 3.1 below.

To overcome this problem, we define a suitable C^∞-smooth function F_m, built upon $\{\phi_m^i\}_i$ and $\{\xi_i\}_i$– see equation (3.6.15) below– and define the regularized set Ω_m as the sublevel set $\{F_m < 0\}$, so that

$$\partial\Omega_m = \{F_m = 0\},$$

and by construction we will have $\omega_m \Subset \Omega \Subset \Omega_m$.
The function F_m is called *boundary defining functions* of Ω_m– see [130, Section 5.4].

In order to show that $\partial\Omega_m$ is a smooth manifold, we prove that the gradient of F_m along the directions of graphicality of ϕ^i is greater than a positive constant depending on L_Ω– see estimate (3.6.21). This property of F_m will be proven by exploiting the so-called *transversality condition* of ϕ^i, which is inherited via convolution by ϕ^i_m as well. Therefore, F_m is strictly monotone along these directions, which entails that its zero-level set $\partial\Omega_m$ is a smooth manifold with local boundary charts ψ^i_m defined on the same reference system as ϕ^i.

Thanks to the properties of convolution, we show that F_m converge to the boundary defining function F of Ω built upon $\{\phi^i\}_i$ and $\{\xi_i\}_i$– see equations (3.6.10) and (3.6.11)– and thus ψ^i_m converge uniformly to ϕ^i.

Then, as in the proof of the implicit function theorem, we differentiate the identity $F_m\big(y', \psi^i_m(y')\big) = 0$, so that we may express the gradient $\nabla\psi^i_m$ (and its Hessian $\nabla^2\psi^i_m$) in terms of $\{\phi^j_m, \nabla\phi^j_m\}_j$ (and $\{\nabla^2\phi^j_m\}_j$), and then (3.2.4), (3.2.5) (and (3.2.6)) will be obtained by exploiting the convergence properties of convolution.

Finally, in order to get the isocapacitary estimate (3.2.7), we make use of the estimates on $|\nabla^2\psi^i_m|$ obtained in the previous steps, as to evaluate weighted Poincaré type quotients of the kind

$$\frac{\int_{\partial\Omega_m} v^2\, |\mathcal{B}_{\Omega_m}|\, d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla v|^2 dx}, \quad v \in C^\infty_c\big(B_r(x^0_m)\big),\, x^0_m \in \partial\Omega_m$$

in terms of the corresponding quotient with weight $|\mathcal{B}_\Omega|$, and then (3.2.7) will follow from the celebrated isocapacitary equivalency Theorem of Maz'ya [147], [150, Theorem 2.4.1] or [170, Propositions 16.1-16.2]

Our next and final result shows the flexibility of our approximation method, which takes into account even higher regularity of the domain Ω.

Theorem 3.2.2. *Under the same notations as Theorem 3.2.1, we have that*

1. *if $\partial\Omega \in C^k$ for some $k \in \mathbb{N}$, then*
$$\psi^i_m \xrightarrow{m\to\infty} \phi^i \quad \text{and} \quad \varphi^i_m \xrightarrow{m\to\infty} \phi^i \quad \text{in } C^k(B'_{R_\Omega-\varepsilon_0});$$
2. *if $\partial\Omega \in C^{k,\alpha}$ for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$, then*
$$\psi^i_m \xrightarrow{m\to\infty} \phi^i \quad \text{and} \quad \varphi^i_m \xrightarrow{m\to\infty} \phi^i \quad \text{in } C^{k,\alpha'}(B'_{R_\Omega-\varepsilon_0}),$$
for all $0 < \alpha' < \alpha$;
3. *if $\partial\Omega \in W^{k,q}$ for some $k \in \mathbb{N}$ and $q \in [1,\infty)$, then*
$$\psi^i_m \xrightarrow{m\to\infty} \phi^i \quad \text{and} \quad \varphi^i_m \xrightarrow{m\to\infty} \phi^i \quad \text{in } W^{k,q}(B'_{R_\Omega-\varepsilon_0}).$$
4. *if $\partial\Omega \in C^{k,1}$ for some $k \in \mathbb{N}$, then*
$$\psi^i_m \xrightarrow{m\to\infty} \phi^i \quad \text{and} \quad \varphi^i_m \xrightarrow{m\to\infty} \phi^i \quad \text{weakly-}* \text{ in } W^{k,\infty}(B'_{R_\Omega-\varepsilon_0}).$$

The proof of Theorem 3.2.2 can be easily carried out by extending the proof and estimates of Theorem 3.2.1 to higher order derivatives, and by using standard compactness theorems such as Ascoli-Arzelá's and weak-$*$ compactness. For this very reason, we decided to omit the proof.

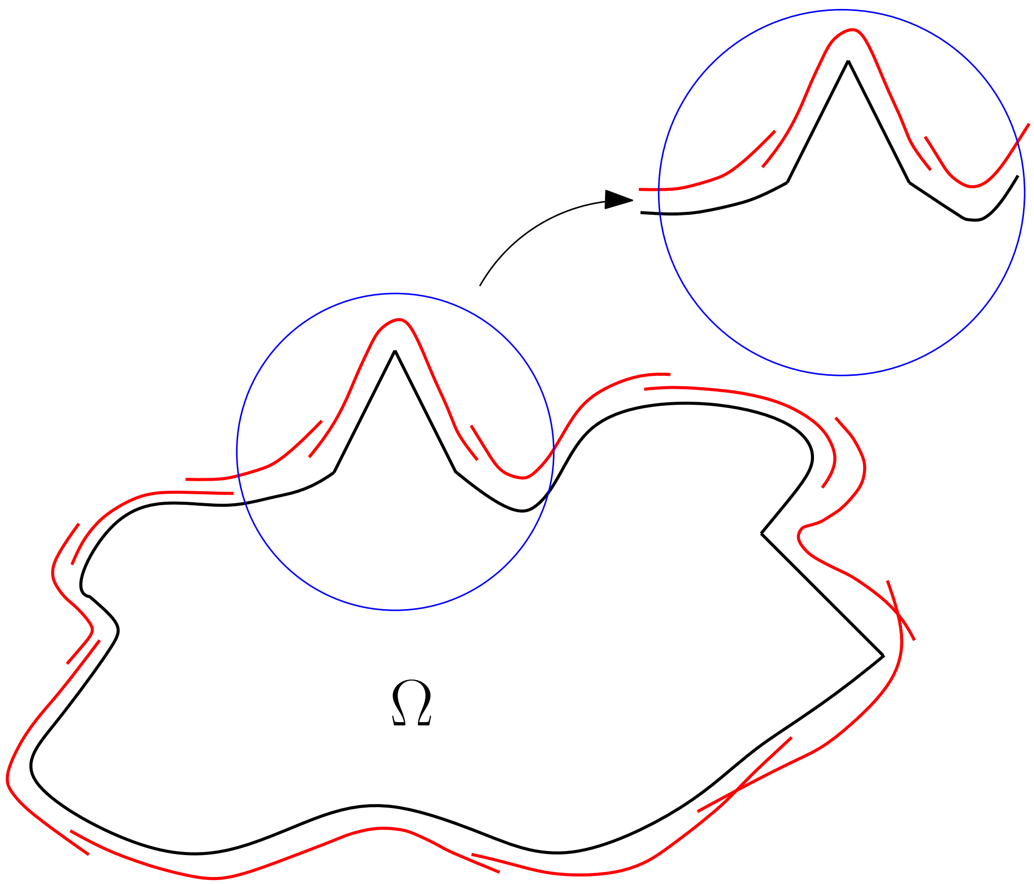

Figure 3.1: In red: the graphs of the regularized local charts (up to isometry)

3.3 Auxiliary results

In this section, we state and prove a useful convergence property regarding the convolution of functions composed with a suitable family of bi-Lipschitz maps.

Proposition 3.3.1. Let $U \subset \mathbb{R}^{n-1}$ be a bounded domain, $K > 0$ be a constant, and $\{\Psi_m\}_{m\in\mathbb{N}}$ be a family of bi-Lipschitz maps on U such that

$$\sup_{m\in\mathbb{N}} \|\nabla\Psi_m^{-1}\|_{L^\infty} \leq K\,, \tag{3.3.1}$$

and there exists a bi-Lipschitz map $\Psi : U \to \Psi(U)$ such that

$$\|\Psi_m - \Psi\|_{L^\infty(U)} \leq \frac{K}{m} \quad \text{for all } m \in \mathbb{N}. \tag{3.3.2}$$

Let $\mathcal{O} \subset \mathbb{R}^{n-1}$ open be such that $\Psi(U) \Subset \mathcal{O}$, and $\phi \in L^p(\mathcal{O})$ for some $p \in [1, \infty)$. Then

$$M_m(\phi) \circ \Psi_m \xrightarrow{m\to\infty} \phi \circ \Psi \quad \mathcal{H}^{n-1}\text{-a.e. in } U \text{ and in } L^p(U). \tag{3.3.3}$$

Proof. Set

$$U_\phi := \big\{x' \in U \,:\, \Psi(x') \text{ is a Lebesgue point of } \phi\big\}$$

By Lebesgue differentiation theorem and since Ψ is a bi-Lipschitz map, we have that U_ϕ is a subset of U with full measure. Also, thanks to (3.3.2) and the fact that $\Psi\big(U\big) \Subset \mathcal{O}$, we have that ϕ and $M_m(\phi)$

are well defined on a neighbourhood of $\Psi_m(U)$ for $m > m_0$ large enough. Then, for all $x' \in U_\phi$ we have

$$\left|M_m(\phi)\big(\Psi_m(x')\big) - \phi\big(\Psi(x')\big)\right| = \left|\int_{B'_{\frac{1}{m}}(\Psi_m(x'))} \Big[\phi(z') - \phi\big(\Psi(x')\big)\Big]\rho_m\big(\Psi_m(x') - z'\big)\,dz\right|$$
$$\leq \big(\sup_{\mathbb{R}^{n-1}} \rho\big)\, m^{n-1} \int_{B'_{\frac{(K+1)}{m}}(\Psi(x'))} \left|\phi(z') - \phi\big(\Psi(x')\big)\right| dz' \xrightarrow{m\to\infty} 0\,.$$

Above we used the fact that $\Psi(x')$ is a Lebesgue point of ϕ, and $B'_{\frac{1}{m}}(\Psi_m(x')) \subset B'_{\frac{(K+1)}{m}}(\Psi(x'))$ as a consequence of (3.3.2).

Now fix $\varepsilon > 0$, and take a function $\widetilde{\phi} \in C_c^\infty(\mathbb{R}^{n-1})$ satisfying

$$\|\phi - \widetilde{\phi}\|^p_{L^p(\mathcal{O})} \leq \varepsilon\,. \tag{3.3.4}$$

Standard properties of convolutions ensure that

$$\|M_m(\widetilde{\phi}) - \widetilde{\phi}\|_{L^\infty(\mathcal{O})} \xrightarrow{m\to\infty} 0\,. \tag{3.3.5}$$

Then we have

$$\begin{aligned} \int_U \left|M_m(\phi)\big(\Psi_m(x')\big) - \phi\big(\Psi(x')\big)\right|^p dx' &\leq c(p) \int_U \left|M_m(\phi - \widetilde{\phi})\big(\Psi_m(x')\big)\right|^p dx' \\ + c(p) \int_U \left|M_m(\widetilde{\phi})\big(\Psi_m(x')\big) - \widetilde{\phi}\big(\Psi(x')\big)\right|^p dx' &+ c(p) \int_U \left|\widetilde{\phi}\big(\Psi(x') - \phi\big(\Psi(x')\big)\right|^p dx' \end{aligned} \tag{3.3.6}$$

By applying Jensen inequality, the change of variables $w' = \Psi_m(x') - z'$ and Fubini-Tonelli's Theorem we obtain

$$\int_U \left|M_m(\phi - \widetilde{\phi})\big(\Psi_m(x')\big)\right|^p dx' \leq \int_U \int_{B'_{1/m}} \left|\phi\big(\Psi_m(x') - z'\big) - \widetilde{\phi}\big(\Psi_m(x') - z'\big)\right|^p \rho_m(z')\,dz'\,dx'$$
$$\leq c(n)\,K^{n-1} \int_{\mathbb{R}^{n-1}} \rho_m(z')\,dz' \int_{\mathcal{O}} \left|\phi(w') - \widetilde{\phi}(w')\right|^p dw' \leq c(n)\,K^{n-1}\,\varepsilon\,,$$

where we also used estimates (3.3.1) and (3.3.4).

Then, by using (3.3.2) and (3.3.5), it is immediate to verify that

$$\lim_{m\to\infty} \int_U \left|M_m(\widetilde{\phi})\big(\Psi_m(x')\big) - \widetilde{\phi}\big(\Psi(x')\big)\right|^p dx' = 0\,,$$

and finally, via a change of variables $y' = \Psi(x')$, and (3.3.4) we get

$$\int_U \left|\widetilde{\phi}\big(\Psi(x') - \phi\big(\Psi(x')\big)\right|^p dx' \leq c(n)\,\|\nabla\Psi^{-1}\|^{n-1}_{L^\infty}\,\varepsilon\,.$$

Henceforth, by plugging the last three estimates into (3.3.6), we find

$$\limsup_{m\to\infty} \int_U \left|M_m(\phi)\big(\Psi_m(x')\big) - \phi\big(\Psi(x')\big)\right|^p dx' \leq c(n,p,L,\Psi)\,\varepsilon\,,$$

and thus (3.3.3) follows by the arbitrariness of ε. □

We close this section recalling a variant of Lebesgue dominated convergence Theorem which will be useful later on.

Theorem 3.3.2 (Dominated convergence Theorem)**.** Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of measurable functions on $E \subset \mathbb{R}^{n-1}$ such that

(i) $f_k \to f$ almost everywhere on E;

(ii) $|f_k| \le g_k$ almost everywhere on E, with $g_k \in L^q(E)$ for some $q \in [1,\infty)$;

(iii) there exists $g \in L^q(E)$ such that $g_k \to g$ a.e. on E, and $\int_E g_k^q\,dx \to \int_E g^q\,dx$.

Then $f \in L^q(E)$, and

$$\int_E |f_k - f|^q\,dx \to 0\,.$$

3.4 Transversality and graphicality

Throughout this section, we shall consider an isometry T of $\mathbb{R}^n$, such that

$$Tx = \mathcal{R}x + x^0\,, \quad x \in \mathbb{R}^n\,, \tag{3.4.1}$$

where $\mathcal{R} = \left\{\mathcal{R}_{ij}\right\}_{i,j=1}^n$ is an orthogonal matrix of $\mathbb{R}^n$, and $x^0 \in \mathbb{R}^n$. Let

$$\mathbf{n} = \mathcal{R}^t e_n \in \mathbb{S}^{n-1}\,,$$

where e_n denotes the n-th canonical vector of $\mathbb{R}^n$, i.e., $e_n = (0,\dots,0,1)$, $\mathcal{R}^t$ is the transpose matrix of $\mathcal{R}$, and $\mathbb{S}^{n-1}$ is the unit sphere on $\mathbb{R}^n$.

Here we introduce the geometric notion of *transversality*, which was already used in [117] in a wider sense. The definition given here suffices to our purposes.

Definition 3.4.1 (Transversality)**.** Let $\phi : U \to \mathbb{R}$ be a Lipschitz continuous function on $U \subset \mathbb{R}^{n-1}$ open. We say that a unit vector $\mathbf{n} \in \mathbb{S}^{n-1}$ is transversal to ϕ if there exists $\kappa > 0$ such that

$$\mathbf{n}\cdot\nu(x') \ge \kappa \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in U,$$

where ν denotes the outward normal to G_ϕ with respect to the subgraph S_ϕ.

The next proposition shows a very interesting feature: the transversality of $\mathbf{n} \in \mathbb{S}^{n-1}$ to a Lipschitz function ϕ is equivalent to the graphicality (and subgraphicality) of ϕ with respect to any reference system having $e_n = \mathbf{n}$, that is after performing a rotation of the axes through $\mathcal{R}$, the graph and subgraph of ϕ are mapped onto the graph and subgraph of another function ψ– see identities (3.4.2) below.

Proposition 3.4.2. Let $U \subset \mathbb{R}^{n-1}$ be open, $\phi : U \to \mathbb{R}$ be a Lipschitz function, let T be an isometry of the form (3.4.1), and let $\mathbf{n} = \mathcal{R}^t e_n$.
(i) If there exists an L-Lipschitz function $\psi : V \to \mathbb{R}$ such that

$$TG_\phi = G_\psi \quad \text{and} \quad TS_\phi = S_\psi \cap T(U\times\mathbb{R})\,, \tag{3.4.2}$$

then we have the transversality condition

$$\mathbf{n}\cdot\nu(x') \ge \frac{1}{\sqrt{1+L^2}} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in U. \tag{3.4.3}$$

(ii) Viceversa, if $\phi \in C^k(U)$ for some $k \in \mathbb{N}$ and (3.4.3) holds, then there exist $V \subset \mathbb{R}^{n-1}$ open, and a function $\psi \in C^k(V)$ such that $\|\nabla\psi\|_{L^\infty(V)} \le L$ and (3.4.2) holds true.

Let us comment on this result. Part (i) states that if G_ϕ and S_ϕ are, respectively, the graph and subgraph of an L-Lipschitz function ψ with respect to the reference system $z = (z', z_n)$ having $\mathbf{n} = e_n$, then the quantitative transversality estimate (3.4.3) holds true.

Part (ii) states the opposite in the C^k case: the transversality condition (3.4.3) implies the graphicality and subgraphicality of ϕ with respect to the coordinate system $z = (z', z_n)$, and it also provides a Lipschitz estimate to ψ.

Before starting the proof, we need to introduce the so-called *transition map* $\mathcal{C}$ from ϕ to ψ. Under the same notation as Proposition 3.4.2, the transition map $\mathcal{C} : U \to V$ is defined as

$$\mathcal{C}x' := \Pi\, T\big(x', \phi(x')\big)\,.$$

Here $\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection map $\Pi(x', x_n) = x'$. Observe that, when identities (3.4.2) hold true, by the very definition of $\mathcal{C}$ we have the equation

$$T\big(x', \phi(x')\big) = \big(\mathcal{C}x', \psi\big(\mathcal{C}x'\big)\big)$$

In particular, this implies that $\mathcal{C}$ is a bijection, with inverse function $\mathcal{C}^{-1} : V \to U$ given by

$$\mathcal{C}^{-1}z' = \Pi\, T^{-1}\big(z', \psi(z')\big)\,.$$

Also, since ϕ, ψ are Lipschitz continuous, then $\mathcal{C}$ is a bi-Lipschitz tranformation from U to V.

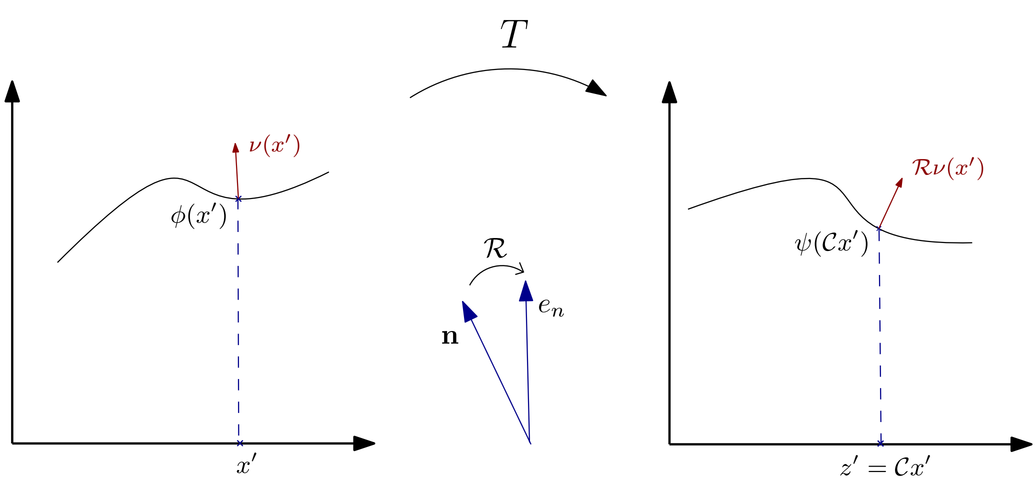

Figure 3.2:

Proof of Proposition 3.4.2. (i) By Rademacher's theorem, the normal vector ν to G_ϕ outward with respect to S_ϕ is well defined $\mathcal{H}^{n-1}$-almost everywhere, and thanks to (3.4.2) and the definition of $\mathcal{C}$, we may write

$$\nu(x') = \frac{(-\nabla\phi(x'), 1)}{\sqrt{1 + |\nabla\phi(x')|^2}} = \mathcal{R}^t\left(\frac{(-\nabla\psi(\mathcal{C}x'), 1)}{\sqrt{1 + |\nabla\psi(\mathcal{C}x')|^2}}\right) \quad \mathcal{H}^{n-1}\text{-a.e. } x' \in U. \tag{3.4.4}$$

Therefore, since $\mathcal{R}\mathbf{n} = e_n$ and $|\nabla\psi| \leq L$, from (3.4.4) we infer

$$\mathbf{n}\cdot\nu(x') = e_n\cdot\mathcal{R}\nu(x') = \frac{1}{\sqrt{1 + |\nabla\psi(\mathcal{C}x')|^2}} \geq \frac{1}{\sqrt{1 + L^2}} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in U. \tag{3.4.5}$$

(ii) Assume $\phi \in C^k(U)$ and that (3.4.3) is in force.

Consider the C^k-function $f : U \times \mathbb{R} \to \mathbb{R}$, defined as $f(x) \coloneqq x_n - \phi(x')$, so that

$$\{f = 0\} = G_\phi \quad \text{and} \quad \{f < 0\} = S_\phi\,. \tag{3.4.6}$$

Now let $\tilde{f} : T(U \times \mathbb{R}) \to \mathbb{R}$ be the function defined as $\tilde{f}(z) = f(x)$ for $z = Tx$. Recalling $\mathcal{R}\mathbf{n} = e_n$, via the chain rule we compute

$$\frac{\partial \tilde{f}(z)}{\partial z_n} = \mathcal{R}_{nn} - \sum_{k=1}^{n-1} \frac{\partial \phi(x')}{\partial x'_k} \mathcal{R}_{nk} = (-\nabla\phi(x'), 1) \cdot \mathbf{n}\,. \tag{3.4.7}$$

Thus, from expression (3.4.4) of $\nu(x')$ and estimate (3.4.3), we obtain

$$\frac{\partial \tilde{f}(z)}{\partial z_n} = \sqrt{1 + |\nabla\phi(x')|^2}\, \nu(x') \cdot \mathbf{n} \geq \frac{1}{\sqrt{1+L^2}} \quad \text{for } z = Tx. \tag{3.4.8}$$

Therefore, owing to (3.4.8) and the implicit function theorem, we immediately infer the existence of a function $\psi \in C^k(V)$, with $V \subset \mathbb{R}^{n-1}$ open, such that

$$\{\tilde{f} = 0\} = G_\psi \quad \text{and} \quad \{\tilde{f} = 0\} = S_\psi \cap T(U \times \mathbb{R}).$$

Thereby, (3.4.2) follows from the very definition of $\tilde{f}$ and (3.4.6).

Finally, by using (3.4.5) we infer that $|\nabla\psi(\mathcal{C}x')| \leq L$ for all $x' \in U$, whence $\|\nabla\psi\|_{L^\infty(V)} \leq L$ since the transition map $\mathcal{C}$ is a bijection.

□

Remark 3.4.3. We point out that inequality (3.4.8), when evaluated at points $z = T\big(x', \phi(x')\big)$, holds true if ϕ and ψ are merely Lipschitz continuous and satisfy (3.4.2).

Indeed, since $\mathcal{C}$ is a bi-Lipschitz map, by Rademacher's Theorem and the chain rule we may perform the same computations as (3.4.7)-(3.4.8) and get

$$\mathcal{R}_{nn} - \sum_{k=1}^{n-1} \frac{\partial \phi(x')}{\partial x'_k} \mathcal{R}_{nk} \geq \frac{1}{\sqrt{1+L^2}} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in U. \tag{3.4.9}$$

By making use of this information, we now show that the transversality condition (3.4.3) is inherited by the regularized function $M_m(\phi)$. This is the content of the following proposition

Proposition 3.4.4. Let U, $V \subset \mathbb{R}^{n-1}$ be open bounded , let T be an isometry of the form (3.4.1), and $\mathbf{n} = \mathcal{R}^t e_n$. Let $\phi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$ be L-Lipschitz functions satisfying (3.4.2). If we set

$$U_m \coloneqq \big\{x' \in U \,:\, \operatorname{dist}(x', \partial U) > \tfrac{1}{m}\big\}$$

and for some sequence $\{c_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ we define

$$\phi_m(x') \coloneqq M_m(\phi)(x') + c_m \quad \text{for } x' \in U_m,$$

then ϕ_m is L-Lipschitz continuous on U_m and

$$\|\phi_m - \phi\|_{L^\infty(U_m)} \leq \frac{L}{m} + |c_m|\,. \tag{3.4.10}$$

In addition, we have the transversality condition

$$\mathcal{R}_{nn} - \sum_{k=1}^{n-1} \frac{\partial \phi_m}{\partial x'_k}(x')\mathcal{R}_{nk} = \big(- \nabla\phi_m(x'), 1\big) \cdot \mathbf{n} \geq \frac{1}{\sqrt{1+L^2}} \quad \text{for all } x' \in U_m, \tag{3.4.11}$$

and

$$\mathbf{n} \cdot \nu_m(x') \geq \frac{1}{1+L^2} \quad \text{for all } x' \in U_m\,, \tag{3.4.12}$$

where ν_m is the outward unit normal to G_{ϕ_m} with respect to the subgraph S_{ϕ_m}.

Proof. Let $x'_0 \in U_m$. By multiplying (3.4.9) with $\rho_m(x'_0 - x')$ and integrating in x' we immediately obtain

$$\mathcal{R}_{nn} - \sum_{k=1}^{n-1} \frac{\partial M_m(\phi)(x'_0)}{\partial x'_k}\,\mathcal{R}_{nk} \geq \frac{1}{\sqrt{1+L^2}} \quad \text{for all } x'_0 \in U_m,$$

and (3.4.11) holds true.

Next, from the L-Lipschitz continuity of ϕ, we have

$$\begin{aligned}\big|M_m(\phi)(x') - M_m(\phi)(y')\big| &\leq \int_{\mathbb{R}^{n-1}} \big|\phi(x'-z') - \phi(y'-z')\big|\,\rho_m(z')\,dz' \\ &\leq L\,|x'-y'| \int_{\mathbb{R}^{n-1}} \rho_m(z')\,dz' = L\,|x'-y'|\end{aligned}$$

for all $x', y' \in U_m$, hence ϕ_m is L-Lipschitz continuous as well. From this and (3.4.11), we get

$$\mathbf{n} \cdot \nu_m(x') = \mathbf{n} \cdot \frac{\big(- \nabla M_m(\phi)(x'), 1\big)}{\sqrt{1 + |\nabla M_m(\phi)(x')|^2}} \geq \frac{1}{1+L^2} \quad \text{for all } x' \in U_m,$$

that is (3.4.12). Next, since ρ_m is radially symmetric and ϕ is L-Lipschitz continuous, for all $x' \in U_m$ we get

$$\begin{aligned}\big|M_m(\phi)(x') - \phi(x')\big| &\leq \int_{B'_{1/m}} \big|\phi(x'+y') - \phi(x')\big|\,\rho_m(y')\,dy' \\ &\leq \int_{B'_{1/m}} L\,|y'|\,\rho_m(y')\,dy' \leq \frac{L}{m}\,,\end{aligned}$$

and thus (3.4.10) follows. □

Since we have proven that the regularized function $M_m(\phi)$ satisfies the transversality condition, Part (ii) of Proposition 3.4.2 entails its "graphicality" with respect to the coordinate system having $\mathbf{n} = e_n$.

Proposition 3.4.5. Under the same assumptions of Proposition 3.4.4, there exist $V_m \subset \mathbb{R}^{n-1}$ open bounded such that

$$\operatorname{dist}_{\mathcal{H}}(V_m, V) \leq \frac{2\sqrt{1+L^2}}{m} + |c_m|\,, \tag{3.4.13}$$

and a function $\psi_m \in C^\infty(V_m)$ satisfying

$$\|\nabla\psi_m\|_{L^\infty(V_m)} \leq 2(1+L^2)\,, \tag{3.4.14}$$

$$TG_{\phi_m} = G_{\psi_m} \quad \text{and} \quad TS_{\phi_m} = S_{\psi_m} \cap T\big(U_m \times \mathbb{R}\big)\,. \tag{3.4.15}$$

If in addition $V_m \cap V \neq \emptyset$, then

$$\|\psi_m - \psi\|_{L^\infty(V_m\cap V)} \leq \frac{L(1+L)}{m} + (1+L)\,|c_m|, \tag{3.4.16}$$

and if $\mathcal{C}_m$ is the transition map of ϕ_m, we have that

$$\|\mathcal{C}_m - \mathcal{C}\|_{L^\infty(U_m)} + \|\mathcal{C}_m^{-1} - \mathcal{C}^{-1}\|_{L^\infty(V_m\cap V)} \leq c(n)\,(1+L^2)\Big(\frac{1}{m} + |c_m|\Big)\,. \tag{3.4.17}$$

Proof. From the results of Part (ii) of Proposition 3.4.2 and (3.4.12), there exist $V_m \subset \mathbb{R}^{n-1}$ open bounded, and a function $\psi_m \in C^\infty(V_m)$ such that (3.4.15) holds. Also, owing to (3.4.3), we immediately obtain (3.4.14).

Now we recall that the transition map of ϕ_m is the function $\mathcal{C}_m : U_m \to V_m$ defined as $\mathcal{C}_m x' = \Pi\, T\big(x', \phi_m(x')\big)$, and for all $x' \in U_m$ we have

$$T\big(x', \phi(x')\big) = \big(\mathcal{C}x', \psi(\mathcal{C}x')\big) \quad \text{and} \quad T\big(x', \phi_m(x')\big) = \big(C_m x', \psi_m(\mathcal{C}_m x')\big)\,,$$

so that from (3.4.10) we infer

$$|c_m| + \frac{L}{m} \geq |\phi_m(x') - \phi(x')| = \big|\big(x', \phi_m(x')\big) - \big(x', \phi(x')\big)\big| = \big|\big(\mathcal{C}_m x', \psi_m(\mathcal{C}_m x')\big) - \big(\mathcal{C}x', \psi(\mathcal{C}x')\big)\big|\,,$$

for all $x' \in U_m$. In particular

$$\begin{cases} |\mathcal{C}_m x' - \mathcal{C}x'| \leq \dfrac{L}{m} + |c_m| \\[2mm] \big|\psi_m\big(\mathcal{C}_m x'\big) - \psi\big(\mathcal{C}x'\big)\big| \leq \dfrac{L}{m} + |c_m| \end{cases} \qquad \text{for all } x' \in U_m \tag{3.4.18}$$

The first inequality in (3.4.18) entails $\mathrm{dist}_{\mathcal{H}}\big(V_m, \mathcal{C}(U_m)\big) \leq \frac{L}{m} + |c_m|$.

On the other hand, by definition of U_m, for any $x' \in U$ we may find $x'_m \in U_m$ such that $|x' - x'_m| \leq \frac{1}{m}$. Since Π and T are 1-Lipschitz continuous, and ϕ is L-Lipschitz continuous, it follows that

$$|\mathcal{C}x' - \mathcal{C}x'_m| \leq \big|\big(x', \phi(x')\big) - \big(x'_m, \phi(x'_m)\big)\big| \leq \frac{\sqrt{1+L^2}}{m}\,,$$

which implies $\mathrm{dist}_{\mathcal{H}}\big(\mathcal{C}(U_m), V\big) \leq \frac{\sqrt{1+L^2}}{m}$ since $\mathcal{C}(U) = V$. Hence, by using the triangle inequality we get

$$\mathrm{dist}_{\mathcal{H}}\big(V_m, V\big) \leq \mathrm{dist}_{\mathcal{H}}\big(V_m, \mathcal{C}(U_m)\big) + \mathrm{dist}_{\mathcal{H}}\big(\mathcal{C}(U_m), V\big) \leq \frac{2\sqrt{1+L^2}}{m} + |c_m|\,,$$

that is (3.4.13).

Next, on assuming that $V_m \cap V \neq \emptyset$, and $\mathcal{C}_m$ being a bijection between U_m and V_m, we may take a point $y' \in V_m \cap V$ such that $y' = \mathcal{C}_m x'$ for some $x' \in U_m$ From (3.4.18) we find

$$|\mathcal{C}_m x' - \mathcal{C}x'| = |y' - \mathcal{C}\mathcal{C}_m^{-1} y'| \leq \frac{L}{m} + |c_m|\,,$$

and

$$\big|\psi\big(\mathcal{C}x'\big) - \psi_m\big(\mathcal{C}_m x'\big)\big| = \big|\psi\big(\mathcal{C}\mathcal{C}_m^{-1} y'\big) - \psi_m(y')\big| \leq \frac{L}{m} + |c_m|\,.$$

By using these two estimates and the L-Lipschitz continuity of ψ, we obtain

$$\begin{aligned} |\psi(y') - \psi_m(y')| &\le |\psi(y') - \psi(\mathcal{C}\mathcal{C}_m^{-1}y')| + |\psi(\mathcal{C}\mathcal{C}_m^{-1}y') - \psi_m(y')| \\ &\le L\,|y' - \mathcal{C}\mathcal{C}_m^{-1}y'| + \frac{L}{m} + |c_m| \le \frac{L(1+L)}{m} + (1+L)\,|c_m| \quad \text{for all } y' \in V_m \cap V, \end{aligned}$$

that is (3.4.16). Finally, by making use of (3.4.16) and a similar argument as in the proof of (3.4.18), we obtain (3.4.17). □

The next proposition shows that if $\phi \in W^{2,q}$, then $\psi \in W^{2,q}$ as well. Namely, graphicality preserves Sobolev second-order regularity for Lipschitz functions.

Proposition 3.4.6. *Under the same assumptions of Propositions 3.4.4-3.4.5, if in addition $\phi \in W^{2,q}_{loc}(U)$ for some $q \in [1,\infty]$, then $\psi \in W^{2,q}_{loc}(V)$.*

Proof. In the following proof, we will make use of Propositions 3.4.4-3.4.5 with $c_m \equiv 0$.

Fix $U_0 \Subset U$ open, and set $V_0 = \mathcal{C}(U_0)$. Since $\operatorname{dist}_{\mathcal{H}}(V_m, V) \to 0$ due to (3.4.13), from [116, Proposition 2.2.17] we may find $m_0 > 0$ large enough such that

$$V_0 \Subset V \cap V_m \quad \text{for all } m > m_0.$$

Now let

$$f_m(x) = x_n - M_m(\phi)(x') \quad \text{for } x \in U_m \times \mathbb{R},$$

and set $\widetilde{f}_m(y) \equiv f_m(x)$ for $y = Tx$. Then owing to (3.4.15), we have that $\widetilde{f}_m\big(y', \psi_m(y')\big) = 0$ for all $y' \in V_m$. By differentiating this expression, we obtain

$$\frac{\partial \psi_m}{\partial y'_k}(y') = -\left(\frac{\partial \widetilde{f}_m}{\partial y_n}\big(y', \psi_m(y')\big)\right)^{-1} \left(\frac{\partial \widetilde{f}_m}{\partial y'_k}\big(y', \psi_m(y')\big)\right), \tag{3.4.19}$$

and from the chain rule, equation $\mathbf{n} = \mathcal{R}^t e_n$, the definition of $\mathcal{C}_m^{-1}$ and (3.4.11), we have

$$\begin{aligned} \frac{\partial \widetilde{f}_m}{\partial y'_k}\big(y', \psi_m(y')\big) &= \mathcal{R}_{kn} - \sum_{l=1}^{n-1} \frac{\partial M_m(\phi)}{\partial x'_l}(\mathcal{C}_m^{-1}y')\,\mathcal{R}_{kl} \\ \frac{\partial \widetilde{f}_m}{\partial y_n}\big(y', \psi_m(y')\big) &= \mathcal{R}_{nn} - \sum_{l=1}^{n-1} \frac{\partial M_m(\phi)}{\partial x'_l}(\mathcal{C}_m^{-1}y')\mathcal{R}_{nl} \ge \frac{1}{\sqrt{1+L^2}}, \end{aligned} \tag{3.4.20}$$

Moreover, thanks to (3.4.14) and the L-Lipschitz continuity of $M_m(\phi)$, the maps $\mathcal{C}_m$ are uniformly bi-Lipschitz, i.e.,

$$\|\nabla \mathcal{C}_m\|_{L^\infty} + \|\nabla \mathcal{C}_m^{-1}\|_{L^\infty} \le C(n, L).$$

Thanks to this piece of information and (3.4.17), we may apply Proposition 3.3.1 and get

$$\nabla M_m(\phi)(\mathcal{C}_m^{-1}y') \to \nabla \phi(\mathcal{C}^{-1}y') \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y' \in V_0 \tag{3.4.21}$$

By combining (3.4.19)-(3.4.21), and by using dominated convergence theorem, we find that $\nabla\psi_m$ converges in $L^p(V_0)$ to some vector-valued function G for all $p \in [1,\infty)$. It then follows from (3.4.16) and the uniqueness of the distributional limit that $G = \nabla\psi$, hence

$$\nabla \psi_m \to \nabla \psi \quad \mathcal{H}^{n-1}\text{-a.e. in } V_0 \text{ and in } L^p(V_0). \tag{3.4.22}$$

Next, we differentiate twice identity $\widetilde{f}_m(y', \psi_m(y')) = 0$, and for $k, r = 1, \dots, n-1$ we obtain

$$
\begin{aligned}
\frac{\partial^2 \psi_m}{\partial y'_k \partial y'_r}(y') = -\Big(\frac{\partial \widetilde{f}}{\partial y_n}\big(y', \psi_m(y')\big)\Big)^{-1} \Big\{ & \frac{\partial^2 \widetilde{f}}{\partial y'_k \partial y'_r}\big(y', \psi_m(y')\big) + \frac{\partial^2 \widetilde{f}}{\partial y'_k \partial y_n}\big(y', \psi_m(y')\big) \frac{\partial \psi_m}{\partial y'_r}(y') \\
& + \frac{\partial^2 \widetilde{f}}{\partial y'_r \partial y_n}\big(y', \psi_m(y')\big) \frac{\partial \psi_m}{\partial y'_k}(y') \\
& + \frac{\partial^2 \widetilde{f}}{\partial y_n \partial y_n}\big(y', \psi_m(y')\big) \frac{\partial \psi_m}{\partial y'_k}(y') \frac{\partial \psi_m}{\partial y'_r}(y') \Big\},
\end{aligned}
\tag{3.4.23}
$$

while from the chain rule and the properties of $\mathcal{C}_m$, we obtain

$$
\frac{\partial^2 \widetilde{f}}{\partial y'_k \partial y'_r}\big(y', \psi_m(y')\big) = -\sum_{l,t=1}^{n-1} \frac{\partial^2 M_m(\phi)}{\partial x'_l \partial x'_t}(\mathcal{C}_m^{-1} y')\, \mathcal{R}_{kl} \mathcal{R}_{rt}\,. \tag{3.4.24}
$$

Then, another application of Proposition 3.3.1 entails that

$$
\nabla^2 M_m(\phi)(\mathcal{C}_m^{-1} y') \to \nabla^2 \phi(\mathcal{C}^{-1} y') \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y' \in V_0 \text{ and in } L^q(V_0),
$$

in the Case $q \in [1, \infty)$. From this, (3.4.20), (3.4.22)-(3.4.24) and by using dominated convegence Theorem 3.3.2, we find that $\nabla^2 \psi_m$ converges in $L^q(V_0)$ to some matrix valued function H. Whence $H = \nabla^2 \psi$ due to the uniqueness of the distributional limit, and the proof in the Case $q \in [1, \infty)$ is complete due to the arbitrariness of U_0.

In the Case $q = \infty$, from (3.4.20), (3.4.23) and (3.4.24) we infer that $\{\psi_m\}_m$ is a sequence uniformly bounded in $W^{2,\infty}(V_0)$ with respect to m. Therefore, up to a subsequence, we have that ψ_m weakly-$*$ converge in $W^{2,\infty}(V_0)$ to ψ, thus completing the proof. □

Remark 3.4.7. Let us point out that, by using the argument of (3.4.19)-(3.4.22), it is possible to extend Part (ii) of Proposition 3.4.2 to merely Lipschitz continuous functions ϕ.

At last, we close this section with the following intrinsic property of $W^{2,q}$ domains. Namely, every Lipschitz local boundary chart ϕ of $\partial\Omega \in W^{2,q}$ is of class $W^{2,q}$.

Corollary 3.4.8. *Let Ω be a bounded Lipschitz domains such that $\partial\Omega \in W^{2,q}$ for some $q \in [1, \infty]$. Then any Lipschitz local chart ψ of $\partial\Omega$ is of class $W^{2,q}$.*

Proof. From Definition 3.1.2, there exists a Lipschitz local chart $\phi \in W^{2,q}$ and an isometry T such that (3.4.2) holds. The thesis then follows from Proposition 3.4.6. □

We conclude by mentioning that both Proposition 3.4.6 and Corollary 3.4.8 can be easily extended to the $W^{k,q}$ Case.

3.5 Trace inequalities in Lipschitz domains

The goal of this section is to study weighted Poincaré trace inequalities on $\mathfrak{L}_\Omega$-Lipschitz domains, which were utilized in the proof of the global quantitative regularity estimates of Chapter 2.

The validity of such inequalities is characterized in terms of weighted isocapacitary inequalities and, as a consequence, of integrability properties of the weight function. The focus of our discussion is on the explicit dependence of the constants in the relevant inequalities on both the weight and the Lipschitz characteristic $\mathfrak{L}_\Omega$ of the domains.

Let Ω be a Lipschitz domain with Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$, and let $\varrho \in L^1(\partial\Omega)$ be a nonnegative function. We set, for $r \in (0, R_\Omega]$,

$$\mathcal{K}_{\Omega,\varrho}(r) = \sup_{\substack{E \subset B_r(x) \\ x \in \partial\Omega}} \frac{\int_{\partial\Omega\cap E} \varrho \, d\mathcal{H}^{n-1}}{\operatorname{cap}(B_r(x), E)}, \tag{3.5.1}$$

and

$$\Psi_{\Omega,\varrho}(r) = \begin{cases} \sup_{x\in\partial\Omega} \|\varrho\|_{L^{n-1,\infty}(\partial\Omega\cap B_r(x))} & \text{if } n \geq 3 \\ \\ \sup_{x\in\partial\Omega} \|\varrho\|_{L^{1,\infty}\log L(\partial\Omega\cap B_r(x))} & \text{if } n = 2. \end{cases} \tag{3.5.2}$$

A bound for the constant in a trace inequality in terms of the quantity $\mathcal{K}_{\Omega,\varrho}(r)$ is provided by the following result.

Proposition 3.5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$. Assume that ϱ is a nonnegative function on $\partial\Omega$ such that $\varrho \in L^1(\partial\Omega)$. Then,*

$$\int_{\partial\Omega\cap B_R(x_0)} v^2 \, \varrho \, d\mathcal{H}^{n-1} \leq 32\,(1+L_\Omega)^4 \, \mathcal{K}_{\Omega,\varrho}(R) \int_{\Omega\cap B_R(x_0)} |\nabla v|^2 \, dx\,, \tag{3.5.3}$$

for every $x_0 \in \partial\Omega$, for every $R \in (0, R_\Omega]$, and for every $v \in W^{1,2}_0(B_R(x_0))$.

Proof. Under the notations of Definition 3.1.1, observe that the function

$$\psi : B'_R \times (-\ell, \ell) \to \psi\big(B'_R \times (-\ell, \ell)\big)$$

given by

$$\psi(x', x_n) = (x', x_n - \phi(x')) \quad \text{for } (x', x_n) \in B'_R \times (-\ell, \ell), \tag{3.5.4}$$

defines a bi-Lipschitz diffeomorphism, whose inverse

$$\psi^{-1} : \psi\big(B'_R \times (-\ell, \ell)\big) \to B'_R \times (-\ell, \ell)$$

obeys:

$$\psi^{-1}(y', y_n) = (y', y_n + \phi(y')) \quad \text{for } (y', y_n) \in \big(B'_R \times (-\ell, \ell)\big).$$

Notice also that $\psi\big(B'_R \times (-\ell, \ell)\big) \subset B'_R \times (-2\ell, 2\ell)$, and

$$\|\nabla\psi\|_{L^\infty} + \|\nabla\psi^{-1}\|_{L^\infty} \leq c\,(1 + L_\Omega) \tag{3.5.5}$$

for some constant $c = c(n)$.

In what follows, with a slight abuse of notation, we shall identify $\mathbb{R}^{n-1} \times \{0\}$ with $\mathbb{R}^{n-1}$, and subsets of the former set with subsets of the latter.

We may assume, without loss of generality, that $x_0 = 0$. Fix a compact set $F \subset \psi\big(B_R\big)$. Hence, the set $E = \psi^{-1}(F)$ is a compact subset of B_R. Moreover,

$$\int_{\partial\Omega\cap E} \varrho \, d\mathcal{H}^{n-1} = \int_{\psi(\partial\Omega\cap E)} (\varrho \circ \psi^{-1})(y', 0)\sqrt{1 + |\nabla\varphi(y')|^2} \, dy' \tag{3.5.6}$$

$$= \int_{F \cap B'_R} (\varrho \circ \psi^{-1})(y', 0)\sqrt{1 + |\nabla\varphi(y')|^2}\, dy' .$$

We claim that

$$\operatorname{cap}\big(E, B_R\big) \le 2\,(1 + L_\Omega)^2 \operatorname{cap}\big(F, \psi(B_R)\big) . \tag{3.5.7}$$

To prove this claim, fix a function $\theta \in C_0^{0,1}\big(\psi(B_R)\big)$ such that $\theta \ge 1$ in F. Therefore, $\theta \circ \psi \in C_0^{0,1}(B_R)$, $\theta \circ \psi \ge 1$ in E, and

$$\begin{aligned} \operatorname{cap}\big(E, B_R\big) \le \int_{B_R} |\nabla(\theta \circ \psi)|^2\, dx &\le 2\,(1 + L_\Omega)^2 \int_{B_R} \big|\nabla\theta(\psi(x))\big|^2\, dx \\ &= 2\,(1 + L_\Omega)^2 \int_{\psi(B_R)} |\nabla\theta|^2\, dy. \end{aligned}$$

Inequality (3.5.7) hence follows by taking the infimum over θ. Combining (3.5.6) and (3.5.7) entails

$$\frac{\int_{F \cap B'_R} (\varrho \circ \psi^{-1})(y', 0)\sqrt{1 + |\nabla\varphi(y')|^2} dy'}{\operatorname{cap}(F, \psi(B_R))} \le 2\,(1 + L_\Omega)^2 \frac{\int_{E \cap \partial\Omega} \varrho\, d\mathcal{H}^{n-1}}{\operatorname{cap}(E, B_R)} \tag{3.5.8}$$

It suffices to prove inequality (3.5.3) for functions $v \in C_0^{0,1}(B_R)$, the general case following via a standard density argument. Since the function $v \circ \psi^{-1} \in C_0^{0,1}\big(\psi\big(B'_R \times (-\ell, \ell)\big)\big) \subset C_0^{0,1}\big(B'_R \times (-2\ell, 2\ell)\big)$, it can be extended to a function $w : \mathbb{R}^n \to \mathbb{R}$ as

$$w(y', y_n) = \begin{cases} v \circ \psi^{-1}(y', y_n) & \text{if } y_n \le 0 \\ v \circ \psi^{-1}(y', -y_n) & \text{if } y_n > 0. \end{cases} \tag{3.5.9}$$

Observe that

$$\|w\|^2_{L^2(\mathbb{R}^n)} = 2\,\|v \circ \psi^{-1}\|^2_{L^2(\mathbb{R}^n_-)} \quad \text{and} \quad \|\nabla w\|^2_{L^2(\mathbb{R}^n)} = 2\,\|\nabla(v \circ \psi^{-1})\|^2_{L^2(\mathbb{R}^n_-)} , \tag{3.5.10}$$

where we have set $\mathbb{R}^n_- = \{(x', x_n) \in \mathbb{R}^n \,:\, x_n \le 0\}$. Denote by Tr the trace operator on $\mathbb{R}^{n-1}$, which, according to the convention above, it is identified with $\partial\mathbb{R}^n_+$. Thus,

$$\operatorname{Tr}(w) = \operatorname{Tr}(v \circ \psi^{-1}) \quad \text{on } \mathbb{R}^{n-1} ,$$

and $\operatorname{supp}\big(\operatorname{Tr}(w)\big) \subset B'_R$. An application of a Poincaré type trace inequality [150, Theorem 2.4.1] tells us that

$$\begin{aligned} \int_{\partial\Omega \cap B_R} v^2 \varrho\, d\mathcal{H}^{n-1} &= \int_{B'_R} |\operatorname{Tr}(w)(y')|^2 (\varrho \circ \psi^{-1})(y', 0)\sqrt{1 + |\nabla\varphi(y')|^2} dy' \\ &\le 4\Bigg(\sup_{\substack{F \subset \psi(B_R) \\ F \text{ compact}}} \frac{\int_{F \cap B'_R} (\varrho \circ \psi^{-1})(y', 0)\sqrt{1 + |\nabla\varphi(y')|^2} dy'}{\operatorname{cap}(F, \psi(B_R))} \Bigg) \int_{\mathbb{R}^n} |\nabla w|^2 dy. \end{aligned}$$

Combining the latter inequality with (3.5.8) and (3.5.10) yields:

$$\begin{aligned} \int_{\partial\Omega \cap B_R} v^2 \varrho\, d\mathcal{H}^{n-1} &\le 16\,(1 + L_\Omega)^2 \Bigg(\sup_{\substack{E \subset B_R \\ E \text{ compact}}} \frac{\int_{\partial\Omega \cap E} \varrho\, d\mathcal{H}^{n-1}}{\operatorname{cap}(E, B_R)} \Bigg) \int_{\{y_n \le 0\} \cap \psi(B_R)} |\nabla(v \circ \psi^{-1})(y)|^2\, dy \\ &\le 32\,(1 + L_\Omega)^4 \Bigg(\sup_{\substack{E \subset B_R \\ E \text{ compact}}} \frac{\int_{\partial\Omega \cap E} \varrho\, d\mathcal{H}^{n-1}}{\operatorname{cap}(E, B_R)} \Bigg) \int_{\Omega \cap B_R} |\nabla v|^2\, dx. \end{aligned}$$

Hence, inequality (3.5.3) follows. □

Proposition 3.5.1 enables one to deduce a parallel result, where the role of the quantity $\mathcal{K}_{\Omega,\varrho}(r)$ is instead played by $\Psi_{\Omega,\varrho}(r)$.

Proposition 3.5.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$. Assume that ϱ is a nonnegative function on $\partial\Omega$ such that $\varrho \in L^1(\partial\Omega)$. Then,*

$$\int_{\partial\Omega\cap B_R(x_0)} v^2\,\varrho\,d\mathcal{H}^{n-1} \leq \begin{cases} c\,(1+L_\Omega)^{8+\frac{n-2}{n-1}}\,\Psi_{\Omega,\varrho}(R)\,\displaystyle\int_{\Omega\cap B_R(x_0)} |\nabla v|^2\,dx & \text{for } n \geq 3 \\ c\,(1+L_\Omega)^{11}\,\Psi_{\Omega,\varrho}(R)\,\displaystyle\int_{\Omega\cap B_R(x_0)} |\nabla v|^2\,dx & \text{for } n = 2 \end{cases} \tag{3.5.11}$$

for some constant $c = c(n)$, for every $x_0 \in \partial\Omega$, for every $R \in (0, R_\Omega]$, and for every $v \in W^{1,2}_0(B_R(x_0))$.

The derivation of Proposition 3.5.2 from Proposition 3.5.1 relies upon some intermediate steps contained in the following lemmas.

In particular, the following inequalities for the fractional Sobolev space $W^{\frac{1}{2},2}(\mathbb{R}^{n-1})$ come into play. In what follows, $\|\cdot\|_{\exp L^2(\mathbb{R})}$ denotes the Luxemburg norm associated with the Young function $A(t) = e^{t^2} - 1$.

Lemma B (Fractional Sobolev-type embedding)**.** *Let $n \geq 2$.*
(i) Assume that $n \geq 3$ and set $q = 2\frac{(n-1)}{(n-2)}$. Then, there exists a constant $c_s = c_s(n)$ such that

$$\|v\|_{L^q(\mathbb{R}^{n-1})} \leq c_s\,\|v\|_{W^{\frac{1}{2},2}(\mathbb{R}^{n-1})} \tag{3.5.12}$$

for every $v \in W^{\frac{1}{2},2}(\mathbb{R}^{n-1})$.
(ii) Assume that $n = 2$. Then, there exists an absolute constant c_s such that

$$\|v\|_{\exp L^2(\mathbb{R})} \leq c_s \|v\|_{W^{\frac{1}{2},2}(\mathbb{R})} \tag{3.5.13}$$

for every $v \in W^{\frac{1}{2},2}(\mathbb{R})$ such that $\operatorname{supp}(v) \subset (-1,1)$.

Lemma C (Trace embedding)**.** *Let $n \geq 2$. Then, there exists an absolute constant c such that*

$$\|\mathrm{Tr}(v)\|_{W^{\frac{1}{2},2}(\mathbb{R}^{n-1})} \leq c\,\|v\|_{W^{1,2}(\mathbb{R}^n)}\,. \tag{3.5.14}$$

for every $v \in W^{1,2}(\mathbb{R}^n)$.

Part (i) of the Lemma B and Lemma C are standard. Part (ii) of Lemma B follows as a special case of [3].

Lemma 3.5.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$. Let $x_0 \in \partial\Omega$ and $R \in (0, R_\Omega]$.*
(i) Assume that $n \geq 3$. Then,

$$\left(\int_{\partial\Omega\cap E} |v|\,d\mathcal{H}^{n-1}\right)^2 \leq c\,(1+L_\Omega)^{\frac{3n-4}{n-1}}(1+\ell^2)\,\mathcal{H}^{n-1}(\partial\Omega\cap E)^{\frac{n}{(n-1)}} \int_{\Omega\cap B_R(x_0)} |\nabla v|^2 dx\,, \tag{3.5.15}$$

for some constant $c = c(n)$, for every $v \in W^{1,2}_0(B_R(x_0))$, and for every compact set $E \subset B_R(x_0)$.
(ii) Assume that $n = 2$. Then,

(3.5.16)
$$\Big(\int_{\partial\Omega\cap E}|v|\,d\mathcal{H}^1\Big)^2 \leq c\,(1+L_\Omega)^5\,(1+\ell^2)\,\big(\mathcal{H}^1(\partial\Omega\cap E)\big)^2\,\log\Big(1+\frac{1}{\mathcal{H}^1(\partial\Omega\cap E)}\Big)\int_{\Omega\cap B_R(x_0)}|\nabla v|^2dx$$

for some absolute constant c, for every $v\in W^{1,2}_0(B_R(x_0))$, and for every compact set $E\subset B_R(x_0)$.

Proof. Without loss of generality, we may assume that $x_0=0$, and hence (3.1.3) is in force. Moreover, one can deal with functions $v\in C^{0,1}_0(B_R)$, since general case follows via a standard density argument.
Part (i). Set $q=2\frac{(n-1)}{(n-2)}$. Since $B_R\subset B'_R\times(-\ell,\ell)$, from inequalities (3.5.12) and (3.5.14) one can deduce that

(3.5.17)
$$\int_{\partial\Omega\cap B_R}|v|^qd\mathcal{H}^{n-1}=\int_{\partial\Omega\cap\big(B'_R\times(-\ell,\ell)\big)}|v|^qd\mathcal{H}^{n-1}=\int_{B'_R}\big|\mathrm{Tr}(v\circ\psi^{-1})(y')\big|^q\sqrt{1+|\nabla\varphi(y')|^2}dy'$$
$$\leq\sqrt{1+L^2_\Omega}\int_{B'_R}|\mathrm{Tr}(w)|^qdy'\leq c(n)\,\sqrt{1+L^2_\Omega}\,\|\mathrm{Tr}(w)\|^q_{W^{\frac12,2}(\mathbb{R}^{n-1})}\leq c'(n)\,\sqrt{1+L^2_\Omega}\,\|w\|^q_{W^{1,2}(\mathbb{R}^n)}\,,$$

where w is the function defined by (3.5.9). Hence, since $\mathrm{supp}(w)\subset B'_R\times(-2\ell,2\ell)\subset(-2\ell,2\ell)^n$, by equations (3.5.10) and (3.5.5) we have that

(3.5.18)
$$\begin{aligned}\int_{\partial\Omega\cap B_R}|v|^qd\mathcal{H}^{n-1}&\leq c(n)(1+L^2_\Omega)^{1/2}(1+\ell^2)^{q/2}\|\nabla w\|^q_{L^2(\mathbb{R}^n)}\\&=c'(n)\,(1+L_\Omega)(1+\ell^2)^{q/2}\Big(\int_{\mathbb{R}^n_-}|\nabla(v\circ\psi^{-1})|^2dy\Big)^{q/2}\\&\leq c''(n)(1+L_\Omega)^{1+q}(1+\ell^2)^{q/2}\Big(\int_{\Omega\cap B_R}|\nabla v|^2dx\Big)^{q/2}.\end{aligned}$$

Notice that, besides inequality (3.5.17), the first inequality in (3.5.18) also relies upon a standard Poincaré inequality on the cube $(-2\ell,2\ell)^n$ (whose constant is $4l$). Inequality (3.5.15) follows from (3.5.18), via Hölder's inequality.
Part (ii). Hölder's inequality in Orlicz spaces [169, Theorem 4.7.5] ensures that

(3.5.19)
$$\begin{aligned}\int_{\partial\Omega\cap E}|v|\,d\mathcal{H}^1&=\int_{\psi(E)\cap B'_R}\big|\mathrm{Tr}(v\circ\psi^{-1})\big|\sqrt{1+|\nabla\varphi(y')|^2}dy'\\&\leq\sqrt{1+L^2_\Omega}\int_{\mathbb{R}}\big|\mathrm{Tr}(v\circ\psi^{-1})(y')\big|\,\chi_{\psi(E)\cap B'_R}(y')dy'\\&\leq2\sqrt{1+L^2_\Omega}\,\|v\circ\psi^{-1}\|_{L^A(\mathbb{R})}\,\|\chi_{\psi(E)\cap B'_R(0')}\|_{L^{\widetilde{A}}(\mathbb{R})},\end{aligned}$$

where χ_F denotes the characteristic function of a set F, and $\widetilde{A}$ the Young conjugate of A. One has that
$$\|\chi_F\|_{L^{\widetilde{A}}(\mathbb{R})}=\frac{1}{\widetilde{A}^{-1}(1/|F|)}\leq|F|\,A^{-1}(1/|F|)$$
for every measurable set $F\subset\mathbb{R}$. Since $A^{-1}(t)=\sqrt{\log(1+t)}$, and
$$|\psi(E)\cap B'_R|\leq\mathcal{H}^1(\partial\Omega\cap E)\leq\sqrt{1+L^2_\Omega}\,|\psi(E)\cap B'_R|\,,$$

we obtain that

$$\|\chi_{\psi(E)\cap B'_R}\|_{L^{\widetilde{A}}(\mathbb{R})} \leq \mathcal{H}^1(\partial\Omega\cap E)\sqrt{\log\left(1+\frac{(1+L_\Omega^2)^{1/2}}{\mathcal{H}^1(\partial\Omega\cap E)}\right)}. \tag{3.5.20}$$

Inequalities (3.5.13), (3.5.14), (3.5.19) and (3.5.20) enable one to infer that

(3.5.21)

$$\begin{aligned}
\int_{\partial\Omega\cap E}|v|\,d\mathcal{H}^1 &\leq \sqrt{1+L_\Omega^2}\,\|v\circ\psi^{-1}\|_{L^A(\mathbb{R})}\,\mathcal{H}^1(\partial\Omega\cap E)\sqrt{\log\left(1+\frac{(1+L_\Omega^2)^{1/2}}{\mathcal{H}^1(\partial\Omega\cap E)}\right)}\\
&\leq c\sqrt{1+L_\Omega^2}\,\|\mathrm{Tr}(w)\|_{W^{\frac12,2}(\mathbb{R})}\,\mathcal{H}^1(\partial\Omega\cap E)\sqrt{\log\left(1+\frac{(1+L_\Omega^2)^{1/2}}{\mathcal{H}^1(\partial\Omega\cap E)}\right)}\\
&\leq c'\sqrt{1+L_\Omega^2}\,\|w\|_{W^{1,2}(\mathbb{R}^n)}\,\mathcal{H}^1(\partial\Omega\cap E)\sqrt{\log\left(1+\frac{(1+L_\Omega^2)^{1/2}}{\mathcal{H}^1(\partial\Omega\cap E)}\right)}\\
&\leq c''\,(1+\ell^2)^{1/2}\,(1+L_\Omega)^2\,\|\nabla v\|_{L^2(\Omega\cap B_R)}\,\mathcal{H}^1(\partial\Omega\cap E)\sqrt{\log\left(1+\frac{(1+L_\Omega^2)^{1/2}}{\mathcal{H}^1(\partial\Omega\cap E)}\right)},
\end{aligned}$$

where c'' is an absolute constant. Note that the last inequality rests upon the inequality

$$\|w\|_{W^{1,2}(\mathbb{R}^2)} \leq c\,(1+\ell^2)^{1/2}\,(1+L_\Omega)\,\|\nabla v\|_{L^2(\Omega\cap B_R)}\,,$$

with c an absolute constant, which is in turn a consequence of (3.5.10), (3.5.5) and Poincaré inequality, as in the case $n\geq 3$. Finally, since

$$1+\frac{(1+L_\Omega^2)^{1/2}}{\mathcal{H}^1(\partial\Omega\cap E)} \leq \left(1+\frac{1}{\mathcal{H}^1(\partial\Omega\cap E)}\right)^{(1+L_\Omega^2)^{1/2}}$$

we have that

$$\log\left(1+\frac{(1+L_\Omega^2)^{1/2}}{\mathcal{H}^1(\partial\Omega\cap E)}\right) \leq (1+L_\Omega^2)^{1/2}\,\log\left(1+\frac{1}{\mathcal{H}^1(\partial\Omega\cap E)}\right).$$

Coupling the latter inequality with (3.5.21) yields (3.5.16). □

Lemma 3.5.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$. Assume that ϱ is a nonnegative function on $\partial\Omega$ such that $\varrho\in L^1(\partial\Omega)$. Then,*

$$\sup_{\substack{E\subset B_R(x_0)\\ E\text{ compact}}}\frac{\int_{\partial\Omega\cap E}\varrho\,d\mathcal{H}^{n-1}}{\mathrm{cap}(E,B_R(x_0))} \leq \begin{cases} c\,(1+L_\Omega)^{\frac{3n-4}{n-1}}\,(1+\ell^2)\,\|\varrho\|_{L^{n-1,\infty}(\partial\Omega\cap B_R(x_0))} & \text{if } n\geq 3\\[2ex] c\,(1+L_\Omega)^5(1+\ell^2)\,\|\varrho\|_{L^{1,\infty}\log L(\partial\Omega\cap B_R(x_0))} & \text{if } n=2\end{cases} \tag{3.5.22}$$

for some constant $c=c(n)$, for every $x_0\in\partial\Omega$ and every $R\in(0,R_\Omega]$.

Proof of Lemma 3.5.4. We may assume that the norms on the right-hand side of inequality (3.5.22) are finite, otherwise there is nothing to prove. Let $\{E_k\}$ be a sequence of compact sets such that $E_k\subset B_R$, and

$$\sup_{\substack{E\subset B_R\\ E\text{ compact}}}\frac{\int_{\partial\Omega\cap E}\varrho\,d\mathcal{H}^{n-1}}{\mathrm{cap}(E,B_R)} = \lim_{k\to\infty}\frac{\int_{\partial\Omega\cap E_k}\varrho\,d\mathcal{H}^{n-1}}{\mathrm{cap}(E_k,B_R)}\,.$$

Applying either inequality (3.5.15) or (3.5.16) with functions $v \in C_0^{0,1}(B_R)$ such that $v \geq 1$ on E_k, and taking the infimum of the ratio of the integrals on their two sides among these functions u tell us that

$$\begin{cases} \mathcal{H}^{n-1}(\partial\Omega \cap E_k)^{\frac{n-2}{n-1}} \leq c(n)\,(1+L_\Omega)^{\frac{3n-4}{n-1}}(1+\ell^2)\,\mathrm{cap}(E_k, B_R) & \text{if } n \geq 3 \\ \log\left(1 + \dfrac{1}{\mathcal{H}^1(\partial\Omega \cap E_k)}\right)^{-1} \leq c\,(1+L_\Omega)^5(1+\ell^2)\,\mathrm{cap}(E_k, B_R) & \text{if } n = 2, \end{cases} \tag{3.5.23}$$

where c is an absolute constant if $n = 2$.
If $n \geq 3$, then inequality (3.5.23) and an application of the Hardy-Littlewood inequality for rearrangements enable one to deduce that

$$\begin{aligned} \frac{\int_{\partial\Omega\cap E_k} \varrho\, d\mathcal{H}^{n-1}}{\mathrm{cap}(E_k, B_R)} &\leq c(n)\,(1+L_\Omega)^{\frac{3n-4}{n-1}}(1+\ell^2)\,\frac{\int_{\partial\Omega\cap E_k} \varrho\, d\mathcal{H}^{n-1}}{\big(\mathcal{H}^{n-1}(\partial\Omega\cap E_k)\big)^{q/2}} \\ &\leq c(n)\,(1+L_\Omega)^{\frac{3n-4}{n-1}}(1+\ell^2)\,\frac{\int_0^{\mathcal{H}^{n-1}(\partial\Omega\cap E_k)} \varrho^*(t)\, dt}{\big(\mathcal{H}^{n-1}(\partial\Omega\cap E_k)\big)^{\frac{n-1}{n-2}}} \\ &\leq c(n)\,(1+L_\Omega)^{\frac{3n-4}{n-1}}(1+\ell^2)\,\sup_{s>0} s^{\frac{1}{n-1}}\varrho^{**}(s) \\ &\leq c(n)\,(1+L_\Omega)^{\frac{3n-4}{n-1}}(1+\ell^2)\,\|\varrho\|_{L^{n-1,\infty}(\partial\Omega\cap B_R)}. \end{aligned} \tag{3.5.24}$$

If $n = 2$, then inequality (3.5.23) and the Hardy-Littlewood inequality again yield:

$$\begin{aligned} \frac{\int_{\partial\Omega\cap E_k} \varrho\, d\mathcal{H}^{n-1}}{\mathrm{cap}(E_k, B_R)} &\leq c\,(1+L_\Omega)^5(1+\ell^2)\,\log\Big(1+\frac{1}{\mathcal{H}^1(\partial\Omega\cap E_k)}\Big)\int_{\partial\Omega\cap E_k} \varrho\, d\mathcal{H}^1 \\ &\leq\, (1+L_\Omega)^5(1+\ell^2)\,\log\Big(1+\frac{1}{\mathcal{H}^1(\partial\Omega\cap E_k)}\Big)\int_0^{\mathcal{H}^1(\partial\Omega\cap E_k)} \varrho^*(t)\, dt \\ &\leq c\,(1+L_\Omega)^5(1+\ell^2)\,\sup_{s>0}\Big(s\,\log\big(1+\tfrac{1}{s}\big)\,\varrho^{**}(s)\Big) \\ &\leq c\,(1+L_\Omega)^5(1+\ell^2)\,\|\varrho\|_{L^{1,\infty}\log L(\partial\Omega\cap B_R)}\,. \end{aligned} \tag{3.5.25}$$

Then, (3.5.22) follows by letting $k \to \infty$ in (3.5.24) and (3.5.25). □

We are now in a position to prove Proposition 3.5.2.

Proof of Proposition 3.5.2. Recalling (3.1.2), from inequality (3.5.22) of Lemma 3.5.4 we infer that, for any $R \in (0, R_\Omega)$,

$$\mathcal{K}_{\Omega,\varrho}(R) \leq \begin{cases} c\,(1+L_\Omega)^{\frac{n-2}{n-1}+4}\,\Psi_{\Omega,\varrho}(R) & \text{if } n \geq 3 \\ c\,(1+L_\Omega)^7\,\Psi_{\Omega,\varrho}(R) & \text{if } n = 2\,, \end{cases} \tag{3.5.26}$$

for some constant $c = c(n)$. Inequality (3.5.11) follows from (3.5.3) and (3.5.26). □

We conclude with an estimate for the function $\mathcal{K}_{\Omega,\varrho}$ in the special case when $\varrho \in L^\infty(\partial\Omega)$, which is particularly useful when dealing with domains with bounded curvature, i.e., whose boundary $\partial\Omega \in C^{1,1}$.

Corollary 3.5.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$. Assume that ϱ is a nonnegative function on $\partial\Omega$ such that $\varrho \in L^\infty(\partial\Omega)$. Then,*

$$\int_{\partial\Omega\cap B_R(x_0)} v^2\,\varrho\,d\mathcal{H}^{n-1} \leq \begin{cases} c(n)\,(1+L_\Omega)^9\,R\,\|\varrho\|_{L^\infty(\partial\Omega)}\,\int_{\Omega\cap B_R(x_0)}|\nabla v|^2\,dx & \text{if } n\geq 3 \\ \\ c(n)\,(1+L_\Omega)^{12}\,R\,\log\big(1+\frac{1}{R}\big)\,\|\varrho\|_{L^\infty(\partial\Omega)}\,\int_{\Omega\cap B_R(x_0)}|\nabla v|^2\,dx & \text{if } n=2 \end{cases} \tag{3.5.27}$$

for $x_0 \in \partial\Omega$, for $R \in (0, R_\Omega]$, and for $v \in W^{1,2}_0(B_R(x_0))$.

Proof. Owing to the Area formula and the L_Ω-Lipschitz continuity of the boundary chart ϕ in Definition 3.1.1, there exist positive constants $c_1 = c_1(n)$ and $c_2 = c_2(n)$ such that

$$c_1(n)\,R^{n-1} \leq \mathcal{H}^{n-1}\big(\partial\Omega\cap B_R(x_0)\big) \leq c_2(n)\,\sqrt{1+L_\Omega^2}\,R^{n-1}$$

for every $x_0 \in \partial\Omega$ and $R \in (0, R_\Omega]$, hence

$$\begin{cases} \|\varrho\|_{L^{n-1,\infty}(\partial\Omega\cap B_R(x_0))} \leq c(n)\,(1+L_\Omega)^{\frac{1}{(n-1)}}\,R\,\|\varrho\|_{L^\infty(\partial\Omega\cap B_R(x_0))} & \text{if } n\geq 3 \\ \\ \|\varrho\|_{L^{1,\infty}\log L(\partial\Omega\cap B_R(x_0))} \leq c\,(1+L_\Omega)\,R\log\big(1+\frac{1}{R}\big)\,\|\varrho\|_{L^\infty(\partial\Omega\cap B_R(x_0))} & \text{if } n=2 \end{cases}$$

for every $x_0 \in \partial\Omega$ and $R \in (0, R_\Omega]$. From inequality (3.5.22) of Lemma 3.5.4 we infer that,

$$\mathcal{K}_{\Omega,\varrho}(R) \leq \begin{cases} c(n)\,(1+L_\Omega)^5\,R\,\|\varrho\|_{L^\infty(\partial\Omega)} & \text{if } n\geq 3 \\ \\ c(n)\,(1+L_\Omega)^8\,R\,\log\big(1+\frac{1}{R}\big)\,\|\varrho\|_{L^\infty(\partial\Omega)} & \text{if } n=2\,, \end{cases} \tag{3.5.28}$$

for $R \in (0, R_\Omega]$. The desired conclusions then follow from these estimates, via Proposition 3.5.1. □

3.6 Proof of Theorem 3.2.1

This section is devoted to the proof of Theorem 3.2.1, which is divided into a few steps.
From here onward, m_0 and k_0 will denote positive integers, possibly changing from line to line.

3.6.1 Covering of $\partial\Omega$

By Definition 3.1.1, for any $x_0 \in \partial\Omega$, we may find an L_Ω-Lipschitz function $\phi^{x_0} : B'_{R_\Omega} \to \mathbb{R}$, and an isometry T^{x_0} of $\mathbb{R}^n$ such that $T^{x_0}x_0 = 0$, and

$$\begin{aligned} &T^{x_0}\partial\Omega \cap \big(B'_{R_\Omega}\times(-\ell,\ell)\big) = \big\{(y',\phi^{x_0}(y'))\,:\,y'\in B'_{R_\Omega}\big\}, \\ &T^{x_0}\Omega \cap \big(B'_{R_\Omega}\times(-\ell,\ell)\big) = \big\{(y',y_n)\,:\,x'\in B'_{R_\Omega}\,,\,-\ell<y_n<\phi^{x_0}(y')\big\}\,, \end{aligned}$$

where $\ell = R_\Omega(1+L_\Omega)$. Let us consider the open covering $\{B_{R_\Omega/8}(x_0)\}_{x_0\in\partial\Omega}$ of $\partial\Omega$ [1]. By compactness, we may find a finite sequence of points $\{x^i\}_{i=1}^N \subset \partial\Omega$ such that

$$\partial\Omega \Subset \bigcup_{i=1}^N B_{\frac{R_\Omega}{8}}(x^i)\,, \tag{3.6.1}$$

[1] Any other open covering is allowed, as long as its sets are strictly contained in the coordinate cylinders $B'_{R_\Omega}\times(-\ell,\ell)$. The open covering here chosen helps simplifying a few computations, especially in the isocapacitary estimate (3.2.7).

as well as L_Ω-Lipschitz functions ϕ^i and isometries T^i satisfying

$$\begin{aligned} T^i\partial\Omega \cap \big(B'_{R_\Omega} \times (-\ell,\ell)\big) &= \big\{(y',\phi^i(y')) \,:\, y' \in B'_{R_\Omega}\big\}, \\ T^i\Omega \cap \big(B'_{R_\Omega} \times (-\ell,\ell)\big) &= \big\{(y',y_n) \,:\, y' \in B'_{R_\Omega}\,,\, -\ell < y_n < \phi^i(y')\big\}. \end{aligned} \tag{3.6.2}$$

We denote by $\mathcal{R}^i$ the orthogonal matrix of T^i, i.e., T^i can be written as

$$T^i x = \mathcal{R}^i(x - x^i) \quad x \in \mathbb{R}^n\,.$$

Notice also that the cardinality N of this covering of $\partial\Omega$ may be chosen satisfying

$$N \le c(n)\left(\frac{d_\Omega}{R_\Omega}\right)^n. \tag{3.6.3}$$

We then set

$$\Omega_t := \{x \in \Omega \,:\, \mathrm{dist}(x,\partial\Omega) > t\}\,, \tag{3.6.4}$$

so that by (3.6.1) we have

$$\overline{\Omega} \Subset W := \bigcup_{i=1}^N B_{\frac{R_\Omega}{8}}(x^i) \cup \Omega_{\frac{R_\Omega}{32}}\,. \tag{3.6.5}$$

Starting from this point, we construct a suitable partition of unity: let

$$\eta_i := \tilde\rho_{\frac{R_\Omega}{32}} * \chi_{B_{\frac{3R_\Omega}{16}}(x^i)} \quad \text{and} \quad \eta_0 := \tilde\rho_{\frac{R_\Omega}{64}} * \chi_{\Omega_{\frac{3R_\Omega}{64}}}\,,$$

where $\tilde\rho_t$ is the standard, radially symmetric convolution kernel on $\mathbb{R}^n$, and χ_A denotes the indicator function of a set A.
Standard properties of convolution ensure that $\eta_i \in C_c^\infty(B_{\frac{R_\Omega}{4}}(x^i))$, $\eta_0 \in C_c^\infty(\Omega_{\frac{R_\Omega}{16}})$, $0 \le \eta_i \le 1$,

$$\eta_i \ge 1 \quad \text{on } B_{\frac{R_\Omega}{8}}(x^i)\,, \quad \eta_0 \ge 1 \quad \text{on } \Omega_{\frac{R_\Omega}{32}}\,,$$

and

$$|\nabla^k \eta_i| \le \frac{c(n,k)}{R_\Omega^k}\,, \quad \text{for all } k \in \mathbb{N}.$$

Therefore, by defining $\xi_i : W \to [0,1]$ as

$$\xi_i := \frac{\eta_i}{\sqrt{\sum_{j=0}^N \eta_j}}\,, \quad i = 0,\dots,N\,,$$

then we have that $\xi_i \in C_c^\infty(B_{\frac{R_\Omega}{4}}(x^i))$ for $i = 1,\dots,N$, $\xi_0 \in C_c^\infty(\Omega_{\frac{R_\Omega}{16}})$,

$$\sum_{i=0}^N \xi_i(x) = 1 \quad \text{for all } x \in W, \tag{3.6.6}$$

and

$$|\nabla^k \xi_i| \le \frac{c(n,k)}{R_\Omega^k} \quad \text{on } W, \text{ for all } k \in \mathbb{N}\,. \tag{3.6.7}$$

3.6.2 Boundary defining function

Starting from the partition of unity $\{\xi_i\}_{i=0}^N$, and the local charts $\{\phi^i\}_{i=1}^N$, we can construct the boundary defining function of $\partial\Omega$ as in [130, Proposition 5.43].

For any $\varepsilon \in [0, R_\Omega)$ and $j = 1, \dots, N$, we define the rotated cylinders

$$K_\varepsilon^j := (T^j)^{-1}\big(B'_{R_\Omega - \varepsilon} \times (-\ell, \ell)\big)\,, \tag{3.6.8}$$

where $\ell = R_\Omega(1 + L_\Omega)$. Let $f^j : K_0^j \to \mathbb{R}$ be the functions defined as

$$f^j(x) := z_n - \phi^j(z')\,, \quad z = T^j x\,,$$

and observe that from (3.6.2) we have

$$\begin{aligned} \{f^j = 0\} &= \partial\Omega \cap K_0^j \\ \{f^j < 0\} &= \Omega \cap K_0^j \end{aligned} \tag{3.6.9}$$

A boundary defining function of $\overline{\Omega}$ is the function $F : W \to \mathbb{R}$ defined as

$$F(x) := \sum_{j=1}^N f^j(x)\,\xi_j(x) - \xi_0(x)\,, \tag{3.6.10}$$

where the product $f^j(x)\,\xi_j(x)$ is set equal to zero if $x \notin \operatorname{supp}\xi_j$. Since each f^j is Lipschitz continuous, so is the function F.

Thanks to the properties of $\{\xi_j\}_{j=0}^N$, (3.6.2) and (3.6.9), it is easily seen that

$$\Omega = \{x \in W \,:\, F(x) < 0\} \quad \text{and} \quad \partial\Omega = \{x \in W \,:\, F(x) = 0\}\,. \tag{3.6.11}$$

3.6.3 Regularization and definition of the smooth approximating sets ω_m, Ω_m

For $i = 1, \dots, N$, we can define the smooth functions $\phi_m^i, \widetilde{\phi}_m^i : B'_{R_\Omega - \frac{1}{m}} \to \mathbb{R}$ as

$$\begin{gathered} \phi_m^i := M_m(\phi^i) + \|M_m(\phi^i) - \phi^i\|_{L^\infty(B'_{R_\Omega - 1/m})} + \frac{L_\Omega}{m} \\ \text{and} \\ \widetilde{\phi}_m^i := M_m(\phi^i) - \|M_m(\phi^i) - \phi^i\|_{L^\infty(B'_{R_\Omega - 1/m})} - \frac{L_\Omega}{m}\,. \end{gathered} \tag{3.6.12}$$

From the results of Proposition 3.4.4, we deduce that $\phi_m^i, \widetilde{\phi}_m^i \in C^\infty$ are L_Ω-Lipschitz functions, and

$$\begin{aligned} \frac{L_\Omega}{m} &\le \phi_m^i(y') - \phi^i(y') \le \frac{3\,L_\Omega}{m} \\ \frac{L_\Omega}{m} &\le \phi^i(y') - \widetilde{\phi}_m^i(y') \le \frac{3\,L_\Omega}{m}\,, \end{aligned} \tag{3.6.13}$$

for all $y' \in B'_{R_\Omega - 1/m}$ and $i = 1, \dots, N$. Taking inspiration from (3.6.9) and (3.6.11), we are led to define the functions

$$\begin{aligned} f_m^j(x) &:= z_n - \phi_m^j(z') \\ \tilde{f}_m^j(x) &:= z_n - \widetilde{\phi}_m^j(z')\,, \quad z = T^j x \in B'_{R_\Omega - \frac{1}{m}} \times (-\ell, \ell)\,, \end{aligned} \tag{3.6.14}$$

and functions $F_m, \widetilde{F}_m : W \to \mathbb{R}$ defined as

$$\begin{aligned} F_m(x) &:= \sum_{j=1}^{N} f_m^j(x)\,\xi_j(x) - \xi_0(x) \\ \widetilde{F}_m(x) &:= \sum_{j=1}^{N} \tilde{f}_m^j(x)\,\xi_j(x) - \xi_0(x)\,, \end{aligned} \tag{3.6.15}$$

where the products $f_m^j(x)\,\xi_j(x)$ and $\tilde{f}_m^j(x)\,\xi_j(x)$ have to be interpreted equal to zero when $x \notin \operatorname{supp}\xi_j$.

Clearly, F_m and $\widetilde{F}_m$ are C^∞-smooth functions on W, and since

$$\frac{L_\Omega}{m} \le f^j(x) - f_m^j(x) < \frac{3\,L_\Omega}{m}\,, \quad \frac{L_\Omega}{m} \le \tilde{f}_m^j(x) - f^j(x) < \frac{3\,L_\Omega}{m} \tag{3.6.16}$$

for all $x \in K^j_{1/m}$ thanks to (3.6.13), we then have

$$\frac{L_\Omega}{m} \le F(x) - F_m(x) \le \frac{3\,L_\Omega}{m}\,, \quad \frac{L_\Omega}{m} \le \widetilde{F}_m(x) - F(x) \le \frac{3\,L_\Omega}{m} \quad \text{for all } x \in W. \tag{3.6.17}$$

The approximating open sets Ω_m, ω_m are thus defined as follows

$$\Omega_m := \{x \in W \,:\, F_m(x) < 0\} \quad \text{and} \quad \omega_m := \{x \in W \,:\, \widetilde{F}_m(x) < 0\}\,, \tag{3.6.18}$$

with boundaries

$$\partial\Omega_m = \{x \in W \,:\, F_m(x) = 0\} \quad \text{and} \quad \partial\omega_m = \{x \in W \,:\, \widetilde{F}_m(x) = 0\}\,. \tag{3.6.19}$$

In particular, since $F_m(x) < F(x) < \widetilde{F}_m(x)$ for all $x \in W$, owing to (3.6.11) we have

$$\omega_m \Subset \Omega \Subset \Omega_m \quad \text{for all } m \in \mathbb{N}.$$

We now proceed to prove the remaining properties of Theorem 3.2.1 for the outer sets Ω_m. The proofs for the inner sets ω_m are analogous.

3.6.4 $\partial\Omega_m, \partial\omega_m$ are smooth manifolds.

Let us show that $\partial\Omega_m$ is a smooth manifold, with local charts $\{\psi_m^i\}_{i=1}^N$ defined on the same coordinate systems as $\{\phi^i\}_{i=1}^N$.

We fix a constant $\varepsilon_0 \in (0, R_\Omega/4)$, and for all $i = 1, \dots, N$ we set

$$F^i(y) = F(x) \quad \text{and} \quad F^i_m(y) = F_m(x) \quad \text{for } y = T^i x,\, x \in W.$$

Owing to (3.6.2) we have

$$\begin{gathered} \partial\Omega \cap K_0^i \cap K_0^j = (T^i)^{-1} G_{\phi^i} \cap K_0^j = (T^j)^{-1} G_{\phi^j} \cap K_0^i \\ \text{and} \\ \Omega \cap K_0^j \cap K_0^i = (T^i)^{-1} S_{\phi^i} \cap K_0^j \cap K_0^i = (T^j)^{-1} S_{\phi^j} \cap K_0^i \cap K_0^j\,, \end{gathered} \tag{3.6.20}$$

whenever $\partial\Omega \cap K_0^i \cap K_0^j \neq \emptyset$.

This piece of information will allow us to use the transversality property. Specifically, thanks to (3.6.20) we may apply Propositions 3.4.2-3.4.4 with functions $\phi = \phi^j$, $\psi = \phi^i$, isometry $T = T^i(T^j)^{-1}$, and defining set

$$U = U^{j,i} = \Pi\Big(G_{\phi^j} \cap T^j K_0^i\Big) \subset B'_{R_\Omega}\,.$$

Claim 1. There exists $m_0 > 0$ such that, for all $i = 1, \dots, N$, for all $m \geq m_0$ and all $x \in \big\{\frac{-3L_\Omega}{m_0} \leq F \leq \frac{3L_\Omega}{m_0}\big\} \cap K^i_{\varepsilon_0}$, we have

$$\frac{\partial F^i_m}{\partial y_n}(y) \geq \frac{1}{2\sqrt{1+L_\Omega^2}}\,, \quad \text{for all } y = T^i x \in B'_{R_\Omega - \varepsilon_0} \times (-\ell, \ell). \tag{3.6.21}$$

Suppose by contradiction this is false; then for every $k \in \mathbb{N}$, we may find $m_k \geq k$ and a sequence $x^k \in \big\{ -\frac{3L_\Omega}{k} \leq F \leq \frac{3L_\Omega}{k}\big\}$ such that $y^k = T^i x^k \in B'_{R_\Omega - \varepsilon_0} \times (-\ell, \ell)$ and

$$\frac{\partial F^i_{m_k}}{\partial y_n}(y^k) < \frac{1}{2\sqrt{1+L_\Omega^2}}\,, \quad \text{for all } k \in \mathbb{N} \tag{3.6.22}$$

By compactness, we may extract a subsequence, still labeled as x^k, such that $x^k \to x^0$, and in particular $x^0 \in \overline{K^i_0}$ and $F(x^0) = 0$, hence $x^0 \in \partial\Omega \cap \overline{K^i_0}$ due to (3.6.11).

Then, by the chain rule we have

$$\frac{\partial f^i_m}{\partial y_n}(x) = 1 \quad \text{and} \quad \frac{\partial f^j_m}{\partial y_n}(x) = \big(\mathcal{R}^j (\mathcal{R}^i)^t\big)_{nn} - \sum_{s=1}^{n-1} \frac{\partial \phi^j_m}{\partial z'_s}(z') \, \big(\mathcal{R}^j (\mathcal{R}^i)^t\big)_{sn}\,, \tag{3.6.23}$$

if $x \in \operatorname{supp} \xi_j$, where $z' = \Pi\, T^j x$. We now distinguish two cases:
(i) $j \in \{1, \dots, N\}$ is such that $x^0 \notin \operatorname{supp} \xi_j$. Then $\operatorname{dist}\big(x^0, \operatorname{supp} \xi_j\big) > 0$, hence $x^k \notin \operatorname{supp} \xi_j$ for all $k \geq k_0$ large enough.
(ii) $j \in \{1, \dots, N\}$ is such that $x^0 \in \operatorname{supp} \xi_j$. In this case, it follows that $x^0 \in \partial\Omega \cap K^i_0 \cap B_{\frac{R_\Omega}{4}}(x^j)$, so that from (3.6.20) we have $T^j x^0 \in G_{\phi^j} \cap B_{\frac{R_\Omega}{4}} \cap T^j \overline{K^i_{\varepsilon_0}}$. By setting $(z^k)' = \Pi\, T^j x^k$, we thus have

$$B'_{\frac{1}{m_k}}\big((z^k)'\big) \Subset \Pi\Big(G_{\phi^j} \cap T^j K^i_0\Big)\,,$$

for all $k \geq k_0$ large enough. Recalling the remarks after (3.6.20), by applying Proposition 3.4.4, and in particular the transversality property (3.4.11) in (3.6.23), we infer

$$\frac{\partial f^j_{m_k}}{\partial y_n}\big(x^k\big) = \big(\mathcal{R}^j (\mathcal{R}^i)^t\big)_{nn} - \sum_{s=1}^{n-1} \frac{\partial \phi^j_{m_k}}{\partial z'_s}\big((z^k)'\big) \, \big(\mathcal{R}^j (\mathcal{R}^i)^t\big)_{sn} \geq \frac{1}{\sqrt{1+L_\Omega^2}}\,,$$

provided $k \geq k_0$ is large enough.
In both cases, we have found that

$$\frac{\partial f^j_{m_k}}{\partial y_n}(x^k)\, \xi_j\big(x^k\big) \geq \frac{\xi_j(x^k)}{\sqrt{1+L_\Omega^2}} \quad \text{for all } j = 1, \dots, N \text{ and } k \geq k_0. \tag{3.6.24}$$

Also, owing to (3.6.16) and (3.6.9) we have

$$\begin{aligned} |f^j_{m_k}(x^k)| \left|\frac{\partial \xi_j(x^k)}{\partial y_n}\right| &\leq |f^j_{m_k}(x^k) - f^j(x^k)|\, |\nabla \xi_j(x^k)| + |f^j(x^k)|\, |\nabla \xi_j(x^k)| \\ &\leq \frac{1}{m_k} + |f^j(x^k)|\, |\nabla \xi_j(x^k)| \xrightarrow{k\to\infty} |f^j(x^0)|\, |\nabla \xi_j(x^0)| = 0\,, \end{aligned}$$

and $|\nabla\xi_0(x^k)| \to |\nabla\xi_0(x^0)| = 0$ since $x^0 \in \partial\Omega$. By coupling this piece of information with (3.6.6), (3.6.22) and (3.6.24), we finally obtain

$$\begin{aligned}\frac{1}{2\sqrt{1+L_\Omega^2}} > \frac{\partial F^i_{m_k}}{\partial y_n}(y^k) &= \sum_{j=1}^N \frac{\partial f^j_{m_k}}{\partial y_n}(x^k)\,\xi_j(x^k) + \sum_{j=1}^N f^j_{m_k}(x^k)\,\frac{\partial \xi_j}{\partial y_n}(x^k) - \frac{\partial \xi_0}{\partial y_n}(x^k)\\ &\geq \sum_{j=1}^N \frac{\xi_j(x^k)}{\sqrt{1+L_\Omega^2}} + \sum_{j=1}^N f^j_{m_k}(x^k)\,\frac{\partial \xi_j}{\partial y_n}(x^k) - \frac{\partial \xi_0}{\partial y_n}(x^k)\\ &\xrightarrow{k\to\infty} \sum_{j=1}^N \frac{\xi_j(x^0)}{\sqrt{1+L_\Omega^2}} = \frac{1}{\sqrt{1+L_\Omega^2}}\,,\end{aligned}$$

which is a contradiction, and thus (3.6.21) holds true.

Claim 2. There exists $m_0 > 0$ such that $\forall y' \in B'_{R_\Omega-\varepsilon_0}$, $\forall m \geq m_0$, $\exists y_n \in (-\ell,\ell)$ with $y = (y', y_n) = T^i x \in T^i W$ satisfying $F^i_m(y) \geq 0$.

Again, assume by contradiction this is false. Then for all $k \in \mathbb{N}$, we may find sequences $m_k \geq k$ and $(y^k)' \in B'_{R_\Omega-\varepsilon_0}$ such that

$$F^i_{m_k}\big((y^k)', y_n\big) < 0 \quad \text{for all } y_n \in (-\ell,\ell) \text{ such that } \big((y^k)', y_n\big) \in T^i W. \tag{3.6.25}$$

By compactness, we may find a subsequence, still labeled as $(y^k)'$, satisfying $(y^k)' \to (y^0)' \in \overline{B}'_{R_\Omega-\varepsilon_0}$. Fix $w_n \in (-\ell,\ell)$ such that $\big((y^0)', w_n\big) \in T^i W$, and let $\{w_n^k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence satisfying $w_n^k \xrightarrow{k\to\infty} w_n$. Then $\big((y^k)', w_n^k\big) \to \big((y^0)', w_n\big)$, so that $\big((y^k)', w_n^k\big) \in T^i W$ for $k \geq k_0$ large enough being W open, and from (3.6.25) we have $F^i_{m_k}\big((y^k)', w_n^k\big) < 0$. By using (3.6.17) and the Lipschitz continuity of F, it is readily shown that

$$\lim_{k\to\infty} F^i_{m_k}\big((y^k)', w_n^k\big) = F^i\big((y^0)', w_n\big)\,,$$

whence $F^i\big((y^0)', w_n\big) \leq 0$ for all w_n as above, but this contradicts the fact that $F^i\big((y^0)', w_n\big) > 0$ whenever $w_n > \phi^i\big((y^0)'\big)$ due to (3.6.11), hence Claim 2 is proven.

Now let $y' \in B'_{R_\Omega-\varepsilon_0}$; by (3.6.17) and since $F^i\big(y', \phi^i(y')\big) = 0$, we have $F^i_m\big(y', \phi^i(y')\big) < 0$. Thus, owing to Claim 2 we may find y_n such that $F^i_m(y', y_n) = 0$.

The monotonicity property (3.6.21) of Claim 1, and the fact that $\partial\Omega_m = \{F_m = 0\} \subset \{\frac{L_\Omega}{m} \leq F \leq \frac{3L_\Omega}{m}\}$ due to (3.6.17) ensure that such point y_n is unique for all $y' \in B'_{R_\Omega-\varepsilon_0}$. This entails the existence of a function $\psi^i_m : B'_{R_\Omega-\varepsilon_0} \to \mathbb{R}$ such that $F^i_m\big(y', \psi^i_m(y')\big) = 0$ for all $y' \in B'_{R_\Omega-\varepsilon_0}$. Furthemore, owing to (3.6.11) and (3.6.17), we have that $\psi^i_m(y') > \phi^i(y')$ for all $y' \in B'_{R_\Omega-\varepsilon_0}$, and from the implicit function theorem we also infer that $\psi^i_m \in C^\infty\big(B'_{R_\Omega-\varepsilon_0}\big)$. Moreover, via a compactness argument as in Claim 1-2 and (3.6.1), one can prove that

$$\begin{aligned}&\Big\{-\frac{3L_\Omega}{m} \leq F \leq \frac{3L_\Omega}{m}\Big\} \subset \bigcup_{i=1}^N B_{\frac{R}{8}}(x^i)\\ &\Big\{-\frac{3L_\Omega}{m} \leq F \leq \frac{3L_\Omega}{m}\Big\} \cap \operatorname{supp}\xi_0 = \emptyset\,, \quad \text{for all } m > m_0,\end{aligned} \tag{3.6.26}$$

so that, in particular, the cylinders $\big\{K^i_{2\varepsilon_0}\big\}_{i=1}^N$ are an open cover of $\partial\Omega_m$, and $\partial\Omega_m \cap \operatorname{supp}\xi_0 = \emptyset$ provided $m > m_0$ is large enough.

We have thus proven that $\partial\Omega_m$ is a C^∞-smooth manifold for $m > m_0$, with local boundary charts $\{\psi_m^i\}_{i=1}^N$ defined on the same coordinate cylinders as $\{\phi^i\}_{i=1}^N$, that is

$$\begin{aligned} T^i\partial\Omega_m \cap \big(B'_{R_\Omega-\varepsilon_0} \times (-\ell,\ell)\big) &= \big\{(y',\psi_m^i(y')) \,:\, y' \in B'_{R_\Omega-\varepsilon_0}\big\}, \\ T^i\Omega_m \cap \big(B'_{R_\Omega-\varepsilon_0} \times (-\ell,\ell)\big) &= \big\{(y',y_n) \,:\, y' \in B'_{R_\Omega-\varepsilon_0}\,,\, -\ell < y_n < \psi_m^i(y')\big\}. \end{aligned} \tag{3.6.27}$$

3.6.5 Approximation properties.

First, we show that there exists $m_0 > 0$ such that

$$\|\psi_m^i - \phi^i\|_{L^\infty(B'_{R_\Omega-2\,\varepsilon_0})} \le \frac{6\,L_\Omega\sqrt{1+L_\Omega^2}}{m} \quad \text{for all } m > m_0. \tag{3.6.28}$$

Assume by contradiction this is false; then we may find sequences $m_k \uparrow \infty$ and $(y^k)' \in B'_{R_\Omega-2\varepsilon_0}$ such that

$$\psi^i_{m_k}\big((y^k)'\big) - \phi^i\big((y^k)'\big) > \frac{6\,L_\Omega\sqrt{1+L_\Omega^2}}{m_k} \tag{3.6.29}$$

Up to a subsequence, we have $(y^k)' \to (y^0)' \in \overline{B}'_{R_\Omega-2\varepsilon_0}$, and $\psi^i_{m_k}\big((y^k)'\big) \to \ell_0 \in \mathbb{R}$. Furthermore, since $\Big((y^k)',\psi_m^i\big((y^k)'\big)\Big) \in \{F^i_{m_k}=0\} \subset T^i\{\frac{L_\Omega}{m_k} \le F \le \frac{3L_\Omega}{m_k}\}$, we readily infer that $F^i\big((y^0)',\ell_0\big) = 0$, whence $\ell_0 = \phi^i\big((y')^0\big)$ due to (3.6.11) and (3.6.2). By continuity we also have $\phi^i\big((y^k)'\big) \to \phi^i\big((y^0)'\big)$, which implies that

$$\psi^i_{m_k}\big((y^k)'\big) - \phi^i\big((y^k)'\big) \xrightarrow{k\to\infty} 0\,.$$

Then, for all $t \in [0,1]$, we have

$$\begin{aligned} &\Big|F^i\Big((y^k)', t\,\psi^i_{m_k}\big((y^k)'\big) + (1-t)\phi^i\big((y^k)'\big)\Big) - F^i\Big((y^k)',\phi^i\big((y^k)'\big)\Big)\Big| \\ &\qquad\qquad \le L_F\, t\,|\psi^i_{m_k}\big((y^k)'\big) - \phi^i\big((y^k)'\big)| \xrightarrow{k\to\infty} 0\,, \end{aligned}$$

where L_F denotes the Lipschitz constant of F. This implies that for all $k \ge k_0$ large enough, the line segment

$$\big\{(y^k)'\big\} \times \big[\phi^i\big((y^k)'\big), \psi^i_{m_k}\big((y^k)'\big)\big] \subset T^i\Big\{-\frac{3\,L_\Omega}{m_0} \le F \le \frac{3\,L_\Omega}{m_0}\Big\}\,.$$

Therefore, by using (3.6.2), (3.6.11) (3.6.17), (3.6.21) and (3.6.29), we obtain

$$\begin{aligned} \frac{3\,L_\Omega}{m_k} &\ge F^i\Big((y^k)',\phi^i\big((y^k)'\big)\Big) - F^i_{m_k}\Big((y^k)',\phi^i\big((y^k)'\big)\Big) = -F^i_{m_k}\Big((y^k)',\phi^i\big((y^k)'\big)\Big) \\ &= F^i_{m_k}\Big((y^k)',\psi^i_{m_k}\big((y^k)'\big)\Big) - F^i_{m_k}\Big((y')^k,\phi^i\big((y')^k\big)\Big) \\ &= \left(\int_0^1 \frac{\partial F^i_{m_k}}{\partial y_n}\Big((y^k)', t\,\psi^i_{m_k}\big((y^k)'\big) + (1-t)\,\phi^i\big((y^k)'\big)\Big)\,dt\right)\Big[\psi^i_{m_k}\big((y^k)'\big) - \phi^i\big((y^k)'\big)\Big] \\ &> \frac{1}{2\sqrt{1+L_\Omega^2}}\,\frac{6\,L_\Omega\sqrt{1+L^2}}{m_k} = \frac{3\,L_\Omega}{m_k}\,, \quad \text{for all } k \ge k_0 \text{ large enough,} \end{aligned}$$

which is a contradiction, hence (3.6.28) holds true.

Now, recalling that $\{K^j_{2\varepsilon_0}\}_{j=1}^N$ is an open cover of $\partial\Omega$ and $\partial\Omega_m$, from (3.6.2), (3.6.27) and (3.6.28), one can easily obtain that

$$\mathrm{dist}_{\mathcal{H}}\big(\partial\Omega_m, \partial\Omega\big) \leq \frac{6\,L_\Omega\sqrt{1+L_\Omega^2}}{m}\,.$$

This convergence property in the sense of Hausdorff immediately implies that $d_{\Omega_m} \leq c(n)\, d_\Omega$, and $\lim_{m\to\infty}|\Omega_m \setminus \Omega| = 0$ –see for instance [116, Proposition 2.2.23]– and thus (3.2.1), (3.2.2) and (3.2.3) are proven.

Let us now prove that Ω_m are connected. First observe that, being Ω connected and $\mathfrak{L}_\Omega$ Lipschitz continuous, the set $\Omega_{\frac{R_\Omega}{32}}$ given by (3.6.4) is connected as well. Then, owing to (3.2.3), (3.6.2) and (3.6.27) and the fact that $\{K^i_{2\varepsilon_0}\}_{i=1}^N$ is an open cover of $\partial\Omega_m$, we may write

$$\Omega_m = \bigcup_{i=1}^N \Big(\Omega_m \cap K^i_{2\varepsilon_0}\Big) \cup \Omega_{\frac{R}{32}}\,, \tag{3.6.30}$$

for all $m > m_0$. On the other hand, since Ω is a Lipschitz domain, and in particular it is connected and property (3.6.2) holds, every connected component of its boundary is a closed $(n-1)$-dimensional (Lipschitz) manifold. It follows that for each $i = 1, \dots, N$, there exists $j \neq i$ such that

$$\Omega \cap K^i_{2\varepsilon_0} \cap K^j_{2\varepsilon_0} \neq \emptyset\,.$$

Also, by construction we have $\Omega_{\frac{R}{32}} \cap (\Omega \cap K^i_{2\varepsilon_0}) \neq \emptyset$ for all $i = 1, \dots, N$.

Then, owing to (3.6.28), the same properties hold true for Ω_m, i.e.,

$$\Omega_m \cap K^i_{2\varepsilon_0} \cap K^j_{2\varepsilon_0} \neq \emptyset\,,$$

and $\Omega_{\frac{R}{32}} \cap (\Omega_m \cap K^i_{2\varepsilon_0}) \neq \emptyset$ for all $i = 1, \dots, N$. Finally, since $\Omega_m \cap K^i_{2\varepsilon_0}$ are connected open sets by (3.6.27), we infer that Ω_m is connected thanks to identity (3.6.30) and classical topological theorems regarding connected sets– see, e.g., [161, Theorem 23.3].

We now introduce the transition maps related to the local charts of $\partial\Omega$ and $\partial\Omega_m$.

First of all, note that thanks to (3.6.27), we have

$$\begin{gathered}\partial\Omega_m \cap K^i_{\varepsilon_0} \cap K^j_{\varepsilon_0} = (T^i)^{-1} G_{\psi^i_m} \cap K^j_{\varepsilon_0} = (T^j)^{-1} G_{\psi^j_m} \cap K^i_{\varepsilon_0}\\ \text{and}\\ \Omega_m \cap K^j_{\varepsilon_0} \cap K^i_{\varepsilon_0} = (T^i)^{-1} S_{\psi^i_m} \cap K^j_{\varepsilon_0} \cap K^i_{\varepsilon_0} = (T^j)^{-1} S_{\psi^j_m} \cap K^i_{\varepsilon_0} \cap K^j_{\varepsilon_0}\,,\end{gathered} \tag{3.6.31}$$

whenever $\partial\Omega_m \cap K^i_{\varepsilon_0} \cap K^j_{\varepsilon_0} \neq \emptyset$.

For all $i \in \{1, \dots, N\}$, we define the set of indexes

$$\mathcal{I}_i := \big\{ j \in \{1, \dots, N\} \,:\, \partial\Omega \cap K^i_{2\varepsilon_0} \cap K^j_{2\varepsilon_0} \neq \emptyset \big\}\,.$$

If $j \in \mathcal{I}_i$, then owing to (3.6.2) there exists $y' \in B'_{R_\Omega - 2\varepsilon_0}$ such that $(T^i)^{-1}\big(y', \phi^i(y')\big) \in \partial\Omega \cap K^j_{2\varepsilon_0}$. Since ϕ^j is L_Ω-Lipschitz continuous and $\phi^j(0') = 0$, we have $|\phi^j(z')| \leq L_\Omega\,|z'|$, so it follows from (3.6.20), (3.6.27) and (3.6.28) that $(T^i)^{-1}\big(y', \psi^i_m(y')\big) \in \partial\Omega_m \cap K^i_{\varepsilon_0} \cap K^j_{\varepsilon_0}$ for all $m \geq m_0$ large enough.

Henceforth, for all $j \in \mathcal{I}_i$, (3.6.20) and (3.6.31) allow us to define the transition maps $\mathcal{C}^{i,j}, \mathcal{C}^{i,j}_m$ from ϕ^i to ϕ^j and from ψ^i_m to ψ^j_m respectively, i.e.,

$$\begin{aligned}\mathcal{C}^{i,j} y' &= \Pi\, T^j (T^i)^{-1}\big(y', \phi^i(y')\big)\\ \mathcal{C}^{i,j}_m y' &= \Pi\, T^j (T^i)^{-1}\big(y', \psi^i_m(y')\big)\,,\end{aligned} \tag{3.6.32}$$

which are defined on the open sets

$$U^{i,j} = \Pi\left(G_{\phi^i} \cap T^i K_0^j\right) \quad \text{and} \quad U_m^{i,j} = \Pi\left(G_{\psi_m^i} \cap T^i K_{\varepsilon_0}^j\right).$$

In particular, by their definitions and the arguments of Section 3.4, we may write

$$\begin{aligned} x &= (T^i)^{-1}\big(y', \phi^i(y')\big) = (T^j)^{-1}\big(\mathcal{C}^{i,j}y', \phi^j(\mathcal{C}^{i,j}y')\big) \quad \text{for } x \in \partial\Omega \cap K_0^i \cap K_0^j \\ x^m &= (T^i)^{-1}\big(y', \psi_m^i(y')\big) = (T^j)^{-1}\big(\mathcal{C}_m^{i,j}y', \psi_m^j(\mathcal{C}_m^{i,j}y')\big) \quad \text{for } x^m \in \partial\Omega_m \cap K_{\varepsilon_0}^i \cap K_{\varepsilon_0}^j. \end{aligned} \tag{3.6.33}$$

and their inverse functions are $(\mathcal{C}^{i,j})^{-1} = \mathcal{C}^{j,i}$ and $(\mathcal{C}_m^{i,j})^{-1} = \mathcal{C}_m^{j,i}$. Observe also that $\mathcal{C}^{i,i} = \mathcal{C}_m^{i,i} = \mathrm{Id}$.

Furthermore, since $\operatorname{supp}\xi_j \Subset B_{R_\Omega/4}(x^j) \Subset K_{2\varepsilon_0}^j$, it follows from the definition of $\mathcal{I}_i$ and (3.6.28) that

$$\xi_j\big((T^i)^{-1}(y', \phi^i(y'))\big) = \xi_j\big((T^i)^{-1}(y', \psi_m^i(y'))\big) = 0 \quad \text{if } j \notin \mathcal{I}_i, \tag{3.6.34}$$

for all $y' \in B'_{R_\Omega - \varepsilon_0}$, and all $m \geq m_0$.

We now claim that for all $j \in \mathcal{I}_i$, there exists an open set $V^{i,j} \subset B'_{R_\Omega - 2\varepsilon_0}$ for which we have

$$\xi_j\big((T^i)^{-1}(y', \phi^i(y'))\big) = \xi_j\big((T^i)^{-1}(y', \psi_m^i(y'))\big) = 0 \quad \text{if } y' \notin V^{i,j}, \tag{3.6.35}$$

and such that $V^{i,j} \subset U^{i,j} \cap U_m^{i,j}$ for all $m > m_0$. This in particular implies that both $\mathcal{C}^{i,j}$ and $\mathcal{C}_m^{i,j}$ are defined on $V^{i,j}$.

To this end, let

$$V^{i,j} := \Pi\Big(G_{\phi^i} \cap T^i K_{2\varepsilon_0}^j\Big) \cap B'_{R_\Omega - 2\varepsilon_0}.$$

Then, owing to (3.6.28) it is immediate to verify that

$$B'_{R_\Omega - 2\varepsilon_0} \cap \left(\Pi\Big(G_{\phi^i} \cap T^i B_{R_\Omega/4}(x^j)\Big) \cup \Pi\Big(G_{\psi_m^i} \cap T^i B_{R_\Omega/4}(x^j)\Big)\right) \Subset V^{i,j}, \tag{3.6.36}$$

whenever $m > m_0$ is large enough, and thus (3.6.35) is satisfied by our choice of set $V^{i,j}$.

Clearly $V^{i,j} \subset U^{i,j}$, so we are left to verify that $V^{i,j} \subset U_m^{i,j}$. To this end, let $y' \in V^{i,j}$; then by (3.6.31) and (3.6.33) we may write

$$T^j(T^i)^{-1}(y', \phi^i(y')) = (\mathcal{C}^{i,j}y', \phi^j(\mathcal{C}^{i,j}y')) \in B'_{R_\Omega - 2\varepsilon_0} \times \big(-L_\Omega(R_\Omega - 2\varepsilon_0), L_\Omega(R_\Omega - 2\varepsilon_0)\big),$$

where in the latter inclusion we made use of the inequality $|\phi^j(z')| \leq L_\Omega\,|z'|$. Therefore, thanks to (3.6.28), for $m > m_0$ we have $(T^i)^{-1}(y', \psi_m^i(y')) \in \partial\Omega_m \cap K_{\varepsilon_0}^i \cap K_{2\varepsilon_0}^j$, hence $y' \in U_m^{i,j}$ by (3.6.31) and the definition of $U_m^{i,j}$, so the claim is proven.

We also remark that

$$\bigcup_{j \in \mathcal{I}_i} V^{i,j} = B'_{R_\Omega - 2\varepsilon_0}, \tag{3.6.37}$$

since $\{T^i K_{2\varepsilon_0}^j\}_{j \in \mathcal{I}_i}$ is an open cover of $G_{\phi^i} \cap K_{2\varepsilon_0}^i$, and the projection map Π is a homeomorphism from G_{ϕ^i} (with the induced topology) to B'_{R_Ω}.

Moreover, owing to (3.6.28) and by proceeding as in the derivation of (3.4.18), we obtain

$$\|\mathcal{C}_m^{i,j} - \mathcal{C}^{i,j}\|_{L^\infty(V^{i,j})} \leq \frac{6\,L_\Omega\sqrt{1 + L_\Omega^2}}{m} \quad \text{for all } m > m_0. \tag{3.6.38}$$

Our next goal is to obtain estimates on $\nabla\psi^i_m$. To this end, we differentiate equation $F^i_m(y',\psi^i_m(y'))=0$ with respect to y'_k, for $k=1,\dots,n-1$, and recalling (3.6.34) we find

$$\frac{\partial\psi^i_m}{\partial y'_k}(y') = -\left(\frac{\partial F^i_m(y',\psi^i_m(y'))}{\partial y_n}\right)^{-1}\sum_{j\in\mathcal{I}_i}\left\{\frac{\partial f^j_m(x^m)}{\partial y'_k}\,\xi_j(x^m)+f^j_m(x^m)\,\frac{\partial\xi_j(x^m)}{\partial y'_k}\right\}, \tag{3.6.39}$$

where $x^m=(T^i)^{-1}\big(y',\psi^i_m(y')\big)$, $y'\in B'_{R_\Omega-2\varepsilon_0}$.

For all $l=1,\dots,n$, by using the chain rule and recalling the definition of $\mathcal{C}^{i,j}_m$, we find

$$\begin{aligned}&\frac{\partial f^i_m}{\partial y'_l}(x^m) = -\frac{\partial\phi^i_m}{\partial y'_l}(y') \quad\text{and}\quad \frac{\partial f^i_m}{\partial y_n}(x^m)=1\\ &\frac{\partial f^j_m}{\partial y_l}(x^m) = \big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{nl}-\sum_{r=1}^{n-1}\frac{\partial\phi^j_m}{\partial z'_r}(\mathcal{C}^{i,j}_m y')\,\big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{rl}\,,\end{aligned} \tag{3.6.40}$$

for all $j\in\mathcal{I}_i$ such that $x^m\in\operatorname{supp}\xi_j$. Since ϕ^j_m are L_Ω-Lipschitz continuous, from (3.6.40) it follows that

$$\sum_{l=1}^{n}\Big|\frac{\partial f^j_m(x^m)}{\partial y_l}\Big|\le c(n)(1+L_\Omega),\quad\text{for all } j\in\mathcal{I}_i. \tag{3.6.41}$$

Moreover, from (3.6.16), (3.6.28) and (3.6.9), we find that $f^j_m(x^m)\,|\nabla\xi_j(x^m)| \xrightarrow{m\to\infty} f^j(x^0)\,|\nabla\xi_j(x^0)|=0$, where $x^0=(T^i)^{-1}\big(y',\phi^i(y')\big)\in\partial\Omega$.

By making use of this piece of information, (3.6.41) and (3.6.21), from (3.6.39) we finally obtain the gradient estimate

$$|\nabla\psi^i_m(y')|\le c(n)\big(1+L^2_\Omega\big)\,,\quad\text{for all } y'\in B'_{R_\Omega-2\varepsilon_0}, \tag{3.6.42}$$

for all $i=1,\dots,N$ and $m>m_0$ large enough. In particular, owing to (3.6.28), (3.6.27) and (3.6.42), it is readily seen that Ω_m are $\mathcal{L}_{\Omega_m}$-Lipschitz domains, with

$$L_{\Omega_m}\le c(n)\big(1+L^2_\Omega\big)\quad\text{and}\quad R_{\Omega_m}\ge\frac{R_\Omega}{c(n)\,\big(1+L^2_\Omega\big)}\,,$$

and (3.2.4) is proven.

Next, the definition of $\mathcal{C}^{i,j}$ and $\mathcal{C}^{i,j}_m$, (3.6.42) and the L_Ω-Lipschitz continuity of ϕ^i imply

$$\sup_{i=1,\dots,N}\sup_{j\in\mathcal{I}_i}\Big\{\|\nabla\mathcal{C}^{i,j}\|_{L^\infty}+\|\nabla\mathcal{C}^{i,j}_m\|_{L^\infty}\Big\}\le c(n)(1+L^2_\Omega)\quad\text{for all } m>m_0, \tag{3.6.43}$$

and in particular $\mathcal{C}^{i,j}$ and $\mathcal{C}^{i,j}_m$ are uniformly bi-Lipschitz transformations.

Hence, thanks to (3.6.38) and (3.6.43), we are in the position to apply Proposition 3.3.1 and get

$$\frac{\partial\phi^j_m}{\partial z'_r}(\mathcal{C}^{i,j}_m y')\xrightarrow{m\to\infty}\frac{\partial\phi^j}{\partial z'_r}(\mathcal{C}^{i,j}y')\quad\text{for }\mathcal{H}^{n-1}\text{-a.e. } y'\in V^{i,j}. \tag{3.6.44}$$

From this, (3.6.21), (3.6.35), (3.6.37), (3.6.40) and identity (3.6.39) we find

$$\nabla\psi^i_m(y')\xrightarrow{m\to\infty}G(y')\quad\text{for }\mathcal{H}^{n-1}\text{-a.e. } y'\in B'_{R_\Omega-2\varepsilon_0}\,,$$

where G is a bounded vector valued function which can be explictly written. From (3.6.42) and on applying dominated convergence theorem, we get that $\nabla\psi^i_m\xrightarrow{m\to\infty}G$ in $L^p(B'_{R-2\varepsilon_0})$ for all $p\in[1,\infty)$. On the other hand, (3.6.28) and the uniqueness of the distributional limit imply that $G=\nabla\phi^i$, hence (3.2.5) is proven.

3.6.6 Curvature convergence

Assume now that $\partial\Omega \in W^{2,q}$ for some $q \in [1,\infty)$. Then the local charts $\phi^i \in W^{2,q}(B'_{R_\Omega})$.

We differentiate twice the identity $F^i_m(y', \psi^i_m(y')) = 0$ with respect to $y'_k\, y'_l$ for $k, l = 1, \dots n-1$, and find

$$\begin{aligned}\frac{\partial^2 \psi^i_m}{\partial y'_k \partial y'_l}(y') = -\Big(\frac{\partial F^i_m(y', \psi^i_m(y'))}{\partial y_n}\Big)^{-1} \Big\{ & \frac{\partial^2 F^i_m(y', \psi^i_m(y'))}{\partial y'_k \partial y'_l} + \frac{\partial^2 F^i_m(y', \psi^i_m(y'))}{\partial y'_l \partial y_n}\, \frac{\partial \psi^i_m}{\partial y'_k}(y') + \\ & + \frac{\partial^2 F^i_m(y', \psi^i_m(y'))}{\partial y'_k \partial y_n}\, \frac{\partial \psi^i_m}{\partial y'_l}(y') + \\ & + \frac{\partial^2 F^i_m(y', \psi^i_m(y'))}{\partial y_n \partial y_n}\, \frac{\partial \psi^i_m}{\partial y'_k}(y')\, \frac{\partial \psi^i_m}{\partial y'_l}(y') \Big\}. \end{aligned} \tag{3.6.45}$$

Elementary computations and (3.6.34) show that, for $l, r = 1, \dots n$, we have

$$\begin{aligned}\frac{\partial^2 F^i_m}{\partial y_r \partial y_l}(y', \psi^i_m(y')) = \sum_{j\in\mathcal{I}_i} \Big\{ & \frac{\partial^2 f^j_m}{\partial y_r \partial y_l}(x^m)\, \xi_j(x^m) + \frac{\partial f^j_m}{\partial y_r}(x^m)\, \frac{\partial \xi_j}{\partial y_l}(x^m) \\ & + \frac{\partial f^j_m}{\partial y_l}(x^m)\, \frac{\partial \xi_j}{\partial y_r}(x^m) + f^j_m(x^m)\, \frac{\partial^2 \xi_j}{\partial y_r \partial y_r}(x^m) \Big\}, \end{aligned} \tag{3.6.46}$$

where $x^m = (T^i)^{-1}(y', \psi^i_m(y'))$. We also have

$$\frac{\partial^2 f^j_m}{\partial y_r \partial y_l}(x^m) = -\sum_{s,t=1}^{n-1} \frac{\partial^2 \phi^j_m}{\partial z'_s \partial z'_t}(\mathcal{C}^{i,j}_m y')\, \big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{sr} \big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{tl} \tag{3.6.47}$$

for all $j \in \mathcal{I}_i$ such that $x^m \in \operatorname{supp} \xi_j$.

Thanks to (3.6.16), (3.6.28) and (3.6.9), we readily find that $f^j_m(x^m)\,|\nabla \xi_j(x^m)| \to 0$ and $f^j_m(x^m)\,|\nabla^2 \xi_j(x^m)| \to 0$. From this, and by using (3.6.7), (3.6.21), (3.6.41), (3.6.42) and (3.6.45)-(3.6.47), we obtain

$$|\nabla^2 \psi^i_m(y')| \le c(n)(1 + L^5_\Omega) \sum_{j\in\mathcal{I}_i} \Big\{ |\nabla^2 \phi^j_m|(\mathcal{C}^{i,j}_m y')\, \xi_j\big((T^i)^{-1}(y', \psi^i_m(y')\big) + \frac{(1+L_\Omega)}{R_\Omega} \Big\}, \tag{3.6.48}$$

for all $y' \in B'_{R_\Omega - 2\varepsilon_0}$, provided $m > m_0$ is large enough.

Then again, thanks to (3.6.38) and(3.6.43), we may apply Proposition 3.3.1 and infer

$$\nabla^2 \phi^j_m(\mathcal{C}^{i,j}_m y') \to \nabla^2 \phi^j(\mathcal{C}^{i,j} y') \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y' \in V^{i,j} \text{ and in } L^q(V^{i,j}). \tag{3.6.49}$$

Finally, recalling (3.6.35) and (3.6.37), the properties (3.6.21), (3.6.28), (3.6.40), (3.6.44), (3.6.45)-(3.6.49) and dominated convergence Theorem 3.3.2 entail

$$\nabla^2 \psi^i_m \to M\,, \quad \mathcal{H}^{n-1}\text{-a.e. on } B'_{R_\Omega - 2\varepsilon_0} \text{ and in } L^q(B'_{R_\Omega - 2\varepsilon_0})\,,$$

for some matrix valued function M, which can be explictly written in terms of $\phi^j, \nabla\phi^j, \nabla^2\phi^j$ and ξ_j. On the other hand, (3.6.28) and the uniqueness of the distributional limit imply that $M = \nabla^2 \phi^i$, hence (3.2.6) is proven.

3.6.7 Proof of the isocapacitary estimate (3.2.7)

In the following subsection, we will denote by $\widetilde{M}_m(h)$ the convolution of a function $h \in L^1_{loc}(\mathbb{R}^n)$ with respect to the first $(n-1)$-variables, i.e.,

$$\widetilde{M}_m(h)(z', z_n) = \int_{\mathbb{R}^{n-1}} h(x', z_n)\, \rho_m(z' - x')\, dx' \,.$$

We then have the following elementary lemma, which will be useful later.

Lemma 3.6.1. *Let $v \in C^\infty_c(\mathbb{R}^n)$. Then, if we set*

$$\widetilde{v}_m := \sqrt{\widetilde{M}_m(v^2)}\,,$$

we have that $\widetilde{v}_m$ is Lipschitz continuous on $\mathbb{R}^n$, and

$$|\nabla \widetilde{v}_m| \le c(n) \sqrt{\widetilde{M}_m\big(|\nabla v|^2\big)} \quad \text{a.e. on } \mathbb{R}^n. \tag{3.6.50}$$

Proof. By Hölder's inequality, for $k = 1, \dots, n$ we have

$$\left|\frac{\partial \widetilde{M}_m(v^2)}{\partial x_k}\right| = \left|\widetilde{M}_m\left(\frac{\partial v^2}{\partial x_k}\right)\right| = 2\left|\widetilde{M}_m\left(v\,\frac{\partial v}{\partial x_k}\right)\right| \le 2\sqrt{\widetilde{M}_m(v^2)}\sqrt{\widetilde{M}_m\left(\left|\frac{\partial v}{\partial x_k}\right|^2\right)}\,.$$

Therefore, on setting $\widetilde{v}_{\varepsilon,m} := \sqrt{\varepsilon^2 + \widetilde{M}_m(v^2)}$, for all $\varepsilon \in (0,1)$ we have that

$$|\nabla \widetilde{v}_{\varepsilon,m}| = \frac{\big|\nabla \widetilde{M}_m(v^2)\big|}{2\sqrt{\varepsilon^2 + \widetilde{M}_m(v^2)}} \le c(n)\,\frac{\sqrt{\widetilde{M}_m(v^2)}\sqrt{\widetilde{M}_m(|\nabla v|^2)}}{\sqrt{\varepsilon^2 + \widetilde{M}_m(v^2)}} \le c(n)\sqrt{\widetilde{M}_m\big(|\nabla v|^2\big)}\,. \tag{3.6.51}$$

Thus, the sequence $\{\widetilde{v}_{\varepsilon,m}\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $C^{0,1}_c(\mathbb{R}^n)$, and since $\widetilde{v}_{\varepsilon,m} \xrightarrow{\varepsilon\to 0^+} \widetilde{v}_m$ on $\mathbb{R}^n$, we deduce that $\widetilde{v}_m \in C^{0,1}_c(\mathbb{R}^n)$ by weak-$*$ compactness, and the thesis follows by letting $\varepsilon \to 0$ in (3.6.51) and by Rademacher's Theorem. □

Now let $x^0_m \in \partial\Omega_m$; then owing to (3.6.26) and (3.6.17), there exists $i \in \{1, \dots, N\}$ such that $x^0_m \in B_{R_\Omega/8}(x^i)$. Therefore, we may write $x^0_m = (T^i)^{-1}\Big((y^0)', \psi^i_m\big((y^0)'\big)\Big)$ for some $(y^0)' \in B'_{R_\Omega/8}$, and we also set $x^0 := (T^i)^{-1}\Big((y^0)', \phi^i\big((y^0)'\big)\Big) \in \partial\Omega$. Let

$$r_0 := \frac{R_\Omega}{C(n)\,\big(1 + L^2_\Omega\big)}\,,$$

for some fixed constant $C(n) > 1$ large enough, and consider $r \le r_0$, and $v \in C^\infty_c\big(B_r(x^0_m)\big)$. Then, since $B_r(x^0_m) \Subset B_{R_\Omega/4}(x^i) \Subset K^i_{2\varepsilon_0}$, we have

$$\int_{\partial\Omega_m} v^2\, |\mathcal{B}_{\Omega_m}|\, d\mathcal{H}^{n-1} = \int_{B'_{R_\Omega/4}} v^2\Big((T^i)^{-1}\big(y', \psi^i_m(y')\big)\Big)|\mathcal{B}_{\Omega_m}(y')|\,\sqrt{1 + |\nabla\psi^i_m(y')|^2}\,dy'.$$

Consider the new set of indices

$$\mathbb{J}^{x^0_m}_r := \big\{j \in \mathcal{I}_i \,:\, B_r(x^0_m) \cap \operatorname{supp}\xi_j \ne \emptyset\big\}\,.$$

Owing to (3.1.11), (3.6.33), (3.6.35), (3.6.42) and the Hessian estimate (3.6.48), we obtain

$$\int_{\partial\Omega_m} v^2\,|\mathcal{B}_{\Omega_m}|\,d\mathcal{H}^{n-1} \le \sqrt{1+L_\Omega^2}\int_{B'_{R_\Omega/4}} v^2\Big((T^i)^{-1}\big(y',\psi^i_m(y')\big)\Big)\,|\nabla^2\psi^i_m(y')|\,dy'$$

(3.6.52)

$$\le c(n)\,(1+L_\Omega^6)\sum_{j\in\mathbb{J}_r^{x_m^0}}\int_{V^{i,j}}\bigg\{v^2\Big((T^j)^{-1}\big(\mathcal{C}_m^{i,j}y',\psi^j_m(\mathcal{C}_m^{i,j}y')\big)\Big)\times$$

$$\times\,\xi_j\Big((T^j)^{-1}\big(\mathcal{C}_m^{i,j}y',\psi^j_m(\mathcal{C}_m^{i,j}y')\big)\Big)\,M_m\big(|\nabla^2\phi^j|\big)(\mathcal{C}_m^{i,j}y')\bigg\}\,dy'$$

$$+\,c(n)\,\frac{(1+L_\Omega^7)}{R_\Omega}\,|\mathbb{J}_r^{x_m^0}|\int_{B'_{R/4}} v^2\Big((T^i)^{-1}\big(y',\psi^i_m(y')\big)\Big)\,dy'\,.$$

By using $|\mathbb{J}_r^{x_m^0}|\le N$, (3.6.3), (3.2.4) and the results of Corollary 3.5.5, we get

$$\frac{(1+L_\Omega^7)}{R_\Omega}\,|\mathbb{J}_r^{x_m^0}|\int_{B'_{R_\Omega/4}} v^2\Big((T^i)^{-1}\big(y',\psi^j_m(y')\big)\Big)\,dy' \le c(n)\frac{(1+L_\Omega^7)\,d_\Omega^n}{R_\Omega^{n+1}}\int_{\partial\Omega_m} v^2\,d\mathcal{H}^{n-1}$$

(3.6.53)

$$\le\begin{cases} c'(n)\,\dfrac{(1+L_\Omega^{25})\,d_\Omega^n}{R_\Omega^{n+1}}\left(\displaystyle\int_{\mathbb{R}^n}|\nabla v|^2\,dx\right) r & \text{if } n\ge 3\\[2ex] c\,\dfrac{(1+L_\Omega^{31})\,d_\Omega^n}{R_\Omega^{n+1}}\left(\displaystyle\int_{\mathbb{R}^2}|\nabla v|^2\,dx\right) r\,\log\left(1+\dfrac{1}{r}\right) & \text{if } n=2.\end{cases}$$

On the other hand, via the change of variables $z'=\mathcal{C}_m^{i,j}y'$, by making use of (3.6.43), (3.6.36), and observing that $B_r(x_m^0)\Subset K^i_{2\varepsilon_0}\cap K^j_{2\varepsilon_0}$ for all $j\in\mathbb{J}_r^{x_m^0}$, $x_m^0\in\partial\Omega_m$ and $r\le r_0$, we find

(3.6.54)

$$\int_{V^{i,j}}\bigg\{v^2\Big((T^j)^{-1}\big(\mathcal{C}_m^{i,j}y',\psi^j_m(\mathcal{C}_m^{i,j}y')\big)\Big)\,\xi_j\Big((T^j)^{-1}\big(\mathcal{C}_m^{i,j}y',\psi^j_m(\mathcal{C}_m^{i,j}y')\big)\Big)\,M_m\big(|\nabla^2\phi^j|\big)(\mathcal{C}_m^{i,j}y')\bigg\}\,dy'$$

$$\le c(n)(1+L_\Omega^{(n-1)})\int_{W^{i,j}} w^2_{j,m}(z',0)\,M_m\big(|\nabla^2\phi^j|\big)(z')\,dz'\,,$$

for some open set $W^{i,j}\Subset\mathcal{C}^{i,j}(U^{i,j})$, where we also set

$$w_{j,m}(z',z_n):=v\Big((T^j)^{-1}\big(z',z_n+\psi^j_m(z')\big)\Big)\,.$$

Since $v\in C_c^\infty(B_r(x_m^0))$ and $x_m^0=(T^j)^{-1}\Big(\mathcal{C}_m^{i,j}\big((y^0)'\big),\psi^j_m\big((y^0)'\big)\Big)$ for all $j\in\mathbb{J}_r^{x_m^0}$, by using (3.6.42) it is readily seen that

$$w_{j,m}\in C_c^\infty\Big(B_{c(n)(1+L_\Omega^2)\,r}\Big(\mathcal{C}_m^{i,j}\big((y^0)'\big),0\Big)\Big)\,,$$

and from the chain rule we find

$$|\nabla w_{j,m}(z',z_n)|\le c(n)(1+L_\Omega^2)\,\Big|\nabla v\Big((T^j)^{-1}\big(z',z_n+\psi^j_m(z')\big)\Big)\Big| \tag{3.6.55}$$

Next, by using Fubini-Tonelli's Theorem we obtain

$$\int_{W^{i,j}} w^2_{j,m}(z',0)\,M_m\big(|\nabla^2\phi^j|\big)(z')\,dz' = \int_{W^{i,j}} w^2_{j,m}(z',0)\int_{B'_{1/m}(z')}|\nabla^2\phi^j(\tilde z')|\,\rho_m(z'-\tilde z')\,d\tilde z'\,dz'$$

$$\le\int_{W^{i,j}+B'_{1/m}}|\nabla^2\phi^j(\tilde z')|\Big(\int_{B'_{1/m(\tilde z')}} w^2_{j,m}(z',0)\,\rho_m(\tilde z'-z')\,dz'\Big)\,d\tilde z'\,.$$

We have thus found that

$$\int_{W^{i,j}} w_{j,m}^2(z',0)\, M_m\big(|\nabla^2\phi^j|\big)(z')\,dz' \le \int_{\widetilde{W}^{i,j}} \widetilde{M}_m(w_{j,m}^2)(z',0)\,|\nabla^2\phi^j(z')|\,dz'\,, \tag{3.6.56}$$

for some open set $\widetilde{W}^{i,j} \Subset \mathcal{C}^{i,j}(U^{i,j})$, provided $m > m_0$ is large enough.

Thanks to Lemma 3.6.1 and inequality (3.6.38), we easily infer

$$\sqrt{\widetilde{M}_m(w_{j,m}^2)} \in C_c^{0,1}\Big(B_{c(n)(1+L_\Omega^2)(r+\frac{1}{m})}\Big(\mathcal{C}^{i,j}\big((y^0)'\big),0\Big)\Big)\,,$$

and

$$\Big|\nabla\sqrt{\widetilde{M}_m(w_{j,m}^2)}\Big| \le c(n)\,\sqrt{\widetilde{M}_m\big(|\nabla w_{j,m}|^2\big)} \quad \text{a.e. on } \mathbb{R}^n. \tag{3.6.57}$$

Finally, set

$$\tilde{h}_{j,m}(x',x_n) := \sqrt{\widetilde{M}_m(w_{j,m}^2)}\Big(T^j\big(x',x_n-\phi^j(x')\big)\Big)$$

so that $\tilde{h}_{j,m}$ is Lipschitz continuous on $\mathbb{R}^n$. Moreover, thanks to (3.6.28), for all $j \in \mathbb{J}_r^{x^0_m}$, we have that

$$B_{c(n)(1+L_\Omega^3)(r+\frac{1}{m})}(x^0) \Subset K_{2\varepsilon_0}^i \cap K_{2\varepsilon_0}^j$$

for all $m > m_0$ sufficiently large and all $r \le r_0$, and thus we may write $x^0 = (T^j)^{-1}\Big(\mathcal{C}^{i,j}\big((y^0)'\big),\phi^j\big((y^0)'\big)\Big)$ due to (3.6.33). Recalling that ϕ^j is L_Ω-Lipschitz continous, it follows that

$$\tilde{h}_{j,m} \in C_c^{0,1}\Big(B_{c(n)(1+L_\Omega^3)(r+\frac{1}{m})}(x^0)\Big)\,,$$

and from the chain rule

$$\big|\nabla\tilde{h}_{j,m}(x',x_n)\big| \le c(n)(1+L_\Omega)\,\Big|\nabla\sqrt{\widetilde{M}_m(w_{j,m}^2)}(x',x_n-\phi^j(x'))\Big| \quad \text{for a.e. } x. \tag{3.6.58}$$

Owing to (3.1.11) and the definition of $\widetilde{h}_{j,m}$, we have

$$\begin{aligned}
\int_{\widetilde{W}^{i,j}} \widetilde{M}_m(w_{j,m}^2)(z',0)\,|\nabla^2\phi^j(z')|\,dz' &= \int_{\widetilde{W}^{i,j}} \tilde{h}_{j,m}^2\big((T^j)^{-1}(z',\phi^j(z'))\big)\,|\nabla^2\phi^j(z')|\,dz' \\
&\le c(n)(1+L_\Omega^3)\int_{\widetilde{W}^{i,j}} \tilde{h}_{j,m}^2\big((T^j)^{-1}(z',\phi^j(z'))\big)\,\big|\mathcal{B}_\Omega(z')\big|\sqrt{1+|\nabla\phi^j(z')|^2}\,dz' \\
&= c(n)(1+L_\Omega^3)\int_{\partial\Omega} \tilde{h}_{j,m}^2\big|\mathcal{B}_\Omega\big|\,d\mathcal{H}^{n-1} \\
&\le c(n)(1+L_\Omega^3)\left(\sup\frac{\int_{\partial\Omega} h^2\,\big|\mathcal{B}_\Omega\big|\,d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n}|\nabla h|^2\,dx}\right)\int_{\mathbb{R}^n}|\nabla\tilde{h}_{j,m}|^2\,dx\,,
\end{aligned} \tag{3.6.59}$$

where the supremum above is taken over all functions $h \in C_c^{0,1}\Big(B_{c(n)(1+L_\Omega^3)(r+\frac{1}{m})}(x^0)\Big)$.

Henceforth, by coupling (3.6.3) and estimates (3.6.52)-(3.6.59), for all $v \in C_c^\infty\big(B_r(x_m^0)\big)$ we obtain

$$\begin{aligned}
\int_{\partial\Omega_m} v^2 \left|\mathcal{B}_{\Omega_m}\right| d\mathcal{H}^{n-1} &\le c(n)\,(1+L_\Omega^{n+4}) \left(\sup \frac{\int_{\partial\Omega} h^2 \left|\mathcal{B}_\Omega\right| d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla h|^2\,dx} \right) \sum_{j\in\mathbb{J}_r^{x_m^0}} \int_{\mathbb{R}^n} \widetilde{M}_m\big(|\nabla w_{j,m}|^2\big)\,dx \\
&\qquad + \tilde{c} \int_{\mathbb{R}^n} |\nabla v|^2 dx \\
&\le c(n)\,(1+L_\Omega^{n+4}) \left(\sup \frac{\int_{\partial\Omega} h^2 \left|\mathcal{B}_\Omega\right| d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla h|^2\,dx} \right) \sum_{j\in\mathbb{J}_r^{x_m^0}} \int_{\mathbb{R}^n} |\nabla w_{j,m}|^2\,dx + \tilde{c}\int_{\mathbb{R}^n} |\nabla v|^2 dx \\
&\le c(n)\,(1+L_\Omega^{n+8})\, N \left(\sup \frac{\int_{\partial\Omega} h^2 \left|\mathcal{B}_\Omega\right| d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla h|^2\,dx} \right) \int_{\mathbb{R}^n} |\nabla v|^2\,dx + \tilde{c}\int_{\mathbb{R}^n} |\nabla v|^2 dx \\
&\le c'(n)\,(1+L_\Omega^{n+8}) \frac{d_\Omega^n}{R_\Omega^n} \left(\sup \frac{\int_{\partial\Omega} h^2 \left|\mathcal{B}_\Omega\right| d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla h|^2\,dx} \right) \int_{\mathbb{R}^n} |\nabla v|^2 dx + \tilde{c}\int_{\mathbb{R}^n} |\nabla v|^2 dx\,,
\end{aligned}$$

where in the second inequality we made use of Fubini-Tonelli's Theorem, the supremum above is taken over all $h \in C_c^{0,1}\Big(B_{c(n)(1+L_\Omega^3)(r+\frac{1}{m})}(x^0)\Big)$, and we set

$$\tilde{c} = \tilde{c}(n, L_\Omega, R_\Omega, d_\Omega, r) = \begin{cases} c(n) \dfrac{(1+L_\Omega^{25})\,d_\Omega^n}{R_\Omega^{n+1}}\, r & \text{if } n \ge 3 \\[2ex] c(n) \dfrac{(1+L_\Omega^{31})\,d_\Omega^n}{R_\Omega^{n+1}}\, r \,\log\Big(1+\dfrac{1}{r}\Big) & \text{if } n = 2. \end{cases} \tag{3.6.60}$$

Therefore, for all $x_m^0 \in \partial\Omega_m$, $r \le r_0$, we have found

$$\begin{aligned}
&\sup_{v\in C_c^\infty(B_r(x_m^0))} \frac{\int_{\partial\Omega_m} v^2 \left|\mathcal{B}_{\Omega_m}\right| d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla v|^2\,dx} \\
&\qquad \le \frac{c(n)\,(1+L_\Omega^{n+8})\,d_\Omega^n}{R_\Omega^n} \left(\sup_{\substack{x^0\,\in\,\partial\Omega \\ v\,\in\, C_c^{0,1}(B_{c(n)(1+L_\Omega^3)(r+1/m)}(x^0))}} \frac{\int_{\partial\Omega} v^2 \left|\mathcal{B}_\Omega\right| d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla v|^2\,dx} \right) + \tilde{c}\,.
\end{aligned}$$

From this, (3.6.60) and the isocapacitary equivalence [150, Theorem 2.4.1], we finally obtain the desired estimates

$$\begin{aligned}
\mathcal{K}_{\Omega_m}(r) \le \frac{c(n)\,(1+L_\Omega^{n+8})\,d_\Omega^n}{R_\Omega^n}\, &\mathcal{K}_\Omega\Big(c(n)(1+L_\Omega^3)(r+\tfrac{1}{m})\Big) \\
&+ \frac{c(n)\,(1+L_\Omega^{25})\,d_\Omega^n}{R_\Omega^{n+1}}\, r\,, \qquad \text{if } n \ge 3
\end{aligned} \tag{3.6.61}$$

and

$$\begin{aligned}
\mathcal{K}_{\Omega_m}(r) \le \frac{c(n)\,(1+L_\Omega^{n+8})\,d_\Omega^n}{R_\Omega^n}\, &\mathcal{K}_\Omega\Big(c(n)(1+L_\Omega^3)(r+\tfrac{1}{m})\Big) \\
&+ \frac{c(n)\,(1+L_\Omega^{31})\,d_\Omega^n}{R_\Omega^{n+1}}\, r\,\log\Big(1+\frac{1}{r}\Big)\,, \quad \text{if } n = 2,
\end{aligned} \tag{3.6.62}$$

for all $r \le r_0$ and $m > m_0$, and the proof is complete.

Chapter 4

Global gradient regularity and a Hopf Lemma for quasilinear operators of mixed local-nonlocal type

4.1 Main results

This final chapter is concerned about quasilinear operators of mixed local-nonlocal type, whose model example is given by $-\Delta_p u + (-\Delta_q)^s u$. Our results apply to a large family of operators of mixed type, which we now proceed to define.

Let $n \geq 2$ be an integer, $p, q \in (1, +\infty)$, and $s \in (0, 1)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We consider the operator

$$Qu := Q_L\, u + Q_N\, u, \tag{4.1.1}$$

defined as the sum of the local term

$$Q_L\, u(x) = Q_L^A\, u(x) := -\mathrm{div} A\big(x, Du(x)\big)$$

and of the nonlocal one

$$Q_N\, u(x) = Q_N^{\phi,\mathfrak{B},s,q}\, u(x) := 2\,\mathrm{P.V.} \int_{\mathbb{R}^n} \phi\big(u(x) - u(y)\big) \frac{\mathfrak{B}(x, y)}{|x - y|^{n+sq}}\, dy.$$

Here, $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field such that $A(x, \cdot) \in C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n)$ for all $x \in \Omega$, $A(\cdot, \xi) \in C^\alpha(\Omega; \mathbb{R}^n)$ for all $\xi \in \mathbb{R}^n$, and which satisfies the p-growth and coercivity conditions

$$\begin{cases} |A(x,\xi)| + |\xi||\nabla_\xi A(x,\xi)| \leq \Lambda \left(|\xi|^2 + \mu^2\right)^{\frac{p-2}{2}} |\xi| & \text{for } x \in \Omega,\ \xi \in \mathbb{R}^n \setminus \{0\}, \\ |A(x,\xi) - A(y,\xi)| \leq \Lambda \left(|\xi|^2 + \mu^2\right)^{\frac{p-1}{2}} |x - y|^\alpha & \text{for } x, y \in \Omega,\ \xi \in \mathbb{R}^n, \\ \langle \nabla_\xi A(x,\xi)\eta, \eta \rangle \geq \Lambda^{-1} \left(|\xi|^2 + \mu^2\right)^{\frac{p-2}{2}} |\eta|^2 & \text{for } x \in \Omega,\ \xi \in \mathbb{R}^n \setminus \{0\},\ \eta \in \mathbb{R}^n, \end{cases} \tag{4.1.2}$$

for some constants $\alpha \in (0, 1)$, $\mu \in [0, 1]$, and $\Lambda \geq 1$, while $\mathfrak{B} : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ is a measurable function satisfying

$$\mathfrak{B}(x, y) = \mathfrak{B}(y, x) \quad \text{and} \quad \Lambda^{-1} \leq \mathfrak{B}(x, y) \leq \Lambda \quad \text{for a.a. } x, y \in \mathbb{R}^n, \tag{4.1.3}$$

and $\phi \in C^0(\mathbb{R})$ is an odd, non-decreasing function fulfilling the q-growth and coercivity assumption

$$\Lambda^{-1}|t|^q \leq \phi(t)\, t \leq \Lambda |t|^q \quad \text{for all } t \in \mathbb{R}. \tag{4.1.4}$$

As already mentioned, the classical example of such an operator is

$$-\Delta_p u + (-\Delta_q)^s u, \tag{4.1.5}$$

which is obtained by taking $A(x,\xi) = |\xi|^{p-2}\xi$, $\phi(t) = |t|^{q-2}t$, and $\mathfrak{B}$ equal to a constant.

Our first result concerns the global differentiability of weak solutions to the Dirichlet problem for the operator Q. The notion of weak solution and the relevant functional spaces will be made precise in Section 4.1.1.

Theorem 4.1.1 (Global $C^{1,\theta}$-regularity)**.** *Let $p, q \in (1, +\infty)$ and $s \in (0, 1)$ be such that*

$$p > sq. \tag{4.1.6}$$

Let $\Omega \subset\subset \Omega' \subset \mathbb{R}^n$ be bounded open sets, with $\partial\Omega$ of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Suppose that A, $\mathfrak{B}$, and ϕ satisfy assumptions (4.1.2), (4.1.3), and (4.1.4). Let $f \in L^d(\Omega)$ for some $d > n$ and $g \in \mathbb{W}^{s,q}(\Omega) \cap W^{1,\infty}(\Omega') \cap C^{1,\alpha}(\partial\Omega)$. Let $u \in W^{1,p}_g(\Omega) \cap \mathbb{W}^{s,q}_g(\Omega)$ be the weak solution of the Dirichlet problem

$$\begin{cases} Qu = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.1.7}$$

Then, $u \in C^{1,\theta}(\overline{\Omega})$ and

$$\|u\|_{C^{1,\theta}(\overline{\Omega})} \leq C,$$

for some constants $\theta \in (0,1)$ and $C > 0$ depending only on n, p, q, s, Λ, d, α, Ω, and Ω', as well as on $\|f\|_{L^d(\Omega)}$, $\|g\|_{\mathbb{W}^{s,q}(\Omega)}$, $\|g\|_{W^{1,\infty}(\Omega')}$, and $\|g\|_{C^{1,\alpha}(\partial\Omega)}$.

Under virtually the same assumptions on Q, interior $C^{1,\theta}$ estimates and boundary almost Lipschitz regularity were established in [75]. Theorem 4.1.1 provides a strengthening of these results, in the case of a sufficiently regular outside datum g. We also point out that, for Q as in (4.1.5) with $p = q = 2$, $f \in L^\infty(\Omega)$, and $g \equiv 0$, global $C^{1,\theta}$-estimates have been obtained in [25, 183].

Like the majority of the results in [75], Theorem 4.1.1 relies crucially on assumption (4.1.6). This requirement ensures that the local operator Q_L is the leading term in (4.1.1), making it increasingly prevailing over Q_N at smaller scales and ultimately becoming the source of regularity. Clearly, (4.1.6) is satisfied if $p = q$, as, for instance, when $Qu = -\Delta_p u + (-\Delta_p)^s u$. If $p < sq$, then the leading term becomes Q_N, from which one should not be able to extract more than the global Hölder continuity of solutions—see [173] and [118]. Different is the case of interior regularity, where, in some cases, $C^{1,\theta}$ estimates are expected. However, to obtain them, one would need to fully understand the regularizing features of Q_N, something which at the moment is still lacking—see [32, 99] for some of the most relevant results in this direction.

Theorem 4.1.1 gives the $C^{1,\theta}$-regularity of the solution u of problem (4.1.7) up to the boundary of Ω, from the interior. However, no matter how nice the outer datum g is, u will in general be no more than Lipschitz across the boundary. This can be deduced as a particular consequence of the second result of this chapter, a Hopf type boundary point lemma for the operator Q.

In order to state and prove this result, we need to impose some additional regularity hypotheses on the operators Q_L and Q_N. Namely, we require that $A(\cdot,\xi) \in C^1(\Omega;\mathbb{R}^n)$ for all $\xi \in \mathbb{R}^n$ and that

$$|\nabla_x A(x,\xi)| \leq \Lambda \left(|\xi|^2 + \mu^2\right)^{\frac{p-2}{2}} |\xi| \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^n. \tag{4.1.8}$$

Note that this is a strengthening of the second line in (4.1.2). Concerning the operator Q_N, we assume that $\mathfrak{B} \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}^n)$, with

$$\big|\mathfrak{B}(x+w, y+z) - \mathfrak{B}(x,y)\big| \le \Lambda\big(|w| + |z|\big) \quad \text{for all } x, y, w, z \in \mathbb{R}^n, \tag{4.1.9}$$

and that $\phi \in C^1(\mathbb{R} \setminus \{0\})$, with

$$\Lambda^{-1}|t|^{q-2} \le \phi'(t) \le \Lambda |t|^{q-2} \quad \text{for all } t \in \mathbb{R} \setminus \{0\}. \tag{4.1.10}$$

Observe that condition (4.1.10) is stronger than (4.1.4) (up to taking a different Λ), as $\phi(0) = 0$—recall that ϕ is an odd continuous function.

Having made these additional assumptions, we can now state our Hopf lemma for Q-superharmonic functions—as before, see Section 4.1.1 for definitions. We recall that ν denotes the unit normal vector field of $\partial\Omega$, pointing outwards from Ω.

Theorem 4.1.2 (Hopf lemma)**.** *Let $p, q \in (1, +\infty)$ and $s \in (0,1)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class $C^{1,\alpha}$, for some $\alpha \in (0,1)$. Suppose that A, $\mathfrak{B}$, and ϕ satisfy assumptions* (4.1.2)*,* (4.1.3)*,* (4.1.8)*,* (4.1.9)*, and* (4.1.10)*. Let $u \in W^{1,p}(\Omega) \cap \mathbb{W}^{s,q}(\Omega) \cap C^0(\overline{\Omega})$ be a non-negative weak supersolution of $Qu = 0$ in Ω, positive in Ω and vanishing at a point $x_0 \in \partial\Omega$. Then,*

$$\liminf_{h \searrow 0} \frac{u(x_0 - h\nu(x_0))}{h} > 0. \tag{4.1.11}$$

We remark that Theorem 4.1.2 holds for every $p, q \in (1, +\infty)$ and $s \in (0,1)$—in particular, assumption (4.1.6) is not required here. Indeed, the result is not of perturbative nature and its proof treats both operators Q_L and Q_N as equals. In consequence of Theorem 4.1.1, the linear growth from the boundary implied by (4.1.11) is optimal when $p > sq$. We believe it is an interesting question to determine whether a stronger condition might hold when $p < sq$, such as

$$\liminf_{h \searrow 0} \frac{u(x_0 - h\nu(x_0))}{h^s} > 0,$$

in agreement with the Hopf lemmas available for fractional Laplacians—see [110, 81]. We point out that in the linear case (i.e., $p = q = 2$) and for domains having the interior ball condition, the Hopf lemma was obtained in [29]–see also [120].

Clearly, supersolutions of $Qu = 0$ might not be differentiable and thus the $\liminf$ in (4.1.11) might not in general be a limit. Of course, this is true unless the supersolution u is a priori assumed to be of class $C^1(\overline{\Omega})$ or if u is an actual solution of the equation and (4.1.6) is in force, thanks to Theorem 4.1.1.

In the following result, we showcase this last possibility and provide a unified statement which can be easily proved by combining Theorems 4.1.1 and 4.1.2 with the weak and the strong maximum principles for Q—see, e.g., the forthcoming Propositions 4.3.1 and 4.5.1.

Corollary 4.1.3. *Let $p, q \in (1, +\infty)$ and $s \in (0,1)$ be such that* (4.1.6) *holds true. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class $C^{1,\alpha}$, for some $\alpha \in (0,1)$. Suppose that A, $\mathfrak{B}$, and ϕ satisfy assumptions* (4.1.2)*,* (4.1.3)*,* (4.1.8)*,* (4.1.9)*, and* (4.1.10)*. Let $f \in L^d(\Omega)$, for some $d > n$, be a non-negative function and $u \in W^{1,p}(\Omega) \cap \mathbb{W}^{s,q}(\Omega)$ be the weak solution of*

$$\begin{cases} Qu = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then, $u \in C^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ depending only on n, p, q, s, Λ, d, α, Ω, and $\|f\|_{L^d(\Omega)}$. Furthermore, either $u \equiv 0$ in $\mathbb{R}^n$, or $u > 0$ in Ω and

$$-\frac{\partial u}{\partial \nu^-}(x_0) := \lim_{h \searrow 0} \frac{u\big(x_0 - h\,\nu(x_0)\big)}{h} > 0,$$

for every $x_0 \in \partial\Omega$.

Outline of the proofs. The proof of Theorem 4.1.1 is mostly based on the perturbation argument described in the Introduction, which was already exploited in [75, Theorem 5] for the interior Hölder continuity of (4.1.1). Namely, after carrying out a suitable flattening of the boundary, we establish Caccioppoli type estimates for the solution u of problem (4.1.7) near said flat parts of the boundary. Then, in the same spirit of [103, 135] for purely local operators, we make use of the perturbation argument and compare u to the solution of the local, autonomous, homogeneous problem in the half-ball, whose gradient regularity is well understood. This allows us to obtain finer estimates on the gradient of u and, in conjunction with the Caccioppoli estimates of the previous step, we ultimately get boundary Campanato type estimates for Du. The Hölder regularity of Du is then recovered as a consequence of the Campanato isomorphism.

For what concerns Theorem 4.1.2, its proof proceeds similarly to those usually employed to establish Hopf lemmas, via the construction of a suitable positive subsolution. Once this barrier is built, the conclusion then follows from the weak comparison principle—see, e.g., the forthcoming Proposition 4.3.1.

A first difficulty to face when building such a barrier comes from the mild regularity assumptions made on the boundary of Ω, which is only required to be $C^{1,\alpha}$—in particular, it might not satisfy the interior ball condition. After flattening the boundary through a specific diffeomorphism, this low regularity translates into a transformed operator having coefficients which may blow up near x_0. To overcome this difficulty, we construct an explicit subsolution v having second derivatives which blow up at a faster rate, with the correct sign. This method is, to the best of our knowledge, rather unexplored even in the case of a single local operator—see [107, 95] for similar approaches. We believe it might be further generalized past the Hölder continuity class and could lead to results for $C^{1,\mathrm{Dini}}$-regular boundaries, the optimal regularity under which the Hopf lemma holds in the local case—see, e.g., [191, 134, 10].

A second difficulty naturally lies in the fact that Q is the sum of two operators having different scaling and homogeneity properties. In order to circumvent this issue, we actually construct v in a way that makes it subharmonic for both Q_L and Q_N at the same time. As a technical remark, we point out that, to prove that v is a subsolution of $Q_N v = 0$ in a neighborhood of x_0, we need both a careful asymptotic analysis of the behavior of the part of $Q_N v$ localized around x_0 (in the mildly nonlocal regime $(1-s)q < 1$) and purely nonlocal techniques, adding a large bump function supported away from the boundary as in [81] (in the strongly nonlocal regime $(1-s)q \geq 1$).

4.1.1 Notation and definitions

Before passing to the proofs, we collect a few additional definitions and fix some of the terminology that we will use in the rest of the chapter. We assume that $p, q \in (1, +\infty)$, $s \in (0,1)$, and that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz continous boundary.

- We recall that $W^{1,p}(\Omega)$ denotes the Sobolev space of $L^p(\Omega)$ weakly differentiable functions having weak gradients in $L^p(\Omega)$, endowed with the usual norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

Given $g \in W^{1,p}(\Omega)$, we indicate with $W^{1,p}_g(\Omega)$ the subset of $W^{1,p}(\Omega)$ made up by those function whose traces on $\partial\Omega$ coincide with that of g. Writing

$$\begin{aligned}\mathscr{C}_\Omega &:= \big(\mathbb{R}^n \times \mathbb{R}^n\big) \setminus \big((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)\big) \\ &= \big(\Omega \times \Omega\big) \cup \big(\Omega \times (\mathbb{R}^n \setminus \Omega)\big) \cup \big((\mathbb{R}^n \setminus \Omega) \times \Omega\big),\end{aligned}$$

we define $\mathbb{W}^{s,q}(\Omega)$ to be the set of measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $u|_\Omega \in L^q(\Omega)$ and the map $(x, y) \mapsto |x - y|^{-n-sq}|u(x) - u(y)|^q$ is integrable over $\mathscr{C}_\Omega$. We norm this space by

$$\|u\|_{\mathbb{W}^{s,q}(\Omega)} := \|u\|_{L^q(\Omega)} + \left(\iint_{\mathscr{C}_\Omega} \frac{|u(x) - u(y)|^q}{|x - y|^{n+sq}}\, dxdy\right)^{\frac{1}{q}}.$$

Also, given $g \in \mathbb{W}^{s,q}(\Omega)$, we denote by $\mathbb{W}^{s,q}_g(\Omega)$ the space composed by all functions in $\mathbb{W}^{s,q}(\Omega)$ which agree with g outside of Ω.

- Let Q be as in the introduction to this chapter. Given $g \in W^{1,p}(\Omega) \cap \mathbb{W}^{s,q}(\Omega)$ and $f \in L^n(\Omega)$, we say that a function $u \in W^{1,p}_g(\Omega) \cap \mathbb{W}^{s,q}_g(\Omega)$ is a weak solution of the Dirichlet problem (4.1.7) if

$$\begin{aligned}&\int_\Omega A(x, Du(x)) \cdot D\varphi(x)\, dx \\ &\quad + \iint_{\mathscr{C}_\Omega} \phi\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big) \frac{\mathfrak{B}(x, y)}{|x - y|^{n+sq}}\, dxdy = \int_\Omega f\varphi\, dx,\end{aligned} \tag{4.1.12}$$

for every $\varphi \in W^{1,p}_0(\Omega) \cap \mathbb{W}^{s,q}_0(\Omega)$. Moreover, given two functions $u, v \in W^{1,p}(\Omega) \cap \mathbb{W}^{s,q}(\Omega)$, we say that $Qu \le Qv$ in Ω in the weak sense if

$$\begin{aligned}&\int_\Omega \Big(A(x, Du(x)) - A(x, Dv(x))\Big) \cdot D\varphi(x)\, dx \\ &\quad + \iint_{\mathscr{C}_\Omega} \Big(\phi\big(u(x) - u(y)\big) - \phi\big(v(x) - v(y)\big)\Big) \left(\varphi(x) - \varphi(y)\right) \frac{\mathfrak{B}(x, y)}{|x - y|^{n+sq}}\, dxdy \le 0,\end{aligned} \tag{4.1.13}$$

for every non-negative function $\varphi \in W^{1,p}_0(\Omega) \cap \mathbb{W}^{s,q}_0(\Omega)$. By taking respectively $v \equiv 0$ or $u \equiv 0$ in the above formulation, we obtain the definition of weak sub- and superharmonic functions for the operator Q in Ω, i.e., of weak sub- and supersolutions of $Qu = 0$ in Ω. We stress that the left-hand sides of (4.1.12) and (4.1.13) are well-defined and finite thanks to assumptions (4.1.2), (4.1.3), (4.1.4) on A, B, ϕ, while the finiteness of the right-hand side of (4.1.12) follows from the embedding of $W^{1,p}(\Omega)$ into $L^{\frac{n}{n-1}}(\Omega)$.

- In the next sections, we denote by C a constant greater than 1 and possibly changing from line to line. Unless otherwise specified, when it appears inside a proof it is assumed to depend on the quantities listed in the corresponding statement.

4.2 Proof of Theorem 4.1.1, global $C^{1,\theta}$-regularity

This section is devoted to the proof of Theorem 4.1.1. In order to make the exposition clearer, we divide it in a few steps.

Step 1: Reduction to nicer outside data

In this first preliminary step we show that, without loss of generality, the outside datum can be assumed to be compactly supported, of class $C^{1,\alpha}$ on $\partial\Omega$, and globally Lipschitz. In order to do this, we first establish the following global L^∞ estimate for the solution u. We stress that here assumption (4.1.6) is not required to hold.

Lemma 4.2.1. *Let $\Omega \subset\subset \Omega' \subset \mathbb{R}^n$ be bounded open sets with $\partial\Omega$ Lipschitz. Given $f \in L^n(\Omega)$ and $g \in W^{1,p}(\Omega) \cap \mathbb{W}^{s,q}(\Omega) \cap L^\infty(\Omega')$, let $u \in W^{1,p}_g(\Omega) \cap \mathbb{W}^{s,q}_g(\Omega)$ be a weak solution of problem (4.1.7). Then, $u \in L^\infty(\Omega)$ and it holds*

$$\|u\|_{W^{1,p}(\Omega)} + \|u\|_{L^\infty(\Omega)} \leq C$$

for some constant $C > 0$ depending only on n, p, q, s, Λ, Ω, and Ω', as well as on $\|f\|_{L^n(\Omega)}$, $\|g\|_{W^{1,p}(\Omega)}$, $\|g\|_{\mathbb{W}^{s,q}(\Omega)}$, and $\|g\|_{L^\infty(\Omega')}$.

The proof of this result is somewhat standard—it is similar, for instance, to that of [75, Proposition 2.1]. We thus postpone it to Section 4.6.

Let $\Omega'' \subset\subset \Omega'$ be an open set with $\Omega \subset\subset \Omega''$ and $\eta \in C^\infty_c(\mathbb{R}^n)$ be a smooth cutoff function satisfying $0 \leq \eta \leq 1$ in $\mathbb{R}^n$, $\eta = 1$ in Ω'', and $\operatorname{supp}(\eta) \subset\subset \Omega'$. Set $\hat{g} \coloneqq \eta g$ and $\hat{u} \coloneqq \eta u$. Then, $\hat{u}$ is a weak solution of

$$\begin{cases} Q\hat{u} = \hat{f} & \text{in } \Omega, \\ \hat{u} = \hat{g} & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\hat{f} = f + \bar{f}$, with

$$\bar{f}(x) \coloneqq 2\int_{\mathbb{R}^n\setminus\Omega''} \Big(\phi\big(u(x) - \eta(y)g(y)\big) - \phi\big(u(x) - g(y)\big)\Big) \frac{\mathfrak{B}(x,y)}{|x-y|^{n+sq}}\, dy \quad \text{for } x \in \Omega.$$

We have that $\bar{f} \in L^\infty(\Omega)$. To see it, we first observe that, since the Hausdorff distance $\operatorname{dist}(\Omega, \mathbb{R}^n \setminus \Omega'')$ is strictly positive and Ω is bounded, there exists a constant $C \geq 1$ depending only on Ω and Ω'' such that

$$C^{-1}(1+|y|) \leq |x-y| \leq C(1+|y|) \quad \text{for all } x \in \Omega \text{ and } y \in \mathbb{R}^n \setminus \Omega''. \tag{4.2.1}$$

Using assumptions (4.1.3)-(4.1.4), (4.2.1), and Hölder's inequality, we easily compute

$$\begin{aligned}
\|\bar{f}\|_{L^\infty(\Omega)} &\leq C \sup_{x\in\Omega} \left(\int_{\mathbb{R}^n\setminus\Omega''} \frac{|u(x)|^{q-1} + |g(y)|^{q-1}}{|x-y|^{n+sq}}\, dy \right) \\
&\leq C \left\{ \|u\|^{q-1}_{L^\infty(\Omega)} + \fint_\Omega \left(\int_{\mathbb{R}^n\setminus\Omega''} \frac{|g(y)|^{q-1}}{(1+|y|)^{n+sq}}\, dy \right) dz \right\} \\
&\leq C \left\{ \|u\|^{q-1}_{L^\infty(\Omega)} + \|g\|^{q-1}_{L^q(\Omega)} + \left(\iint_{\mathscr{C}_\Omega} \frac{|g(z)-g(y)|^q}{|z-y|^{n+sq}}\, dz dy \right)^{\frac{q-1}{q}} \right\},
\end{aligned}$$

for some $C \geq 1$ depending only on n, q, s, Λ, Ω, and Ω''. Thus, $\tilde{f}$ is bounded in Ω.

Also notice that $\hat{g} \in C^{1,\alpha}(\partial\Omega) \cap W^{1,\infty}(\mathbb{R}^n)$ and thus, since $\operatorname{supp}(\hat{g}) \subset \Omega'$, that $\hat{g} \in W^{a,\chi}(\mathbb{R}^n)$ for all $a \in (0,1)$ and $\chi \geq 1$, with corresponding norms in these spaces bounded only in terms of $\|g\|_{W^{1,\infty}(\Omega')}$ and $\|g\|_{C^{1,\alpha}(\partial\Omega)}$.

Step 2: Straightening of the boundary

We now proceed with the actual proof of Theorem 4.1.1. To do this, it is convenient to locally straighten the boundary around any given point $x_0 \in \partial\Omega$. Following the argument of [75, Section 5] (and mostly adopting its notation), we see that there exists a global $C^{1,\alpha}$-diffeomorphism $\mathcal{T}$ of $\mathbb{R}^n$ such that [1]

$$\begin{gathered}\mathcal{T}(x_0) = x_0, \qquad B^+_{r_0}(x_0) \subset \mathcal{T}\big(\Omega_{3r_0}(x_0)\big) \subset B^+_{4r_0}(x_0),\\ \Gamma_{r_0}(x_0) \subset \mathcal{T}\big(\partial\Omega \cap B_{3r_0}(x_0)\big) \subset \Gamma_{4r_0}(x_0),\end{gathered}$$

for some small radius $r_0 \in (0,1]$. Here $\Omega_r(x_0) = \Omega \cap B_r(x_0)$ and $\Gamma_r(x_0) = B_r(x_0) \cap \{x_n = 0\}$. Write $\mathcal{S} := \mathcal{T}^{-1}$ and $\mathfrak{c} := \big|\mathcal{J}_{\mathcal{S}}\big|$, with $\mathcal{J}_{\mathcal{S}}$ denoting the Jacobian determinant of the inverse $\mathcal{S}$. Let $\widetilde{\Omega} := \mathcal{T}(\Omega)$, $\tilde{g} := g \circ \mathcal{S}$, $\tilde{f} := \mathfrak{c}(f \circ \mathcal{S})$, and

$$\tilde{u} := u \circ \mathcal{S}. \tag{4.2.2}$$

It is easy to see that $\tilde{f} \in L^d(\widetilde{\Omega})$, $\tilde{g} \in W^{1,\infty}(\mathbb{R}^n) \cap C^{1,\alpha}(\Gamma_{r_0}(x_0))$, and $\tilde{u} \in W^{1,p}_{\tilde{g}}(\widetilde{\Omega}) \cap \mathbb{W}^{s,q}_{\tilde{g}}(\widetilde{\Omega})$. Moreover, $\tilde{u}$ is a weak solution of

$$\begin{cases} -\mathrm{div}\widetilde{A}(\cdot, D\tilde{u}) + \widetilde{Q}_N\, \tilde{u} = \tilde{f} & \text{in } \widetilde{\Omega},\\ \tilde{u} = \tilde{g} & \text{in } \mathbb{R}^n \setminus \widetilde{\Omega}, \end{cases} \tag{4.2.3}$$

where

$$\widetilde{A}(x,\xi) := \mathfrak{c}(x)\, A\big(\mathcal{S}(x), \xi(D\mathcal{T} \circ \mathcal{S})(x)\big)\, \big(D\mathcal{T} \circ \mathcal{S}\big)(x)^t$$

and

$$\widetilde{Q}_N\, u(x) := 2\,\mathrm{P.V.} \int_{\mathbb{R}^n} \phi\,(u(x) - u(y))\, \widetilde{K}(x,y)\, dy,$$

with

$$\widetilde{K}(x,y) := \mathfrak{c}(x)\mathfrak{c}(y)\, \frac{\mathfrak{B}\big(\mathcal{S}(x), \mathcal{S}(y)\big)}{|\mathcal{S}(x) - \mathcal{S}(y)|^{n+sq}}.$$

From assumptions (4.1.2)-(4.1.3) and the regularity of $\mathcal{T}$, we infer that

$$\begin{cases} 0 < \widetilde{\Lambda}^{-1} \le \mathfrak{c}(x) \le \widetilde{\Lambda} & \text{for all } x \in \mathbb{R}^n,\\ |\mathfrak{c}(x) - \mathfrak{c}(y)| \le \widetilde{\Lambda}|x-y|^\alpha & \text{for all } x, y \in B_{r_0}(x_0),\\ \widetilde{K}(x,y) = \widetilde{K}(y,x) & \text{for a.a. } x, y \in \mathbb{R}^n,\\ \dfrac{\widetilde{\Lambda}^{-1}}{|x-y|^{n+s\gamma}} \le \widetilde{K}(x,y) \le \dfrac{\widetilde{\Lambda}}{|x-y|^{n+s\gamma}} & \text{for a.a. } x, y \in \mathbb{R}^n. \end{cases}$$

and that the p-growth and coercivity conditions are preserved, namely

$$\begin{cases} |\widetilde{A}(x,\xi)| + |\xi||\nabla_\xi \widetilde{A}(x,\xi)| \le \widetilde{\Lambda}\, \left(|\xi|^2 + \mu^2\right)^{\frac{p-2}{2}} |\xi| & \text{for all } x \in B^+_{r_0}(x_0),\\ \big|\widetilde{A}(x,\xi) - \widetilde{A}(y,\xi)\big| \le \widetilde{\Lambda}(|\xi|^2 + \mu^2)^{\frac{p-1}{2}} |x-y|^\alpha & \text{for all } x, y \in B^+_{r_0}(x_0),\\ \langle \nabla_\xi \widetilde{A}(x,\xi)\eta, \eta\rangle \ge \widetilde{\Lambda}^{-1} \left(|\xi|^2 + \mu^2\right)^{\frac{p-2}{2}} |\eta|^2 & \text{for all } x \in B^+_{r_0}(x_0),\ \eta \in \mathbb{R}^n, \end{cases} \tag{4.2.4}$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and for some constant $\widetilde{\Lambda} \ge 1$ depending only on n, p, α, Λ, and Ω.

[1] In order to be consistent with [40, 75], throughout the rest of this chapter we will use that Lipschitz domains Ω, and in particular $C^{1,\alpha}$-domains, can be locally described as the supergraph of a boundary chart. Clearly, this only involves a simple change of orientation with respect to the coordinate system given by Definition 3.1.1.

Step 3: Preliminary estimates on $\tilde{u}$

To prove Theorem 4.1.1, we need a few lower order estimates on $\tilde{u}$, which mostly follow from the results of [75]. In order to obtain them, we first need to introduce the following "Caccioppoli" control quantity.

Given any point $\tilde{x}_0 \in \Gamma_{r_0/2}(x_0)$, radius $\varrho \in \big(0, \frac{r_0}{4}\big]$, and constants a, χ, γ satisfying

$$a \in (0,1), \quad \gamma > \max\{p, n\}, \quad \chi > \gamma, \quad a\chi > n, \tag{4.2.5}$$

we define

$$\begin{aligned} \mathtt{ccp}^{+}_{\gamma,a,\chi}(\varrho) &:= \varrho^{-p} \fint_{B^+_\varrho(\tilde{x}_0)} |\tilde{u} - \tilde{g}|^p \, dx + \int_{\mathbb{R}^n \setminus B_\varrho(\tilde{x}_0)} \frac{|\tilde{u}(y) - (\tilde{u})_{B_\varrho(\tilde{x}_0)}|^\gamma}{|y - \tilde{x}_0|^{n+s\gamma}} \, dy \\ &\quad + \left(\|\tilde{f}\|^{\frac{p}{p-1}}_{L^n(B^+_\varrho(\tilde{x}_0))} + 1 \right) + \left(\fint_{B^+_\varrho(\tilde{x}_0)} |D\tilde{g}|^\gamma \, dx \right)^{p/\gamma} \\ &\quad + \left(\varrho^{\chi(a-s)} \int_{B_\varrho(\tilde{x}_0)} \fint_{B_\varrho(\tilde{x}_0)} \frac{|\tilde{g}(x) - \tilde{g}(y)|^\chi}{|x-y|^{n+a\chi}} \, dx dy \right)^{\gamma/\chi}. \end{aligned} \tag{4.2.6}$$

We then have the following preliminary estimates. From now on, we assume the validity of condition (4.1.6) and all constants to depend on the quantities declared in the statement of Theorem 4.1.1.

Lemma 4.2.2. *The function $\tilde{u}$ defined by (4.2.2) belongs to $C^\beta(\mathbb{R}^n)$ for every $\beta \in (0,1)$ and it holds*

$$\|\tilde{u}\|_{C^\beta(\mathbb{R}^n)} \le C_\beta. \tag{4.2.7}$$

Moreover, it satisfies

$$\int_{B_t(\tilde{x}_0)} \fint_{B_t(\tilde{x}_0)} \frac{|\tilde{u}(x) - \tilde{u}(y)|^\gamma}{|x-y|^{n+s\gamma}} \, dx dy \le C_\beta \, t^{(\beta - s)\gamma} \quad \text{for all } \beta \in (s,1) \text{ and } t \in \left(0, \frac{r_0}{4}\right), \tag{4.2.8}$$

$$\int_{\mathbb{R}^n \setminus B_t(\tilde{x}_0)} \frac{|\tilde{u}(y) - (\tilde{u})_{B_t(\tilde{x}_0)}|^\gamma}{|y - \tilde{x}_0|^{n+s\gamma}} \, dy \le C \quad \text{for all } t \in \left(0, \frac{r_0}{4}\right), \tag{4.2.9}$$

$$\fint_{B^+_{\varrho/2}(\tilde{x}_0)} \left(|D\tilde{u}|^2 + \mu^2\right)^{p/2} dx \le C_\lambda \, \varrho^{-\lambda p} \quad \text{for all } \lambda > 0 \text{ and } \varrho \in \left(0, \frac{r_0}{4}\right). \tag{4.2.10}$$

The constant C_β may also depend on β, while C_λ also on λ.

Proof. The statement concerning the Hölder regularity of $\tilde{u}$ is the content of [75, Theorem 4 and Proposition 5.1]—see also Theorem 6 there and the discussion preceding its statement. To establish (4.2.8), it suffices to apply (4.2.7). Indeed,

$$\begin{aligned} \int_{B_t(\tilde{x}_0)} \fint_{B_t(\tilde{x}_0)} \frac{|\tilde{u}(x) - \tilde{u}(y)|^\gamma}{|x-y|^{n+s\gamma}} \, dx dy &\le [\tilde{u}]^q_{C^\beta(\mathbb{R}^n)} \int_{B_t(\tilde{x}_0)} \fint_{B_t(\tilde{x}_0)} \frac{dx dy}{|x-y|^{n+(s-\beta)\gamma}} \\ &\le C_\beta \int_{B_{2t}(\tilde{x}_0)} \frac{dz}{|z|^{n+(s-\beta)\gamma}} \le C_\beta \, t^{(\beta-s)\gamma}, \end{aligned}$$

where we made the change of variables $z = x - y$ and used the fact $B_t(\tilde{x}_0) - y \subset B_{2t}(\tilde{x}_0)$ for every $y \in B_t(\tilde{x}_0)$.

Regarding (4.2.9), we also use (4.2.7) and estimate

$$\int_{\mathbb{R}^n \setminus B_t(\tilde{x}_0)} \frac{|\tilde{u}(y) - (\tilde{u})_{B_t(\tilde{x}_0)}|^\gamma}{|y - \tilde{x}_0|^{n+s\gamma}} \, dx dy$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^n\setminus B_{r_0}(\tilde{x}_0)} \frac{|\tilde{u}(y)-(\tilde{u})_{B_t(\tilde{x}_0)}|^\gamma}{|y-\tilde{x}_0|^{n+s\gamma}}\,dxdy + 2^{q-1}\int_{B_{r_0}(\tilde{x}_0)\setminus B_t(\tilde{x}_0)} \frac{|\tilde{u}(y)-\tilde{u}(\tilde{x}_0)|^\gamma}{|y-\tilde{x}_0|^{n+s\gamma}}\,dy \\
&\quad + 2^{q-1}|\tilde{u}(\tilde{x}_0)-(\tilde{u})_{B_t(\tilde{x}_0)}|^\gamma \int_{B_{r_0}(\tilde{x}_0)\setminus B_t(\tilde{x}_0)} \frac{dy}{|y-\tilde{x}_0|^{n+s\gamma}} \\
&\le C\left\{ r_0^{-sq}\|\tilde{u}\|^q_{L^\infty(\mathbb{R}^n)} + [\tilde{u}]^q_{C^\beta(B_{r_0}(\tilde{x}_0))}\left(\int_{B_{r_0}} \frac{dz}{|z|^{n+(s-\beta)\gamma}} + t^{\beta q}\int_{\mathbb{R}^n\setminus B_t}\frac{dz}{|z|^{n+s\gamma}}\right)\right\} \\
&\le C_\beta\left(r_0^{-s\gamma} + r_0^{(\beta-s)\gamma} + t^{(\beta-s)\gamma}\right),
\end{aligned}
$$

for every $\beta\in(s,1)$. By choosing, e.g., $\beta=(1+s)/2$, we find the desired inequality (4.2.9).

Finally, to prove (4.2.10) we recall the boundary Caccioppoli inequality of [75, Lemma 5.1]: for every γ, a, χ satisfying (4.2.5), we have

$$
\fint_{B^+_{\varrho/2}(\tilde{x}_0)} \left(|D\tilde{u}|^2+\mu^2\right)^{p/2} dx + \int_{B_{\varrho/2}(\tilde{x}_0)}\fint_{B_{\varrho/2}(\tilde{x}_0)} \frac{|\tilde{u}(x)-\tilde{u}(y)|^\gamma}{|x-y|^{n+s\gamma}}\,dxdy \le C\,\mathtt{ccp}^+_{\gamma,a,\chi}(\varrho). \tag{4.2.11}
$$

Therefore, to obtain (4.2.10) we only need to estimate each term of (4.2.6). To this end, by using (4.2.7), the regularity of $\tilde{g}$, and the fact that $\tilde{u}(\tilde{x}_0)=\tilde{g}(\tilde{x}_0)$, we compute

$$
\begin{aligned}
\varrho^{-p}\fint_{B^+_\varrho(\tilde{x}_0)} |\tilde{u}-\tilde{g}|^p\,dx &\le 2^{p-1}\varrho^{-p}\fint_{B^+_\varrho(\tilde{x}_0)}\left(|\tilde{u}-\tilde{u}(\tilde{x}_0)|^p + |\tilde{g}-\tilde{g}(\tilde{x}_0)|^p\right)dx \\
&\le 2^{p-1}\left([\tilde{u}]^p_{C^\beta(B^+_\varrho(\tilde{x}_0))} + [\tilde{g}]^p_{C^\beta(B^+_\varrho(\tilde{x}_0))}\right)\varrho^{(\beta-1)\,p} \le C_\beta\varrho^{(\beta-1)\,p}.
\end{aligned} \tag{4.2.12}
$$

Clearly, for any fixed constants γ, a, χ satisfying (4.2.5), we have

$$
\left(\fint_{B^+_\varrho(\tilde{x}_0)} |D\tilde{g}|^\gamma\,dx\right)^{p/\gamma} \le C\,[\tilde{g}]^p_{W^{1,\infty}(B^+_\varrho(\tilde{x}_0))} \tag{4.2.13}
$$

and

$$
\begin{aligned}
&\left(\varrho^{\chi(a-s)}\int_{B_\varrho(\tilde{x}_0)}\fint_{B_\varrho(\tilde{x}_0)} \frac{|\tilde{g}(x)-\tilde{g}(y)|^\chi}{|x-y|^{n+a\chi}}\,dxdy\right)^{\gamma/\chi} \\
&\qquad \le \left(\varrho^{\chi(a-s)}[\tilde{g}]^\chi_{W^{1,\infty}(B_\varrho(\tilde{x}_0))}\int_{B_\varrho(\tilde{x}_0)}\fint_{B_\varrho(\tilde{x}_0)} \frac{dxdy}{|x-y|^{n+(a-1)\chi}}\right)^{\gamma/\chi} \le C\varrho^{(1-s)\gamma}.
\end{aligned} \tag{4.2.14}
$$

Therefore, by recalling that $\varrho\in(0,1]$ and $\tilde{f}\in L^n(\widetilde{\Omega})$, plugging (4.2.9), (4.2.12), (4.2.13), and (4.2.14) into (4.2.11), and choosing $\beta\ge 1-\lambda$, we are led to (4.2.10). □

Step 4: Boundary p-harmonic functions

We will obtain the Hölder continuity of the gradient of $\tilde{u}$ by comparing it to the solution $\tilde{h}\in W^{1,p}_{\tilde{u}}(B^+_{\varrho/4}(\tilde{x}_0))$ of the homogeneous Dirichlet problem

$$
\begin{cases}
\operatorname{div}\widetilde{A}(\tilde{x}_0, D\tilde{h}) = 0 & \text{in } B^+_{\varrho/4}(\tilde{x}_0),\\
\tilde{h}=\tilde{u} & \text{on } \partial B^+_{\varrho/4}(\tilde{x}_0).
\end{cases} \tag{4.2.15}
$$

Within this step, ϱ is a fixed radius in $(0, r_0/4)$.

In the next lemma we collect some useful properties of $\tilde{h}$, which are essentially all contained in [135, Lemma 5]. We also point out that the existence and uniqueness of $\tilde{h}$ is classical—it is mentioned for instance in [135] and it can be established via the theory of monotone operators (see, e.g., [193, Theorem 26.A]).

Lemma 4.2.3. *Let $\tilde{h}$ be the solution of problem* (4.2.15)*. Then, there exist constants $\sigma \in (0,1)$ and $C > 0$ such that,*

$$\fint_{B^+_{\varrho/4}(\tilde{x}_0)} \left(|D\tilde{h}|^2 + \mu^2\right)^{p/2} dx \le C \fint_{B^+_{\varrho/4}(\tilde{x}_0)} \left(|D\tilde{u}|^2 + \mu^2\right)^{p/2} dx, \tag{4.2.16}$$

$$\|\tilde{h}\|_{L^\infty(B^+_{\varrho/4}(\tilde{x}_0))} \le \|\tilde{u}\|_{L^\infty(B^+_{\varrho/4}(\tilde{x}_0))}, \qquad \operatorname*{osc}_{B^+_{\varrho/4}(\tilde{x}_0)} \tilde{h} \le \operatorname*{osc}_{B^+_{\varrho/4}(\tilde{x}_0)} \tilde{u}, \tag{4.2.17}$$

and

$$\operatorname*{osc}_{B^+_t(\tilde{x}_0)} D\tilde{h} \le C \left(\frac{t}{\varrho}\right)^\sigma \left\{ \fint_{B^+_{\varrho/4}(\tilde{x}_0)} \left(|D\tilde{h}|^2 + \mu^2\right)^{p/2} dx + \|\tilde{g}\|^p_{C^{1,\alpha}(\Gamma_{r_0}(x_0))} \right\}^{\frac{1}{p}}, \tag{4.2.18}$$

for all $t \in \left(0, \frac{\varrho}{8}\right]$.

Proof. Estimate (4.2.18) is established in [135, Lemma 5], while inequalities (4.2.17) are an immediate consequence of the weak maximum principle for the elliptic operator $h \mapsto \operatorname{div}\widetilde{A}(\tilde{x}_0, Dh)$. Estimate (4.2.16) can also be obtained by arguing as in the proof of [135, Lemma 5]. We provide here a complete proof for the reader's convenience.

By testing the weak formulation of (4.2.15) with $\tilde{h} - \tilde{u} \in W^{1,p}_0(B^+_t(\tilde{x}_0))$ and taking advantage of estimates (4.2.4), we find that

$$\begin{aligned}\int_{B^+_t(\tilde{x}_0)} \widetilde{A}(\tilde{x}_0, D\tilde{h}) \cdot D\tilde{h}\, dx &= \int_{B^+_t(\tilde{x}_0)} \widetilde{A}(\tilde{x}_0, D\tilde{h}) \cdot D\tilde{u}\, dx \\ &\le \widetilde{\Lambda} \int_{B^+_t(\tilde{x}_0)} \left(|D\tilde{h}|^2 + \mu^2\right)^{\frac{p-2}{2}} |D\tilde{h}||D\tilde{u}|\, dx.\end{aligned}$$

Using again hypothesis (4.2.4), we see that $\widetilde{A}(\tilde{x}_0, \xi) \cdot \xi \ge \min\left\{1, \frac{1}{p-1}\right\} \widetilde{\Lambda}^{-1} \left(|\xi|^2 + \mu^2\right)^{\frac{p-2}{2}} |\xi|^2$ for every $\xi \in \mathbb{R}^n$, so that

$$\int_{B^+_t(\tilde{x}_0)} \widetilde{A}(\tilde{x}_0, D\tilde{h}) \cdot D\tilde{h}\, dx \ge \frac{\widetilde{\Lambda}^{-1}}{p} \int_{B^+_t(\tilde{x}_0)} \left(|D\tilde{h}|^2 + \mu^2\right)^{\frac{p-2}{2}} |D\tilde{h}|^2\, dx.$$

Thus,

$$\int_{B^+_t(\tilde{x}_0)} \left(|D\tilde{h}|^2 + \mu^2\right)^{\frac{p-2}{2}} |D\tilde{h}|^2\, dx \le p\, \widetilde{\Lambda}^2 \int_{B^+_t(\tilde{x}_0)} \left(|D\tilde{h}|^2 + \mu^2\right)^{\frac{p-2}{2}} |D\tilde{h}||D\tilde{u}|\, dx. \tag{4.2.19}$$

Now, if $p \ge 2$ this yields

$$\int_{B^+_t(\tilde{x}_0)} |D\tilde{h}|^p\, dx \le p\, \widetilde{\Lambda}^2 \int_{B^+_t(\tilde{x}_0)} \left(|D\tilde{h}|^2 + \mu^2\right)^{\frac{p-1}{2}} \left(|D\tilde{u}|^2 + \mu^2\right)^{\frac{1}{2}} dx,$$

which immediately leads to (4.2.16) after an application of Hölder's inequality. If $p \in (1,2)$, we also exploit Hölder's inequality along with the fact that

$$\frac{t^{\frac{p}{p-1}}}{\left(t^2 + \mu^2\right)^{\frac{(2-p)p}{2(p-1)}}} \le \left(t^2 + \mu^2\right)^{\frac{p-2}{2}} t^2, \quad \text{for all } t \ge 0,$$

to deduce from (4.2.19) that

$$\begin{aligned}\int_{B_t^+(\tilde{x}_0)} & \big(|D\tilde{h}|^2+\mu^2\big)^{\frac{p-2}{2}}|D\tilde{h}|^2\,dx \\ & \leq p\,\widetilde{\Lambda}^2\Bigg(\int_{B_t^+(\tilde{x}_0)} \frac{|D\tilde{h}|^{\frac{p}{p-1}}}{\big(|D\tilde{h}|^2+\mu^2\big)^{\frac{(2-p)p}{2(p-1)}}}\,dx\Bigg)^{\frac{p-1}{p}}\bigg(\int_{B_t^+(\tilde{x}_0)}|D\tilde{u}|^p\,dx\bigg)^{\frac{1}{p}} \\ & \leq 2\widetilde{\Lambda}^2\bigg(\int_{B_t^+(\tilde{x}_0)}\big(|D\tilde{h}|^2+\mu^2\big)^{\frac{p-2}{2}}|D\tilde{h}|^2\,dx\bigg)^{\frac{p-1}{p}}\bigg(\int_{B_t^+(\tilde{x}_0)}|D\tilde{u}|^p\,dx\bigg)^{\frac{1}{p}}.\end{aligned}$$

This gives

$$\int_{B_t^+(\tilde{x}_0)}\big(|D\tilde{h}|^2+\mu^2\big)^{\frac{p-2}{2}}|D\tilde{h}|^2\,dx \leq 2^p\widetilde{\Lambda}^{2p}\int_{B_t^+(\tilde{x}_0)}|D\tilde{u}|^p\,dx \leq 4\widetilde{\Lambda}^{2p}\int_{B_t^+(\tilde{x}_0)}\big(|D\tilde{u}|^2+\mu^2\big)^{p/2}\,dx,$$

which, together with the trivial estimate

$$\int_{B_t^+(\tilde{x}_0)}\big(|D\tilde{h}|^2+\mu^2\big)^{\frac{p-2}{2}}\mu^2\,dx \leq \int_{B_t^+(\tilde{x}_0)}\mu^p \leq \int_{B_t^+(\tilde{x}_0)}\big(|D\tilde{u}|^2+\mu^2\big)^{p/2}\,dx,$$

readily yields (4.2.16). The proof of (4.2.16) is thus complete. □

Next, we consider the function $\widetilde{w} := \tilde{u}-\tilde{h} \in W_0^{1,p}(B_{\varrho/4}^+(\tilde{x}_0))$ and extend it to $\mathbb{R}^n$ by setting $\widetilde{w}\equiv 0$ in $\mathbb{R}^n \setminus B_{\varrho/4}^+(\tilde{x}_0)$. Note that this new function $\widetilde{w}$ belongs to $W^{s,q}(\mathbb{R}^n)$ and thus to $\mathbb{W}_0^{s,q}(B_{\varrho/4}^+(\tilde{x}_0))$. This is a consequence of its boundedness and of the fact that $p \geq sq$—see, e.g., [75, Lemma 2.4], [80, Lemma 5.1], and also the discussion at the beginning of the proof of [75, Lemma 5.2]. Furthermore, by (4.2.7) and (4.2.17), we infer that

$$\|\widetilde{w}\|_{L^\infty(B_{\varrho/4}^+(\tilde{x}_0))} \leq \operatorname*{osc}_{B_{\varrho/4}^+(\tilde{x}_0)}\tilde{u} + \operatorname*{osc}_{B_{\varrho/4}^+(\tilde{x}_0)}\tilde{h} \leq 2\operatorname*{osc}_{B_{\varrho/4}^+(\tilde{x}_0)}\tilde{u} \leq 2\,[\tilde{u}]_{C^\beta(B_{\varrho/4}^+(\tilde{x}_0))}\,\varrho^\beta \leq C_\beta\,\varrho^\beta, \tag{4.2.20}$$

for every $\beta\in(0,1)$ and for some constant $C_\beta>0$ depending also on β.

In order to continue with the proof of Theorem 4.1.1, we need to introduce a few more important quantities and recall a couple of useful inequalities. We set

$$V_\mu(\xi) := \big(|\xi|^2+\mu^2\big)^{\frac{p-2}{4}}\xi \quad \text{for } \xi\in\mathbb{R}^n.$$

It is not hard to see that there exists a constant $C>0$, depending only on n, p, and $\widetilde{\Lambda}$, for which

$$|V_\mu(\xi_1)-V_\mu(\xi_2)|^2 \leq C\left(\widetilde{A}(\tilde{x}_0,\xi_1)-\widetilde{A}(\tilde{x}_0,\xi_2)\right)\cdot(\xi_1-\xi_2) \quad \text{for all } \xi_1,\xi_2\in\mathbb{R}^n.$$

This is a consequence of the structural hypotheses (4.2.4)—see, e.g., [75, (2.10)]. As a consequence, defining $\widetilde{\mathcal{V}}^2 := |V_\mu(D\tilde{u})-V_\mu(D\tilde{h})|^2$, we see that

$$\widetilde{\mathcal{V}}^2 \leq C\left(\widetilde{A}(\tilde{x}_0,D\tilde{u})-\widetilde{A}(\tilde{x}_0,D\tilde{h})\right)\cdot D\widetilde{w} \quad \text{a.e. in } \mathbb{R}^n. \tag{4.2.21}$$

On the other hand, by using [75, (2.9)] and Hölder's inequality it follows that

$$\begin{aligned}\frac{1}{C}&\fint_{B_{\varrho/4}^+(\tilde{x}_0)}|D\tilde{u}-D\tilde{h}|^p\,dx \\ &\leq \begin{cases}\displaystyle\fint_{B_{\varrho/4}^+(\tilde{x}_0)}\widetilde{\mathcal{V}}^2\,dx & \text{if } p\geq 2,\\[2ex] \displaystyle\bigg(\fint_{B_{\varrho/4}^+(\tilde{x}_0)}\widetilde{\mathcal{V}}^2\,dx\bigg)^{\frac{p}{2}}\bigg(\fint_{B_{\varrho/4}^+(\tilde{x}_0)}\Big(|D\tilde{u}|^2+|D\tilde{h}|^2+\mu^2\Big)^{p/2}\,dx\bigg)^{\frac{2-p}{2}} & \text{if } p\in(1,2).\end{cases}\end{aligned} \tag{4.2.22}$$

Using these inequalities we may quantify the closeness of the gradients of $\tilde{u}$ and $\tilde{h}$, as described by the following result.

Lemma 4.2.4. *Let $\tilde{u}$ and $\tilde{h}$ be the functions defined in* (4.2.2) *and* (4.2.15)*, respectively. Then there exist constants $C > 0$ and $\bar{\sigma} \in (0,1)$ such that,*

$$\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |D\tilde{u} - D\tilde{h}|^p \, dx \le C\varrho^{\bar{\sigma}p}. \tag{4.2.23}$$

Proof. First we notice that, by definition of $\widetilde{w}$, (4.2.10), and (4.2.16), it holds

$$\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |D\widetilde{w}|^p \, dx \le C \fint_{B^+_{\varrho/4}(\tilde{x}_0)} \big(|D\tilde{u}|^2 + \mu^2\big)^{p/2} \, dx \le C_\lambda \, \varrho^{-\lambda p} \tag{4.2.24}$$

for every $\lambda > 0$ and for some constant $C_\lambda > 0$ depending also on λ. By plugging $\widetilde{w}$ in the weak formulations of both (4.2.3) and (4.2.15), taking advantage of (4.2.21), and arguing as in the proof of [75, Lemma 5.2], we estimate

$$\fint_{B^+_{\varrho/4}(\tilde{x}_0)} \widetilde{\mathcal{V}}^2 \, dx \le C\Big(I_1 + I_2 + I_3 + I_4\Big), \tag{4.2.25}$$

where

$$\begin{aligned}
I_1 &:= \varrho^\alpha \fint_{B^+_{\varrho/4}(\tilde{x}_0)} \big(|D\tilde{u}|^2 + \mu^2\big)^{(p-1)/2} |D\widetilde{w}| \, dx, \\
I_2 &:= \fint_{B^+_{\varrho/4}(\tilde{x}_0)} |\tilde{f}\widetilde{w}| \, dx, \\
I_3 &:= \int_{B_{\varrho/2}(\tilde{x}_0)} \fint_{B_{\varrho/2}(\tilde{x}_0)} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{\gamma-1} |\widetilde{w}(x) - \widetilde{w}(y)|}{|x-y|^{n+s\gamma}} \, dxdy, \\
I_4 &:= \int_{\mathbb{R}^n \setminus B_{\varrho/2}(\tilde{x}_0)} \left(\fint_{B_{\varrho/2}(\tilde{x}_0)} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{\gamma-1} |\widetilde{w}(x)|}{|x-y|^{n+s\gamma}} \, dx \right) dy.
\end{aligned} \tag{4.2.26}$$

By Hölder's inequality and (4.2.24), we get

$$I_1 \le C\varrho^\alpha \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} \big(|D\tilde{u}|^2 + \mu^2\big)^{p/2} \, dx \right)^{\frac{p-1}{p}} \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |D\widetilde{w}|^p \, dx \right)^{1/p} \le C_\lambda \, \varrho^{\alpha - \lambda p}, \tag{4.2.27}$$

for every $\lambda > 0$. Next, by using Hölder and Sobolev inequalities—recall that $\widetilde{w}$ vanishes on the boundary of $B^+_{\varrho/4}(\tilde{x}_0)$—together with (4.2.24), we infer

$$\begin{aligned}
I_2 &\le \frac{C}{\varrho^n} \|\tilde{f}\|_{L^n(B^+_{\varrho/4}(\tilde{x}_0))} \|\widetilde{w}\|_{L^{\frac{n}{n-1}}(B^+_{\varrho/4}(\tilde{x}_0))} \le \frac{C}{\varrho^n} \|\tilde{f}\|_{L^n(B^+_{\varrho/4}(\tilde{x}_0))} \|D\widetilde{w}\|_{L^1(B^+_{\varrho/4}(\tilde{x}_0))} \\
&\le C\varrho^{1-\frac{n}{d}} \|\tilde{f}\|_{L^d(B^+_{\varrho/4}(\tilde{x}_0))} \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |D\widetilde{w}|^p \, dx \right)^{1/p} \le C_\lambda \, \varrho^{1-\frac{n}{d}-\lambda},
\end{aligned} \tag{4.2.28}$$

for every $\lambda > 0$. We now take advantage of Hölder's inequality once again, estimate (4.2.8), and the interpolation inequality of [75, Lemma 2.4] in the ball $B_{\varrho/2}(\tilde{x}_0)$ to find

$$I_3 \le C \left(\int_{B_{\varrho/2}(\tilde{x}_0)} \fint_{B_{\varrho/2}(\tilde{x}_0)} \frac{|\tilde{u}(x) - \tilde{u}(y)|^\gamma}{|x-y|^{n+s\gamma}} \, dxdy \right)^{\frac{q-1}{q}} \left(\int_{B_{\varrho/2}(\tilde{x}_0)} \fint_{B_{\varrho/2}(\tilde{x}_0)} \frac{|\widetilde{w}(x) - \widetilde{w}(y)|^\gamma}{|x-y|^{n+s\gamma}} \, dxdy \right)^{\frac{1}{\gamma}}$$

$$
\begin{aligned}
&\leq C_\beta\, \varrho^{(\beta-s)(\gamma-1)+\vartheta-s}\, \|\widetilde{w}\|^{1-\vartheta}_{L^\infty(B^+_{\varrho/4}(\tilde{x}_0))} \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |D\widetilde{w}|^p\, dx \right)^{\vartheta/p} \\
&\leq C_{\beta,\lambda}\, \varrho^{(\beta-s)(\gamma-1)+\vartheta-s+\beta(1-\vartheta)-\vartheta\,\lambda},
\end{aligned}
$$

for every $\beta \in (s,1)$ and $\lambda > 0$, where $C_{\beta,\lambda}$ is a constant possibly depending on λ and β, and ϑ is defined by

$$
\vartheta := \begin{cases} s & \text{if } \gamma > p, \\ 1 & \text{if } \gamma \leq p. \end{cases}
$$

Note that in the last inequality we applied (4.2.20) and (4.2.24). From this estimate and the definition of ϑ, we deduce in particular that

$$
I_3 \leq C_{\beta,\lambda}\, \varrho^{(\beta-s)(\gamma-1)-\lambda}, \tag{4.2.29}
$$

for every $\beta \in (s,1)$ and $\lambda > 0$.

Finally, we estimate I_4. Since $\widetilde{w}$ is supported in $B^+_{\varrho/4}(\tilde{x}_0)$ and it holds

$$
\frac{|y-\tilde{x}_0|}{|x-y|} \leq 2 \quad \text{for every } x \in B^+_{\varrho/4}(\tilde{x}_0) \text{ and } y \in \mathbb{R}^n \setminus B_{\varrho/2}(\tilde{x}_0),
$$

we have

$$
\begin{aligned}
I_4 &\leq C \int_{\mathbb{R}^n \setminus B_{\varrho/2}(\tilde{x}_0)} \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} \frac{\left(|\tilde{u}(x) - (\tilde{u})_{B_{\varrho/2}(\tilde{x}_0)}|^{\gamma-1} + |\tilde{u}(y) - (\tilde{u})_{B_{\varrho/2}(\tilde{x}_0)}|^{\gamma-1} \right) |\widetilde{w}(x)|}{|y-\tilde{x}_0|^{n+s\gamma}}\, dx \right) dy \\
&\leq C\varrho^{-s\gamma} \fint_{B^+_{\varrho/4}(\tilde{x}_0)} |\tilde{u}(x) - (\tilde{u})_{B_{\varrho/2}(\tilde{x}_0)}|^{\gamma-1}\, |\widetilde{w}(x)|\, dx \\
&\quad + C \left(\int_{\mathbb{R}^n \setminus B_{\varrho/2}(\tilde{x}_0)} \frac{|\tilde{u}(y) - (\tilde{u})_{B_{\varrho/2}(\tilde{x}_0)}|^{\gamma-1}}{|y-\tilde{x}_0|^{n+s\gamma}}\, dy \right) \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |\widetilde{w}(x)|\, dx \right),
\end{aligned} \tag{4.2.30}
$$

where in the second inequality we used that

$$
\int_{\mathbb{R}^n \setminus B_{\varrho/2}(\tilde{x}_0)} \frac{dy}{|y-\tilde{x}_0|^{n+s\gamma}} \leq C\varrho^{-s\gamma}. \tag{4.2.31}
$$

By Hölder's inequality, (4.2.7), and (4.2.20), we obtain

$$
\begin{aligned}
&\varrho^{-s\gamma} \fint_{B^+_{\varrho/4}(\tilde{x}_0)} |\tilde{u}(x) - (\tilde{u})_{B_{\varrho/2}(\tilde{x}_0)}|^{\gamma-1}\, |\widetilde{w}(x)|\, dx \\
&\qquad \leq C\varrho^{-s\gamma} \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |\tilde{u} - (\tilde{u})_{B_{\varrho/2}(\tilde{x}_0)}|^{\gamma}\, dx \right)^{\frac{q-1}{q}} \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |\widetilde{w}|^{\gamma}\, dx \right)^{\frac{1}{\gamma}} \\
&\qquad \leq C\varrho^{-s\gamma+\beta(\gamma-1)} [\tilde{u}]^{\gamma-1}_{C^\beta(B_{\varrho/2}(\tilde{x}_0))}\, \|\widetilde{w}\|_{L^\infty(B^+_{\varrho/4}(\tilde{x}_0))} \leq C_\beta\, \varrho^{(\beta-s)\gamma},
\end{aligned}
$$

whereas, by Hölder's inequality, (4.2.31), (4.2.9), and (4.2.20), we get

$$
\left(\int_{\mathbb{R}^n \setminus B_{\varrho/2}(\tilde{x}_0)} \frac{|\tilde{u}(y) - (\tilde{u})_{B_{\varrho/2}(\tilde{x}_0)}|^{\gamma-1}}{|y-\tilde{x}_0|^{n+s\gamma}}\, dy \right) \left(\fint_{B^+_{\varrho/4}(\tilde{x}_0)} |\widetilde{w}(x)|\, dx \right)
$$

$$
\begin{aligned}
&\leq C\varrho^{-s}\left(\int_{\mathbb{R}^n\setminus B_{\varrho/2}(\tilde{x}_0)}\frac{|\tilde{u}(y)-(\tilde{u})_{B_{\varrho/2}(\tilde{x}_0)}|^\gamma}{|y-\tilde{x}_0|^{n+s\gamma}}\,dy\right)^{1-\frac{1}{\gamma}}\|\widetilde{w}\|_{L^\infty(B^+_{\varrho/4}(\tilde{x}_0))}\\
&\leq C_\beta\,\varrho^{\beta-s}.
\end{aligned}
$$

By inserting these two inequalities into (4.2.30) and recalling that $\varrho\in(0,1]$ and $\gamma>1$, we find that

$$
I_4\leq C_\beta\,\varrho^{\beta-s}, \tag{4.2.32}
$$

for every $\beta\in(s,1)$.

All in all, by plugging (4.2.27), (4.2.28), (4.2.29), and (4.2.32) into (4.2.25)-(4.2.26), we obtain the integral inequality

$$
⨍_{B^+_{\varrho/4}(\tilde{x}_0)}\widetilde{\mathcal{V}}^2\,dx\leq C_{\beta,\lambda}\Big(\varrho^{\alpha-\lambda p}+\varrho^{1-\frac{n}{d}-\lambda}+\varrho^{(\beta-s)(\gamma-1)-\lambda}+\varrho^{\beta-s}\Big), \tag{4.2.33}
$$

for every $\beta\in(s,1)$ and $\lambda>0$. We now choose the constants λ and β as follows:

$$
\beta:=\frac{1+s}{2}\quad\text{and}\quad\lambda:=\min\left\{\frac{\alpha}{2p},\frac{1}{2}\Big(1-\frac{n}{d}\Big),\frac{(1-s)(\gamma-1)}{4}\right\},
$$

so that (4.2.33) becomes just

$$
⨍_{B^+_{\varrho/4}(\tilde{x}_0)}\widetilde{\mathcal{V}}^2\,dx\leq C\varrho^{\sigma_0\,p}, \tag{4.2.34}
$$

with $\sigma_0:=\frac{1}{p}\min\left\{\frac{\alpha}{2},\frac{1}{2}\big(1-\frac{n}{d}\big),\frac{(1-s)(\gamma-1)}{4},\frac{1-s}{2}\right\}$.

We are now in position to conclude, using (4.2.34) in combination with (4.2.22). When $p\geq 2$, estimate (4.2.23) follows immediately with $\bar{\sigma}=\sigma_0$. On the other hand, when $p\in(1,2)$ we estimate the second factor in (4.2.22) through (4.2.16) and (4.2.10), obtaining

$$
⨍_{B^+_{\varrho/4}(\tilde{x}_0)}|D\tilde{u}-D\tilde{h}|^p\,dx\leq C_\lambda\,\varrho^{\frac{\sigma_0 p-(2-p)\lambda}{2}\,p}\quad\text{for every }\lambda>0.
$$

Therefore, by choosing $\lambda:=\frac{\sigma_0\,p}{2(2-p)}$, we obtain the desired estimate (4.2.23) with $\bar{\sigma}=\frac{\sigma_0\,p}{4}$. The proof is thus complete. □

Step 5: Conclusion

Having Lemma 4.2.4, we are now ready to prove a Campanato type boundary estimate and thus, with it, Theorem 4.1.1.

Proposition 4.2.5. *Let $\tilde{u}$ be the function defined in (4.2.2). Then, there exist a radius $\varrho_0\in(0,1)$ and costants $C>0$, $\sigma_1\in(0,1)$ such that,*

$$
\sup_{\tilde{x}_0\in\Gamma_{r_0/2}}⨍_{B^+_\varrho(\tilde{x}_0)}\big|D\tilde{u}-(D\tilde{u})_{B^+_\varrho(\tilde{x}_0)}\big|^p\,dx\leq C\varrho^{\sigma_1\,p}\quad\textit{for every }\varrho\in(0,\varrho_0]. \tag{4.2.35}
$$

Proof. Let $t\in\big(0,\frac{\varrho}{8}\big]$, with $\varrho\in\big(0,\frac{r_0}{4}\big]$. For every $\tilde{x}_0\in\Gamma_{r_0/2}$, we have

$$
⨍_{B^+_t(\tilde{x}_0)}\big|D\tilde{u}-(D\tilde{u})_{B^+_t(\tilde{x}_0)}\big|^p\,dx
$$

$$
\begin{aligned}
&\leq 2^{p-1} \fint_{B_t^+(\tilde{x}_0)} \big|D\tilde{u} - D\tilde{h}\big|^p \, dx + 4^{p-1} \fint_{B_t^+(\tilde{x}_0)} \big|D\tilde{h} - (D\tilde{h})_{B_t^+(\tilde{x}_0)}\big|^p \, dx \\
&\quad + 4^{p-1} \big|(D\tilde{u})_{B_t^+(\tilde{x}_0)} - (D\tilde{h})_{B_t^+(\tilde{x}_0)}\big|^p \\
&\leq C \left\{ \fint_{B_t^+(\tilde{x}_0)} \big|D\tilde{u} - D\tilde{h}\big|^p \, dx + \fint_{B_t^+(\tilde{x}_0)} \big|D\tilde{h} - (D\tilde{h})_{B_t^+(\tilde{x}_0)}\big|^p \, dx \right\} \\
&\leq C \left\{ \left(\frac{\varrho}{t}\right)^n \fint_{B_{\varrho/4}^+(\tilde{x}_0)} \big|D\tilde{u} - D\tilde{h}\big|^p \, dx + \left(\operatorname*{osc}_{B_t^+(\tilde{x}_0)} D\tilde{h} \right)^p \right\}.
\end{aligned}
$$

Recalling (4.2.18), (4.2.23), (4.2.16), and (4.2.10), this yields

$$
\begin{aligned}
&\fint_{B_t^+(\tilde{x}_0)} \big|D\tilde{u} - (D\tilde{u})_{B_t^+(\tilde{x}_0)}\big|^p \, dx \\
&\quad \leq C \left\{ \left(\frac{\varrho}{t}\right)^n \varrho^{\bar{\sigma} p} + \left(\frac{t}{\varrho}\right)^{\sigma p} \left(\fint_{B_{\varrho/4}^+(\tilde{x}_0)} \big(|D\tilde{h}|^2 + \mu^2\big)^{p/2} \, dx + \|\tilde{g}\|^p_{C^{1,\alpha}(\Gamma_{r_0}(x_0))} \right) \right\} \\
&\quad \leq C_\lambda \left\{ \left(\frac{\varrho}{t}\right)^n \varrho^{\bar{\sigma} p} + \left(\frac{t}{\varrho}\right)^{\sigma p} \left(\varrho^{-\lambda p} + \|\tilde{g}\|^p_{C^{1,\alpha_b}(\Gamma_{r_0}(x_0))} \right) \right\} \leq C_\lambda \left\{ \left(\frac{\varrho}{t}\right)^n \varrho^{\bar{\sigma} p} + \left(\frac{t}{\varrho}\right)^{\sigma p} \varrho^{-\lambda p} \right\},
\end{aligned}
$$

for every $\lambda > 0$. By choosing $t := \frac{\varrho^{1+\frac{\bar{\sigma} p}{2n}}}{8}$ and $\lambda := \frac{\sigma \, \bar{\sigma} \, p}{4n}$, we then obtain

$$
\fint_{B_t^+(\tilde{x}_0)} \big|D\tilde{u} - (D\tilde{u})_{B_t^+(\tilde{x}_0)}\big|^p \, dx \leq C \, t^{\sigma_1 p} \quad \text{for every } t \in (0, \varrho_0),
$$

with $\varrho_0 := \frac{1}{8}\big(\frac{r_0}{4}\big)^{1+\frac{\bar{\sigma} p}{2n}}$ and $\sigma_1 := \min\left\{ \frac{n\bar{\sigma}}{2n+\bar{\sigma} p}, \frac{\sigma \, \bar{\sigma} \, p}{2(2n+\bar{\sigma} p)} \right\}$. This concludes the proof of (4.2.35), up to relabeling t as ϱ. □

Proof of Theorem 4.1.1. By combining the interior Campanato estimate of [75, Theorem 5] and the boundary estimate (4.2.35), the result follows via a standard covering argument and Campanato's characterization of Hölder spaces [41, 43]–see also [100, Section 5]. □

4.3 A weak comparison principle

The aim of this very brief section is to establish a weak comparison principle for the operator Q, which will be used shortly to prove Theorem 4.1.2. The precise statement is as follows.

Proposition 4.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Assume that A, B, and ϕ satisfy hypotheses* (4.1.2), (4.1.3), *and* (4.1.4). *Let $u, v \in W^{1,p}(\Omega) \cap \mathbb{W}^{s,q}(\Omega)$ be satisfying $Qu \leq Qv$ in Ω in the weak sense. If $u \leq v$ in $\mathbb{R}^n \setminus \Omega$, then $u \leq v$ in Ω as well.*

Proof. By plugging $\varphi = (u - v)_+$ in the weak formulation (4.1.13) and observing that, by the monotonicity of ϕ,

$$
\Big(\phi\big(u(x) - u(y)\big) - \phi\big(v(x) - v(y)\big)\Big)\Big(\big(u(x) - v(x)\big)_+ - \big(u(y) - v(y)\big)_+\Big) \geq 0,
$$

for a.e. $x, y \in \mathbb{R}^n$, we obtain that

$$
\int_{\Omega_+} \Big(A(x, Du(x)) - A(x, Dv(x))\Big) \cdot \big(Du(x) - Dv(x)\big) \, dx \leq 0,
$$

where $\Omega_+ := \{x \in \Omega : u(x) > v(x)\}$. From the third line of assumption (4.1.2) on A, it is immediate to deduce that the integrand above is non-negative and vanishes only at those points $x \in \Omega_+$ where $Du(x) = Dv(x)$—see, e.g., [71, Lemma 2.1 and Theorem 1.2].

Therefore, we conclude that $Du = Dv$ in Ω_+, and thus that $u \leq v$ in Ω. □

4.4 Proof of Theorem 4.1.2, Hopf Lemma

In this section we establish Theorem 4.1.2, whose proof will be divided into a few steps. Note that, given $r, \rho > 0$, we write $\mathcal{C}^+_{r,\rho} := B'_r \times (0, \rho)$ and $\mathcal{C}^+_r = \mathcal{C}^+_{r,r}$.

Step 1: Straightening of the boundary

Differently from Section 4.2, here we need to consider a more specific diffeomorphism of $\mathbb{R}^n$ in order to pointwise evaluate the operator Q.

Up to a rigid movement, we may assume that $x_0 = 0$ and $\nu(0) = -e_n$. Therefore, since $\partial\Omega$ is of class $C^{1,\alpha}$, there exist a radius $R \in (0, 1)$ and a function $h \in C^{1,\alpha}(\mathbb{R}^{n-1})$ vanishing outside of B'_{4R}, satisfying

$$h(0') = 0, \quad D'h(0') = 0', \tag{4.4.1}$$

and such that

$$\begin{aligned} \Omega \cap B_{2R} &= \Big\{(x', x_n) \in B_{2R} : x_n > h(x')\Big\}, \\ \partial\Omega \cap B_{2R} &= \Big\{(x', x_n) \in B_{2R} : x_n = h(x')\Big\}. \end{aligned} \tag{4.4.2}$$

Here, we denoted by $D'h$ the gradient of h with respect to the first $(n-1)$-variable x'.

Then, by suitably modifying h in $B'_{4R} \setminus B'_{3R}$, we may also assume that

$$(-\Delta)^{\frac{1}{2}} h(0) = \pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \text{P.V.} \int_{\mathbb{R}^{n-1}} \frac{h(0) - h(z')}{|z'|^n} \, dz' = 0. \tag{4.4.3}$$

Indeed, it suffices to replace h by the function $h + \ell\phi$, for an arbitrary $\phi \in C^\infty_c(B'_{4R} \setminus B'_{3R})$ and with $\ell := -\Big((-\Delta)^{\frac{1}{2}}\phi(0)\Big)^{-1}(-\Delta)^{\frac{1}{2}} h(0)$.

We straighten the boundary of Ω inside B_{2R} via a suitable diffeomorphism $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ globally of class $C^{1,\alpha}$, but actually smooth inside $\Omega \cap B_{2R}$. In order to do this, we first consider a nice extension of h to the whole space $\mathbb{R}^n$.

Lemma 4.4.1. *Given $\alpha \in (0, 1)$, let $h \in C^{1,\alpha}(\mathbb{R}^{n-1})$ be a compactly supported function satisfying (4.4.1) and (4.4.3). Then, there exists a function $\mathscr{H} \in C^{1,\alpha}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n_+)$ such that $\mathscr{H}(x', 0) = h(x')$ for all $x' \in \mathbb{R}^{n-1}$, $D\mathscr{H}(0) = 0$,*

$$\|\mathscr{H}\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq C \, \|h\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} \tag{4.4.4}$$

and

$$\left|D^2\mathscr{H}(y', y_n)\right| \leq C \, [D'h]_{C^\alpha(\mathbb{R}^{n-1})} \, y_n^{\alpha-1} \quad \text{for all } (y', y_n) \in \mathbb{R}^n_+, \tag{4.4.5}$$

for some constant $C > 0$ depending only on n and α.

We take as $\mathscr{H}$ a suitable $C^{1,\alpha}(\mathbb{R}^n)$-continuation of the harmonic extension of h to the upper half-space. The proof of Lemma 4.4.1 is then rather natural and follows from the Poisson representation for $\mathscr{H}$. For this reason, we postpone it to Section 4.7 and resume here the proof of Theorem 4.1.2.

Let

$$\eta := \left(1 + 2\,\|D\mathscr{H}\|_{L^\infty(\mathbb{R}^n)}\right)^{-1}, \tag{4.4.6}$$

and define $\mathcal{S} : \mathbb{R}^n \to \mathbb{R}^n$ by setting

$$\mathcal{S}(y', y_n) := \left(y', y_n + \mathscr{H}(y', \eta\, y_n)\right) \quad \text{for all } (y', y_n) \in \mathbb{R}^n.$$

Clearly, its Jacobian matrix is given by

$$D\mathcal{S}(y', y_n) = \left(\begin{array}{c|c} \mathrm{Id}_{n-1} & 0' \\ \hline D'\mathscr{H}(y', \eta\, y_n)^t & 1 + \eta\, \partial_{y_n}\mathscr{H}(y', \eta\, y_n) \end{array}\right). \tag{4.4.7}$$

Denoting with $\mathfrak{c} := \mathcal{J}_{\mathcal{S}}$ its Jacobian determinant, we have

$$\mathfrak{c}(y) = \partial_{y_n}\mathcal{S}^n(y', y_n) = 1 + \eta\, \partial_{y_n}\mathscr{H}(y', \eta\, y_n),$$

so that (4.4.4), (4.4.5), and (4.4.6) entail

$$\mathfrak{c}(y) \in \left[\frac{1}{2}, \frac{3}{2}\right] \quad \text{for all } y \in \mathbb{R}^n, \tag{4.4.8}$$

and

$$\begin{cases} \|\mathfrak{c}\|_{C^\alpha(\mathbb{R}^n)} \le C \\ |D\mathfrak{c}(y)| \le C\, y_n^{\alpha-1} \quad \text{for all } y \in \mathbb{R}^n_+. \end{cases} \tag{4.4.9}$$

In this step, C indicates a constant depending only on n, α, and $\|h\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}$. Therefore, it is immediate to see that $\mathcal{S}$ is a $C^{1,\alpha}$-diffeomorphism of $\mathbb{R}^n$ onto itself, such that

$$\mathcal{S}(\mathbb{R}^n_+) = \left\{(x', x_n) \in \mathbb{R}^n : x_n > h(x')\right\}, \quad \mathcal{S}(\partial\mathbb{R}^n_+) = \left\{(x', x_n) \in \mathbb{R}^n : x_n = h(x')\right\}. \tag{4.4.10}$$

In particular, setting $\mathcal{T} = \mathcal{S}^{-1}$, explicit computations show that

$$(D\mathcal{T} \circ \mathcal{S})(y) = D\mathcal{S}(y)^{-1} = \left(\begin{array}{c|c} \mathrm{Id}_{n-1} & 0' \\ \hline \dfrac{-D'\mathscr{H}(y', \eta\, y_n)^t}{1 + \eta\, \partial_{y_n}\mathscr{H}(y', \eta\, y_n)} & \dfrac{1}{1 + \eta\, \partial_{y_n}\mathscr{H}(y', \eta\, y_n)} \end{array}\right). \tag{4.4.11}$$

Then, from (4.4.4), (4.4.6), (4.4.7), and (4.4.11), we infer that

$$\|D\mathcal{S}\|_{C^\alpha(\mathbb{R}^n)} + \|D\mathcal{T} \circ \mathcal{S}\|_{C^\alpha(\mathbb{R}^n)} \le C. \tag{4.4.12}$$

In particular, these estimates yield the global Lipschitz bounds

$$C^{-1}|y - z| \le |\mathcal{S}(y) - \mathcal{S}(z)| \le C\,|y - z| \quad \text{for all } y, z \in \mathbb{R}^n. \tag{4.4.13}$$

Since $DH(0) = 0$, we have that $D\mathcal{S}(0) = (D\mathcal{T} \circ \mathcal{S})(0) = \mathrm{Id}_n$ and thus, by (4.4.12),

$$\big|D\mathcal{S}(y) - \mathrm{Id}_n\big| + \big|(D\mathcal{T} \circ \mathcal{S})(y) - \mathrm{Id}_n\big| \le C\,|y|^\alpha \quad \text{for all } y \in \mathbb{R}^n.$$

This also implies that

$$\begin{cases} \frac{1}{2}\,|\xi|^2 \le \big\langle D\mathcal{S}(y)\,\xi, \xi\big\rangle \le 2\,|\xi|^2, \\ \frac{1}{2}\,|\xi|^2 \le \big\langle (D\mathcal{T} \circ \mathcal{S})(y)\,\xi, \xi\big\rangle \le 2\,|\xi|^2, \qquad \text{for all } y \in B_{r_0},\ \xi \in \mathbb{R}^n, \\ \frac{1}{2}\,|\xi| \le \big|D\mathcal{S}(y)\,\xi\big| \le 2\,|\xi|, \end{cases} \tag{4.4.14}$$

for some $r_0 \in (0,1)$ suitably small, in dependence of n, α, and $\|h\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}$ only. Moreover, by differentiating (4.4.7) and (4.4.11), taking advantage of estimate (4.4.5), and recalling definition (4.4.6), we find

$$\big|D^2\mathcal{S}(y)\big| + \big|D_y(D\mathcal{T} \circ \mathcal{S})(y)\big| \le C\,y_n^{\alpha-1} \quad \text{for all } y \in \mathbb{R}^n_+. \tag{4.4.15}$$

We now transform the operator Q via $\mathcal{S}$. Recalling (4.4.2), (4.4.10), and since $\mathcal{S}(0) = 0$, thanks to (4.4.12) we can find $\tau \in (0,1)$, depending only on n, α, and $\|h\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}$, such that

$$\mathcal{S}\big(\mathcal{C}^+_{4r}\big) \subset \Omega \cap B_R \tag{4.4.16}$$

for all $r \in (0, \tau R)$. Let $r \in (0, \frac{r_0}{8})$ be as such and define $\tilde{u} := u \circ \mathcal{S}$. As u is a weak supersolution of $Qu = 0$ in Ω, simple computations show that $\tilde{u} \in W^{1,p}(\mathcal{C}^+_{2r}) \cap \mathbb{W}^{s,q}(\mathcal{C}^+_{2r}) \cap C^0\big(\overline{\mathcal{C}^+_{2r}}\big)$ is a weak supersolution of $\widetilde{Q}\tilde{u} = 0$ in $\mathcal{C}^+_{2r}$, where $\widetilde{Q}$ is defined by $\widetilde{Q} := \widetilde{Q}_{\mathrm{L}} + \widetilde{Q}_{\mathrm{N}}$ and

$$\widetilde{Q}_{\mathrm{L}}\,\tilde{u}(y) := -\mathrm{div}\widetilde{A}(y, D\tilde{u}(y)), \quad \text{with } \widetilde{A}(y,\xi) := \mathfrak{c}(y) A\big(\mathcal{S}(y), \xi(D\mathcal{T} \circ \mathcal{S})(y)\big)(D\mathcal{T} \circ \mathcal{S})(y)^t,$$
$$\widetilde{Q}_{\mathrm{N}}\,\tilde{u}(y) := 2\,\mathfrak{c}(y)\,\mathrm{P.V.}\int_{\mathbb{R}^n} \phi\big(\tilde{u}(y) - \tilde{u}(z)\big)\frac{\mathfrak{B}\big(\mathcal{S}(y), \mathcal{S}(z)\big)}{|\mathcal{S}(y) - \mathcal{S}(z)|^{n+sq}}\,\mathfrak{c}(z)\,dz,$$

for all $y \in \mathcal{C}^+_{2r}$.

Step 2: Definition of a subsolution v for $\widetilde{Q}_L$

To establish (4.1.11), we need to construct a suitable subsolution. Let $\beta \in (0,1)$, $\delta \in \big(0, \frac{1}{4}\big]$, and define

$$\varphi(y) := -\frac{\delta}{r^2}\,|y'|^2 + \frac{y_n}{2r} + \frac{y_n^{1+\beta}}{2r^{1+\beta}} \quad \text{for } y \in \mathcal{C}^+_{2r,r}.$$

Note that $\varphi \in C^\infty(\mathcal{C}^+_{2r,r})$ and

$$D\varphi(y) = \left(-\frac{2\delta}{r^2}\,y', \frac{1}{2r} + \frac{1+\beta}{2r^{1+\beta}}\,y_n^\beta\right) \quad \text{for all } y \in \mathcal{C}^+_{2r,r},$$

so that, in particular, $D\varphi \ne 0$ in $\mathcal{C}^+_{2r,r}$. Also, the matrix $D^2\varphi$ is diagonal and

$$\partial^2_{y_i' y_i'}\varphi(y) = -\frac{2\delta}{r^2} \quad \text{and} \quad \partial^2_{y_n y_n}\varphi(y) = \frac{\beta(1+\beta)}{2\,r^{1+\beta}}\,y_n^{\beta-1}, \tag{4.4.17}$$

for every $i = 1, \dots, n-1$. For $\varepsilon \in (0,1]$ to be chosen later, we set $v = v_\varepsilon := \varepsilon\varphi$.

We claim that, if $\beta \in (0,\alpha)$ and δ is small enough, in dependence of n, p, Λ, α, β, and $\mathcal{S}$ only, it holds

$$\widetilde{Q}_L v_\varepsilon(y) \le 0 \quad \text{for all } y \in \mathcal{C}^+_{r,\delta r} \text{ and } \varepsilon \in (0,1]. \tag{4.4.18}$$

To verify this, we first observe that, by exploiting the structural assumptions (4.1.2) and (4.1.8), together with (4.4.8), (4.4.12), (4.4.14), and (4.4.15), the function $\widetilde{A}$ satisfies

$$\begin{cases} \left|\nabla_y \widetilde{A}(y,\xi)\right| \le C\left(|\xi|^2+\mu^2\right)^{\frac{p-2}{2}} |\xi|\, y_n^{\alpha-1} \\ \left|\partial_z \widetilde{A}(y,\xi)\right| \le C\left(|\xi|^2+\mu^2\right)^{\frac{p-2}{2}} \\ \langle \nabla_\xi \widetilde{A}(y,\xi)\eta,\eta\rangle \ge C^{-1}\left(|\xi|^2+\mu^2\right)^{\frac{p-2}{2}} |\eta|^2 \end{cases} \quad \text{for all } y \in \mathcal{C}^+_{2r},\ \xi \in \mathbb{R}^n\setminus\{0\},\ \eta \in \mathbb{R}^n. \tag{4.4.19}$$

Within this step, C depends only on n, p, Λ, α, β, and $\mathcal{S}$. Since $Dv \neq 0$, $D^2 v$ is diagonal, and $\widetilde{A} \in C^1\big(\mathcal{C}^+_{2r} \times (\mathbb{R}^n \setminus \{0\})\big)$, the chain rule entails

$$\widetilde{Q}_L v = -\sum_{i=1}^n \partial_{y_i} \widetilde{A}^i(y,Dv) - \sum_{i=1}^n \partial_{z_i} \widetilde{A}^i(y,Dv)\, \partial^2_{y_i y_i} v \quad \text{in } \mathcal{C}^+_{2r,r}.$$

Therefore, by using (4.4.17), (4.4.19), and the fact that

$$\frac{\varepsilon}{2r} \le |Dv| \le \frac{C\,\varepsilon}{r} \quad \text{in } \mathcal{C}^+_{2r,r},$$

we obtain

$$\begin{aligned} \widetilde{Q}_L v(y) &\le -\varepsilon\, \frac{y_n^{\beta-1}}{r^{1+\beta}} \left(\frac{\varepsilon^2}{r^2}+\mu^2\right)^{\frac{p-2}{2}} \left\{\frac{\beta(1+\beta)}{C} - C\,\delta\left(\frac{y_n}{r}\right)^{1-\beta} - C\, r^{1+\beta} y_n^{\alpha-\beta}\right\} \\ &\le -\varepsilon\, \frac{y_n^{\beta-1}}{r^{1+\beta}} \left(\frac{\varepsilon^2}{r^2}+\mu^2\right)^{\frac{p-2}{2}} \left\{\frac{1}{C} - C\,\delta^{2-\beta} - C\, r^{1+\alpha}\delta^{\alpha-\beta}\right\} \quad \text{for all } y \in \mathcal{C}^+_{r,\delta r}, \end{aligned}$$

From this, claim (4.4.18) immediately follows by taking δ sufficiently small.

Step 3: Extending v to a subsolution for $\widetilde{Q}_N$

Next, we extend v to a bounded function $\tilde{v}$ defined on the whole $\mathbb{R}^n$ satisfying

$$\widetilde{Q}_N \tilde{v}(y) \le 0 \quad \text{for all } y \in \mathcal{C}^+_{r,\delta r}, \tag{4.4.20}$$

provided δ is sufficiently small. We stress that the nonlocal operator $\widetilde{Q}_N \tilde{v}$ is well-defined in $\mathcal{C}^+_{r,\delta r}$ in the pointwise sense, as $\tilde{v}$ is globally bounded and smooth inside $\mathcal{C}^+_{r,\delta r}$ with non-vanishing gradient—this can be easily justified through the computations made, for instance, in [122, Section 3].

In order to achieve this, we let $\widetilde{\varphi}$ be any bounded, Lipschitz continuous, and compactly supported extension of φ to $\mathbb{R}^n$ satisfying

$$\begin{cases} \widetilde{\varphi}(y) = -\dfrac{\delta}{r^2}|y'|^2 & \text{for all } y \in B'_{2r} \times (-r,0], \\ \widetilde{\varphi}(y) = M & \text{for all } y \in B_{\frac{r}{4}}\left(\dfrac{3r}{2}\, e_n\right), \\ \widetilde{\varphi}(y) \le 0 & \text{for all } y \in \mathbb{R}^n \setminus \Big(\mathcal{C}^+_{r,\delta r} \cup \big(B'_{3r} \times [\delta r, 2r)\big)\Big), \\ -2 \le \widetilde{\varphi}(y) \le M & \text{for all } y \in \mathbb{R}^n, \end{cases} \tag{4.4.21}$$

for some $M \geq 2$ to be chosen suitably large. As before, we also set $\tilde{v} = \tilde{v}_\varepsilon := \varepsilon\widetilde{\varphi}$.

We write

$$\widetilde{Q}_N \tilde{v}(y) = 2\,\mathfrak{c}(y)\Big(I(y) + E_1(y) + E_2(y)\Big), \tag{4.4.22}$$

where

$$\begin{aligned}
I(y) &:= \text{P.V.}\int_{B_{\frac{r}{2}}(y)} \phi\big(\tilde{v}(y) - \tilde{v}(z)\big)\frac{\mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(z)\big)}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}}\,\mathfrak{c}(z)\,dz,\\
E_1(y) &:= \int_{B_{\frac{r}{4}}\left(\frac{3r}{2}e_n\right)} \phi\big(\tilde{v}(y) - \tilde{v}(z)\big)\frac{B\big(\mathcal{S}(y),\mathcal{S}(z)\big)}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}}\,\mathfrak{c}(z)\,dz,\\
E_2(y) &:= \int_{\mathbb{R}^n\setminus\left(B_{\frac{r}{2}}(y)\cup B_{\frac{r}{4}}\left(\frac{3r}{2}e_n\right)\right)} \phi\big(\tilde{v}(y) - \tilde{v}(z)\big)\frac{B\big(\mathcal{S}(y),\mathcal{S}(z)\big)}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}}\,\mathfrak{c}(z)\,dz.
\end{aligned}$$

By using that $\widetilde{\varphi} = \varphi \leq 1$ in $\mathcal{C}^+_{r,\delta r}$, $\widetilde{\varphi} = M$ in $B_{\frac{r}{4}}\big(\frac{3r}{2}\,e_n\big)$, and $\widetilde{\varphi} \geq -2$ in $\mathbb{R}^n$, in combination with the monotonicity of ϕ, bounds (4.4.8) and (4.4.13), as well as assumptions (4.1.3)-(4.1.4), we obtain

$$\begin{aligned}
E_1(y) &\leq -\frac{\varepsilon^{q-1}(M-1)^{q-1}}{\Lambda^2}\int_{B_{\frac{r}{4}}\left(\frac{3r}{2}e_n\right)} \frac{\mathfrak{c}(z)}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}}\,dz\\
&\leq -\frac{\varepsilon^{q-1}M^{q-1}}{C\,r^{sq}} \quad \text{for all } y \in \mathcal{C}^+_{r,\delta r}
\end{aligned} \tag{4.4.23}$$

and

$$\begin{aligned}
E_2(y) &\leq \Lambda^2 3^{q-1}\varepsilon^{q-1}\int_{\mathbb{R}^n\setminus\left(B_{\frac{r}{2}}(y)\cup B_{\frac{r}{4}}\left(\frac{3r}{2}e_n\right)\right)} \frac{\mathfrak{c}(z)}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}}\,dz\\
&\leq \frac{C\,\varepsilon^{q-1}}{r^{sq}} \quad \text{for all } y \in \mathcal{C}^+_{r,\delta r}.
\end{aligned} \tag{4.4.24}$$

Here, C is a constant depending only on n, q, s, Λ, α, β, and $\mathcal{S}$.

We now inspect the term I. We write

$$\frac{\mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(z)\big)\,\mathfrak{c}(z)}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}} = \frac{\mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(y)\big)\,\mathfrak{c}(y)}{|D\mathcal{S}(y)\,(y-z)|^{n+sq}} + \mathcal{R}_1(y,z) + \mathcal{R}_2(y,z) + \mathcal{R}_3(y,z), \tag{4.4.25}$$

with

$$\begin{aligned}
\mathcal{R}_1(y,z) &:= \mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(y)\big)\,\mathfrak{c}(y)\left(\frac{1}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}} - \frac{1}{|D\mathcal{S}(y)\,(y-z)|^{n+sq}}\right),\\
\mathcal{R}_2(y,z) &:= \mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(y)\big)\,\frac{\mathfrak{c}(z)-\mathfrak{c}(y)}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}},\\
\mathcal{R}_3(y,z) &:= \mathfrak{c}(z)\,\frac{\mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(z)\big) - \mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(y)\big)}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq}}.
\end{aligned}$$

We claim that, for $i = 1, 2, 3$ and for all $y \in \mathcal{C}^+_{r,\delta r}$, it holds

$$|\mathcal{R}_i(y,z)| \leq C\begin{cases} |y-z|^{-n-sq+\alpha} & \text{for all } z \in B_{\frac{r}{2}}(y),\\ y_n^{\alpha-1}|y-z|^{-n-sq+1} & \text{for all } z \in B_{\frac{y_n}{2}}(y).\end{cases} \tag{4.4.26}$$

By taking advantage of (4.1.3), (4.1.9), (4.4.8), (4.4.9), and (4.4.13), we immediately deduce the validity of (4.4.26) for $i = 2$ as well as the following stronger inequality for $i = 3$:

$$|\mathcal{R}_3(y,z)| \leq C\,|y-z|^{-n-sq+1} \quad \text{for all } y \in \mathcal{C}^+_{r,\delta r},\ z \in B_{\frac{r}{2}}(y).$$

On the other hand, using (4.1.3), (4.4.8), (4.4.13), and (4.4.14), together with the numerical inequality $\left|A^P - B^P\right| \leq P(A+B)^{P-1}|A-B|$, valid for every $P > 1$ and $A, B \geq 0$, we find

$$\begin{aligned} |\mathcal{R}_1(y,z)| &\leq C\, \frac{\Big| |D\mathcal{S}(y)(z-y)|^{n+sq} - |\mathcal{S}(z)-\mathcal{S}(y)|^{n+sq} \Big|}{|\mathcal{S}(y)-\mathcal{S}(z)|^{n+sq} |D\mathcal{S}(y)(z-y)|^{n+sq}} \\ &\leq C\, \frac{\big|\mathcal{S}(z)-\mathcal{S}(y)-D\mathcal{S}(y)(z-y)\big|}{|y-z|^{n+sq+1}}, \end{aligned}$$

from which (4.4.26) for $i = 1$ follows at once by noticing that

$$\begin{aligned} \big|\mathcal{S}(z)-\mathcal{S}(y)-D\mathcal{S}(y)(z-y)\big| &\leq |y-z| \int_0^1 \big|D\mathcal{S}(tz+(1-t)y) - D\mathcal{S}(y)\big|\,dt \\ &\leq C \begin{cases} |y-z|^{1+\alpha} & \text{for all } z \in B_{\frac{r}{2}}(y), \\ y_n^{\alpha-1}|y-z|^2 & \text{for all } z \in B_{\frac{y_n}{2}}(y), \end{cases} \end{aligned}$$

thanks to (4.4.12), (4.4.15), and the fact that the segment joining y and $tz+(1-t)y$ lies in the half-space $\left\{w \in \mathbb{R}^n : w_n \geq \frac{y_n}{2}\right\}$ for every $t \in [0,1]$.

Observe now that

$$y_n^{\alpha-1} \int_{B_{\frac{y_n}{2}}(y)} \frac{dz}{|y-z|^{n+sq-q}} + \int_{B_{\frac{r}{2}}(y)\setminus B_{\frac{y_n}{2}}(y)} \frac{dz}{|y-z|^{n+sq-\alpha-q+1}} \leq C\, r^{(1-s)q+\alpha-1} L_\alpha\Big(\frac{y_n}{r}\Big),$$

where, for $t \in (0,1)$ and $\gamma \in \mathbb{R}$, we set

$$L_\gamma(t) := \begin{cases} t^{(1-s)q+\gamma-1} & \text{if } (1-s)q+\gamma < 1, \\ -\log t & \text{if } (1-s)q+\gamma = 1, \\ 1 & \text{if } (1-s)q+\gamma > 1. \end{cases}$$

Also, by means of (4.1.4), of the fundamental theorem of calculus, and of the estimate

$$|D\tilde{v}(y)| \leq \frac{C\,\varepsilon}{r} \quad \text{for all } y \in B'_{2r} \times (-r,r),$$

we see that

$$\big|\phi\big(\tilde{v}(y)-\tilde{v}(z)\big)\big| \leq C\,\frac{\varepsilon^{q-1}}{r^{q-1}}\,|y-z|^{q-1} \quad \text{for all } y \in \mathcal{C}^+_{r,\delta r},\ z \in B_{\frac{r}{2}}(y).$$

In light of these facts, (4.4.25), (4.4.26), and recalling the definition of I, we have that

$$I(y) = \mathfrak{c}(y)\,\mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(y)\big)\ \text{P.V.}\int_{B_{\frac{r}{2}}(y)} \frac{\phi\big(\tilde{v}(y)-\tilde{v}(z)\big)}{|D\mathcal{S}(y)(y-z)|^{n+sq}}\,dz + \mathcal{E}(y),$$

with

$$|\mathcal{E}(y)| \leq C\,\varepsilon^{q-1} r^{-sq+\alpha} L_\alpha\Big(\frac{y_n}{r}\Big) \quad \text{for every } y \in \mathcal{C}^+_{r,\delta r}. \tag{4.4.27}$$

Since, by symmetry,

$$\text{P.V.}\int_{B_{\frac{r}{2}}(y)} \frac{\phi\big(D\tilde{v}(y)\cdot(y-z)\big)}{|D\mathcal{S}(y)(y-z)|^{n+sq}}\,dz=0,$$

the previous identity can be rewritten as

$$I(y)=\mathfrak{c}(y)\,\mathfrak{B}\big(\mathcal{S}(y),\mathcal{S}(y)\big)I_1(y)+\mathcal{E}(y)\quad\text{for every } y\in\mathcal{C}^+_{r,\delta r}, \tag{4.4.28}$$

with $\mathcal{E}$ satisfying (4.4.27) and

$$I_1(y):=\int_{B_{\frac{r}{2}}(y)} \frac{\phi\big(\tilde{v}(y)-\tilde{v}(z)\big)-\phi\big(D\tilde{v}(y)\cdot(y-z)\big)}{|D\mathcal{S}(y)(y-z)|^{n+sq}}\,dz.$$

We now claim that

$$I_1(y)\le-\frac{1}{C}\frac{\varepsilon^{q-1}L_0\big(\frac{y_n}{r}\big)}{r^{sq}}\quad\text{for all } y\in\mathcal{C}^+_{r,\delta r}, \tag{4.4.29}$$

for some constant $C\ge1$, provided δ is small enough, all in dependence of n, q, s, Λ, and β only.

The remaining of Step 3 is essentially occupied by the proof of this claim. First, we apply the change of variables $\ell:=\frac{z-y}{y_n}$ and observe that (4.4.29) is equivalent to showing that

$$\begin{aligned}\int_{B_{\frac{1}{2x_n}}}&\bigg\{\widetilde{\phi}\left(-2\delta\Big(2x'\cdot\ell'+x_n|\ell'|^2\Big)+(1+\ell_n)_+-1+x_n^\beta\left((1+\ell_n)_+^{1+\beta}-1\right)\right)\\&-\widetilde{\phi}\left(\Big(-4\delta x',1+(1+\beta)x_n^\beta\Big)\cdot\ell\right)\bigg\}\frac{d\ell}{|D\mathcal{S}(rx)\,\ell|^{n+sq}}\ge\frac{1}{C}\frac{L_0(x_n)}{x_n^{(1-s)q-1}},\end{aligned} \tag{4.4.30}$$

for all $x:=\frac{y}{r}\in\mathcal{C}^+_{1,\delta}$, and where $\widetilde{\phi}:=(\varepsilon x_n)^{1-q}\,\phi\big(\frac{\varepsilon x_n}{2}\,\cdot\big)$. We make a further substitution and consider the new variables w defined by $\ell=(w',d\cdot w)$, with

$$d':=\frac{4\delta x'}{1+(1+\beta)x_n^\beta}\quad\text{and}\quad d_n:=\frac{1}{1+(1+\beta)x_n^\beta}. \tag{4.4.31}$$

In particular, it defines a bi-Lipschitz map on $\mathbb{R}^n$ such that

$$\frac{1}{2}\,|w|\le\big|(w',d\cdot w)\big|\le2\,|w|\quad\text{for all } w\in\mathbb{R}^n, \tag{4.4.32}$$

provided δ is sufficiently small. Inequality (4.4.30) then becomes

$$\int_{\mathbb{R}^n}\mathcal{F}(w)\,dw\ge\frac{1}{C}\frac{L_0(x_n)}{d_nx_n^{(1-s)q-1}}, \tag{4.4.33}$$

where

$$\begin{aligned}\mathcal{F}(w):=\bigg\{&\widetilde{\phi}\left(-2\delta\Big(2x'\cdot w'+x_n|w'|^2\Big)+(1+d\cdot w)_+-1+x_n^\beta\left((1+d\cdot w)_+^{1+\beta}-1\right)\right)\\&-\widetilde{\phi}(w_n)\bigg\}\frac{\chi_{[0,1)}\Big(2x_n\sqrt{|w'|^2+(d\cdot w)^2}\Big)}{|D\mathcal{S}(rx)\,(w',d\cdot w)|^{n+sq}}.\end{aligned}$$

We now look for a lower bound on $\mathcal{F}$. First, let $w \in B_{\frac{1}{2}}$. In this case, we observe that

$$|d \cdot w| = \left| \frac{4\delta x' \cdot w' + w_n}{1 + (1+\beta) x_n^\beta} \right| \le \frac{4\delta |w'| + |w_n|}{1 + (1+\beta) x_n^\beta} \le (1 + 4\delta)|w| \le \frac{3}{4},$$

if we take $\delta \in \big(0, \frac{1}{8}\big]$. Hence, writing

$$N_q(w) := \begin{cases} |w|^{q-2} & \text{if } q \ge 2, \\ |w_n|^{q-2} & \text{if } q \in (1,2), \end{cases}$$

exploiting the monotonicity of ϕ, inequalities (4.4.14), (4.4.32), and

$$\Big| (1 + d \cdot w)^{1+\beta} - 1 \Big| \le 4\,|d \cdot w| \le 6\,|w| \quad \text{for every } w \in B_{\frac{1}{2}},$$

as well as the bound

$$|\widetilde{\phi}(a) - \widetilde{\phi}(b)| \le C_q \Lambda \big(|a| + |b|\big)^{q-2} |a - b| \quad \text{for all } (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}, \tag{4.4.34}$$

which holds for some constant $C_q > 0$ depending only on q thanks to the fact that ϕ fulfils assumption (4.1.10)—see, e.g., [71, Lemma 2.1]—, and ultimately recalling (4.4.31) and the fact that $x \in \mathcal{C}_{1,\delta}^+$, we estimate

$$\begin{aligned} |\mathcal{F}(w)| &\le C\, \frac{N_q(w)}{|w|^{n+sq}} \left| -2\delta x_n |w'|^2 + x_n^\beta \Big((1 + d\cdot w)^{1+\beta} - 1 - (1+\beta) d \cdot w \Big) \right| \\ &\le C\, \frac{N_q(w)}{|w|^{n+sq}} \, \Big(\delta^2 |w'|^2 + x_n^\beta \, (d \cdot w)^2 \Big) \le C \delta^\beta \frac{N_q(w)}{|w|^{n-2+sq}} \qquad \text{for a.e. } w \in B_{\frac{1}{2}}. \end{aligned}$$

On the other hand, the monotonicity of ϕ and again (4.4.14), (4.4.32), and (4.4.34) lead us to

$$\begin{aligned} \mathcal{F}(w) &\ge \frac{\widetilde{\phi}\Big(-5\delta |w'| + d \cdot w - x_n^\beta\Big) - \widetilde{\phi}(w_n)}{\big| D\mathcal{S}(rx)\,(w', d \cdot w) \big|^{n+sq}} \, \chi_{[0,1)}\Big(2x_n \sqrt{|w'|^2 + (d\cdot w)^2} \Big) \\ &\ge -C\, \frac{N_q(w)}{|w|^{n+sq}} \Big(\delta |w| + x_n^\beta + |d \cdot w - w_n| \Big)\, \chi_{B_{\frac{1}{x_n}}}(w) \\ &\ge -C\delta^\beta \frac{N_q(w)}{|w|^{n-1+sq}}\, \chi_{B_{\frac{1}{x_n}}}(w) \qquad \text{for a.e. } w \in \mathbb{R}^n \setminus B_{\frac{1}{2}}. \end{aligned}$$

These two estimates yield that

$$\int_{\{w_n \ge -10\}} \mathcal{F}(w)\, dw \ge -C\delta^\beta \int_{B_{\frac{1}{x_n}}} \frac{N_q(w) \min\{|w|, 1\}}{|w|^{n-1+sq}}\, dw \ge -C\delta^\beta \frac{L_0(x_n)}{x_n^{(1-s)q-1}}. \tag{4.4.35}$$

The last inequality is straightforward if $q \ge 2$. When $q \in (1,2)$, it is a consequence of the following calculation, valid for every $R \ge 2$:

$$\begin{aligned} \int_{B_R} \frac{|w_n|^{q-2} \min\{|w|, 1\}}{|w|^{n-1+sq}}\, dw &\le C \int_0^R \left(\int_0^R \frac{\min\Big\{ \sqrt{\rho^2 + w_n^2}, 1 \Big\}}{(\rho^2 + w_n^2)^{\frac{n-1+sq}{2}}} \rho^{n-2}\, d\rho \right) \frac{dw_n}{w_n^{2-q}} \\ &\le C \int_0^{+\infty} \left(\sqrt{1+t^2} \int_0^{\frac{1}{\sqrt{1+t^2}}} \frac{dw_n}{w_n^{1-(1-s)q}} + \int_{\frac{1}{\sqrt{1+t^2}}}^{\frac{R}{t}} \frac{dw_n}{w_n^{2-(1-s)q}} \right) \frac{t^{n-2}}{(1+t^2)^{\frac{n-1+sq}{2}}}\, dt \end{aligned}$$

$$
\begin{aligned}
&\leq C\left\{\int_0^{+\infty}\frac{t^{n-2}}{(1+t^2)^{\frac{n+q-2}{2}}}\,dt+\int_0^{+\infty}\left(\int_{\frac{1}{\sqrt{1+t^2}}}^{\frac{R}{t}}\frac{dw_n}{w_n^{2-(1-s)q}}\right)\frac{t^{n-2}}{(1+t^2)^{\frac{n-1+sq}{2}}}\,dt\right\}\\
&\leq CR^{(1-s)q-1}L_0\left(\frac{1}{R}\right).
\end{aligned}
$$

In light of (4.4.35), we are left with bounding the integral over $\{w_n<-10\}$. Using that ϕ is odd and monotone, along with the bounds (4.4.14) and (4.4.32), for $w\in\{w_n<-10\}\cap B_{1/\delta}$ we have

$$
\begin{aligned}
\mathcal{F}(w)&\geq\frac{\widetilde{\phi}(-w_n)-\widetilde{\phi}\,(5\delta|w|+2)}{|D\mathcal{S}(rx)\,(w',d\cdot w)|^{n+sq}}\,\chi_{[0,1)}\Big(2x_n\sqrt{|w'|^2+(d\cdot w)^2}\Big)\\
&\geq\frac{1}{C}\frac{\widetilde{\phi}(-w_n)-\widetilde{\phi}(10)}{|w|^{n+sq}}\,\chi_{B_{\frac{1}{4x_n}}}(w),
\end{aligned}
$$

while, using also that ϕ satisfies the growth assumption (4.1.4), for $w\in\{w_n<-10\}\setminus B_{1/\delta}$ it holds

$$
\begin{aligned}
\mathcal{F}(w)\geq{}&\frac{\widetilde{\phi}(-w_n)-\widetilde{\phi}(10)+\widetilde{\phi}(10)-\widetilde{\phi}\left(5\delta|w|+1+x_n^{\beta}\right)}{|D\mathcal{S}(rx)\,(w',d\cdot w)|^{n+sq}}\,\chi_{[0,1)}\Big(2x_n\sqrt{|w'|^2+(d\cdot w)^2}\Big)\\
\geq{}&\frac{1}{C}\frac{\widetilde{\phi}(-w_n)-\widetilde{\phi}(10)}{|w|^{n+sq}}\,\chi_{B_{\frac{1}{4x_n}}}(w)-\frac{C\delta^{q-1}}{|w|^{n+1-(1-s)q}}\,\chi_{B_{\frac{1}{x_n}}}(w).
\end{aligned}
$$

Putting these two estimates together and using the fact that, thanks to assumption (4.1.10),

$$
\widetilde{\phi}(-w_n)-\widetilde{\phi}(10)\geq\frac{2^{1-q}}{(q-1)\Lambda}\Big((-w_n)^{q-1}-10^{q-1}\Big)\quad\text{for every } w_n<-10,
$$

we compute, after a change of coordinates,

$$
\begin{aligned}
&\int_{\{w_n<-10\}}\mathcal{F}(w)\,dw\\
&\qquad\geq\frac{1}{C}\int_{B_{\frac{1}{4x_n}}\cap\{w_n<-10\}}\frac{(-w_n)^{q-1}-10^{q-1}}{|w|^{n+sq}}\,dw-C\delta^{q-1}\int_{B_{\frac{1}{x_n}}\setminus B_{\frac{1}{\delta}}}\frac{dw}{|w|^{n+1-(1-s)q}}\\
&\qquad\geq\frac{10^{(1-s)q-1}}{C}\int_{B_{\frac{1}{40x_n}}\cap\{z>1\}}\frac{z_n^{q-1}-1}{|z|^{n+sq}}\,dz-C\delta^{q-1}\frac{L_0(x_n)}{x_n^{(1-s)q-1}}.
\end{aligned}
\tag{4.4.36}
$$

Through a further changing variables, we see that

$$
\begin{aligned}
\int_{B_{\frac{1}{40x_n}}\cap\{z>1\}}\frac{z_n^{q-1}-1}{|z|^{n+sq}}\,dz&\geq\int_1^{\frac{1}{80x_n}}\left(\int_{B'_{z_n}}\frac{dz'}{\left(|z'|^2+z_n^2\right)^{\frac{n+sq}{2}}}\right)\left(z_n^{q-1}-1\right)dz_n\\
&=\mathcal{H}^{n-2}(\partial B'_1)\left(\int_0^1\frac{t^{n-2}}{(1+t^2)^{\frac{n+sq}{2}}}\,d\rho\right)\left(\int_1^{\frac{1}{80x_n}}\frac{z_n^{q-1}-1}{z_n^{1+sq}}\,dz_n\right)\\
&\geq\frac{1}{C}\frac{L_0(x_n)}{x_n^{(1-s)q-1}}.
\end{aligned}
$$

From this, (4.4.35), and (4.4.36) it follows that inequality (4.4.33) holds for some constant $C\geq1$, provided δ is sufficiently small, all in dependence of n, q, s, Λ, and β only. Consequently, claim (4.4.29) is proved.

From (4.1.3), (4.4.8), (4.4.22), (4.4.23), (4.4.24), (4.4.27), (4.4.28), and (4.4.29) we immediately infer that

$$\widetilde{Q}_N \tilde{v}(y) \leq -\frac{\varepsilon^{q-1}}{r^{sq}} \left\{ \frac{M^{q-1}}{C} + \frac{1}{C} L_0\Big(\frac{y_n}{r}\Big) - C - C\, r^{\alpha} L_{\alpha}\Big(\frac{y_n}{r}\Big) \right\} \quad \text{for all } y \in \mathcal{C}^+_{r,\delta r},$$

whence (4.4.20) holds true, provided we take δ sufficiently small (when $(1-s)q < 1$) or M sufficiently large (when $(1-s)q \geq 1$).

Step 4: Conclusion

Taking advantage of (4.4.18), (4.4.20), and of the locality of $\widetilde{Q}_L$, we see that

$$\widetilde{Q}\tilde{v}_\varepsilon = \widetilde{Q}_L v_\varepsilon + \widetilde{Q}_N \tilde{v}_\varepsilon \leq 0 \quad \text{in } \mathcal{C}^+_{r,\delta r},$$

for every $\varepsilon \in (0,1)$.

We now show that, by taking ε tiny enough, we can make $\tilde{v}_\varepsilon = \varepsilon\widetilde{\varphi}$ smaller than $\tilde{u}$ in the whole of $\mathbb{R}^n$. Indeed, by (4.4.21) we have that $\tilde{v}_\varepsilon \leq 0$ in $\mathbb{R}^n \setminus \big(\mathcal{C}^+_{r,\delta r} \cup (B'_{3r} \times [\delta r, 2r))\big)$ and that $\tilde{v}_\varepsilon \leq \varepsilon M$ in $B'_{3r} \times [\delta r, 2r)$. As $\mathcal{S}\big(B'_{3r} \times [\delta r, 2r)\big) \subset\subset \Omega$, thanks to (4.4.2), (4.4.10), (4.4.16), and since u is positive and continuous in Ω, then $m := \inf_{B'_{3r} \times [\delta r, 2r)} \tilde{u} > 0$. By choosing $\varepsilon \leq \frac{m}{M}$ and recalling that $\tilde{u}$ is non-negative in the whole of $\mathbb{R}^n$, we infer that $\tilde{v}_\varepsilon \leq \tilde{u}$ in $\mathbb{R}^n \setminus \mathcal{C}^+_{r,\delta r}$.

Thanks to the weak comparison principle of Proposition 4.3.1, we then conclude that $\tilde{v}_\varepsilon \leq \tilde{u}$ in $\mathbb{R}^n$. This yields in particular that

$$\tilde{u}(0', y_n) \geq \varepsilon\, \varphi(0', y_n) \geq \frac{\varepsilon}{2r}\, y_n \quad \text{for all } y_n \in (0, r).$$

Recalling that $D\mathcal{T}(0) = \mathrm{Id}_n$, by rephrasing this inequality in terms of the original variable x and of the function u, we are easily led to (4.1.11). The proof is thus complete.

4.5 A strong maximum principle

In this short section, we show how the Hopf lemma of Theorem 4.1.2 yields the following strong maximum (or, better, minimum) principle for the operator Q.

Proposition 4.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that A, $\mathfrak{B}$, and ϕ satisfy hypotheses* (4.1.2), (4.1.3), *and* (4.1.4). *Let $u \in W^{1,p}(\Omega) \cap \mathbb{W}^{s,q}(\Omega) \cap C^1(\Omega)$ be a non-negative weak supersolution of $Qu = 0$ in Ω. Then, either $u > 0$ in Ω or $u \equiv 0$ in $\mathbb{R}^n$.*

Proof. We already know from Proposition 4.3.1 that $u \geq 0$ in $\mathbb{R}^n$. Thus, to prove the proposition we suppose that there exists a point $x_0 \in \Omega$ at which $u(x_0) = 0$ and show that u must vanish identically in $\mathbb{R}^n$. Let U be the connected component of Ω containing x_0. Note that U is a bounded connected open set. We begin by establishing that

$$u = 0 \quad \text{in } U. \tag{4.5.1}$$

The proof of this claim is standard. Nevertheless, we provide the details for the sake of completeness. Let $U' := \{x \in U : u(x) = 0\}$. Clearly, U' is relatively closed in U and non-empty, as u is continuous and $x_0 \in U'$. Hence, its complement $U \setminus U'$ is open. If it were non-empty, then there would exist a point x_1 in it such that $\mathrm{dist}(x_1, U') < \mathrm{dist}(x_1, \mathbb{R}^n \setminus U)$ and a radius $r > 0$ for which $B_r(x_1) \subset U \setminus U'$ and $\partial B_r(x_1) \cap U' \neq \varnothing$. Applying Theorem 4.1.2, we would deduce that $\frac{\partial u}{\partial \nu}(x_2) < 0$ at every point $x_2 \in$

$\partial B_r(x_1) \cap U'$. But this is a contradiction, since x_2 is an interior minimum point for u and $\nabla u(x_2)$ must therefore vanish, u being of class C^1 inside U. We conclude that (4.5.1) holds true.

We now show that u must vanish outside of U as well. Here, the presence of the nonlocal operator Q_N plays a crucial role. In view of (4.5.1) and of the fact that u is a weak supersolution, we infer that

$$\iint_{\mathscr{C}_U} \phi\big(u(x) - u(y)\big) \left(\varphi(x) - \varphi(y)\right) \frac{\mathfrak{B}(x,y)}{|x-y|^{n+sq}}\, dxdy \geq 0,$$

for every non-negative $\varphi \in C_c^\infty(U)$. Being ϕ odd and $\mathfrak{B}$ symmetric, by taking advantage of (4.5.1) once again we find

$$\int_U \left(\int_{\mathbb{R}^n \setminus U} \phi\big(u(y)\big) \frac{\mathfrak{B}(x,y)}{|x-y|^{n+sq}}\, dy \right) \varphi(x)\, dx \leq 0 \quad \text{for every non-negative } \varphi \in C_c^\infty(U).$$

Since u is non-negative, $\mathfrak{B}$ is strictly positive in $\mathbb{R}^n \times \mathbb{R}^n$, and ϕ is strictly positive in $(0, +\infty)$, thanks to assumptions (4.1.3) and (4.1.4), we deduce that $u = 0$ in $\mathbb{R}^n \setminus U$, thus concluding the proof. □

4.6 Proof of Lemma 4.2.1

We include here a proof of Lemma 4.2.1, claiming global $W^{1,p}$ and L^∞ bounds for the solutions of problem (4.1.7). We begin by establishing the $W^{1,p}$ estimate.

By testing the weak formulation of (4.1.7) with $\varphi = u - g$, we get

$$\begin{aligned}
&\int_\Omega A(x, Du) \cdot \big(Du - Dg\big)\, dx \\
&\quad + \iint_{\mathscr{C}_\Omega} \phi\big(u(x) - u(y)\big)\big(u(x) - u(y) - g(x) + g(y)\big) \frac{\mathfrak{B}(x,y)}{|x-y|^{n+sq}}\, dxdy = \int_\Omega f(u-g)\, dx.
\end{aligned}$$

Taking advantage of assumption (4.1.2) and arguing similarly as in the proof of Lemma 4.2.3, we obtain the following estimate for the first summand on the left-hand side:

$$\int_\Omega A(x, Du) \cdot (Du - Dg)\ dx \geq \frac{1}{C} \|Du\|_{L^p(\Omega)}^p - C \left(1 + \|Dg\|_{L^p(\Omega)}^p\right).$$

The second summand can be handled using hypothesis (4.1.3) and (4.1.4) along with the weighted Young's inequality. We get

$$\begin{aligned}
&\iint_{\mathscr{C}_\Omega} \phi\big(u(x) - u(y)\big)\big(u(x) - u(y) - g(x) + g(y)\big) \frac{\mathfrak{B}(x,y)}{|x-y|^{n+sq}}\, dxdy \\
&\qquad \geq \frac{1}{C} \iint_{\mathscr{C}_\Omega} \frac{|u(x) - u(y)|^q}{|x-y|^{n+sq}}\, dxdy - C \iint_{\mathscr{C}_\Omega} \frac{|u(x) - u(y)|^{q-1}|g(x) - g(y)|}{|x-y|^{n+sq}}\, dxdy \\
&\qquad \geq \frac{1}{C} \iint_{\mathscr{C}_\Omega} \frac{|u(x) - u(y)|^q}{|x-y|^{n+sq}}\, dxdy - C \iint_{\mathscr{C}_\Omega} \frac{|g(x) - g(y)|^q}{|x-y|^{n+sq}}\, dxdy.
\end{aligned}$$

Finally, thanks to the Sobolev (or Morrey) and Poincaré inequalities, the right-hand is estimated by

$$\int_\Omega f(u-g)\, dx \leq \|f\|_{L^n(\Omega)} \|u - g\|_{L^{\frac{n}{n-1}}(\Omega)} \leq C \|f\|_{L^n(\Omega)} \big(\|Du\|_{L^p(\Omega)} + \|Dg\|_{L^p(\Omega)}\big).$$

Hence, using again the weighted Young's inequality, we find that

$$\|Du\|_{L^p(\Omega)}^p + \iint_{\mathscr{C}_\Omega} \frac{|u(x) - u(y)|^q}{|x-y|^{n+sq}}\, dxdy$$

$$\leq C \left(1 + \|Dg\|_{L^p(\Omega)}^p + \iint_{\mathscr{C}_\Omega} \frac{|g(x)-g(y)|^q}{|x-y|^{n+sq}} \, dxdy + \|f\|_{L^n(\Omega)}^{\frac{p}{p-1}} \right).$$

The bound for $\|u\|_{W^{1,p}(\Omega)}$ immediately follows from this and Poincaré's inequality.

We now deal with the global boundedness of u in Ω, which we establish it via the De Giorgi-Stampacchia method. Clearly, in the supercritical case $p > n$ the bound is a consequence of Morrey's inequality and the previous estimate for $\|u\|_{W^{1,p}(\Omega)}$. Assume then that $n \geq p$.

Let $k > M > M_1$, with

$$M_1 := 1 + \|g\|_{L^\infty(\Omega')} + \left(\int_{\mathbb{R}^n} \frac{|g(y)|^{q-1}}{(1+|y|)^{n+sq}} \, dy \right)^{\frac{1}{p-1}} + \|f\|_{L^n(\Omega)}^{\frac{1}{p-1}}.$$

Observe that the quantity on the right-hand side is finite, thanks to the inequality

$$\int_{\mathbb{R}^n} \frac{|g(y)|^{q-1}}{(1+|y|)^{n+sq}} \, dy \leq C \left(\|g\|_{L^q(\Omega')} + \iint_{\mathscr{C}_\Omega} \frac{|g(x)-g(y)|^q}{|x-y|^{n+sq}} \, dxdy \right)^{q-1}, \tag{4.6.1}$$

which is easily established by using (4.2.1). As $k > \|g\|_{L^\infty(\Omega')}$, the function $\varphi := (u-k)_+ \chi_\Omega$ lies in $W_0^{1,p}(\Omega) \cap \mathbb{W}_0^{s,q}(\Omega)$, and can thus be plugged in the weak formulation of problem (4.1.7). Setting $\Omega_k := \{x \in \Omega : u(x) > k\}$, we obtain

$$\begin{aligned} \int_{\Omega_k} f(u-k) \, dx = & \int_{\Omega_k} A(x, Du) \cdot Du \, dx \\ & + \iint_{\mathscr{C}_\Omega} \frac{\phi\big(u(x)-u(y)\big)\big((u(x)-k)_+\chi_\Omega(x) - (u(y)-k)_+\chi_\Omega(y)\big)\mathfrak{B}(x,y)}{|x-y|^{n+sq}} \, dxdy. \end{aligned} \tag{4.6.2}$$

On the one hand, we clearly have that

$$\begin{aligned} \int_{\Omega_k} A(x, Du) \cdot Du \, dx & \geq \frac{1}{C} \int_{\Omega_k} \big(|Du|^2 + \mu^2\big)^{\frac{p-2}{2}} |Du|^2 \, dx \\ & \geq \frac{1}{C} \Big(\big[(u-k)_+\big]_{W^{1,p}(\Omega)}^p - \mu^p |\Omega_k| \Big). \end{aligned} \tag{4.6.3}$$

Regarding the nonlocal term, thanks to the oddness of ϕ we observe that

$$\begin{aligned} & \phi\big(u(x)-u(y)\big)\big((u(x)-k)_+\chi_\Omega(x) - (u(y)-k)_+\chi_\Omega(y)\big) \\ & \qquad = \begin{cases} \phi\big(u(x)-u(y)\big)\big(u(x)-u(y)\big) & \text{if } x, y \in \Omega_k, \\ \phi\big(u(x)-u(y)\big)\big(u(x)-k\big) & \text{if } x \in \Omega_k, \, y \in \mathbb{R}^n \setminus \Omega_k, \\ \phi\big(u(y)-u(x)\big)\big(u(y)-k\big) & \text{if } x \in \mathbb{R}^n \setminus \Omega_k, \, y \in \Omega_k, \\ 0 & \text{if } x, y \in \mathbb{R}^n \setminus \Omega_k. \end{cases} \end{aligned}$$

Recalling (4.1.3)-(4.1.4) and using that $u(y) = g(y) \leq k$ for a.e. $y \in \Omega' \setminus \Omega$, we conclude from the previous identity that

$$\begin{aligned} & \iint_{\mathscr{C}_\Omega} \frac{\phi\big(u(x)-u(y)\big)\big((u(x)-k)_+\chi_\Omega(x) - (u(y)-k)_+\chi_\Omega(y)\big)\mathfrak{B}(x,y)}{|x-y|^{n+sq}} \, dxdy \\ & \qquad \geq \frac{1}{C} \int_{\Omega'} \int_{\Omega'} \frac{\big|\big(u(x)-k\big)_+ - \big(u(y)-k\big)_+\big|^q}{|x-y|^{n+sq}} \, dxdy \\ & \qquad \quad - C \int_{\Omega_k} \left(\int_{\mathbb{R}^n \setminus \Omega'} \frac{\big(u(x)-k\big)\big(g(y)-u(x)\big)_+^{q-1}}{|x-y|^{n+sq}} \, dy \right) dx \\ & \qquad \geq -C \, k^{p-1} \, \|(u-k)_+\|_{L^1(\Omega)}, \end{aligned} \tag{4.6.4}$$

where we used that

$$\begin{aligned}\int_{\Omega_k}\left(\int_{\mathbb{R}^n\setminus\Omega'}\frac{(u(x)-k)\big(g(y)-u(x)\big)_+^{q-1}}{|x-y|^{n+sq}}\,dy\right)dx &\le \int_{\Omega_k}(u(x)-k)\left(\int_{\mathbb{R}^n\setminus\Omega'}\frac{|g(y)|^{q-1}}{|x-y|^{n+sq}}\,dy\right)dx\\ &\le C\|(u-k)_+\|_{L^1(\Omega)}\int_{\mathbb{R}^n}\frac{|g(y)|^{q-1}}{(1+|y|)^{n+sq}}\,dy.\end{aligned}$$

Finally, we estimate

$$\int_{\Omega_k} f(u-k)\,dx \le \|f\|_{L^n(\Omega)}\|(u-k)_+\|_{L^{\frac{n}{n-1}}(\Omega)} \le C\,k^{p-1}\,\|(u-k)_+\|_{L^{\frac{n}{n-1}}(\Omega)}.$$

Combining this with (4.6.2), (4.6.3), (4.6.4), and the Poincaré-Sobolev inequality in $W_0^{1,p}(\Omega)$, we get that

$$\|(u-k)_+\|^p_{L^{mp}(\Omega)} \le C\Big(k^p|\Omega_k| + k^{p-1}\|(u-k)_+\|_{L^1(\Omega)} + k^{p-1}\|(u-k)_+\|_{L^{\frac{n}{n-1}}(\Omega)}\Big),$$

where m is equal to $\frac{n}{n-p}$ when $n>p$ or to any number strictly larger than $\frac{n}{n-1}$ when $n=p$. From this and the inequalities

$$\begin{aligned}\|(u-k)_+\|_{L^{\frac{n}{n-1}}(\Omega)} &\le (k-h)^{1-\frac{(n-1)mp}{n}}\|(u-h)_+\|^{\frac{(n-1)mp}{n}}_{L^{mp}(\Omega)},\\ |\Omega_k| &\le |\Omega|^{\frac{1}{n}}(k-h)^{-\frac{(n-1)mp}{n}}\|(u-h)_+\|^{\frac{(n-1)mp}{n}}_{L^{mp}(\Omega)},\\ \|(u-k)_+\|_{L^1(\Omega)} &\le |\Omega|^{\frac{1}{n}}(k-h)^{1-\frac{(n-1)mp}{n}}\|(u-h)_+\|^{\frac{(n-1)mp}{n}}_{L^{mp}(\Omega)},\end{aligned}$$

valid for any $h\in(0,k)$, we find

$$\|(u-k)_+\|^p_{L^{mp}(\Omega)} \le C\,\frac{k^p}{(k-h)^{\frac{(n-1)mp}{n}}}\,\|(u-h)_+\|^{\frac{(n-1)mp}{n}}_{L^{mp}(\Omega)}.$$

Letting now $\delta := \frac{(n-1)m-n}{n} > 0$, $k_i := (2-2^{-i})M$, and $\Psi_i := \|(u-k_i)_+\|^p_{L^{mp}(\Omega)}$ for every $i\in\mathbb{N}\cup\{0\}$, we infer that

$$\Psi_{i+1} \le \frac{C}{M^{\delta p}}\,2^{(1+\delta)pi}\,\Psi_i^{1+\delta}\quad\text{for every } i\in\mathbb{N}\cup\{0\}.$$

By taking advantage of [108, Lemma 7.1], we conclude that $\|(u-2M)_+\|^p_{L^{mp}(\Omega)} = \lim\limits_{i\to+\infty}\Psi_i = 0$, i.e., $u\le 2M$ in Ω, provided $\|(u-M)_+\|^p_{L^{mp}(\Omega)} = \Psi_0 \le C^{-1}M^p$ for some constant $C\ge 1$ large enough. Clearly, this can be achieved by taking $M := M_1 + M_2$ with

$$M_2 := C\|u\|_{W^{1,p}(\Omega)}$$

and $C\ge 1$ sufficiently large, thanks to the Sobolev inequality.

We thus established an upper bound for u. Since a corresponding lower bound can be obtained analogously, we conclude that the proof of Lemma 4.2.1 is complete—also recall the tail estimate (4.6.1).

4.7 Proof of Lemma 4.4.1

This section is devoted to the proof of the extension Lemma 4.4.1, used within Section 4.4. We begin by constructing a $C^{1,\alpha}(\overline{\mathbb{R}^n_+})\cap C^\infty(\mathbb{R}^n_+)$-extension $\mathscr{H}$ of h satisfying $D\mathscr{H}(0)=0$, (4.4.5), and

$$\|\mathscr{H}\|_{C^{1,\alpha}(\overline{\mathbb{R}^n_+})} \le C\,\|h\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}. \tag{4.7.1}$$

We take as $\mathscr{H}$ the harmonic extension of h to $\mathbb{R}^n_+$, namely the unique bounded solution of

$$\begin{cases} \Delta\mathscr{H} = 0 & \text{in } \mathbb{R}^n_+, \\ \mathscr{H} = h & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{4.7.2}$$

Standard regularity theory yields that $\mathscr{H}$ is of class $C^{1,\alpha}_{loc}(\overline{\mathbb{R}^n_+}) \cap C^\infty(\mathbb{R}^n_+)$. Furthermore, since $\partial_{y_n}\mathscr{H}(\cdot,0) = -(-\Delta)^{1/2}h(\cdot)$, from (4.4.1) and (4.4.3) we infer that $D\mathscr{H}(0) = 0$.

We now address the proof of estimates (4.7.1) and (4.4.5), which will both follow from the Poisson representation formula

$$\begin{aligned} \mathscr{H}(y', y_n) &= \frac{2y_n}{n\omega_n}\int_{\mathbb{R}^{n-1}} \frac{h(z')}{(|y'-z'|^2+y_n^2)^{\frac{n}{2}}}\,dz' \\ &= \frac{2y_n}{n\omega_n}\int_{\mathbb{R}^{n-1}} \frac{h(y'+\ell')}{(|\ell'|^2+y_n^2)^{\frac{n}{2}}}\,d\ell' \\ &= \frac{2}{n\omega_n}\int_{\mathbb{R}^{n-1}} \frac{h(y'+y_n\tau')}{(1+|\tau'|^2)^{\frac{n}{2}}}\,d\tau', \end{aligned} \tag{4.7.3}$$

valid for $(y', y_n) \in \mathbb{R}^n_+$. Here we set $\omega_n = |B_1|$. From (4.7.3) and the fact that

$$\frac{2y_n}{n\omega_n}\int_{\mathbb{R}^{n-1}} \frac{d\ell'}{(|\ell'|^2+y_n^2)^{\frac{n}{2}}} = 1 \quad \text{for every } y_n > 0, \tag{4.7.4}$$

we immediately infer that

$$\|\mathscr{H}\|_{L^\infty(\mathbb{R}^n_+)} \le \|h\|_{L^\infty(\mathbb{R}^{n-1})}. \tag{4.7.5}$$

By differentiating the second identity in (4.7.3), we get

$$D'\mathscr{H}(y', y_n) = \frac{2y_n}{n\omega_n}\int_{\mathbb{R}^{n-1}} \frac{D'h(y'+\ell')}{(|\ell'|^2+y_n^2)^{\frac{n}{2}}}\,d\ell'.$$

From this and (4.7.4), it readily follows that

$$\big|D'\mathscr{H}(y', y_n)\big| \le \|D'h\|_{L^\infty(\mathbb{R}^{n-1})} \quad \text{for all } y' \in \mathbb{R}^{n-1},\ y_n > 0 \tag{4.7.6}$$

and

$$\big|D'\mathscr{H}(y', y_n) - D'\mathscr{H}(z', y_n)\big| \le [D'h]_{C^\alpha(\mathbb{R}^{n-1})}\,|y'-z'|^\alpha \quad \text{for all } y', z' \in \mathbb{R}^{n-1},\ y_n > 0. \tag{4.7.7}$$

Through a suitable change of variables, we then compute

$$\begin{aligned} \partial_{y_n}\mathscr{H}(y', y_n) &= \frac{2}{n\omega_n}\int_{\mathbb{R}^{n-1}} \frac{|\ell'|^2-(n-1)y_n^2}{(|\ell'|^2+y_n^2)^{\frac{n+2}{2}}}\,h(y'+\ell')\,d\ell' \\ &= \frac{2}{n\omega_n y_n}\int_{\mathbb{R}^{n-1}} \frac{|\tau'|^2-(n-1)}{(1+|\tau'|^2)^{\frac{n+2}{2}}}\,h(y'+y_n\tau')\,d\tau' \\ &= \frac{2}{n\omega_n y_n}\left\{\int_{\mathbb{R}^{n-1}\setminus B'_M} \frac{|\tau'|^2-(n-1)}{(1+|\tau'|^2)^{\frac{n+2}{2}}}\Big(h(y'+y_n\tau')-h(y')\Big)\,d\tau' \right. \\ &\quad \left. + \int_{B'_M} \frac{|\tau'|^2-(n-1)}{(1+|\tau'|^2)^{\frac{n+2}{2}}}\Big(h(y'+y_n\tau')-h(y')-D'h(y')\cdot y_n\tau'\Big)\,d\tau'\right\}, \end{aligned} \tag{4.7.8}$$

for $M > 0$ to be chosen. Note that for the last identity we took advantage of the identities

$$\int_{\mathbb{R}^{n-1}} \frac{|\tau'|^2-(n-1)}{(1+|\tau'|^2)^{\frac{n+2}{2}}}\,d\tau' = 0 \quad \text{and} \quad \int_{B'_M} \frac{|\tau'|^2-(n-1)}{(1+|\tau'|^2)^{\frac{n+2}{2}}}\,w'\cdot\tau'\,d\tau' = 0, \tag{4.7.9}$$

valid for every $w' \in \mathbb{R}^{n-1}$ and $M > 0$. By taking $M = y_n^{-1}$ in (4.7.8), we deduce that

$$\begin{aligned}\big|\partial_{y_n}\mathscr{H}(y',y_n)\big| &\le \frac{C}{y_n}\left\{\|h\|_{L^\infty(\mathbb{R}^{n-1})}\int_M^\infty \frac{d\rho}{\rho^2} + [D'h]_{C^\alpha(\mathbb{R}^{n-1})}y_n^{1+\alpha}\int_0^M \frac{\rho^{n-1+\alpha}}{(1+\rho)^n}\,d\rho\right\}\\ &\le C\left\{\|h\|_{L^\infty(\mathbb{R}^{n-1})}(My_n)^{-1} + [D'h]_{C^\alpha(\mathbb{R}^{n-1})}(My_n)^\alpha\right\}\\ &\le C\,\|h\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} \quad \text{for all } y' \in \mathbb{R}^{n-1},\ y_n > 0.\end{aligned} \tag{4.7.10}$$

Arguing as for (4.7.8), we also have that

$$\begin{aligned}&\big|\partial_{y_n}\mathscr{H}(y',y_n) - \partial_{y_n}\mathscr{H}(z',y_n)\big|\\ &\quad\le \frac{C}{y_n}\left\{\int_{\mathbb{R}^{n-1}\setminus B'_N} \frac{\big|h(y'+y_n\tau')-h(y')-h(z'+y_n\tau')+h(z')\big|}{(1+|\tau'|)^n}\,d\tau'\right.\\ &\qquad\left.+\int_{B'_N}\frac{\big|h(y'+y_n\tau')-h(y')-h(z'+y_n\tau')+h(z')-\big(D'h(y')-D'h(z')\big)\cdot y_n\tau'\big|}{(1+|\tau'|)^n}\,d\tau'\right\},\end{aligned}$$

for any radius $N > 0$. Using the fundamental theorem of calculus, it is not difficult to see that the numerator of the fraction inside the first integral is bounded by $\|D'h\|_{C^\alpha(\mathbb{R}^{n-1})}|y'-z'|\min\big\{2, y_n^\alpha|\tau'|^\alpha\big\}$, while that pertaining to the second integral by $2[D'h]_{C^\alpha(\mathbb{R}^{n-1})}y_n^{1+\alpha}|\tau'|^{1+\alpha}$. In light of these estimates, choosing $N = |y'-z'|y_n^{-1}$ we get that

$$\begin{aligned}&\big|\partial_{y_n}\mathscr{H}(y',y_n) - \partial_{y_n}\mathscr{H}(z',y_n)\big|\\ &\quad\le \frac{C\,\|D'h\|_{C^\alpha(\mathbb{R}^{n-1})}}{y_n}\left\{|y'-z'|\int_{y_n^{-1}}^{+\infty}\frac{d\rho}{\rho^2} + y_n^\alpha|y'-z'|\int_N^{y_n^{-1}}\frac{d\rho}{\rho^{2-\alpha}} + y_n^{1+\alpha}\int_0^N \frac{\rho^{n-1+\alpha}}{(1+\rho)^n}\,d\rho\right\}\\ &\quad\le C\,\|D'h\|_{C^\alpha(\mathbb{R}^{n-1})}\left\{|y'-z'| + \frac{|y'-z'|}{(Ny_n)^{1-\alpha}} + (Ny_n)^\alpha\right\} \le C\,\|D'h\|_{C^\alpha(\mathbb{R}^{n-1})}|y'-z'|^\alpha,\end{aligned}$$

for every $y', z' \in \mathbb{R}^{n-1}$ such that $|y'-z'| < 1$ and for every $y_n > 0$. Combining this with (4.7.10), we conclude that

$$\big|\partial_{y_n}\mathscr{H}(y',y_n) - \partial_{y_n}\mathscr{H}(z',y_n)\big| \le C\|h\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}|y'-z'|^\alpha \quad \text{for } y', z' \in \mathbb{R}^{n-1},\ y_n > 0. \tag{4.7.11}$$

Now, differentiating the first identity in (4.7.3), we obtain the following alternative expression for the horizontal gradient of $\mathscr{H}$:

$$D'\mathscr{H}(y',y_n) = \frac{2y_n}{\omega_n}\int_{\mathbb{R}^{n-1}} \frac{h(z')\,(z'-y')}{(|y'-z'|^2+y_n^2)^{\frac{n+2}{2}}}\,dz' = \frac{2}{\omega_n y_n}\int_{\mathbb{R}^{n-1}} \frac{h(y'+y_n\tau')\,\tau'}{(1+|\tau'|^2)^{\frac{n+2}{2}}}\,d\tau'.$$

Therefore, for all $i, j = 1, \ldots, n-1$ we have, by symmetry,

$$\partial^2_{y'_i y'_j}\mathscr{H}(y',y_n) = \frac{2}{\omega_n y_n}\int_{\mathbb{R}^{n-1}} \frac{\partial_{y'_i}h(y'+y_n\tau')\,\tau_j}{(1+|\tau'|^2)^{\frac{n+2}{2}}}\,d\tau'$$

$$= \frac{2}{\omega_n y_n} \int_{\mathbb{R}^{n-1}} \frac{\Big(\partial_{y_i'} h(y' + y_n\tau') - \partial_{y_i'} h(y')\Big)\tau_j}{(1+|\tau'|^2)^{\frac{n+2}{2}}}\, d\tau',$$

so that

$$\begin{aligned}\left|\partial^2_{y_i' y_j'} \mathscr{H}(y', y_n)\right| &\le C\,[D'h]_{C^\alpha(\mathbb{R}^{n-1})}\, y_n^{\alpha-1} \int_{\mathbb{R}^{n-1}} \frac{|\tau'|^{1+\alpha}}{(1+|\tau'|)^{n+2}}\, d\tau' \\ &\le C\,[D'h]_{C^\alpha(\mathbb{R}^{n-1})}\, y_n^{\alpha-1}.\end{aligned} \tag{4.7.12}$$

Next, from the second identity in (4.7.8), we get

$$\begin{aligned}\partial^2_{y_i' y_n} \mathscr{H}(y', y_n) &= \frac{2}{n\omega_n y_n} \int_{\mathbb{R}^{n-1}} \frac{|\tau'|^2 - (n-1)}{\big(1+|\tau'|^2\big)^{\frac{n+2}{2}}}\, \partial_{y_i'} h(y' + y_n\tau')\, d\tau' \\ &= \frac{2}{n\omega_n y_n} \int_{\mathbb{R}^{n-1}} \frac{|\tau'|^2 - (n-1)}{\big(1+|\tau'|^2\big)^{\frac{n+2}{2}}} \big(\partial_{y_i'} h(y' + y_n\tau) - \partial_{y_i'} h(y')\big)\, d\tau',\end{aligned}$$

where we also made use of the first identity in (4.7.9). Therefore, we estimate

$$\left|\partial^2_{y_i' y_n} \mathscr{H}(y', y_n)\right| \le C\,[D'h]_{C^\alpha(\mathbb{R}^{n-1})}\, y_n^{\alpha-1} \int_{\mathbb{R}^{n-1}} \frac{|\tau'|^\alpha}{(1+|\tau'|)^n}\, d\tau' \le C\,[D'h]_{C^\alpha(\mathbb{R}^{n-1})}\, y_n^{\alpha-1}. \tag{4.7.13}$$

Since from the equation in (4.7.2) we know that $\partial^2_{y_n y_n} \mathscr{H} = -\sum_{i=1}^{n-1} \partial^2_{y_i' y_i'} \mathscr{H}$ in $\mathbb{R}^n_+$, from (4.7.12) it also follows that $\left|\partial^2_{y_n y_n} \mathscr{H}(y', y_n)\right| \le C\,[D'h]_{C^\alpha(\mathbb{R}^{n-1})}\, y_n^{\alpha-1}$. By combining this with (4.7.12) and (4.7.13), we conclude that (4.4.5) holds true.

Finally, from (4.4.5) and the fundamental theorem of calculus, we find that

$$\left|D\mathscr{H}(y', y_n) - D\mathscr{H}(y', z_n)\right| = \left|\int_{z_n}^{y_n} \partial_{y_n} D\mathscr{H}(y', t)\, dt\right| \le C\,[D'h]_{C^\alpha(\mathbb{R}^{n-1})}\, |y_n - z_n|^\alpha.$$

By putting together this with (4.7.5), (4.7.6), (4.7.7), (4.7.10), and (4.7.11), we obtain (4.7.1).

Finally, in order to conclude the proof it suffices to consider any $C^{1,\alpha}(\mathbb{R}^n)$-extension of $\mathscr{H}$ to the whole $\mathbb{R}^n$ having $C^{1,\alpha}(\mathbb{R}^n)$ norm bounded by that of $\mathscr{H}$, up to a factor. This can be done, for instance, via the elegant approach of [178]–see also [150, Theorem 1.1.17].

Bibliography

[1] R.A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975, xviii+268 pp.

[2] A. Alberico, A. Cianchi, I. Chlebicka, A. Zatorska-Goldstein, *Fully anisotropic elliptic problems with minimally integrable data*, Calc. Var. Partial Differential Equations 58 (2019), no. 6, Paper No. 186, 50 pp.

[3] A. Alberico, A. Cianchi, L. Pick, L. Slaviková, *Continuous and compact fractional Orlicz-Sobolev embeddings on domains*, in preparation.

[4] A. Alvino, V. Ferone, G. Trombetti, P.-L. Lions, *Convex symmetrization and applications*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 14 (1997), no. 2, 275-293.

[5] L. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. xviii+434 pp.

[6] C.A. Antonini, *Smooth approximation of Lipschitz domains, weak curvatures and isocapacitary estimates*, ArXiv preprint (2023) arXiv:2311.00100.

[7] C.A. Antonini, A. Cianchi, G. Ciraolo, A. Farina, V.G. Maz'ya, *Global second-order estimates in anisotropic elliptic problems*, ArXiv preprint (2023) arXiv:2307.03052.

[8] C.A. Antonini, G. Ciraolo, A. Farina, *Interior regularity results for inhomogeneous anisotropic quasilinear equations*, Math. Ann. (2023), https://doi.org/10.1007/s00208-022-02500-x.

[9] C.A. Antonini, M. Cozzi, *Global gradient regularity and a Hopf lemma for quasilinear operators of mixed local-nonlocal type*, Arxiv preprint (2023), arXiv:2308.06075

[10] D. E. Apushkinskaya, A. I. Nazarov, *A counterexample to the Hopf-Oleinik lemma (elliptic case)*, Anal. PDE 9 (2016), no. 2, 439-458.

[11] E. Arbel, J.W. Cahn, *method for the absolute measurement of anisotropic surface free energies*, Surf. Sci., 66:14-24 (1977).

[12] G. Aronsson, *Representation of a p-harmonic function near a critical point in the plane*, Manuscripta Math. 66 (1989), no. 1, 73-95.

[13] B. Avelin, T. Kuusi, G. Mingione, *Nonlinear Calderón-Zygmund theory in the limiting case*, Arch. Ration. Mech. Anal. 227 (2018), no. 2, 663-714.

[14] A. Kh. Balci, A. Cianchi, L. Diening, V. G. Maz'ya, *A pointwise differential inequality and second-order regularity for nonlinear elliptic systems*, Math. Ann. 383 (2022), no. 3-4, 1775-1824.

[15] A. Kh. Balci, L. Diening, R. Giova, A. Passarelli di Napoli, *Elliptic equations with degenerate weights*, SIAM J. Math. Anal. 54 (2022), no. 2, 2373-2412.

[16] A. Kh. Balci, L. Diening, M. Weimar, *Higher order Calderón-Zygmund estimates for the p-Laplace equation*, J. Differential Equations 268 (2020), no. 2, 590-635.

[17] A. Kh. Balci, Sun-Sig Byun, L. Diening, Ho-Sik Lee, *Global maximal regularity for equations with degenerate weights*, J. Math. Pures Appl. Volume 177, September (2023), pp 484-530.

[18] J. M. Ball, A. Zarnescu, *Partial regularity and smooth topology-preserving approximations of rough domains*, Calc. Var. Partial Differential Equations 56 (2017), no. 1, Paper No. 13, 32 pp.

[19] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vázquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22 (1995), no. 2, 241-273.

[20] A. Bensoussan, J, Frehse, *Regularity results for nonlinear elliptic systems and applications*, Applied Mathematical Sciences, Vol. 151, Springer-Verlag, Berlin (2022), pp. xii+441.

[21] S. Bernstein, *Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre*, Math. Ann. 59 (1904), no. 1-2, 20-76.

[22] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, *Semilinear elliptic equations involving mixed local and nonlocal operators*, Proc. Roy. Soc. Edinburgh Sect. A 151 (2021), no. 5, 1611–1641.

[23] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, *Mixed local and nonlocal elliptic operators: regularity and maximum principles*, Comm. Partial Differential Equations 47 (2022), no. 3, 585–629.

[24] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, *A Hong-Krahn-Szegö inequality for mixed local and nonlocal operators*, Math. Eng. 5 (2023), no. 1, Paper No. 014, 25 pp.

[25] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, *A Faber-Krahn inequality for mixed local and nonlocal operators*, J. Anal. Math. 150 (2023), no. 2, 405-448.

[26] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, *A Brezis-Nirenberg type result for mixed local and nonlocal operators*, arXiv preprint, arXiv:2209.07502.

[27] S. Biagi, D. Mugnai, E. Vecchi, *A Brezis-Oswald approach for mixed local and nonlocal operators*, to appear in Commun. Contemp. Math., DOI:10.1142/S0219199722500572.

[28] C. Bianchini, G. Ciraolo, *Wulff shape characterizations in overdetermined anisotropic elliptic problems*, Comm. Partial Differential Equations 43 (2018), no. 5, 790-820.

[29] A. Biswas, M. Modasiya, *Mixed local-nonlocal operators: maximum principles, eigenvalue problems and their applications*, ArXiv preprint (2021), arXiv:2110.06746.

[30] N. Biswas, H. Prasad, *Lipschitz potential estimates for diffusion with jumps*, ArXiv preprint (2023), arXiv:2307.02803.

[31] L. Boccardo, T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. 87 (1989), no. 1, 149-169.

[32] L. Brasco, E. Lindgren, A. Schikorra, *Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case*, Adv. Math. 338 (2018), 782–846.

[33] D. Breit, A. Cianchi, L. Diening, S. Schwarzacher, *Global Schauder estimates for the p-Laplace system*, Arch. Ration. Mech. Anal. 243 (2022), no. 1, 201-255.

[34] D. Breit, A. Cianchi, L. Diening, T. Kuusi, S. Schwarzacher, *Pointwise Calderón-Zygmund gradient estimates for the p-Laplace system*, J. Math. Pures Appl. (9) 114 (2018), 146-190.

[35] D. Breit, A. Cianchi, L. Diening, S. Schwarzacher, *Global Schauder Estimates for the p-Laplace System*, Arch. Rational Mech. Anal. 243 (2022) 201-255.

[36] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext. Springer, New York, 2011, xiv+599 pp.

[37] F.E. Browder, *Strongly elliptic systems of differential equations*, Contributions to the Theory of Partial Differential Equations, pp. 15-51, Princeton University Press (1955).

[38] Sun-Sig Byun, S. Ryu, *Global weighted estimates for the gradient of solutions to nonlinear elliptic equations*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 30 (2) (2013) 291-313.

[39] Sun-Sig Byun, P. Shin, Y. Youn, *Fractional differentiability results for nonlinear measure data problems with coefficients in C^{α}_{γ}*, J. Differential Equations 270 (2021), 390-434.

[40] L. Caffarelli, L. Silvestre, *An extension Problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32, 1245-1260 (2007).

[41] S. Campanato, *Proprietá di hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 17 (1963), 175-188.

[42] S. Campanato, *Proprietá di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 18 (1964), 137-160.

[43] S. Campanato, *Equazioni ellittiche del II ordine e spazi $\mathcal{L}^{(2,\lambda)}$*, Ann. Mat. Pura Appl. (4) 69 (1965), 321-381.

[44] A. Cellina, *The regularity of solutions to some variational problems, including the p-Laplace equation for* $2 \leq p < 3$, ESAIM Control Optim. Calc. Var. 23 (2017), no. 4, 1543-1553.

[45] A. Cellina, *The regularity of solutions to some variational problems, including the p-Laplace equation for* $3 \leq p < 4$, Discrete Contin. Dyn. Syst. 38 (2018), no. 8, 4071-4085.

[46] S. Challal, A. Lyaghfouri, *Second Order Regularity for the A-Laplace Operator*, Mediterr. J. Math. 7 (2010), 283-296.

[47] Z.-Q. Chen, P. Kim, R. Song, Z. Vondraček, *Sharp Green function estimates for $\Delta+\Delta^{\alpha/2}$ in $C^{1,1}$ open sets and their applications*, Illinois J. Math. 54 (2010), no. 3, 981–1024.

[48] Z.-Q. Chen, P. Kim, R. Song, Z. Vondraček, *Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$*, Trans. Amer. Math. Soc. 364 (2012), no. 8, 4169–4205.

[49] I. Chlebicka, *A pocket guide to nonlinear differential equations in Musielak-Orlicz spaces*, Nonlinear Anal. 175 (2018), 1-27.

[50] A. Cianchi, *Elliptic equations on manifolds and isoperimetric inequalities*, Proc. R. Soc. Edinburgh 114 (1990), non. 3-4, 213-227.

[51] A. Cianchi, *A sharp embedding theorem for Orlicz-Sobolev spaces*, Indiana Univ. Math. J. 45 (1996), no. 1, 39-65.

[52] A. Cianchi, *Boundedness of solutions to variational problems under general growth conditions*, Comm. Partial Differential Equations 22 (1997), no. 9-10, 1629-1646.

[53] A. Cianchi, V.G. Maz'ya, *Global Lipschitz regularity for a class of quasilinear elliptic equations*, Comm. Partial Differential Equations 36 (2011), no. 1, 100-133.

[54] A. Cianchi, V.G. Maz'ya, *Global boundedness of the gradient for a class of nonlinear elliptic systems*, Arch. Ration. Mech. Anal. 212 (2014), no. 1, 129-177.

[55] A. Cianchi, V.G. Maz'ya, *Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems*, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 3, 571-595.

[56] A. Cianchi, V.G. Maz'ya, *Global gradient estimates in elliptic problems under minimal data and domain regularity*, Commun. Pure Appl. Anal. 14 (2015), no. 1, 285-311.

[57] A. Cianchi, V.G. Maz'ya, *Quasilinear elliptic problems with general growth and merely integrable, or measure, data*, Nonlinear Anal. 164 (2017), 189-215.

[58] A. Cianchi, V.G. Maz'ya, *Second-order two-sided estimates in nonlinear elliptic problems*, Arch. Ration. Mech. Anal. 229 (2018), no. 2, 569-599.

[59] A. Cianchi, V.G. Maz'ya, *Optimal second-order regularity for the p-Laplace system*, J. Math. Pures Appl. (9) 132 (2019), 41-78.

[60] A. Cianchi, P. Salani, *Overdetermined anisotropic elliptic problems*, Math. Ann. 345 (2009), no. 4, 859-881.

[61] S. Cingolani, M. Degiovanni, G. Vannella, *Amann-Zehnder type results for p-Laplace problems*, Ann. Mat. Pura Appl. (4) 197 (2018), no. 2, 605-640.

[62] G. Ciraolo, *A weak comparison principle for solutions of very degenerate elliptic equations*, Ann. Mat. Pura Appl., 193 (2014) 1485-1490.

[63] G. Ciraolo, A. Figalli, A. Roncoroni, *Symmetry results for critical anisotropic p-Laplacian equations in convex cones*, Geom. Funct. Anal. 30 (2020), no. 3, 770-803.

[64] U. Clarenz, *Enclosure theorems for extremals of elliptic parametric functional*, Calc. Var. Partial Differential Equations 15.3 (2002), pp. 313-324.

[65] U. Clarenz, *Enclosure theorems for extremals of elliptic parametric funcionals*, Calc. Var. Partial Differential Equations 15 (2002), 313-324.

[66] U. Clarenz, H. Von der Mosel, *Compactness theorems and an isoperimetric inequality for critical points of elliptic parametric functional*, Calc. Var. Partial Differential Equations 12.1 (2001), pp. 85-107.

[67] M. Cozzi, A. Farina, E. Valdinoci, *Gradient bounds and rigidity results for singular, degenerate, anisotropic partial differential equations*, Comm. Math. Phys. 331 (2014), no. 1, 189-214.

[68] M. Cozzi, A. Farina, E. Valdinoci, *Monotonicity formulae and classification results for singular, degenerate, anisotropic PDEs*, Adv. Math. 293 (2016), 343-381.

[69] K. V. da Silva, A. M. Salort, *A limiting problem for local/non-local p-Laplacians with concave-convex nonlinearities*, Z. Angew. Math. Phys. 71 (2020), no. 6, Paper No. 191, 27 pp.

[70] A. Dall'Aglio, *Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations*, Ann. Mat. Pura Appl. (4) 170 (1996), 207-240.

[71] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 15 (1998), no. 4, 493-516.

[72] L. Damascelli, B. Sciunzi, *Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations*, J. Differential Equations 206 (2004), no. 2, 483-515.

[73] S. Dipierro, E.P. Lippi, E. Valdinoci, *(Non)local logistic equations with Neumann conditions.* Ann. Inst. H. Poincaré C Anal. Non Linéaire 40 (2023), no. 5, 1093-1166.

[74] S. Dipierro, E. Valdinoci, *Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy processes*, Phys. A 575 (2021), Paper No. 126052, 20 pp.

[75] C. De Filippis, G. Mingione, *Gradient regularity in mixed local and nonlocal problems*, to appear in Math. Ann (2023), DOI:10.1007/s00208-022-02512-7.

[76] C. De Filippis, G. Mingione, *Nonuniformly elliptic Schauder theory*, Inventiones mathematicae (2023) 234:1109-1196 https://doi.org/10.1007/s00222-023-01216-2.

[77] C. De Filippis, M. Piccinini, *Borderline global regularity for nonuniformly elliptic systems*, International Mathematics Research Notices (2022), https://doi.org/10.1093/imrn/rnac283.

[78] E. De Giorgi, *Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957), 25-43.

[79] F. De Téhlin, *Local regularity properties for the solutions of a nonlinear partial differential equation*, Nonlin. Anal. TMA 6 (1982), 839-844.

[80] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[81] L.M. Del Pezzo, A. Quaas, *A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian*, J. Differential Equations 263 (2017), no. 1, 765–778.

[82] M. G. Delgadino, F. Maggi, C. Mihaila, R. Neumayer, *Bubbling with L^2-almost constant mean curvature and an Alexandrov-type theorem for crystals*, Arch. Ration. Mech. Anal. 230 (2018), no. 3, 1131-1177.

[83] E. Di Benedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. 7 (1983) 827-850.

[84] L. Diening, P. Kaplický, S. Schwarzacher, *BMO estimates for the p-Laplacian*, Nonlinear Anal. 75 (2012), no. 2, 637-650.

[85] A. Dinghas, *Über einen geometrischen Satz von Wulff für die Gleichgewichts- form von Kristallen*, Z. Kristallogr., 105:304-314 (1944).

[86] P. Doktor, *Approximation of domains with Lipschitzian boundary*, Časopis Pěst. Mat. 101 (1976), no. 3, 237-255.

[87] F. Duzaar, G. Mingione, *Gradient estimates via linear and nonlinear potentials*, J. Funct. Anal. 259 (2010), no. 11, 2961-2998.

[88] F. Duzaar, G. Mingione, *Gradient continuity estimates*, Calc. Var. Partial Differential Equations 39 (2010), no. 3-4, 379-418.

[89] F. Duzaar, G. Mingione, *Gradient estimates via non-linear potentials*, Amer. J. Math. 133 (2011), no. 4, 1093-1149.

[90] S. Esedoḡlu, S.J. Osher, *Decomposition of images by the anisotropic Rudin-Osher-Fatemi model*, Commun. Pure Appl. Math., 57:1609-1626 (2004).

[91] L.C. Evans, *A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e.*, J. Differential Equations 45 (1982), no. 3, 356-373.

[92] X. Fan, *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations 235 (2007) 397-417.

[93] V. Ferone, B. Kawohl, *Remarks on a Finsler-Laplacian*, Proc. Amer. Math. Soc. 137 (2009), no. 1, 247-253.

[94] A. Figalli, F. Maggi, A. Pratelli, *A mass transportation approach to quantitative isoperimetric inequalities*, Invent. Math. 182 (2010), no. 1, 167-211.

[95] R. Finn, D. Gilbarg, *Asymptotic behavior and uniquenes of plane subsonic flows*, Comm. Pure Appl. Math. 10 (1957), 23–63.

[96] K.O. Friedrichs, *On the differentiability of the solutions of linear elliptic differential equations*, Comm. Pure Appl. Math. 6, 299-326 (1953).

[97] P. Garain, J. Kinnunen, *On the regularity theory for mixed local and nonlocal quasilinear elliptic equations*, Trans. Am. Math. Soc. 375 (2022), no. 8, 5393–5423.

[98] P. Garain, E. Lindgren, *Higher Hölder regularity for mixed local and nonlocal degenerate elliptic equations*, Calc. Var. Partial Differential Equations 62 (2023), no. 2, Paper No. 67, 36 pp.

[99] P. Garain, E. Lindgren, *Higher Hölder regularity for the fractional p-Laplace equation in the subquadratic case*, ArXiv preprint (2023), arXiv:2310.03600.

[100] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies 105, Princeton University Press, Princeton, NJ, 1983, vii+297.

[101] M. Giaquinta, E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math. 148 (1982), 31-46.

[102] M. Giaquinta, E. Giusti, *Differentiability of minima of nondifferentiable functionals*, Invent. Math. 72 (1983), no. 2, 285-298.

[103] M. Giaquinta, E. Giusti, *Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form*, J. Reine Angew. Math. 351 (1984), 55-65.

[104] M. Giaquinta, L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Second edition, Appunti, Scuola Normale Superiore di Pisa (Nuova Serie), 11. Edizioni della Normale, Pisa, 2012, xiv+366 pp.

[105] Y. Giga, *Surface evolution equations. A level set approach*, , Birkhäuser Verlag, Basel, 2006.

[106] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, xiv+517.

[107] G. Giraud, *Problémes de valeurs à la frontière relatifs à certaines données discontinues*, Bull. Soc. Math. France 61 (1933), 1–54.

[108] E. Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003. viii+403 pp.

[109] M. Goering, L. Koch, *A note on improved differentiability for the Banach-space valued Finsler g-Laplacian*, to appear in Comptes Rendus Mathématique (2023).

[110] A. Greco, R. Servadei, *Hopf's lemma and constrained radial symmetry for the fractional Laplacian*, Math. Res. Lett. 23 (2016), no. 3, 863–885.

[111] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics 24, Pitman (Advanced Publishing Program), Boston, MA, 1985, xiv+410.

[112] U. Guarnotta, S. Mosconi, *A general notion of uniform ellipticity and the regularity of the stress field for elliptic equations in divergence form*, Analysis & PDE, Vol. 16 (2023), No. 8, 1955-1988, DOI: 10.2140/apde.2023.16.1955.

[113] C. Gui, Y. Hu, Q. Li, *On smooth interior approximation of sets of finite perimeter*, Proc. Amer. Math. Soc. 151 (2023), no. 5, 1949-1962.

[114] A. Haarala, S. Sarsa, *Global second order Sobolev-regularity of p-harmonic functions*, J. Differential Equations 339 (2022), 232-260.

[115] P. Harjulehto, P. Hästö, *Orlicz spaces and generalized Orlicz spaces*, Lecture Notes in Mathematics, 2236. Springer, Cham, 2019, x+167 pp.

[116] A. Henrot, M. Pierre, *Shape variation and optimization*, European Mathematical Society (EMS), Zürich, 2018. xi+365 pp.

[117] S. Hofmann, M. Mitrea, T. Michael, *Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains*, J. Geom. Anal. 17 (2007), no. 4, 593-647.

[118] A. Iannizzotto, S. Mosconi, M. Squassina, *Global Hölder regularity for the fractional p-Laplacian*, Rev. Mat. Iberoam. 32 (2016), no. 4, 1353–1392.

[119] T. Iwaniec, J.J. Manfredi, *Regularity of p-harmonic functions on the plane*, Rev. Mat. Iberoamericana 5 (1989), no. 1-2, 1-19.

[120] T. Klimsiak, T. Komorowski, *Hopf type lemmas for subsolutions of integro-differential equations*, Bernoulli 29 (2023), no. 2, 1435-1463.

[121] A.G. Korolev, *On boundedness of generalized solutions of elliptic differential equations with nonpower nonlinearities*, Mathematics of the USSR-Sbornik 66 (1990), no. 1, 83-106

[122] J. Korvenpää, T. Kuusi, E. Lindgren, *Equivalence of solutions to fractional p-Laplace type equations*, J. Math. Pures Appl. (9) 132 (2019), 1–26.

[123] I. N. Krol' , V. G. Maz'ya, *The absence of the continuity and Hölder continuity of the solutions of quasilinear elliptic equations near a nonregular boundary*, Trudy Moskov. Mat. Obšč 26 (1972), 75-94

[124] T. Kuusi, G. Mingione, *Linear potentials in nonlinear potential theory*, Arch. Ration. Mech. Anal. 207 (2013), no. 1, 215-246.

[125] T. Kuusi, G. Mingione, *A nonlinear Stein theorem*, Calc. Var. Partial Differential Equations 51 (2014), no. 1-2, 45-86.

[126] T. Kuusi, G. Mingione, *Partial regularity and potentials*, J. Éc. polytech. Math. 3 (2016), 309-363.

[127] T. Kuusi, G. Mingione, *Vectorial nonlinear potential theory*, J. Eur. Math. Soc. (JEMS) 20 (2018), no. 4, 929-1004.

[128] O.A. Ladyzhenskaya, N. N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York-London, 1968, xviii+495

[129] P. Lax, *On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations*, Comm. Pure Appl. Math 8, 615-633 (1955).

[130] J.M. Lee, *Introduction to smooth manifolds*, Second edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013. xvi+708 pp.

[131] J.M. Lee, *Introduction to Riemannian manifolds*, Second edition, Graduate Texts in Mathematics, 176. Springer, Cham, 2018. xiii+437 pp.

[132] G. Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics 181, Second Edition, American Mathematical Society, Providence, RI, 2017, xxii+734

[133] J. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. 32 (1983), no. 6, 849-858.

[134] G. M. Lieberman, *Regularized distance and its applications*, Pacific J. Math. 117 (1985), no. 2, 329-352

[135] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. 12 (1988), no. 11, 1203-1219

[136] G. M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations*, Comm. Partial Differential Equations 16 (1991), no. 2-3, 311-361

[137] G. M. Lieberman, *Sharp forms of estimates for subsolutions and supersolutions of quasilinear elliptic equations involving measures*, Communications in Partial Differential Equations 18 (1993), no. 7-8, 1191-1212

[138] P.-L. Lions, F.Murat, *Sur les solutions renormalisées d'équations elliptiques non linéaires*, manuscript

[139] H. Lou, *On singular sets of local solutions to p-Laplace equations*, Chinese Ann. Math. Ser. B 29 (2008), no. 5, 521-530

[140] F. Maggi, *Sets of finite perimeter and geometric variational problems*, An introduction to geometric measure theory. Cambridge Studies in Advanced Mathematics, 135. Cambridge University Press, Cambridge, 2012. xx+454 pp.

[141] J.J. Manfredi, *Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations*, Ph.D. Thesis, University of Washington, St. Louis (1986).

[142] J.J. Manfredi, *Regularity for minima of functionals with p-growth*, . J. Differ. Equ. 76 (1988) 203-212.

[143] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p, q-growth conditions*, J. Diff. Equ. 90, 1-30, (1991).

[144] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions*, Arch. Rat. Mech. Anal. 105, 267-284, (1989).

[145] M. Marcus, V. J. Mizel, *Absolute continuity on tracks and mappings of Sobolev spaces*, Arch. Rational Mech. Anal. 45 (1972), 294-320

[146] U. Massari, L. Pepe, *Sull'approssimazione degli aperti lipschitziani di $\mathbb{R}^n$ con varietá differenziabili*, Boll. Un. Mat. Ital. (4) 10 (1974), 532-544.

[147] V.G. Maz'ya, *p-conductivity and theorems on imbedding certain functional spaces into a C-space*, Dokl. Akad. Nauk SSSR 140 (1961), 299-302.

[148] V.G. Maz'ya, *Solvability in $\dot{W}_2{}^2$ of the Dirichlet problem in a region with a smooth irregular boundary*, Vestnik Leningrad. Univ. 22 (1967), no. 7, 87-95.

[149] V.G. Maz'ya, *The coercivity of the Dirichlet problem in a domain with irregular boundary*, Izv. Vysš. Učebn. Zaved. Matematika (1973), no. 4 (131), 64-76.

[150] V.G. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 342, Springer, Heidelberg, 2011, xxviii+866.

[151] V.G. Maz'ya, T. O. Shaposhnikova, *Theory of Sobolev multipliers*, Grundlehren Math. Wiss., 337 [Fundamental Principles of Mathematical Sciences] Springer-Verlag, Berlin, 2009. xiv+609 pp.

[152] C. Mercuri, G. Riey, B. Sciunzi, *A regularity result for the p-Laplacian near uniform ellipticity*, SIAM J. Math. Anal. 48 (2016), 2059-2075.

[153] Q. Miao, Fa Peng, Y. Zhou, *A global second order Sobolev regularity for p-Laplacian type equations with variable coefficients in bounded domains*, Calc. Var. Partial Differential Equations 62 (2023), no. 7, Paper No. 191, 36 pp.

[154] G. Mingione, *Gradient estimates below the duality exponent*, Math. Ann. 346 (2010), no. 3, 571-627.

[155] G. Mingione, *Gradient potential estimates*, J. Eur. Math. Soc. (JEMS) 13 (2011), no. 2, 459-486.

[156] M. Miskiewicz, *Fractional differentiability for solutions of the inhomogeneous p-Laplace system*, Proc. Amer. Math. Soc. 146 (2018), no. 7, 3009-3017.

[157] E. Montefusco, B. Pellacci, G. Verzini, *Fractional diffusion with Neumann boundary conditions: the logistic equation*, Discrete Contin. Dyn. Syst. Ser. B 18 (8), 2175-2202 (2013)

[158] L. Montoro, L. Muglia, B. Sciunzi, *Optimal second order boundary regularity for solutions to p-Laplace equations*, ArXiv preprint arXiv:2305.08447

[159] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. 13 (1960), 457-468.

[160] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. 14 (1961), 577-591.

[161] J.R. Munkres, *Topology*, Second edition. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. xvi+537 pp.

[162] R. Musina, *On the continuity of the Nemitsky operator induced by a Lipschitz continuous map*, Proc. Amer. Math. Soc. 111 (1991), no. 4, 1029-1041

[163] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. 80 (1958), 931-954.

[164] J. Nečas, *On domains of type $\mathcal{N}$*, Czechoslovak Math. J. 12 (87) (1962), 274-287.

[165] L. Nirenberg, *On nonlinear elliptic partial differential equations and Hölder continuity*, Comm. Pure Appl. Math. 6, 103-156 (1953).

[166] L. Nirenberg, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. 8, 649-675 (1955).

[167] S. Osher, M. Burger, D. Goldfarb, J. Xu, W. Yin, *An iterative regularization method for total variation-based image restoration*, Multiscale Model. Simul., 4:460-489 (2005).

[168] B. Pellacci, G. Verzini, *Best dispersal strategies in spatially heterogeneous environments: optimization of the principal eigenvalue for indefinite fractional Neumann problems*, J. Math. Biol. 76(6), 1357-1386 (2018).

[169] L. Pick, A. Kufner, O. John, S. Fučik, *Function spaces Vol. 1*, De Gruyter Series in Nonlinear Analysis and Applications 14, Walter de Gruyter & Co., Berlin, 2013, xvi+479

[170] C.A. Ponce, *Elliptic PDEs, measures and capacities*, EMS Tracts in Mathematics, 23. European Mathematical Society (EMS), Zürich, 2016. x+453 pp.

[171] M.M. Rao, Z.D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991, xii+449 pp.

[172] R.C. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. 26 (1977), no. 3, 459-472

[173] X. Ros-Oton, J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. (9) 101 (2014), no. 3, 275–302.

[174] W. Rudin, *Functional analysis*, Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, xviii+424 (1991)

[175] J. Schauder, *Über den Zusammenhang zwischen der Eindeutigkeit und Lösbarkeit partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Math. Ann. 107 (1933), no. 1, 164 pp.

[176] T. Schmidt, *Strict interior approximation of sets of finite perimeter and functions of bounded variation*, Proc. Amer. Math. Soc. 143 (2015), no. 5, 2069-2084.

[177] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, (1993) xiv+490 pp.

[178] R. T. Seeley, *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. 15 (1964), 625–626.

[179] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. 11 (1964), 247-302

[180] J. Simon, *Régularité de solutions de problémes nonlinéaires*, C. R. Acad. Sci. Paris Sér. A-B 282 (1976), no. 23, Aii, A1351-A1354.

[181] G. Stampacchia, *On some regular multiple integral problems in the calculus of variations*, Comm. Pure Appl. Math. 16 (1963), 383-421

[182] G. Stampacchia, *Le probléme de Dirichlet pour les équations elliptiques du second ordre á coefficients discontinus*, Ann. Inst. Fourier, 15 (1965), 189-258.

[183] X. Su, E. Valdinoci, Y. Wei, J. Zhang, *Regularity results for solutions of mixed local and nonlocal elliptic equations*, Math. Z. 302 (2022), no. 3, 1855–1878.

[184] G. Talenti, *Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl. (4) 120 (1979), 160-184

[185] G. Talenti, *Boundedness of minimizers*, Hokkaido Math. J. 19 (1990), no. 2, 259-279

[186] J. E. Taylor, *Crystalline variational problems*, Bull. Amer. Math. Soc., 84:568- 588 (1978).

[187] P. Tolksdorf, *Regularity for a more general class of Quasilinear Elliptic Equations*, Journal of Differential Equations 51, 126-150 (1984)

[188] N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. 20 (1967), 721-747.

[189] G. Wang, C. Xia, *A characterization of the Wulff shape by an overdetermined anisotropic PDE*, Arch. Ration. Mech. Anal. 199 (2011), no. 1, 99-115.

[190] G. Wang, C. Xia, *An optimal anisotropic Poincaré inequality for convex domains*, Pacific J. Math. 258.2 (2012), pp. 305-325.

[191] K.-O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations*, Math. Scand. 21 (1967), 17–37.

[192] G. Wulff, *Zur Frage der Geschwindigkeit des Wachsthums und der Auflösung der Krystallflachen*, Z. Kristallogr. Mineral., 34:449-530, (1901)

[193] E. Zeidler, *Nonlinear functional analysis and its applications. II/B. Nonlinear monotone operators*, Springer-Verlag, New York, 1990. pp. i-xvi and 469-1202.

Ringraziamenti

Ringrazio i miei relatori prof. Giulio Ciraolo e prof. Alberto Farina, per avermi guidato lungo questo percorso con i vostri consigli, incoraggiamenti, e per l'infinita pazienza che avete avuto, soprattutto nei confronti del mio "leggero" disordine.

Ringrazio il prof. Andrea Cianchi, per la sua grandissima disponibilità, il suo supporto, e per le stimolanti discussioni di ricerca che abbiamo avuto, anche di Domenica mattina!

Ringrazio il prof. Matteo Cozzi, per l'interessante collaborazione avuta soprattutto durante il mio ultimo anno di dottorato.

Ringrazio tutti i miei colleghi del dipartimento di matematica per la loro compagnia, specialmente i "figli di Giulio" ovvero Luigi, Camilla, ed il nuovo arrivato Michele.

Ringrazio i miei amici: Pierluigi, Michelangelo (Mix), Livia, Marika, Emil, Marco (il Becca), Stefano (il grande Sté), Lorenzo (Lancio pancio), Ludovica, Antonio, Federica, e tutti gli altri. Grazie per avermi supportato, o meglio sopportato, ed essermi stato vicino nonostante tutto!

Dedico questa tesi ai miei due cuginetti, Pier Francesco e Stefano (Stefanuccio), che per me sono come dei fratelli; a mia zia Mirella "zialella" e zio Pasqualino "zioa", i quali hanno sempre creduto in me e nelle mie capacità.

La dedico ai miei carissimi nonni Giancarlo e Lucia, per tutto il bene che mi vogliono, e ai quali si illuminano gli occhi solo nel vedermi.

La dedico anche ai miei zii Giuseppe (zio Peppe), Franco, Luisa e Tonino, oltre che ai miei cugini Danilo, Vincenzina, Laura (la neodottoressa), Loris (con i suoi due nuovi arrivati) e Roberto.

Ed infine il mio ringraziamento più grande va ai miei genitori, che mi hanno sempre supportato, anche nei miei momenti più difficili, e che tuttora continuano a farlo.

Senza di voi, certamente non ce l'avrei fatta.

A tutti voi: GRAZIE!!!