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A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation

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Abstract: We describe how the nested soft-collinear subtraction scheme [1] can be used to compute the next-to-next-to-leading order (NNLO) QCD corrections to the production of *an arbitrary number of gluonic jets* in hadron collisions. We show that the infrared subtraction terms can be combined into recurring structures that in many cases are simple iterations of those terms known from next-to-leading order. The way that these recurring structures are identified and computed is fairly general, and can be applied to any partonic process. As an example, we explicitly demonstrate the cancellation of *all* singularities in the fully-differential cross section for the $q\bar{q} \to X + Ng$ process at NNLO in QCD. The finite remainder of the NNLO QCD contribution, which arises upon cancellation of all ϵ-poles, is expressed via relatively simple formulas, which can be implemented in a numerical code in a straightforward way. Our approach can be extended to describe arbitrary processes at NNLO in QCD; the largest remaining challenge at this point is the combinatorics of quark and gluon collinear limits.

Keywords: Higher-Order Perturbative Calculations, Renormalization and Regularization, Specific QCD Phenomenology

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1 Introduction

The theoretical description of hard scattering processes at the LHC is based, almost entirely, on perturbative QCD. Because of this, the development of theoretical methods that can be used to provide predictions at progressively higher orders of perturbation theory has been one of the most active and exciting topics in theoretical particle physics in the past decade (see refs. [2–5] for the recent reviews).

An important part of the theoretical toolbox that allows the description of infrared-safe observables at high orders of perturbative QCD is the treatment of infrared singularities. It is well-known that these singularities cancel upon combining virtual corrections, unresolved real-radiation contributions, and the collinear renormalization of parton distribution functions (PDFs). An important question is then how to organize this cancellation in a process-independent way and how to arrive at finite remainders that are suitable for numerical evaluations.

This problem was fully solved at next-to-leading order (NLO) in perturbative QCD many years ago [6–11] (see also ref. [12] for more recent work), but its extension to next-to-next-to-leading order (NNLO) and beyond has proved to be difficult. In fact, there are many NNLO subtraction and slicing schemes [1, 13–34] that have been used to perform the many impressive computations at this perturbative order,[1] but it is fair to say that the complete generality achieved at NLO is still elusive at NNLO.

A peculiar illustration of this statement is the fact that the cancellation of $1/\epsilon^n$ infrared poles[2] for a *generic* hadron collider process has not been demonstrated in any NNLO slicing or subtraction scheme up to now, although important work in this direction, focusing on gluonic states, has recently been presented in ref. [59], and including other partonic channels in ref. [60]. For e^+e^- collisions such a cancellation for arbitrary final states has been shown only in the context of the so-called local analytic sector subtraction scheme [31, 32].

The goal of this paper is to partially address this issue in the context of the nested soft-collinear subtraction scheme [1]. This scheme has already been successfully applied to compute the NNLO QCD corrections to a variety of processes such as color singlet production [61] and decay [62], deep inelastic scattering [63], Higgs production in WBF [64], non-factorizable corrections to t-channel single-top production [65] as well as mixed QCD-electroweak corrections to the production of electroweak gauge bosons and dilepton pairs [66–68]. This suggests that the nested soft-collinear subtraction scheme possesses the flexibility and the simplicity that is needed for studies of multi-particle final states.

[1]See refs. [35–58] for a representative list of NNLO calculations by different collaborations.

[2]Throughout the paper we use dimensional regularization and work in $d = 4 - 2\epsilon$ dimensions.

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Moreover, the computations of double-unresolved soft and collinear contributions for arbitrary kinematics are usually considered to be some of the most challenging calculations required to develop a particular subtraction scheme. Interestingly, in the case of the nested soft-collinear subtraction scheme, such computations were completed several years ago [69, 70], but this has not led to immediate applications of this scheme to high-multiplicity QCD processes. Understanding the reasons for that is essential for further developing the nested soft-collinear subtraction scheme and for making it applicable to the description of arbitrary collider processes.

In this paper, we take a step in that direction by describing the application of this scheme to the study of NNLO QCD corrections to the production of an *arbitrary* number of gluons and a colorless final state X in $q\bar{q}$ annihilation. We emphasize that we restrict ourselves to the case where all resolved and unresolved final-state partons are gluons, i.e. splittings of the form $g^* \to q\bar{q}$ are not considered in this paper. Practically, this can easily be achieved by setting the number of light quark flavors $n_f = 0$ in e.g. the QCD β function. Nevertheless, we will report all such formulas with their n_f contributions, with an eye on a future extension to unresolved quark final states, and with the understanding that we take $n_f = 0$ throughout the paper.

Therefore we are interested the process $1_a + 2_b \to X + N\,g$ with $a, b \in \{q, \bar{q}\}$, i.e. the process $1_a + 2_b \to X + N\,g$ with $a, b \in \{q, \bar{q}\}$.[3] However, we will keep the generic notation of a and b for the initial-state partons, in order to make a future generalization easier. In particular, we stress that the extension of this result to gg annihilation into $X + N\,g$ is straightforward since many of our arguments apply *verbatim* to this case as well, and the problem reduces to repeating certain steps of the calculation using different splitting functions and replacing a few color factors.

Moreover, although our results are currently restricted to gluonic final states, they require the analysis of matrix elements containing the richest singularity structures that can possibly arise, and we are confident that the new insights into the mechanisms of infrared cancellations at NNLO in QCD that we obtain in this paper are useful for generic final states. In fact, the outstanding challenge in generalizing the results from all-gluonic to arbitrary final states is the combinatorics of various collinear limits. This aspect of the problem does not show up prominently for all-gluonic final states because of the symmetry of the relevant matrix elements under permutations of final-state gluons.

There is multiple evidence suggesting that infrared subtraction terms can be organized into clear structures that iterate from NLO to NNLO and possibly, beyond. This is rather obvious in case of leading collinear singularities where the highest collinear poles at each perturbative order are described by convolutions of leading-order splitting functions. The fact that a similar iterative description should hold for soft emissions as well follows from Catani's formula for ϵ-poles of one- and two-loop amplitudes [71]. However, the iterative nature of the subtraction terms is not manifest in many NNLO subtraction schemes because, following the idea of FKS subtraction at NLO, one often splits real-emission phase spaces into partitions and sectors to project matrix elements onto the minimal number of singular kinematic configurations that one has to deal with at any point in the calculation.

[3] A prototypical physical process is the gluonic contribution to $q\bar{q}$ annihilation into an electroweak vector boson and a large number of jets.

In this paper we show how these iterative structures can be recognized and constructed in the context of the nested soft-collinear subtraction scheme. We also demonstrate that the existence of these iterative structures provides a strong guide for organizing NNLO QCD computations and leads to the reduction of the computational complexity, allowing us to deal with final states of *arbitrary* multiplicity.

The main result of this paper is a formula that allows the computation of NNLO QCD corrections to a process where a $q\bar{q}$ initial state annihilates into N final-state hard gluons and an arbitrary number of colorless particles, through a fully local subtraction procedure. This formula can be implemented in a computer code in a straightforward way; it requires finite remainders of two- and one-loop scattering amplitudes for a particular process and the corresponding Born amplitudes. Since the cancellation of all $1/\epsilon^n$ singularities is proved analytically, all required numerical integrations can be performed in four-dimensional space-time.

The rest of the paper is organized as follows. After preliminary remarks in the next section, we present the computation of NLO QCD corrections to the process $1_a + 2_b \to X + N\,g$ with $a, b \in \{q, \bar{q}\}$ in section 3. The reader might also find it useful to refer to appendix C, where we elaborate on the cancellation of poles at NLO. This discussion allows us to introduce the iterative structures that are crucial for the subsequent analyses of the NNLO QCD corrections in section 4. There we show how to rewrite the double-real contribution as a sum of terms with well-defined partonic multiplicities, and how to express these through operators corresponding to soft or collinear limits or virtual corrections. The reader who is more interested in the mechanism of the pole cancellation at NNLO can skim over this section and focus instead on section 5. The final results for the finite remainders of the NNLO QCD corrections are presented in section 6. This section is quite self-contained so that the reader who is only interested in these results can skip to this section right away. We conclude in section 7.

Finally, we note that the discussion of many technical details is relegated to multiple appendices. In particular, we collect the definitions of the various constants, splitting functions and fundamental operators used throughout the manuscript in appendix A. For the readers' convenience, the many different notations that we use in the paper are summarized in an alphabetic index that can be found at the end of the paper and used to identify the place in the paper where a particular notation has been introduced for the first time.

2 Preliminary considerations

Subtraction schemes should enable calculations of hard processes at lepton and hadron colliders at higher orders in QCD perturbation theory. In this paper, we will consider the process where N jets and a color-singlet system X are produced in hadron collisions, $pp \to X + N$ jets. The cross section of this process is given by the following formula

$$\mathrm{d}\sigma = \sum_{a,b} \int \mathrm{d}x_1 \mathrm{d}x_2 \; f_a(x_1, \mu_F) f_b(x_2, \mu_F)\, \mathrm{d}\hat{\sigma}_{ab}(x_1, x_2, \mu_R, \mu_F; O)\,. \tag{2.1}$$

Here $\mathrm{d}\hat{\sigma}_{ab}$ is the cross section in the ab partonic channel, $f_{a,b}$ are the parton distribution functions (PDFs), μ_R and μ_F are the renormalization and factorization scales, respectively, and O is an observable, which provides (among other things) an infrared-safe definition of the N-jet final state.

The partonic cross section can be expanded in the strong coupling α_s. We write

$$\mathrm{d}\hat{\sigma}_{ab} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{LO}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NNLO}} + \mathcal{O}(\alpha_s^{q+3})\,, \tag{2.2}$$

where the LO term is proportional to α_s^q, and we have suppressed the arguments of all the functions for brevity.

The computation of partonic cross sections and kinematic distributions requires integrating matrix elements squared over phase spaces of relevant final states. For a generic process, we find it convenient to treat matrix elements as vectors in color space [9]. A matrix element where N_p partons[4] are assigned definite color indices is then written as a projection on a particular color-space basis vector

$$\mathcal{M}^{c_1,\ldots,c_{N_p}}(p_1,\ldots,p_{N_p}) = {}_c\langle c_1,\ldots,c_{N_p}|\mathcal{M}(p_1,\ldots,p_{N_p})\rangle_c\,. \tag{2.3}$$

The square of the amplitude summed over all possible color assignments is then

$$|\mathcal{M}(p_1,\ldots,p_{N_p})|^2 = {}_c\langle\mathcal{M}(p_1,\ldots,p_{N_p})|\mathcal{M}(p_1,\ldots,p_{N_p})\rangle_c\,. \tag{2.4}$$

Although it is sufficient to use the summed-over-colors amplitude squared to compute leading-order cross sections, in higher orders of QCD perturbation theory color-correlated matrix elements appear. For example, at NLO, one encounters $\langle\mathcal{M}|\boldsymbol{T}_i\cdot\boldsymbol{T}_j|\mathcal{M}\rangle$, where $\boldsymbol{T}_{i(j)}$ is the color charge operator of parton $i(j)\in\{1,\ldots,N_p\}$. To address this possibility, it is convenient to introduce a tensor product of leading-order matrix elements $|\mathcal{M}_0\rangle$ in color space. We therefore define the function

$$\begin{aligned}\widetilde{F}_{\mathrm{LM}}(1_a,2_b;3,\ldots N_p;X) = {}&|\mathcal{M}_0(1_a,2_b;3,\ldots,N_p;X)\rangle_c\otimes{}_c\langle\mathcal{M}_0(1_a,2_b;3,\ldots,N_p;X)|\\ &\times\mathrm{dLips_X}\,O(p_3,\ldots,p_{N_p};p_X)\,,\end{aligned} \tag{2.5}$$

to describe the partonic process $1_a+2_b\to X+N$ jets at leading order. In eq. (2.5), $N_p=N+2$ is the number of initial- and final-state partons, the symbol $\otimes$ indicates a tensor product in color space, and $\mathrm{dLips_X}$ is the Lorentz-invariant phase space for the colorless system X, including the momentum-conserving delta function. Furthermore, we always assume $f_i=g$ for $i=3,\ldots,N_p$, where f_i is the flavor of parton i, and hence we do not show a flavor index for the final state partons.

The matrix element squared is obtained by taking the trace in color space

$$\mathrm{Tr}\left[\widetilde{F}_{\mathrm{LM}}\right]_c = \mathrm{dLips_X}\,|\mathcal{M}_0|^2\,O \equiv F_{\mathrm{LM}}\,, \tag{2.6}$$

where the arguments of all functions have been suppressed. As we already mentioned, in the course of NLO and NNLO calculations we will need to act on $\widetilde{F}_{\mathrm{LM}}$ with a function of operators in color space, and take the trace in color space after that. Denoting such a function as A, we introduce the notation

$$A\cdot F_{\mathrm{LM}} \equiv \mathrm{Tr}\left[A\,\widetilde{F}_{\mathrm{LM}}\right]_c = {}_c\langle\mathcal{M}_0|A|\mathcal{M}_0\rangle_c\,\mathrm{dLips_X}\,O\,. \tag{2.7}$$

[4]In the case of $pp\to X+N$ jets we have $N_p=N+2$.

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The LO partonic cross section can be obtained by integrating $F_{\rm LM}(1_a, 2_b; 3, \ldots, N_p; X)$ over the phase space of the final-state partons. We write

$$2s\, {\rm d}\hat{\sigma}_{ab}^{\rm LO} = \mathcal{N} \int \prod_{i=3}^{N_p} [{\rm d}p_i]\; F_{\rm LM}(1_a, 2_b; 3, \ldots, N_p; X) = \langle F_{\rm LM} \rangle\,, \tag{2.8}$$

where $s = 2p_1 \cdot p_2$ is the partonic center-of-mass energy squared, and the angular brackets $\langle \ldots \rangle$ indicate the integration over the final-state phase space. In eq. (2.8) $\mathcal{N}$ is a normalization factor that takes into account color and spin averages as well as symmetry factors, and $[{\rm d}p_i]$ is the phase-space element of a final-state parton i

$$[{\rm d}p_i] = \frac{{\rm d}^3 p_i}{(2\pi)^3 2E_i}\,. \tag{2.9}$$

3 Calculations at next-to-leading order

In this section we discuss the calculation of the partonic cross section of the process $1_a + 2_b \to X + N\, g$ at next-to-leading order in perturbative QCD. Our main goal is to introduce an infrared finite-operator $I_{\rm T}$, see eq. (3.2), that describes the sum of virtual, soft and certain collinear contributions and, as we explain later, is important for simplifying NNLO QCD calculations.

Computation of NLO corrections requires the one-loop (virtual) contribution, the real-emission contribution and the contribution of the collinear renormalization of parton distribution functions[5]

$${\rm d}\hat{\sigma}_{ab}^{\rm NLO} = {\rm d}\hat{\sigma}_{ab}^{\rm V} + {\rm d}\hat{\sigma}_{ab}^{\rm R} + {\rm d}\hat{\sigma}_{ab}^{\rm pdf}\,. \tag{3.1}$$

It is well-known that the virtual contribution contains explicit poles in ϵ that arise from the integration over the loop momentum. For a generic process, these poles can be written in a closed form using Catani's function $I_1(\epsilon)$ [71]. On the other hand, the real-emission contributions do not contain explicit poles in ϵ until the integration over the phase space of final-state partons is performed. Such an integration extends over singular kinematic regions that correspond to soft and/or collinear emissions and generates the $1/\epsilon^n$ poles. Eventually, many of these poles will cancel with poles in the one-loop contribution; therefore, we would like to parametrize them in a manner similar to Catani's function for the virtual corrections. Hence, we define soft and hard-collinear analogs of Catani's function, which we call $I_{\rm S}(\epsilon)$ and $I_{\rm C}(\epsilon)$, respectively, as well as a function $I_{\rm V}(\epsilon)$ which is related to $I_1(\epsilon)$. These functions will multiply terms with leading order kinematics, such that the sum

$$I_{\rm T}(\epsilon) = I_{\rm V}(\epsilon) + I_{\rm S}(\epsilon) + I_{\rm C}(\epsilon)\,, \tag{3.2}$$

is ϵ-finite.

To define all the I-operators in eq. (3.2) and to explain how their combination arises, we begin by considering the real-emission contribution to the NLO cross section. This contribution refers to the process $1_a + 2_b \to X + (N+1)\, g$. We write

$$2s\, {\rm d}\hat{\sigma}_{ab}^{\rm R} = \langle F_{\rm LM}(1_a, 2_b; 3, \ldots, N_p + 1; X) \rangle\,. \tag{3.3}$$

[5] Throughout this paper we work with UV-renormalized matrix elements.

Since the observable O in the definition of $F_{\rm LM}$ requires at least N resolved partons, one and only one parton among the $N+1$ final-state ones in the above equation can become unresolved, i.e. soft and/or collinear to another parton. To identify the unresolved parton, we introduce damping factors $\Delta^{(i)}$ such that they provide a partition of unity,

$$\sum_{i=3}^{N_p+1} \Delta^{(i)} = 1\,. \tag{3.4}$$

The explicit form of the damping factors can be found in appendix B. They are constructed in such a way that a damping factor $\Delta^{(i)}$ vanishes when any parton, with the exception of parton i, becomes either soft or collinear to any other parton, including the incoming ones. This implies that in the combination $\Delta^{(i)} F_{\rm LM}$, only soft and collinear limits of parton i can lead to non-integrable singularities and, eventually, to the appearance of $1/\epsilon^n$ poles.

We then write

$$\left\langle F_{\rm LM}(1_a,2_b;3,\ldots,N_p+1;X)\right\rangle = \sum_{i=3}^{N_p+1} \left\langle \Delta^{(i)} F_{\rm LM}(1_a,2_b;3,\ldots,N_p+1;X)\right\rangle. \tag{3.5}$$

Since we focus on the all-gluon final state, $F_{\rm LM}$ is unchanged under any permutation of the final-state partons. Then we obtain

$$\sum_{i=3}^{N_p+1} \left\langle \Delta^{(i)} F_{\rm LM}(1_a,2_b;3,\ldots,N_p+1;X)\right\rangle = \left\langle (N_p-1)\Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\right\rangle. \tag{3.6}$$

In the above result, we have relabelled the arguments of $F_{\rm LM}$ in such a way that the damping factors become identical for each term in the sum and we denote the potentially-unresolved gluon as $\mathfrak{m}$. The remaining $N = N_p - 2$ final-state gluons are resolved. For simplicity, we do not show the dependence of $F_{\rm LM}$ on their momenta and polarizations. We also omit the dependence of $F_{\rm LM}$ on the kinematics of color-singlet final-state particles.

We note that in eq. (3.6) the functions $F_{\rm LM}$ include $1/(N_p-1)!$ symmetry factors for the all-gluon final state. The factor (N_p-1) on the right hand side of that equation combines with $1/(N_p-1)!$ and turns into $1/(N_p-2)! = 1/N!$ where N is the minimal required number of *resolved* jets. This is the same symmetry factor as in e.g. the virtual contribution and we will simply not write it explicitly in what follows. Thus, by an abuse of notation, we will write the right-hand side of eq. (3.6) as $\langle \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle$, with the understanding that symmetry factors in $F_{\rm LM}$ refer to *resolved final-state gluons only.*

To deal with matrix elements and phase spaces in soft and collinear limits we need the corresponding operators. These operators were introduced earlier [1] and we repeat their definitions here for completeness. The actions of soft S_i and collinear C_{ij} operators on a function A are described by the following formulas

$$S_i A = \lim_{E_i \to 0} A\,, \qquad C_{ij} A = \lim_{\rho_{ij}\to 0} A\,, \tag{3.7}$$

where E_i is the energy of parton i and $\rho_{ij} = 1-\cos\theta_{ij}$, with θ_{ij} is the angle between the three-momenta of partons i and j.[6] When these operators appear in the formulas for cross

[6] Since our primary variables are energies and angles, we need to fix a reference frame at the beginning of the calculation.

sections, it is understood that they act on all quantities to the right of them; when limits in the conventional sense do not exist, they extract the most singular contributions.

The soft and collinear operators acting on the damping factors lead to the following results[7]

$$S_{\mathfrak{m}}\Delta^{(\mathfrak{m})} = 1\,, \qquad C_{a\mathfrak{m}}\Delta^{(\mathfrak{m})} = 1\,, \qquad C_{i\mathfrak{m}}\Delta^{(\mathfrak{m})} = \frac{E_{\mathfrak{m}}}{E_i + E_{\mathfrak{m}}} \equiv z_{\mathfrak{m},i}\,, \tag{3.8}$$

for $a = 1, 2$ and $i \geq 3$.

We will now use these operators to isolate and subtract the singular contributions, starting with the soft one. We write

$$\langle \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle = \langle S_{\mathfrak{m}} F_{\rm LM}(\mathfrak{m})\rangle + \langle \overline{S}_{\mathfrak{m}}\Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle\,, \tag{3.9}$$

where we introduced the handy notation

$$\overline{S}_{\mathfrak{m}} \equiv \mathbb{1} - S_{\mathfrak{m}}\,. \tag{3.10}$$

The soft limit of the matrix element squared reads

$$S_{\mathfrak{m}} F_{\rm LM}(\mathfrak{m}) = -g_{s,b}^2 \sum_{(ij)}^{N_p} \frac{p_i\cdot p_j}{(p_i\cdot p_{\mathfrak{m}})(p_j\cdot p_{\mathfrak{m}})}\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\,, \tag{3.11}$$

where $g_{s,b}$ is the bare coupling constant, and we have used eq. (2.7) to write the color-correlated matrix element squared in a convenient way. In eq. (3.11), the sum runs over distinct indices i and j. We remind the reader that the color-charge operators of different particles $\boldsymbol{T}_i$ commute with each other. Furthermore, we use the Casimir operators to compute squares of color-charge operators with $\boldsymbol{T}_q^2 = \boldsymbol{T}_{\bar{q}}^2 = C_F$ and $\boldsymbol{T}_g^2 = C_A$.

Since the unresolved gluon $\mathfrak{m}$ decouples from $F_{\rm LM}$, we can integrate eq. (3.11) over its d-dimensional phase space. To do so, we introduce an upper bound on the soft gluon energy, $E_{\mathfrak{m}} \leq E_{\rm max}$.[8] Performing this integration, we find

$$\begin{aligned}\langle S_{\mathfrak{m}} F_{\rm LM}(\mathfrak{m})\rangle &= -[\alpha_s]\frac{(2E_{\rm max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{(ij)}^{N_p}\langle \eta_{ij}^{-\epsilon} K_{ij}\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\rangle \\ &\equiv [\alpha_s]\langle I_{\rm S}(\epsilon)\cdot F_{\rm LM}\rangle\,,\end{aligned} \tag{3.12}$$

where

$$[\alpha_s] = \frac{\alpha_s(\mu)}{2\pi}\,\frac{e^{\epsilon\gamma_{\rm E}}}{\Gamma(1-\epsilon)}\,, \tag{3.13}$$

and

$$K_{ij} = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{ij}^{1+\epsilon}\,{}_2F_1(1,1,1-\epsilon,1-\eta_{ij})\,, \qquad \eta_{ij} = \rho_{ij}/2\,. \tag{3.14}$$

We now return to eq. (3.9) and focus on the second term on the right-hand side. This term is soft-regulated, but contains collinear singularities. In order to remove them, we introduce angular partitions of unity $\omega^{\mathfrak{m}i}$, which satisfy the following equations

$$\sum_{i=1}^{N_p}\omega^{\mathfrak{m}i} = 1; \qquad C_{j\mathfrak{m}}\omega^{\mathfrak{m}i} = \delta^{ij}\,. \tag{3.15}$$

[7] Derivation of these results can be found in appendix B.

[8] $E_{\rm max}$ is an arbitrary quantity that should be larger than the largest energy that a particle in a particular process can have. For additional information, see ref. [1].

Generic expressions which satisfy these constraints are presented in eq. (B.21). We thus write

$$\begin{aligned}\langle \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle = \langle S_{\mathfrak{m}} F_{\rm LM}(\mathfrak{m})\rangle + \sum_{i=1}^{N_p} \langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle \\ + \sum_{i=1}^{N_p} \langle \overline{S}_{\mathfrak{m}} \overline{C}_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i} \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle \,,\end{aligned} \tag{3.16}$$

where

$$\overline{C}_{i\mathfrak{m}} \equiv \mathbb{1} - C_{i\mathfrak{m}}\,. \tag{3.17}$$

The last term on the right-hand side of eq. (3.16) is fully regulated and can be integrated in four dimensions. In the hard-collinear limits that appear in the second term on the right-hand side in eq. (3.16), the gluon decouples from $F_{\rm LM}$ either partially or fully, allowing us to integrate over its phase space in d dimensions.

We continue with the second term on the right-hand side of eq. (3.16), and consider the situation where the gluon $\mathfrak{m}$ becomes collinear to the final-state gluon i and produces a single final-state gluon that we label as $[i\mathfrak{m}]$. Integrating over the phase space of gluon $\mathfrak{m}$ and renaming $[i\mathfrak{m}] \mapsto i$, we find

$$\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle = [\alpha_s] \left\langle \frac{\Gamma_{i,g}}{\epsilon} F_{\rm LM} \right\rangle \,. \tag{3.18}$$

In eq. (3.18) we have introduced the *generalized energy-dependent final-state gluon anomalous dimension*

$$\Gamma_{i,g} = \left(\frac{2E_i}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\, \gamma^{22}_{z,g\to gg}(\epsilon, L_i)\,, \qquad i = 3,\ldots,N_p\,, \tag{3.19}$$

where, for any function $f(z)$ regular at $z=1$, we define

$$\begin{aligned}\gamma^{nk}_{f(z),g\to gg}(\epsilon, L_i) = & -\int_0^1 {\rm d}z\,(1-S_z)\left[z^{-n\epsilon}(1-z)^{-k\epsilon}\, f(z)\, P_{gg}(z)\right] \\ & + 2\boldsymbol{T}_g^2 \frac{1-e^{k\epsilon L_i}}{k\epsilon} f(1)\,,\end{aligned} \tag{3.20}$$

and $L_i = \log(E_{\rm max}/E_i)$. In eq. (3.20), we introduced an operator S_z which extracts the (soft) $z \to 1$ limit of the expression it acts upon, and used P_{gg} to denote the spin-averaged gluon splitting function defined in eq. (A.23). We emphasize that $\Gamma_{i,g}$ depends on the energy of the hard-collinear parton and on $E_{\rm max}$, but we do not show these dependencies in what follows.

We continue with the case when the gluon $\mathfrak{m}$ becomes collinear to one of the initial-state partons, say 1_a. The matrix element squared that enters the definition of the function $F_{\rm LM}$ depends on the energy fraction $z = 1 - E_{\mathfrak{m}}/E_1$, which implies that one cannot integrate over the energy of the collinear gluon. However, integrating over the relative angle between $\mathfrak{m}$ and a is possible; performing this integration, we find

$$\langle \overline{S}_{\mathfrak{m}} C_{a\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle = [\alpha_s] \left\langle \frac{\Gamma_{1,a}}{\epsilon} F_{\rm LM} \right\rangle + \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}^{\rm gen}_{aa}(\epsilon) \otimes F_{\rm LM}\rangle \,. \tag{3.21}$$

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In eq. (3.21) $\Gamma_{1,a}$ is the *generalized initial-state anomalous dimension* which reads

$$\Gamma_{1,a} = \left(\frac{2E_1}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\gamma_a + \boldsymbol{T}_a^2 \frac{1-e^{-2\epsilon L_1}}{\epsilon}\right), \tag{3.22}$$

where γ_a is the anomalous dimensions of the initial-state parton a.[9] When writing eq. (3.21) we have used the fact that we only consider final-state gluons; because of that the parton type does not change after the collinear splitting. The function $\mathcal{P}_{aa}^{\rm gen}$ in eq. (3.21) is the *generalized splitting function*

$$\mathcal{P}_{aa}^{\rm gen}(z, E_1) = \left(\frac{2E_1}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\hat{P}_{aa}^{(0)}(z) + \epsilon\,\mathcal{P}_{aa}^{\rm fin}(z)\right], \tag{3.23}$$

where $\hat{P}_{aa}^{(0)}$ are the Altarelli-Parisi splitting functions which can be found in appendix A, together with the definition of the function $\mathcal{P}_{aa}^{\rm fin}$.[10] Furthermore, in eq. (3.21) we also used the shorthand notation

$$\mathcal{P}_{aa}^{\rm gen} \otimes F_{\rm LM} \equiv \int_0^1 {\rm d}z\, \mathcal{P}_{aa}^{\rm gen}(z)\, \frac{F_{\rm LM}(z\cdot 1_a, 2_b; \ldots)}{z}. \tag{3.24}$$

The case when the gluon $\mathfrak{m}$ becomes collinear to the initial-state parton 2_b is described by an equation which is analogous to eq. (3.21) but contains $\Gamma_{2,b}$ instead of $\Gamma_{1,a}$, and the "right" convolution

$$F_{\rm LM} \otimes \mathcal{P}_{bb}^{\rm gen} \equiv \int_0^1 {\rm d}z\, \mathcal{P}_{bb}^{\rm gen}(z)\, \frac{F_{\rm LM}(1_a, z\cdot 2_b; \ldots)}{z}. \tag{3.25}$$

We can now combine the various contributions and write the real-emission part of the NLO cross section. We find[11]

$$\begin{aligned} 2s\, {\rm d}\hat{\sigma}_{ab}^{\rm R} &= [\alpha_s]\langle (I_{\rm S}(\epsilon) + I_{\rm C}(\epsilon)) \cdot F_{\rm LM}\rangle + \frac{[\alpha_s]}{\epsilon}\Big[\langle \mathcal{P}_{aa}^{\rm gen} \otimes F_{\rm LM}\rangle + \langle F_{\rm LM} \otimes \mathcal{P}_{bb}^{\rm gen}\rangle\Big] \\ &+ \sum_{i=1}^{N_p} \langle \overline{S}_{\mathfrak{m}} \overline{C}_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i} \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle, \end{aligned} \tag{3.26}$$

where we introduced the hard-collinear operator

$$I_{\rm C}(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}, \tag{3.27}$$

with $f_1 = a$ and $f_2 = b$.

[9] We remind the reader that the quark and gluon anomalous dimensions read $\gamma_q = 3/2\, C_F$ and $\gamma_g = \beta_0 = 11/6\, C_A - 2/3\, T_R n_f$. Since in this paper we only deal with gluon final states, we systematically set n_f to zero in what follows.

[10] We note that $\mathcal{P}_{aa}^{\rm fin}$ is a function of ϵ; for brevity, we do not show this dependence.

[11] We note that, since we consider gluonic final states, $\mathcal{P}_{aa}^{\rm gen}$ is the same as $\mathcal{P}_{bb}^{\rm gen}$. Nevertheless, we find it convenient to distinguish between these two.

The infrared poles in eq. (3.26) cancel against those in the virtual contribution and the collinear renormalization of the PDFs, producing a finite remainder proportional to terms with lower parton multiplicities. To show this, we note that the infrared poles of the one-loop amplitude $\mathcal{M}_1$ can be written using Catani's formula [71]

$$\begin{aligned}\mathcal{M}_1(1_a,2_b;3,\ldots,N_p;X) = \frac{\alpha_s(\mu)}{2\pi}\, I_1(\epsilon)\, \mathcal{M}_0(1_a,2_b;3,\ldots,N_p;X) \\ + \mathcal{M}_1^{\text{fin}}(1_a,2_b;3,\ldots,N_p;X)\,,\end{aligned} \tag{3.28}$$

where $\mathcal{M}_1^{\text{fin}}$ is the infrared finite one-loop amplitude and

$$I_1(\epsilon) = \frac{1}{2}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_{(ij)}^{N_p}\frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\boldsymbol{T}_i^2}\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}\,, \quad \mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}\,. \tag{3.29}$$

The parameters λ_{ij} in eq. (3.29) are 1 if i and j are both incoming or outgoing partons and zero otherwise. Therefore, we can write

$$2s\, \mathrm{d}\hat{\sigma}_{ab}^{V} = \langle F_{\text{LV}}\rangle = [\alpha_s]\langle I_{\text{V}}(\epsilon)\cdot F_{\text{LM}}\rangle + \langle F_{\text{LV}}^{\text{fin}}\rangle\,, \tag{3.30}$$

where

$$I_{\text{V}}(\epsilon) = \overline{I}_1(\epsilon) + \overline{I}_1^{\dagger}(\epsilon)\,. \tag{3.31}$$

In the equation above we have introduced the operator $\overline{I}_1$ in place of Catani's original operator to factor out the same strong coupling $[\alpha_s]$ that appears in the real-emission contribution. It is defined by the following equation

$$\overline{I}_1(\epsilon) = \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}}\, I_1(\epsilon)\,, \tag{3.32}$$

such that

$$[\alpha_s]\,\overline{I}_1(\epsilon) = \frac{\alpha_s(\mu)}{2\pi}\, I_1(\epsilon)\,. \tag{3.33}$$

Furthermore, $F_{\text{LV}}^{\text{fin}}$ in eq. (3.30) is analogous to F_{LM} in eq. (2.6) but with $2\,\mathrm{Re}\left[\mathcal{M}_1^{\text{fin}}\,\mathcal{M}_0^*\right]$ instead of $|\mathcal{M}_0|^2$.

The collinear renormalization of parton distribution functions is standard. The NLO contribution to the cross section reads

$$2s\, \mathrm{d}\hat{\sigma}_{ab}^{\text{pdf}} = \frac{\alpha_s(\mu)}{2\pi\epsilon}\left[\langle \hat{P}_{aa}^{(0)}\otimes F_{\text{LM}}\rangle + \langle F_{\text{LM}}\otimes \hat{P}_{bb}^{(0)}\rangle\right]\,. \tag{3.34}$$

Finally, combining virtual (see eq. (3.30)), real-emission (see eq. (3.26)) and PDF-renormalization (see eq. (3.34)) contributions, we derive the following *finite* formula for the NLO cross section

$$\begin{aligned}2s\, \mathrm{d}\hat{\sigma}_{ab}^{\text{NLO}} &= \mathrm{d}\hat{\sigma}_{ab}^{\text{V}} + \mathrm{d}\hat{\sigma}_{ab}^{\text{R}} + \mathrm{d}\hat{\sigma}_{ab}^{\text{pdf}} = \frac{\alpha_s(\mu)}{2\pi}\langle I_{\text{T}}^{(0)}\cdot F_{\text{LM}}\rangle + \langle F_{\text{LV}}^{\text{fin}}\rangle \\ &+ \frac{\alpha_s(\mu)}{2\pi}\left[\left\langle \mathcal{P}_{aa}^{\text{NLO}}\otimes F_{\text{LM}}\right\rangle + \left\langle F_{\text{LM}}\otimes \mathcal{P}_{bb}^{\text{NLO}}\right\rangle\right] + \langle \mathcal{O}_{\text{NLO}}\,\Delta^{(\mathfrak{m})}F_{\text{LM}}(\mathfrak{m})\rangle\,,\end{aligned} \tag{3.35}$$

where $I_{\text{T}}^{(0)}$ is the $\mathcal{O}(\epsilon^0)$ coefficient in the expansion of $I_{\text{T}}(\epsilon)$, displayed in eq. (3.2).

A few comments about this result are in order. First, as we have anticipated at the beginning of this section, we have defined an infrared-finite sum[12] of the virtual, soft, and collinear I-operators that appears in the fully-unresolved part of $\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}}$

$$\left\langle I_{\mathrm{T}}(\epsilon)\cdot F_{\mathrm{LM}}\right\rangle = \left\langle \left[I_{\mathrm{V}}(\epsilon)+I_{\mathrm{S}}(\epsilon)+I_{\mathrm{C}}(\epsilon)\right]\cdot F_{\mathrm{LM}}\right\rangle = \mathcal{O}(\epsilon^0)\,. \tag{3.36}$$

As we show in the next section, *iterations* of this operator will appear in the result for the NNLO contribution to the cross section; this fact will play an important role in proving the cancellation of poles at NNLO as well. Second, we have denoted the subtraction operator for the fully-regulated real-emission contribution as

$$\mathcal{O}_{\mathrm{NLO}} = \sum_{i=1}^{N_p} \overline{S}_{\mathfrak{m}}\overline{C}_{i\mathfrak{m}}\,\omega^{\mathfrak{m}i}\,. \tag{3.37}$$

Finally we have exploited the expansion of $\mathcal{P}_{aa}^{\mathrm{gen}}$

$$\mathcal{P}_{aa}^{\mathrm{gen}}(z,E_i) = -\hat{P}_{aa}^{(0)}(z) + \epsilon\,\mathcal{P}_{aa}^{\mathrm{NLO}}(z,E_i) + \mathcal{O}(\epsilon^2)\,, \tag{3.38}$$

to obtain a manifestly finite quantity once we combine the hard-collinear subtraction terms with the PDF-renormalization contributions. The function $\mathcal{P}_{aa}^{\mathrm{NLO}}$ is defined in eq. (I.3). When using this function it is understood that E_i should be set to E_1 in $\left\langle \mathcal{P}_{aa}^{\mathrm{NLO}}\otimes F_{\mathrm{LM}}\right\rangle$ and to E_2 in $\left\langle F_{\mathrm{LM}}\otimes \mathcal{P}_{bb}^{\mathrm{NLO}}\right\rangle$.

For the reader's convenience, the definitions introduced in this section are repeated in appendix A. A more detailed discussion of the NLO calculation, including expansions of the various functions in powers of ϵ and a demonstration of the cancellation of the ϵ-poles, is presented in appendix C.

4 Calculations at next-to-next-to-leading order

In this section we extend the NLO QCD analysis described in the previous section to NNLO. At this order of perturbation theory we have to combine the double-virtual, the real-virtual, the double-real and the PDF renormalization contributions to compute the differential cross section. Hence, we write

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NNLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{VV}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{RV}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{RR}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}}\,. \tag{4.1}$$

Although the NNLO computation is significantly more involved than the NLO one, our aim is to replicate the latter as much as possible. In doing so, we face the following dilemma. On the one hand, the double-real contributions need to be split into partitions and sectors in order to define the approach to collinear singular limits in a unique way. On the other hand, this "sectoring" destroys the emergence of structures that can be combined in a natural way with the double-virtual and real-virtual corrections. Hence, finding an optimal balance between splitting the real-emission contributions into many well-defined pieces and identifying proper structures early in the calculation is the central challenge to organizing the NNLO computation in an efficient way. We explain how we address this challenge in this section.

[12] We show that this sum is ϵ-finite in appendix C.

Similar to the NLO case, we distinguish between resolved and potentially unresolved partons with the help of the partitions $\Delta^{(i)}$ and $\Delta^{(ij)}$ defined in appendix B. We use symmetries of the final-state gluons to define the NNLO contribution to the cross section without the PDFs renormalization in the following way

$$
\begin{aligned}
2s\, \mathrm{d}\bar{\sigma}^{\mathrm{NNLO}} &= \langle F_{\mathrm{VV}}(1_a, 2_b; 3, \ldots, N_p)\rangle + \langle \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}(1_a, 2_b; 3, \ldots, N_p, \mathfrak{m}_g)\rangle \\
&+ \frac{1}{2!}\langle \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}(1_a, 2_b; 3, \ldots, N_p, \mathfrak{m}_g, \mathfrak{n}_g)\rangle\,.
\end{aligned}
\tag{4.2}
$$

Here, F_{VV} and F_{RV} are defined analogously to eq. (2.6), but using double-virtual and real-virtual matrix elements, while $\mathfrak{m}$ and $\mathfrak{n}$ are potentially-unresolved partons. Furthermore, all the functions F_{VV}, F_{RV} and F_{RR} include the symmetry factor $1/(N_p-2)!$ arising from the $N = N_p - 2$ identical *resolved* gluons in the final state. The dependence of the matrix elements and phase spaces on colorless final-state particles is not shown.

It is convenient to remove the (remaining) symmetry factor $1/2!$ from the double-real contribution by introducing the energy ordering of the unresolved gluons $\mathfrak{m}$ and $\mathfrak{n}$

$$
\frac{1}{2!}\langle \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}(\ldots, \mathfrak{m}_g, \mathfrak{n}_g)\rangle = \langle \Delta^{(\mathfrak{m}\mathfrak{n})} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}(\ldots, \mathfrak{m}_g, \mathfrak{n}_g)\rangle\,,
\tag{4.3}
$$

where $\Theta_{\mathfrak{m}\mathfrak{n}} = \Theta(E_\mathfrak{m} - E_\mathfrak{n})$. We obtain

$$
\begin{aligned}
2s\, \mathrm{d}\bar{\sigma}^{\mathrm{NNLO}} &= \langle F_{\mathrm{VV}}(1_a, 2_b; 3, \ldots, N_p)\rangle + \langle \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}(1_a, 2_b; 3, \ldots, N_p, \mathfrak{m}_g)\rangle \\
&+ \langle \Delta^{(\mathfrak{m}\mathfrak{n})} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}(1_a, 2_b; 3, \ldots, N_p, \mathfrak{m}_g, \mathfrak{n}_g)\rangle\,.
\end{aligned}
\tag{4.4}
$$

The above equation provides the starting point for our calculation. It follows that the NNLO QCD corrections to the cross section contain contributions that exist in three distinct phase spaces. These phase spaces overlap in configurations where the gluons labelled as $\mathfrak{m}$ and $\mathfrak{n}$ become unresolved. When this happens, the corresponding amplitudes become singular and integrating over unresolved phase spaces leads to the appearance of $1/\epsilon^n$ poles, similar to the NLO case. Our goal is to isolate and remove these singularities locally in the phase space, demonstrate the cancellation of poles between the different contributions in eq. (4.4), and determine the finite remainder.

We begin by isolating the soft limits of the real-emission contributions. As already discussed in ref. [1], two soft limits are needed: one to describe the double-soft limit $E_\mathfrak{m} \sim E_\mathfrak{n} \to 0$, which we denote as $S_{\mathfrak{m}\mathfrak{n}}$, and one for the single-soft limit $E_\mathfrak{n} \to 0$ at fixed $E_\mathfrak{m}$, which we denote as $S_\mathfrak{n}$. We write

$$
\begin{aligned}
2s\, \mathrm{d}\bar{\sigma}^{\mathrm{NNLO}} &= \langle F_{\mathrm{VV}}\rangle + \langle S_{\mathfrak{m}\mathfrak{n}} \Delta^{(\mathfrak{m}\mathfrak{n})} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}(\mathfrak{m}, \mathfrak{n})\rangle \\
&+ \langle \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}(\mathfrak{m})\rangle + \langle \overline{S}_{\mathfrak{m}\mathfrak{n}} S_\mathfrak{n}\, \Delta^{(\mathfrak{m}\mathfrak{n})} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}(\mathfrak{m}, \mathfrak{n})\rangle \\
&+ \langle \overline{S}_{\mathfrak{m}\mathfrak{n}} \overline{S}_\mathfrak{n}\, \Delta^{(\mathfrak{m}\mathfrak{n})} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}(\mathfrak{m}, \mathfrak{n})\rangle\,,
\end{aligned}
\tag{4.5}
$$

where the operator $\overline{S}_x = I - S_x$ has already been introduced in the context of the NLO QCD computation. Furthermore, when writing eq. (4.5), we have dropped the arguments related to the resolved partons, i.e.

$$
F_{\mathrm{LM}}(\mathfrak{m}, \mathfrak{n}) \equiv F_{\mathrm{LM}}(1_a, 2_b, 3, \ldots, N_p, \mathfrak{m}_g, \mathfrak{n}_g)\,.
\tag{4.6}
$$

Next, we take the fourth term on the right-hand side of eq. (4.5)

$$\langle \overline{S}_{\mathfrak{mn}} S_{\mathfrak{n}}\, \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle\,, \tag{4.7}$$

make use of the fact that

$$\langle \overline{S}_{\mathfrak{mn}} S_{\mathfrak{n}}\, \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle = \langle \overline{S}_{\mathfrak{m}} S_{\mathfrak{n}}\, \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle\,, \tag{4.8}$$

and add collinear subtractions for the gluon $\mathfrak{m}$. We find

$$\begin{aligned}\langle \overline{S}_{\mathfrak{mn}} S_{\mathfrak{n}}\, \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle &= \langle \mathcal{O}_{\mathrm{NLO}} \Delta^{(\mathfrak{m})} S_{\mathfrak{n}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle \\ &+ \sum_{i=1}^{N_p} \langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} S_{\mathfrak{n}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle\,.\end{aligned} \tag{4.9}$$

We remind the reader that the operator $\mathcal{O}_{\mathrm{NLO}}$, defined in eq. (3.37), subtracts singularities associated with parton $\mathfrak{m}$, and we have used eq. (B.13) to simplify eq. (4.9). To obtain a similar structure for the real-virtual contribution, we rewrite F_{RV} as

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}(\mathfrak{m}) \rangle = \langle S_{\mathfrak{m}} F_{\mathrm{RV}}(\mathfrak{m}) \rangle + \sum_{i=1}^{N_p} \langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}(\mathfrak{m}) \rangle + \langle \mathcal{O}_{\mathrm{NLO}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}(\mathfrak{m}) \rangle\,. \tag{4.10}$$

Since the cancellation of infrared singularities can only occur among terms with similar kinematics of the hard final-state partons, we would like to write the NNLO QCD cross section in such a way that contributions with the same number of *resolved* final-state partons are combined. At NNLO this number varies between N and $N+2$, so there are three terms that need to be considered. Hence, we aim to write the cross section in the following way

$$2s\, \mathrm{d}\bar{\sigma}^{\mathrm{NNLO}} = \Sigma_N + \Sigma_{N+1} + \Sigma_{N+2}\,. \tag{4.11}$$

Most of the contributions to the above equation are yet to be determined. However, as a first step, we can use eq. (4.5) and the rearrangement of terms that led to eqs. (4.9) and (4.10) to write[13]

$$2s\, \mathrm{d}\bar{\sigma}^{\mathrm{NNLO}} = \Sigma_N^{(1)} + \Sigma_{N+1}^{(1)} + \Sigma_{\mathrm{RR}}\,, \tag{4.12}$$

where

$$\begin{aligned}\Sigma_N^{(1)} &= \langle F_{\mathrm{VV}} \rangle + \langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle + \langle S_{\mathfrak{m}} F_{\mathrm{RV}}(\mathfrak{m}) \rangle \\ &+ \sum_{i=1}^{N_p} \langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} \left[F_{\mathrm{RV}}(\mathfrak{m}) + S_{\mathfrak{n}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \right] \rangle\,, \\ \Sigma_{N+1}^{(1)} &= \langle \mathcal{O}_{\mathrm{NLO}} \Delta^{(\mathfrak{m})} \left[F_{\mathrm{RV}}(\mathfrak{m}) + S_{\mathfrak{n}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \right] \rangle\,, \\ \Sigma_{\mathrm{RR}} &= \langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}}\, \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle\,.\end{aligned} \tag{4.13}$$

The quantity $\Sigma_N^{(1)}$ is double-unresolved, in the sense that both gluons $\mathfrak{m}$ and $\mathfrak{n}$ are either soft or collinear. The superscript indicates that this is the *first* of several contributions to

[13] All the steps that are needed for the rearrangements can be found in figure 1.

Σ_N that has been identified. Similarly, the quantity $\Sigma_{N+1}^{(1)}$ is the first single-unresolved term contributing to Σ_{N+1} that we identify. On the contrary, $\Sigma_{\rm RR}$ is a mix of various contributions as it contains unregulated collinear singularities. As we will see, upon extracting these singularities, some parts of $\Sigma_{\rm RR}$ will contribute to Σ_N and Σ_{N+1} and will play an important role in the cancellation of infrared poles.

It is well-known that extracting all singularities from the double-real contribution is a complicated problem as many of them overlap. To disentangle them, we partition the angular phase space [1, 20, 21, 61]. Further details are given in appendices B and D. Using these results, we split $\Sigma_{\rm RR}$ into four distinct terms. We write

$$\Sigma_{\rm RR} = \Sigma_{N+2}^{\rm fin} + \Sigma_N^{(2)} + \Sigma_{\rm RR,2c} + \Sigma_{\rm RR,1c}\,, \tag{4.14}$$

where, as we already mentioned, the subscripts of the first two terms on the right-hand side indicate the number of resolved partons. In brief, the first term on the right-hand side in eq. (4.14) is fully resolved, the second is the triple-collinear subtraction term, the third is the double-collinear term and the last term is the single-collinear contribution. To elaborate further, the first term $\Sigma_{N+2}^{\rm fin}$ is the fully-regulated contribution given by

$$\Sigma_{N+2}^{\rm fin} = \left\langle \overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}\,\Omega_1\,\Delta^{(\mathfrak{mn})}\,\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle, \tag{4.15}$$

where Ω_1 is a function of collinear-subtraction operators and partition functions defined in eq. (D.5).[14] The quantity $\Sigma_{N+2}^{\rm fin}$ is the only contribution to the NNLO cross section with $N+2$ resolved final-state partons and it can be implemented in a numerical code without further ado.

The second term $\Sigma_N^{(2)}$ is the triple-collinear contribution. It reads

$$\Sigma_N^{(2)} = \left\langle \overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}\,\Omega_2\,\Delta^{(\mathfrak{mn})}\,\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle, \tag{4.16}$$

where Ω_2 is a triple-collinear projection operator that can be found in eq. (D.6). We note that $\Sigma_N^{(2)}$ was computed in ref. [69] and can be immediately borrowed from there. It represents the second contribution to the fully-unresolved term Σ_N that we have identified.

The third term $\Sigma_{\rm RR,2c}$ is the double-collinear contribution where gluons are emitted from different legs

$$\begin{aligned}\Sigma_{\rm RR,2c} &= \left\langle \overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}\,\Omega_3\,\Delta^{(\mathfrak{mn})}\,\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle \\ &= -\sum_{(ij)}^{N_p}\left\langle \overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}C_{j\mathfrak{n}}C_{i\mathfrak{m}}[{\rm d}p_{\mathfrak{m}}][{\rm d}p_{\mathfrak{n}}]\,\omega^{\mathfrak{m}i,\mathfrak{n}j}\Delta^{(\mathfrak{mn})}\,\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle,\end{aligned} \tag{4.17}$$

where the angular partition functions $\omega^{\mathfrak{m}i,\mathfrak{n}j}$ are defined in eq. (B.26). Although this contribution is fairly simple, it is useful to rewrite it before proceeding further. According to eq. (4.17) both collinear operators $C_{i\mathfrak{m}}$ and $C_{j\mathfrak{n}}$ act on the phase space of partons $\mathfrak{m}$ and $\mathfrak{n}$. This is necessary to be able to use the results for $\Sigma_N^{(2)}$ from ref. [69]. Eventually, we will have to combine these double-collinear contributions with collinear limits of the single-soft, the real-virtual and other terms, where by definition the collinear operators do not act on the potentially

[14]We note that this contribution will contribute to final states with $N+2$, $N+1$ and N gluons, as a result of the subtracted limits contained in the definition of Ω_1.

unresolved phase spaces. Hence, it is convenient to rewrite eq. (4.17) in the same way, ensuring that $C_{i\mathfrak{m}}$ and $C_{j\mathfrak{n}}$ do not act on the phase space of the unresolved partons. We explain how to do this in appendix E.2. Here, we just state the result and write $\Sigma_{\rm RR,2c}$ as follows

$$\Sigma_{\rm RR,2c} = \Sigma_N^{(3)} + \Sigma_N^{\rm fin,(1)}\,, \tag{4.18}$$

where $\Sigma_N^{(3)}$ is the third (divergent) double-unresolved contribution that we have extracted. Likewise, $\Sigma_N^{\rm fin,(1)}$ is the first ϵ-finite contribution to Σ_N that we have encountered. We stress that this is *not* the same as the finite part of $\Sigma_N^{(1)}$ defined previously. The two terms read

$$\begin{aligned}\Sigma_N^{(3)} &= -\sum_{(ij)}^{N_p} \langle \overline{S}_{\mathfrak{n}} C_{j\mathfrak{n}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})}\, \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle\,,\\ \Sigma_N^{\rm fin,(1)} &= -\left[\left(\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\right)^2 - 1\right]\sum_{(ij)}^{N_p} \langle \overline{S}_{\mathfrak{n}} C_{j\mathfrak{n}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{mn})}\, \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle\,.\end{aligned} \tag{4.19}$$

We note that the unresolved phase space $[{\rm d}p_{\mathfrak{m}}][{\rm d}p_{\mathfrak{n}}]$ does not appear in the above formulas, indicating that collinear operators do not act on it anymore. In addition, we have used $C_{j\mathfrak{n}} C_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i,\mathfrak{n}j} = 1$ to remove the partition functions. Furthermore, if the gluons are emitted off different external legs (which is ensured by the two collinear operators), we have

$$S_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}}\,[\ldots] = 0\,, \tag{4.20}$$

allowing us to write $\overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}} = \overline{S}_{\mathfrak{n}}$. Finally, to see that $\Sigma_N^{\rm fin,(1)}$ is finite, we observe that eq. (4.19) is completely soft-regulated, while the two collinear operators $C_{i\mathfrak{m}}$ and $C_{j\mathfrak{n}}$ each produce an $\mathcal{O}(\epsilon^{-1})$ singularity upon integrating over the phase space of gluons $\mathfrak{m}$ and $\mathfrak{n}$. This is compensated by the prefactor $\left(\Gamma(1-2\epsilon)/\Gamma^2(1-\epsilon)\right)^2 - 1 \sim \mathcal{O}(\epsilon^2)$, leading to an infrared finite quantity. To summarize, we have written $\Sigma_{\rm RR,2c}$ as the sum of two double-unresolved contributions, one of which contains poles and one of which is ϵ-finite.

We are left with $\Sigma_{\rm RR,1c}$, which is the double-real single-collinear contribution. It reads

$$\begin{aligned}\Sigma_{\rm RR,1c} &= \langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}}\, \Omega_4\, \Delta^{(\mathfrak{mn})}\, \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle\\ &= \sum_{(ij)}^{N_p} \Big\langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}} \Big[C_{i\mathfrak{m}}[{\rm d}p_{\mathfrak{m}}] + C_{j\mathfrak{n}}[{\rm d}p_{\mathfrak{n}}]\Big]\, \omega^{\mathfrak{m}i,\mathfrak{n}j} \Delta^{(\mathfrak{mn})}\, \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big\rangle\\ &\quad + \sum_{i=1}^{N_p} \Big\langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}} \Big[C_{i\mathfrak{n}}\theta^{(a)} + C_{\mathfrak{mn}}\theta^{(b)} + C_{i\mathfrak{m}}\theta^{(c)} + C_{\mathfrak{mn}}\theta^{(d)}\Big]\\ &\quad \times [{\rm d}p_{\mathfrak{m}}][{\rm d}p_{\mathfrak{n}}]\, \omega^{\mathfrak{m}i,\mathfrak{n}i} \Delta^{(\mathfrak{mn})}\, \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big\rangle\,,\end{aligned} \tag{4.21}$$

where the partitions $\omega^{\mathfrak{m}i,\mathfrak{n}j}$ and $\omega^{\mathfrak{m}i,\mathfrak{n}i}$ can be found in eq. (B.26) and eq. (B.27), respectively. the functions $\theta^{(\alpha)}$ with $\alpha = a,b,c,d$ indicate that a particular contribution is confined to a certain phase-space *sector*. These sectors, together with the corresponding phase-space parameterizations, are defined in appendices D and E, respectively. The challenge therefore is to write $\Sigma_{\rm RR,1c}$ as a sum of terms with a well-defined number of resolved partons. To

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do this, we need to extract the remaining collinear singularities from $\Sigma_{\rm RR,1c}$.[15] We do so in the next section. We do so in the next section.

4.1 Analyzing single-collinear contributions

The $1/\epsilon^n$ singularities in $\Sigma_{\rm RR,1c}$ simplify if the contributions of different partitions and sectors are *combined*. To appreciate why doing so is non-trivial, we need to remind ourselves why partitions and sectors were introduced in the first place. The reason was to disentangle overlapping singular limits, making them uniquely defined. However, it also complicates the identification of physical quantities such as e.g. collinear anomalous dimensions and splitting functions. We emphasize that the ability to recognize these universal structures in the early stages of the calculation is very useful for canceling the infrared divergences in an efficient and transparent manner. Hence, our strategy will be to remove sectors in a *controlled* way, eventually getting to the point where various contributions can be rearranged into recognizable universal structures.

As a result of this analysis we are able to represent $\Sigma_{\rm RR,1c}$ by a sum of five divergent $\Sigma_N^{(4,\dots,8)}$ and four finite ($\Sigma_N^{{\rm fin},(2,\dots,5)}$) double-unresolved quantities, and two divergent $\Sigma_{N+1}^{(2,3)}$ and two finite $\Sigma_{N+1}^{{\rm fin}(1,2)}$ single-unresolved quantities, see figure 1. These quantities are used in eq. (4.67) and eq. (5.1), respectively, to construct relevant contributions to the NNLO cross section. The remainder of this section describes manipulations of $\Sigma_{\rm RR,1c}$ that lead to such a representation.

We begin by separating sectors $\theta^{(b)}$ and $\theta^{(d)}$ from the remaining contributions to $\Sigma_{\rm RR,1c}$. We write

$$\Sigma_{\rm RR,1c} = \Sigma_{\rm RR,1c}^{(a,c,{\rm dc})} + \Sigma_{\rm RR,1c}^{(b,d)}\,, \tag{4.22}$$

where[16]

$$\begin{aligned}\Sigma_{\rm RR,1c}^{(a,c,{\rm dc})} = \Big\langle \overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}\Big[\sum_{(ij)}^{N_p}(C_{i\mathfrak{m}}+C_{j\mathfrak{n}})\omega^{\mathfrak{m}i,\mathfrak{n}j} + \sum_{i=1}^{N_p}(C_{i\mathfrak{n}}\theta^{(a)}+C_{i\mathfrak{m}}\theta^{(c)})\omega^{\mathfrak{m}i,\mathfrak{n}i}\Big] \\ \times [{\rm d}p_{\mathfrak{m}}][{\rm d}p_{\mathfrak{n}}]\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big\rangle\,,\end{aligned} \tag{4.23}$$

and

$$\Sigma_{\rm RR,1c}^{(b,d)} = \sum_{i=1}^{N_p}\Big\langle \overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}C_{\mathfrak{mn}}(\theta^{(b)}+\theta^{(d)})[{\rm d}p_{\mathfrak{m}}][{\rm d}p_{\mathfrak{n}}]\omega^{\mathfrak{m}i,\mathfrak{n}i}\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big\rangle\,. \tag{4.24}$$

[15]The label "single-collinear" for this contribution refers to the one collinear limit appearing in eq. (4.21). Such collinear limit is relevant for only one potentially unresolved parton, but the remaining one is still unregulated. For this reason we need to further extract the singularities associated to the second extra emission. This procedure will lead to double-unresolved terms.

[16]The superscript $(a,c,{\rm dc})$ reminds us that $\Sigma_{\rm RR,1c}^{(a,c,{\rm dc})}$ includes contributions of sectors a and c and of the double-collinear partitions.

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We first consider $\Sigma^{(a,c,\mathrm{dc})}_{\mathrm{RR,1c}}$. In this case eq. (4.20) holds, so that $\overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}$ can be replaced by $\overline{S}_{\mathfrak{n}}$. We then write

$$\begin{aligned}\Sigma^{(a,c,\mathrm{dc})}_{\mathrm{RR,1c}} = \Big\langle \overline{S}_{\mathfrak{n}}\Big[\sum_{(ij)}^{N_p}(C_{i\mathfrak{m}}+C_{j\mathfrak{n}})\omega^{\mathfrak{m}i,\mathfrak{n}j} \\ + \sum_{i=1}^{N_p}(C_{i\mathfrak{n}}\theta^{(a)}+C_{i\mathfrak{m}}\theta^{(c)})\omega^{\mathfrak{m}i,\mathfrak{n}i}\Big][\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle .\end{aligned} \tag{4.25}$$

We can simplify this expression by renaming gluons $\mathfrak{m}$ and $\mathfrak{n}$ in such a way that the collinear operators always refer to the gluon $\mathfrak{m}$. We also exploit the fact that under such a relabelling sector $\theta^{(a)}$ becomes sector $\theta^{(c)}$, see eq. (D.1). Hence, we obtain

$$\begin{aligned}\Sigma^{(a,c,\mathrm{dc})}_{\mathrm{RR,1c}} = \Big\langle \mathcal{S}(\mathfrak{m},\mathfrak{n})\Big[\sum_{(ij)}^{N_p}C_{i\mathfrak{m}}\,\omega^{\mathfrak{m}i,\mathfrak{n}j} \\ + \sum_{i=1}^{N_p}C_{i\mathfrak{m}}\theta^{(c)}\omega^{\mathfrak{m}i,\mathfrak{n}i}\Big][\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle ,\end{aligned} \tag{4.26}$$

where the soft-regulating operator $\mathcal{S}(\mathfrak{m},\mathfrak{n})$ reads

$$\mathcal{S}(\mathfrak{m},\mathfrak{n}) = \overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}} + \overline{S}_{\mathfrak{m}}\Theta_{\mathfrak{nm}} . \tag{4.27}$$

We note that we can rewrite the operator $\mathcal{S}(\mathfrak{m},\mathfrak{n})$ in several equivalent ways

$$\begin{aligned}\mathcal{S}(\mathfrak{m},\mathfrak{n}) &= \overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}} + \overline{S}_{\mathfrak{m}}\Theta_{\mathfrak{nm}} = \mathbb{1} - S_{\mathfrak{n}}\Theta_{\mathfrak{mn}} - S_{\mathfrak{m}}\Theta_{\mathfrak{nm}} \\ &= \overline{S}_{\mathfrak{m}}\overline{S}_{\mathfrak{n}} + S_{\mathfrak{m}}\overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}} + S_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}\Theta_{\mathfrak{nm}} = \overline{S}_{\mathfrak{n}}(\mathbb{1} - S_{\mathfrak{m}}\Theta_{\mathfrak{nm}}) + S_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}\Theta_{\mathfrak{nm}} ,\end{aligned} \tag{4.28}$$

and we will use the different representations displayed above in what follows.

To simplify $\Sigma^{(a,c,\mathrm{dc})}_{\mathrm{RR,1c}}$ we need to extract the remaining collinear singularities. Since we relabeled gluons so that the collinear operators refer to the gluon $\mathfrak{m}$, the unregulated singularities affect gluon $\mathfrak{n}$ only. However, there is an additional technical detail that should be highlighted before proceeding.

As we already mentioned, the many single-collinear contributions will have to be combined with collinear limits from single-soft, real-virtual and other terms where the collinear operators do not act on the phase space. Therefore, it is useful to rewrite $\Sigma^{(a,c,\mathrm{dc})}_{\mathrm{RR,1c}}$ in such a way that: *i*) $C_{i\mathfrak{m}}$ does not act on the phase space and *ii*) restrictions imposed by the presence of sector $\theta^{(c)}$ are lifted. We explain how to do this in appendix E.2. Here, we just report the final result, which is obtained once we insert $\mathbb{1} = \overline{C}_{i\mathfrak{m}} + C_{i\mathfrak{m}}$ in the equation for $\Sigma^{(a,c,\mathrm{dc})}_{\mathrm{RR,1c}}$. We find

$$\Sigma^{(a,c,\mathrm{dc})}_{\mathrm{RR,1c}} = \Sigma^{(a,c,\mathrm{dc}),1}_{\mathrm{RR,1c}} + \Sigma^{(a,c,\mathrm{dc}),2}_{\mathrm{RR,1c}} , \tag{4.29}$$

where

$$\begin{aligned}\Sigma^{(a,c,\mathrm{dc}),1}_{\mathrm{RR,1c}} = \Big\langle \mathcal{S}(\mathfrak{m},\mathfrak{n})\Big[\sum_{(ij)}^{N_p}\overline{C}_{j\mathfrak{n}}C_{i\mathfrak{m}}\,\omega^{\mathfrak{m}i,\mathfrak{n}j} \\ + \sum_{i=1}^{N_p}(\eta_{i\mathfrak{n}}/2)^{-\epsilon}\overline{C}_{i\mathfrak{n}}C_{i\mathfrak{m}}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}\Big]\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle ,\end{aligned} \tag{4.30}$$

and

$$\Sigma_{\mathrm{RR,1c}}^{(a,c,\mathrm{dc}),2} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \left\langle \mathcal{S}(\mathfrak{m},\mathfrak{n}) \left[\sum_{(ij)}^{N_p} C_{j\mathfrak{n}} C_{i\mathfrak{m}} + \sum_{i=1}^{N_p} (\eta_{i\mathfrak{n}}/2)^{-\epsilon} C_{i\mathfrak{n}} C_{i\mathfrak{m}} \right] \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \right\rangle . \tag{4.31}$$

As was the case in eq. (4.19), the phase space $[\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]$ does not appear in these formulas anymore, indicating that collinear operators there do not act on it. We also note that the sector function $\theta^{(c)}$ disappeared from eq. (4.30), leaving as a remnant the factor of $(\eta_{i\mathfrak{n}}/2)^{-\epsilon}$. Furthermore, eq. (4.31) becomes *potentially ambiguous* because the collinear operators $C_{i\mathfrak{n}}$ and $C_{i\mathfrak{m}}$ do not commute in general. Therefore the order in which they appear in the above formula (and in similar formulas) is important.[17] On the other hand, since the operator $\mathcal{S}(\mathfrak{m},\mathfrak{n})$ represents a soft subtraction, it commutes with the collinear operators. Finally, we have omitted an overall factor $\Gamma(1-2\epsilon)/\Gamma^2(1-\epsilon)$ in $\Sigma_{\mathrm{RR,1c}}^{(a,c,\mathrm{dc}),1}$ because it would only generate $\mathcal{O}(\epsilon)$ terms in the result.

We will continue with the discussion of the contribution $\Sigma_{\mathrm{RR,1c}}^{(a,c,\mathrm{dc}),1}$. It is convenient to rewrite the factor $(\eta_{i\mathfrak{n}}/2)^{-\epsilon}$ in eq. (4.30) as follows

$$(\eta_{i\mathfrak{n}}/2)^{-\epsilon} = \left[(\eta_{i\mathfrak{n}}/2)^{-\epsilon} - 1\right] + 1 , \tag{4.32}$$

and combine the second term with the $i \neq j$ sum in that equation. We find

$$\Sigma_{\mathrm{RR,1c}}^{(a,c,\mathrm{dc}),1} = \left\langle \mathcal{S}(\mathfrak{m},\mathfrak{n}) \left[\sum_{i,j=1}^{N_p} \overline{C}_{j\mathfrak{n}} C_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i,\mathfrak{n}j} + \sum_{i=1}^{N_p} \left[(\eta_{i\mathfrak{n}}/2)^{-\epsilon} - 1\right] \overline{C}_{i\mathfrak{n}} C_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i,\mathfrak{n}i} \right] \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \right\rangle , \tag{4.33}$$

where we emphasize that the first sum includes terms with $i = j$. We also note that the comment concerning the non-commutativity of operators $C_{i\mathfrak{n}}$ and $C_{i\mathfrak{m}}$ that we just made applies to eq. (4.33) as well.

Another important point is that the second term in eq. (4.33) is *finite* in the limit $\epsilon \to 0$. The reason for this is that the only singularity present in this term comes from the collinear limit $i||\mathfrak{m}$, which gives an $\mathcal{O}(\epsilon^{-1})$ contribution once integrated over the phase space of gluon $\mathfrak{m}$. On the other hand, the presence of $\overline{C}_{i\mathfrak{n}}$ allows us to expand the difference $\left[(\eta_{i\mathfrak{n}}/2)^{-\epsilon} - 1\right]$, giving an $\mathcal{O}(\epsilon)$ quantity.

Furthermore, we note that, in the first term on the right-hand side of eq. (4.33), the partitioning can be replaced with another, more suitable one. Indeed, since by construction

$$\sum_{j=1}^{N_p} C_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i,\mathfrak{n}j} \equiv \sum_{j=1}^{N_p} \omega^{\mathfrak{m}i,\mathfrak{n}j}_{i||\mathfrak{m}}\, C_{i\mathfrak{m}} = C_{i\mathfrak{m}} , \qquad C_{j\mathfrak{n}} C_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i,\mathfrak{n}j} = C_{j\mathfrak{n}} C_{i\mathfrak{m}} , \tag{4.34}$$

one finds

$$\sum_{i,j=1}^{N_p} \overline{C}_{j\mathfrak{n}} C_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i,\mathfrak{n}j} = \sum_{i=1}^{N_p} C_{i\mathfrak{m}} - \sum_{i,j=1}^{N_p} C_{j\mathfrak{n}} C_{i\mathfrak{m}} \equiv \sum_{i,j=1}^{N_p} \overline{C}_{j\mathfrak{n}}\, \omega^{\mathfrak{n}j}\, C_{i\mathfrak{m}} , \tag{4.35}$$

where $\omega^{\mathfrak{n}j}$ is, e.g., a NLO partition where the unresolved gluon is $\mathfrak{n}$.

[17] For the all-gluonic final states that we consider, these limits will always commute.

Finally, it is convenient to split the soft subtraction operator $\mathcal{S}(\mathfrak{m},\mathfrak{n})$ acting on the first term in eq. (4.33) in a particular way. Employing the following representation (cf. eq. (4.28))

$$\mathcal{S}(\mathfrak{m},\mathfrak{n}) = S_{\mathfrak{n}}\Theta_{\mathfrak{mn}} + S_{\mathfrak{m}}\Theta_{\mathfrak{nm}} = \overline{S}_{\mathfrak{n}}(\mathbb{1} - S_{\mathfrak{m}}\Theta_{\mathfrak{nm}}) + S_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}\Theta_{\mathfrak{nm}}\,, \tag{4.36}$$

we rewrite the formula for $\Sigma^{(a,c,\mathrm{dc}),1}_{\mathrm{RR,1c}}$ in such a way that partonic multiplicities are clearly separated

$$\Sigma^{(a,c,\mathrm{dc}),1}_{\mathrm{RR,1c}} = \Sigma^{(2)}_{N+1} + \Sigma^{(4)}_{N} + \Sigma^{\mathrm{fin},(1)}_{N+1} + \Sigma^{\mathrm{fin},(2)}_{N}\,. \tag{4.37}$$

We note that in eq. (4.37), the first ϵ-finite contribution to the single-unresolved cross section is denoted as $\Sigma^{\mathrm{fin},(1)}_{N+1}$. We emphasize again that this does not correspond to the finite part of $\Sigma^{(1)}_{N+1}$. The individual contributions read

$$\begin{aligned}
\Sigma^{(2)}_{N+1} &= \sum_{i,j=1}^{N_p} \Big\langle \overline{S}_{\mathfrak{n}}(\mathbb{1} - S_{\mathfrak{m}}\Theta_{\mathfrak{nm}})\overline{C}_{j\mathfrak{n}}\,\omega^{\mathfrak{n}j}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle \\
&= \sum_{i=1}^{N_p} \Big\langle \mathcal{O}_{\mathrm{NLO}}(\mathbb{1} - S_{\mathfrak{n}}\Theta_{\mathfrak{mn}})C_{i\mathfrak{n}}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle\,, \\
\Sigma^{(4)}_{N} &= \sum_{i,j=1}^{N_p} \Big\langle S_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}\overline{C}_{j\mathfrak{n}}\,\omega^{\mathfrak{n}j}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle\,, \\
\Sigma^{\mathrm{fin},(1)}_{N+1} &= \sum_{i=1}^{N_p} \Big\langle \big[(\eta_{i\mathfrak{n}}/2)^{-\epsilon} - 1\big]\overline{S}_{\mathfrak{n}}(\mathbb{1} - S_{\mathfrak{m}}\Theta_{\mathfrak{nm}})\overline{C}_{i\mathfrak{n}}C_{i\mathfrak{m}}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle \\
&= \sum_{i=1}^{N_p} \Big\langle \mathcal{O}^{(i)}_{\mathrm{NLO}}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}_{i\|\mathfrak{m}}\,\big[(\eta_{i\mathfrak{m}}/2)^{-\epsilon} - 1\big](\mathbb{1} - S_{\mathfrak{n}}\Theta_{\mathfrak{mn}})C_{i\mathfrak{n}}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle\,, \\
\Sigma^{\mathrm{fin},(2)}_{N} &= \sum_{i=1}^{N_p} \Big\langle \big[(\eta_{i\mathfrak{n}}/2)^{-\epsilon} - 1\big]S_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}\overline{C}_{i\mathfrak{n}}C_{i\mathfrak{m}}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle\,,
\end{aligned} \tag{4.38}$$

where we define $\mathcal{O}^{(i)}_{\mathrm{NLO}} = \overline{S}_{\mathfrak{m}}\overline{C}_{i\mathfrak{m}}$ so that $\mathcal{O}_{\mathrm{NLO}} = \sum_{i=1}^{N_p}\mathcal{O}^{(i)}_{\mathrm{NLO}}\omega^{\mathfrak{m}i}$. We note that when moving from the first to the second line in $\Sigma^{(2)}_{N+1}$ and $\Sigma^{\mathrm{fin},(1)}_{N+1}$ we have relabelled $\mathfrak{m}$ to $\mathfrak{n}$ and *vice versa*.

We now return to $\Sigma^{(a,c,\mathrm{dc}),2}_{\mathrm{RR,1c}}$ (see eq. (4.30)) and rewrite it as follows

$$\Sigma^{(a,c,\mathrm{dc}),2}_{\mathrm{RR,1c}} = \Sigma^{(5)}_{N} + \Sigma^{\mathrm{fin},(3)}_{N}\,, \tag{4.39}$$

where

$$\begin{aligned}
\Sigma^{(5)}_{N} &= \Big\langle \mathcal{S}(\mathfrak{m},\mathfrak{n})\Big[\sum_{(ij)}^{N_p} C_{j\mathfrak{n}}C_{i\mathfrak{m}} + \sum_{i=1}^{N_p}(\eta_{i\mathfrak{n}}/2)^{-\epsilon}C_{i\mathfrak{n}}C_{i\mathfrak{m}}\Big]\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle, \\
\Sigma^{\mathrm{fin},(3)}_{N} &= \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1\right]\Big\langle \mathcal{S}(\mathfrak{m},\mathfrak{n})\Big[\sum_{(ij)}^{N_p} C_{j\mathfrak{n}}C_{i\mathfrak{m}} + \sum_{i=1}^{N_p}(\eta_{i\mathfrak{n}}/2)^{-\epsilon}C_{i\mathfrak{n}}C_{i\mathfrak{m}}\Big] \\
&\quad \times \Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle\,.
\end{aligned} \tag{4.40}$$

Again, we note that $\Sigma_N^{\text{fin},(3)}$ is finite because the soft-regulated collinear limits $C_{j\mathfrak{n}}C_{i\mathfrak{m}}$ produce an $\mathcal{O}(\epsilon^{-2})$ pole when integrated over the angles of $\mathfrak{m}$ and $\mathfrak{n}$, and the prefactor $\Gamma(1-2\epsilon)/\Gamma^2(1-\epsilon)-1$ is $\mathcal{O}(\epsilon^2)$. This concludes our discussion of all single-collinear limits, except for those in triple-collinear sectors (b) and (d).

We now turn to $\Sigma_{\text{RR,1c}}^{(b,d)}$, defined in eq. (4.24). We start by mapping sector $\theta^{(d)}$ onto sector $\theta^{(b)}$ by renaming gluons $\mathfrak{m}$ to $\mathfrak{n}$ and *vice versa* where appropriate.[18] We find

$$
\begin{aligned}
\Sigma_{\text{RR,1c}}^{(b,d)} &= \sum_{i=1}^{N_p} \Big\langle \overline{S}_{\mathfrak{mn}}(\overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}}+\overline{S}_{\mathfrak{m}}\Theta_{\mathfrak{nm}})C_{\mathfrak{mn}}\,\theta^{(b)}[dp_{\mathfrak{m}}][dp_{\mathfrak{n}}]\omega^{\mathfrak{m}i,\mathfrak{n}i}\Delta^{(\mathfrak{mn})}F_{\text{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle \\
&= \sum_{i=1}^{N_p} \Big\langle \overline{S}_{\mathfrak{mn}}(\mathbb{1}-S_{\mathfrak{n}}\Theta_{\mathfrak{mn}}-S_{\mathfrak{m}}\Theta_{\mathfrak{nm}})C_{\mathfrak{mn}}\,\theta^{(b)}[\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\,\omega^{\mathfrak{m}i,\mathfrak{n}i}\Delta^{(\mathfrak{mn})}F_{\text{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle\,.
\end{aligned}
\tag{4.41}
$$

Making use of the fact that the action of the collinear operator $C_{\mathfrak{mn}}$ on the function $F_{\text{LM}}(\mathfrak{m},\mathfrak{n})$ is symmetric in $\mathfrak{m}$ and $\mathfrak{n}$, we can exchange $\mathfrak{m}\leftrightarrow\mathfrak{n}$ in the term with $\Theta_{\mathfrak{nm}}$ in eq. (4.41). We obtain

$$
\Sigma_{\text{RR,1c}}^{(b,d)} = \sum_{i=1}^{N_p} \Big\langle \overline{S}_{\mathfrak{mn}}(\mathbb{1}-2S_{\mathfrak{n}}\Theta_{\mathfrak{mn}})C_{\mathfrak{mn}}\,\theta^{(b)}[\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\,\omega^{\mathfrak{m}i,\mathfrak{n}i}\Delta^{(\mathfrak{mn})}F_{\text{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle\,. \tag{4.42}
$$

The action of the collinear operator $C_{\mathfrak{mn}}$ on the phase space of two unresolved partons leads to a non-trivial result. To derive it, we consider the specific phase-space parametrization described in appendix E and find

$$
\begin{aligned}
&C_{\mathfrak{mn}}[\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}][\mathrm{d}\Omega_{\mathfrak{n}}^{(d-1)}]\theta^{(b)}\omega^{\mathfrak{m}i,\mathfrak{n}i}F_{\text{LM}}(\mathfrak{m},\mathfrak{n}) \\
&= N_\epsilon^{(b,d)}\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}}\eta_{i[\mathfrak{mn}]}^{-\epsilon}(1-\eta_{i[\mathfrak{mn}]})^{\epsilon}[\mathrm{d}\Omega_{[\mathfrak{mn}]}^{(d-1)}]\left[\rho_{\mathfrak{mn}}\frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}}\frac{[\mathrm{d}\Omega_a^{(d-3)}]}{[\Omega^{(d-3)}]}\mathrm{d}\Lambda\right]C_{\mathfrak{mn}}F_{\text{LM}}(\mathfrak{m},\mathfrak{n})\,.
\end{aligned}
\tag{4.43}
$$

Here $[\mathfrak{mn}]$ labels a clustered gluon whose momentum is $p_{[\mathfrak{mn}]}=p_{\mathfrak{m}}+p_{\mathfrak{n}}$ calculated *in the strict collinear limit* and the expression for $C_{\mathfrak{mn}}F_{\text{LM}}(\mathfrak{m},\mathfrak{n})$ is reported in eq. (F.1). From this equation, it follows that $C_{\mathfrak{mn}}F_{\text{LM}}(\mathfrak{m},\mathfrak{n})\sim F_{\text{LM}}([\mathfrak{mn}])$. Since it depends on the kinematics of the clustered parton $[\mathfrak{mn}]$ only, we can integrate over $\mathrm{d}x_4$, $\mathrm{d}\Omega_a^{(d-3)}$ and $\mathrm{d}\Lambda$. We find (see appendix F for details)

$$
\begin{aligned}
\Sigma_{\text{RR,1c}}^{(b,d)} = &-\sum_{i=1}^{N_p}\frac{[\alpha_s]}{2\epsilon}N_\epsilon^{(b,d)}\Big\langle\int_0^{E_{\max}}\frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{2\epsilon-1}}\frac{\mathrm{d}E_{\mathfrak{n}}}{E_{\mathfrak{n}}^{2\epsilon-1}}\int[\mathrm{d}\Omega_{[\mathfrak{mn}]}^{(d-1)}]\,\sigma_{i[\mathfrak{mn}]}^{-\epsilon}\,\omega_{\mathfrak{m}||\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i} \\
&\times\overline{S}_{\mathfrak{mn}}(\mathbb{1}-2S_{\mathfrak{n}}\Theta_{\mathfrak{mn}})\Delta^{([\mathfrak{mn}])}\frac{1}{E_{\mathfrak{m}}E_{\mathfrak{n}}}\Big[P_{gg}(z)F_{\text{LM}}([\mathfrak{mn}]) \\
&+\epsilon\Big[P_{gg}^{\perp}(z)(r_{i,(b)}^{\mu}r_{i,(b)}^{\nu}+g^{\mu\nu})-P_{gg}^{\perp,r}(z)g^{\mu\nu}\Big]F_{\text{LM},\mu\nu}([\mathfrak{mn}])\Big]\Big\rangle\,.
\end{aligned}
\tag{4.44}
$$

In eq. (4.44) we use $z=E_{\mathfrak{m}}/(E_{\mathfrak{m}}+E_{\mathfrak{n}})$ and $P_{gg}^{\perp}$ and $P_{gg}^{\perp,r}$ are splitting functions defined in eqs. (A.24) and (A.25), respectively. Furthermore, we have introduced

$$
\sigma_{ij}=\frac{\eta_{ij}}{1-\eta_{ij}}\,. \tag{4.45}
$$

[18]We note that this exchange of sectors b and d is only possible at the level of integrated subtraction terms, and is not possible for the fully-regulated term $\Sigma_{N+2}^{\text{fin}}$.

The four-vector $r_{i,(b)}$ describes spin correlations that arise in the collinear limit, see appendix E.2 for further details. In particular, we note that $r_{i,(b)}$ is *partition-dependent* as indicated by the subscript i (cf. eq. (E.39)).

Following the discussion in ref. [1], it is convenient to split eq. (4.44) into two terms

$$\Sigma_{\mathrm{RR,1c}}^{(b,d)} = \Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa}} + \Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sc}}\,, \tag{4.46}$$

where the first term on the right-hand side is *spin-averaged*, while the second is *spin-correlated*. The spin-averaged contribution depends on the spin-averaged splitting function P_{gg}. It provides the most divergent part of $\Sigma_{\mathrm{RR,1c}}^{(b,d)}$, with its Laurent expansion starting at $\mathcal{O}(\epsilon^{-2})$. The spin-correlated contribution $\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sc}}$ refers to all terms in eq. (4.44) that are proportional to $F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])$. Since such terms are multiplied by ϵ, the spin-correlated part is less divergent than the spin-averaged one; its Laurent expansion starts at $\mathcal{O}(\epsilon^{-1})$. For this reason, in the following paragraphs we focus on the spin-averaged contribution $\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa}}$ and relegate a detailed discussion of $\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sc}}$ to appendix F.

Our starting point is the following expression for the spin-averaged contribution

$$\begin{aligned}\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa}} = &-\sum_{i=1}^{N_p}\frac{[\alpha_s]}{2\epsilon}N_\epsilon^{(b,d)}\Bigg\langle\int\limits_0^{E_{\max}}\frac{\mathrm{d}E_\mathfrak{m}}{E_\mathfrak{m}^{2\epsilon-1}}\frac{\mathrm{d}E_\mathfrak{n}}{E_\mathfrak{n}^{2\epsilon-1}}\int[\mathrm{d}\Omega_{[\mathfrak{mn}]}^{(d-1)}]\,\sigma_{i[\mathfrak{mn}]}^{-\epsilon}\,\omega_{\mathfrak{m}||\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i}\\&\times\overline{S}_{\mathfrak{mn}}(\mathbb{1}-2S_\mathfrak{n}\Theta_{\mathfrak{mn}})\Delta^{([\mathfrak{mn}])}\frac{1}{E_\mathfrak{m}E_\mathfrak{n}}P_{gg}(z)F_{\mathrm{LM}}([\mathfrak{mn}])\Bigg\rangle.\end{aligned}\tag{4.47}$$

To rewrite it, it is convenient to “undo” the collinear limit. We find

$$-\frac{[\alpha_s]}{2\epsilon}N_\epsilon^{(b,d)}\frac{1}{E_\mathfrak{m}E_\mathfrak{n}}P_{gg}(z)F_{\mathrm{LM}}([\mathfrak{mn}]) \equiv \frac{N_{\mathfrak{m}||\mathfrak{n}}(\epsilon)}{2}\int[\mathrm{d}\Omega_\mathfrak{n}^{(d-1)}]\,C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\,, \tag{4.48}$$

where

$$N_{\mathfrak{m}||\mathfrak{n}}(\epsilon) = 2^{2\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}\,. \tag{4.49}$$

Note that the integration on the right-hand side of eq. (4.48) is performed over the angular phase space of the unresolved parton $\mathfrak{n}$ only. As a result, $\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa}}$ becomes

$$\begin{aligned}\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa}} = &\frac{N_{\mathfrak{m}||\mathfrak{n}}(\epsilon)}{2}\sum_{i=1}^{N_p}\Bigg\langle\int\limits_0^{E_{\max}}\frac{\mathrm{d}E_\mathfrak{m}}{E_\mathfrak{m}^{2\epsilon-1}}\frac{\mathrm{d}E_\mathfrak{n}}{E_\mathfrak{n}^{2\epsilon-1}}\int[\mathrm{d}\Omega_{[\mathfrak{mn}]}^{(d-1)}][\mathrm{d}\Omega_\mathfrak{n}^{(d-1)}]\,\sigma_{i[\mathfrak{mn}]}^{-\epsilon}\,\omega_{\mathfrak{m}||\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i}\\&\times\overline{S}_{\mathfrak{mn}}(\mathbb{1}-2S_\mathfrak{n}\Theta_{\mathfrak{mn}})\Delta^{([\mathfrak{mn}])}C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Bigg\rangle.\end{aligned}\tag{4.50}$$

Next, we note that the action of $S_\mathfrak{mn}$ on $C_\mathfrak{mn}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})$ is equivalent to the action of a soft operator $S_{[\mathfrak{mn}]}$, which refers to the zero-energy limit of a clustered parton $[\mathfrak{mn}]$. We also note that the joint action of $S_\mathfrak{n}$ and $S_\mathfrak{mn}$ can also be described as $S_\mathfrak{mn}S_\mathfrak{n} \equiv S_\mathfrak{m}S_\mathfrak{n}$, and that the action of $S_{\mathfrak{n}}$ on the clustered parton $[\mathfrak{mn}]$ gives $\mathfrak{m}$. Following these observations, we find

$$\begin{aligned}\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa}} = \frac{N_{\mathfrak{m}||\mathfrak{n}}(\epsilon)}{2}\sum_{i=1}^{N_p}\Bigg[\Bigg\langle \int_0^{E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{2\epsilon-1}}\frac{\mathrm{d}E_{\mathfrak{n}}}{E_{\mathfrak{n}}^{2\epsilon-1}} \int [\mathrm{d}\Omega_{[\mathfrak{mn}]}^{(d-1)}][\mathrm{d}\Omega_{\mathfrak{n}}^{(d-1)}]\,\sigma_{i[\mathfrak{mn}]}^{-\epsilon}\,\omega_{\mathfrak{m}||\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i} \\ \times \overline{S}_{[\mathfrak{mn}]}\Delta^{([\mathfrak{mn}])}C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Bigg\rangle \\ - \Big\langle 2\Theta_{\mathfrak{mn}}\overline{S}_{\mathfrak{m}}S_{\mathfrak{n}}\,\sigma_{i\mathfrak{m}}^{-\epsilon}\Delta^{(\mathfrak{m})}\omega_{\mathfrak{m}||\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i}C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle\Bigg].\end{aligned} \tag{4.51}$$

We focus on the first term on the right-hand side in eq. (4.51). Thanks to the constraints on the energies of $\mathfrak{m}$ and $\mathfrak{n}$, the energy of the clustered parton $E_{[\mathfrak{mn}]}$ may exceed $E_{\max}$ and go all the way up to $2E_{\max}$. The two regions for the energy of the clustered particle, namely $E_{[\mathfrak{mn}]} \in [0, E_{\max}]$ and $E_{[\mathfrak{mn}]} \in [E_{\max}, 2E_{\max}]$, are very different: the first one is physical whereas the second one is not. By this we mean that $F_{\mathrm{LM}}([\mathfrak{mn}]) = 0$ for $E_{[\mathfrak{mn}]} > E_{\max}$, since $E_{\max}$ is chosen to exceed the maximal energy that a parton can have in a physical process. On the other hand, this unphysical region gives a non-zero contribution in the soft limit because the parton $[\mathfrak{mn}]$ does not appear in the matrix element.[19] Following this discussion, we write $\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa}}$ as the sum of two terms

$$\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa}} = \Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa},I} + \Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa},II}. \tag{4.52}$$

The first term $\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa},I}$ includes the contribution where the energy of the clustered particle $[\mathfrak{mn}]$ does not exceed $E_{\max}$ as well as the last term on the right-hand side of eq. (4.51), while $\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa},II}$ accommodates the contribution with the energy of the clustered particle exceeding $E_{\max}$.

The term $\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa},I}$ can be written in the following way

$$\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa},I} = \frac{N_{\mathfrak{m}||\mathfrak{n}}(\epsilon)}{2}\sum_{i=1}^{N_p}\Big\langle \overline{S}_{\mathfrak{m}}(\mathbb{1} - 2\Theta_{\mathfrak{mn}}S_{\mathfrak{n}})\sigma_{i\mathfrak{m}}^{-\epsilon}\Delta^{(\mathfrak{m})}\omega_{\mathfrak{m}||\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i}C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle, \tag{4.53}$$

where in the first ($\Theta_{\mathfrak{mn}}$-independent) term we renamed $[\mathfrak{mn}] \to [\mathfrak{m}]$. The above expression contains divergences which arise when gluon $\mathfrak{m}$ becomes collinear to parton i. We extract these divergences by introducing collinear operators and write

$$\Sigma_{\mathrm{RR,1c}}^{(b,d),\mathrm{sa},I} = \Sigma_{N+1}^{\mathrm{fin},(2)} + \Sigma_{N}^{(6)} + \Sigma_{N+1}^{(3)}, \tag{4.54}$$

where

$$\begin{aligned}\Sigma_{N+1}^{\mathrm{fin},(2)} &= \sum_{i=1}^{N_p}\frac{1}{2}\Big\langle \mathcal{O}_{\mathrm{NLO}}^{(i)}\omega_{\mathfrak{m}||\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i}(\mathbb{1} - 2\Theta_{\mathfrak{mn}}S_{\mathfrak{n}})\big[N_{\mathfrak{m}||\mathfrak{n}}(\epsilon)\sigma_{i\mathfrak{m}}^{-\epsilon} - 1\big]\Delta^{(\mathfrak{m})}C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle, \\ \Sigma_{N}^{(6)} &= \sum_{i=1}^{N_p}\frac{N_{\mathfrak{m}||\mathfrak{n}}(\epsilon)}{2}\Big\langle \overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\sigma_{i\mathfrak{m}}^{-\epsilon}(\mathbb{1} - 2\Theta_{\mathfrak{mn}}S_{\mathfrak{n}})\Delta^{(\mathfrak{m})}C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\Big\rangle, \\ \Sigma_{N+1}^{(3)} &= \frac{1}{2}\big\langle \mathcal{O}_{\mathrm{NLO}}(\mathbb{1} - 2\Theta_{\mathfrak{mn}}S_{\mathfrak{n}})\Delta^{(\mathfrak{m})}C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\big\rangle.\end{aligned} \tag{4.55}$$

[19] We note that this term combines with soft subtractions in other sectors such that the final result is not affected by unphysical contributions. We refer the reader to ref. [1] for a full discussion of this issue.

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The first term in the above formula is finite in the limit $\epsilon \to 0$, the second term is double-unresolved, and the last one is single-unresolved. We remind the reader that $\mathcal{O}^{(i)}_{\rm NLO} = \overline{S}_{\mathfrak{m}}\overline{C}_{i\mathfrak{m}}$ and $\mathcal{O}_{\rm NLO} = \sum_i \mathcal{O}^{(i)}_{\rm NLO}\omega^{\mathfrak{m}i}$. We also note that in $\Sigma^{(3)}_{N+1}$ we replaced the NNLO partition functions $\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}}$ with NLO partion functions $\omega^{\mathfrak{m}i}$, cf. eq. (4.35).

We continue with the discussion of $\Sigma^{(b,d),{\rm sa},II}_{\rm RR,1c}$. It can be obtained from eq. (4.51) upon neglecting the last term on the right-hand side and restricting the integration over energies to the region $E_{[\mathfrak{mn}]} > E_{\rm max}$. We find

$$\begin{aligned}\Sigma^{(b,d),{\rm sa},II}_{\rm RR,1c} &= \frac{N_{\mathfrak{m}||\mathfrak{n}}(\epsilon)}{2}\sum_{i=1}^{N_p}\Bigg\langle \int\limits_0^{E_{\rm max}} \frac{{\rm d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{2\epsilon-1}}\frac{{\rm d}E_{\mathfrak{n}}}{E_{\mathfrak{n}}^{2\epsilon-1}}\int [{\rm d}\Omega^{(d-1)}_{[\mathfrak{mn}]}][{\rm d}\Omega^{(d-1)}_{\mathfrak{n}}]\,\sigma^{-\epsilon}_{i[\mathfrak{mn}]}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}} \\ &\quad\times\Theta(E_{\mathfrak{m}}+E_{\mathfrak{n}}-E_{\rm max})\overline{S}_{[\mathfrak{mn}]}\Delta^{([\mathfrak{mn}])}C_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Bigg\rangle .\end{aligned} \tag{4.56}$$

We can also replace $\overline{S}_{[\mathfrak{mn}]}$ with $-S_{[\mathfrak{mn}]}$ in the above equation as $F_{\rm LM}([\mathfrak{mn}])$ has zero support if the energy of the clustered parton exceeds $E_{\rm max}$. Finally, changing the integration variables to $E_{[\mathfrak{mn}]} = E_{\mathfrak{m}} + E_{\mathfrak{n}}$ and $z = E_{\mathfrak{m}}/(E_{\mathfrak{m}} + E_{\mathfrak{n}})$, computing the collinear $[\mathfrak{mn}]||\mathfrak{n}$ limit of $F_{\rm LM}$ and integrating over the angular phase space of the gluon $\mathfrak{n}$, we obtain

$$\begin{aligned}\Sigma^{(b,d),{\rm sa},II}_{\rm RR,1c} &= \frac{N^{(b,d)}_{\epsilon}}{2}\sum_{i=1}^{N_p}\Bigg\langle \int\limits_{E_{\rm max}}^{2E_{\rm max}} \frac{{\rm d}E_{[\mathfrak{mn}]}}{E_{[\mathfrak{mn}]}^{4\epsilon-1}}\int\limits_{1-\frac{E_{\rm max}}{E_{[\mathfrak{mn}]}}}^{\frac{E_{\rm max}}{E_{[\mathfrak{mn}]}}} {\rm d}z\,[z(1-z)]^{-2\epsilon}P_{gg}(z) \\ &\quad\times\int [{\rm d}\Omega^{(d-1)}_{[\mathfrak{mn}]}]\,\sigma^{-\epsilon}_{i[\mathfrak{mn}]}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}}S_{[\mathfrak{mn}]}F_{\rm LM}([\mathfrak{mn}])\Bigg\rangle .\end{aligned} \tag{4.57}$$

Using the standard result for the remaining soft limit $S_{[\mathfrak{mn}]}F_{\rm LM}([\mathfrak{mn}])$ in eq. (4.57), we find

$$\begin{aligned}\Sigma^{(b,d),{\rm sa},II}_{\rm RR,1c} &= -\frac{[\alpha_s]^2\,\delta^{\rm sa}_g(\epsilon)(E_{\rm max}/\mu)^{-2\epsilon}}{\epsilon} \\ &\quad\times\sum_{i=1}^{N_p}\sum_{(kl)}^{N_p}\int\frac{[{\rm d}\Omega^{(d-1)}_{[\mathfrak{mn}]}]}{[\Omega^{(d-2)}]}\Bigg\langle \sigma^{-\epsilon}_{i[\mathfrak{mn}]}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}}\frac{\rho_{kl}}{\rho_{k[\mathfrak{mn}]}\,\rho_{l[\mathfrak{mn}]}}\,(\boldsymbol{T}_k\cdot\boldsymbol{T}_l)\,F_{\rm LM}\Bigg\rangle ,\end{aligned} \tag{4.58}$$

where

$$\delta^{\rm sa}_g(\epsilon) = \frac{N^{(b,d)}_{\epsilon}E^{4\epsilon}_{\rm max}}{2}\int\limits_{E_{\rm max}}^{2E_{\rm max}}\frac{{\rm d}E_{[\mathfrak{mn}]}}{E^{1+4\epsilon}_{[\mathfrak{mn}]}}\int\limits_{1-\frac{E_{\rm max}}{E_{[\mathfrak{mn}]}}}^{\frac{E_{\rm max}}{E_{[\mathfrak{mn}]}}}{\rm d}z\,[z(1-z)]^{-2\epsilon}P_{gg}(z)\,. \tag{4.59}$$

The integration over the angle of the clustered gluon $[\mathfrak{mn}]$ in eq. (4.58) is described in appendix G. The result reads

$$\Sigma^{(b,d),{\rm sa},II}_{\rm RR,1c} = \Sigma^{\rm sa}_N + \Sigma^{\rm sa,fin}_N\,, \tag{4.60}$$

where $\Sigma^{\rm sa}_N$ is given by

$$\begin{aligned}\Sigma^{\rm sa}_N &= 2[\alpha_s]^2\delta^{\rm sa}_g(\epsilon)\left(\frac{E_{\rm max}}{\mu}\right)^{-2\epsilon} \\ &\quad\times\left[-\langle I_{\rm S}(\epsilon)\cdot F_{\rm LM}\rangle + \frac{(2E_{\rm max}/\mu)^{-2\epsilon}}{2\epsilon^2}N_c(\epsilon)\sum_{i=1}^{N}\boldsymbol{T}_i^2\,\langle F_{\rm LM}\rangle\right] ,\end{aligned} \tag{4.61}$$

with $N_c(\epsilon)$ reported in eq. (A.5). The quantity $\Sigma_N^{\rm sa,fin}$ is finite and reads

$$\Sigma_N^{\rm sa,fin} = [\alpha_s]^2\, 2^{-2\epsilon}\, \delta_g^{\rm sa}(\epsilon) \left(\frac{E_{\rm max}}{\mu}\right)^{-4\epsilon} \sum_{i=1}^{N_p} \langle \mathcal{W}_i^{\mathfrak{m}\|\mathfrak{n},\rm fin} \cdot F_{\rm LM}\rangle\,, \tag{4.62}$$

where $\mathcal{W}_a^{\mathfrak{m}\|\mathfrak{n},\rm fin}$ is computed in appendix G with the result given in eq. (G.10).

The final contribution to consider is the spin-correlated term $\Sigma_{\rm RR,1c}^{(b,d),\rm sc}$ in eq. (4.44). In appendix F, we show (see eq. (F.44)) that among the contributions that the spin-correlated term of eq. (4.44) can produce, there are two that are *identical* to $\Sigma_N^{\rm sa}$ and $\Sigma_N^{\rm sa,fin}$, provided we substitute $\delta_g^{\rm sa} \mapsto \delta_g^{\perp,r}$, where $\delta_g^{\perp,r}$ is defined in eq. (A.30). We call these contributions $\Sigma_N^{\rm sc}$ and $\Sigma_N^{\rm sc,fin}$. Combining them with $\Sigma_N^{\rm sa}$ and $\Sigma_N^{\rm sa,fin}$, respectively, we define the following quantities

$$\begin{aligned}\Sigma_N^{(7)} = \Sigma_N^{\rm sa} + \Sigma_N^{\rm sc} = 2[\alpha_s]^2 \delta_g(\epsilon) \left(\frac{E_{\rm max}}{\mu}\right)^{-2\epsilon} \\ \times \left[-\langle I_{\rm S}(\epsilon)\cdot F_{\rm LM}\rangle + \frac{(2E_{\rm max}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i=1}^{N} \boldsymbol{T}_i^2 \,\langle F_{\rm LM}\rangle \right],\end{aligned} \tag{4.63}$$

and

$$\Sigma_N^{{\rm fin},(4)} = \Sigma_N^{\rm sa,fin} + \Sigma_N^{\rm sc,fin} = [\alpha_s]^2\, 2^{-2\epsilon}\, \delta_g(\epsilon) \left(\frac{E_{\rm max}}{\mu}\right)^{-4\epsilon} \sum_{i=1}^{N_p} \langle \mathcal{W}_i^{\mathfrak{m}\|\mathfrak{n},\rm fin} \cdot F_{\rm LM}\rangle\,, \tag{4.64}$$

with $\delta_g(\epsilon) = \delta_g^{\rm sa}(\epsilon) + \delta_g^{\perp,r}(\epsilon)$, see eq. (A.30). We denote the remaining spin-correlated terms as

$$\Sigma_N^{(8)} = \Sigma_{\rm RR,1c}^{(b,d),{\rm sc},I,1}\,, \tag{4.65}$$

and

$$\Sigma_N^{{\rm fin},(5)} = [\alpha_s]^2\, \delta_g^{\perp} \left(\frac{E_{\rm max}}{\mu}\right)^{-4\epsilon} \sum_{i=1}^{N_p} \langle \mathcal{W}_r^{(i)} \cdot F_{\rm LM}\rangle\,, \tag{4.66}$$

where $\Sigma_{\rm RR,1c}^{(b,d),{\rm sc},I,1}$ is given in eq. (F.38).

To recapitulate, we have succeeded in writing $\Sigma_{\rm RR,1c}$ as a sum of contributions to the single- and double-unresolved terms Σ_{N+1} and Σ_N. We can combine them with the corresponding contributions of $\Sigma_{\rm RR,2c}$ as well as those of eq. (4.13), and explore the cancellation of the ϵ-poles in Σ_{N+1} and Σ_N. We study such cancellations in section 5 but before diving into this discussion we need to rearrange double-unresolved terms to make the investigation of the pole cancellation easier. We discuss a suitable rearrangement in the next subsections.

4.2 Rearranging double-unresolved terms

We now turn our attention to the question of how the double-unresolved terms can be rearranged. Once this is accomplished, the preparatory work will be complete and the cancellation of singularities between the different contributions can be explored.

We have seen that the contributions with two unresolved partons can be written as a sum of eight divergent and five finite terms, i.e.

$$\Sigma_N = \sum_{i=1}^{8} \Sigma_N^{(i)} + \sum_{i=1}^{5} \Sigma_N^{{\rm fin},(i)}\,. \tag{4.67}$$

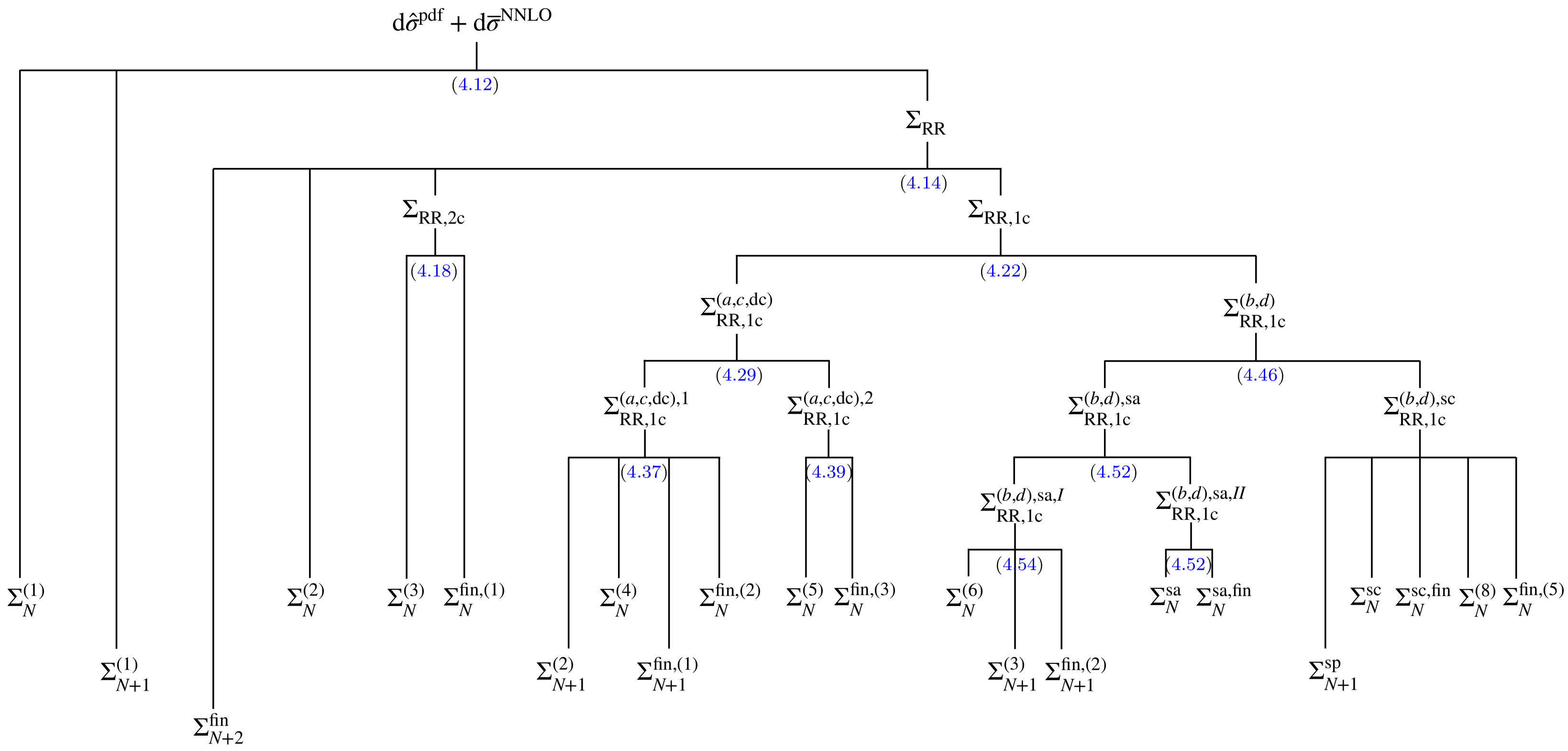

Figure 1. A chart illustrating rearrangement of the subtraction terms according to final-state multiplicities. Clickable references to relevant equations are provided. Terms with same parton multiplicities are shown at the same heights.

The contributions are in eqs. ((4.13), (4.16), (4.19), (4.38), (4.40), (4.55), (4.63), (4.64), (4.65), (4.66)). Three of the divergent contributions, namely $\Sigma_N^{(3,4,5)}$, contain various collinear limits and we find that combining and rearranging them is helpful for understanding the cancellation of poles.

To make the required manipulations more transparent, in $\Sigma_N^{(4)}$ we write $\overline{C}_{j\mathfrak{n}}$ as $(\mathbb{1} - C_{j\mathfrak{n}})$, use the fact that $\sum\limits_{j=1} \omega^{j\mathfrak{n}} = 1$ and separate the $i \neq j$ and $i = j$ sums. We find

$$
\begin{aligned}
&\Sigma_N^{(3)} + \Sigma_N^{(4)} + \Sigma_N^{(5)} \\
&= -\sum_{(ij)}^{N_p} \Big\langle \Big(\overline{S}_\mathfrak{n} C_{j\mathfrak{n}} C_{i\mathfrak{m}} - (\overline{S}_\mathfrak{n}\Theta_{\mathfrak{mn}} + \overline{S}_\mathfrak{m}\Theta_{\mathfrak{nm}}) C_{j\mathfrak{n}} C_{i\mathfrak{m}}\Big)\, \Delta^{(\mathfrak{mn})} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big\rangle \\
&\quad + \sum_{i=1}^{N_p} \langle S_\mathfrak{n}\overline{S}_\mathfrak{m} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle - \sum_{(ij)}^{N_p} \langle S_\mathfrak{n}\overline{S}_\mathfrak{m} C_{j\mathfrak{n}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle \\
&\quad - \sum_{i=1}^{N_p} \Big\langle \Big(S_\mathfrak{n}\overline{S}_\mathfrak{m} C_{i\mathfrak{n}} C_{i\mathfrak{m}}\Theta_{\mathfrak{nm}} - (\eta_{i\mathfrak{n}}/2)^{-\epsilon}\mathcal{S}(\mathfrak{m},\mathfrak{n}) C_{i\mathfrak{n}} C_{i\mathfrak{m}}\Big)\, \Delta^{(\mathfrak{mn})} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big\rangle\,.
\end{aligned}
\tag{4.68}
$$

Combining terms with $i \neq j$ sums in the above equation, we obtain

$$
\begin{aligned}
&\Sigma_N^{(3)} + \Sigma_N^{(4)} + \Sigma_N^{(5)} = \\
&= \sum_{i=1}^{N_p} \langle S_\mathfrak{n}\overline{S}_\mathfrak{m} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle + \sum_{(ij)}^{N_p} \langle \overline{S}_\mathfrak{n}\overline{S}_\mathfrak{m} C_{j\mathfrak{n}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle \\
&\quad - \sum_{i=1}^{N_p} \Big\langle \Big(S_\mathfrak{n}\overline{S}_\mathfrak{m} C_{i\mathfrak{n}} C_{i\mathfrak{m}} - (\eta_{i\mathfrak{n}}/2)^{-\epsilon}\mathcal{S}(\mathfrak{m},\mathfrak{n})\, C_{i\mathfrak{n}} C_{i\mathfrak{m}}\Big)\, \Delta^{(\mathfrak{mn})} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big\rangle\,.
\end{aligned}
\tag{4.69}
$$

We can further simplify the above equation if we rewrite the $i \neq j$ sum as follows

$$
\begin{aligned}
&\sum_{(ij)}^{N_p} \langle \overline{S}_\mathfrak{n}\overline{S}_\mathfrak{m} C_{j\mathfrak{n}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle \\
&= \frac{1}{2}\sum_{i,j=1}^{N_p} \langle \overline{S}_\mathfrak{n}\overline{S}_\mathfrak{m} C_{j\mathfrak{n}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle - \frac{1}{2}\sum_{i=1}^{N_p} \langle \overline{S}_\mathfrak{n}\overline{S}_\mathfrak{m} C_{i\mathfrak{n}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle\,.
\end{aligned}
\tag{4.70}
$$

We now take the last term on the right-hand side of the above equation and combine it with the next-to-last term in eq. (4.69). We find

$$
-\frac{1}{2}\sum_{i=1}^{N_p} \langle (\overline{S}_\mathfrak{n}\overline{S}_\mathfrak{m} + 2S_\mathfrak{n}\overline{S}_\mathfrak{m}\Theta_{\mathfrak{nm}})\, C_{i\mathfrak{n}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{mn})} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle\,.
\tag{4.71}
$$

We split the second term under the sum sign in eq. (4.71) into two identical ones, and change

$\mathfrak{m} \leftrightarrow \mathfrak{n}$ in one of them. We obtain

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{N_p}\left\langle\left(\overline{S}_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}+S_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}\Theta_{\mathfrak{nm}}+S_{\mathfrak{m}}\overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}}\right)C_{i\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle \\
&+\frac{1}{2}\sum_{i=1}^{N_p}\left\langle S_{\mathfrak{m}}\overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}}\left[C_{i\mathfrak{n}},C_{i\mathfrak{m}}\right]\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle \\
&=-\frac{1}{2}\sum_{i=1}^{N_p}\left\langle\left(\mathcal{S}(\mathfrak{m},\mathfrak{n})\,C_{i\mathfrak{n}}C_{i\mathfrak{m}}-S_{\mathfrak{m}}\overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}}\left[C_{i\mathfrak{n}},C_{i\mathfrak{m}}\right]\right)\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle .
\end{aligned}
\tag{4.72}
$$

Putting everything together, we find

$$
\begin{aligned}
&\Sigma_N^{(3)}+\Sigma_N^{(4)}+\Sigma_N^{(5)}= \\
&=\sum_{i=1}^{N_p}\left\langle S_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle+\frac{1}{2}\sum_{i,j=1}^{N_p}\left\langle\overline{S}_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}C_{j\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle \\
&\quad+\frac{1}{2}\sum_{i=1}^{N_p}\left\langle\left[2(\eta_{i\mathfrak{n}}/2)^{-\epsilon}-1\right]\mathcal{S}(\mathfrak{m},\mathfrak{n})\,C_{i\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle \\
&\quad+\frac{1}{2}\sum_{i=1}^{N_p}\left\langle S_{\mathfrak{m}}\overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}}\left[C_{i\mathfrak{n}},C_{i\mathfrak{m}}\right]\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle .
\end{aligned}
\tag{4.73}
$$

We can now combine this result with the remaining double-unresolved contributions. We find

$$
\begin{aligned}
\Sigma_N=&\left\langle F_{\mathrm{VV}}\right\rangle+\left\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle+\left\langle S_{\mathfrak{m}}F_{\mathrm{RV}}(\mathfrak{m})\right\rangle \\
&+\sum_{i=1}^{N_p}\left\langle\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{m})}\left[F_{\mathrm{RV}}(\mathfrak{m})+S_{\mathfrak{n}}\Theta_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right]\right\rangle \\
&+\sum_{i=1}^{N_p}\left\langle S_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{nm}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle+\frac{1}{2}\sum_{i,j=1}^{N_p}\left\langle\overline{S}_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}C_{j\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle \\
&+\frac{1}{2}\sum_{i=1}^{N_p}\left\langle\left[2(\eta_{i\mathfrak{n}}/2)^{-\epsilon}-1\right]\mathcal{S}(\mathfrak{m},\mathfrak{n})\,C_{i\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle \\
&+\frac{1}{2}\sum_{i=1}^{N_p}\left\langle S_{\mathfrak{m}}\overline{S}_{\mathfrak{n}}\Theta_{\mathfrak{mn}}\left[C_{i\mathfrak{n}},C_{i\mathfrak{m}}\right]\Delta^{(\mathfrak{mn})}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle \\
&-2[\alpha_s]^2\,\delta_g(\epsilon)\left(\frac{E_{\max}}{\mu}\right)^{-2\epsilon}\left[\left\langle I_{\mathrm{S}}(\epsilon)\cdot F_{\mathrm{LM}}\right\rangle-\frac{(2E_{\max}/\mu)^{-2\epsilon}}{2\epsilon^2}N_c(\epsilon)\sum_{i=1}^{N_p}\boldsymbol{T}_i^2\left\langle F_{\mathrm{LM}}\right\rangle\right] \\
&+\sum_{i=1}^{N_p}\frac{N_{\mathfrak{m}||\mathfrak{n}}(\epsilon)}{2}\left\langle\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\,\sigma_{i\mathfrak{m}}^{-\epsilon}\,(\mathbb{1}-2\Theta_{\mathfrak{mn}}S_{\mathfrak{n}})\Delta^{(\mathfrak{m})}C_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle \\
&+\Sigma_N^{(2)}+\Sigma_N^{(8)}+\sum_{i=1}^{5}\Sigma_N^{\mathrm{fin},(i)} .
\end{aligned}
\tag{4.74}
$$

It is clear from the above formula that Σ_N contains a large number of terms of different physical origin that exhibit infrared and collinear singularities, which will cancel when

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combined with the PDFs renormalization contributions. To simplify the discussion of how this happens, we will identify groups of terms which exhibit shared features. These features include quartic, triple and quadratic correlations of color-charge operators, which originate from exchanges of soft real and virtual gluons, as well as double- and single-boosted kinematics that are generated by hard-collinear initial-state emissions. We will focus on these different categories in turn, since the cancellation of ϵ-poles has to occur independently for each of them.

In subsection 4.3 we describe some manipulations of the virtual and soft contributions to eq. (4.74), which set the stage for the discussion of the cancellation of poles in color-correlated contributions that can be found in subsections 5.2 and 5.3. With color-correlated infrared singularities out of the way, we are left with terms that are proportional to squares of color charges of the resolved partons, which include both boosted and unboosted contributions. Such terms primarily come from collinear emissions. We discuss such contributions and the cancellation of the corresponding singularities in subsections 5.4 and 5.5.

4.3 Simplifying virtual and soft corrections

In this subsection we focus on the color-correlated contributions to the fully-unresolved quantity Σ_N. To this end, we will examine those terms in eq. (4.74) that contain soft limits and/or loop amplitudes. Similar to what will be done in section 5.1, we will write the results in terms of generalizations of the operators $I_{\rm S}$, $I_{\rm V}$ and $I_{\rm C}$, with an eye on combining these into manifestly-finite $I_{\rm T}$ structures. Furthermore, we will observe the appearance of terms involving triple correlators of color charges, which we will discuss separately in section 5.2.

We begin by considering the double-virtual contribution $\langle F_{\rm VV}\rangle$ to eq. (4.74). We write the loop expansion of the amplitude of the $1_a + 2_b \to X + N\, g$ process to $\mathcal{O}(\alpha_s^2)$ with respect to the LO as

$$|\mathcal{M}\rangle_c = |\mathcal{M}_0\rangle_c + \left[\frac{\alpha_s(\mu)}{2\pi}\right] |\mathcal{M}_1\rangle_c + \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 |\mathcal{M}_2\rangle_c + \mathcal{O}(\alpha_s^3)\,. \tag{4.75}$$

The double-virtual contribution to the cross section is obtained by squaring the amplitude and retaining the $\mathcal{O}(\alpha_s^2)$ terms. The result reads[20]

$$\langle\mathcal{M}|\mathcal{M}\rangle_{\alpha_s^2} = \langle\mathcal{M}_0|\mathcal{M}_2\rangle + \langle\mathcal{M}_2|\mathcal{M}_0\rangle + \langle\mathcal{M}_1|\mathcal{M}_1\rangle\,. \tag{4.76}$$

Following ref. [71], we extract the infrared poles of $|\mathcal{M}_1\rangle$ and $|\mathcal{M}_2\rangle$ and write them as

$$\begin{aligned}|\mathcal{M}_1\rangle &= I_1(\epsilon)\,|\mathcal{M}_0\rangle + |\mathcal{M}_1^{\rm fin}\rangle\,,\\ |\mathcal{M}_2\rangle &= I_1(\epsilon)\,|\mathcal{M}_1\rangle + I_2(\epsilon)\,|\mathcal{M}_0\rangle + |\mathcal{M}_2^{\rm fin}\rangle\,,\end{aligned} \tag{4.77}$$

where $|\mathcal{M}_1^{\rm fin}\rangle$ and $|\mathcal{M}_2^{\rm fin}\rangle$ are infrared-finite. The operator $I_1(\epsilon)$ was introduced in the context of the NLO calculation and is given in eq. (3.29). The operator $I_2(\epsilon)$ reads

$$I_2(\epsilon) = -\frac{1}{2}I_1(\epsilon)\left(I_1(\epsilon) + \frac{2\beta_0}{\epsilon}\right) + c_\epsilon\left(\frac{\beta_0}{\epsilon} + K\right) I_1(2\epsilon) + \mathcal{H}_2\,, \tag{4.78}$$

[20]We drop the subscript "c" in the notation for the color vector of a matrix element.

with[21]

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f \,, \qquad c_\epsilon = \frac{e^{-\epsilon\gamma_E}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \,. \tag{4.79}$$

The operator $\mathcal{H}_2$ contains $\mathcal{O}(\epsilon^{-1})$ poles only. We split this function into a term containing triple color correlations and a color-diagonal term

$$\mathcal{H}_2(\epsilon) = \mathcal{H}_{2,\text{tc}}(\epsilon) + \mathcal{H}_{2,\text{cd}}(\epsilon) \,. \tag{4.80}$$

The two quantities $\mathcal{H}_{2,\text{tc}}$ and $\mathcal{H}_{2,\text{cd}}$ were explicitly computed in refs. [72, 73]. The triple color-correlated term $\mathcal{H}_{2,\text{tc}}$ is given in eq. (5.27). The color-diagonal piece reads

$$\mathcal{H}_{2,\text{cd}}(\epsilon) = \frac{1}{2\epsilon}\sum_{i=1}^{N_p} H_{f_i} \,, \tag{4.81}$$

where f_i denotes the flavor of parton i. Explicitly one has

$$H_g = C_A^2\left(\frac{5}{12} + \frac{11}{144}\pi^2 + \frac{\zeta_3}{2}\right) + C_A n_f\left(-\frac{29}{27} - \frac{\pi^2}{72}\right) + \frac{C_F n_f}{2} + \frac{5}{27}n_f^2 \,, \tag{4.82}$$

and

$$H_q = C_F^2\left(\frac{\pi^2}{2} - 6\zeta_3 - \frac{3}{8}\right) + C_A C_F\left(\frac{245}{216} - \frac{23}{48}\pi^2 + \frac{13}{2}\zeta_3\right) + C_F n_f\left(\frac{\pi^2}{24} - \frac{25}{108}\right) . \tag{4.83}$$

The matrix element squared that appears in the double-virtual term F_{VV} is then

$$\begin{aligned}\langle\mathcal{M}|\mathcal{M}\rangle_{\alpha_s^2} = &\left\langle\mathcal{M}_0\middle|\frac{1}{2}I_1^2(\epsilon) + \frac{1}{2}\left(I_1^\dagger(\epsilon)\right)^2 + I_1^\dagger(\epsilon)I_1(\epsilon) + \left(\mathcal{H}_2 + \mathcal{H}_2^\dagger\right)\middle|\mathcal{M}_0\right\rangle \\ &+ \left\langle\mathcal{M}_0\middle| -\frac{\beta_0}{\epsilon}\left(I_1(\epsilon) + I_1^\dagger(\epsilon)\right) + c_\epsilon\left(\frac{\beta_0}{\epsilon} + K\right)\left(I_1(2\epsilon) + I_1^\dagger(2\epsilon)\right)\middle|\mathcal{M}_0\right\rangle \\ &+ 2\text{Re}\left[\langle\mathcal{M}_0|I_1(\epsilon) + I_1^\dagger(\epsilon)|\mathcal{M}_1^{\text{fin}}\rangle\right] + 2\text{Re}\left[\langle\mathcal{M}_0|\mathcal{M}_2^{\text{fin}}\rangle\right] + \langle\mathcal{M}_1^{\text{fin}}|\mathcal{M}_1^{\text{fin}}\rangle \,.\end{aligned} \tag{4.84}$$

The one-loop operators I_1 in the second and third lines appear as the sum of I_1 and $I_1^\dagger$; for this reason, they can immediately be written using the function I_{V} defined in eq. (3.31). However, this does not happen automatically for entries in the first line in eq. (4.84). To force the appearance of I_{V}, we write

$$\frac{1}{2}\overline{I}_1^2(\epsilon) + \frac{1}{2}\left(\overline{I}_1^\dagger(\epsilon)\right)^2 + \overline{I}_1^\dagger(\epsilon)\overline{I}_1(\epsilon) = \frac{1}{2}I_{\text{V}}^2(\epsilon) - \frac{1}{2}\left[\overline{I}_1, \overline{I}_1^\dagger\right] \,. \tag{4.85}$$

As we will see, in the general case the commutator in the above equation contains triple color-correlated poles. We will study them in detail in section 5.2. For now, we use eq. (4.85) and write the double-virtual contribution as follows

$$\begin{aligned}\langle F_{\text{VV}}\rangle = &[\alpha_s]^2\left\langle\left[\frac{1}{2}I_{\text{V}}^2(\epsilon) - \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\text{E}}}}\left(\frac{\beta_0}{\epsilon}I_{\text{V}}(\epsilon) - \left(\frac{\beta_0}{\epsilon} + K\right)I_{\text{V}}(2\epsilon)\right)\right]\cdot F_{\text{LM}}\right\rangle \\ &+ [\alpha_s]^2\left\langle\left[-\frac{1}{2}\left[\overline{I}_1(\epsilon), \overline{I}_1^\dagger(\epsilon)\right] + \mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger + \mathcal{H}_{2,\text{cd}} + \mathcal{H}_{2,\text{cd}}^\dagger\right]\cdot F_{\text{LM}}\right\rangle \\ &+ [\alpha_s]\langle I_{\text{V}}(\epsilon)\cdot F_{\text{LV}}^{\text{fin}}\rangle + \langle F_{\text{LV}^2}^{\text{fin}}\rangle + \langle F_{\text{VV}}^{\text{fin}}\rangle \,.\end{aligned} \tag{4.86}$$

[21] We remind the reader that in this paper we are accounting for gluonic final states only. For this reason n_f should be set to 0, and β_0 to $11/6\,C_A$.

In eq. (4.86) $F^{\rm fin}_{\rm LV^2}$ and $F^{\rm fin}_{\rm VV}$ contain the finite remainders of the one-loop squared and two-loop amplitudes interfered with the tree level, respectively. Furthermore, we have made use of the fact that $\mathcal{H}_2 \sim \mathcal{O}(\epsilon^{-1})$ to replace the coupling $\alpha_s(\mu)/(2\pi)$ with $[\alpha_s]$ in front of it. This concludes our discussion of the double-virtual contribution, and we will make use of eq. (4.86) in section 5 to discuss the cancellation of poles.

Next, we consider the double-soft term $\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}\rangle$ in eq. (4.74). As was mentioned earlier, it was computed in ref. [70] for an arbitrary opening angle between the hard radiators. We can write the result in terms of a double color-correlated and a quartic color-correlated component

$$
\big\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\big\rangle = \big\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\big\rangle_{T^2} + \big\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\big\rangle_{T^4} . \tag{4.87}
$$

The quartic color-correlated component has a simple (factorized) form

$$
\begin{aligned}
\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^4} = 2g_{s,b}^4 \sum_{(ij),(kl)}^{N_p} \Big\langle \int [{\rm d}p_{\mathfrak{m}}][{\rm d}p_{\mathfrak{n}}]\Theta(E_{\mathfrak{m}}-E_{\mathfrak{n}}) S_{ij}(p_{\mathfrak{m}})S_{kl}(p_{\mathfrak{n}}) \\
\times \{\boldsymbol{T}_i\cdot\boldsymbol{T}_j, \boldsymbol{T}_k\cdot\boldsymbol{T}_l\}\cdot F_{\rm LM}\Big\rangle \\
= [\alpha_s]^2 \frac{1}{2}\Big\langle I_{\rm S}^2(\epsilon)\cdot F_{\rm LM}\Big\rangle .
\end{aligned} \tag{4.88}
$$

In the above, we have introduced the short-hand notation $S_{ij}(p_{\mathfrak{m}})$ for the eikonal function

$$
S_{ij}(p_{\mathfrak{m}}) = \frac{p_i\cdot p_j}{2(p_i\cdot p_{\mathfrak{m}})(p_j\cdot p_{\mathfrak{m}})} . \tag{4.89}
$$

The (double) color-correlated term appears to be significantly more complex [70]. However, upon careful inspection, we find that its *poles* can be written in a reasonably simple manner. We obtain

$$
\begin{aligned}
&\big\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\big\rangle_{T^2} \\
&= g_{s,b}^4 \sum_{i<j}^{N_p} \int [{\rm d}p_{\mathfrak{m}}][{\rm d}p_{\mathfrak{n}}]\Theta(E_{\mathfrak{m}}-E_{\mathfrak{n}})\langle \widetilde{S}_{ij}(p_{\mathfrak{m}},p_{\mathfrak{n}})\ (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\rangle \\
=&[\alpha_s]^2\left[\frac{C_A}{\epsilon^2}c_1(\epsilon) + \frac{\beta_0}{\epsilon}c_2(\epsilon) + \beta_0\, c_3(\epsilon)\right]\langle \widetilde{I}_{\rm S}(2\epsilon)\cdot F_{\rm LM}\rangle + \langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle^{\rm fin}_{T^2} ,
\end{aligned} \tag{4.90}
$$

where $\widetilde{S}_{ij}$ is the double-soft current defined in ref. [74]. We note that the last term in eq. (4.90) is ϵ-finite and can be found in eq. (I.17). Furthermore, the quantities $c_{1,2,3}$ are polynomials in ϵ and are given in eq. (A.8). Additionally, we have introduced

$$
\widetilde{I}_{\rm S}(2\epsilon) = -\frac{(2E_{\rm max}/\mu)^{-4\epsilon}}{(2\epsilon)^2} \sum_{\substack{i,j=1\\ i\neq j}}^{N_p} \eta_{ij}^{-2\epsilon}\widetilde{K}_{ij}(\epsilon)\ (\boldsymbol{T}_i\cdot\boldsymbol{T}_j) , \tag{4.91}
$$

where

$$
\widetilde{K}_{ij}(\epsilon) = \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\,\eta_{ij}^{1+3\epsilon}\,{}_2F_1(1+\epsilon,1+\epsilon,1-\epsilon,1-\eta_{ij}) . \tag{4.92}
$$

We note apparent similarities between $\widetilde{I}_{\rm S}$ and $\widetilde{K}_{ij}$ and $I_{\rm S}$ and K_{ij} defined in eqs. (3.12) and (3.14). In fact, one can use the following property of the hypergeometric functions

$$
{}_2F_1(a,b,c,z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b,c,z)\,, \tag{4.93}
$$

to show that

$$
\widetilde{K}_{ij}(\epsilon) = K_{ij}(2\epsilon)\,\frac{{}_2F_1(-2\epsilon,-2\epsilon;1-\epsilon,1-\eta_{ij})}{{}_2F_1(-2\epsilon,-2\epsilon,1-2\epsilon,1-\eta_{ij})} = K_{ij}(2\epsilon) + \mathcal{O}(\epsilon^3)\,. \tag{4.94}
$$

It follows that

$$
\widetilde{I}_{\rm S}(2\epsilon) = I_{\rm S}(2\epsilon) + \mathcal{O}(\epsilon)\,. \tag{4.95}
$$

This relation will be very helpful for demonstrating the cancellation of poles in color-correlated terms. Following this discussion, we write the double-soft term as

$$
\begin{aligned}
&\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle \\
&= [\alpha_s]^2\Big\langle\Big[\frac{1}{2}I_{\rm S}^2(\epsilon) + \Big(\frac{C_A}{\epsilon^2}c_1(\epsilon) + \frac{\beta_0}{\epsilon}c_2(\epsilon) + \beta_0\,c_3(\epsilon)\Big)\,\widetilde{I}_{\rm S}(2\epsilon)\Big]\cdot F_{\rm LM}\Big\rangle \\
&\quad + \langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}^{\rm fin}\,.
\end{aligned} \tag{4.96}
$$

This concludes our discussion of the double-soft limits.

We now move on to the third term on the right-hand side of eq. (4.74), which involves the soft limit of the real-virtual contribution. This limit reads [75, 76]

$$
\begin{aligned}
&S_{\mathfrak{m}}\,F_{\rm RV}(\mathfrak{m}) \\
&= -g_{s,b}^2\sum_{(ij)}^{N_p}\Big\{2\,S_{ij}(p_{\mathfrak{m}})\;(\boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_j)\cdot F_{\rm LV} - \frac{\alpha_s(\mu)}{2\pi}\,\frac{\beta_0}{\epsilon}\,2\,S_{ij}(p_{\mathfrak{m}})\;(\boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_j)\cdot F_{\rm LM} \\
&\quad - 2\,\frac{[\alpha_s]}{\epsilon^2}\,C_A\,A_K(\epsilon)\,\Big(S_{ij}(p_{\mathfrak{m}})\Big)^{1+\epsilon}\;(\boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_j)\cdot F_{\rm LM} \\
&\quad - [\alpha_s]\,\frac{4\pi\,\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)}\sum_{\substack{k=1\\k\neq i,j}}^{N_p}\kappa_{ij}\,S_{ki}(p_{\mathfrak{m}})\,\Big(S_{ij}(p_{\mathfrak{m}})\Big)^{\epsilon}f_{abc}\,T_k^a\,T_i^b\,T_j^c\,F_{\rm LM}\Big\}\,,
\end{aligned} \tag{4.97}
$$

where $\kappa_{ij} \equiv (\lambda_{ij} - \lambda_{i\mathfrak{m}} - \lambda_{j\mathfrak{m}}) = +1$ when both i and j are incoming momenta and $\kappa_{ij} = -1$ otherwise. We point out that κ_{ij} is symmetric under the exchange $i \leftrightarrow j$. Moreover, we have introduced the constant (cf. eq. (A.9))

$$
A_K(\epsilon) = \frac{\Gamma^3(1+\epsilon)\,\Gamma^5(1-\epsilon)}{\Gamma(1+2\epsilon)\,\Gamma^2(1-2\epsilon)} = 1 + \mathcal{O}(\epsilon^2)\,. \tag{4.98}
$$

The terms in eq. (4.97) that include $S_{ij}(p_{\mathfrak{m}})$ can be integrated over the unresolved phase space along the same lines as the soft subtraction term at NLO (see eq. (3.12)), giving rise to the operator $I_{\rm S}$. The term with $F_{\rm LV}$ in eq. (4.97) can be further simplified using Catani's formula (eq. (3.28)) to extract divergences from the loop amplitude. However, care is needed since the operators I_1 and $I_{\rm S}$ do not commute in general. Hence, upon integrating the first term on the right-hand side of eq. (4.97) over the phase space of gluon $\mathfrak{m}$, we find the following expression for the combination of divergent loop and soft-emission contributions

$$[\alpha_s]^2 \left\langle \left[I_{\rm S}(\epsilon) \cdot \overline{I}_1(\epsilon) + \overline{I}_1^\dagger(\epsilon) \cdot I_{\rm S}(\epsilon) \right] \cdot F_{\rm LM} \right\rangle . \tag{4.99}$$

We can rewrite the above quantity using the identity

$$I_{\rm S}\overline{I}_1 + \overline{I}_1^\dagger I_{\rm S} = \frac{1}{2}\left(\left(\overline{I}_1 + \overline{I}_1^\dagger \right) I_{\rm S} + I_{\rm S}\left(\overline{I}_1 + \overline{I}_1^\dagger \right) + \left[I_{\rm S} \, , \, \overline{I}_1 - \overline{I}_1^\dagger \right] \right) , \tag{4.100}$$

where the first and second terms can be expressed through $I_{\rm V}$ and $I_{\rm S}$, and the third term contains triple color correlations and will be discussed in detail in section 5.2.

The integration of the third term on the right-hand side of eq. (4.97), which includes the factor $(S_{ij}(p_{\mathfrak{m}}))^{1+\epsilon}$, leads to

$$\begin{aligned}
&- 2g_{s,b}^2 \sum_{(ij)}^{N_p} \left\langle (S_{ij}(p_{\mathfrak{m}}))^{1+\epsilon} \, (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\rm LM} \right\rangle \\
&= - \frac{[\alpha_s]}{4\epsilon^2} \left(\frac{2E_{\rm max}}{\mu} \right)^{-4\epsilon} \sum_{(ij)}^{N_p} \left\langle \eta_{ij}^{-2\epsilon} \widetilde{K}_{ij}(\epsilon) \, (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\rm LM} \right\rangle \\
&= [\alpha_s] \left\langle \widetilde{I}_{\rm S}(2\epsilon) \cdot F_{\rm LM} \right\rangle .
\end{aligned} \tag{4.101}$$

The last term on the right-hand side of eq. (4.97) contains explicit triple color correlators. Integrating this term over the phase space of gluon $\mathfrak{m}$ is non-trivial and is discussed at length in appendix H. In what follows we will refer to it as the triple color-correlated real-virtual subtraction term, $I_{\rm tri}^{\rm RV}$. Putting everything together, we find that the soft limit of the real-virtual correction can be written in the following way

$$\begin{aligned}
\langle S_{\mathfrak{m}} \, F_{\rm RV}(\mathfrak{m}) \rangle = [\alpha_s]^2 & \left\langle \frac{1}{2} \left[I_{\rm S}(\epsilon) \cdot I_{\rm V}(\epsilon) + I_{\rm V}(\epsilon) \cdot I_{\rm S}(\epsilon) \right] \cdot F_{\rm LM} \right\rangle \\
& + [\alpha_s] \left\langle I_{\rm S}(\epsilon) \cdot F_{\rm LV}^{\rm fin} \right\rangle - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \left\langle I_{\rm S}(\epsilon) F_{\rm LM} \right\rangle \\
& - \frac{[\alpha_s]^2}{\epsilon^2} C_A \, A_K(\epsilon) \left\langle \widetilde{I}_{\rm S}(2\epsilon) \cdot F_{\rm LM} \right\rangle \\
& + [\alpha_s]^2 \left\langle \left(\frac{1}{2} \left[I_{\rm S}(\epsilon) \, , \, \overline{I}_1(\epsilon) - \overline{I}_1^\dagger(\epsilon) \right] + I_{\rm tri}^{\rm RV}(\epsilon) \right) \cdot F_{\rm LM} \right\rangle .
\end{aligned} \tag{4.102}$$

We have now analyzed all terms with quartic and triple-color correlators. These arose due to soft limits of real emission amplitudes and virtual corrections; because of that, they are associated with unboosted kinematics. We have also found a number of terms with double-color correlations. Further terms of this kind emerge when a soft or virtual operator appears in conjunction with a collinear limit, and such terms can also lead to unboosted kinematic configurations. Our next goal is to identify such contributions in eq. (4.74).

We begin with the term that describes the hard-collinear limits of the real-virtual amplitude squared $\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \omega^{\mathfrak{m}i} \Delta^{(\mathfrak{m})} F_{\rm RV}(\mathfrak{m}) \rangle$. These limits were studied in refs. [76, 77]. They involve both the tree-level splitting function P_{ii} as well as the one-loop splitting function $P_{ii}^{1\rm L}$, whose explicit form can be found in appendix A. Even though $P_{ii}^{1\rm L}$ is more complicated than the corresponding tree-level splitting function, the integration over unresolved phase space of the gluon $\mathfrak{m}$ proceeds in exactly the same way as in the NLO computation.

Similar to the NLO case, it is useful to distinguish between the initial-state and the final-state splittings. When the unresolved parton $\mathfrak{m}$ becomes collinear to a final-state parton i we find

$$\begin{aligned}\left\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \omega^{\mathfrak{m}i} \Delta^{(\mathfrak{m})} F_{\rm RV}(\mathfrak{m}) \right\rangle = [\alpha_s]^2 \left\langle \frac{\Gamma_{i,g}}{\epsilon} I_{\rm V}(\epsilon) \cdot F_{\rm LM} \right\rangle \\ - [\alpha_s]^2 \frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\rm E}}} \left\langle \frac{\Gamma_{i,g}}{\epsilon} F_{\rm LM} \right\rangle - \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left\langle \frac{\Gamma^{\rm 1L}_{i,g}}{2\epsilon} F_{\rm LM} \right\rangle + [\alpha_s] \left\langle \frac{\Gamma_{i,g}}{\epsilon} F^{\rm fin}_{\rm LV} \right\rangle ,\end{aligned} \tag{4.103}$$

where

$$h_c(\epsilon) = \frac{\Gamma^2(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-3\epsilon)} = 1 + \mathcal{O}(\epsilon^3) . \tag{4.104}$$

Furthermore, the one-loop generalized anomalous dimension for the final-state splitting reads

$$\Gamma^{\rm 1L}_{i,g} = \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \frac{\epsilon^2 \cos(\pi\epsilon)}{C_A} \gamma^{33,\rm 1L}_{z,g\to gg}(\epsilon, L_i) , \qquad i = 3, \ldots, N_p , \tag{4.105}$$

where $\gamma^{33,\rm 1L}_{z,g\to gg}$ is defined analogously to eq. (3.20), but with the splitting function $P^{\rm 1L}_{gg}$ instead of P_{gg}. The ϵ-expansion of the one-loop generalized anomalous dimension reads

$$\Gamma^{\rm 1L}_{i,g} = \gamma_i + 2\boldsymbol{T}_i^2 L_i + \mathcal{O}(\epsilon) , \qquad i = 3, \ldots, N_p . \tag{4.106}$$

We continue with the case where the unresolved parton $\mathfrak{m}$ becomes collinear to an initial state parton, say 1_a. In this case we find

$$\begin{aligned}\left\langle \overline{S}_{\mathfrak{m}} C_{1\mathfrak{m}} \omega^{\mathfrak{m}1} \Delta^{(\mathfrak{m})} F_{\rm RV}(\mathfrak{m}) \right\rangle &= [\alpha_s]^2 \left\langle \frac{\Gamma_{1,f_1}}{\epsilon} I_{\rm V}(\epsilon) \cdot F_{\rm LM} \right\rangle + [\alpha_s] \left\langle \frac{\Gamma_{1,f_1}}{\epsilon} F^{\rm fin}_{\rm LV} \right\rangle \\ &+ \frac{[\alpha_s]^2}{\epsilon} \langle \mathcal{P}^{\rm gen}_{aa} \otimes (I_{\rm V}(\epsilon) \cdot F_{\rm LM}) \rangle + \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}^{\rm gen}_{aa} \otimes F^{\rm fin}_{\rm LV} \rangle \\ &- [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \left[\left\langle \frac{\Gamma_{1,f_1}}{\epsilon} F_{\rm LM} \right\rangle + \frac{1}{\epsilon} \langle \mathcal{P}^{\rm gen}_{aa} \otimes F_{\rm LM} \rangle \right] \\ &- \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left\langle \mathcal{P}^{\rm gen}_{aa} \otimes \left(\frac{\Gamma^{\rm 1L}_{1,f_1}(\epsilon)}{2\epsilon} F_{\rm LM} \right) \right\rangle \\ &- \frac{[\alpha_s]^2}{2\epsilon^3} C_A h_c(\epsilon) \langle \mathcal{P}^{\rm 1L,gen}_{aa} \otimes F_{\rm LM} \rangle ,\end{aligned} \tag{4.107}$$

where the one-loop initial-state generalized anomalous dimension is

$$\begin{aligned}\Gamma^{\rm 1L}_{1,f_1}(\epsilon) &= \left[\left(\frac{2E_1}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[\gamma_{f_1} + 2\boldsymbol{T}^2_{f_1} \frac{1 - e^{-4\epsilon L_1}}{4} \pi \cot(\pi\epsilon) \right] \\ &= \gamma_{f_1} + 2\boldsymbol{T}^2_{f_1} L_1 + \mathcal{O}(\epsilon) ,\end{aligned} \tag{4.108}$$

and we have also introduced a generalized splitting function at one-loop

$$\mathcal{P}^{\rm 1L,gen}_{aa}(z, E_1) = \left[\left(\frac{2E_1}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[- \hat{P}^{(0)}_{aa}(z) + \epsilon \hat{P}^{\rm 1L,fin}_{aa}(z) \right] . \tag{4.109}$$

We observe that the one-loop generalized anomalous dimension $\Gamma^{\rm 1L}_{i,g}$ coincides with its tree-level counterpart $\Gamma_{i,g}$ at $\mathcal{O}(\epsilon^0)$, cf. eq. (C.17). Similarly, the one-loop and tree-level generalized splitting functions $\mathcal{P}_{aa}^{\rm 1L,gen}$ and $\mathcal{P}_{aa}^{\rm gen}$ have the same expansion at $\mathcal{O}(\epsilon^0)$. Further details concerning these one-loop generalized anomalous dimensions and splitting functions can be found in appendix A. Finally, we note that in eq. (4.107) some terms involve the convolution of a splitting function with the product of $I_{\rm V}$ or the anomalous dimensions and $F_{\rm LM}$. In these cases, the relevant energy in $I_{\rm V}$ or $\Gamma^{\rm 1L}_{1,f_1}$ is also multiplied by a factor of z.

Summing the initial and final state collinear limits we find

$$\begin{aligned}\sum_{i=1}^{N_p}\langle\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\omega^{\mathfrak{m}i}\Delta^{(\mathfrak{m})}F_{\rm RV}\rangle = [\alpha_s]^2\langle I_{\rm C}(\epsilon)I_{\rm V}(\epsilon)\cdot F_{\rm LM}\rangle + [\alpha_s]\langle I_{\rm C}(\epsilon)\cdot F_{\rm LV}^{\rm fin}\rangle \\ +\frac{[\alpha_s]^2}{\epsilon}\langle\mathcal{P}_{aa}^{\rm gen}\otimes(I_{\rm V}(\epsilon)\cdot F_{\rm LM})\rangle + \frac{[\alpha_s]}{\epsilon}\langle\mathcal{P}_{aa}^{\rm gen}\otimes F_{\rm LV}^{\rm fin}\rangle \\ +\frac{[\alpha_s]^2}{\epsilon}\langle(I_{\rm V}(\epsilon)\cdot F_{\rm LM})\otimes\mathcal{P}_{bb}^{\rm gen}\rangle + \frac{[\alpha_s]}{\epsilon}\langle F_{\rm LV}^{\rm fin}\otimes\mathcal{P}_{bb}^{\rm gen}\rangle \\ -[\alpha_s]^2\frac{\beta_0}{\epsilon}\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\rm E}}}\left[\frac{1}{\epsilon}\langle\mathcal{P}_{aa}^{\rm gen}\otimes F_{\rm LM}\rangle + \frac{1}{\epsilon}\langle F_{\rm LM}\otimes\mathcal{P}_{bb}^{\rm gen}\rangle\right. \\ \left. +\langle I_{\rm C}(\epsilon)\cdot F_{\rm LM}\rangle\right] - \frac{[\alpha_s]^2}{\epsilon^2}C_A h_c(\epsilon)\langle\widetilde{I}_{\rm C}(2\epsilon)\cdot F_{\rm LM}\rangle \\ -\frac{[\alpha_s]^2}{2\epsilon^3}C_A h_c(\epsilon)\langle\mathcal{P}_{aa}^{\rm 1L,gen}\otimes F_{\rm LM} + F_{\rm LM}\otimes\mathcal{P}_{bb}^{\rm 1L,gen}\rangle\,,\end{aligned}\tag{4.110}$$

with

$$\widetilde{I}_{\rm C}(2\epsilon) = \sum_{i=1}^{N_p}\frac{\Gamma^{\rm 1L}_{i,f_i}(\epsilon)}{2\epsilon}\,.\tag{4.111}$$

We point out that the relation between the one-loop and tree-level hard-collinear operators

$$\widetilde{I}_{\rm C}(2\epsilon) \equiv I_{\rm C}(2\epsilon) + \mathcal{O}(\epsilon)\,,\tag{4.112}$$

is analogous to that of the soft operators, see eq. (4.95).

We now consider the fifth and sixth terms in eq. (4.74)

$$\langle\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{m})}(S_{\mathfrak{n}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n}))\rangle + \langle S_{\mathfrak{n}}(\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\,\Delta^{(\mathfrak{m})}\Theta_{\mathfrak{nm}}F_{\rm LM}(\mathfrak{m},\mathfrak{n}))\rangle\,,\tag{4.113}$$

where we have used $S_{\mathfrak{n}}\Delta^{(\mathfrak{mn})} = \Delta^{(\mathfrak{m})}$. At first glance, it may seem that the two terms in eq. (4.113) can be trivially combined, since the first contains an energy-ordering theta-function which enforces $E_{\mathfrak{m}} > E_{\mathfrak{n}}$, while the second requires $E_{\mathfrak{n}} > E_{\mathfrak{m}}$. However, one should be careful about the order in which the various operators act on $F_{\rm LM}$. In the first term, one should compute the soft limit $S_{\mathfrak{n}}$ of $F_{\rm LM}$ first, then integrate over the unresolved phase space of $\mathfrak{n}$, and then compute the hard-collinear limit $\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}$ and integrate over the phase space of $\mathfrak{m}$. In the second term, the hard-collinear limit $\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}$ is evaluated first, followed by the integration over the phase space of $\mathfrak{m}$. Then we take the soft limit $S_{\mathfrak{n}}$ and integrate over the phase space of $\mathfrak{n}$. We emphasize that these operations do not commute. Indeed, one can show by explicit calculation that the following holds true

$$\begin{aligned}&\langle S_{\mathfrak{n}}(\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{m})}\Theta_{\mathfrak{nm}}F_{\rm LM}(\mathfrak{m},\mathfrak{n}))\rangle \\ &= \langle\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{m})}(S_{\mathfrak{n}}\Theta_{\mathfrak{nm}}F_{\rm LM}(\mathfrak{m},\mathfrak{n}))\rangle - \frac{[\alpha_s]}{\epsilon^2}C_A\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \\ &\quad\times\left\langle\eta_{i\mathfrak{m}}^{-\epsilon}\,\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\left[\left(\frac{2E_{\rm max}}{\mu}\right)^{-2\epsilon} - \left(\frac{2E_{\mathfrak{m}}}{\mu}\right)^{-2\epsilon}\right]\Delta^{(\mathfrak{m})}F_{\rm LM}(\mathfrak{m})\right\rangle.\end{aligned}\tag{4.114}$$

Thus we can rewrite eq. (4.113) as follows

$$\begin{aligned}
&\left\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} S_{\mathfrak{n}} \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \right\rangle + \left\langle S_{\mathfrak{n}} \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} \Theta_{\mathfrak{nm}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \right\rangle \\
&= \left\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} S_{\mathfrak{n}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \right\rangle - \frac{[\alpha_s]}{\epsilon^2} C_A \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \\
&\quad \times \left\langle \eta_{i\mathfrak{m}}^{-\epsilon}\, \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \left[\left(\frac{2E_{\rm max}}{\mu}\right)^{-2\epsilon} - \left(\frac{2E_{\mathfrak{m}}}{\mu}\right)^{-2\epsilon} \right] \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m}) \right\rangle .
\end{aligned} \tag{4.115}$$

It is straightforward to integrate the second term on the right-hand side of eq. (4.115) over the phase space of parton $\mathfrak{m}$ since the required calculation is NLO-like. On the contrary, the first term on the right-hand side in eq. (4.115) requires some discussion. We begin by acting with the soft operator $S_{\mathfrak{n}}$ on $F_{\rm LM}(\mathfrak{m},\mathfrak{n})$ and integrating over the phase space of $\mathfrak{n}$. We find

$$\begin{aligned}
&\left\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} S_{\mathfrak{n}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \right\rangle \\
&= -\frac{[\alpha_s]}{\epsilon^2} \left(\frac{2E_{\rm max}}{\mu}\right)^{-2\epsilon} \sum_{(kl)}^{N_p+1} \left\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \eta_{kl}^{-\epsilon}\, K_{kl}\, \Delta^{(\mathfrak{m})}\, (\boldsymbol{T}_k \cdot \boldsymbol{T}_l) \cdot F_{\rm LM}(\mathfrak{m}) \right\rangle .
\end{aligned} \tag{4.116}$$

The important point is that the sum in the above expression runs over $N_p + 1$ partons which includes the parton $\mathfrak{m}$. To simplify such an expression, we split the sum into the following contributions

$$\begin{aligned}
\sum_{(kl)}^{N_p+1} A_{kl} \boldsymbol{T}_k \cdot \boldsymbol{T}_l &= \sum_{\substack{k,l\neq i\\ k\neq l}}^{N_p} A_{kl} \boldsymbol{T}_k \cdot \boldsymbol{T}_l + \sum_{k\neq i}^{N_p} (A_{ik} \boldsymbol{T}_i + A_{\mathfrak{m}k} \boldsymbol{T}_{\mathfrak{m}}) \cdot \boldsymbol{T}_k \\
&\quad + \sum_{k\neq i}^{N_p} \boldsymbol{T}_k \cdot (A_{ki} \boldsymbol{T}_i + A_{k\mathfrak{m}} \boldsymbol{T}_{\mathfrak{m}}) + 2 A_{i\mathfrak{m}} \boldsymbol{T}_i \cdot \boldsymbol{T}_{\mathfrak{m}} ,
\end{aligned} \tag{4.117}$$

for an arbitrary symmetric A_{ij}. We consider the action of the operator $\overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}}$ in each of the terms in eq. (4.117). In the first term, these operators act directly on $F_{\rm LM}(\mathfrak{m})$. In the second term, the factor $A_{\mathfrak{m}k}$ becomes A_{ik} because of the collinear $i||\mathfrak{m}$ limit. Thus the corresponding color factors combine into $(\boldsymbol{T}_i + \boldsymbol{T}_{\mathfrak{m}}) \cdot \boldsymbol{T}_k = \boldsymbol{T}_{[i\mathfrak{m}]} \cdot \boldsymbol{T}_k$. The same occurs in the third term, leading to $\boldsymbol{T}_k \cdot \boldsymbol{T}_{[i\mathfrak{m}]}$. Finally, in the last term, the product of the color charges is $2\boldsymbol{T}_i \cdot \boldsymbol{T}_{\mathfrak{m}} = -C_A$, because the parton $\mathfrak{m}$ is a gluon. Using the limit

$$\lim_{\eta_{ij}\to 0} K_{ij} = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} , \tag{4.118}$$

we find

$$\begin{aligned}
\sum_{(kl)}^{N_p+1} \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \eta_{kl}^{-\epsilon} K_{kl} \Big[(\boldsymbol{T}_k \cdot \boldsymbol{T}_l) \cdot F_{\rm LM}(\mathfrak{m}) \Big] &= \sum_{(kl)}^{N_p} \eta_{kl}^{-\epsilon} K_{kl}\, (\boldsymbol{T}_k \cdot \boldsymbol{T}_l)\, \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \cdot F_{\rm LM}(\mathfrak{m}) \\
&\quad - C_A \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \eta_{i\mathfrak{m}}^{-\epsilon}\, \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} F_{\rm LM}(\mathfrak{m}) ,
\end{aligned} \tag{4.119}$$

where in the first term on the right-hand side the sum over partons k and l includes a clustered parton $[i\mathfrak{m}]$ in place of parton i.

Putting everything together and including the sum over all unresolved partons, we find

$$\begin{aligned}
&\sum_{i=1}^{N_p}\Big[\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} S_{\mathfrak{n}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}\rangle + \langle S_{\mathfrak{n}} \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} \Theta_{\mathfrak{nm}} F_{\mathrm{LM}}\rangle\Big] \\
&= [\alpha_s]^2 \langle I_{\mathrm{S}}(\epsilon)\cdot I_{\mathrm{C}}(\epsilon)\cdot F_{\mathrm{LM}}\rangle + \frac{[\alpha_s]^2}{\epsilon^2} h_c(\epsilon) C_A \langle I_{\mathrm{C}}^{(4)}(\epsilon)\cdot F_{\mathrm{LM}}\rangle \\
&\quad + \frac{[\alpha_s]^2}{\epsilon}\langle \mathcal{P}_{aa}^{\mathrm{gen}} \otimes I_{\mathrm{S}}(\epsilon)\cdot F_{\mathrm{LM}} + I_{\mathrm{S}}(\epsilon)\cdot F_{\mathrm{LM}} \otimes \mathcal{P}_{bb}^{\mathrm{gen}}\rangle \\
&\quad + \frac{[\alpha_s]^2}{2\epsilon^3} C_A h_c(\epsilon) \langle \mathcal{P}_{aa}^{(4),\mathrm{gen}} \otimes F_{\mathrm{LM}} + F_{\mathrm{LM}} \otimes \mathcal{P}_{bb}^{(4),\mathrm{gen}}\rangle\,.
\end{aligned} \tag{4.120}$$

In the above formula, we have employed generalizations of I_{C} and $\mathcal{P}_{ab}^{\mathrm{gen}}$. They are defined in appendix A. For the specific case that we are interested in here, we have

$$I_{\mathrm{C}}^{(4)}(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}^{(4)}(\epsilon)}{2\epsilon}\,, \tag{4.121}$$

where

$$\begin{aligned}
\Gamma_{i,f_i}^{(4)} &= \left(\frac{2E_i}{\mu}\right)^{-4\epsilon} \frac{\Gamma^4(1-\epsilon)}{\Gamma^2(1-2\epsilon)} \left[\gamma_{f_i} + \boldsymbol{T}_{f_i}^2 \frac{1-e^{-4\epsilon L_i}}{2\epsilon}\right], \qquad && i = 1,2\,, \\
\Gamma_{i,g}^{(4)} &= \left(\frac{2E_i}{\mu}\right)^{-4\epsilon} \frac{\Gamma^4(1-\epsilon)}{\Gamma^2(1-2\epsilon)} \gamma_{z,g\to gg}^{24}(\epsilon, L_i)\,, && i = 3,\ldots,N_p\,,
\end{aligned} \tag{4.122}$$

and

$$\mathcal{P}_{ab}^{(4),\mathrm{gen}}(z, E_a) = \left[\left(\frac{2E_a}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^2 \left[-\hat{P}_{ab}^{(0)}(z) + \epsilon\, \mathcal{P}_{ab}^{(4),\mathrm{fin}}(z)\right]. \tag{4.123}$$

The function $\mathcal{P}_{ab}^{(4),\mathrm{fin}}$ is given in eq. (A.35). It follows from the above formulas that $\Gamma_{i,f_i}^{(4)}$ and $\mathcal{P}_{ab}^{(4),\mathrm{gen}}$ coincide with Γ_{i,f_i} and $\mathcal{P}_{ab}^{\mathrm{gen}}$ to $\mathcal{O}(\epsilon^0)$. Similarly, I_{C} and $I_{\mathrm{C}}^{(4)}$ have the same pole structure

$$I_{\mathrm{C}}^{(4)}(\epsilon) \equiv I_{\mathrm{C}}(2\epsilon) + \mathcal{O}(\epsilon^0)\,. \tag{4.124}$$

Before closing this section, we make a brief comment about the term on the third-to-last line of eq. (4.74), which is proportional to $\delta_g(\epsilon)$. It turns out that one can rewrite it in the following way

$$\begin{aligned}
&2[\alpha_s]^2 \delta_g(\epsilon) \left(\frac{E_{\max}}{\mu}\right)^{-2\epsilon} \left[-\langle I_{\mathrm{S}}(\epsilon)\cdot F_{\mathrm{LM}}\rangle + \frac{(2E_{\max}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i=1}^{N_p} \boldsymbol{T}_i^2\, \langle F_{\mathrm{LM}}\rangle\right] \\
&= -[\alpha_s]^2\, 2^{2+2\epsilon} \left(C_A \delta_g^{C_A}(\epsilon) + \beta_0 \delta_g^{\beta_0}(\epsilon)\right) \left\langle \tilde{I}_{\mathrm{S}}(2\epsilon)\cdot F_{\mathrm{LM}}\right\rangle + \mathcal{O}(\epsilon^0)\,,
\end{aligned} \tag{4.125}$$

where

$$\delta_g^{C_A}(\epsilon) = \left(-\frac{131}{72} + \frac{\pi^2}{6}\right) + \mathcal{O}(\epsilon)\,; \qquad \delta_g^{\beta_0}(\epsilon) = \log 2 + \mathcal{O}(\epsilon)\,. \tag{4.126}$$

The reason why this rewriting is useful will become clear when we discuss the cancellation of color-correlated contributions with unboosted kinematics.

In summary, we have derived expressions for all the divergent terms in eq. (4.74) that involve virtual amplitudes and the various soft limits. Such contributions involve infrared poles in color-correlated matrix elements that don't appear in other parts of the calculations. Thus, we anticipate that the poles of the color-correlated contributions cancel amongst themselves. We describe this cancellation, as well as the cancellation of the poles of the single-unresolved and color-uncorrelated double-unresolved contributions, in the following section.

5 Cancellation of poles

We begin our discussion of the infrared poles by focusing on the single-unresolved contribution. We show that the cancellation of poles there is *equivalent* to that in the NLO QCD contribution to the process $q\bar{q} \to X + (N+1)g$. We then continue with the discussion of the various contributions to the double-unresolved term Σ_N, starting from the color-correlated ones.

5.1 Single-unresolved terms

As explained in the previous section, when extracting singularities from the double-real and real-virtual contributions, we find terms featuring $N+1$ resolved partons. In this section we will show that, once combined, these terms exhibit significant simplifications, allowing us to cancel the poles in the same way as we did for the NLO contribution. We consider $\Sigma^{(1)}_{N+1}$, $\Sigma^{(2)}_{N+1}$ and $\Sigma^{(3)}_{N+1}$, given in eqs. (4.13), (4.38) and (4.55), respectively. We will refer to the sum of these contributions as $\Sigma^{\rm div}_{N+1}$. It reads

$$
\begin{aligned}
\Sigma^{\rm div}_{N+1} = \sum_{i=1}^{3} \Sigma^{(i)}_{N+1} = \big\langle \mathcal{O}_{\rm NLO} \Delta^{(\mathfrak{m})} \big[F_{\rm RV}(\mathfrak{m}) + S_{\mathfrak{n}} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \big] \big\rangle \\
+ \sum_{i=1}^{N_p} \big\langle \mathcal{O}_{\rm NLO} (\mathbb{1} - S_{\mathfrak{n}} \Theta_{\mathfrak{m}\mathfrak{n}}) C_{i\mathfrak{n}} \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \big\rangle \\
+ \frac{1}{2} \big\langle \mathcal{O}_{\rm NLO} \Delta^{(\mathfrak{m})} (\mathbb{1} - 2 S_{\mathfrak{n}} \Theta_{\mathfrak{m}\mathfrak{n}}) C_{\mathfrak{m}\mathfrak{n}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \big\rangle \, .
\end{aligned}
\tag{5.1}
$$

In the equation above, gluon $\mathfrak{m}$ is resolved, since all the singularities associated with its emission are regulated by the $\mathcal{O}_{\rm NLO}$ operator (see eq. (3.37)). The gluon $\mathfrak{n}$, on the other hand, plays the role of an unresolved parton in NLO computations. Such a structure suggests a close relation between $\Sigma^{\rm div}_{N+1}$ and the NLO cross section for the production of $(N+1)$ jets. In order to make this correspondence transparent, we need to rewrite eq. (5.1) in terms of virtual, soft and collinear operators defined in the phase space for $(N+1)$ partons.

We begin our analysis with the first term in eq. (5.1). It contains the one-loop amplitudes with $(N+1)$ final-state partons and a contribution from the soft limit of gluon $\mathfrak{n}$. The former term can be treated analogously to what has been done in section 3; its infrared singularities can be written with the help of Catani's formula. The latter contribution, once integrated over the $\mathfrak{n}$-parton phase space, returns the same structure as in eq. (3.12), up to replacing $E_{\rm max}$ with $E_{\mathfrak{m}}$. This is due to the energy-ordering factor $\Theta_{\mathfrak{m}\mathfrak{n}}$ appearing in eq. (5.1), which forces the energy of gluon $\mathfrak{m}$, rather than $E_{\max}$, to serve as the upper cut-off for the integration over the energy of gluon $\mathfrak{n}$ in the soft limit. We thus find

$$\begin{aligned}&\langle \mathcal{O}_{\rm NLO}\Delta^{(\mathfrak{m})}\left[F_{\rm RV}(\mathfrak{m})+S_{\mathfrak{n}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right]\rangle=\\&=[\alpha_s]\left\langle \mathcal{O}_{\rm NLO}\Delta^{(\mathfrak{m})}\left[I_{\rm V}^{N_p+1}+I_{\rm S}^{N_p+1}(E_{\mathfrak{m}})\right]\cdot F_{\rm LM}(\mathfrak{m})\right\rangle+\langle \mathcal{O}_{\rm NLO}\Delta^{(\mathfrak{m})}F_{\rm RV}^{\rm fin}(\mathfrak{m})\rangle\,,\end{aligned}\tag{5.2}$$

where $I_{\rm V}^{N_p+1}$ is constructed in analogy with eq. (3.31), but starting from Catani's operator I_1 in eq. (3.29) with $N_p\mapsto N_p+1$. Similarly, $I_{\rm S}^{N_p+1}(E_{\mathfrak{m}})$ can be obtained by replacing $N_p\mapsto N_p+1$ in eq. (3.12) and using $E_{\mathfrak{m}}$ in place of $E_{\max}$.

We then address the contributions shown in the second and third lines in eq. (5.1). Both of these contributions describe soft-subtracted collinear limits; as such they provide either generalized anomalous dimensions (in case of final state splittings) or generalized anomalous dimensions and splitting functions (in case of initial state splittings). It follows from eq. (5.1) that in both of these cases integrations over the energy of the soft-collinear parton $\mathfrak{n}$ extends to $E_{\mathfrak{m}}$ and not to $E_{\max}$.

We would like to assemble these two terms to create the collinear operator $I_{\rm C}$ for the process with (N_p+1) partons, which could then be combined with the terms in eq. (5.2) to produce an infrared-finite operator $I_{\rm T}$, similar to what we did when describing the NLO calculation in section 3. At first glance it appears simple to do that. Indeed, the second line of eq. (5.1) contains terms with collinear limits of N_p (and not N_p+1) partons, and the required collinear limit of one additional parton is supplied by the third line of this equation. However, there seems to be a mismatch between these terms because the final state collinear operators acting on $\Delta^{(\mathfrak{mn})}$ in the second line produce $z_{i,\mathfrak{n}}\Delta^{(\mathfrak{m})}$, whereas in the third line the collinear operator does not act on $\Delta^{(\mathfrak{m})}$ and, therefore, cannot produce such a factor. The resolution of this hypothetical problem boils down to the fact that we consider a gluon-only final state, which is highly symmetric. The additional factor of $z_{i,\mathfrak{n}}$ effectively lowers this symmetry, and hence plays the same role as the factor 1/2 in the last term in eq. (5.1).[22] We can thus write the second and third lines on the right-hand side of eq. (5.1) as

$$\begin{aligned}&\sum_{i=1}^{N_p}\langle \mathcal{O}_{\rm NLO}\,(\mathbb{1}-S_{\mathfrak{n}}\Theta_{\mathfrak{mn}})C_{i\mathfrak{n}}\Delta^{(\mathfrak{mn})}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle\\&\qquad+\frac{1}{2}\langle \mathcal{O}_{\rm NLO}\,\Delta^{(\mathfrak{m})}(\mathbb{1}-2S_{\mathfrak{n}}\Theta_{\mathfrak{mn}})C_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle\\&\quad=[\alpha_s]\Big[\Big\langle \mathcal{O}_{\rm NLO}\,\Delta^{(\mathfrak{m})}(\mathcal{P}_{aa}^{\rm gen}\otimes F_{\rm LM}(\mathfrak{m}))\Big\rangle+\Big\langle \mathcal{O}_{\rm NLO}\,\Delta^{(\mathfrak{m})}(F_{\rm LM}(\mathfrak{m})\otimes\mathcal{P}_{bb}^{\rm gen})\Big\rangle\Big]\\&\qquad+[\alpha_s]\Big\langle \mathcal{O}_{\rm NLO}\left[I_{\rm C}^{N_p+1}(E_{\mathfrak{m}})\cdot\Delta^{(\mathfrak{m})}F_{\rm LM}(\mathfrak{m})\right]\Big\rangle\,.\end{aligned}\tag{5.3}$$

Similar to the (N_p+1) virtual and soft operators, $I_{\rm C}^{N_p+1}(E_{\mathfrak{m}})$ is defined as in eq. (3.27), but with $N_p\mapsto N_p+1$ and setting $E_{\max}\mapsto E_{\mathfrak{m}}$ in the definition of Γ_{i,f_i}. We emphasize that the $\mathcal{O}_{\rm NLO}$ operator does not commute with the collinear operator $I_{\rm C}^{N_p+1}$ or the splitting function $\mathcal{P}_{ab}^{\rm gen}$. Indeed the latter depends on the energy of parton $\mathfrak{m}$, which is sensitive to the action of the soft limit encoded in $\mathcal{O}_{\rm NLO}$.

The expression for $\Sigma_{N+1}^{\rm div}$ is the sum of eqs. (5.2) and (5.3). We note that this quantity still contains hard-collinear singularities related to initial state emissions. To remove them,

[22] More explicitly, the additional factor $z_{i,\mathfrak{n}}$ produces an additional factor of 1/2 upon integrating over the final-state phase space.

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we need to add the PDF renormalization contribution proportional to the $\mathcal{O}_{\rm NLO}$ operator, i.e.

$$\Sigma^{\rm div,pdf}_{N+1} = \frac{\alpha_s(\mu)}{2\pi\epsilon}\Big[\langle \hat{P}^{(0)}_{aa} \otimes \mathcal{O}_{\rm NLO}\Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m})\rangle + \langle \mathcal{O}_{\rm NLO}\Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m}) \otimes \hat{P}^{(0)}_{bb}\rangle\Big]\,. \tag{5.4}$$

In contrast with the observation made below eq. (5.3), in the expression of $\Sigma^{\rm div,pdf}_{N+1}$ we can exchange the order of the Altarelli-Parisi splitting functions and the $\mathcal{O}_{\rm NLO}$ operator. In fact, $\hat{P}^{(0)}_{qq}$ is independent of any energy variables, and thus can be moved "inside" the fully-resolved operator. Given this, we can write eq. (5.4) as

$$\Sigma^{\rm div,pdf}_{N+1} = \frac{\alpha_s(\mu)}{2\pi\epsilon}\Big[\langle \mathcal{O}_{\rm NLO}\Delta^{(\mathfrak{m})}(\hat{P}^{(0)}_{aa} \otimes F_{\rm LM}(\mathfrak{m}))\rangle + \langle \mathcal{O}_{\rm NLO}\Delta^{(\mathfrak{m})}(F_{\rm LM}(\mathfrak{m}) \otimes \hat{P}^{(0)}_{bb})\rangle\Big]\,, \tag{5.5}$$

and combine it with $\Sigma^{\rm div}_{N+1}$. We obtain

$$\begin{aligned}
\Sigma^{\rm fin,(3)}_{N+1} &= \Sigma^{\rm div}_{N+1} + \Sigma^{\rm div,pdf}_{N+1} \\
&= [\alpha_s]\left\langle \mathcal{O}_{\rm NLO}\,\Delta^{(\mathfrak{m})}\left(I_{\rm T}^{N_p+1}(E_{\mathfrak{m}}) \cdot F_{\rm LM}(\mathfrak{m})\right)\right\rangle \\
&\quad + [\alpha_s]\left[\left\langle \mathcal{O}_{\rm NLO}\,\Delta^{(\mathfrak{m})}\left(\mathcal{P}^{\rm NLO}_{aa} \otimes F_{\rm LM}(\mathfrak{m})\right)\right\rangle + \left\langle \mathcal{O}_{\rm NLO}\,\Delta^{(\mathfrak{m})}\left(F_{\rm LM}(\mathfrak{m}) \otimes \mathcal{P}^{\rm NLO}_{bb}\right)\right\rangle\right] \\
&\quad + \left\langle \mathcal{O}_{\rm NLO}\,\Delta^{(\mathfrak{m})}\,F^{\rm fin}_{\rm RV}(\mathfrak{m})\right\rangle\,.
\end{aligned} \tag{5.6}$$

As expected, eq. (5.6) contains a generalized version of the ϵ-finite operator $I_{\rm T}$ given in eq. (3.36). It reads

$$I_{\rm T}^{N_p+1}(E_{\mathfrak{m}}) \equiv I_{\rm V}^{N_p+1} + I_{\rm S}^{N_p+1}(E_{\mathfrak{m}}) + I_{\rm C}^{N_p+1}(E_{\mathfrak{m}})\,. \tag{5.7}$$

Note also that, as we mentioned at the beginning of this section, $\Sigma^{\rm fin,(3)}_{N+1}$ contains almost exactly the NLO contribution to the $(N+1)$-jet production cross section; the only missing piece is the fully-regulated term with up to $N+2$ resolved jets.

In addition to $\Sigma^{\rm fin,(3)}_{N+1}$, there are three other contributions with $N+1$ resolved final state partons that are explicitly ϵ-finite; they appeared in the course of simplifying $\Sigma_{\rm RR,1c}$, discussed in the previous section. We combine all these contributions into a single quantity that we will refer to as ${\rm d}\hat{\sigma}^{\rm NNLO}_{N+1}$. It is given by

$$2s\,{\rm d}\hat{\sigma}^{\rm NNLO}_{N+1} = \Sigma^{\rm fin,(1)}_{N+1} + \Sigma^{\rm fin,(2)}_{N+1} + \Sigma^{\rm fin,(3)}_{N+1} + \Sigma^{\rm sp}_{N+1}\,, \tag{5.8}$$

where $\Sigma^{\rm fin,(1)}_{N+1}$ is given in eq. (4.38) and $\Sigma^{\rm fin,(2)}_{N+1}$ in eq. (4.55). The final term originates from the spin-correlated contributions discussed in appendix F; in particular, it describes the $\mathcal{O}_{\rm NLO}$ piece of the expression given in eq. (F.18). We can expand these three terms in ϵ, leading to the following $\mathcal{O}(\epsilon^0)$ result

$$
\begin{aligned}
\Sigma_{N+1}^{\mathrm{fin},(1)} &= [\alpha_s] \left\langle \hat{P}_{qq}^{(0)} \otimes \left[\mathcal{O}_{\mathrm{NLO}}^{(1)}\, \omega_{1\|\mathfrak{n}}^{\mathfrak{m}1,\mathfrak{n}1} \log\left(\frac{\eta_{1\mathfrak{m}}}{2}\right) \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m}) \right] \right\rangle \\
&\quad + [\alpha_s] \left\langle \left[\mathcal{O}_{\mathrm{NLO}}^{(2)}\, \omega_{2\|\mathfrak{n}}^{\mathfrak{m}2,\mathfrak{n}2} \log\left(\frac{\eta_{2\mathfrak{m}}}{2}\right) \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m}) \right] \otimes \hat{P}_{qq}^{(0)} \right\rangle \\
&\quad - \sum_{i=1}^{N_p} [\alpha_s] \left\langle \mathcal{O}_{\mathrm{NLO}}^{(i)}\, \omega_{i\|\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i}\, \Gamma_{i,f_i} \log\left(\frac{\eta_{i\mathfrak{m}}}{2}\right) \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m}) \right\rangle_{E_{\max}\mapsto E_{\mathfrak{m}}} , \\
\Sigma_{N+1}^{\mathrm{fin},(2)} &= - \sum_{i=1}^{N_p} [\alpha_s]\, \gamma_{z,g\to gg}^{22} \left\langle \mathcal{O}_{\mathrm{NLO}}^{(i)}\, \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i} \log\left(\frac{\eta_{i\mathfrak{m}}}{4(1-\eta_{i\mathfrak{m}})}\right) \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m}) \right\rangle , \\
\Sigma_{N+1}^{\mathrm{sp}} &= \sum_{i=1}^{N_p} \frac{[\alpha_s]}{2} \gamma_{\perp,g\to gg}^{22} \left\langle \mathcal{O}_{\mathrm{NLO}}^{(i)}\, \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i} \Delta^{(\mathfrak{m})} (r_i^\mu r_i^\nu + g^{\mu\nu}) F_{\mathrm{LM},\mu\nu}(\mathfrak{m}) \right\rangle \\
&\quad + \sum_{i=1}^{N_p} \frac{[\alpha_s]}{2} \gamma_{\perp,g\to gg}^{22,r} \left\langle \mathcal{O}_{\mathrm{NLO}}^{(i)}\, \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}i,\mathfrak{n}i} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m}) \right\rangle ,
\end{aligned}
\tag{5.9}
$$

where $\gamma_{z,g\to gg}^{22}$ is equal to the function reported in eq. (3.20) upon setting $L_i = 0$, and $\gamma_{\perp,g\to gg}^{22}$ and $\gamma_{\perp,g\to gg}^{22,r}$ in eq. (A.29).

5.2 Double-unresolved triple color-correlated contributions

Having demonstrated how ϵ-poles in single-unresolved terms disappear, we continue with the discussion of poles in the double-unresolved contribution Σ_N. We begin with the investigation of ϵ-poles that involve matrix elements of triple correlators of color-charge operators $\langle \mathcal{M}_0 | f_{abc} T_i^a T_j^b T_k^c | \mathcal{M}_0 \rangle$. Such terms vanish for processes with three or fewer partons at tree level, but are non-zero in general.

As we explained in the previous subsection, triple color-correlated terms arise in two distinct ways. First, there are two contributions that contain triple color correlators explicitly. One of these is the $\mathcal{H}_{2,\mathrm{tc}}$ term of the double-virtual contribution in eq. (4.86) and the other one was denoted by $I_{\mathrm{tri}}^{\mathrm{RV}}$ in the integrated soft limit of the real-virtual correction in eq. (4.102).

Second, triple correlators of color charges appear in *commutators* of various I-operators. Such commutators are present in eqs. (4.86), (4.102); they arise because we expressed the double-virtual contribution and the soft limit of the real-virtual corrections through an operator I_{V}. All in all, combining the relevant terms, we find[23]

$$
\begin{aligned}
\Sigma_N^{\mathrm{tri}} &= [\alpha_s]^2 \left\langle \left(\frac{1}{2} \left[I_{\mathrm{S}}(\epsilon)\,,\, \overline{I}_1(\epsilon) - \overline{I}_1^\dagger(\epsilon) \right] + I_{\mathrm{tri}}^{\mathrm{RV}}(\epsilon) \right) \cdot F_{\mathrm{LM}} \right\rangle , \\
&\quad + [\alpha_s]^2 \left\langle \left(-\frac{1}{2} \left[\overline{I}_1(\epsilon), \overline{I}_1^\dagger(\epsilon) \right] + \mathcal{H}_{2,\mathrm{tc}} + \mathcal{H}_{2,\mathrm{tc}}^\dagger \right) \cdot F_{\mathrm{LM}} \right\rangle .
\end{aligned}
\tag{5.10}
$$

We find it convenient to rewrite the commutators that appear in eq. (5.10) as follows

$$
\frac{1}{2} \left[I_{\mathrm{S}}\,,\, \overline{I}_1 - \overline{I}_1^\dagger \right] - \frac{1}{2} \left[\overline{I}_1, \overline{I}_1^\dagger \right] = - \left[I_+, I_- \right] + \left[2 I_+ + I_{\mathrm{S}}, I_- \right] , \tag{5.11}
$$

[23] In general, there are triple color correlators in ϵ-finite terms present in Σ_N that are not included in eq. (5.10).

where we introduced two additional I-operators

$$I_+(\epsilon) = \frac{\overline{I}_1(\epsilon) + \overline{I}_1^\dagger(\epsilon)}{2}\,, \qquad I_-(\epsilon) = \frac{\overline{I}_1(\epsilon) - \overline{I}_1^\dagger(\epsilon)}{2}\,, \tag{5.12}$$

such that

$$I_{\rm V}(\epsilon) = \overline{I}_1(\epsilon) + \overline{I}_1^\dagger(\epsilon) \equiv 2I_+(\epsilon)\,. \tag{5.13}$$

We combine the commutators and the operator $\mathcal{H}_{2,\rm tc}$, and write

$$\Sigma_N^{\rm tri} = [\alpha_s]^2 \left\langle \left(I_{\rm tri}^{\rm RV} + I_{\rm tri}^{\rm (cc)} \right) \cdot F_{\rm LM} \right\rangle\,, \tag{5.14}$$

where $I_{\rm tri}^{\rm (cc)}$ is defined as

$$I_{\rm tri}^{\rm (cc)} = -\,[I_+, I_-] + [2I_+ + I_{\rm S}, I_-] + \mathcal{H}_{2,\rm tc} + \mathcal{H}_{2,\rm tc}^\dagger\,. \tag{5.15}$$

Eq. (5.14) collects all potentially divergent terms where the triple color-correlated contributions can appear and provides the starting point for their analysis.

To proceed, we need to compute the commutators of the various I-operators that appear in eq. (5.15). To do that, we write $\overline{I}_1$ as (see eqs. (3.29) and (3.33))

$$\overline{I}_1 = -\frac{1}{2}\sum_{i=1}^{N_p}\left(\frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}\right) + \frac{1}{2}\sum_{(ij)}^{N_p}\left(\frac{1}{\epsilon^2} + \frac{\gamma_i}{\boldsymbol{T}_i^2\epsilon}\right)(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\left(e^{i\lambda_{ij}\pi\epsilon}e^{\epsilon L_{ij}} - 1\right), \tag{5.16}$$

where $L_{ij} = \log\left(\mu^2/s_{ij}\right)$ with $s_{ij} = 2p_i\cdot p_j$, and $\lambda_{ij} = 1$ if both i and j are either incoming or outgoing, and $\lambda_{ij} = 0$ otherwise. Since we are interested in commutators of I-operators, in general the only non-vanishing contributions come from color-correlated terms. Therefore, the first term on the right hand side in eq. (5.16) is irrelevant, and only the term with the $\boldsymbol{T}_i\cdot\boldsymbol{T}_j$ product can play a role. Hence, we define

$$\overline{I}_1^{\rm (cc)} = \frac{1}{2}\sum_{(ij)}^{N_p}\left(\frac{1}{\epsilon^2} + \frac{\gamma_i}{\boldsymbol{T}_i^2\epsilon}\right)\ (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\ \left(e^{i\lambda_{ij}\pi\epsilon}e^{\epsilon L_{ij}} - 1\right), \tag{5.17}$$

and we can use this operator instead of $\overline{I}_1$ to compute the commutators in eq. (5.14). To this end, we compute the color-correlated versions of $I_\pm$ using $\overline{I}_1^{\rm (cc)}$ and find[24]

$$\begin{aligned} I_+^{\rm (cc)} &= \frac{1}{2}\sum_{(ij)}^{N_p}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\left(\frac{1}{\epsilon}L_{ij} + \delta_{ij}^+\right) + \mathcal{O}(\epsilon)\,,\\ I_-^{\rm (cc)} &= \frac{i\pi}{2}\sum_{(ij)}^{N_p}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\left(\frac{1}{\epsilon}\lambda_{ij} + \delta_{ij}^-\right) + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{5.18}$$

where

$$\delta_{ij}^+ = \frac{1}{2}L_{ij}^2 + \frac{\gamma_i}{\boldsymbol{T}_i^2}\,L_{ij} - \frac{1}{2}\pi^2\lambda_{ij}^2\,, \qquad \delta_{ij}^- = \frac{\gamma_i}{\boldsymbol{T}_i^2}\,\lambda_{ij} + L_{ij}\,\lambda_{ij}\,. \tag{5.19}$$

[24] We defined the color-correlated versions of $I_\pm$ operators $2I_\pm^{\rm (cc)} = \overline{I}_1^{\rm (cc)} \pm \left(\overline{I}_1^{\rm (cc)}\right)^\dagger$.

We note that the objects shown in eq. (5.18) are sufficient to compute the poles in the triple color-correlated contributions to Σ_N.

We can now proceed with the calculation of the commutators in eq. (5.14). Since they involve objects such as $[\boldsymbol{T}_k\cdot\boldsymbol{T}_l, \boldsymbol{T}_i\cdot\boldsymbol{T}_j]$, it is convenient to report the following general relation: given two operators A and B defined as

$$A = \sum_{(ij)}^{N_p} a_{ij}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\,, \qquad B = \sum_{(ij)}^{N_p} b_{ij}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\,, \tag{5.20}$$

where a_{ij} and b_{ij} are symmetric tensors,[25] their commutator reads

$$[A,B] = i\sum_{(ijk)}^{N_p}(a_{kj}+a_{jk})(b_{ij}+b_{ji})F^{(kij)}\,, \qquad F^{(kij)} = f_{abc}\,T_k^a\,T_i^b\,T_j^c\,. \tag{5.21}$$

Note that in the above equation, we introduced the handy notation (ijk) to label triplets with different i, j and k in the sum.

Eq. (5.21) can be used to compute the commutators in eq. (5.14), replacing $I_\pm$ with their color-correlated analogues $I_\pm^{(\mathrm{cc})}$, as discussed above. We find

$$\begin{aligned}\left[I_+^{(\mathrm{cc})}, I_-^{(\mathrm{cc})}\right] = &-\frac{\pi}{2}\sum_{(ijk)}^{N_p}F^{(kij)}\left[\frac{2L_{kj}\,\lambda_{ij}}{\epsilon^2}+\frac{\lambda_{ij}(\delta_{kj}^+ + \delta_{jk}^+)}{\epsilon}+\frac{L_{kj}(\delta_{ij}^- + \delta_{ji}^-)}{\epsilon}\right.\\ &\left.+2(\delta_{kj}^+ + \delta_{jk}^+)(\delta_{ij}^- + \delta_{ji}^-)\right] + \mathcal{O}(\epsilon)\,.\end{aligned} \tag{5.22}$$

The second commutator that we need is $[2I_+ + I_{\mathrm{S}}, I_-]$. To compute it, we extract the color-correlated contributions to I_{S}. Proceeding along the same lines as in the derivation of $I_\pm^{(\mathrm{cc})}$, we obtain

$$I_{\mathrm{S}}^{(\mathrm{cc})} = \sum_{(ij)}^{N_p}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left[\frac{\log(\eta_{ij})}{\epsilon}+\phi_{ij}\right]+\mathcal{O}(\epsilon)\,, \tag{5.23}$$

with

$$\phi_{ij} = -2\log\left(\frac{2E_{\mathrm{max}}}{\mu}\right)\log(\eta_{ij}) - \frac{1}{2}\log^2(\eta_{ij}) - \mathrm{Li}_2(1-\eta_{ij})\,. \tag{5.24}$$

Considering the expressions in eq. (5.18) and eq. (5.23), and following the discussion in appendix C, it is easy to show that the equation

$$2I_+^{(\mathrm{cc})} + I_{\mathrm{S}}^{(\mathrm{cc})} = \sum_{(ij)}^{N_p}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left(\delta_{ij}^+ + \phi_{ij}\right)+\mathcal{O}(\epsilon)\,, \tag{5.25}$$

holds. With this representation at hand, we can calculate the second color-correlated commutator required in eq. (5.15)

$$\begin{aligned}\left[2I_+^{(\mathrm{cc})} + I_{\mathrm{S}}^{(\mathrm{cc})}, I_-^{(\mathrm{cc})}\right] = &-\frac{\pi}{2}\sum_{(ijk)}^{N_p}F^{(kij)}\left[\frac{2\lambda_{ij}}{\epsilon}\left(\delta_{kj}^+ + \phi_{kj} + \delta_{jk}^+ + \phi_{jk}\right)\right.\\ &\left.+\left(\delta_{ij}^- + \delta_{ji}^-\right)\left(\delta_{kj}^+ + \phi_{kj} + \delta_{jk}^+ + \phi_{jk}\right)\right] + \mathcal{O}(\epsilon)\,.\end{aligned} \tag{5.26}$$

[25] If one starts with non-symmetric tensors, as is the case for the $\delta_{ij}^\pm$ functions, then it is clear that only their symmetric components will contribute to the sums of the type shown in eq. (5.20).

It remains to determine a suitable representation for the triple color-correlated part of the operator $\mathcal{H}_2$, which we denote as $\mathcal{H}_{2,\mathrm{tc}}$. According to ref. [73], one can write $\mathcal{H}_{2,\mathrm{tc}}$ as a commutator

$$\mathcal{H}_{2,\mathrm{tc}} = \frac{1}{2\epsilon}\left[\Gamma, C\right] , \tag{5.27}$$

where the two operators Γ and C are related to the ϵ-expansion of the $\overline{I}_1^{(\mathrm{cc})}$ operator

$$\overline{I}_1^{(\mathrm{cc})} = \frac{\Gamma}{\epsilon} + C + \mathcal{O}(\epsilon)\,. \tag{5.28}$$

Since $\overline{I}_1^{(\mathrm{cc})} = I_+ + I_-$, we easily obtain

$$\Gamma = \frac{1}{2}\sum_{(ij)} \boldsymbol{T}_i\cdot\boldsymbol{T}_j \left(L_{ij} + i\pi\lambda_{ij}\right), \qquad C = \frac{1}{2}\sum_{(ij)} \boldsymbol{T}_i\cdot\boldsymbol{T}_j \left(\delta_{ij}^{+} + i\pi\,\delta_{ij}^{-}\right). \tag{5.29}$$

Here, in analogy with eq. (5.21), we have used the shorthand notation (ij) to indicate that the sum runs over all possible pairs of distinct partons. It is then straightforward to compute the commutator of these two operators following the preceding discussion. The result reads

$$\mathcal{H}_{2,\mathrm{tc}} + \mathcal{H}_{2,\mathrm{tc}}^{\dagger} = -\frac{\pi}{2\epsilon}\sum_{(ijk)}^{N_p} F^{(kij)}\Big[L_{kj}(\delta_{ij}^{-} + \delta_{ji}^{-}) + \lambda_{kj}(\delta_{ij}^{+} + \delta_{ji}^{+})\Big]. \tag{5.30}$$

We can now combine the three triple color-correlated terms in eqs. (5.22), (5.26) and (5.30) to obtain the final expression the operator $I_{\mathrm{tri}}^{(\mathrm{cc})}$ of eq. (5.15), i.e.

$$I_{\mathrm{tri}}^{(\mathrm{cc})}(\epsilon) = \frac{\pi}{2}\sum_{(ijk)}^{N_p} F^{(kij)}\left[\frac{2L_{kj}\,\lambda_{ij}}{\epsilon^2} - \frac{4\phi_{jk}\lambda_{ij}}{\epsilon} + \left(\delta_{ij}^{-} + \delta_{ji}^{-}\right)\left(\delta_{kj}^{+} + \delta_{jk}^{+} - 2\phi_{jk}\right)\right], \tag{5.31}$$

where we have used $\phi_{jk} = \phi_{kj}$ and have omitted $\mathcal{O}(\epsilon)$ terms.

The calculation of Σ_N^{tri} requires us to compute $I_{\mathrm{tri}}^{\mathrm{RV}}$ up to finite terms in ϵ. Such a calculation is non-trivial; we describe it in appendix H. The result for $I_{\mathrm{tri}}^{\mathrm{RV}}$ is given in eq. (H.15). Once $I_{\mathrm{tri}}^{\mathrm{RV}}$ is computed, it is then possible to show that Σ_N^{tri} is free of ϵ-poles. To do that, we need to rewrite eq. (5.31) to make the role of the factors λ_{ij} clear. We recall that λ_{ij} are phase factors that distinguish between time-like and space-like processes. In fact, $\lambda_{ij} = 1$ if partons i and j are both either incoming or outgoing, and zero otherwise. Furthermore, important simplifications in eq. (5.31) occur because ϕ_{jk} and $L_{kj} = \log(\mu^2/s_{kj})$ are symmetric and $F_{\mathrm{LM}}^{(kij)}$ is antisymmetric with respect to $k \leftrightarrow j$ exchange. Thus, for a process with only outgoing (or only incoming) partons, we have $\lambda_{ij} = 1$ for all i, j and hence the triple color-correlated poles in eq. (5.31) vanish. A similar analysis shows that ϵ-poles in $I_{\mathrm{tri}}^{\mathrm{RV}}$ also vanish if all resolved partons are in the final state.

To understand what happens in processes where both incoming and outgoing partons are present, it is convenient to write λ_{ij} in the following way

$$\lambda_{ij} = 1 - \delta_{i1} - \delta_{i2} - \delta_{j1} - \delta_{j2} + 2\delta_{i1}\,\delta_{j2} + 2\delta_{i2}\,\delta_{j1}\,, \tag{5.32}$$

where 1 and 2 label the initial state partons. We have already argued that the first term on the right-hand side provides a vanishing contribution to eq. (5.31). Terms in eq. (5.32) that depend on the index i only also do not contribute since they do not break the $k \leftrightarrow j$ (anti)symmetry. The terms that depend on the index j also vanish. To see this, we write

$$\begin{aligned}\sum_{(ijk)} \langle \mathcal{M}|F^{(kij)} A_{kj} C_j|\mathcal{M}\rangle &= \sum_{(jk)}^{N_p} \sum_{i\neq j,k}^{N_p} \langle \mathcal{M}| f_{abc}\, A_{kj}\, C_j\, T_k^a T_j^c T_i^b |\mathcal{M}\rangle \\ &= -\sum_{(jk)}^{N_p} \langle \mathcal{M}| f_{abc}\, A_{kj}\, C_j\, T_k^a T_j^c (T_j^b + T_k^b)|\mathcal{M}\rangle \\ &= \frac{iC_A}{2} \sum_{(jk)}^{N_p} \langle \mathcal{M}| A_{kj} C_j \left(\boldsymbol{T}_k \cdot \boldsymbol{T}_j - \boldsymbol{T}_j \cdot \boldsymbol{T}_k\right)|\mathcal{M}\rangle = 0\,,\end{aligned} \tag{5.33}$$

where A_{kj} stands for L_{kj} or $\phi_{kj} + \phi_{jk}$. Furthermore, we have used color conservation

$$\sum_{i\neq j,k}^{N_p} T_i^b |\mathcal{M}\rangle = -(T_j^b + T_k^b)|\mathcal{M}\rangle\,, \tag{5.34}$$

to go from the first line to the second in eq. (5.33), and the SU(3) commutation relations for color charges in the next step.

Finally, we write $L_{jk} = \log(\mu/(2E_j)) + \log(\mu/(2E_k)) - \log(\eta_{jk})$. Using the same arguments as above, it is easy to show that the first two of these terms do not contribute to $I_{\rm tri}^{\rm (cc)}$. The only terms that remain include $\log(\eta_{jk})$ and the final two terms of eq. (5.32). Combining all these results, we finally arrive at an expression for the triple color-correlated poles

$$\begin{aligned}I_{\rm tri}^{\rm (cc)} = \sum_{k\neq 1,2}^{N_p} F^{(k12)} \Bigg[-\frac{2\pi}{\epsilon^2} \log\left(\frac{\eta_{2k}}{\eta_{1k}}\right) - \frac{2\pi}{\epsilon}\Bigg(2\log\left(\frac{4E_{\rm max}^2}{\mu^2}\right)\log\left(\frac{\eta_{1k}}{\eta_{2k}}\right) \\ + \log^2\eta_{1k} - \log^2\eta_{2k} + 2{\rm Li}(1-\eta_{1k}) - 2{\rm Li}(1-\eta_{2k})\Bigg)\Bigg] + \mathcal{O}(\epsilon^0)\,.\end{aligned} \tag{5.35}$$

Comparing this result with the expression for $I_{\rm tri}^{\rm RV}$ in eq. (H.15), we find that their poles are equal and opposite in sign.[26] This establishes the cancellation of ϵ-poles in triple color-correlated contributions for a generic $1_a + 2_b \to X + N\,g$ process.

5.3 Other color-correlated double-unresolved contributions

We continue with the discussion of divergent contributions to Σ_N that contain double and quartic color-correlated matrix elements squared with double-unresolved kinematics. As these contributions must involve either a loop amplitude or a soft limit, we are interested in those terms in eqs. ((4.86), (4.96), (4.102), (4.110), (4.120), (4.125)) that contain either $I_{\rm V}$ or $I_{\rm S}$ or both.

[26] One could equally well understand this as the $\mathcal{O}(\epsilon^{-2})$ poles cancelling between the commutator terms in $I_{\rm tri}^{\rm (cc)}$ (see eq. (5.15)) and $I_{\rm tri}^{\rm RV}$, leaving a simple pole which cancels against the contribution originating from the double-virtual amplitude, cf. eq. (5.30).

The sum of the elastic (i.e. unboosted) terms involving color correlations, which we denote as $\Sigma_N^{(\mathrm{V+S}),\mathrm{el}}$, reads

$$\begin{aligned}\Sigma_N^{(\mathrm{V+S}),\mathrm{el}} = {} & [\alpha_s]^2\frac{1}{2}\Big\langle \Big[I_\mathrm{V}^2+I_\mathrm{V}I_\mathrm{S}+I_\mathrm{S}I_\mathrm{V}+I_\mathrm{S}^2+2I_\mathrm{C}I_\mathrm{V}+2I_\mathrm{C}I_\mathrm{S}\Big]\cdot F_\mathrm{LM}\Big\rangle \\ & +[\alpha_s]^2\frac{\beta_0}{\epsilon}\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_\mathrm{E}}}\Big\langle\Big[-[I_\mathrm{S}(\epsilon)+I_\mathrm{V}(\epsilon)]+I_\mathrm{V}(2\epsilon)+\tilde{c}(\epsilon)\,\widetilde{I}_\mathrm{S}(2\epsilon)\Big]\cdot F_\mathrm{LM}\Big\rangle \\ & +[\alpha_s]^2\Big\langle\Big[K\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_\mathrm{E}}}\,I_\mathrm{V}(2\epsilon)+C_A\Big(\frac{c_1(\epsilon)}{\epsilon^2}-\frac{A_K(\epsilon)}{\epsilon^2}-2^{2+2\epsilon}\delta_g^{C_A}(\epsilon)\Big) \\ & \times\widetilde{I}_\mathrm{S}(2\epsilon)\Big]\cdot F_\mathrm{LM}\Big\rangle+[\alpha_s]\big\langle[I_\mathrm{V}(\epsilon)+I_\mathrm{S}(\epsilon)]\cdot F_\mathrm{LV}^\mathrm{fin}\big\rangle\,,\end{aligned} \tag{5.36}$$

where

$$\tilde{c}(\epsilon)=\frac{e^{\epsilon\gamma_\mathrm{E}}}{\Gamma(1-\epsilon)}\Big(c_2(\epsilon)+\epsilon\,c_3(\epsilon)-2^{2+2\epsilon}\,\delta_g^{\beta_0}(\epsilon)\Big)\,. \tag{5.37}$$

Before continuing, we recall that the soft and virtual operators I_S and I_V have *color-correlated* poles starting at $\mathcal{O}(\epsilon^{-1})$, while I_C does not contain any color-correlated terms and I_T is finite. It follows that the combination $I_\mathrm{V+S}=I_\mathrm{V}+I_\mathrm{S}=I_\mathrm{T}-I_\mathrm{C}$ contains color-correlated contributions starting at $\mathcal{O}(\epsilon^0)$.

Using these properties, it is easy to see that the first and last lines of eq. (5.36) do not contain divergent color-correlated contributions. Indeed, the sum of I-operators in the first line gives $I_\mathrm{T}^2-I_\mathrm{C}^2$, while the final line yields $I_\mathrm{V+S}$. Further details about this rearrangement and the origin of each term can be found in ref. [78]. We note that all quartic color correlations $\sim(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\,(\boldsymbol{T}_k\cdot\boldsymbol{T}_l)$ appear in the first line, so this demonstrates the complete cancellation of infrared singularities associated with this color structure.

We continue with the discussion of terms proportional to β_0 that appear in the second line of eq. (5.36). Here we can reconstruct two different versions of $I_\mathrm{V+S}$. Indeed, the first two terms in square brackets return $I_\mathrm{V+S}(\epsilon)$, while the third and fourth terms suggest that the combination $I_\mathrm{V+S}(2\epsilon)$ can be assembled. To do so, we add and subtract the soft operators $I_\mathrm{S}(2\epsilon)$ and $\widetilde{I}_\mathrm{S}(2\epsilon)$ such that

$$\begin{aligned}\Sigma_N^{(\mathrm{V+S}),\mathrm{el},\beta_0} & = [\alpha_s]^2\frac{\beta_0}{\epsilon}\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_\mathrm{E}}}\Big\langle\Big[-[I_\mathrm{S}(\epsilon)+I_\mathrm{V}(\epsilon)]+I_\mathrm{V}(2\epsilon)+\tilde{c}(\epsilon)\widetilde{I}_\mathrm{S}(2\epsilon)\Big]\cdot F_\mathrm{LM}\Big\rangle \\ & = [\alpha_s]^2\frac{\beta_0}{\epsilon}\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_\mathrm{E}}}\Big\langle\Big[-I_\mathrm{V+S}(\epsilon)+I_\mathrm{V+S}(2\epsilon)+\big(\tilde{c}(\epsilon)-1\big)\,\widetilde{I}_\mathrm{S}(2\epsilon) \\ & \quad +\widetilde{I}_\mathrm{S}(2\epsilon)-I_\mathrm{S}(2\epsilon)\Big]\cdot F_\mathrm{LM}\Big\rangle\,.\end{aligned} \tag{5.38}$$

We now argue that this contribution does not contain divergent color-correlated terms. First, since $I_\mathrm{V+S}(2\epsilon)$ and $I_\mathrm{V+S}(\epsilon)$ must coincide at $\mathcal{O}(\epsilon^0)$, the difference $I_\mathrm{V+S}(2\epsilon)-I_\mathrm{V+S}(\epsilon)$ contains color-correlated terms at $\mathcal{O}(\epsilon)$ only. Second, it is easy to check that

$$\tilde{c}(\epsilon)-1=\mathcal{O}(\epsilon^2)\,, \tag{5.39}$$

and since color-correlated terms in $\widetilde{I}_\mathrm{S}(2\epsilon)$ appear for the first time at order $\mathcal{O}(\epsilon^{-1})$, the third term in eq. (5.38) also does not give rise to color-correlated poles. Finally, as we have mentioned previously (cf. eq. (4.95)), the difference

$$\widetilde{I}_\mathrm{S}(2\epsilon)-I_\mathrm{S}(2\epsilon)=\mathcal{O}(\epsilon)\,, \tag{5.40}$$

which implies that the combination of the fourth and the fifth term in eq. (5.38) is also finite. Hence, we have proved that all terms proportional to β_0 in eq. (5.36) are free of divergent color-correlated contributions. Finally, for future purposes, it is convenient to introduce the following decomposition

$$\Sigma_N^{(\mathrm{V+S}),\mathrm{el},\beta_0} = [\alpha_s]^2 \frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\mathrm{E}}}} \left\langle \left[-I_{\mathrm{V+S}}(\epsilon) + I_{\mathrm{V+S}}(2\epsilon) + (\tilde{c}(\epsilon)-1)\, \widetilde{I}_{\mathrm{S}}(2\epsilon) \right] \cdot F_{\mathrm{LM}} \right\rangle \\ + \Sigma_N^{\mathrm{fin},(6)} , \tag{5.41}$$

where

$$\Sigma_N^{\mathrm{fin},(6)} = [\alpha_s]^2 \frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\mathrm{E}}}} \left\langle \left[\widetilde{I}_{\mathrm{S}}(2\epsilon) - I_{\mathrm{S}}(2\epsilon) \right] \cdot F_{\mathrm{LM}} \right\rangle . \tag{5.42}$$

The term in the third line in eq. (5.36) can be analyzed in a similar manner. We write

$$\begin{aligned} & K\, I_{\mathrm{V}}(2\epsilon) + C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon} \delta_g^{C_A}(\epsilon) \right) \widetilde{I}_{\mathrm{S}}(2\epsilon) \\ & = K\, I_{\mathrm{V+S}}(2\epsilon) + \left[C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon} \delta_g^{C_A}(\epsilon) \right) - K \right] \widetilde{I}_{\mathrm{S}}(2\epsilon) + K \left(\widetilde{I}_{\mathrm{S}}(2\epsilon) - I_{\mathrm{S}}(2\epsilon) \right) , \end{aligned} \tag{5.43}$$

where we dropped the factor $\Gamma(1-\epsilon)/e^{\epsilon\gamma_{\mathrm{E}}}$ as it contributes at $\mathcal{O}(\epsilon^0)$ only. We observe that the first and the third terms on the right-hand side of the above equation do not contain singular color-correlated terms for the reasons discussed above. The second term on the right-hand side in eq. (5.43) also does not contain divergent color-correlated contributions because

$$C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon} \delta_g^{C_A}(\epsilon) \right) - K = \mathcal{O}(\epsilon) . \tag{5.44}$$

This completes the analysis of the unboosted color-correlated contributions.

Additionally, there are boosted terms with color correlations in eqs. (4.110) and (4.120). It is straightforward to show that the sum of these terms assumes a particularly simple form

$$\Sigma_N^{(\mathrm{V+S}),\mathrm{boost}} = \frac{[\alpha_s]^2}{\epsilon} \langle \mathcal{P}_{aa}^{\mathrm{gen}} \otimes [I_{\mathrm{V+S}}(\epsilon) \cdot F_{\mathrm{LM}}] + [I_{\mathrm{V+S}}(\epsilon) \cdot F_{\mathrm{LM}}] \otimes \mathcal{P}_{bb}^{\mathrm{gen}} \rangle . \tag{5.45}$$

Given the properties of $I_{\mathrm{V+S}}(\epsilon)$ stated above, it is clear that $\Sigma_N^{(\mathrm{V+S}),\mathrm{boost}}$ contains color-correlated divergences at $\mathcal{O}(\epsilon^{-1})$. These divergences get canceled upon combining eq. (5.45) with similar contributions that arise as the result of the collinear renormalization of parton distribution functions. We briefly discuss this point at the end of section 5.4, after eq. (5.60).

Hence, the analysis performed in the current and previous sections proves the cancellation of *all* color-correlated divergent terms in a generic process $1_a + 2_b \to X + N\, g$. The remaining divergences in the double-unresolved contribution Σ_N are *not* color-correlated and, instead, are proportional to the squares of color charges of the external partons. These are related to collinear emissions and we continue with their analysis in the next section.

5.4 Collinear double-unresolved contributions

Having demonstrated the cancellation of poles in the color-correlated contributions to Σ_N in the previous two sections, we need to discuss the remaining terms in this quantity. Such terms are related to collinear emissions and, therefore, are proportional to the squares of color charges of the external hard partons. In this subsection we manipulate the corresponding contributions to eq. (4.74) in order to write them in terms of collinear operators $I_{\rm C}$ and splitting functions $\mathcal{P}_{ab}^{\rm gen}$. This will pave the way for demonstrating the cancellation of the poles, which we undertake in subsections 5.5.

The first term that we have yet to discuss is the last one in the third line of eq. (4.74). We find it convenient to split it into two pieces

$$\begin{aligned}&\frac{1}{2}\sum_{i,j=1}^{N_p}\left\langle \overline{S}_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}C_{j\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle\\&=\sum_{\substack{i,j=1\\i<j}}^{N_p}\left\langle \overline{S}_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}C_{j\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle+\frac{1}{2}\sum_{i=1}^{N_p}\left\langle \overline{S}_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}C_{i\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle .\end{aligned}\tag{5.46}$$

In the first term on the right-hand side of eq. (5.46) the unresolved partons $\mathfrak{m}$ and $\mathfrak{n}$ become collinear to two different resolved partons i and j, and we have used the symmetry of the limits to remove the factor $1/2$. In the second term in eq. (5.46) both $\mathfrak{m}$ and $\mathfrak{n}$ become collinear to the same parton i. It is straightforward to evaluate the first term since all we need to do is perform the NLO-like computation twice. The result reads

$$\begin{aligned}&\sum_{\substack{i,j=1\\i<j}}^{N_p}\left\langle \overline{S}_{\mathfrak{n}}\overline{S}_{\mathfrak{m}}C_{j\mathfrak{n}}C_{i\mathfrak{m}}\Delta^{(\mathfrak{mn})}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle\\&=\frac{[\alpha_s]^2}{\epsilon^2}\Bigg\{\frac{1}{2}\sum_{(ij)}^{N_p}\langle\Gamma_{i,f_i}\Gamma_{j,f_j}\cdot F_{\rm LM}\rangle+\langle\mathcal{P}_{aa}^{\rm gen}\otimes F_{\rm LM}\otimes\mathcal{P}_{bb}^{\rm gen}\rangle\\&\quad+\sum_{\substack{i=1\\i\neq 1}}^{N_p}\langle\mathcal{P}_{aa}^{\rm gen}\otimes[\Gamma_{i,f_i}\cdot F_{\rm LM}]\rangle+\sum_{\substack{i=1\\i\neq 2}}^{N_p}\langle[\Gamma_{i,f_i}\cdot F_{\rm LM}]\otimes\mathcal{P}_{bb}^{\rm gen}\rangle\Bigg\}.\end{aligned}\tag{5.47}$$

The last term in eq. (5.46) requires more care, as it involves a product of two operators that describe the soft-subtracted collinear limits of gluons $\mathfrak{m}$ and $\mathfrak{n}$ relative to the same hard parton. We would like to relate this contribution to the iteration of two collinear emissions and write it in terms of the functions $\mathcal{P}_{ab}^{\rm gen}$ and Γ_{i,f_i}, as done in eq. (5.47). It turns out that this is *nearly* possible but that the intertwined phase space of the two collinear gluons leads to one additional term when such a rewriting is performed. Indeed, for $i=1$ we find

$$\begin{aligned}\frac{1}{2}\left\langle \overline{S}_{\mathfrak{m}}\overline{S}_{\mathfrak{n}}C_{1\mathfrak{m}}C_{1\mathfrak{n}}\Delta^{(\mathfrak{mn})}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\right\rangle=&\frac{[\alpha_s]^2}{2\epsilon^2}\left\langle\Gamma_{1,a}^2\cdot F_{\rm LM}\right\rangle\\&+\frac{[\alpha_s]^2}{\epsilon^2}\langle\mathcal{P}_{aa}^{\rm gen}\otimes[\Gamma_{1,a}\cdot F_{\rm LM}]\rangle+\frac{[\alpha_s]^2}{2\epsilon^2}\langle[\mathcal{P}_{aa}^{\rm gen}\,\bar{\otimes}\,\mathcal{P}_{aa}^{\rm gen}]\otimes F_{\rm LM}\rangle\\&+\frac{[\alpha_s]^2}{2\epsilon^2}\langle G_1(z)\otimes F_{\rm LM}\rangle .\end{aligned}\tag{5.48}$$

The "bar"-convolution $[f\bar{\otimes}g]$ is defined as

$$[f(z_1,E_i)\,\bar{\otimes}\,g(z_2,E_i)](z,E_i)=\int_0^1{\rm d}z_1\,{\rm d}z_2\,f(z_1,E_i)g(z_2,z_1E_i)\delta(z-z_1z_2)\,.\tag{5.49}$$

The first three terms on the right-hand side of eq. (5.48) represent the "naive" product of two soft-subtracted collinear limits and the function G_1 incorporates the modifications required by the non-trivial dependence of the double-collinear phase space of two unresolved gluons on their energies. To obtain the results for $i = 2$ we can use eq. (5.48) and replace $i = 1$ with $i = 2$ and exchange "left" and "right" convolutions. The functions G_i read

$$G_i(z, E_i) = [\Gamma_{i,f_i} - \Gamma_{i,f_i}(z)]\, \mathcal{P}^{\rm gen}_{f_i f_i}(z, E_i)\,, \qquad i = 1, 2\,, \tag{5.50}$$

with

$$\Gamma_{i,f_i}(z) = \left(\frac{2zE_i}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\gamma_{f_i} + \boldsymbol{T}^2_{f_i} \frac{1 - e^{-2\epsilon L_{z\cdot i}}}{\epsilon}\right]. \tag{5.51}$$

In the above equation $L_{z\cdot i} = \log E_{\rm max}/(zE_i)$. A similar computation for the final-state parton i yields

$$\frac{1}{2}\langle \overline{S}_{\mathfrak{m}} \overline{S}_{\mathfrak{n}} C_{i\mathfrak{m}} C_{i\mathfrak{n}} \Delta^{(\mathfrak{mn})} F_{\rm LM}(\mathfrak{m}, \mathfrak{n})\rangle = \frac{[\alpha_s]^2}{2\epsilon^2} \langle \Gamma^2_{i,f_i} F_{\rm LM}\rangle + \frac{[\alpha_s]^2}{2\epsilon^2}\langle G_i F_{\rm LM}\rangle\,, \tag{5.52}$$

where

$$\begin{aligned} G_i = & \left[\left(\frac{2E_i}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^2 \left[\gamma^{22}_{z,g\to gg}(\epsilon, L_i) + \frac{\boldsymbol{T}^2_g}{\epsilon} e^{-2\epsilon L_i}\right] \\ & \times \left[\gamma^{42}_{z,g\to gg}(\epsilon, L_i) - \gamma^{22}_{z,g\to gg}(\epsilon, L_i)\right], \end{aligned} \tag{5.53}$$

and $i = 3, \ldots, N_p$. Combining eqs. (5.47), (5.48) and (5.52) and summing over the final-state partons, we find the following result for the last term in the third line of eq. (4.74)

$$\begin{aligned} & \frac{1}{2}\sum_{i,j=1}^{N_p} \left\langle \overline{S}_{\mathfrak{m}} \overline{S}_{\mathfrak{n}} C_{i\mathfrak{m}} C_{j\mathfrak{n}} \Delta^{(\mathfrak{mn})} F_{\rm LM}(\mathfrak{m}, \mathfrak{n})\right\rangle \\ & = [\alpha_s]^2 \left\{ \frac{1}{2}\left\langle I^2_{\rm C}(\epsilon)\cdot F_{\rm LM}\right\rangle \right. \\ & \quad + \frac{1}{2\epsilon^2}\sum_{i=3}^{N_p} \langle G_i F_{\rm LM}\rangle + \frac{1}{\epsilon}\left[\langle \mathcal{P}^{\rm gen}_{aa} \otimes [I_{\rm C}(\epsilon)\cdot F_{\rm LM}]\rangle + \langle [I_{\rm C}(\epsilon)\cdot F_{\rm LM}] \otimes \mathcal{P}^{\rm gen}_{bb}\rangle\right] \\ & \quad + \frac{1}{2\epsilon^2}\left(\langle [\mathcal{P}^{\rm gen}_{aa} \bar{\otimes} \mathcal{P}^{\rm gen}_{aa}] \otimes F_{\rm LM}\rangle + \langle F_{\rm LM} \otimes [\mathcal{P}^{\rm gen}_{bb} \bar{\otimes} \mathcal{P}^{\rm gen}_{bb}]\rangle\right) \\ & \quad \left. + \frac{1}{2\epsilon^2}\left[\langle G_1 \otimes F_{\rm LM}\rangle + \langle F_{\rm LM} \otimes G_2\rangle\right] + \frac{1}{\epsilon^2}\langle \mathcal{P}^{\rm gen}_{aa} \otimes F_{\rm LM} \otimes \mathcal{P}^{\rm gen}_{bb}\rangle \right\}. \end{aligned} \tag{5.54}$$

As we will show in the next subsection, the above equation is already in a suitable form to discuss the cancellation of some $1/\epsilon$ collinear contributions to Σ_N.

We now briefly discuss the terms in the fourth and fifth lines of eq. (4.74). The term in the fourth line contains two soft-subtracted collinear operators $C_{i\mathfrak{n}} C_{i\mathfrak{m}}$ and a factor $[2(\eta_{i\mathfrak{n}}/2)^{-2\epsilon} - 1]$. The two soft-subtracted collinear limits produce an $\mathcal{O}(\epsilon^{-2})$ term but the prefactor is arranged in such a way that the actual singularity is just $\mathcal{O}(\epsilon^{-1})$. In what follows we will mostly focus on the cancellation of $1/\epsilon^2$ collinear singularities and for this reason we do not need to discuss how this term can be rewritten. Furthermore, the term on the fifth line includes a commutator of the limits $C_{i\mathfrak{m}}$ and $C_{i\mathfrak{n}}$. Since we consider final-state gluons only, this contribution is identically zero for the purposes of this paper. However, we note that this term would no longer vanish when one considers processes with both quarks and gluons in the final state. The only term in eq. (4.74) that we have yet to consider is the one on the penultimate line, which originates from the soft-regulated double collinear limits in sectors (b) and (d). The first part of the computation proceeds similarly to the NLO case, and results in

$$\begin{aligned}&\Big\langle C_{\mathfrak{mn}}\Big[F_{\rm LM}(\mathfrak{m},\mathfrak{n})-2S_{\mathfrak{n}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big]\Big\rangle\\&=\frac{[\alpha_s]}{\epsilon}\left(\frac{2E_{\mathfrak{m}}}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\Big\langle 2\,\gamma^{22}_{z,g\to gg}(\epsilon)F_{\rm LM}(\mathfrak{m})\Big\rangle\,,\end{aligned}\tag{5.55}$$

where $\gamma^{22}_{z,g\to gg}(\epsilon)=\gamma^{22}_{z,g\to gg}(\epsilon,L_i=0)$ and we have renamed the clustered parton $[\mathfrak{mn}]\to\mathfrak{m}$.[27] To complete the calculation, we need to evaluate the soft-regulated collinear limit $\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}$. Recalling that $\sigma_{ij}=\eta_{ij}/(1-\eta_{ij})$, we find $C_{i\mathfrak{m}}\sigma_{i\mathfrak{m}}^{-\epsilon}=\eta_{i\mathfrak{m}}^{-\epsilon}$ and obtain

$$\begin{aligned}&\sum_{i=1}^{N_p}\frac{N_{\mathfrak{m}\|\mathfrak{n}}(\epsilon)}{2}\Big\langle\overline{S}_{\mathfrak{m}}C_{i\mathfrak{m}}\sigma_{i\mathfrak{m}}^{-\epsilon}\Delta^{(\mathfrak{m})}C_{\mathfrak{mn}}\Big[F_{\rm LM}(\mathfrak{m},\mathfrak{n})-2S_{\mathfrak{n}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\Big]\Big\rangle\\&=\frac{[\alpha_s]^2}{\epsilon}N_{sc}^{(b,d)}\langle\gamma^{22}_{z,g\to gg}(\epsilon)\big[I_{\rm C}^{(4)}(\epsilon)\cdot F_{\rm LM}\big]\rangle\\&\quad+\frac{[\alpha_s]^2}{2\epsilon^2}N_{sc}^{(b,d)}\Big\langle\gamma^{22}_{z,g\to gg}(\epsilon)\Big[\mathcal{P}^{(4),{\rm gen}}_{aa}\otimes F_{\rm LM}+F_{\rm LM}\otimes\mathcal{P}^{(4),{\rm gen}}_{bb}\Big]\Big\rangle\,,\end{aligned}\tag{5.56}$$

where $I_{\rm C}^{(4)}$ and $\mathcal{P}^{(4),{\rm gen}}_{ab}$ are defined in eqs. (4.121) and (4.123), respectively, and the normalization constants are collected in appendix A.1.

This concludes the discussion of the collinear contributions to Σ_N; through $\mathcal{O}(\epsilon^{-2})$ they are given by the sum of eqs. (5.54) and (5.56). In addition to these terms, there are also remnants of virtual and soft contributions that are not color correlated. All these terms will have to be combined together with the collinear renormalizations of parton distribution functions to demonstrate the cancellation of singularities.

Before discussing the details of this cancellation, we will write down the term in eq. (4.1) that arises from the collinear renormalization of the parton distribution functions at $\mathcal{O}(\alpha_s^2)$. It reads

$$\begin{aligned}{\rm d}\hat\sigma^{\rm pdf}_{ab}=&\left[\frac{\alpha_s(\mu)}{2\pi}\right]\sum_x\Big[\hat{\mathbb{P}}_{1,xa}\otimes{\rm d}\bar\sigma^{\rm NLO}_{xb}+{\rm d}\bar\sigma^{\rm NLO}_{ax}\otimes\hat{\mathbb{P}}_{1,xb}\Big]\\&+\left[\frac{\alpha_s(\mu)}{2\pi}\right]^2\sum_{x,y}\Big[\hat{\mathbb{P}}_{1,xa}\otimes{\rm d}\bar\sigma^{\rm LO}_{xy}\otimes\hat{\mathbb{P}}_{1,yb}+\hat{\mathbb{P}}_{2,xa}\otimes{\rm d}\bar\sigma^{\rm LO}_{xb}+{\rm d}\bar\sigma^{\rm LO}_{ax}\otimes\hat{\mathbb{P}}_{2,xb}\Big]\,,\end{aligned}\tag{5.57}$$

where we have used the following short-hand notation

$$\hat{\mathbb{P}}_{1,ab}(z)=\frac{\hat P^{(0)}_{ab}(z)}{\epsilon}\,,\qquad\hat{\mathbb{P}}_{2,ab}(z)=\frac{[\hat P^{(0)}_{ax}\otimes\hat P^{(0)}_{xb}](z)-\beta_0\hat P^{(0)}_{ab}(z)}{2\epsilon^2}+\frac{\hat P^{(1)}_{ab}(z)}{2\epsilon}\,.\tag{5.58}$$

We note that at variance with eq. (3.35), ${\rm d}\bar\sigma^{\rm NLO}$ does not include the PDFs renormalization. Furthermore, the summation in eq. (5.57) is performed over all initial-state parton flavors.

[27]We are free to do so because both $\mathfrak{m}$ and $\mathfrak{n}$ are gluons, and hence the clustered parton $[\mathfrak{mn}]$ is also a gluon.

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However, since we consider processes with $q\bar{q}$ initial states and gluonic final states, the Altarelli-Parisi splitting functions always have identical indices. We can therefore write the contribution from the PDFs renormalization as follows

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}} = {} & \left[\frac{\alpha_s(\mu)}{2\pi}\right]\left[\frac{\hat{P}_{aa}^{(0)}\otimes \mathrm{d}\bar{\sigma}_{ab}^{\mathrm{NLO}}}{\epsilon}+\frac{\mathrm{d}\bar{\sigma}_{ab}^{\mathrm{NLO}}\otimes \hat{P}_{bb}^{(0)}}{\epsilon}\right] \\
& +\left[\frac{\alpha_s(\mu)}{2\pi}\right]^2\left\{\left[\frac{\hat{P}_{aa}^{(0)}\otimes \hat{P}_{aa}^{(0)}-\beta_0\hat{P}_{aa}^{(0)}}{2\epsilon^2}-\frac{\hat{P}_{aa}^{(1)}}{2\epsilon}\right]\otimes\langle F_{\mathrm{LM}}\rangle\right. \\
& \left.+\langle F_{\mathrm{LM}}\rangle\otimes\left[\frac{\hat{P}_{bb}^{(0)}\otimes \hat{P}_{bb}^{(0)}-\beta_0\hat{P}_{bb}^{(0)}}{2\epsilon^2}-\frac{\hat{P}_{bb}^{(1)}}{2\epsilon}\right]+\frac{\hat{P}_{aa}^{(0)}\otimes\langle F_{\mathrm{LM}}\rangle\otimes\hat{P}_{bb}^{(0)}}{\epsilon^2}\right\}.
\end{aligned}
\tag{5.59}
$$

The NLO cross section $\mathrm{d}\bar{\sigma}_{ab}^{\mathrm{NLO}}$ can be obtained from the results of section 3 and reads

$$
\begin{aligned}
\mathrm{d}\bar{\sigma}_{ab}^{\mathrm{NLO}} = {} & [\alpha_s]\langle I_{\mathrm{T}}(\epsilon)\cdot F_{\mathrm{LM}}\rangle+\langle F_{\mathrm{LV}}^{\mathrm{fin}}\rangle+\frac{[\alpha_s]}{\epsilon}\left[\langle\mathcal{P}_{aa}^{\mathrm{gen}}\otimes F_{\mathrm{LM}}\rangle+\langle F_{\mathrm{LM}}\otimes\mathcal{P}_{bb}^{\mathrm{gen}}\rangle\right] \\
& +\left\langle\mathcal{O}_{\mathrm{NLO}}\,\Delta^{(\mathfrak{m})}F_{\mathrm{LM}}(\mathfrak{m})\right\rangle .
\end{aligned}
\tag{5.60}
$$

As mentioned earlier, we do not include the $\mathcal{O}(\alpha_s)$ contribution of the collinear renormalization of PDFs in the definition of $\mathrm{d}\bar{\sigma}_{ab}^{\mathrm{NLO}}$, and therefore this quantity still contains unsubtracted hard-collinear poles. We also note that we already used the convolution of the Altarelli-Parisi splitting function with the $\mathcal{O}_{\mathrm{NLO}}$ term in eq. (5.60) to cancel ϵ-poles in single-unresolved contributions to $\mathrm{d}\hat{\sigma}_{N+1}^{\mathrm{NNLO}}$ shown in eq. (5.8).

Before continuing with the discussion of the double-unresolved collinear contributions, we can use eq. (5.59) to complete the demonstration of the cancellation of the color-correlated divergences, see the discussion after eq. (5.45). We note that terms in the first line of eq. (5.59) contain divergent contributions that involve a convolution of a splitting function and a next-to-leading order cross section. The latter contains the I_{T} operator which has color-correlated terms at $\mathcal{O}(\epsilon^0)$. These terms are identical to those that appear in the operator $I_{\mathrm{V+S}}$ in eq. (5.45). Using the relation between $\mathcal{P}_{ab}^{\mathrm{gen}}$ and $\hat{P}_{ab}^{(0)}$ shown in eq. (3.38), it is easy to check that the color-correlated contribution to the ϵ-poles cancel when eq. (5.45) and the first line of eq. (5.59) are combined.

5.5 Pole cancellation in double-unresolved color-uncorrelated contributions

We are now in the position to discuss the double-unresolved terms that are free of color correlations. These terms must be collected from eqs. ((4.86), (4.96), (4.102), (4.110), (4.120), (4.125), (5.54), (5.56)) and eq. (5.59). They include terms with double-boosted kinematics (db), terms with a single boost from either the right (rb) or the left (lb), as well as elastic terms (el). We discuss these contributions separately. We write

$$
\Sigma_N^{\mathrm{coll}} = \Sigma_N^{\mathrm{c,el}}+\Sigma_N^{\mathrm{lb}}+\Sigma_N^{\mathrm{rb}}+\Sigma_N^{\mathrm{db}}\,,
\tag{5.61}
$$

where the superscript "c" emphasizes that the first term on the right-hand side originates from collinear limits. We begin by considering the double-boosted term, which only receives contributions from the double-collinear limits in eq. (5.54) and the PDFs renormalization in eq. (5.59). Their sum reads

$$\Sigma_N^{\rm db} = \frac{[\alpha_s]^2}{\epsilon^2}\langle \mathcal{P}_{aa}^{\rm gen}\otimes F_{\rm LM}\otimes \mathcal{P}_{bb}^{\rm gen}\rangle + \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2\frac{1}{\epsilon^2}\langle \hat{P}_{aa}^{(0)}\otimes F_{\rm LM}\otimes \hat{P}_{bb}^{(0)}\rangle \\ + \frac{[\alpha_s]}{\epsilon^2}\left[\frac{\alpha_s(\mu)}{2\pi}\right]\Big[\langle \hat{P}_{aa}^{(0)}\otimes F_{\rm LM}\otimes \mathcal{P}_{bb}^{\rm gen}\rangle + \langle \mathcal{P}_{aa}^{\rm gen}\otimes F_{\rm LM}\otimes \hat{P}_{bb}^{(0)}\rangle\Big]. \tag{5.62}$$

Using the expansion

$$\mathcal{P}_{aa}^{\rm gen} = -\hat{P}_{aa}^{(0)} + \epsilon\,\mathcal{P}_{aa}^{\rm NLO} + \mathcal{O}(\epsilon^2)\,, \tag{5.63}$$

and the fact that $\alpha_s(\mu)/(2\pi) = [\alpha_s] + \mathcal{O}(\epsilon^2)$, we can simplify the expression of $\Sigma_N^{\rm db}$ and find

$$\Sigma_N^{\rm db} = [\alpha_s]^2\,\langle \mathcal{P}_{aa}^{\rm NLO}\otimes F_{\rm LM}\otimes \mathcal{P}_{bb}^{\rm NLO}\rangle\,, \tag{5.64}$$

which is finite in ϵ.

We continue with the single-boosted terms, and demonstrate the pole cancellation up to $\mathcal{O}(\epsilon^{-1})$. Focusing on the left boost, i.e. the boost applied to the initial-state parton with momentum p_1, and combining selected contributions from eqs. ((4.110), (4.120), (5.54), (5.56)) and eq. (5.59), we obtain the following result

$$\Sigma_N^{\rm lb} = [\alpha_s]^2\left\langle \mathcal{P}_{aa}^{\rm NLO}\otimes[I_{\rm T}(\epsilon)\cdot F_{\rm LM}]\right\rangle + [\alpha_s]\langle \mathcal{P}_{aa}^{\rm NLO}\otimes F_{\rm LV}^{\rm fin}\rangle \\ + \frac{1}{2\epsilon^2}\left\langle\left\{[\alpha_s]^2[\mathcal{P}_{aa}^{\rm gen}\,\bar{\otimes}\,\mathcal{P}_{aa}^{\rm gen}] + \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2[\hat{P}_{aa}^{(0)}\otimes\hat{P}_{aa}^{(0)}]\right.\right. \\ \left.\left. + 2[\alpha_s]\left[\frac{\alpha_s(\mu)}{2\pi}\right][\hat{P}_{aa}^{(0)}\,\bar{\otimes}\,\mathcal{P}_{aa}^{\rm gen}]\right\}\otimes F_{\rm LM}\right\rangle \\ + \frac{[\alpha_s]^2}{2\epsilon^3}\left\langle\left[C_A h_c(\epsilon)\left(\mathcal{P}_{aa}^{(4),\rm gen} - \mathcal{P}_{aa}^{\rm 1L,gen}\right) + \epsilon\,G_1\right]\otimes F_{\rm LM}\right\rangle \\ - \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2\frac{\beta_0}{2\epsilon^2}\langle \hat{P}_{aa}^{(0)}\otimes F_{\rm LM}\rangle - [\alpha_s]\left[\frac{\alpha_s(\mu)}{2\pi}\right]\frac{\beta_0}{\epsilon^2}\langle \mathcal{P}_{aa}^{\rm gen}\otimes F_{\rm LM}\rangle \\ + \frac{[\alpha_s]^2}{2\epsilon^2}N_{sc}^{(b,d)}\left\langle \gamma_{z,g\to gg}^{22}(\epsilon)\mathcal{P}_{aa}^{(4),\rm gen}\otimes F_{\rm LM}\right\rangle\,, \tag{5.65}$$

where we have dropped irrelevant $\mathcal{O}(\epsilon)$ terms in the first line, and we used the bar-convolution, defined in eq. (5.49).

The two terms on the first line of eq. (5.65) are clearly finite in ϵ. As for the sum of the second and third lines, we recall that

$$\mathcal{P}_{aa}^{(k),\rm gen} = -\hat{P}_{aa}^{(0)} + \mathcal{O}(\epsilon)\,, \tag{5.66}$$

and hence

$$[\alpha_s]^2[\mathcal{P}_{aa}^{\rm gen}\,\bar{\otimes}\,\mathcal{P}_{aa}^{\rm gen}] = [\alpha_s]^2[\hat{P}_{aa}^{(0)}\bar{\otimes}\hat{P}_{aa}^{(0)}] + \mathcal{O}(\epsilon)\,, \\ 2[\alpha_s]\left[\frac{\alpha_s(\mu)}{2\pi}\right][\hat{P}_{aa}^{(0)}\,\bar{\otimes}\,\mathcal{P}_{aa}^{\rm gen}] = -2[\alpha_s]^2[\hat{P}_{aa}^{(0)}\bar{\otimes}\hat{P}_{aa}^{(0)}] + \mathcal{O}(\epsilon)\,. \tag{5.67}$$

The two convolutions of Altarelli-Parisi splitting functions are related by

$$\left[\frac{\alpha_s(\mu)}{2\pi}\right]^2[\hat{P}_{aa}^{(0)}\otimes\hat{P}_{aa}^{(0)}](z) - [\alpha_s]^2[\hat{P}_{aa}^{(0)}\bar{\otimes}\hat{P}_{aa}^{(0)}](z) \\ = [\alpha_s]^2\int_0^1 {\rm d}z_1\,{\rm d}z_2\,(1-z_1^{-2\epsilon})\,\hat{P}_{aa}^{(0)}(z_1)\,\hat{P}_{aa}^{(0)}(z_2)\,\delta(z-z_1z_2) + \mathcal{O}(\epsilon) \equiv \mathcal{O}(\epsilon)\,. \tag{5.68}$$

It follows that the $\mathcal{O}(\epsilon^{-2})$ poles on the second and third lines of eq. (5.65) vanish.

To discuss the cancellation of the poles in the fourth line on the right-hand side of eq. (5.65) we require the functions $P_{aa}^{1\mathrm{L,gen}}$, $P_{aa}^{(4)\mathrm{,gen}}$, and G_1. These quantities are defined in eqs. (4.109), (4.123) and (5.50), respectively. Using both eq. (5.66) and the relation

$$\mathcal{P}_{aa}^{1\mathrm{L,gen}} = -\hat{P}_{aa}^{(0)} + \mathcal{O}(\epsilon)\,, \tag{5.69}$$

we see that $\mathcal{O}(\epsilon^{-3})$ poles disappears. Furthermore, using

$$\begin{aligned} &h_c(\epsilon)\Big(\mathcal{P}_{aa}^{(4)\mathrm{,gen}}(z,E_1) - \mathcal{P}_{aa}^{1\mathrm{L,gen}}(z,E_1)\Big) = 2\epsilon\log(z)\hat{P}_{aa}^{(0)}(z) + \mathcal{O}(\epsilon^2)\,,\\ &\epsilon\, G_i(z,E_1) = -2\epsilon\log(z)\hat{P}_{aa}^{(0)}(z) + \mathcal{O}(\epsilon^2)\,, \end{aligned} \tag{5.70}$$

we observe the cancellation of $\mathcal{O}(\epsilon^{-2})$ poles in the fourth line of eq. (5.65). The cancellation of the $\mathcal{O}(\epsilon^{-2})$ poles in the last two lines of eq. (5.65) follows from the expansions $\gamma_{z,g\to gg}^{22}(\epsilon) = 11/6\,C_A + \mathcal{O}(\epsilon)$ and $N_{sc}^{(b,d)} = 1 + \mathcal{O}(\epsilon)$, and recalling that $\beta_0 = 11/6\,C_A$ in our setup. Demonstrating the complete cancellation of the single poles takes more effort. We comment on this point at the end of this section.

Finally, we discuss the pole cancellation in elastic terms. We begin by summing terms that arise from hard-collinear limits and that do not involve contributions from virtual loops. These terms can be found in eqs. (4.110), (4.120), (5.54) and (5.56). The result reads

$$\begin{aligned} \Sigma_N^{\mathrm{c,el}} = [\alpha_s]^2 \bigg\{ &\bigg\langle \bigg[\frac{I_{\mathrm{C}}^2(\epsilon)}{2} - \frac{\beta_0}{\epsilon}\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\mathrm{E}}}} I_{\mathrm{C}}(\epsilon) \bigg] \cdot F_{\mathrm{LM}} \bigg\rangle \\ &+ \frac{1}{\epsilon^2} \bigg\langle \bigg[C_A h_c(\epsilon) \left(I_{\mathrm{C}}^{(4)}(\epsilon) - \widetilde{I}_{\mathrm{C}}(2\epsilon) \right) + \frac{1}{2}\sum_{i=3}^{N_p} G_i \bigg] \cdot F_{\mathrm{LM}} \bigg\rangle \\ &+ \frac{1}{\epsilon} \Big\langle N_{sc}^{(b,d)} \Big[\gamma_{z,g\to gg}^{22}(\epsilon) I_{\mathrm{C}}^{(4)}(\epsilon) \Big] \cdot F_{\mathrm{LM}} \Big\rangle \bigg\} \\ &+ [\alpha_s] \langle I_{\mathrm{C}}(\epsilon) \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \rangle\,. \end{aligned} \tag{5.71}$$

In eq. (5.36) we defined the color-correlated component of the elastic term $\Sigma_N^{(\mathrm{V+S}),\mathrm{el}}$, and in the discussion that followed we demonstrated that the color-correlated poles vanish. However, this still left color-uncorrelated poles in $\Sigma_N^{(\mathrm{V+S}),\mathrm{el}}$, starting at $\mathcal{O}(\epsilon^{-2})$. Combining this term with $\Sigma_N^{\mathrm{c,el}}$ we find

$$\begin{aligned} \Sigma_N^{(\mathrm{V+S}),\mathrm{el}} + \Sigma_N^{\mathrm{c,el}} = [\alpha_s]\langle I_{\mathrm{T}}(\epsilon)\cdot F_{\mathrm{LV}}^{\mathrm{fin}}\rangle + [\alpha_s]^2 \bigg\{ &\frac{1}{2}\langle I_{\mathrm{T}}^2(\epsilon)\cdot F_{\mathrm{LM}}\rangle + K\langle I_{\mathrm{T}}(2\epsilon)\cdot F_{\mathrm{LM}}\rangle \\ &+ \frac{\beta_0}{\epsilon}\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\mathrm{E}}}} \Big\langle \Big(I_{\mathrm{T}}(2\epsilon) - I_{\mathrm{T}}(\epsilon) \Big) \cdot F_{\mathrm{LM}} \Big\rangle \\ &+ \frac{\beta_0}{\epsilon}\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\mathrm{E}}}} \Big\langle (\tilde{c}(\epsilon) - 1)\, \widetilde{I}_{\mathrm{S}}(2\epsilon)\cdot F_{\mathrm{LM}} \Big\rangle + \Sigma_N^{\mathrm{fin},(6)} \\ &+ \bigg\langle \bigg[C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon} - 2^{2+2\epsilon}\delta_g^{C_A}(\epsilon) \right) - K \bigg] \widetilde{I}_{\mathrm{S}}(2\epsilon)\cdot F_{\mathrm{LM}} \bigg\rangle \\ &+ \Big\langle K \left(\widetilde{I}_{\mathrm{S}}(2\epsilon) - I_{\mathrm{S}}(2\epsilon) \right) \cdot F_{\mathrm{LM}} \Big\rangle \\ &+ \frac{1}{\epsilon^2} \bigg\langle \bigg[C_A h_c(\epsilon) \left(I_{\mathrm{C}}^{(4)}(\epsilon) - \widetilde{I}_{\mathrm{C}}(2\epsilon) \right) - K\epsilon^2 I_{\mathrm{C}}(2\epsilon) + \frac{1}{2}\sum_{i=3}^{N_p} G_i \bigg] \cdot F_{\mathrm{LM}} \bigg\rangle \\ &+ \frac{1}{\epsilon} \bigg\langle \bigg[N_{sc}^{(b,d)} \gamma_{z,g\to gg}^{22}(\epsilon) I_{\mathrm{C}}^{(4)}(\epsilon) - \beta_0 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\mathrm{E}}}} I_{\mathrm{C}}(2\epsilon) \bigg] \cdot F_{\mathrm{LM}} \bigg\rangle \bigg\}\,. \end{aligned} \tag{5.72}$$

The terms in the first line are manifestly finite. We explained in section 5.3 that $I_{\rm T}(2\epsilon) - I_{\rm T}(\epsilon) = \mathcal{O}(\epsilon)$; thus the second line is finite as well. The third, fourth, and fifth lines give rise to $\mathcal{O}(\epsilon^{-1})$ poles only; this follows from the fact that the highest pole in $\widetilde{I}_{\rm S}$ is $\mathcal{O}(\epsilon^{-2})$, but the coefficients of $\widetilde{I}_{\rm S}$ suppress this singularity as can be seen by using eqs. (5.39), (5.40) and (5.44). Likewise, the fifth line contains poles of $\mathcal{O}(\epsilon^{-1})$, since

$$C_A h_c(\epsilon)\left(I_{\rm C}^{(4)}(\epsilon) - \widetilde{I}_{\rm C}(2\epsilon)\right) = \sum_{i=3}^{N_p} C_A\,\boldsymbol{T}_i^2\left(-\frac{65}{72} + \frac{\pi^2}{3}\right) + \mathcal{O}(\epsilon)\,, \tag{5.73}$$

and

$$G_i = 2C_A\,\boldsymbol{T}_i^2\left(\frac{65}{72} - \frac{\pi^2}{3}\right) + \mathcal{O}(\epsilon)\,, \tag{5.74}$$

while $\epsilon^2 I_{\rm C}(2\epsilon) = \mathcal{O}(\epsilon)$. Finally, using eq. (4.124) and expansions already employed in this section, we can easily check that the last line of eq. (5.72) contains $\mathcal{O}(\epsilon^{-1})$ poles only.

At this point, it is useful to review what we have accomplished regarding the double-unresolved contributions. In sections 5.2 and 5.3, we have combined contributions of soft limits of real-emission amplitudes and contributions of loop amplitudes to demonstrate the cancellation of all ϵ-poles that contain correlators of color-charge operators. We are then left with ϵ-poles proportional to squares of the color charges of the external partons. In this section, we combined these remaining divergences with the ones from hard-collinear limits and showed that all poles multiplying double-boosted matrix elements vanish, and that poles multiplying single-boosted and elastic contributions vanish up to $\mathcal{O}(\epsilon^{-1})$.

We have done this by combining structures that emerge from virtual, soft, and collinear singularities into finite operators such as $I_{\rm T}$, or, where this has not been possible, we have used simple relationships between the ϵ-expansions of the various operators. This dramatically simplifies the cancellation of the singularities. As a result we are able to demonstrate the cancellation of poles without resorting to excessive evaluations of multiple singular terms, which would have been needed had we followed the approach of refs. [1, 61].

In order to investigate how the remaining $\mathcal{O}(\epsilon^{-1})$ color-uncorrelated poles cancel, we need to consider the $\mathcal{O}(\epsilon^{-1})$ terms from eqs. (5.65) and (5.72), the triple-collinear and spin-correlated terms $\Sigma_N^{(2)}$ and $\Sigma_N^{(8)}$ in eq. (4.74), the term in the fourth line in eq. (4.74), and the contribution from the NLO Altarelli-Parisi kernel $\hat{P}_{qq}^{(1)}$ in the collinear renormalization of parton distribution functions.[28] Although it should be possible to organize the cancellation of the remaining $1/\epsilon$ terms following what has been done for higher poles, it becomes much more cumbersome to do so. For this reason, we simply note that the cancellation of the remaining $\mathcal{O}(\epsilon^{-1})$ poles has been checked by means of a straightforward, but tedious, algebraic computation. We emphasize that this computation is done with N as a parameter, and thus holds for an *arbitrary* number of gluons. Everything that is needed to confirm this cancellation is provided in the main body of this paper and the relevant appendices.

Having cancelled all the poles, we can take the $\epsilon \to 0$ limit and obtain a finite result for the NNLO contribution to the cross section $\mathrm{d}\hat{\sigma}_{q\bar{q}}^{\rm NNLO}$ for the process $1_q + 2_{\bar{q}} \to X + Ng$. We present this result in the following section.

[28] Since we consider gluonic final states only, we need to remove the contribution of final state quarks from $P_{qq}^{(1)}$. The resulting expression $\hat{P}_{qq,\widetilde{\rm NS}}^{(1)}$ is shown in eq. (A.20).

6 Final result

In this section we present a formula for the *finite* NNLO QCD contribution $\mathrm{d}\hat{\sigma}_{q\bar{q}}^{\mathrm{NNLO}}$ to the partonic cross section of the process $1_q + 2_{\bar{q}} \to X + Ng$. This formula is the main result of this paper. As explained in the preceding sections, we arrive at this result by considering double-real, double-virtual, real-virtual and PDF-renormalization contributions to $\mathrm{d}\hat{\sigma}_{q\bar{q}}^{\mathrm{NNLO}}$ and manipulating them to remove all singularities without impacting the fully-differential nature of the result. An important feature of our approach is the organization of the subtraction terms into iterations of NLO-like structures, which allows us to ameliorate the proliferation of subtraction terms that plagues NNLO calculations. As a result, the NNLO remainder can be written in a very compact form.

We split $\mathrm{d}\hat{\sigma}_{q\bar{q}}^{\mathrm{NNLO}}$ into contributions with $N+2$, $N+1$ and N resolved final-state partons (cf. eq. (4.11)) and write

$$\mathrm{d}\hat{\sigma}_{q\bar{q}}^{\mathrm{NNLO}} = \mathrm{d}\hat{\sigma}_{N+2}^{\mathrm{NNLO}} + \mathrm{d}\hat{\sigma}_{N+1}^{\mathrm{NNLO}} + \mathrm{d}\hat{\sigma}_{N}^{\mathrm{NNLO}}. \tag{6.1}$$

The first term on the right-hand side is the finite, fully-regulated contribution given in eq. (4.15). The single-unresolved cross section $\mathrm{d}\hat{\sigma}_{N+1}^{\mathrm{NNLO}}$ can be found in eq. (5.8). The double-unresolved contribution $\mathrm{d}\hat{\sigma}_{N}^{\mathrm{NNLO}}$ is obtained by combining the many different terms calculated in the previous sections. As was explained there, it is convenient to write $\mathrm{d}\hat{\sigma}_{N}^{\mathrm{NNLO}}$ as the sum of double-boosted, single-boosted and elastic terms

$$\mathrm{d}\hat{\sigma}_{N}^{\mathrm{NNLO}} = \mathrm{d}\hat{\sigma}_{\mathrm{db}}^{\mathrm{NNLO}} + \mathrm{d}\hat{\sigma}_{\mathrm{sb}}^{\mathrm{NNLO}} + \mathrm{d}\hat{\sigma}_{\mathrm{el}}^{\mathrm{NNLO}}\,. \tag{6.2}$$

We now present each contribution separately, using several functions that we collect in appendix I. The double-boosted contribution is described by the very simple expression

$$2s\;\mathrm{d}\hat{\sigma}_{\mathrm{db}}^{\mathrm{NNLO}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \left\langle \mathcal{P}_{qq}^{\mathrm{NLO}} \otimes F_{\mathrm{LM}} \otimes \mathcal{P}_{qq}^{\mathrm{NLO}} \right\rangle, \tag{6.3}$$

where $\mathcal{P}_{qq}^{\mathrm{NLO}}$ is the finite remainder of NLO splitting functions, and can be found in eq. (I.3). As expected, this contribution is independent of the multiplicity of the final state.

The expression for the single-boosted contribution is slightly more complex and corresponds to

$$\begin{aligned}
2s\;\mathrm{d}\hat{\sigma}_{\mathrm{sb}}^{\mathrm{NNLO}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \Big\{ & \langle \mathcal{P}_{qq}^{\mathrm{NLO}} \otimes [I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LM}}] \rangle + \langle [I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LM}}] \otimes \mathcal{P}_{qq}^{\mathrm{NLO}} \rangle \\
& + \langle \mathcal{P}_{qq}^{\mathcal{W}} \otimes [\mathcal{W}_1^{1\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}}] \rangle + \langle [\mathcal{W}_2^{2\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}}] \otimes \mathcal{P}_{qq}^{\mathcal{W}} \rangle \\
& + \langle \mathcal{P}_{qq}^{\mathrm{NNLO}} \otimes F_{\mathrm{LM}} \rangle + \langle F_{\mathrm{LM}} \otimes \mathcal{P}_{qq}^{\mathrm{NNLO}} \rangle \\
& + \langle \mathcal{P}_{qq}^{\mathrm{NLO}} \otimes F_{\mathrm{LV}}^{\mathrm{fin}} \rangle + \langle F_{\mathrm{LV}}^{\mathrm{fin}} \otimes \mathcal{P}_{qq}^{\mathrm{NLO}} \rangle \Big\},
\end{aligned} \tag{6.4}$$

Here, we remind the reader that $I_{\mathrm{T}}^{(0)}$ is the $\epsilon \to 0$ limit of the finite operator $I_{\mathrm{T}}(\epsilon)$. Its explicit expression is reported in eq. (A.66). The function $\mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}}$, appearing in the second line of eq. (6.4), is given in eq. (G.12), while the NNLO splitting function $\mathcal{P}_{qq}^{\mathrm{NNLO}}$ is reported in eq. (I.5).

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Finally, the elastic contribution reads

$$
\begin{aligned}
2s\,\mathrm{d}\hat{\sigma}_{\mathrm{el}}^{\mathrm{NNLO}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \Bigg\{ & \left\langle \left[I_{\mathrm{cc}}^{\mathrm{fin}} + I_{\mathrm{tri}}^{\mathrm{fin}} + I_{\mathrm{unc}}^{\mathrm{fin}}\right] \cdot F_{\mathrm{LM}} \right\rangle \\
& + \sum_{i=1}^{N_p} \left\langle \left[\gamma^{\mathcal{W}}(L_i)\, \theta_{i2}\, \mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}} + \delta_g^{(0)}\, \mathcal{W}_i^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}} + \delta_g^{\perp} \mathcal{W}_r^{(i)} \right] \cdot F_{\mathrm{LM}} \right\rangle \Bigg\} \\
& + \left[\frac{\alpha_s(\mu)}{2\pi}\right] \left\langle I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \right\rangle + \left\langle S_{\mathfrak{m}\mathfrak{n}} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \right\rangle_{T^2}^{\mathrm{fin}} + \left\langle F_{\mathrm{LV}^2}^{\mathrm{fin}} \right\rangle + \left\langle F_{\mathrm{VV}}^{\mathrm{fin}} \right\rangle .
\end{aligned}
\tag{6.5}
$$

In this equation $\theta_{i2} = 1$ if i is the final-state parton ($i > 2$) and 0 otherwise. In the first line we have the combination of a double color-correlated contribution, a triple color-correlated component, and a color-uncorrelated part. They are presented in eq. (I.8), (I.9), and (I.12) respectively. In the second line of eq. (6.5), the functions $\gamma^{\mathcal{W}}$, $\mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}}$, $\mathcal{W}_i^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}}$ and $\mathcal{W}_r^{(i)}$ appear. They are given in eqs. (I.15), (G.12), (G.10) and (F.41). The constants $\delta_g^{(0)}$ and $\delta_g^{\perp}$ are reported in eq. (I.16). The term $\langle S_{\mathfrak{m}\mathfrak{n}} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}^{\mathrm{fin}}$ in eq. (6.5) refers to the finite remainder of the double-soft integrated subtraction term. It can be extracted from ref. [70], and its explicit expression is reported in eq. (I.17). Finally, $F_{\mathrm{LV}^2}^{\mathrm{fin}}$ and $F_{\mathrm{VV}}^{\mathrm{fin}}$ are the process-dependent finite remainders of virtual amplitudes.

We claim that the above result for $\mathrm{d}\hat{\sigma}_{q\bar{q}}^{\mathrm{NNLO}}$ can be used, without further ado, to implement the finite remainder of NNLO QCD corrections to a process $q\bar{q} \to X + Ng$ in a computer code. In theory, this can be done for *arbitrary* N, but the practical realization of this idea will have to wait until finite remainders for two-loop amplitudes for such processes become available.

Nevertheless, it is important to emphasize that the form of the final results is well-suited for numerical implementation, in the sense that the parameter N that controls the final state multiplicity only appears in relatively few places. Indeed, the splitting functions that appear in the boosted contributions are universal and are determined only by the flavor of the external partons and their energies. In the elastic contribution, the final state multiplicity only affects the upper limit in the sum over partons, see e.g. eqs. (6.5), (I.8), (I.9) and (I.12). It follows that implementing the color-uncorrelated elastic terms in a numerical code is also quite simple for any N. It could be less trivial to implement contributions containing color correlations (e.g. $I_{\mathrm{T}}^{(0)}$), as these require one to evaluate color-correlated matrix elements for high-multiplicity processes. However, even in this case a numerical implementation for a given N should be straightforward, using e.g. the ideas of color ordering.

Results of the general computation reported here can be compared with those obtained for specific values of N. The $N = 0$ case corresponds to the Drell-Yan process, and the $N = 1$ case to the gluonic contribution to the $V +$jet production. It is well-known that, in both cases, the correlators of color-charge operators can be expressed through Casimir operators. For example, in the case of $q_1\bar{q}_2 \to V + g_3$, we find

$$
\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 = \frac{C_A}{2} - C_F\,, \qquad \boldsymbol{T}_1 \cdot \boldsymbol{T}_3 = \boldsymbol{T}_2 \cdot \boldsymbol{T}_3 = -\frac{C_A}{2}\,. \tag{6.6}
$$

Using such expressions it is straightforward to replace all products of color-charge operators in eqs. (6.3), (6.4), (6.5) with the corresponding Casimir operators. One can also easily check that the partition functions defined for generic N turn into structures already used in earlier computations. It follows from the definition in eq. (B.10) that $\Delta^{(ij)} = 1$ for the Drell-Yan process and

$$\Delta^{(\mathfrak{m}\mathfrak{n})} = \frac{p_{\perp,3}}{p_{\perp,3} + p_{\perp,\mathfrak{m}} + p_{\perp,\mathfrak{n}}}\,, \tag{6.7}$$

for $V +$jet partitioning. Similarly, it is easy to see that ω-partitions are the same as those used in refs. [1, 61] for $N = 0$ and ref. [79] for $N = 1$.

We have reproduced the analytic results for the finite NNLO remainders for Drell-Yan production that were reported in ref. [61] starting from eqs. (6.3), (6.4), (6.5), and setting $N = 0$. We have also checked that, upon setting $N = 1$, the general formulas reproduce the results of a dedicated computation of the NNLO QCD corrections to the process $q\bar{q} \to V + g$ that we performed earlier. Although this computation was also based on the nested soft-collinear subtraction scheme, it was organized very differently, with an emphasis on separately integrating all the different subtraction terms over unresolved phase spaces *before* combining and simplifying them. The two approaches are sufficiently independent to provide an important check of the general-N formula that we reported in this section.

7 Conclusions

In this paper, we have shown how to use the nested soft-collinear subtraction scheme to describe the production of a generic color-singlet state accompanied by an *arbitrary* number of gluons in quark-antiquark annihilation at NNLO QCD. We have identified recurring structures associated with the sums of single-soft, single-collinear and one-loop virtual corrections. We have also shown that by organizing the calculation in such a way that the iterative nature of these finite contributions is fully exposed, much of the complexity of NNLO computations related to an interplay of soft and collinear singularities can be ameliorated. This has allowed us to demonstrate the cancellation of all color-correlated poles, as well as color-uncorrelated poles through $\mathcal{O}(\epsilon^{-2})$, in a straightforward manner. We have also confirmed the cancellation of the remaining ϵ-poles, and obtained compact expressions for the finite subtraction terms, which we have checked, where possible, against previous results and independent calculations. To the best of our knowledge, it is the first time that such expressions have been presented for the production of an arbitrary number of gluons at a hadron collider.[29]

Although we considered a $q\bar{q}$ initial state in this paper, many of our arguments apply to gg annihilation as well; the only modifications required for this channel would be the use of gluon splitting functions in place of the quark ones as well as the necessary changes in the color charges where appropriate. These modifications are clearly minor and do not impact the logic of the computation that we report in this paper.

The results of this study provide a necessary step towards the complete generalization of the nested soft-collinear subtraction scheme to arbitrary initial and final states. Indeed, on the one hand, the gluonic final state ensures that the maximal number of infrared and collinear singularities are present, so processes with final state quarks should have a simpler singularity structure. On the other hand, we relied on the symmetries of the final state and particular features of the initial state, and this will not be possible if generic processes are considered.

[29] See ref. [59] for a related analysis in the context of the antenna subtraction scheme.

Although nothing will change as a matter of principle, the combinatorics of collinear limits will become more complicated. We look forward to addressing these issues in future studies.

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A Constants, angular integrals, splitting functions, anomalous dimensions and fundamental operators

In this section we provide a collection of formulas that are used throughout the main text of this paper. They include:

(i) various constants in appendix A.1;

(ii) angular integrals in appendix A.2;

(iii) the relevant Altarelli-Parisi splitting functions in appendix A.3;

(iv) generalized splitting functions and anomalous dimensions in appendix A.4;

(v) operators arising from soft and collinear limits as well as from virtual corrections, and useful relations between them in appendix A.5;

A.1 Useful constants

Here we summarize the various constants that we have introduced throughout the manuscript. First we discuss the notations related to color. Following ref. [80], we denote the color-charge operators with $\boldsymbol{T}_i$; squares of color-charge operators are the Casimir operators of the corresponding representations of SU(3). They read $\boldsymbol{T}_q^2 = \boldsymbol{T}_{\bar{q}}^2 = C_F$, $\boldsymbol{T}_g^2 = C_A$, where $C_F = (N_c^2-1)/(2N_c)$, $C_A = N_c$, and $N_c = 3$ is the number of colors. Quark and gluon anomalous dimensions read $\gamma_q = 3/2\,C_F$ and $\gamma_g = 11/6\,C_A - 2/3\,T_R n_f$, where $T_R = 1/2$ and n_f is the number of massless quark flavors.

We renormalize the strong coupling in the $\overline{\text{MS}}$ scheme, i.e.

$$g_{s,b}^2 = g_s^2 S_\epsilon \mu^{2\epsilon}\left[1 - \frac{\alpha_s(\mu)}{2\pi}\frac{\beta_0}{\epsilon} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right) + \mathcal{O}(\alpha_s^3)\right], \tag{A.1}$$

where $S_\epsilon = (4\pi)^{-\epsilon} e^{\epsilon\gamma_{\rm E}}$ and

$$\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_R n_f = \gamma_g\,, \qquad \beta_1 = \frac{17}{6}C_A^2 - \frac{5}{3}C_A T_R n_f - C_F T_R n_f\,. \tag{A.2}$$

We note that we only consider gluons in the final state, so that n_f is set to zero throughout this paper. Furthermore, it is convenient to define the following coupling

$$[\alpha_s] \equiv \frac{\alpha_s(\mu)}{2\pi}\frac{e^{\epsilon\gamma_{\rm E}}}{\Gamma(1-\epsilon)}\,. \tag{A.3}$$

Then, combining eqs. (A.1) and (A.3), we find

$$g_{s,b}^2 = 8\pi^2[\alpha_s]\,\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\Big[1+\mathcal{O}(\alpha_s)\Big]\,. \tag{A.4}$$

In the main text of this paper we encounter a number of angular integrals, for which we introduce the following normalization constants:

$$\begin{aligned}
N_\epsilon^{(b,d)} &= \frac{\Gamma(1-\epsilon)\,\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)} = 1+\frac{\pi^2}{3}\epsilon^2+\mathcal{O}(\epsilon^3)\,,\\
N_{\mathfrak{m}||\mathfrak{n}}(\epsilon) &= 2^{2\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)} = 1+2\epsilon\log 2+\frac{1}{2}\epsilon^2(\pi^2+4\log^2 2)+\mathcal{O}(\epsilon^3)\,,\\
N_c(\epsilon) &= -\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}+\frac{2\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = 1+\mathcal{O}(\epsilon^3)\,.
\end{aligned} \tag{A.5}$$

We note that all the above normalization constants are equal to one to zeroth order in ϵ. To describe virtual corrections we have used the convention of refs. [71, 75]

$$\begin{aligned}
\lambda_{ij} &= \begin{cases} +1 & i \text{ and } j \text{ are both incoming or outgoing}\,,\\ 0 & \text{otherwise}\,,\end{cases}\\
\kappa_{ij} &= (\lambda_{ij}-\lambda_{i\mathfrak{m}}-\lambda_{j\mathfrak{m}}) = \begin{cases} +1 & i \text{ and } j \text{ are both incoming or outgoing}\,,\\ -1 & \text{otherwise}\,.\end{cases}
\end{aligned} \tag{A.6}$$

For double-virtual amplitudes we have used the following constants [71]

$$\begin{aligned}
K &= \left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A - \frac{10}{9}T_R n_f\,,\\
c_\epsilon &= \frac{e^{-\epsilon\gamma_E}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} = 1+\frac{\pi^2}{4}\epsilon^2+\frac{7}{3}\zeta_3\,\epsilon^3+\mathcal{O}(\epsilon^4)\,.
\end{aligned} \tag{A.7}$$

To describe integrated double-soft limits (see eq. (4.90)), we have introduced

$$\begin{aligned}
c_1(\epsilon) &= 1+\left(\frac{\pi^2}{6}-\frac{32}{9}\right)\epsilon^2+\left(\frac{217}{27}-\frac{137}{9}\log 2-22\log^2 2+\frac{11\zeta_3}{2}\right)\epsilon^3\,,\\
c_2(\epsilon) &= 1+\frac{\pi^2}{3}\epsilon^2\,,\\
c_3(\epsilon) &= 4\log 2+8\epsilon\log^2 2\,.
\end{aligned} \tag{A.8}$$

We emphasize that $c_{1,2,3}$ do not contain powers of ϵ beyond those shown above.

To compute soft and collinear limits of the real-virtual contribution $F_{\rm RV}$, we used

$$
\begin{aligned}
A_K(\epsilon) &= \frac{\Gamma^3(1+\epsilon)\,\Gamma^5(1-\epsilon)}{\Gamma(1+2\epsilon)\,\Gamma^2(1-2\epsilon)} = 1 - \frac{\pi^2}{3}\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \\
h_c(\epsilon) &= \frac{\Gamma^2(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-3\epsilon)} = 1 + \mathcal{O}(\epsilon^3)\,.
\end{aligned}
\tag{A.9}
$$

We also have defined (see eq. (4.125))

$$
\delta_g^{C_A}(\epsilon) = \left(-\frac{131}{72} + \frac{\pi^2}{6}\right) + \mathcal{O}(\epsilon)\,, \qquad \delta_g^{\beta 0}(\epsilon) = \log 2 + \mathcal{O}(\epsilon)\,. \tag{A.10}
$$

When combining the unboosted terms involving color correlations (see section 5.3), we require the following combinations of some of the above constants

$$
\begin{aligned}
&\tilde{c}(\epsilon) = \frac{e^{\epsilon\gamma_{\rm E}}}{\Gamma(1-\epsilon)}\left(c_2(\epsilon) + \epsilon c_3(\epsilon) - 2^{2+2\epsilon}\,\delta_g^{\beta_0}(\epsilon)\right) = 1 + \mathcal{O}(\epsilon^2)\,, \\
&C_A\left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon}\delta_g^{C_A}(\epsilon)\right) - K = \mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{A.11}
$$

A.2 A collection of simple angular integrals

Throughout the manuscript we make use of various integrals over the angles of unresolved gluons. We summarize some of the useful formulas here. First, we define the normalized element of the solid angle in $(d-1)$- and $(d-2)$-dimensions

$$
[\mathrm{d}\Omega_i^{(d-1)}] \equiv \frac{\mathrm{d}\Omega_i^{(d-1)}}{2(2\pi)^{d-1}}\,, \qquad [\mathrm{d}\Omega_i^{(d-2)}] \equiv \frac{\mathrm{d}\Omega_i^{(d-2)}}{2(2\pi)^{d-1}}\,. \tag{A.12}
$$

Then, we find

$$
[\Omega^{(d-2)}] \equiv \int [\mathrm{d}\Omega^{(d-2)}] = \frac{1}{8\pi^2}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,. \tag{A.13}
$$

Furthermore, we use

$$
\begin{aligned}
\int \frac{[\mathrm{d}\Omega_a^{(d-1)}]}{[\Omega^{(d-2)}]}\,\frac{\rho_{ij}}{\rho_{ia}\rho_{ja}} &= -\frac{2^{1-2\epsilon}}{\epsilon}\,\eta_{ij}^{-\epsilon}\,K_{ij}\,, \\
\int \frac{[\mathrm{d}\Omega_a^{(d-1)}]}{[\Omega^{(d-2)}]}\,\frac{1}{\rho_{ia}} &= -\frac{2^{-2\epsilon}}{\epsilon}\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,, \\
\int \frac{[\mathrm{d}\Omega_a^{(d-1)}]}{[\Omega^{(d-2)}]}\,\left(\frac{\rho_{ia}}{4}\right)^{-\epsilon}\,\frac{1}{\rho_{ia}} &= -\frac{2^{-2\epsilon}}{2\epsilon}\,\frac{2^\epsilon\,\Gamma(1-\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\,,
\end{aligned}
\tag{A.14}
$$

where K_{ij} is given by (cf. eq. (3.14))

$$
K_{ij} = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\eta_{ij}^{1+\epsilon}\,{}_2F_1(1,1,1-\epsilon,1-\eta_{ij})\,. \tag{A.15}
$$

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Other integrals that we require involve the collinear limits acting on the angular phase space measure; they can be computed using the phase space parametrization described in appendix E. Here we just give two examples that appear frequently

$$\int [C_{ij}\,\mathrm{d}\Omega_j^{(d-1)}]\,\frac{1}{\rho_{ij}} = -\frac{2^{-2\epsilon}}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right], \tag{A.16}$$

and

$$\int [C_{ij}\,\mathrm{d}\Omega_j^{(d-1)}]\,\frac{1}{\rho_{ij}}\,\Theta\left(\eta_{ij}<\frac{\eta_{ik}}{2}\right) = -\frac{1}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\rho_{ik}^{-\epsilon}\,. \tag{A.17}$$

A.3 Altarelli-Parisi splitting functions

In this section we report the Altarelli-Parisi splitting functions that we use in this paper. The only leading order splitting function that we require reads

$$\hat{P}_{qq}^{(0)}(z) = C_F\left[2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z)\right], \tag{A.18}$$

where

$$\mathcal{D}_n(z) \equiv \left[\frac{\log^n(1-z)}{1-z}\right]_+ . \tag{A.19}$$

At NLO, we need the non-singlet splitting function from which the contribution of identical quarks has been subtracted, which reads

$$\begin{aligned}\hat{P}^{(1)}_{qq,\widetilde{\mathrm{NS}}}(z) = C_A C_F\Bigg[&\frac{\pi^2}{6}(1+z) - \frac{62}{9}z - \frac{19}{18} + \left(\frac{67}{9}-\frac{\pi^2}{3}\right)\mathcal{D}_0(z)\\ &+\frac{2+11z^2}{6(1-z)}\log z - \frac{1+z^2}{1-z}\mathrm{Li}_2(1-z) + \delta(1-z)\left(\frac{17}{24}+\frac{11}{18}\pi^2 - 3\zeta_3\right)\Bigg]\\ &+C_F^2\Bigg[3-2z-2\frac{1+z^2}{1-z}\log(1-z)\log z + 2\log z + \frac{1+3z^2}{2(1-z)}\log^2 z\\ &+2\frac{1+z^2}{1-z}\mathrm{Li}_2(1-z) + \delta(1-z)\left(\frac{3}{8}-\frac{\pi^2}{2}+6\zeta_3\right)\Bigg]\,.\end{aligned} \tag{A.20}$$

A.4 Generalized splittings and anomalous dimensions

A.4.1 Tree-level

We start by introducing the two tree-level splitting functions needed throughout the paper

$$\begin{aligned} P_{qq}(z) &= C_F\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right],\\ P_{gg}^{\mu\nu}(z) &= 2C_A\left[-g^{\mu\nu}\left(\frac{1-z}{z}+\frac{z}{1-z}\right) + 2(1-\epsilon)z(1-z)\kappa_\perp^\mu\kappa_\perp^\nu\right], \end{aligned} \tag{A.21}$$

where $\kappa_\perp^\mu$ is a transverse momentum defined as

$$\kappa_\perp^\mu = \frac{k_\perp^\mu}{\sqrt{-k_\perp^2}}, \qquad \kappa_\perp^2 = -1\,. \tag{A.22}$$

We also need the gluon spin-averaged splitting function

$$P_{gg}(z) = 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right] . \tag{A.23}$$

To describe the spin-correlated component arising from sectors (b) and (d) we have introduced the functions

$$P_{gg}^{\perp}(z) = 4C_A(1-\epsilon)\, z\, (1-z)\,, \tag{A.24}$$

$$P_{gg}^{\perp,r}(z) = 2C_A\, z(1-z)(1-2\epsilon)\,, \tag{A.25}$$

$$P_{gg}^{(0)}(z) = 2C_A \left(\frac{z}{1-z} + \frac{1-z}{z}\right) , \tag{A.26}$$

$$P_{gg}(z,\epsilon) = P_{gg}^{(0)}(z) + \frac{1}{2} P_{gg}^{\perp}(z) = 2C_A \left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\,(1-\epsilon)\right) . \tag{A.27}$$

We also require the following integral of the soft-subtracted function P_{gg} over z

$$\begin{aligned} \gamma^{nk}_{f(z),g\to gg}(\epsilon, L_i) = & -\int_0^1 \mathrm{d}z\, (1-S_z) \left[z^{-n\epsilon}(1-z)^{-k\epsilon}\, f(z)\, P_{gg}(z)\right] \\ & + 2C_A \frac{1-e^{k\epsilon L_i}}{k\epsilon} f(1)\,, \end{aligned} \tag{A.28}$$

where S_z stands for the soft $z \to 1$ limit and $L_i = \log(E_{\max}/E_i)$. We also define the following integrals over z

$$\gamma^{22}_{\perp,g\to gg} = -\int_0^1 \mathrm{d}z\, \frac{P_{gg}^{\perp}(z)}{[z(1-z)]^{2\epsilon}}\,, \qquad \gamma^{22,r}_{\perp,g\to gg} = -\int_0^1 \mathrm{d}z\, \frac{P_{gg}^{\perp,r}(z)}{[z(1-z)]^{2\epsilon}}\,, \tag{A.29}$$

as well as integrals over z and the energy of the unresolved parton

$$\begin{aligned} \delta_g^{\mathrm{sa}}(\epsilon) &= \frac{N_\epsilon^{(b,d)}}{2} E_{\max}^{4\epsilon} \int_{E_{\max}}^{2E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{1+4\epsilon}} \int_{1-\xi}^{\xi} \mathrm{d}z\, [z(1-z)]^{-2\epsilon} P_{gg}(z)\,, \\ \delta_g^{\perp,r}(\epsilon) &= \frac{N_\epsilon^{(b,d)}}{2} E_{\max}^{4\epsilon} \int_{E_{\max}}^{2E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{1+4\epsilon}} \int_{1-\xi}^{\xi} \mathrm{d}z\, [z(1-z)]^{-2\epsilon}\, \epsilon\, P_{gg}^{\perp,r}(z)\,, \\ \delta_g^{\perp}(\epsilon) &= \frac{N_\epsilon^{(b,d)}}{2} E_{\max}^{4\epsilon} \int_{E_{\max}}^{2E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{1+4\epsilon}} \int_{1-\xi}^{\xi} \mathrm{d}z\, [z(1-z)]^{-2\epsilon}\, P_{gg}^{\perp}(z)\,, \\ \delta_g(\epsilon) &= \frac{N_\epsilon^{(b,d)}}{2} E_{\max}^{4\epsilon} \int_{E_{\max}}^{2E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{1+4\epsilon}} \int_{1-\xi}^{\xi} \mathrm{d}z\, [z(1-z)]^{-2\epsilon} \left(P_{gg}(z,\epsilon) + \epsilon\, P_{gg}^{\perp}(z)\right) , \end{aligned} \tag{A.30}$$

where we have defined $\xi = E_{\max}/E_{\mathfrak{m}}$.

For the configurations where a final state gluon becomes collinear to an initial state parton 1_a, we require convolutions of the type

$$\int_0^1 \mathrm{d}z\, \mathcal{P}_{aa}^{(k)}(z,E_1)\, g(z) = \int_0^1 \mathrm{d}z \left[(1-S_z)\Big[(1-z)^{-k\epsilon} P_{aa}(z)\Big] - 2\boldsymbol{T}_a^2 \frac{1-e^{-k\epsilon L_1}}{k\epsilon} \delta(1-z) \right] g(z)\,, \tag{A.31}$$

where $k=2$ at NLO and $k=4$ at NNLO. It is worth rewriting the above splitting function as

$$-\left[\left(\frac{2E_1}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^{\frac{k}{2}} \mathcal{P}_{aa}^{(k)}(z,E_1) = \Gamma_{1,a}^{(k)}\, \delta(1-z) + \mathcal{P}_{aa}^{(k),\mathrm{gen}}(z,E_1)\,, \tag{A.32}$$

with

$$\Gamma_{1,a}^{(k)} = \left[\left(\frac{2E_1}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^{\frac{k}{2}} \left[\gamma_a + 2\boldsymbol{T}_a^2 \frac{1-e^{-k\epsilon L_1}}{k\epsilon}\right], \tag{A.33}$$

$$\mathcal{P}_{aa}^{(k),\mathrm{gen}}(z,E_1) = \left[\left(\frac{2E_1}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^{\frac{k}{2}} \left[-\hat{P}_{aa}^{(0)}(z) + \epsilon\, \mathcal{P}_{aa}^{(k),\mathrm{fin}}(z)\right]. \tag{A.34}$$

Here $\hat{P}_{aa}^{(0)}$ is the Altarelli-Parisi splitting function given in eq. (A.18), while $\mathcal{P}_{aa}^{(k),\mathrm{fin}}$ is an $\mathcal{O}(\epsilon^0)$ function that can be obtained by comparing eqs. (A.31) and (A.34), namely

$$\mathcal{P}_{aa}^{(k),\mathrm{fin}}(z) = \frac{1}{\epsilon}\left[2\sum_{n=1}^{\infty} \frac{(-1)^n (k\epsilon)^n}{n!} \mathcal{D}_n(z) + (1-z)^{-k\epsilon} P_{aa}^{\mathrm{reg}}(z) + (1-z)\right], \tag{A.35}$$

with

$$P_{aa}^{\mathrm{reg}}(z) = -\left[(1+z) + \epsilon(1-z)\right]. \tag{A.36}$$

If the unresolved final state gluon goes collinear to another final state parton i_g, the generalized gluon final-state anomalous dimension reads

$$\Gamma_{i,g}^{(k)} = \left[\left(\frac{2E_i}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^{\frac{k}{2}} \gamma_{z,g\to gg}^{2k}(\epsilon, L_i)\,, \qquad i \in [3, N_p]\,, \tag{A.37}$$

where $\gamma_{f(z),g\to gg}^{nk}$ is defined in eq. (3.20) and repeated in eq. (A.28). Throughout the paper, we use

$$\mathcal{P}_{aa}^{\mathrm{gen}} = \mathcal{P}_{aa}^{(2),\mathrm{gen}}\,, \qquad \mathcal{P}_{aa}^{\mathrm{fin}} = \mathcal{P}_{aa}^{(2),\mathrm{fin}}\,, \qquad \Gamma_{i,f_i} = \Gamma_{i,f_i}^{(2)}\,, \tag{A.38}$$

to lighten the notation.

A.4.2 One-loop

When computing the real-virtual contributions, one finds a convolution similar to the one in eq. (A.31) for the case when a final-state gluon is collinear to initial state parton 1_a. It reads

$$\int_0^1 \mathrm{d}z\, \mathcal{P}_{aa}^{(k),1\mathrm{L}}(z,E_1) g(z) = \int_0^1 \mathrm{d}z \left[(1-S_z)\Big[(1-z)^{-k\epsilon} P_{aa,\mathrm{i}}^{1\mathrm{L}}(z)\Big] + 2\boldsymbol{T}_a^2 \frac{1-e^{-(2+k)\epsilon L_1}}{(2+k)} \pi \cot(\pi\epsilon) \delta(1-z) \right] g(z)\,. \tag{A.39}$$

The initial-state one-loop splitting function for a $q \to q$ splitting is given by [76, 77, 81]

$$
\begin{aligned}
P^{1\mathrm{L}}_{qq,\mathrm{i}}(z) = &- \frac{C_A}{\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)}(1-z)^{-\epsilon} + 2\sum_{n=1}^{\infty}\epsilon^{2n}\,\mathrm{Li}_{2n}(1-z)\right] \\
&\times (1-z)^{-\epsilon}P_{qq}(z) + \frac{2C_F}{\epsilon^2}(1-z)^{-\epsilon}P_{qq}(z)\sum_{n=1}^{\infty}\epsilon^n\,\mathrm{Li}_n(1-z) \\
&- C_F(C_A-C_F)\frac{z+\epsilon(1-z)}{1-2\epsilon}(1-z)^{-\epsilon}\,.
\end{aligned}
\tag{A.40}
$$

We rewrite $\mathcal{P}^{(k),1\mathrm{L}}_{aa}$ in analogy with eq. (A.32), getting

$$
\left[\left(\frac{2E_1}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^k \mathcal{P}^{(k),1\mathrm{L}}_{aa}(z,E_1) = \frac{C_A}{\epsilon^2}\left[\Gamma^{(k),1\mathrm{L}}_{1,a}\delta(1-z)+\mathcal{P}^{(k),1\mathrm{L,gen}}_{aa}(z,E_1)\right], \tag{A.41}
$$

where

$$
\Gamma^{(k),1\mathrm{L}}_{1,a} = \left[\left(\frac{2E_1}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^k\left[\gamma_a + 2\boldsymbol{T}^2_a\,\frac{1-e^{-(2+k)\epsilon L_1}}{(2+k)}\pi\frac{\cos(\pi\epsilon)}{\sin(\pi\epsilon)}\right], \tag{A.42}
$$

$$
\mathcal{P}^{(k),1\mathrm{L,gen}}_{aa}(z,E_1) = \left[\left(\frac{2E_1}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^k\left[-\hat{P}^{(0)}_{aa}(z)+\epsilon\,\mathcal{P}^{(k),1\mathrm{L,fin}}_{aa}(z)\right]. \tag{A.43}
$$

In eq. (A.43), the function $\mathcal{P}^{(k),1\mathrm{L,fin}}_{aa}$ is finite in ϵ and can be extracted from refs. [76, 77, 81].

For the final state collinear limits, the equivalent of eq. (A.37) is the generalized gluon one-loop, final-state anomalous dimension

$$
\Gamma^{(k),1\mathrm{L}}_{i,g} = -\left[\left(\frac{2E_i}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]^k\frac{\epsilon^2\cos(\pi\epsilon)}{C_A}\,\gamma^{3(k+1),1\mathrm{L}}_{z,g\to gg}(\epsilon,L_i)\,, \qquad i\in[3,N_p]\,, \tag{A.44}
$$

with

$$
\begin{aligned}
\gamma^{n(k+1),1\mathrm{L}}_{f(z),g\to gg}(\epsilon,L_i) = &-\int_0^1 \mathrm{d}z\,(1-S_z)\left[z^{-n\epsilon}(1-z)^{-(k+1)\epsilon}f(z)\,P^{1\mathrm{L}}_{gg}(z)\right] \\
&- 2C_A^2\,\frac{1-e^{-(2+k)\epsilon L_i}}{(2+k)}\,\frac{\pi}{\epsilon^2\sin(\pi\epsilon)}f(1)\,.
\end{aligned}
\tag{A.45}
$$

The above formula requires the following splitting function

$$
P^{1\mathrm{L}}_{gg}(z) = C_A P_{gg}(z)\left[\frac{a(z)+\tilde{b}(z)}{2}\right] + \widetilde{P}^{\mathrm{new}}_{gg}(z)\left[\frac{n_f - C_A(1-\epsilon)}{(1-\epsilon)(1-2\epsilon)(3-2\epsilon)}\right], \tag{A.46}
$$

where

$$
\begin{aligned}
a(z) &= (1-z)F_1(1-z)\,, \\
\tilde{b}(z) &= \frac{2}{\epsilon^2} + zF_1(z)\,,
\end{aligned}
\tag{A.47}
$$

with

$$
F_1(z) = \frac{2}{z\epsilon^2}\Big[-\Gamma(1-\epsilon)\Gamma(1+\epsilon)z^{-\epsilon}(1-z)^{\epsilon} - 1 + (1-z)^{\epsilon}\,{}_2F_1(\epsilon,\epsilon,1+\epsilon,z)\Big], \tag{A.48}
$$

and

$$
\widetilde{P}^{\mathrm{new}}_{gg}(z) = -C_A\left[\frac{1-2z(1-z)\epsilon}{1-\epsilon}\right]. \tag{A.49}
$$

A.5 Definitions of the main operators, commutators, and expansions

Throughout this paper we have used virtual, soft, and collinear operators to encode singularities, and have made use of various relations between them. For the reader's convenience we list these definitions and relations here.

We begin with Catani's operator [71]

$$I_1(\epsilon) = \frac{1}{2}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_{(ij)}^{N_p}\frac{\mathcal{V}_i^{\rm sing}(\epsilon)}{\boldsymbol{T}_i^2}\,\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}\,, \tag{A.50}$$

where the relevant constants are defined in subsection A.1. We find it convenient to modify the normalization slightly, yielding

$$\overline{I}_1(\epsilon) = \frac{1}{2}\sum_{(ij)}^{N_p}\frac{\mathcal{V}_i^{\rm sing}(\epsilon)}{\boldsymbol{T}_i^2}\,\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}\,, \tag{A.51}$$

from which we define the operators for amplitudes-squared

$$I_\pm(\epsilon) = \frac{\overline{I}_1(\epsilon)\pm\overline{I}_1^\dagger(\epsilon)}{2}\,, \qquad I_{\rm V}(\epsilon) = \overline{I}_1(\epsilon)+\overline{I}_1^\dagger(\epsilon)\equiv 2I_+(\epsilon)\,. \tag{A.52}$$

The Laurent expansion for $I_{\rm V}(\epsilon)$ reads

$$I_{\rm V}(\epsilon) = \sum_{n=-2}^{\infty}\epsilon^n I_{\rm V}^{(n)}\,, \tag{A.53}$$

where

$$I_{\rm V}^{(-2)} = -\sum_{i=1}^{N_p}\boldsymbol{T}_i^2\,, \qquad I_{\rm V}^{(-1)} = \sum_{(ij)}^{N_p}\boldsymbol{T}_i\cdot\boldsymbol{T}_j L_{ij} - \sum_{i=1}^{N_p}\gamma_i\,. \tag{A.54}$$

The soft operator is equal to

$$I_{\rm S}(\epsilon) = -\frac{(2E_{\rm max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{(ij)}^{N_p}\eta_{ij}^{-\epsilon}K_{ij}\left(\boldsymbol{T}_i\cdot\boldsymbol{T}_j\right), \tag{A.55}$$

where K_{ij} is defined in eq. (3.14). The Laurent expansion of $I_{\rm S}$ reads

$$I_{\rm S}(\epsilon) = \sum_{n=-2}^{\infty}\epsilon^n I_{\rm S}^{(n)}\,, \tag{A.56}$$

and we require the following terms in the above expansion

$$I_{\rm S}^{(-2)} = \sum_{i=1}^{N_p}\boldsymbol{T}_i^2\,,$$
$$I_{\rm S}^{(-1)} = \sum_{(ij)}^{N_p}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\log\eta_{ij} - 2L_{\rm max}\sum_{i=1}^{N_p}\boldsymbol{T}_i^2\,,$$

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$$
\begin{aligned}
I_{\rm S}^{(0)} &= -\sum_{(ij)}^{N_p} \boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_j \left[2L_{\max}\log\eta_{ij} + \frac{1}{2}\log^2\eta_{ij} + {\rm Li}_2(1-\eta_{ij})\right] \\
&\quad + \left[2L_{\max}^2 - \frac{\pi^2}{6}\right]\sum_{i=1}^{N_p}\boldsymbol{T}_i^2\,, \qquad\qquad \text{(A.57)}\\
I_{\rm S}^{(1)} &= \sum_{(ij)}^{N_p} \boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_j \Bigg[2L_{\max}^2\log\eta_{ij} + \left(L_{\max} - \frac{1}{2}\log(1-\eta_{ij})\right)\log^2\eta_{ij} \\
&\quad + \frac{1}{6}\log^3\eta_{ij} + 2L_{\max}\,{\rm Li}_2(1-\eta_{ij}) - {\rm Li}_3(1-\eta_{ij}) - {\rm Li}_3(\eta_{ij})\Bigg] \\
&\quad - \left[L_{\max}\left(\frac{4}{3}L_{\max}^2 - \frac{\pi^2}{3}\right) + 3\zeta_3\right]\sum_{i=1}^{N_p}\boldsymbol{T}_i^2\,,
\end{aligned}
$$

where $L_{\max} = \log(2E_{\max}/\mu)$.

The computation of the soft contributions requires a variant of the soft operator, namely

$$
\widetilde{I}_{\rm S}(2\epsilon) = -\frac{(2E_{\max}/\mu)^{-4\epsilon}}{(2\epsilon)^2}\sum_{(ij)}^{N_p}\eta_{ij}^{-2\epsilon}\widetilde{K}_{ij}\left(\boldsymbol{T}_i\cdot\boldsymbol{T}_j\right), \qquad \text{(A.58)}
$$

where $\widetilde{K}_{ij}$ is defined in eq. (4.92). The following property relates $I_{\rm S}$ and $\widetilde{I}_{\rm S}$

$$
\widetilde{I}_{\rm S}(2\epsilon) = I_{\rm S}(2\epsilon) + \mathcal{O}(\epsilon)\,. \qquad \text{(A.59)}
$$

We also require an ϵ-expansion for $\widetilde{I}_{\rm S}$. Given eq. (A.59), the first three coefficients $\widetilde{I}_{\rm S}^{(n)}$ with $n = -2, -1, 0$ can be directly obtained from those in eq. (A.57), up to a rescaling by factors of $1/4, 1/2$ and 1 respectively. The coefficient at $\mathcal{O}(\epsilon)$ reads

$$
\begin{aligned}
\widetilde{I}_{\rm S}^{(1)} &= \sum_{(ij)}^{N_p} \boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_j \Bigg[\left(2L_{\max} - \frac{3}{2}\log(1-\eta_{ij})\right)\log^2\eta_{ij} + \frac{1}{3}\log^3\eta_{ij} \\
&\quad + \left(\frac{\pi^2}{6} + 4L_{\max}^2 - {\rm Li}_2(1-\eta_{ij})\right)\log\eta_{ij} \\
&\quad + 4L_{\max}\,{\rm Li}_2(1-\eta_{ij}) - {\rm Li}_3(1-\eta_{ij}) - 3{\rm Li}_3(\eta_{ij})\Bigg] \qquad \text{(A.60)}\\
&\quad + \left[\frac{2}{3}L_{\max}\left(\pi^2 - 4L_{\max}^2\right) - 7\zeta_3\right]\sum_{i=1}^{N_p}\boldsymbol{T}_i^2\,.
\end{aligned}
$$

Moving to collinear limits, we define the hard-collinear operator as

$$
I_{\rm C}^{(k)}(\epsilon) = \sum_{i=1}^{N_p}\frac{2}{k}\frac{\Gamma_{i,f_i}^{(k)}}{\epsilon}\,, \qquad \text{(A.61)}
$$

where $\Gamma_{i,f_i}^{(k)}$ is given in eq. (A.33) if $i = 1, 2$ and in eq. (A.37) if $i \in [3, N_p]$. To treat the hard-collinear limits of the real-virtual matrix element we have introduced

$$
\widetilde{I}_{\rm C}(2\epsilon) = \sum_{i=1}^{N_p}\frac{\Gamma_{i,f_i}^{\rm 1L}}{2\epsilon}\,, \qquad \text{(A.62)}
$$

where Γ^{1L}_{i,f_i} is given in eq. (A.42) if $i = 1, 2$ and in eq. (A.44) if $i \in [3, N_p]$. We note that the following relations hold

$$\begin{aligned} \widetilde{I}_{\rm C}(2\epsilon) &= I_{\rm C}(2\epsilon) + \mathcal{O}(\epsilon)\,, \\ I^{(4)}_{\rm C}(\epsilon) &= I_{\rm C}(2\epsilon) + \mathcal{O}(\epsilon^0)\,. \end{aligned} \tag{A.63}$$

Furthermore, we have used the ϵ-finite operator $I_{\rm T}$ defined as

$$I_{\rm T}(\epsilon) = I_{\rm V}(\epsilon) + I_{\rm S}(\epsilon) + I_{\rm C}(\epsilon)\,, \tag{A.64}$$

to simplify the NLO and NNLO calculations. Its expansion in ϵ reads

$$I_{\rm T}(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n I^{(n)}_{\rm T}\,, \tag{A.65}$$

with expansion coefficients given by

$$\begin{aligned} I^{(0)}_{\rm T} =& -\sum_{(ij)}^{N_p} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \bigg[\left(2L_{\rm max} + \frac{1}{2}\log\eta_{ij}\right)\log\eta_{ij} - \frac{1}{2}L_{ij}\left(L_{ij} + \frac{2\gamma_i}{\boldsymbol{T}_i^2}\right) \\ & + {\rm Li}_2(1-\eta_{ij}) + \frac{\pi^2}{2}\lambda_{ij}\bigg] \\ & + \sum_{i=1}^{N_p} \boldsymbol{T}_i^2 \bigg[2\widetilde{L}_i^2 - \frac{\pi^2}{6} - \frac{2\gamma_i}{\boldsymbol{T}_i^2}\,\widetilde{L}_i\,\bar{\theta}_{i2} + \left(\frac{67}{9} - \frac{11}{3}\widetilde{L}_i - \frac{2\pi^2}{3}\right)\theta_{i2}\bigg]\,, \\ I^{(1)}_{\rm T} =& \sum_{(ij)}^{N_p} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \bigg[\frac{1}{6}\left(L_{ij}^3 + \log^3\eta_{ij}\right) + 2L_{\rm max}\log\eta_{ij} - \frac{\pi^2}{2}\lambda_{ij}L_{ij} \\ & + \left(L_{\rm max} - \frac{1}{2}\log(1-\eta_{ij})\right)\log^2\eta_{ij} + 2L_{\rm max}\,{\rm Li}_2(1-\eta_{ij}) - {\rm Li}_3(\eta_{ij}) \\ & - {\rm Li}_3(1-\eta_{ij}) + \frac{\gamma_i}{2\boldsymbol{T}_i^2}\left(L_{ij}^2 - \pi^2\lambda_{ij}\right)\bigg] \\ & + \sum_{i=1}^{N_p} \boldsymbol{T}_i^2 \bigg[-\frac{4}{3}\widetilde{L}_i^3 + \frac{\pi^2}{3}\widetilde{L}_i - 3\zeta_3 + \left(2\widetilde{L}_i^2 - \frac{\pi^2}{6}\right)\gamma_i\,\bar{\theta}_{i2} + \bigg(\frac{808}{27} - \frac{134}{9}\widetilde{L}_i \\ & + \frac{11}{3}\widetilde{L}_i^2 + \pi^2\left(\frac{4}{3}\widetilde{L}_i - \frac{55}{36}\right) - 16\zeta_3\bigg)\theta_{i2}\bigg]\,, \end{aligned} \tag{A.66}$$

where $\theta_{i2} = 1$ if $i > 2$ and 0 otherwise, and $\bar{\theta}_{i2} = 1 - \theta_{i2}$.

While discussing the rearrangement of the single-unresolved terms (cf. section 5.1), we have introduced a variant of the virtual, soft and collinear operators, valid in the case of $N+1$ final-state partons. In particular, we have defined

$$I_{\rm V}^{N_p+1}(\epsilon) = \overline{I}_1^{N_p+1}(\epsilon) + \left(\overline{I}_1^{N_p+1}(\epsilon)\right)^{\dagger}\,, \tag{A.67}$$

with $\overline{I}_1^{N_p+1}$ defined as in eq. (A.51), up to replacing $N_p \mapsto N_p+1$. Similarly, we have also used

$$I_{\rm S}^{N_p+1}(E_{\mathfrak{m}}) = -\frac{(2E_{\mathfrak{m}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p+1} \eta_{ij}^{-\epsilon} K_{ij}\left(\boldsymbol{T}_i\cdot\boldsymbol{T}_j\right)\,, \tag{A.68}$$

and

$$I_{\rm C}^{N_p+1}(E_{\mathfrak{m}}) = \sum_{i=1}^{N_p+1} \frac{\Gamma_{i,f_i}}{\epsilon}\bigg|_{E_{\rm max}\mapsto E_{\mathfrak{m}}}\,, \tag{A.69}$$

where one needs to set $E_{\rm max} \mapsto E_{\mathfrak{m}}$ in the definition of Γ_{i,f_i}, see eqs. (3.19), (3.22).

B Partitions at NLO and NNLO for an arbitrary number of final-state particles

To treat the infrared singularities of a process with a large number of final state particles, we require partitions that separate resolved and potentially unresolved partons. To construct them, we consider a process that involves N_p partons at leading order and an arbitrary colorless final state

$$f_1(p_1) + f_2(p_2) \to f_3(p_3) + \ldots + f_{N_p}(p_{N_p}) + X\,. \tag{B.1}$$

At next-to-leading order, we need to add another particle to the final state to describe the real-emission process. We denote the corresponding list of final-state partons in this case as $\psi_{N+1} = \{f_3, f_4, \ldots, f_{N_p}, f_{N_p+1}\}$, where $N = N_p - 2$ is the number of final-state partons at leading order.

In principle, any of these final state partons can become unresolved. Suppose we want to describe a situation when this happens with a parton i. We then write the set of $N+1$ partons as

$$\psi_{N+1} = \{i, \psi_N^{(i)}\}\,, \tag{B.2}$$

where $\psi_N^{(i)} = \psi_{N+1}/\{i\}$ and introduce the function

$$d^{(i)} = \prod_{k\in\psi_N^{(i)}} p_{k,\perp} \prod_{\substack{l,m\in\psi_N^{(i)}\\ l<m}} (1-\cos\theta_{lm})\,, \tag{B.3}$$

where $p_{k,\perp}$ is the transverse momentum of parton k.[30] These functions are used to construct the partitions

$$\Delta^{(i)} = \frac{d^{(i)}}{\sum\limits_{j\in\psi_{N+1}} d^{(j)}}\,, \tag{B.4}$$

where $i \in \psi_{N+1}$. It follows from their definition that the functions $\Delta^{(i)}$ provide a partition of unity

$$\sum_{i\in\psi_{N+1}} \Delta^{(i)} = 1\,. \tag{B.5}$$

It is straightforward to determine the action of soft and collinear operators on the partition functions. In the soft limit of parton k, described by the operator S_k, we find

$$S_k\Delta^{(i)} = \delta_{ki}\,. \tag{B.6}$$

[30]We note that in the case of only one hard jet, k, $d^{(i)}$ reduces to $p_{k,\perp}$.

In the limit where partons l and m become collinear, we have

$$C_{lm}\Delta^{(i)} = \begin{cases} 0\,, & l,m \neq i\,, \\ 1\,, & l=i,\ m\in\{1,2\}\,, \\ z_{i,m}\,, & l=i, m\in\psi_N\,, \end{cases} \tag{B.7}$$

where $z_{i,m} = E_i/(E_i+E_m)$ and we assumed that partons 1 and 2 are in the initial state. The limits obtained by the interchange of l and m assignments follow naturally from the above formulas and are not shown for this reason.

A new element required for NNLO computations is the double-real emission process. To construct the corresponding partition functions, we consider an extended set of final-state partons

$$\psi_{N+2} = \{f_3, f_4, \ldots, f_{N_p+1}, f_{N_p+2}\}\,. \tag{B.8}$$

Two of these final-state partons can become unresolved and we assume that this happens with partons i and j. We then write $\psi_{N+2} = \{(i,j), \psi_N^{(ij)}\}$, define functions $d^{(ij)}$ as follows

$$d^{(ij)} = \prod_{k\in\psi_N^{(ij)}} p_{k,\perp} \prod_{\substack{l,m\in\psi_N^{(ij)} \\ l<m}} (1-\cos\theta_{lm})\,, \tag{B.9}$$

and use them to construct the NNLO partitions

$$\Delta^{(ij)} = \frac{d^{(ij)}}{\sum\limits_{(lm)\in\psi_{N+2}} d^{(lm)}}\,. \tag{B.10}$$

Similar to the NLO case the functions $\Delta^{(ij)}$ provide partition of unity

$$\sum_{(ij)\in\psi_{N+2}} \Delta^{(ij)} = 1\,, \tag{B.11}$$

where the sum is over unordered pairs (ij).

For the NNLO computation, we require the double-soft (S_{lm}), the single-soft (S_l), the collinear (C_{lk}) and the triple-collinear ($C_{lk,m}$) limits of the partition functions $\Delta^{(ij)}$. The double-soft limit reads

$$S_{lm}\,\Delta^{(ij)} = \delta_{(ij),(lm)}\,, \tag{B.12}$$

where the Kronecker delta indicates that the unordered pair (ij) should coincide with the unordered pair (lm) for this limit to be different from zero. The single-soft limit is

$$S_l\,\Delta^{(ij)} = \begin{cases} 0\,, & l\neq i,\ l\neq j\,, \\ \Delta^{(j)}\,, & l=i\,, \\ \Delta^{(i)}\,, & l=j\,, \end{cases} \tag{B.13}$$

where $\Delta^{(i)}$ and $\Delta^{(j)}$ in the above formulas are NLO partitions constructed for sets $\psi_{N+1} = \{j, \psi_N^{(ij)}\}$ and $\psi_{N+1} = \{i, \psi_N^{(ij)}\}$, respectively.

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Next, we consider the collinear limits. We find

$$C_{lk}\,\Delta^{(ij)}=\begin{cases}0\,, & l\neq i,j,\;\; k\neq i,j\,,\\ \Delta^{(k')}\,, & l\in\{1,2\},\;\; k\in\{i,j\}\,,\\ \Delta^{([ij])}\,, & \{l,k\}=\{i,j\}\,,\\ z_{k,i}\Delta^{(j)}\,, & l=i,\;\; k\neq j\,,\end{cases}\tag{B.14}$$

where $k'=i$ if $k=j$ and $k'=j$ if $k=i$, and $[ij]$ represents the "clustered particle" whose four-momentum is given by $p_{[ij]}=(1+E_j/E_i)\,p_i$ and the function $\Delta^{([ij])}$ is constructed from the set $\psi_{N+1}=\{[ij],\psi_N^{(ij)}\}$. In the final line of eq. (B.14), the Δ-function is constructed using the transverse momentum of the clustered particle $[kl]$.

It is instructive to explain how the last formula in eq. (B.14) is derived, since the other formulas in that equation can be computed in a similar way. To describe the collinear $i||k$ limit, where k is a final state particle, we write $\Delta^{(ij)}$ as follows

$$\Delta^{(ij)}=\frac{d^{(ij)}}{d^{(ij)}+d^{(ik)}+\sum\limits_{m\neq k,j}d^{(im)}+d^{(kj)}+\sum\limits_{m\neq i,j}d^{(km)}+\sum\limits_{m,n\neq i,k}d^{(mn)}}\,.\tag{B.15}$$

We now study what happens to the various entries in the above formula when the relevant limit is taken. First, we note that the numerator $d^{(ij)}$ does not contain i but contains k. We replace $p_{\perp,k}$ with $p_{\perp,[ik]}$ and write the resulting expression as

$$C_{ik}\,d^{(ij)}=\frac{E_k}{E_k+E_i}\,d^{(j)}=z_{k,i}\,d^{(j)}\,,\tag{B.16}$$

where $d^{(j)}$ is constructed using the list $\{j,\psi_{N+2}^{(ij)}(k\to[ki])\}$. The various entries in the denominator of eq. (B.15) behave as follows

$$\begin{aligned}
&C_{ik}\,d^{(ik)}=d^{([ik])}\,, && C_{ik}\sum_{m\neq k,j}^{N}d^{(im)}=z_{k,i}\sum_{m\neq k,j}^{N}d^{(m)}\,,\\
&C_{ik}\,d^{(kj)}=z_{i,k}\,d^{(j)}\,, && C_{ik}\sum_{m\neq i,j}^{N}d^{(km)}=z_{i,k}\sum_{m\neq i,j}^{N}d^{(m)}\,.
\end{aligned}\tag{B.17}$$

Therefore

$$C_{ik}\,\Delta^{(ij)}=z_{k,i}\frac{d^{(j)}}{d^{([ik])}+d^{(j)}+\sum_{m\neq j,i,k}d^{(m)}}=z_{k,i}\,\Delta^{(j)}\,,\tag{B.18}$$

with $\Delta^{(j)}$ being a NLO partition where partons i and k that appear in the original list of partons are clustered together.

Finally, formulas for triple-collinear limits can be derived in a similar way. We find that the only non-vanishing limits are

$$C_{kij}\,\Delta^{(ij)}=\begin{cases}1\,, & k\in\{1,2\}\,,\\ z_{k,ij}\Delta^{(ij)}\,, & k\in\{3,\ldots\}\,,\end{cases}\tag{B.19}$$

where $z_{k,ij}=E_k/(E_k+E_i+E_j)$.

In addition to Δ-partitions, which allow us to separate resolved and potentially unresolved partons, we require angular partition functions ω. These functions are supposed to define possible collinear singular directions between unresolved partons. Below we give an example of how such functions can be designed.

We begin with the construction of these angular partition functions at NLO. To this end, we consider a situation where parton $\mathfrak{m}$ is potentially unresolved, so that $\psi_{N+1} = \{\mathfrak{m}, \psi_N^{(\mathfrak{m})}\}$. We define the quantities

$$g_{kl} = \rho_{kl}^{-1}\,, \tag{B.20}$$

and use them to write the function $\omega^{\mathfrak{m}i}$ as

$$\omega^{\mathfrak{m}i} = \frac{g_{i\mathfrak{m}}}{\sum\limits_{j\in\psi_N^{(\mathfrak{m})}} g_{j\mathfrak{m}}}\,, \qquad i \in \psi_N^{(\mathfrak{m})}\,. \tag{B.21}$$

Since

$$\sum_{i\in\psi_N^{(\mathfrak{m})}} \omega^{\mathfrak{m}i} = 1\,, \qquad C_{k\mathfrak{m}}\,\omega^{\mathfrak{m}i} = \delta_{ki}\,, \tag{B.22}$$

the functions $\omega^{\mathfrak{m}i}$ possess the required properties to be used as angular partitions in NLO computations.

We continue with the discussion of the NNLO case, where partons $\mathfrak{m}$ and $\mathfrak{n}$ are potentially unresolved and the remaining N_p hard partons are described by the set $\psi_N^{(\mathfrak{mn})}$. We proceed as follows. First, we employ the NLO partitions to construct a partition of unity in the following way

$$1 = \sum_{\substack{i,j=1\\ i\neq j}}^{N_p} \omega^{\mathfrak{m}i}\omega^{\mathfrak{n}j} + \sum_{i,j=1}^{N_p} \delta_{ij}\,\omega^{\mathfrak{m}i}\omega^{\mathfrak{n}j}\,. \tag{B.23}$$

The two sums on the right-hand side are almost the right partitions for double- and triple-collinear limits except for the fact that the collinear $\mathfrak{m}||\mathfrak{n}$ singularity is present in both terms of this formula. However, we would like to move it into the triple-collinear partition. To achieve this, we introduce yet another partition of unity which involves $\rho_{\mathfrak{mn}}$, $\rho_{i\mathfrak{m}}$ and $\rho_{j\mathfrak{n}}$ only and write

$$1 = \frac{\rho_{\mathfrak{mn}}}{d_{\mathfrak{mn}ij}} + \frac{\rho_{i\mathfrak{m}} + \rho_{j\mathfrak{n}}}{d_{\mathfrak{mn}ij}}\,, \tag{B.24}$$

where

$$d_{\mathfrak{mn}ij} = \rho_{\mathfrak{mn}} + \rho_{i\mathfrak{m}} + \rho_{j\mathfrak{n}}\,. \tag{B.25}$$

We now employ these expressions to define the double-collinear partition

$$\omega^{\mathfrak{m}i,\mathfrak{n}j} = \omega^{\mathfrak{m}i}\omega^{\mathfrak{n}j}\,\frac{\rho_{\mathfrak{mn}}}{d_{\mathfrak{mn}ij}}\,, \qquad i \neq j\,, \tag{B.26}$$

and the triple-collinear partition

$$\omega^{\mathfrak{m}i,\mathfrak{n}i} = \omega^{\mathfrak{m}i}\omega^{\mathfrak{n}i} + \omega^{\mathfrak{n}i}\sum_{\substack{j=1\\ j\neq i}}^{N_p} \frac{\rho_{j\mathfrak{m}}\,\omega^{\mathfrak{m}j}}{d_{\mathfrak{mn}ji}} + \omega^{\mathfrak{m}i}\sum_{\substack{j=1\\ j\neq i}}^{N_p} \frac{\rho_{j\mathfrak{n}}\,\omega^{\mathfrak{n}j}}{d_{\mathfrak{mn}ij}}\,. \tag{B.27}$$

It is easy to check that the following identity holds

$$1 = \sum_{\substack{i,j=1 \\ i\neq j}}^{N_p} \omega^{\mathfrak{m}i,\mathfrak{n}j} + \sum_{i=1}^{N_p} \omega^{\mathfrak{m}i,\mathfrak{n}i} \,. \tag{B.28}$$

The partitions constructed in eqs. (B.26) and (B.27) satisfy all the properties that we need for NNLO QCD computations. In particular, each partition selects a minimal number of collinear singularities and satisfies the following relations

$$\begin{aligned}
C_{i\mathfrak{m}}\omega^{\mathfrak{m}i,\mathfrak{n}j} &= \omega^{\mathfrak{m}i,\mathfrak{n}j}_{\mathfrak{m}\|i} = \lim_{\rho_{i\mathfrak{m}}\to 0} \omega^{\mathfrak{m}i,\mathfrak{n}j} \,, & C_{j\mathfrak{n}}\omega^{\mathfrak{m}i,\mathfrak{n}j} &= \omega^{\mathfrak{m}i,\mathfrak{n}j}_{\mathfrak{n}\|j} = \lim_{\rho_{j\mathfrak{n}}\to 0} \omega^{\mathfrak{m}i,\mathfrak{n}j} \,, \\
C_{\mathfrak{m}\mathfrak{n}}\omega^{\mathfrak{m}i,\mathfrak{n}j} &= \delta_{ij}\, \omega^{\mathfrak{m}i,\mathfrak{n}j}_{\mathfrak{m}\|\mathfrak{n}} = \lim_{\rho_{\mathfrak{m}\mathfrak{n}}\to 0} \omega^{\mathfrak{m}i,\mathfrak{n}i} \,, & & \\
C_{i\mathfrak{m}}\, C_{\mathfrak{m}\mathfrak{n}}\omega^{\mathfrak{m}i,\mathfrak{n}j} &= \delta_{ij}\, C_{i\mathfrak{m}}\, C_{\mathfrak{m}\mathfrak{n}} \,, & C_{j\mathfrak{n}}\, C_{\mathfrak{m}\mathfrak{n}}\, \omega^{\mathfrak{m}i,\mathfrak{n}j} &= \delta_{ij}\, C_{j\mathfrak{n}}\, C_{\mathfrak{m}\mathfrak{n}} \,, \\
C_{i\mathfrak{m}}\, C_{j\mathfrak{n}}\, \omega^{\mathfrak{m}k,\mathfrak{n}l} &= \delta_{ki}\, \delta_{lj}\, C_{\mathfrak{m}i}\, C_{\mathfrak{n}j} \,, & C_{i\mathfrak{m}}\, C_{i\mathfrak{n}}\, \omega^{\mathfrak{m}j,\mathfrak{n}j} &= \delta_{ij}\, C_{i\mathfrak{m}}\, C_{i\mathfrak{n}} \,. \\
C_{\mathfrak{m}\mathfrak{n},i}\, \omega^{\mathfrak{m}j\mathfrak{n}j} &= \delta_{ij}\, C_{\mathfrak{m}\mathfrak{n},i} \,, & &
\end{aligned} \tag{B.29}$$

We note that these relations are important for simplifying the required subtraction terms. The partitions in eqs. (B.26) and (B.27) correspond to those defined in eq. (B.14) in ref. [1] when we restrict them to the case of color-singlet production, i.e. $N_p = 2$.

C Details of the NLO calculation

The goal of this appendix is to provide further details about the NLO computation described in section 3. In particular, we would like to show that the operator $I_{\rm T}(\epsilon)$ introduced in eq. (3.2) does not contain poles in ϵ. According to eq. (3.2), $I_{\rm T}(\epsilon)$ is given by a sum of three terms that describe virtual, soft and hard-collinear contributions.

We begin with the ϵ-expansion of the operator $I_{\rm S}$ defined in eq. (3.12). We report its definition here for convenience

$$I_{\rm S}(\epsilon) = -\frac{(2E_{\rm max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij}\, (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \,. \tag{C.1}$$

The function K_{ij} is defined in eqs. (3.14). We note that its expansion in ϵ reads

$$K_{ij} = 1 + K_{ij}^{(2)}\epsilon^2 + \mathcal{O}(\epsilon^3) \,, \qquad K_{ij}^{(2)} = {\rm Li}_2(1-\eta_{ij}) - \frac{\pi^2}{6} \,. \tag{C.2}$$

Although it is straightforward to construct the expansion of $I_{\rm S}$, arranging it in a particular way is helpful for an efficient demonstration of the cancellations of infrared poles.

We note that the I-operators include quantities raised to ϵ-dependent powers. For example, in the case of $I_{\rm S}$, there are factors $(2E_{\rm max}/\mu)^{-\epsilon}$ and $\eta_{ij}^{-\epsilon}$. The expansion of such quantities in ϵ starts with 1 and it is convenient to make this explicit. To this end, we introduce the function

$$f_k(x) = \frac{x^{-k\epsilon} - 1}{\epsilon} \,, \tag{C.3}$$

such that $f_k(x) \sim \mathcal{O}(\epsilon^0)$ as $\epsilon \to 0$. We then use this function to write $I_{\rm S}$ as

$$\begin{aligned}\left\langle I_{\rm S}(\epsilon)\cdot F_{\rm LM}\right\rangle = &-\frac{1}{\epsilon^2}\left\langle\Big[1+\epsilon f_2(2E_{\rm max}/\mu)\Big]\right.\\ &\left.\times\sum_{(ij)}^{N_p}\Big[1+\epsilon f_1(\eta_{ij})+\epsilon^2 K_{ij}^{(2)}\Big]\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle,\end{aligned} \tag{C.4}$$

where $\mathcal{O}(\epsilon)$ terms have been neglected. Since we only need terms through $\mathcal{O}(\epsilon^0)$, we can simplify the above equation further. We find

$$\begin{aligned}\left\langle I_{\rm S}(\epsilon)\cdot F_{\rm LM}\right\rangle = &-\frac{1}{\epsilon^2}\sum_{(ij)}^{N_p}\left\langle\Big[1+\epsilon f_2(2E_{\rm max}/\mu)\Big]\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle\\ &-\sum_{(ij)}^{N_p}\left\langle\frac{f_1(\eta_{ij})}{\epsilon}\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle\\ &-\sum_{(ij)}^{N_p}\left\langle\Big[f_1(\eta_{ij})f_2(2E_{\rm max}/\mu)+K_{ij}^{(2)}\Big]\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle.\end{aligned} \tag{C.5}$$

Next, we note that in the first term on the right-hand side in eq. (C.5), only the color charge operators depend on the summation indices i and j. For this reason, the summation over one of the indices can be performed using the color conservation condition

$$\sum_{k=1}^{N_p}\boldsymbol{T}_k|\mathcal{M}\rangle_c = 0\,. \tag{C.6}$$

It follows that

$$\sum_{j\neq i}^{N_p}{}_c\langle\mathcal{M}|\boldsymbol{T}_i\cdot\boldsymbol{T}_j|\mathcal{M}\rangle_c = -\boldsymbol{T}_i^2|\mathcal{M}|^2\,, \tag{C.7}$$

and we obtain

$$\begin{aligned}\left\langle I_{\rm S}(\epsilon)\cdot F_{\rm LM}\right\rangle = &\sum_{i=1}^{N_p}\left\langle\Big[1+\epsilon f_2(2E_{\rm max}/\mu)\Big]\,\frac{\boldsymbol{T}_i^2}{\epsilon^2}\,F_{\rm LM}\right\rangle\\ &-\sum_{(ij)}^{N_p}\left\langle\frac{f_1(\eta_{ij})}{\epsilon}\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle\\ &-\sum_{(ij)}^{N_p}\left\langle\Big[f_1(\eta_{ij})f_2(2E_{\rm max}/\mu)+K_{ij}^{(2)}\Big]\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle.\end{aligned} \tag{C.8}$$

It is seen from the above equation that the residue of the $1/\epsilon^2$ pole is proportional to the sum of the Casimir factors $\boldsymbol{T}_i^2$. We recall that the infrared poles of the one-loop amplitude described by Catani's function exhibit a similar feature. The $1/\epsilon$ pole in the second line of eq. (C.8) contains color correlations, while the terms in the third line are ϵ-finite.

We turn to the virtual corrections. We have introduced the operator $I_{\rm V}(\epsilon)$ in eq. (3.31), and we display it here for convenience

$$I_{\rm V}(\epsilon) = \overline{I}_1(\epsilon)+\overline{I}_1^\dagger(\epsilon)\,,\qquad \overline{I}_1(\epsilon) = \frac{1}{2}\sum_{(ij)}^{N_p}\frac{\mathcal{V}_i^{\rm sing}(\epsilon)}{\boldsymbol{T}_i^2}\,(\boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_j)\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon}e^{i\pi\lambda_{ij}\epsilon}\,. \tag{C.9}$$

The quantities λ_{ij} and $\mathcal{V}_i^{\rm sing}(\epsilon)$ are defined in eq. (3.29). Expanding in ϵ, we find

$$\begin{aligned}\langle I_{\rm V}(\epsilon)\cdot F_{\rm LM}\rangle = \sum_{(ij)}^{N_p}\frac{\mathcal{V}_i^{\rm sing}(\epsilon)}{\boldsymbol{T}_i^2}\Big\langle\Big[1+\epsilon f_1(s_{ij}/\mu^2)-\frac{\pi^2}{2}\lambda_{ij}\epsilon^2+\mathcal{O}(\epsilon^3)\Big]\\ \times(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\Big\rangle .\end{aligned} \tag{C.10}$$

In the first term on the right-hand side of eq. (C.10), we can use color conservation to sum over the index j. Doing so allows us to write the virtual contributions as follows

$$\begin{aligned}\langle I_{\rm V}(\epsilon)\cdot F_{\rm LM}\rangle = -\sum_{i=1}^{N_p}\left[\frac{\boldsymbol{T}_i^2}{\epsilon^2}+\frac{\gamma_i}{\epsilon}\right]\langle F_{\rm LM}\rangle+\sum_{(ij)}^{N_p}\left\langle\frac{f_1(s_{ij}/\mu^2)}{\epsilon}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle\\ +\sum_{\substack{i,j=1\\ j\neq i}}^{N_p}\left\langle\left[\frac{\gamma_i}{\boldsymbol{T}_i^2}f_1(s_{ij}/\mu^2)-\frac{\pi^2}{2}\lambda_{ij}\right](\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle ,\end{aligned} \tag{C.11}$$

where we have dropped all terms beyond $\mathcal{O}(\epsilon^0)$. Since $f_n(x)\sim\mathcal{O}(\epsilon^0)$, poles in the color-correlated structures appear only at $\mathcal{O}(\epsilon^{-1})$, while all terms in the last line are finite.

Comparing eqs. (C.8) and (C.11), we observe that the $\mathcal{O}(\epsilon^{-2})$ poles cancel among these two contributions. Furthermore, we note that the function $f_1(s_{ij}/\mu^2)$ in eq. (C.11) can be written as

$$f_1(s_{ij}/\mu^2)=f_1(\eta_{ij})+f_1(2E_i/\mu)+f_1(2E_j/\mu)+\epsilon\, g_{ij}\,. \tag{C.12}$$

The first term on the right-hand side above is the function that appears in the soft contribution $I_{\rm S}$, the next two terms depend on one of the two indices i or j, and the last term

$$g_{ij}=f_1(2E_i/\mu)f_1(2E_j/\mu)+f_1(4E_iE_j/\mu^2)f_1(\eta_{ij})\,, \tag{C.13}$$

is $\mathcal{O}(\epsilon^0)$. Thus we can further simplify the expression for $I_{\rm V}$ by making use of color conservation. We find

$$\sum_{(ij)}^{N}\left\langle\frac{f_1(2E_i/\mu)+f_1(2E_j/\mu)}{\epsilon}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\right\rangle=-2\sum_{i=1}^{N}\frac{\boldsymbol{T}_i^2}{\epsilon}\langle f_1(2E_i/\mu)\,F_{\rm LM}\rangle\,. \tag{C.14}$$

Upon combining soft and virtual I-operators, we obtain the following result

$$\begin{aligned}&\langle[I_{\rm V}(\epsilon)+I_{\rm S}(\epsilon)]\cdot F_{\rm LM}\rangle\\ &=\sum_{i=1}^{N_p}\left\langle\left[\frac{\boldsymbol{T}_i^2}{\epsilon}\Big(f_2(2E_{\rm max}/\mu)-2f_1(2E_i/\mu)\Big)-\frac{\gamma_i}{\epsilon}\right]F_{\rm LM}\right\rangle+\mathcal{O}(\epsilon^0)\\ &=-\sum_{i=1}^{N_p}\left\langle\left(2L_i\frac{\boldsymbol{T}_i^2}{\epsilon}+\frac{\gamma_i}{\epsilon}\right)F_{\rm LM}\right\rangle+\mathcal{O}(\epsilon^0)\,,\end{aligned} \tag{C.15}$$

where we substituted the expansion of $f_{1,2}(x)$ in ϵ and used $L_i=\log(E_{\rm max}/E_i)$. The above equation implies that the ϵ-divergences proportional to correlators of color charges cancel in the sum of the virtual and soft functions, $I_{\rm V}$ and $I_{\rm S}$.

To understand the cancellation of the remaining poles, we need to combine the above result with the operator $I_{\rm C}(\epsilon)$ defined in eq. (3.27). We repeat its definition here for convenience

$$I_{\rm C}(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}\,. \tag{C.16}$$

The generalized collinear anomalous dimension Γ_{i,f_i} that appears in the above equation can be found in eq. (3.22). Expanding it in powers of ϵ, we find

$$\Gamma_{i,f_i} = \gamma_i + 2\boldsymbol{T}_i^2 L_i + \mathcal{O}(\epsilon)\,, \qquad i = 1, \dots, N_p\,, \tag{C.17}$$

so that $I_{\rm C}(\epsilon)$ becomes

$$I_{\rm C}(\epsilon) = \sum_{i=1}^{N_p} \left(2L_i \frac{\boldsymbol{T}_i^2}{\epsilon} + \frac{\gamma_i}{\epsilon} \right) + \mathcal{O}(\epsilon^0)\,. \tag{C.18}$$

Comparing this result with eq. (C.15), we conclude that the following combination of I-operators

$$\langle I_{\rm T}(\epsilon) \cdot F_{\rm LM} \rangle = \langle [I_{\rm V}(\epsilon) + I_{\rm S}(\epsilon) + I_{\rm C}(\epsilon)] \cdot F_{\rm LM} \rangle\,, \tag{C.19}$$

is finite, as stated in the main text. Finally, we note that the cancellation between the initial-state collinear singularities and the PDFs renormalization has been discussed in detail in section 3.

D Partitions and sectors for the NNLO collinear limits

In section 4 we defined the soft-subtracted double-real contribution $\Sigma_{\rm RR}$, and we discussed the extraction of its collinear singularities. To do so, we first split the angular phase space into partitions using the functions $\omega^{\mathfrak{m}i,\mathfrak{n}j}$ defined in appendix B, and then further split the triple-collinear angular partitions into sectors using

$$\begin{aligned} \theta^{(a)} &= \Theta\left(\eta_{i\mathfrak{n}} < \frac{\eta_{i\mathfrak{m}}}{2}\right)\,, & \theta^{(c)} &= \Theta\left(\eta_{i\mathfrak{m}} < \frac{\eta_{i\mathfrak{n}}}{2}\right)\,, \\ \theta^{(b)} &= \Theta\left(\frac{\eta_{i\mathfrak{m}}}{2} < \eta_{i\mathfrak{n}} < \eta_{i\mathfrak{m}}\right)\,, & \theta^{(d)} &= \Theta\left(\frac{\eta_{i\mathfrak{n}}}{2} < \eta_{i\mathfrak{m}} < \eta_{i\mathfrak{n}}\right)\,. \end{aligned} \tag{D.1}$$

It follows that

$$\theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)} \equiv 1\,. \tag{D.2}$$

A parametrization of the angular phase space that naturally achieves this sectoring is given in ref. [20] and is detailed in appendix E.1. This procedure ensures that each partition and sector contains the minimal number of singular collinear limits. We then apply the appropriate collinear operators and write $\Sigma_{\rm RR}$ as the sum of four distinct contributions

$$\Sigma_{\rm RR} = \sum_{i=1}^{4} \Sigma_{\rm RR}^{(i)} \equiv \sum_{i=1}^{4} \langle \overline{S}_{\mathfrak{m}\mathfrak{n}}\, \overline{S}_{\mathfrak{n}}\, \Omega_i\, \Delta^{(\mathfrak{m}\mathfrak{n})}\, \Theta_{\mathfrak{m}\mathfrak{n}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \rangle\,, \tag{D.3}$$

where the four quantities Ω_i provide the partition of unity

$$\sum_{i=1}^{4} \Omega_i \equiv 1\,. \tag{D.4}$$

They read (cf. refs. [1, 61, 62])

$$\begin{aligned}
\Omega_1 &= \sum_{(ij)}^{N_p} \overline{C}_{i\mathfrak{m}} \overline{C}_{j\mathfrak{n}} [\mathrm{d}p_\mathfrak{m}][\mathrm{d}p_\mathfrak{n}]\, \omega^{\mathfrak{m}i,\mathfrak{n}j} \\
&\quad + \sum_{i=1}^{N_p} \left[\overline{C}_{i\mathfrak{n}}\theta^{(a)} + \overline{C}_{\mathfrak{m}\mathfrak{n}}\theta^{(b)} + \overline{C}_{i\mathfrak{m}}\theta^{(c)} + \overline{C}_{\mathfrak{m}\mathfrak{n}}\theta^{(d)}\right] [\mathrm{d}p_\mathfrak{m}][\mathrm{d}p_\mathfrak{n}]\, \overline{C}_{\mathfrak{m}\mathfrak{n},i}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}\,,
\end{aligned} \tag{D.5}$$

$$\Omega_2 = \sum_{i=1}^{N_p} \left[\overline{C}_{i\mathfrak{n}}\theta^{(a)} + \overline{C}_{\mathfrak{m}\mathfrak{n}}\theta^{(b)} + \overline{C}_{i\mathfrak{m}}\theta^{(c)} + \overline{C}_{\mathfrak{m}\mathfrak{n}}\theta^{(d)}\right] [\mathrm{d}p_\mathfrak{m}][\mathrm{d}p_\mathfrak{n}]\, C_{\mathfrak{m}\mathfrak{n},i}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}\,, \tag{D.6}$$

$$\Omega_3 = -\sum_{(ij)}^{N_p} C_{j\mathfrak{n}} C_{i\mathfrak{m}} [\mathrm{d}p_\mathfrak{m}][\mathrm{d}p_\mathfrak{n}]\, \omega^{\mathfrak{m}i,\mathfrak{n}j}\,, \tag{D.7}$$

$$\begin{aligned}
\Omega_4 &= \sum_{(ij)}^{N_p} \Big[C_{i\mathfrak{m}}[\mathrm{d}p_\mathfrak{m}] + C_{j\mathfrak{n}}[\mathrm{d}p_\mathfrak{n}]\Big]\, \omega^{\mathfrak{m}i,\mathfrak{n}j} \\
&\quad + \sum_{i=1}^{N_p} \left[C_{i\mathfrak{n}}\theta^{(a)} + C_{\mathfrak{m}\mathfrak{n}}\theta^{(b)} + C_{i\mathfrak{m}}\theta^{(c)} + C_{\mathfrak{m}\mathfrak{n}}\theta^{(d)}\right] [\mathrm{d}p_\mathfrak{m}][\mathrm{d}p_\mathfrak{n}]\, \omega^{\mathfrak{m}i,\mathfrak{n}i}\,,
\end{aligned} \tag{D.8}$$

where we have introduced the triple-collinear operator $C_{\mathfrak{m}\mathfrak{n},i}$, which extracts the singular behavior in the limit $\rho_{i\mathfrak{m}} \sim \rho_{i\mathfrak{n}} \sim \rho_{\mathfrak{m}\mathfrak{n}} \to 0$. We note that in the above definitions of Ω_i, $[\mathrm{d}p_\mathfrak{m}]$ and $[\mathrm{d}p_\mathfrak{n}]$ are phase-space elements for partons $\mathfrak{m}$ and $\mathfrak{n}$, and that they appear to the *right* of the single collinear operators ($C_{i\mathfrak{m}}$, $C_{\mathfrak{m}\mathfrak{n}}$, etc.) but to the *left* of the triple-collinear operators $C_{\mathfrak{m}\mathfrak{n},i}$. Therefore, the single-collinear operators act on the phase-space elements, while the triple-collinear operators do not [61]. This allows us to use the results of ref. [82] for Ω_2.

E Phase-space parametrization and collinear limits

E.1 Phase-space parametrizations for unresolved partons

In this subsection we describe phase-space parametrizations for two unresolved partons that naturally achieve the angular sectoring required for NNLO computations [20]. We recall that there are two distinct kinematic configurations that require different parametrizations. The first is a triple-collinear configuration which requires a genuine NNLO parametrization to describe strongly-ordered collinear limits. The second is the case where the two partons are emitted by different hard legs and can be described by two independent NLO-like parametrizations.

In both cases, we begin by separating the energy and the angular parts of the phase space and write

$$[\mathrm{d}p_\mathfrak{m}][\mathrm{d}p_\mathfrak{n}] = (\mathrm{d}E_\mathfrak{m}\, E_\mathfrak{m}^{1-2\epsilon})\, (\mathrm{d}E_\mathfrak{n}\, E_\mathfrak{n}^{1-2\epsilon}) [\mathrm{d}\Omega_{\mathfrak{m}\mathfrak{n}}^{(d-1)}]\,, \tag{E.1}$$

where

$$[\mathrm{d}\Omega_{\mathfrak{mn}}^{(d-1)}] = [\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}][\mathrm{d}\Omega_{\mathfrak{n}}^{(d-1)}]\,, \qquad [\mathrm{d}\Omega_{i}^{(d-1)}] = \frac{\mathrm{d}\Omega_i^{(d-1)}}{2(2\pi)^{d-1}}\,. \tag{E.2}$$

We first focus on the triple-collinear sectors and assume that the unresolved partons $\mathfrak{m}$ and $\mathfrak{n}$ are emitted by a hard parton i, with $i \in [1, N_p]$. It is convenient to choose the momentum of parton i as the reference direction. We then write

$$\begin{aligned} p_{\mathfrak{m}}^{\mu} &= E_{\mathfrak{m}}\left(t^{\mu} + \cos\theta_{i\mathfrak{m}}\, e_i^{\mu} + \sin\theta_{i\mathfrak{m}}\, b^{\mu}\right)\,, \\ p_{\mathfrak{n}}^{\mu} &= E_{\mathfrak{n}}\left(t^{\mu} + \cos\theta_{i\mathfrak{n}}\, e_i^{\mu} + \sin\theta_{i\mathfrak{n}}\left(\cos\phi_{\mathfrak{mn}}\, b^{\mu} + \sin\phi_{\mathfrak{mn}}\, a^{\mu}\right)\right)\,, \end{aligned} \tag{E.3}$$

where

$$t^{\mu} = (1, \vec{0})\,, \qquad e_i^{\mu} = (0, \vec{n}_i)\,, \qquad p_i^{\mu} = E_i(t^{\mu} + e_i^{\mu})\,. \tag{E.4}$$

Here $\vec{n}_i$ is a unit vector in $(d-1)$ spatial dimensions and a and b are d-dimensional unit vectors such that

$$t\cdot a = e_i \cdot a = t\cdot b = e_i\cdot b = a\cdot b = 0\,. \tag{E.5}$$

We can use this parametrization to express the angular part of the phase space as [20]

$$\begin{aligned} [\mathrm{d}\Omega_{\mathfrak{mn}}^{(d-1)}] = {} & \frac{\mathrm{d}\Omega_b^{(d-2)}\mathrm{d}\Omega_a^{(d-3)}}{2^{6\epsilon}(2\pi)^{2d-2}}[\eta_{i\mathfrak{m}}(1-\eta_{i\mathfrak{m}})]^{-\epsilon}[\eta_{i\mathfrak{n}}(1-\eta_{i\mathfrak{n}})]^{-\epsilon} \\ & \times \frac{|\eta_{i\mathfrak{m}}-\eta_{i\mathfrak{n}}|^{1-2\epsilon}}{D^{1-2\epsilon}}\frac{\mathrm{d}\eta_{i\mathfrak{m}}\,\mathrm{d}\eta_{i\mathfrak{n}}\,\mathrm{d}\lambda}{[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}\,, \end{aligned} \tag{E.6}$$

where

$$D = \eta_{i\mathfrak{m}} + \eta_{i\mathfrak{n}} - 2\eta_{i\mathfrak{m}}\,\eta_{i\mathfrak{n}} + 2(2\lambda-1)\sqrt{\eta_{i\mathfrak{m}}\,\eta_{i\mathfrak{n}}(1-\eta_{i\mathfrak{m}})(1-\eta_{i\mathfrak{n}})}\,. \tag{E.7}$$

The variable λ parametrizes the dependence on the azimuthal angle $\phi_{\mathfrak{mn}}$ through the relation

$$\sin^2\phi_{\mathfrak{mn}} = 4\lambda(1-\lambda)\frac{|\eta_{i\mathfrak{m}}-\eta_{i\mathfrak{n}}|^2}{D^2}\,. \tag{E.8}$$

The phase space can be split into four different sectors that we will refer to as (a), (b), (c), (d). The following parametrizations are chosen for each of the four sectors

$$a)\ \eta_{i\mathfrak{m}} = x_3\,, \qquad \eta_{i\mathfrak{n}} = x_3x_4/2\,, \tag{E.9}$$

$$b)\ \eta_{i\mathfrak{m}} = x_3\,, \qquad \eta_{i\mathfrak{n}} = x_3(1-x_4/2)\,, \tag{E.10}$$

$$c)\ \eta_{i\mathfrak{m}} = x_3x_4/2\,, \qquad \eta_{i\mathfrak{n}} = x_3\,, \tag{E.11}$$

$$d)\ \eta_{i\mathfrak{m}} = x_3(1-x_4/2)\,, \qquad \eta_{i\mathfrak{n}} = x_3\,, \tag{E.12}$$

with $0 \le x_{3,4} \le 1$. We use them to obtain explicit expressions for

$$[\mathrm{d}\Omega_{\mathfrak{mn}}^{(i)}] = [\mathrm{d}\Omega_{\mathfrak{mn}}^{(d-1)}]\,\theta^{(i)}\,, \qquad i = a,b,c,d\,, \tag{E.13}$$

with $\theta^{(i)}$ defined in eq. (D.1). It turns out that the angular phase spaces for sectors (a) and (c) and for sectors (b) and (d) are identical. For sectors (a) and (c) we find

$$\begin{aligned} [\mathrm{d}\Omega_{\mathfrak{mn}}^{(a,c)}] = {} & \left[\frac{1}{8\pi^2}\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\right]^2\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\frac{[\mathrm{d}\Omega_b^{(d-2)}]}{[\Omega_b^{(d-2)}]}\frac{[\mathrm{d}\Omega_a^{(d-3)}]}{[\Omega_a^{(d-3)}]} \\ & \times \frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}}\frac{\mathrm{d}x_4}{x_4^{1+\epsilon}}\frac{\mathrm{d}\lambda}{\pi[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}\left(256F_{\epsilon}^{(a,c)}\right)^{-\epsilon}4F_0^{(a,c)}x_3^2x_4\,. \end{aligned} \tag{E.14}$$

where

$$F_\epsilon^{(a,c)} = \frac{(1-x_3)(1-x_3x_4/2)(1-x_4/2)^2}{4[N(x_3,x_4/2,\lambda)]^2}\,, \qquad F_0^{(a,c)} = \frac{1-x_4/2}{2N(x_3,x_4/2,\lambda)}\,. \tag{E.15}$$

For sectors (b) and (d) we obtain

$$\begin{aligned}[\mathrm{d}\Omega_{\mathfrak{mn}}^{(b,d)}] = {}& \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2 \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] \frac{[\mathrm{d}\Omega_b^{(d-2)}]}{[\Omega_b^{(d-2)}]}\frac{[\mathrm{d}\Omega_a^{(d-3)}]}{[\Omega_a^{(d-3)}]} \\ & \times \frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}}\frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}}\frac{\mathrm{d}\lambda}{\pi[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}\left(256F_\epsilon^{(b,d)}\right)^{-\epsilon} 4F_0^{(b,d)}x_3^2x_4^2\,,\end{aligned} \tag{E.16}$$

where

$$F_\epsilon^{(b,d)} = \frac{(1-x_3)(1-x_4/2)(1-x_3(1-x_4/2))}{4[N(x_3,1-x_4/2,\lambda)]^2}\,, \quad F_0^{(b,d)} = \frac{1}{4N(x_3,1-x_4/2,\lambda)}\,. \tag{E.17}$$

The function $N(x_3,x_4,\lambda)$ introduced in the above equations reads

$$N(x_3,x_4,\lambda) = 1 + x_4(1-2x_3) - 2(1-2\lambda)\sqrt{x_4(1-x_3)(1-x_3x_4)}\,. \tag{E.18}$$

To simplify the subtraction terms, we need particular collinear limits of the unresolved phase space. To obtain those, we note that the following identities hold

$$\begin{aligned} &\lim_{x_4\to 0} F_\epsilon^{(a,c)} = \frac{1-x_3}{2}\,, && \lim_{x_4\to 0} F_0^{(a,c)} = \frac{1}{2}\,, \\ &\lim_{x_4\to 0} F_\epsilon^{(b,d)} = \frac{1}{64\lambda^2}\,, && \lim_{x_4\to 0} F_0^{(b,d)} = \frac{1}{16\lambda(1-x_3)}\,. \end{aligned} \tag{E.19}$$

The $x_4 \to 0$ limit corresponds to the $\mathfrak{n}||i$ and $\mathfrak{m}||i$ collinear limits in sectors (a) and (c), respectively, and to the $\mathfrak{m}||\mathfrak{n}$ limit in sectors (b) and (d). The singular quantities in sectors (a) and (c) are $\eta_{i\mathfrak{n}}$ and $\eta_{i\mathfrak{m}}$, respectively, and they are given in eqs. (E.9) and (E.11). For sectors (b) and (d), the limit of the corresponding singular variable is more complex. It reads

$$\lim_{x_4\to 0}\eta_{\mathfrak{mn}} = \lim_{x_4\to 0}\frac{x_3x_4^2}{4N(x_3,1-x_4,\lambda)} = \frac{x_3x_4^2}{16\lambda(1-x_3)} \equiv \bar{\eta}_{\mathfrak{mn}}\,. \tag{E.20}$$

The phase-space parametrization is significantly simpler for the double-collinear partitions. Consider the case when parton $\mathfrak{m}$ is collinear to parton i and parton $\mathfrak{n}$ to parton j, with $i \neq j$. We parametrize the momenta $p_\mathfrak{m}$ and $p_\mathfrak{n}$ using the momenta of partons i and j respectively, i.e.

$$\begin{aligned} p_\mathfrak{m}^\mu &= E_\mathfrak{m}(t^\mu + \cos\theta_{i\mathfrak{m}}\, e_i^\mu + \sin\theta_{i\mathfrak{m}}\, b_\mathfrak{m}^\mu)\,, \\ p_\mathfrak{n}^\mu &= E_\mathfrak{n}(t^\mu + \cos\theta_{j\mathfrak{n}}\, e_j^\mu + \sin\theta_{j\mathfrak{m}}\, b_\mathfrak{n}^\mu)\,, \end{aligned} \tag{E.21}$$

and set

$$\eta_{i\mathfrak{m}} = x_3\,, \qquad \eta_{j\mathfrak{n}} = x_4\,. \tag{E.22}$$

We then write the angular phase space for the double-collinear partition $[\mathrm{d}\Omega_{\mathfrak{mn}}^{\mathrm{dc}}]$ as

$$[\mathrm{d}\Omega_{\mathfrak{mn}}^{\mathrm{dc}}] \equiv [\mathrm{d}\Omega_\mathfrak{m}^{(d-1)}][\mathrm{d}\Omega_\mathfrak{n}^{(d-1)}]\,, \tag{E.23}$$

where

$$\begin{aligned} [\mathrm{d}\Omega_\mathfrak{m}^{(d-1)}] &= \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] 2^{4-4\epsilon}\frac{[\mathrm{d}\Omega_\mathfrak{m}^{(d-2)}]}{[\Omega^{(d-2)}]}\frac{\mathrm{d}x_3}{x_3^{1+\epsilon}}(1-x_3)^{-\epsilon}x_3\,, \\ [\mathrm{d}\Omega_\mathfrak{n}^{(d-1)}] &= \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] 2^{4-4\epsilon}\frac{[\mathrm{d}\Omega_\mathfrak{n}^{(d-2)}]}{[\Omega^{(d-2)}]}\frac{\mathrm{d}x_4}{x_4^{1+\epsilon}}(1-x_4)^{-\epsilon}x_4\,. \end{aligned} \tag{E.24}$$

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E.2 Action of the collinear operators on the phase space

In our definitions of the angular terms $\Omega_{1,\ldots,4}$ in eqs. (D.5)-(D.8), the collinear operators act on the phase space of the two unresolved partons. As we have seen, it is useful to rewrite the subtraction terms in such a way that these operators do not act on the phase-space measure. We have quoted the results in the main text of the paper without deriving them, see e.g. eq. (4.30). The goal of this subsection is to provide the omitted details.

We begin by considering the double-collinear partitioning with a collinear operator $C_{i\mathfrak{m}}$; an example can be found in the first term on the right-hand side of eq. (4.26). Since in the double-collinear parametrization of the phase space the collinear limit $i||\mathfrak{m}$ is controlled by the variable x_3 (see eq. (E.22)), we find

$$\begin{aligned}\int C_{i\mathfrak{m}} \frac{[\mathrm{d}\Omega^{\mathrm{dc}}_{\mathfrak{mn}}]}{\rho_{i\mathfrak{m}}}[\ldots] &= \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] 2^{3-4\epsilon}\int \frac{[\mathrm{d}\Omega^{(d-2)}_{\mathfrak{m}}]}{[\Omega^{(d-2)}]}[\mathrm{d}\Omega^{(d-1)}_{\mathfrak{n}}]\int\limits_0^1 \frac{\mathrm{d}x_3}{x_3^{1+\epsilon}}\, C_{i\mathfrak{m}}[\ldots] \\ &= -\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\frac{2^{3-4\epsilon}}{\epsilon}\int \frac{[\mathrm{d}\Omega^{(d-2)}_{\mathfrak{m}}]}{[\Omega^{(d-2)}]}[\mathrm{d}\Omega^{(d-1)}_{\mathfrak{n}}]\, C_{i\mathfrak{m}}[\ldots]\,,\end{aligned} \tag{E.25}$$

where $[\ldots]$ stands for generic non-singular contributions whose exact form is not relevant for the following discussion. If we repeat the above steps without acting with $C_{i\mathfrak{m}}$ on the phase space, we find

$$\begin{aligned}&\int \frac{[\mathrm{d}\Omega^{\mathrm{dc}}_{\mathfrak{mn}}]}{\rho_{i\mathfrak{m}}}\, C_{i\mathfrak{m}}\,[\ldots] \\ &= \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right] 2^{3-4\epsilon}\int \frac{[\mathrm{d}\Omega^{(d-2)}_{\mathfrak{m}}]}{[\Omega^{(d-2)}]}[\mathrm{d}\Omega^{(d-1)}_{\mathfrak{n}}]\int\limits_0^1 \frac{\mathrm{d}x_3}{x_3^{1+\epsilon}}(1-x_3)^{-\epsilon} C_{i\mathfrak{m}}\,[\ldots] \\ &= -\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\frac{2^{3-4\epsilon}}{\epsilon}\int \frac{[\mathrm{d}\Omega^{(d-2)}_{\mathfrak{m}}]}{[\Omega^{(d-2)}]}[\mathrm{d}\Omega^{(d-1)}_{\mathfrak{n}}]\, C_{i\mathfrak{m}}\,[\ldots]\,.\end{aligned} \tag{E.26}$$

Comparing the two formulas, we conclude that

$$\int C_{i\mathfrak{m}} \frac{[\mathrm{d}\Omega^{\mathrm{dc}}_{\mathfrak{mn}}]}{\rho_{i\mathfrak{m}}}[\ldots] = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\int \frac{[\mathrm{d}\Omega^{\mathrm{dc}}_{\mathfrak{mn}}]}{\rho_{i\mathfrak{m}}}\, C_{i\mathfrak{m}}\,[\ldots]\,. \tag{E.27}$$

We can use the above relation when rewriting eq. (4.17) as eq. (4.19). Since in this case we have two collinear operators $C_{j\mathfrak{n}}C_{i\mathfrak{m}}$, we need to apply it twice, i.e.

$$\int C_{j\mathfrak{n}}C_{i\mathfrak{m}} \frac{[\mathrm{d}\Omega^{\mathrm{dc}}_{\mathfrak{mn}}]}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{n}}}[\ldots] = \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\right]^2\int \frac{[\mathrm{d}\Omega^{\mathrm{dc}}_{\mathfrak{mn}}]}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{n}}}\, C_{j\mathfrak{n}}C_{i\mathfrak{m}}\,[\ldots]\,, \tag{E.28}$$

so that eq. (4.17) becomes

$$\Sigma_{\mathrm{RR,2c}} = -\left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\right]^2 \sum_{(ij)}^{N_p}\left\langle \overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}C_{j\mathfrak{n}}C_{i\mathfrak{m}}\,\omega^{\mathfrak{m}i,\mathfrak{n}j}\Delta^{(\mathfrak{mn})}\,\Theta_{\mathfrak{mn}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\right\rangle. \tag{E.29}$$

We stress that the absence of the phase space $[\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]$ in the above equation indicates that the collinear operators $C_{j\mathfrak{n}}C_{i\mathfrak{m}}$ no longer act on it.

Similar formulas can also be derived for the triple-collinear partitions that involve sector $\theta^{(c)}$. As an example, we discuss the second term on the right-hand side of eq. (4.26). In this case, the collinear limit $i||\mathfrak{m}$ corresponds to the $x_4 \to 0$ limit in the phase space parametrization in eq. (E.14). We use eq. (E.19) to compute this limit and find

$$\int C_{i\mathfrak{m}} \frac{[\mathrm{d}\Omega^{(c)}_{\mathfrak{m}\mathfrak{n}}]}{\rho_{i\mathfrak{m}}} [\ldots] = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \int \frac{[\mathrm{d}\Omega^{(d-1)}_{\mathfrak{m}\mathfrak{n}}]}{\rho_{i\mathfrak{m}}} (\eta_{i\mathfrak{n}}/2)^{-\epsilon} C_{i\mathfrak{m}} [\ldots] , \tag{E.30}$$

where the integration over the angular variables of parton $\mathfrak{m}$ on the right hand side of eq. (E.30) is *not restricted* to sector (c) anymore. It follows from the above discussion that eq. (4.26) can be rewritten as

$$\begin{aligned} \Sigma^{(a,c,\mathrm{dc})}_{\mathrm{RR},1\mathrm{c}} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \Big\langle \mathcal{S}(\mathfrak{m},\mathfrak{n}) \Big[\sum_{(ij)}^{N_p} C_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i,\mathfrak{n}j} \\ + \sum_{i=1}^{N_p} (\eta_{i\mathfrak{n}}/2)^{-\epsilon} C_{i\mathfrak{m}}\, \omega^{\mathfrak{m}i,\mathfrak{n}i} \Big] \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \Big\rangle . \end{aligned} \tag{E.31}$$

This expression is the starting point to obtain eqs. (4.30) and (4.31).

Finally, we perform similar manipulations for sector (b) where the collinear limit of interest is $\mathfrak{m}||\mathfrak{n}$. This limit corresponds to $x_4 \to 0$ in the phase space parametrization given in eq. (E.16). Using eq. (E.19) we find

$$\begin{aligned} \int C_{\mathfrak{m}\mathfrak{n}} \frac{[\mathrm{d}\Omega^{(b,d)}_{\mathfrak{m}\mathfrak{n}}]}{\rho_{\mathfrak{m}\mathfrak{n}}} [\ldots] = \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] N^{(b,d)}_\epsilon \int [\mathrm{d}\Omega^{(d-1)}_{[\mathfrak{m}\mathfrak{n}]}] \\ \times \eta^{-\epsilon}_{i[\mathfrak{m}\mathfrak{n}]} \left(1-\eta_{i[\mathfrak{m}\mathfrak{n}]}\right)^\epsilon \mathrm{d}\Lambda \frac{\mathrm{d}\Omega^{(d-3)}_a}{[\Omega^{(d-3)}]} \frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}} C_{\mathfrak{m}\mathfrak{n}}[\ldots] . \end{aligned} \tag{E.32}$$

The normalization constants $N^{(b,d)}_\epsilon$ that appear in eq. (E.32) can be found in eq. (A.5), while $[\mathrm{d}\Omega^{(d-1)}_{[\mathfrak{m}\mathfrak{n}]}]$ is the (exact) angular phase space of the clustered parton $[\mathfrak{m}\mathfrak{n}]$, whose momentum $p_{[\mathfrak{m}\mathfrak{n}]} = p_\mathfrak{m} + p_\mathfrak{n}$ must be computed in the strict collinear limit. Furthermore we have introduced a new variable Λ such that

$$\mathrm{d}\Lambda = \frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)}{\Gamma(1+2\epsilon)\,\Gamma(1-2\epsilon)} \frac{\lambda^{-1/2+\epsilon}(1-\lambda)^{-1/2-\epsilon}}{\pi} \mathrm{d}\lambda . \tag{E.33}$$

We note that the action of the operator $C_{\mathfrak{m}\mathfrak{n}}$ on the matrix element squared is non-trivial because it can lead to integrands that depend on the parameter λ and the transverse vector a^μ. This phenomenon, known as spin correlations, is discussed in the next appendix. Here we consider only those terms for which the action of $C_{\mathfrak{m}\mathfrak{n}}$ in eq. (4.24) does not lead to such terms. In this case we can integrate over x_4, the directions of a^μ, and the azimuthal variable Λ using

$$\int \mathrm{d}\Lambda = 1 . \tag{E.34}$$

Comparing the result with the one that is obtained when the collinear operator $C_{\mathfrak{m}\mathfrak{n}}$ does not act on the phase space, we find

$$\int C_{\mathfrak{m}\mathfrak{n}} \frac{[\mathrm{d}\Omega^{(b,d)}_{\mathfrak{m}\mathfrak{n}}]}{\rho_{\mathfrak{m}\mathfrak{n}}} [\ldots] = 2^{2\epsilon-1} \frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)} \int \frac{[\mathrm{d}\Omega^{(d-1)}_{\mathfrak{m}\mathfrak{n}}]}{\rho_{\mathfrak{m}\mathfrak{n}}} C_{\mathfrak{m}\mathfrak{n}}[\ldots] , \tag{E.35}$$

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where the integration over the angular variables of partons on the right-hand side is unrestricted. We use this relation in eq. (4.48) and the analysis that follows.

As we just mentioned, the action of the collinear operator $C_{\mathfrak{mn}}$ on matrix elements may result in a limit that depends on λ and a^μ. In such cases eq. (E.35) cannot be used. To understand how to proceed, we write (see appendix F)

$$C_{\mathfrak{mn}}\,F_{\rm LM}(\mathfrak{m},\mathfrak{n}) = \frac{g_{s,b}^2}{E_{\mathfrak{n}}E_{\mathfrak{m}}\,\rho_{\mathfrak{mn}}}\left[-\,g^{\mu\nu}P_{gg}^{(0)}(z) + P_{gg}^{\perp}(z)\,\kappa_{\perp,(b)}^{\mu}\kappa_{\perp,(b)}^{\nu}\right]F_{{\rm LM},\mu\nu}([\mathfrak{mn}])\,, \tag{E.36}$$

where vector $\kappa_{\perp,(b)}$ is a unit space-like vector which is orthogonal to $p_{\mathfrak{m}}$

$$\kappa_{\perp,(b)}\cdot p_{\mathfrak{m}} = 0\,. \tag{E.37}$$

Using the phase space parametrization for sector (b), we can write this vector as

$$\kappa_{\perp,(b)}^{\mu} = a^{\mu}\sqrt{1-\lambda} + r_{i,(b)}^{\mu}\,\sqrt{\lambda}\,, \tag{E.38}$$

where vectors a and b were introduced in eq. (E.3) and $r_{i,(b)}$ is the auxiliary spacelike vector ($r_{i,(b)}\cdot r_{i,(b)} = -1$) defined as

$$r_{i,(b)}^{\mu} = \sin\theta_{i\mathfrak{m}}\,e_i^{\mu} - \cos\theta_{i\mathfrak{m}}\,b^{\mu}\,. \tag{E.39}$$

The momentum of the clustered parton $[\mathfrak{mn}]$ is aligned with the momentum $p_{\mathfrak{m}}$, which does not depend on λ and a^μ. Since $F_{\rm LM}([\mathfrak{mn}])$ is independent of λ and a^μ, we can integrate over $\mathrm{d}\Omega_a^{(d-3)}$ and $\mathrm{d}\Lambda$. Specifically, we need to calculate

$$\left\langle\kappa_{\perp,(b)}^{\mu}\,\kappa_{\perp,(b)}^{\nu}\right\rangle = \int\mathrm{d}\Lambda\,\frac{\mathrm{d}\Omega_a^{(d-3)}}{\Omega^{(d-3)}}\;\kappa_{\perp,(b)}^{\mu}\,\kappa_{\perp,(b)}^{\nu}\,. \tag{E.40}$$

To compute this integral, we use eq. (E.34) together with

$$\begin{aligned}
&\int\mathrm{d}\Lambda\,\frac{\mathrm{d}\Omega_a^{(d-3)}}{\Omega^{(d-3)}}\,a^{\mu} = 0\,, \qquad &&\int\mathrm{d}\Lambda\,\frac{\mathrm{d}\Omega_a^{(d-3)}}{\Omega^{(d-3)}}\,a^{\mu}\,a^{\nu} = -\frac{g_{\perp,(d-3)_a}^{\mu\nu}}{d-3}\,,\\
&\int\mathrm{d}\Lambda\,\lambda = \frac{1+2\epsilon}{2}\,, &&\int\mathrm{d}\Lambda\,(1-\lambda) = \frac{1-2\epsilon}{2}\,,
\end{aligned} \tag{E.41}$$

and find

$$\begin{aligned}
\left\langle\kappa_{\perp,(b)}^{\mu}\,\kappa_{\perp,(b)}^{\nu}\right\rangle &= -\frac{g_{\perp,(d-3)}^{\mu\nu}}{2} + \frac{1+2\epsilon}{2}r_{i,(b)}^{\mu}r_{i,(b)}^{\nu}\\
&= \frac{1}{2}\left[g_{\perp,(d-3)}^{\mu\nu} + r_{i,(b)}^{\mu}r_{i,(b)}^{\nu}\right] + \epsilon\,r_{i,(b)}^{\mu}r_{i,(b)}^{\nu}\\
&\equiv -\frac{g_{\perp,(d-2)}^{\mu\nu}}{2} + \epsilon\,r_{i,(b)}^{\mu}\,r_{i,(b)}^{\nu}\,.
\end{aligned} \tag{E.42}$$

We then obtain

$$\left\langle\kappa_{\perp,(b)}^{\mu}\,\kappa_{\perp,(b)}^{\nu}\right\rangle F_{{\rm LM},\mu\nu} = \frac{1}{2}F_{\rm LM} + \epsilon\,r_{i,(b)}^{\mu}\,r_{i,(b)}^{\nu}\,F_{{\rm LM},\mu\nu}\,, \tag{E.43}$$

where we used

$$-g_{\perp,(d-2)}^{\mu\nu}F_{{\rm LM},\mu\nu} = -g^{\mu\nu}\,F_{{\rm LM},\mu\nu} = F_{\rm LM}\,, \tag{E.44}$$

as allowed by the transversality of scattering amplitudes.

F Spin correlations

In this appendix we discuss the double-real contributions where the so-called spin correlations appear. These effects arise in sectors (b) and (d) in the limits when gluons $\mathfrak{m}$ and $\mathfrak{n}$ become collinear to each other. To make this appendix self-contained, we start by considering the $\mathfrak{m}||\mathfrak{n}$ limit, which is described by the following expression (see also eq. (E.36))

$$\begin{aligned}C_{\mathfrak{mn}}\,F_{\rm LM}(\mathfrak{m},\mathfrak{n}) &= \frac{g_{s,b}^2}{E_{\mathfrak{m}}E_{\mathfrak{n}}\,\rho_{\mathfrak{mn}}}\,P_{gg}^{\mu\nu}(z)\,F_{{\rm LM},\mu\nu}([\mathfrak{mn}])\\ &= \frac{g_{s,b}^2}{E_{\mathfrak{n}}E_{\mathfrak{m}}\,\rho_{\mathfrak{mn}}}\left[P_{gg}^{(0)}(z)F_{\rm LM}([\mathfrak{mn}])+P_{gg}^{\perp}(z)\,\kappa_{\perp,(b)}^{\mu}\kappa_{\perp,(b)}^{\nu}\,F_{{\rm LM},\mu\nu}([\mathfrak{mn}])\right],\end{aligned} \tag{F.1}$$

where the splitting functions were introduced in appendix A.4.1, $z = E_{\mathfrak{m}}/(E_{\mathfrak{m}}+E_{\mathfrak{n}})$, and $\kappa_{\perp,(b)}$ is defined in eq. (E.38). The four-momentum of the clustered parton $[\mathfrak{mn}]$ is equal to

$$p^{\mu}_{\mathfrak{mn}} = (E_{\mathfrak{m}}+E_{\mathfrak{n}})\,n^{\mu}_{\mathfrak{m}} = \frac{E_{\mathfrak{m}}+E_{\mathfrak{n}}}{E_{\mathfrak{m}}}\,p^{\mu}_{\mathfrak{m}}\,, \tag{F.2}$$

where the vector $n_{\mathfrak{m}}$ is a light-like vector defined as $n_{\mathfrak{m}} = p_{\mathfrak{m}}/E_{\mathfrak{m}}$. To proceed further, we assume that the collinear limit $\mathfrak{m}||\mathfrak{n}$ occurs in a particular triple-collinear partitioning, characterized by the partition function $\omega^{\mathfrak{m}i,\mathfrak{n}i}$, and to restrict our analysis to sector (b). The contribution that we are interested in reads (see eq. (4.24))

$$\begin{aligned}\Sigma^{(b)}_{{\rm RR},1{\rm c},i} &= \langle \overline{S}_{\mathfrak{mn}}\,\overline{S}_{\mathfrak{n}}\,\Theta_{\mathfrak{mn}}\,C_{\mathfrak{mn}}\,\theta^{(b)}[{\rm d}p_{\mathfrak{m}}][{\rm d}p_{\mathfrak{n}}]\,\omega^{\mathfrak{m}i,\mathfrak{n}i}\,\Delta^{(\mathfrak{mn})}\,F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle\\ &= -\frac{[\alpha_s]}{2\epsilon}\,N_{\epsilon}^{(b,d)}\left\langle \overline{S}_{\mathfrak{mn}}\,\overline{S}_{\mathfrak{n}}\int\limits_{0}^{E_{\max}}\frac{{\rm d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{2\epsilon-1}}\,\frac{{\rm d}E_{\mathfrak{n}}}{E_{\mathfrak{n}}^{2\epsilon-1}}\,\Theta_{\mathfrak{mn}}\int[{\rm d}\Omega_{[\mathfrak{mn}]}]\,\sigma_{i[\mathfrak{mn}]}^{-\epsilon}\,\Delta^{([\mathfrak{mn}])}\right.\\ &\quad\left.\times\,\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}}\,\frac{1}{E_{\mathfrak{m}}E_{\mathfrak{n}}}\left[P_{gg}(z,\epsilon)\,F_{\rm LM}([\mathfrak{mn}])+\epsilon\,P_{gg}^{\perp}(z)\,r^{\mu}_{i,(b)}\,r^{\nu}_{i,(b)}\,F_{{\rm LM},\mu\nu}([\mathfrak{mn}])\right]\right\rangle,\end{aligned} \tag{F.3}$$

where we recall that $\sigma_{ij} = \eta_{ij}/(1-\eta_{ij})$. To derive eq. (F.3) we exploited the parametrization presented in appendix E, integrated over the angles of parton $\mathfrak{n}$ and used the relation displayed in eq. (E.43). All the splitting functions that appear in eq. (F.3) can be found in appendix A.

We note that the hard matrix element squared appears in eq. (F.3) in two distinct ways: once as $F_{\rm LM}([\mathfrak{mn}])$ and once as $F_{{\rm LM},\mu\nu}([\mathfrak{mn}])$, where the open spin indices refer to the clustered parton. In fact, the relation between the two contributions reads

$$F_{\rm LM}([\mathfrak{mn}]) = \sum_{\lambda_{[\mathfrak{mn}]}}\varepsilon_{\mu}^{\lambda_{[\mathfrak{mn}]}}\varepsilon_{\nu}^{\lambda_{[\mathfrak{mn}]},*}F_{{\rm LM},\mu\nu}([\mathfrak{mn}]) = -g_{\mu\nu}\,F_{{\rm LM},\mu\nu}([\mathfrak{mn}])\,, \tag{F.4}$$

where the sum runs over the physical polarizations of the clustered parton $[\mathfrak{mn}]$ and the last step follows from the transversality of $F_{{\rm LM},\mu\nu}$.

In eq. (F.3) the only term that requires further discussion is the one proportional to $F_{{\rm LM},\mu\nu}([\mathfrak{mn}])$. In fact, we find it convenient to split these terms in such a way that the coefficient of $F_{\rm LM}([\mathfrak{mn}])$ in eq. (F.3) is the spin-averaged $g\to gg$ splitting function P_{gg} (cf. eq. (A.23)) and the soft subtraction term associated with it. We will refer to all other contributions that appear in the expression for $\Sigma^{(b)}_{\mathrm{RR},1\mathrm{c},i}$ as "spin-correlated". Hence, we write

$$\Sigma^{(b)}_{\mathrm{RR},1\mathrm{c},i} = -\frac{[\alpha_s]}{2\epsilon} N_\epsilon^{(b,d)} \left\langle \int\limits_0^{E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{2\epsilon-1}} \frac{\mathrm{d}E_{\mathfrak{n}}}{E_{\mathfrak{n}}^{2\epsilon-1}} \Theta_{\mathfrak{mn}} \int [\mathrm{d}\Omega_{[\mathfrak{mn}]}]\, \sigma_{i[\mathfrak{mn}]}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}} \right.$$
$$\times \left\{ \frac{1}{E_{\mathfrak{m}} E_{\mathfrak{n}}} P_{gg}(z)\, \overline{S}_{\mathfrak{mn}}\, \Delta^{([\mathfrak{mn}])} F_{\mathrm{LM}}([\mathfrak{mn}]) - \frac{2C_A}{E_{\mathfrak{n}}^2} \overline{S}_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m}) \right. \tag{F.5}$$
$$\left.\left. + \frac{\epsilon}{E_{\mathfrak{m}} E_{\mathfrak{n}}} \left[P_{gg}^{\perp}(z)\, (r^\mu_{i,(b)}\, r^\nu_{i,(b)} + g^{\mu\nu}) - P_{gg}^{\perp,r}(z)\, g^{\mu\nu} \right] \overline{S}_{\mathfrak{mn}}\, \Delta^{([\mathfrak{mn}])} F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}]) \right\} \right\rangle,$$

where we have used the relation

$$\left[P_{gg}(z,\epsilon) + \epsilon\, P_{gg}^{\perp}(z) \right] F_{\mathrm{LM}}([\mathfrak{mn}]) = P_{gg}(z)\, F_{\mathrm{LM}}([\mathfrak{mn}]) - \epsilon\, g^{\mu\nu}\, P_{gg}^{\perp,r}(z) F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])\,, \tag{F.6}$$

with $P_{gg}^{\perp,r}$ defined in eq. (A.25). The second line in eq. (F.5) contains "spin-averaged" and the third line "spin-correlated" contributions. They read

$$\Sigma^{(b),\mathrm{sa}}_{\mathrm{RR},1\mathrm{c},i} = -\frac{[\alpha_s]}{2\epsilon} N_\epsilon^{(b,d)} \left\langle \int\limits_0^{E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{2\epsilon-1}} \frac{\mathrm{d}E_{\mathfrak{n}}}{E_{\mathfrak{n}}^{2\epsilon-1}} \Theta_{\mathfrak{mn}} \int [\mathrm{d}\Omega_{[\mathfrak{mn}]}]\, \sigma_{i[\mathfrak{mn}]}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}} \right.$$
$$\left. \times \left[\frac{1}{E_{\mathfrak{m}} E_{\mathfrak{n}}} P_{gg}(z)\, \overline{S}_{\mathfrak{mn}}\, \Delta^{([\mathfrak{mn}])} F_{\mathrm{LM}}([\mathfrak{mn}]) - \frac{2C_A}{E_{\mathfrak{n}}^2} \overline{S}_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m}) \right] \right\rangle, \tag{F.7}$$

and

$$\Sigma^{(b),\mathrm{sc}}_{\mathrm{RR},1\mathrm{c},i} = -\frac{[\alpha_s]}{2} N_\epsilon^{(b,d)} \left\langle \int\limits_0^{E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{2\epsilon-1}} \frac{\mathrm{d}E_{\mathfrak{n}}}{E_{\mathfrak{n}}^{2\epsilon-1}} \Theta_{\mathfrak{mn}} \int [\mathrm{d}\Omega_{[\mathfrak{mn}]}]\, \sigma_{i[\mathfrak{mn}]}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}} \right.$$
$$\left. \times \frac{1}{E_{\mathfrak{m}} E_{\mathfrak{n}}} \left[P_{gg}^{\perp}(z)\, (r^\mu_{i,(b)}\, r^\nu_{i,(b)} + g^{\mu\nu}) - P_{gg}^{\perp,r}(z)\, g^{\mu\nu} \right] \overline{S}_{\mathfrak{mn}}\, \Delta^{([\mathfrak{mn}])} F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}]) \right\rangle. \tag{F.8}$$

We continue with the discussion of the spin-correlated collinear limits. We find that after adding the contribution of sector (d) to $\Sigma^{(b),\mathrm{sc}}_{\mathrm{RR},1\mathrm{c},i}$, the energy-ordering constraint disappears and we obtain the following expression for the full spin-correlated part

$$\Sigma^{(b,d),\mathrm{sc}}_{\mathrm{RR},1\mathrm{c},i} = -\frac{[\alpha_s]}{2} N_\epsilon^{(b,d)} \left\langle \int\limits_0^{E_{\max}} \frac{\mathrm{d}E_{\mathfrak{m}}}{E_{\mathfrak{m}}^{2\epsilon-1}} \frac{\mathrm{d}E_{\mathfrak{n}}}{E_{\mathfrak{n}}^{2\epsilon-1}} \int [\mathrm{d}\Omega_{[\mathfrak{mn}]}]\, \sigma_{i[\mathfrak{mn}]}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}} \right.$$
$$\left. \times \frac{1}{E_{\mathfrak{m}} E_{\mathfrak{n}}} \left[P_{gg}^{\perp}(z)\, (r^\mu_{i}\, r^\nu_{i} + g^{\mu\nu}) - P_{gg}^{\perp,r}(z)\, g^{\mu\nu} \right] \overline{S}_{\mathfrak{mn}}\, \Delta^{([\mathfrak{mn}])} F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}]) \right\rangle, \tag{F.9}$$

where we have relabelled $r_{i,(b)}$ as r_i for brevity. We note that the energy integration for each of the two particles $\mathfrak{m}$ and $\mathfrak{n}$ extends to $E_{\max}$. As we discussed in section 4.1, this leads to a possible contribution of the unphysical region $E_{[\mathfrak{mn}]} > E_{\max}$. Since $E_{\max}$ is chosen in such a way that $F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])$ has no support for $E_{[\mathfrak{mn}]} > E_{\max}$, only the soft subtraction term contributes in this case. Hence, in the above formula we can write

$$\overline{S}_{\mathfrak{mn}} = \Theta_{[\mathfrak{mn}],\max}\, \overline{S}_{[\mathfrak{mn}]} - \Theta_{\max,[\mathfrak{mn}]} S_{[\mathfrak{mn}]}\,, \tag{F.10}$$

where in the first (second) term on the right hand side the energy of the clustered particle is restricted to be smaller (larger) than $E_{\rm max}$, respectively. The integration over the energies of partons $\mathfrak{m}$ and $\mathfrak{n}$ can be rearranged to conform with the above splitting of the soft operator

$$\int_0^{E_{\rm max}} {\rm d}E_{\mathfrak{m}} \int_0^{E_{\rm max}} {\rm d}E_{\mathfrak{n}} = \int_0^{E_{\rm max}} {\rm d}E_{[\mathfrak{mn}]}\, E_{[\mathfrak{mn}]} \int_0^1 {\rm d}z + \int_{E_{\rm max}}^{2E_{\rm max}} {\rm d}E_{[\mathfrak{mn}]}\, E_{[\mathfrak{mn}]} \int_{1-\frac{E_{\rm max}}{E_{[\mathfrak{mn}]}}}^{\frac{E_{\rm max}}{E_{[\mathfrak{mn}]}}} {\rm d}z\,. \tag{F.11}$$

Following this rearrangement, we have (cf. eq. (4.52))

$$\Sigma^{(b,d),{\rm sc}}_{{\rm RR},1{\rm c},i} = \Sigma^{(b,d),{\rm sc},I}_{{\rm RR},1{\rm c},i} + \Sigma^{(b,d),{\rm sc},II}_{{\rm RR},1{\rm c},i}\,, \tag{F.12}$$

where

$$\begin{aligned}\Sigma^{(b,d),{\rm sc},I}_{{\rm RR},1{\rm c},i} =& -\frac{[\alpha_s]}{2} N_\epsilon^{(b,d)} \left\langle \int [{\rm d}p_{[\mathfrak{mn}]}]\, E_{[\mathfrak{mn}]}^{-2\epsilon}\, \Theta_{{\rm max},[\mathfrak{mn}]}\, \sigma_{i[\mathfrak{mn}]}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}} \right.\\ &\times \int_0^1 \frac{{\rm d}z}{\big[z(1-z)\big]^{2\epsilon}} \Big[P^{\perp}_{gg}(z)\,(r_i^\mu\, r_i^\nu + g^{\mu\nu}) - P^{\perp,r}_{gg}(z)\, g^{\mu\nu}\Big]\\ &\left.\times\, \overline{S}_{[\mathfrak{mn}]}\, \Delta^{([\mathfrak{mn}])}\, F_{{\rm LM},\mu\nu}([\mathfrak{mn}]) \right\rangle, \end{aligned} \tag{F.13}$$

and

$$\begin{aligned}\Sigma^{(b,d),{\rm sc},II}_{{\rm RR},1{\rm c},i} =& -\frac{[\alpha_s]}{2} N_\epsilon^{(b,d)} \left\langle \int [{\rm d}p_{[\mathfrak{mn}]}]\, E_{[\mathfrak{mn}]}^{-2\epsilon}\, \Theta_{{\rm max},[\mathfrak{mn}]}\, \sigma_{i[\mathfrak{mn}]}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}} \right.\\ &\times \int_{1-\frac{E_{\rm max}}{E_{[\mathfrak{mn}]}}}^{\frac{E_{\rm max}}{E_{[\mathfrak{mn}]}}} \frac{{\rm d}z}{\big[z(1-z)\big]^{2\epsilon}} \Big[P^{\perp}_{gg}(z)\,(r_i^\mu\, r_i^\nu + g^{\mu\nu}) - P^{\perp,r}_{gg}(z)\, g^{\mu\nu}\Big]\\ &\left.\times\, S_{[\mathfrak{mn}]}\, \Delta^{([\mathfrak{mn}])}\, F_{{\rm LM},\mu\nu}([\mathfrak{mn}]) \right\rangle, \end{aligned} \tag{F.14}$$

where $[{\rm d}p_{[\mathfrak{mn}]}]$ identifies the phase space of the clustered parton $[\mathfrak{mn}]$. We first discuss $\Sigma^{(b,d),{\rm sc},I}_{{\rm RR},1{\rm c},i}$, where the integration over z decouples from the rest and can be easily performed. We find

$$\begin{aligned}\Sigma^{(b,d),{\rm sc},I}_{{\rm RR},1{\rm c},i} =& \frac{[\alpha_s]}{2} N_\epsilon^{(b,d)} \left\langle \int [{\rm d}p_{[\mathfrak{mn}]}]\, E_{[\mathfrak{mn}]}^{-2\epsilon}\, \Theta_{{\rm max},[\mathfrak{mn}]}\, \sigma_{i[\mathfrak{mn}]}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}||\mathfrak{n}} \right.\\ &\left.\times \Big[\gamma^{22}_{\perp,g\to gg}\,(r_i^\mu\, r_i^\nu + g^{\mu\nu}) - \gamma^{22,r}_{\perp,g\to gg}\, g^{\mu\nu}\Big] \overline{S}_{[\mathfrak{mn}]}\, \Delta^{([\mathfrak{nn}])}\, F_{{\rm LM},\mu\nu}([\mathfrak{mn}]) \right\rangle, \end{aligned} \tag{F.15}$$

where the functions $\gamma^{22}_{\perp,g\to gg}$ and $\gamma^{22,r}_{\perp,g\to gg}$ are given in eq. (A.29). Since this contribution is soft-regulated, the only singularity left there is $[\mathfrak{mn}]||i$. To regularize and extract this collinear divergence, we insert $1 = \overline{C}_{i[\mathfrak{mn}]} + C_{i[\mathfrak{mn}]}$ into the above formula and obtain

$$\Sigma^{(b,d),{\rm sc},I}_{{\rm RR},1{\rm c},i} = \Sigma^{(b,d),{\rm sc},I,1}_{{\rm RR},1{\rm c},i} + \Sigma^{(b,d),{\rm sc},I,2}_{{\rm RR},1{\rm c},i}\,, \tag{F.16}$$

where

$$\Sigma^{(b,d),\mathrm{sc},I,1}_{\mathrm{RR},1\mathrm{c},i} = \frac{[\alpha_s]}{2} N_\epsilon^{(b,d)} \left\langle \int [dp_{\mathfrak{m}}]\, E_{\mathfrak{m}}^{-2\epsilon}\, \Theta_{\mathrm{max},\mathfrak{m}}\, C_{i\mathfrak{m}}\, \sigma_{i\mathfrak{m}}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}} \right.$$
$$\left. \times \left[\gamma^{22}_{\perp,g\to gg}\, (r_i^\mu\, r_i^\nu + g^{\mu\nu}) - \gamma^{22,r}_{\perp,g\to gg}\, g^{\mu\nu}\right] \overline{S}_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})}\, F_{\mathrm{LM},\mu\nu}(\mathfrak{m}) \right\rangle , \tag{F.17}$$

and

$$\Sigma^{(b,d),\mathrm{sc},I,2}_{\mathrm{RR},1\mathrm{c},i} = \frac{[\alpha_s]}{2} N_\epsilon^{(b,d)} \left\langle \mathcal{O}^{(i)}_{\mathrm{NLO}} E_{\mathfrak{m}}^{-2\epsilon}\, \theta_{\mathrm{max},\mathfrak{m}}\, \sigma_{i\mathfrak{m}}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}} \right.$$
$$\left. \times \left[\gamma^{22}_{\perp,g\to gg}\, (r_i^\mu\, r_i^\nu + g^{\mu\nu}) - \gamma^{22,r}_{\perp,g\to gg}\, g^{\mu\nu}\right] \Delta^{(\mathfrak{m})}\, F_{\mathrm{LM},\mu\nu}(\mathfrak{m}) \right\rangle . \tag{F.18}$$

We note that we have relabelled $[\mathfrak{mn}] \mapsto \mathfrak{m}$ when writing the above equations. The function $\Sigma^{(b,d),\mathrm{sc},I,2}_{\mathrm{RR},1\mathrm{c},i}$ is a fully-regulated single-unresolved contribution which is finite in the limit $\epsilon \to 0$ and can be numerically integrated in four space-time dimensions.

On the other hand, the quantity $\Sigma^{(b,d),\mathrm{sc},I,1}_{\mathrm{RR},1\mathrm{c},i}$ will include a $1/\epsilon$ pole once we integrate over the unresolved parton $\mathfrak{m}$. To do this, we need to evaluate the soft and collinear limits of $r_i^\mu r_i^\nu F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])$, which we have not encountered before. Doing so requires us to revisit the construction of the vectors r_i. We recall that, following eq. (E.3), the angular parametrization employs the direction of parton i as a reference axis, so that (cf. eq. (E.4))

$$p_i^\mu = E_i (t^\mu + e_i^\mu) , \tag{F.19}$$

where t is a time-like vector with $t^2 = 1$ and e_i is a space-like vector with $e_i^2 = -1$. The momentum of the clustered particle $[\mathfrak{mn}]$ is defined as

$$p^\mu_{[\mathfrak{mn}]} = E_{[\mathfrak{mn}]} \left(t^\mu + \cos\theta_{[\mathfrak{mn}]i}\, e_i^\mu + \sin\theta_{[\mathfrak{mn}]i}\, b_i^\mu \right) , \tag{F.20}$$

with

$$b_i^\mu\, e_{i,\mu} = 0 . \tag{F.21}$$

The vector r_i reads

$$r_i^\mu = \sin\theta_{[\mathfrak{mn}]i}\, e_i^\mu - \cos\theta_{[\mathfrak{mn}]i}\, b_i^\mu , \tag{F.22}$$

from which it follows that

$$p^\mu_{[\mathfrak{mn}]}\, r_{i,\mu} = 0 . \tag{F.23}$$

This implies that r_i is a valid polarization vector for the clustered gluon $[\mathfrak{mn}]$. Armed with this understanding, it is straightforward to write a general expression for the soft limits $S_{[\mathfrak{mn}]}$ of spin-correlated amplitudes-squared. We find

$$r_i^\mu r_i^\nu\, S_{[\mathfrak{mn}]}\, F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}]) = -g_{s,b}^2 \sum_{k,l=1}^{N_p} \frac{(p_k \cdot r_i)\,(p_l \cdot r_i)}{(p_k \cdot p_{[\mathfrak{mn}]})\,(p_l \cdot p_{[\mathfrak{mn}]})}\, (\boldsymbol{T}_k \cdot \boldsymbol{T}_l) \cdot F_{\mathrm{LM}} . \tag{F.24}$$

One also needs to consider the limit $C_{i[\mathfrak{mn}]}$ of this expression, which develops singularities arising from two contributions in the sum: first from $k = i, l = i$, and second from $k = i, l \neq i$ or $k \neq i, l = i$.

We begin with the first one and write

$$\frac{(p_i\cdot r_i)(p_i\cdot r_i)}{(p_i\cdot p_{[\mathfrak{mn}]})\,(p_i\cdot p_{[\mathfrak{mn}]})}=\frac{1}{E^2_{[\mathfrak{mn}]}}\frac{\sin^2\theta_{[\mathfrak{mn}]i}}{(1-\cos\theta_{[\mathfrak{mn}]i})^2}=\frac{1}{E^2_{[\mathfrak{mn}]}}\frac{(2-\rho_{[\mathfrak{mn}]i})}{\rho_{[\mathfrak{mn}]i}}\,, \tag{F.25}$$

where we used the explicit parametrization of momenta p_i and $p_{[\mathfrak{mn}]}$ and the vector r_i. The collinear limit of the term in eq. (F.24) with $k=i, l=i$ therefore reads

$$-g^2_{s,b}\,C_{i[\mathfrak{mn}]}\,\frac{(p_i\cdot r_i)\,(p_i\cdot r_i)}{(p_i\cdot p_{[\mathfrak{mn}]})(p_i\cdot p_{[\mathfrak{mn}]})}\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_i)\cdot F_{\mathrm{LM}}=\frac{-2g^2_{s,b}}{E^2_{[\mathfrak{mn}]}\,\rho_{[\mathfrak{mn}]i}}\,\boldsymbol{T}_i^2\,F_{\mathrm{LM}}\,. \tag{F.26}$$

Next, we consider terms with $k=i$ and $l\neq i$

$$-g^2_{s,b}\sum_{l\neq i}\frac{(p_i\cdot r_i)\,(p_l\cdot r_i)}{(p_i\cdot p_{[\mathfrak{mn}]})\,(p_l\cdot p_{[\mathfrak{mn}]})}\,(\boldsymbol{T}_i\cdot\boldsymbol{T}_l)\cdot F_{\mathrm{LM}}\,. \tag{F.27}$$

Since $p_i\cdot r_i\sim\sin\theta_{[\mathfrak{mn}]i}$ and $p_i\cdot p_{[\mathfrak{mn}]}\sim(1-\cos\theta_{[\mathfrak{mn}]i})$, and all other factors in the above expression are regular in the limit $\theta_{[\mathfrak{mn}]i}\to 0$, we conclude that this contribution is actually integrable in the collinear limit $[\mathfrak{mn}]\,||\,i$. The same conclusion holds for the symmetric $k\neq i$ and $l=i$ terms. Hence, we find the following result

$$C_{i[\mathfrak{mn}]}\,S_{[\mathfrak{mn}]}\,r_i^\mu r_i^\nu\,F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])=-\frac{2g^2_{s,b}}{E^2_{[\mathfrak{mn}]}\,\rho_{[\mathfrak{mn}]i}}\,\boldsymbol{T}_i^2\,F_{\mathrm{LM}}\,. \tag{F.28}$$

This coincides with the limit without spin correlations, $C_{i[\mathfrak{mn}]}\,S_{[\mathfrak{mn}]}F_{\mathrm{LM}}([\mathfrak{mn}])$, so that

$$C_{i[\mathfrak{mn}]}\,S_{[\mathfrak{mn}]}(g^{\mu\nu}+r_i^\mu r_i^\nu)\,F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])=0\,. \tag{F.29}$$

We can use this cancellation to simplify $\Sigma^{(b,d),\mathrm{sc},I,1}_{\mathrm{RR,1c},i}$ in eq. (F.17). We write

$$\begin{aligned}\Sigma^{(b,d),\mathrm{sc},I,1}_{\mathrm{RR,1c},i}=&\frac{[\alpha_s]}{2}\,N_\epsilon^{(b,d)}\,\bigg\langle\int[\mathrm{d}p_{[\mathfrak{mn}]}]\,E^{-2\epsilon}_{[\mathfrak{mn}]}\,\Theta_{\mathrm{max},[\mathfrak{mn}]}\,\eta^{-\epsilon}_{i[\mathfrak{mn}]}\\&\times\Big[\gamma^{22}_{\perp,g\to gg}\,C_{i[\mathfrak{mn}]}\,(r_i^\mu\,r_i^\nu+g^{\mu\nu})-\gamma^{22,r}_{\perp,g\to gg}\,g^{\mu\nu}C_{i[\mathfrak{mn}]}\,\overline{S}_{[\mathfrak{mn}]}\Big]\\&\times\Delta^{([\mathfrak{mn}])}\,F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])\bigg\rangle\,.\end{aligned} \tag{F.30}$$

The only limit in the above equation that we have not yet encountered is

$$C_{i[\mathfrak{mn}]}\,r_i^\mu\,r_i^\nu\,F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])\,. \tag{F.31}$$

As we will show later, it evaluates to

$$C_{i[\mathfrak{mn}]}\,r_i^\mu\,r_i^\nu\,F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])=\frac{g^2_{s,b}}{p_i\cdot p_{[\mathfrak{mn}]}}\cdot\begin{cases}P^{\mathrm{spin}}_{f_if_{[i\mathfrak{m}]}}(z)\otimes F^{(i)}_{\mathrm{LM}}(z\cdot[\mathfrak{mn}i])\,, & i\leq 2\,,\\ P^{\mathrm{spin}}_{f_if_{[i\mathfrak{m}]}}(z)\,F_{\mathrm{LM}}\,, & i>2\,,\end{cases} \tag{F.32}$$

where the splitting functions are given by the following equations

$$P^{\mathrm{spin}}_{f_if_{[i\mathfrak{m}]}}(z)=\begin{cases}\frac{1}{2}\,C_F\,\frac{(1+z)^2}{1-z}\,, & f_i=f_{[i\mathfrak{m}]}=\{q,\bar{q}\}\,,\\ 2\,C_A\left[\frac{z}{1-z}+\frac{(1-z)/z+z(1-z)}{2(1-\epsilon)}\right]\,, & f_i=f_{[i\mathfrak{m}]}=g\,,\end{cases} \tag{F.33}$$

and we have adopted the convention that $F_{\rm LM}^{(1)}(z\cdot 1)\equiv F_{\rm LM}(z\cdot 1_a, 2_b,\ldots)/z$ and $F_{\rm LM}^{(2)}(z\cdot 2)\equiv F_{\rm LM}(1_a, z\cdot 2_b,\ldots)/z$.

We are now in the position to evaluate the limits of eq. (F.30) and to integrate over the angular phase space. For the final-state emissions ($i>2$) we find

$$\begin{aligned}\Sigma^{(b,d),{\rm sc},I,1}_{{\rm RR},1{\rm c},i}\Big|_{i>2} = \frac{[\alpha_s]}{2}\,N_\epsilon^{(b,d)}\,\Big\langle\int[{\rm d}p_i][{\rm d}p_{[\mathfrak{mn}]}]\,E_{[\mathfrak{mn}]}^{-2\epsilon}\,\Theta_{{\rm max},[\mathfrak{mn}]}\,\eta_{i[\mathfrak{mn}]}^{-\epsilon}\\ \times\Big[\gamma^{22}_{\perp,g\to gg}\,\frac{g_{s,b}^2}{E_{[\mathfrak{mn}]}E_i\rho_{i[\mathfrak{mn}]}}\,\Big(P_{gg}^{\rm spin}(z)-P_{gg}(z)\Big)\,z\,F_{\rm LM}([\mathfrak{mn}i])\\ +\gamma^{22,r}_{\perp,g\to gg}\,C_{i[\mathfrak{mn}]}\overline{S}_{[\mathfrak{mn}]}\,z\,F_{\rm LM}([\mathfrak{mn}])\Big]\Big\rangle\,,\end{aligned}\tag{F.34}$$

while for $i\le 2$ we find

$$\begin{aligned}\Sigma^{(b,d),{\rm sc},I,1}_{{\rm RR},1{\rm c},i}\Big|_{i\le 2} = \frac{[\alpha_s]}{2}\,N_\epsilon^{(b,d)}\,\Big\langle\int[{\rm d}p_{[\mathfrak{mn}]}]\,E_{[\mathfrak{mn}]}^{-2\epsilon}\,\Theta_{{\rm max},[\mathfrak{mn}]}\,\eta_{i[\mathfrak{mn}]}^{-\epsilon}\\ \times\Big[\gamma^{22}_{\perp,g\to gg}\,\frac{g_{s,b}^2}{E_{[\mathfrak{mn}]}E_i\rho_{i[\mathfrak{mn}]}}\,\Big(P_{qq}^{\rm spin}(z)-P_{qq}(z)\Big)\otimes F_{\rm LM}^{(i)}(z\cdot[\mathfrak{mn}i])\\ +\gamma^{22,r}_{\perp,g\to gg}\,C_{i[\mathfrak{mn}]}\overline{S}_{[\mathfrak{mn}]}\,F_{\rm LM}([\mathfrak{mn}])\Big]\Big\rangle.\end{aligned}\tag{F.35}$$

We can then integrate over the remaining energy and angular variables using the formulas in appendix A.2 and obtain

$$\begin{aligned}\Sigma^{(b,d),{\rm sc},I,1}_{{\rm RR},1{\rm c},i}\Big|_{i>2} &= \frac{[\alpha_s]^2}{4\epsilon}\,\frac{2^{2\epsilon}\,\Gamma(1-\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}N_\epsilon^{(b,d)}\,\Big\langle\Big(\frac{2E_i}{\mu}\Big)^{-4\epsilon}\\ &\times\Big[-\gamma^{22}_{\perp,g\to gg}\Big[\gamma^{24}_{z,g\to gg}-\gamma^{24,\,{\rm spin}}_{z,g\to gg}\Big]+\gamma^{22,r}_{\perp,g\to gg}\,\gamma^{24}_{z,g\to gg}(\epsilon,L_i)\Big]\,F_{\rm LM}\Big\rangle\,,\\ \Sigma^{(b,d),{\rm sc},I,1}_{{\rm RR},1{\rm c},i}\Big|_{i\le 2} &= \frac{[\alpha_s]^2}{4\epsilon}\,\frac{2^{2\epsilon}\,\Gamma(1-\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}N_\epsilon^{(b,d)}\,\Big\langle\Big(\frac{2E_i}{\mu}\Big)^{-4\epsilon}\\ &\times\Big\{\gamma^{22}_{\perp,g\to gg}\int_0^1{\rm d}z\,(1-z)^{-4\epsilon}\Big[P_{qq}(z)-P_{qq}^{\rm spin}(z)\Big]\otimes F_{\rm LM}^{(i)}(z\cdot i)\\ &-\gamma^{22,r}_{\perp,g\to gg}\int_0^1{\rm d}z\,\mathcal{P}_{qq}^{(4)}(z,L_i)\otimes F_{\rm LM}^{(i)}(z\cdot i)\Big\}\Big\rangle\,,\end{aligned}\tag{F.36}$$

where we have defined

$$\gamma^{24,\,{\rm spin}}_{z,g\to gg} = -\int_0^1 dz\Big[\,z^{-2\epsilon}\,(1-z)^{-4\epsilon}\,z\,P_{gg}^{\rm spin}(z)-2\,C_A\,(1-z)^{-1-4\epsilon}\Big]\,.\tag{F.37}$$

Finally, combining emissions off different legs, we write $\Sigma^{(b,d),\mathrm{sc},I,1}_{\mathrm{RR,1c}}$ as

$$\begin{aligned}\Sigma^{(b,d),\mathrm{sc},I,1}_{\mathrm{RR,1c}} = &\frac{[\alpha_s]^2}{4\epsilon}\frac{2^{2\epsilon}\,\Gamma(1-\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}N_\epsilon^{(b,d)} \\ &\times\Bigg\{-\sum_{i=3}^{N_p}\Bigg\langle\left(\frac{2E_i}{\mu}\right)^{-4\epsilon}\gamma^{22}_{\perp,g\to gg}\left[\gamma^{24}_{z,g\to gg}-\gamma^{24,\,\mathrm{spin}}_{z,g\to gg}\right]F_{\mathrm{LM}}\Bigg\rangle \\ &+\sum_{i=1}^{2}\Bigg\langle\left(\frac{2E_i}{\mu}\right)^{-4\epsilon}\gamma^{22}_{\perp,g\to gg}\int_0^1\mathrm{d}z\,(1-z)^{-4\epsilon}\Big[P_{qq}(z)-P^{\mathrm{spin}}_{qq}(z)\Big]\otimes F^{(i)}_{\mathrm{LM}}(z\cdot i)\Bigg\rangle\Bigg\} \\ &+\frac{[\alpha_s]^2}{4\epsilon}N^{(b,d)}_{sc}\,\gamma^{22,r}_{\perp,g\to gg}\sum_{i=1}^{2}\Bigg\langle\int_0^1\mathrm{d}z\,\mathcal{P}^{(4),\mathrm{gen}}_{qq}(z)\otimes F^{(i)}_{\mathrm{LM}}(z\cdot i)\Bigg\rangle \\ &+\frac{[\alpha_s]^2}{2}\,N^{(b,d)}_{sc}\,\gamma^{22,r}_{\perp,g\to gg}\langle I^{(4)}_{\mathrm{C}}(\epsilon)\cdot F_{\mathrm{LM}}\rangle\,,\end{aligned} \tag{F.38}$$

where we have introduced

$$N^{(b,d)}_{\mathrm{sc}} = \frac{2^{2\epsilon}\,\Gamma^3(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma^3(1-\epsilon)}N^{(b,d)}_\epsilon\,. \tag{F.39}$$

We return to the "unphysical" contribution $\Sigma^{(b,d),\mathrm{sc},II}_{\mathrm{RR,1c},i}$ of eq. (F.12). Using eq. (F.24), we can immediately obtain the soft limit $S_{[\mathfrak{mn}]}F_{\mathrm{LM},\mu\nu}[\mathfrak{mn}]$. Integrating over $E_{[\mathfrak{mn}]}$ and z, we find

$$\begin{aligned}\Sigma^{(b,d),\mathrm{sc},II}_{\mathrm{RR,1c},i} =&[\alpha_s]\,g^2_{s,b}\left(\frac{E_{\max}}{\mu}\right)^{-4\epsilon}\Bigg\langle\int\mathrm{d}\Omega_{[\mathfrak{mn}]}\,\sigma^{-\epsilon}_{i[\mathfrak{mn}]}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}}\Big[\delta^\perp_g\,(r_i^\mu r_i^\nu+g^{\mu\nu}) \\ &-\delta^{\perp,r}_g\,g^{\mu\nu}\Big]\sum_{k,l=1}^{N_p}\frac{n_{k,\mu}\,n_{l,\nu}}{(n_k\cdot n_{[\mathfrak{mn}]})\,(n_l\cdot n_{[\mathfrak{mn}]})}\,(\boldsymbol{T}_k\cdot\boldsymbol{T}_l)\cdot F_{\mathrm{LM}}\Bigg\rangle\,,\end{aligned} \tag{F.40}$$

where $\delta^\perp_g$ and $\delta^{\perp,r}_g$ are given in eq. (A.30). At this point, we introduce the functions

$$\begin{aligned}\langle\mathcal{W}^{(i)}_r\cdot F_{\mathrm{LM}}\rangle \equiv&\int\frac{[\mathrm{d}\Omega^{(d-1)}_{[\mathfrak{mn}]}]}{[\Omega^{(d-2)}]}\Bigg\langle\sigma^{-\epsilon}_{i[\mathfrak{mn}]}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}} \\ &\times(r_i^\mu\,r_i^\nu+g^{\mu\nu})\sum_{k,l=1}^{N_p}\frac{n_{k,\mu}\,n_{l,\nu}}{(n_k\cdot n_{[\mathfrak{mn}]})(n_l\cdot n_{[\mathfrak{mn}]})}\,F^{(kl)}_{\mathrm{LM}}\Bigg\rangle\,, \\ \langle\mathcal{W}^{i\|\mathfrak{n}}_i\cdot F_{\mathrm{LM}}\rangle \equiv&-\epsilon\,2^{2\epsilon}\int\frac{[\mathrm{d}\Omega^{(d-1)}_{[\mathfrak{mn}]}]}{[\Omega^{(d-2)}]}\Bigg\langle\sigma^{-\epsilon}_{i[\mathfrak{mn}]}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}} \\ &\times\sum_{\substack{k,l=1\\k\neq l}}^{N_p}\frac{n_k\cdot n_l}{(n_k\cdot n_{[\mathfrak{mn}]})(n_l\cdot n_{[\mathfrak{mn}]})}\,F^{(kl)}_{\mathrm{LM}}\Bigg\rangle\,,\end{aligned} \tag{F.41}$$

where we have used the shorthand notation $F^{(ij)}_{\mathrm{LM}}=(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\mathrm{LM}}$, and write eq. (F.40) as

$$\Sigma^{(b,d),\mathrm{sc},II}_{\mathrm{RR,1c},i} = [\alpha_s]^2\left(\frac{E_{\max}}{\mu}\right)^{-4\epsilon}\Bigg\langle\delta^\perp_g\,\mathcal{W}^{(i)}_r\cdot F_{\mathrm{LM}}+\delta^{\perp,r}_g\,\frac{2^{-2\epsilon}}{\epsilon}\mathcal{W}^{i\|\mathfrak{n}}_i\cdot F_{\mathrm{LM}}\Bigg\rangle\,. \tag{F.42}$$

The function $\mathcal{W}_r^{(i)}$ is finite in ϵ because the pole arising from the term proportional to $r_i^\mu r_i^\nu$ cancels with that arising from the $g^{\mu\nu}$ term. This can be understood as follows: the most singular contribution affecting the term proportional to $r_i^\mu r_i^\nu$ stems from the combination $k = l = i$, since the partition functions damp all other potential collinear configurations. In this case, the singularity is proportional to $2C_{f_i}/\rho_{[\mathfrak{mn}]i}$, as we already saw in eq. (F.26). On the other hand, the singularity proportional to $g^{\mu\nu}$ can only arise when $k = i, l \neq i$ and $k \neq i, l = 1$, given that $n_l^2 = n_k^2 = 0$. We can then isolate the divergent ratio $1/(n_i \cdot n_{[\mathfrak{mn}]})$ and sum over colors, obtaining precisely $-2C_{f_i}/\rho_{[\mathfrak{mn}]i}$. We conclude that $\mathcal{W}_r^{(i)}$ does not contribute to the pole content of $\Sigma^{(b,d),\mathrm{sc},II}_{\mathrm{RR},1\mathrm{c},i}$.

By contrast, the term in eq. (F.42) containing the function $\mathcal{W}_i^{i\|\mathfrak{n}}$ does contain singularities of $\mathcal{O}(\epsilon^{-1})$, which could (in principle) be dependent on the partitions $\omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}}$. This would imply that the pole structure of $\Sigma^{(b,d),\mathrm{sc},II}_{\mathrm{RR},1\mathrm{c},i}$ would depend on the choice of partition functions. However, in appendix G we will show that the sum over all the external legs of $\mathcal{W}_i^{i\|\mathfrak{n}}$ can be written as

$$
\begin{aligned}
\sum_{i=1}^{N_p} \langle \mathcal{W}_i^{i\|\mathfrak{n}} \cdot F_{\mathrm{LM}} \rangle = & -\epsilon\, 2^{2\epsilon} \sum_{i=1}^{N_p} \sum_{\substack{k,l=1\\ k\neq l}}^{N_p} \int \frac{[\mathrm{d}\Omega^{(d-1)}_{\mathfrak{m}}]}{[\Omega^{(d-2)}]} \left\langle \frac{\rho_{kl}}{\rho_{k\mathfrak{m}}\,\rho_{l\mathfrak{m}}}\, \sigma_{i\mathfrak{m}}^{-\epsilon}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}_{\mathfrak{m}\|\mathfrak{n}}\, F_{\mathrm{LM}}^{(kl)} \right\rangle \\
= & \; 2 \sum_{\substack{i,j=1\\ i\neq j}}^{N_p} \left\langle \eta_{ij}^{-\epsilon} K_{ij} F_{\mathrm{LM}}^{(ij)} \right\rangle \\
& + \sum_{i=1}^{N_p} \left[N_c(\epsilon)\, \boldsymbol{T}_i^2 \,\langle F_{\mathrm{LM}} \rangle + \epsilon^2 \left\langle \mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}} \right\rangle \right],
\end{aligned}
\tag{F.43}
$$

where we have relabelled $[\mathfrak{mn}] \mapsto \mathfrak{m}$. It is clear from the above equation that the poles of $\Sigma^{(b,d),\mathrm{sc},II}_{\mathrm{RR},1\mathrm{c},i}$ are in fact *independent* of the partition functions, whose explicit form only affects the finite remainder $\mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}}$ given in eq. (G.12).[31] Summing over emissions from all legs, we find

$$
\begin{aligned}
\Sigma^{(b,d),\mathrm{sc},II}_{RR,1c} = \; & 2[\alpha_s]^2 \delta_g^{\perp,r}(\epsilon) \left(\frac{E_{\max}}{\mu}\right)^{-2\epsilon} \\
& \times \left[-\langle I_{\mathrm{S}}(\epsilon) \cdot F_{\mathrm{LM}} \rangle + \frac{(2E_{\max}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i=1}^{N} \boldsymbol{T}_i^2 \,\langle F_{\mathrm{LM}} \rangle \right] \\
& + [\alpha_s]^2\, 2^{-2\epsilon} \delta_g^{\perp,r}(\epsilon) \left(\frac{E_{\max}}{\mu}\right)^{-4\epsilon} \sum_{i=1}^{N_p} \langle \mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}} \rangle \\
& + [\alpha_s]^2 \delta_g^{\perp} \left(\frac{E_{\max}}{\mu}\right)^{-4\epsilon} \sum_{i=1}^{N_p} \langle \mathcal{W}_r^{(i)} \cdot F_{\mathrm{LM}} \rangle \Bigg].
\end{aligned}
\tag{F.44}
$$

The complete result for spin-correlated contributions is obtained upon combining eqs. (F.18), (F.38) and (F.44).

[31] Changing the form of the partition functions would also change the value obtained from numerical integration for the fully-regulated term $\Sigma^{\mathrm{fin}}_{N+2}$ (cf. eq. (4.15)). These changes would compensate each other such that the physical cross section does not depend on the explicit expression for the partition functions.

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a	b	c	$F_{3g}(z,\varepsilon_a,\varepsilon_b,\varepsilon_c)$
in	in	in	$(1-z)/z+z/(1-z)+z(1-z)$
in	out	out	$z(1-z)$
out	in	out	$z/(1-z)$
out	out	in	$(1-z)/z$

Table 1. The table from ref. [83], page 160. Note that we use $z=1-E_b/E_a$ at variance with ref. [83].

It remains to prove the results for spin-correlated splitting functions introduced in eq. (F.32). To this end, we consider the cases where i is the initial-state or the final-state parton separately. We begin with the discussion of the final-state splitting, in which case i is a gluon. Since r_i can be considered to be the polarization vector of the clustered gluon, the calculation of the collinear limit in eq. (F.32) is equivalent to the computation of a $g\to gg$ splitting for *polarized* gluons. The corresponding results can be found in ref. [83]. To understand how they can be used, we note that ref. [83] defines polarization vectors relative to the decay plane formed by the momenta of the final state particles, there called b and c. Their momenta define a two-dimensional plane in $(d-1)$-dimensional space (we discard the temporal component for obvious reasons). We need $(d-2)$ polarization vectors to fully describe the quantum state of a gluon. Hence, for each of the gluons, we choose one polarization vector to lie in the plane defined by the momenta and $(d-3)$ to be orthogonal to that plane. It is clear that we can choose the "out-of-the-plane" polarization vectors to be the same for the three gluons a,b,c.

The dependence of the $g\to gg$ splitting on the polarization of the partons is characterized by the function $F_{3g}(z)$ shown in table 1. One can use this function to write the collinear limit of the scattering amplitude as follows [83]

$$|\mathcal{M}_{n+1}(\varepsilon_b,\varepsilon_c)|^2\sim\frac{4g_{s,b}^2\,C_A}{(p_{[\mathfrak{mn}]}+p_i)^2}\,F_{3g}(z;\varepsilon_a,\varepsilon_b,\varepsilon_c)|\mathcal{M}_n(\varepsilon_a)|^2\,. \tag{F.45}$$

As explained in ref. [83], this formula implies that the polarizations of the parent and daughter partons are kept fixed. For our purposes, we identify parton $[\mathfrak{mn}]$ with parton b and parton i with c. Therefore we need to sum over the polarizations of partons a and c and keep the polarization of the gluon b fixed and equal to r_i. Note that, since this polarization is composed of vectors e_i and b_i, it is "in-plane", according to the language of ref. [83]. Hence, for our purposes we require

$$\begin{aligned}C_{i[\mathfrak{mn}]}r_i^\mu r_i^\nu F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}])=&\frac{4g_{s,b}^2C_A^2}{(p_{[\mathfrak{mn}]}+p_i)^2}F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}i])\\&\times\left\{\varepsilon_a^\mu(\mathrm{in})\,\varepsilon_a^\nu(\mathrm{in})F_{3g}(a_{\mathrm{in}},b_{\mathrm{in}},c_{\mathrm{in}})+\sum_{\mathrm{out}}\varepsilon_a^\mu(\mathrm{out})\,\varepsilon_a^\nu(\mathrm{out})F_{3g}(a_{\mathrm{out}},b_{\mathrm{in}},c_{\mathrm{out}})\right\}.\end{aligned} \tag{F.46}$$

The "in-plane" polarization for the gluon a in the collinear limit is b. It remains to write the sum for the "out-of-plane" polarizations, which reads

$$\sum_{\mathrm{out}}\varepsilon_a^\mu(\mathrm{out})\,\varepsilon_a^\nu(\mathrm{out})=-g^{\mu\nu}+t^\mu t^\nu+\frac{e_i^\mu e_i^\nu}{e_i^2}+\frac{b_i^\mu b_i^\nu}{b_i^2}\,. \tag{F.47}$$

λ_a	ε_b	λ_c	$F_{qqg}(z,\lambda_a,\varepsilon_b,\lambda_c)$
$\pm$	in	$\pm$	$(1+z)^2/(1-z)$
$\pm$	out	$\pm$	$(1-z)$

Table 2. The table from ref. [83], page 160, that can be used to compute $q\to qg$ splittings.

Thanks to the transversality of $F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}i])$ w.r.t. $p_{[\mathfrak{mn}i]}$, we find

$$t^\mu F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}i]) = -e_i^\mu F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}i])\,. \tag{F.48}$$

This implies that

$$\left(t^\mu t^\nu + \frac{e_i^\mu e_i^\nu}{e_i^2}\right) F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}i]) = 0\,. \tag{F.49}$$

Hence, we obtain

$$\begin{aligned} C_{i[\mathfrak{mn}]}\, r_i^\mu r_i^\nu\, F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}]) &= \frac{2g_{s,b}^2\, C_A}{E_{[\mathfrak{mn}]}E_i\rho_{i[\mathfrak{mn}]}} F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}i])\Big\{b^\mu b^\nu F_{3g}(a_{\mathrm{in}},b_{\mathrm{in}},c_{\mathrm{in}}) \\ &\quad + (-g^{\mu\nu}+b^\mu b^\nu)F_{3g}(a_{\mathrm{out}},b_{\mathrm{in}},c_{\mathrm{out}})\Big\} \\ &= \frac{g_{s,b}^2}{E_{[\mathfrak{mn}]}E_i\rho_{i[\mathfrak{mn}]}} F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}i])\; P_{gg}^{r,\mu\nu}(z)\,, \end{aligned} \tag{F.50}$$

where $z = E_i/(E_i+E_{[\mathfrak{mn}]})$ and

$$P_{gg}^{r,\mu\nu}(z) = 2C_A\left[-\frac{z}{1-z}\,g^{\mu\nu} + \left(\frac{1-z}{z} + z(1-z)\right) b^\mu b^\nu\right]. \tag{F.51}$$

Since we will have to use this result in eq. (F.34), where the integration over directions of b decouples from the rest, we will only require the spin-averaged version of $P_{gg}^{r,\mu\nu}$, that is

$$\langle P_{gg}^{r,\mu\nu}(z)\rangle = (-g^{\mu\nu})P_{gg}^{\mathrm{spin}}(z)\,, \tag{F.52}$$

where (cf. eq. (F.33))

$$P_{gg}^{\mathrm{spin}}(z) = 2C_A\left[\frac{z}{1-z} + \frac{(1-z)/z + z(1-z)}{2(1-\epsilon)}\right]. \tag{F.53}$$

Since the spin-averaging also applies to the standard collinear limit $C_{i[\mathfrak{mn}]}F_{\mathrm{LM}}([\mathfrak{mn}])$, we obtain

$$\begin{aligned} C_{i[\mathfrak{mn}]}(r_i^\mu r_i^\nu + g^{\mu\nu})F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}]) &= \frac{g_{s,b}^2}{E_{[\mathfrak{mn}]}E_i\rho_{i[\mathfrak{mn}]}}\left(P_{gg}^{\mathrm{spin}}(z) - P_{gg}(z)\right) F_{\mathrm{LM}}([\mathfrak{mn}i]) \\ &= -\frac{1-2\epsilon}{1-\epsilon}\,\frac{g_{s,b}^2\, C_A}{E_{[\mathfrak{mn}]}E_i\rho_{i[\mathfrak{mn}]}}\,\frac{(1-z)(1+z^2)}{z} F_{\mathrm{LM}}([\mathfrak{mn}i])\,. \end{aligned} \tag{F.54}$$

To describe the initial-state splitting, we require the $q\to q^* g$ splitting. To compute it, we start from the final state $q^*\to qg$ and then perform the parton crossing. Similar to the gluon case, we need to keep the gluon polarized. The polarization-dependent splitting functions can again be found in ref. [83]; they are reproduced in table 2. We only need to consider the "in plane" polarization of the gluon and sum over quark polarizations. Performing the crossing, we find

$$C_{[\mathfrak{mn}]i} r_i^\mu r_i^\nu \, F_{\mathrm{LM},\mu\nu}([\mathfrak{mn}]) = \frac{4 g_{s,b}^2}{E_{[\mathfrak{mn}]} E_i \, \rho_{i[\mathfrak{mn}]}} \, P_{qg}^{\mathrm{spin}}(z) \, \frac{F_{\mathrm{LM}}(z \cdot i, \ldots)}{z}, \tag{F.55}$$

where $z = 1 - E_{[\mathfrak{mn}]}/E_i$, $i = 1, 2$, and P_{qg}^{spin} is given in eq. (F.33).

G Partition-dependent contribution

In this appendix, we discuss two contributions that appear in the computation of double-unresolved limits. They are required to obtain terms in the final result in the second line of eq. (6.4) and in the third line of eq. (6.5), respectively. They read

$$\begin{aligned} \sum_{k=1}^{N_p} \langle \mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n}} \cdot F_{\mathrm{LM}} \rangle &\equiv \sum_{k=1}^{N_p} \sum_{(ij)}^{N_p} \langle \mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n},(ij)} F_{\mathrm{LM}}^{(ij)} \rangle \\ &= -\epsilon \, 2^{2\epsilon} \sum_{k=1}^{N_p} \sum_{(ij)}^{N_p} \int \frac{[\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \left\langle \sigma_{k\mathfrak{m}}^{-\epsilon} \, \frac{\rho_{ij}}{\rho_{i\mathfrak{m}} \, \rho_{j\mathfrak{m}}} \, \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k} \, F_{\mathrm{LM}}^{(ij)} \right\rangle , \end{aligned} \tag{G.1}$$

and

$$\langle \mathcal{W}_k^{k\|\mathfrak{n}} \cdot F_{\mathrm{LM}} \rangle = -\epsilon \, 2^{2\epsilon} \sum_{(ij)}^{N_p} \int \frac{[\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \left\langle \left[(\eta_{k\mathfrak{m}}/2)^{-\epsilon} - 1 \right] \frac{\rho_{ij}}{\rho_{i\mathfrak{m}} \, \rho_{j\mathfrak{m}}} \, \omega_{k\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k} F_{\mathrm{LM}}^{(ij)} \right\rangle , \tag{G.2}$$

where we have used the shorthand notation $F_{\mathrm{LM}}^{(ij)} = (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LM}}$, which will appear in this appendix.

Extracting singularities from $\mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n}}$. We first investigate eq. (G.1). We note that the contribution of $\langle \mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n}} \cdot F_{\mathrm{LM}} \rangle$ to cross sections will be multiplied by $1/\epsilon^2$ which originates from the integration over gluon energies. For this reason, we require the expansion of eq. (G.1) through $\mathcal{O}(\epsilon^2)$. We also note that, thanks to the partition functions $\omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k}$ that appear in eq. (G.1), the only allowed collinear singularities correspond to the kinematic configurations where $\mathfrak{m}||k$. To isolate such divergences, we write

$$\begin{aligned} \sum_{k=1}^{N_p} \langle \mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n}} \cdot F_{\mathrm{LM}} \rangle = & -\epsilon \, 2^{2\epsilon} \sum_{k=1}^{N_p} \sum_{(ij)}^{N_p} \int \frac{[\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \Big\langle \Big[(\mathbb{1} - C_{k\mathfrak{m}}) \left(\sigma_{k\mathfrak{m}}^{-\epsilon} - 1 \right) \\ & + 1 - C_{k\mathfrak{m}} \left(1 - \sigma_{k\mathfrak{m}}^{-\epsilon} \right) \Big] \frac{\rho_{ij}}{\rho_{i\mathfrak{m}} \, \rho_{j\mathfrak{m}}} \, \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k} \, F_{\mathrm{LM}}^{(ij)} \Big\rangle . \end{aligned} \tag{G.3}$$

Next, we note that the first term in the above equation is $\mathcal{O}(\epsilon^2)$ already. The second term allows us to sum over index k using the relation

$$\sum_{k=1}^{N_p} \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k} = 1 \,, \tag{G.4}$$

and the last one can be simplified since the collinear $C_{k\mathfrak{m}}$ limit selects particular contributions from the sum.

We now consider the second and the third term in more detail. The former reads

$$-\epsilon\, 2^{2\epsilon} \sum_{k=1}^{N_p} \sum_{(ij)}^{N_p} \int \frac{[\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \left\langle \frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{m}}}\, \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k} F_{\mathrm{LM}}^{(ij)} \right\rangle = 2 \sum_{(ij)}^{N_p} \langle \eta_{ij}^{-\epsilon} K_{ij}\, F_{\mathrm{LM}}^{(ij)} \rangle \,. \tag{G.5}$$

To compute the contribution of the third term, we note that

$$C_{k\mathfrak{m}} \left(1 - \sigma_{k\mathfrak{m}}^{-\epsilon}\right) \frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{m}}}\, \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k} = (1 - \eta_{k\mathfrak{m}}^{-\epsilon}) \frac{1}{\rho_{k\mathfrak{m}}} \left(\delta_{ik} + \delta_{jk}\right) . \tag{G.6}$$

Using this expression in eq. (G.3), it becomes possible to sum either over j or i using the color conservation condition. We obtain

$$\epsilon\, 2^{2\epsilon} \sum_{k=1}^{N} \sum_{(ij)}^{N_p} \int \frac{[\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \left\langle C_{k\mathfrak{m}} \left(1 - \sigma_{k\mathfrak{m}}^{-\epsilon}\right) \frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{m}}} \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k} F_{\mathrm{LM}}^{(ij)} \right\rangle = N_c(\epsilon) \sum_{i=1}^{N_p} \boldsymbol{T}_i^2 \,\langle F_{\mathrm{LM}} \rangle \,, \tag{G.7}$$

where

$$N_c(\epsilon) = \frac{2\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} = 1 + \mathcal{O}(\epsilon^3) \,. \tag{G.8}$$

Combining all the relevant terms, we find

$$\begin{aligned} \sum_{k=1}^{N_p} \langle \mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n}} \cdot F_{\mathrm{LM}} \rangle = 2 \sum_{(ij)}^{N_p} \langle \eta_{ij}^{-\epsilon} K_{ij} F_{\mathrm{LM}}^{(ij)} \rangle + N_c(\epsilon) \sum_{i=1}^{N_p} \boldsymbol{T}_i^2 \langle F_{\mathrm{LM}} \rangle \\ + \epsilon^2 \sum_{k=1}^{N_p} \langle \mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}} \rangle \,, \end{aligned} \tag{G.9}$$

where

$$\langle \mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}} \rangle = \sum_{(ij)}^{N_p} \int \frac{\mathrm{d}\Omega_{\mathfrak{m}}^{(3)}}{2\pi} \left\langle (\mathbb{1} - C_{k\mathfrak{m}}) \log\left(\sigma_{k\mathfrak{m}}\right) \frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{m}}}\, \omega_{\mathfrak{m}\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k}\, F_{\mathrm{LM}}^{(ij)} \right\rangle . \tag{G.10}$$

Notice that $\mathcal{W}_k^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}}$ is finite in ϵ, thus we evaluate it in $d = 4$ dimensions.

Extracting singularities from $\boldsymbol{\mathcal{W}_k^{k\|\mathfrak{n}}}$. We can compute the second contribution $\langle \mathcal{W}_k^{k\|\mathfrak{n}} \cdot F_{\mathrm{LM}} \rangle$ shown in eq. (G.2) in the same way. As in the previous case, we introduce collinear subtraction operators as

$$\begin{aligned} \langle \mathcal{W}_k^{k\|\mathfrak{n}} \cdot F_{\mathrm{LM}} \rangle = & - \epsilon\, 2^{2\epsilon} \sum_{(ij)}^{N_p} \int \frac{[\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \\ & \times \left\langle (\mathbb{1} - C_{k\mathfrak{m}} + C_{k\mathfrak{m}}) \left[(\eta_{k\mathfrak{m}}/2)^{-\epsilon} - 1 \right] \frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{m}}}\, \omega_{k\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k}\, F_{\mathrm{LM}}^{(ij)} \right\rangle . \end{aligned} \tag{G.11}$$

The term with $(\mathbb{1} - C_{k\mathfrak{m}})$ leads to an $\mathcal{O}(\epsilon^2)$ contribution that we express through

$$\langle \mathcal{W}_k^{k\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}} \rangle = \sum_{(ij)}^{N_p} \int \frac{\mathrm{d}\Omega_{\mathfrak{m}}^{(3)}}{2\pi} \left\langle (\mathbb{1} - C_{k\mathfrak{m}}) \log\left(\frac{\eta_{k\mathfrak{m}}}{2}\right) \frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{m}}}\, \omega_{k\|\mathfrak{n}}^{\mathfrak{m}k,\mathfrak{n}k}\, F_{\mathrm{LM}}^{(ij)} \right\rangle . \tag{G.12}$$

The calculation of the term with $C_{k\mathfrak{m}}$ proceeds exactly as already explained in the previous subsection. We use

$$C_{k\mathfrak{m}}\,\omega^{\mathfrak{m}k,\mathfrak{n}k}_{k\|\mathfrak{n}}\,\frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\,\rho_{j\mathfrak{m}}}=\frac{1}{\rho_{k\mathfrak{m}}}\,(\delta_{ik}+\delta_{jk})\;, \tag{G.13}$$

sum over one of the color indices and employ the following integral

$$\begin{aligned}\int\frac{[\mathrm{d}\Omega^{(d-1)}_{\mathfrak{m}}]}{[\Omega^{(d-2)}]}\,\frac{2}{\rho_{k\mathfrak{m}}}\left[\left(\frac{\eta_{k\mathfrak{m}}}{2}\right)^{-\epsilon}-1\right]&=\frac{2^{-2\epsilon}}{\epsilon}\left[\frac{2\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}-\frac{2^{\epsilon}\,\Gamma(1-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-3\epsilon)}\right]\\&\equiv\frac{2^{-2\epsilon}}{\epsilon}N_k(\epsilon)\,,\end{aligned} \tag{G.14}$$

to find the final result

$$\langle\mathcal{W}^{k\|\mathfrak{n}}_{k}\cdot F_{\mathrm{LM}}\rangle=\epsilon^2\langle\mathcal{W}^{k\|\mathfrak{n},\mathrm{fin}}_{k}\cdot F_{\mathrm{LM}}\rangle-N_k(\epsilon)\,\boldsymbol{T}^2_k\langle F_{\mathrm{LM}}\rangle\,. \tag{G.15}$$

H Triple color-correlated contributions to real-virtual corrections

In this appendix we discuss the computation of the triple color-correlated component arising from the integrated soft limit of the real-virtual contribution. The relevant factorization formula in the soft limit is given in eq. (4.97), and we are interested in the final term

$$S^{\mathrm{tri}}_{\mathfrak{m}}\,F_{\mathrm{RV}}(\mathfrak{m})=-[\alpha_s]\,\frac{4\pi\,\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)}\sum_{(ijk)}^{N_p}\kappa_{ij}\,S_{ki}(p_{\mathfrak{m}})\,\Big(S_{ij}(p_{\mathfrak{m}})\Big)^{\epsilon}F^{(kij)}_{\mathrm{LM}}\,, \tag{H.1}$$

where (ijk) labels triplets with different i, j and k and we have used the notation

$$F^{(kij)}_{\mathrm{LM}}=\langle\mathcal{M}_0|f_{abc}\,T^a_k\,T^b_i\,T^c_j\,|\mathcal{M}_0\rangle\,, \tag{H.2}$$

to indicate the triple color-correlated matrix element. The phase factor κ_{ij} is reported in eq. (A.6), and the eikonal factor S_{ij} in eq. (4.89). Here we just recall that κ_{ij} is completely symmetric under the exchange $i\leftrightarrow j$ and (obviously) is independent of k.

We begin by pointing out that the triple color-correlated matrix element gives a non-zero contribution only when there are at least four colored particles in the Born-level process. Indeed, with three colored particles one can use color conservation to obtain the following identity

$$f_{abc}\,T^a_1\,T^b_2\,T^c_3\,|\mathcal{M}_0\rangle=-f_{abc}\,T^a_1\,T^b_2\,(T^c_1+T^c_2)\,|\mathcal{M}_0\rangle=0\,. \tag{H.3}$$

Our goal is to integrate eq. (H.1) over the phase space of the soft gluon with momentum $p_{\mathfrak{m}}$. We begin by integrating over the energy $E_{\mathfrak{m}}$ and obtain

$$\begin{aligned}\langle S^{\mathrm{tri}}_{\mathfrak{m}}\,F_{\mathrm{RV}}\rangle=&-[\alpha_s]^2\,\frac{4\pi^{3-\epsilon}\,2^{\epsilon}\,\Gamma(1+\epsilon)\Gamma^4(1-\epsilon)}{\epsilon^2\,\Gamma(1-2\epsilon)}\left(\frac{4E^2_{\max}}{\mu^2}\right)^{-2\epsilon}\\&\times\sum_{(ijk)}\Big\langle\kappa_{ij}\,G^{kij}\,F^{(kij)}_{\mathrm{LM}}\Big\rangle\,,\end{aligned} \tag{H.4}$$

In eq. (H.4) we have defined

$$G^{kij} = \int \frac{\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{k\mathfrak{m}}\rho_{i\mathfrak{m}}} \left(\frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\rho_{j\mathfrak{m}}}\right)^{\epsilon}, \tag{H.5}$$

which is a function of the angular variables ρ_{ij}, ρ_{ik} and ρ_{jk}. We note that since κ_{ij} is a symmetric tensor and $F_{\mathrm{LM}}^{(kij)}$ is fully anti-symmetric, only the anti-symmetric contribution $G^{kij} - G^{kji}$ can contribute to the sum, whereas the symmetric part drops out. We will use this result when writing intermediate expressions for $\langle S_{\mathfrak{m}}^{\mathrm{tri}}\, F_{\mathrm{RV}}(\mathfrak{m})\rangle$.

To perform the remaining integration over the soft-gluon angle, we employ the Mellin-Barnes representation of d-dimensional angular integrals presented in ref. [84], and write the integral as

$$\begin{aligned} G^{kij} &= \int \frac{\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{k\mathfrak{m}}\rho_{i\mathfrak{m}}} \left(\frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\rho_{j\mathfrak{m}}}\right)^{\epsilon} \\ &= \rho_{ki}\, \rho_{ij}^{\epsilon} \int\limits_{-i\infty}^{+i\infty} \frac{dz_{ij}\, dz_{jk}\, dz_{ki}}{(2\pi i)^3} \frac{\pi^{-2+\epsilon}}{2^{4+2\epsilon}} \Gamma(-z_{ij})\Gamma(-z_{ki})\Gamma(-z_{jk}) \\ &\quad \times \Gamma(1+\epsilon+z_{ij}+z_{ki})\Gamma(-1-3\epsilon-z_{ij}-z_{ki}-z_{jk})\Gamma(\epsilon+z_{ij}+z_{jk}) \\ &\quad \times \Gamma(1+z_{ki}+z_{jk}) \frac{1}{\Gamma(-4\epsilon)\Gamma(\epsilon)\Gamma(1+\epsilon)}\, \eta_{ij}^{z_{ij}}\, \eta_{ki}^{z_{ki}}\, \eta_{jk}^{z_{jk}}\,. \end{aligned} \tag{H.6}$$

In the above equation we have introduced the three complex Mellin-Barnes variables z_{ij}, z_{ki}, z_{jk}, and $\eta_{ij} = \rho_{ij}/2$. The integration contour has to be chosen in such a way that the poles of $\Gamma(\ldots + x)$ are separated from the poles of $\Gamma(\ldots - x)$, with x being a generic integration variable. In order to resolve the singularity structure in ϵ we employ the packages `MBresolve` [85] and `MB` [86], which allow us to express our original integral as a linear combination of integrals that can be safely expanded in ϵ under the integration sign, and whose integration contours are straight vertical lines in the complex plane. Upon applying this procedure we find that it is possible to express the function G^{kij} up to $\mathcal{O}(\epsilon^0)$ in terms of classical and generalized polylogarithms (GPLs) [87, 88] up to weight three. It is convenient to write the final result for the angular integral as follows

$$G^{kij} = \int \frac{\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{k\mathfrak{m}}\rho_{i\mathfrak{m}}} \left(\frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\rho_{j\mathfrak{m}}}\right)^{\epsilon} = -\frac{\epsilon^2}{4\pi^2}\left[\frac{2^{-\epsilon}\pi^{\epsilon}\,\Gamma(1-\epsilon)}{\Gamma(1-4\epsilon)\Gamma^2(1+\epsilon)}\right] \overline{G}^{kij}, \tag{H.7}$$

where

$$\begin{aligned} \overline{G}^{kij} &= \frac{3}{4\epsilon^3} + \frac{1}{2\epsilon^2}\Big[\log(\eta_{ij}) - 3\log(\eta_{ik}) - \log(\eta_{jk})\Big] + \frac{1}{\epsilon}\Big[\frac{1}{2}\log^2(\eta_{ij}) \\ &\quad + \log(\eta_{ij})\left(-\log(\eta_{ik}) - \log(\eta_{jk})\right) + \log(\eta_{ik})\left(\log(\eta_{jk}) - 2\log(1-\eta_{ik})\right) \\ &\quad - 2\mathrm{Li}_2(\eta_{ik}) + \frac{3}{2}\log^2(\eta_{ik}) + \frac{1}{2}\log^2(\eta_{jk}) + \pi^2\Big] + \mathcal{O}(\epsilon^0)\,. \end{aligned} \tag{H.8}$$

Note that the ϵ-dependent prefactor in eq. (H.7) starts at $\mathcal{O}(\epsilon^2)$, so that the whole angular integral is effectively $\mathcal{O}(\epsilon^{-1})$, as expected.

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Inserting the result for G^{kij} into eq. (H.4) we get the following final result for the triple color-correlated contribution to the real-virtual counterterm

$$\left\langle S_{\mathfrak{m}}^{\rm tri}\, F_{\rm RV}\right\rangle = [\alpha_s]^2\, \frac{\pi\Gamma^5(1-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1-4\epsilon)\Gamma(1+\epsilon)} \left(\frac{4E_{\rm max}^2}{\mu^2}\right)^{-2\epsilon} \sum_{(ijk)} \left\langle \kappa_{ij}\, \overline{G}^{kij}\, F_{\rm LM}^{(kij)}\right\rangle . \tag{H.9}$$

To proceed further it is convenient to split the function $\overline{G}^{kij}$ into contributions using its symmetry properties under the $i \leftrightarrow j$ permutations. We write

$$\overline{G}^{kij} = \overline{G}_s^{kij} + \overline{G}_r^{kij}\,, \tag{H.10}$$

with

$$\overline{G}_s^{kij} = \frac{3}{4}\frac{1}{\epsilon^3} + \frac{1}{2\epsilon^2}\log\left(\frac{\eta_{ij}}{\eta_{jk}\eta_{ik}}\right) + \frac{1}{\epsilon}\left[\frac{2\pi^2}{3} + \frac{1}{2}\log^2\left(\frac{\eta_{ij}}{\eta_{jk}\eta_{ik}}\right)\right] + \mathcal{O}(\epsilon^0)\,, \tag{H.11}$$

and

$$\overline{G}_r^{kij} = -\frac{\log\eta_{ik}}{\epsilon^2} + \frac{1}{\epsilon}\Big[\log^2\eta_{ik} + 2{\rm Li}_2(1-\eta_{ik})\Big] + \mathcal{O}(\epsilon^0)\,. \tag{H.12}$$

The function $\overline{G}_s^{kij}$ is symmetric under $i \leftrightarrow j$ permutations; hence, it does not contribute to $\langle S_{\mathfrak{m}}^{\rm tri}\, F_{\rm RV}(\mathfrak{m})\rangle$ and can be dropped. Note also that the function $\overline{G}_r^{kij}$, up to $\mathcal{O}(\epsilon^{-1})$, is symmetric under the $i \leftrightarrow k$ permutation. It follows that $\langle S_{\mathfrak{m}}^{\rm tri}\, F_{\rm RV}(\mathfrak{m})\rangle$ is free of poles for processes with a color-singlet initial state, as in this case we have $\kappa_{ij} = -1$.

For a hadron collider process with two incoming and any number of outgoing partons, the function κ_{ij} reads

$$\kappa_{ij} = -1 + 2\delta_{i1}\delta_{j2} + 2\delta_{i2}\delta_{j1}\,, \tag{H.13}$$

from which it follows that

$$\begin{aligned}\sum_{(ijk)} \left\langle \kappa_{ij}\, \overline{G}^{kij}\, F_{\rm LM}^{(kij)}\right\rangle &= \sum_{(ijk)} \left\langle \kappa_{ij}\, \overline{G}_r^{kij}\, F_{\rm LM}^{(kij)}\right\rangle \\ &= 2\sum_{k\neq 1,2} \left\langle \left(\overline{G}_r^{k12} - \overline{G}_r^{k21}\right) F_{\rm LM}^{(k12)}\right\rangle .\end{aligned} \tag{H.14}$$

Using this result together with eq. (H.12) and eq. (H.9), we obtain the final formula for the poles in the triple color-correlated contribution to the soft limit of the real-virtual corrections. It reads

$$\begin{aligned}\left\langle S_{\mathfrak{m}}^{\rm tri}\, F_{\rm RV}\right\rangle = [\alpha_s]^2 \sum_{k\neq 1,2} \Bigg\langle F_{\rm LM}^{(k12)} \Bigg\{ \frac{2\pi}{\epsilon^2}\log\frac{\eta_{2k}}{\eta_{1k}} + \frac{2\pi}{\epsilon}\Bigg[\log^2\eta_{1k} - \log^2\eta_{2k} \\ + 2\log\left(\frac{4E_{\rm max}^2}{\mu^2}\right)\log\left(\frac{\eta_{1k}}{\eta_{2k}}\right) + 2{\rm Li}_2(1-\eta_{1k}) - 2{\rm Li}_2(1-\eta_{2k})\Bigg] + \mathcal{O}(\epsilon^0)\Bigg\}\Bigg\rangle .\end{aligned} \tag{H.15}$$

We now present the formula for the $\mathcal{O}(\epsilon^0)$ terms of eq. (H.8). We exploit once again the symmetry properties of the triple color-correlated contribution under the exchange of $i \leftrightarrow j$

indices and therefore only present results for the antisymmetric part. The result reads

$$\begin{aligned}
\overline{G}^{kij}_{r,\text{fin}} = {}& \text{Li}_2(\eta_{ij})\log\left(\frac{\eta_{ik}}{\eta_{jk}}\right) - \text{Li}_2(\eta_{ik})\log\left(\frac{\eta_{jk}}{\eta_{ij}\eta_{ik}}\right) + \text{Li}_2(\eta_{jk})\log\left(\frac{\eta_{ik}}{\eta_{ij}\eta_{jk}}\right) \\
& + \log(\eta_{ik})\text{Li}_2\left(-\frac{\eta_{ik}-\eta_{jk}}{1-\eta_{ik}}\right) + \log(\eta_{ik})\text{Li}_2\left(-\frac{\eta_{ik}-\eta_{jk}}{\eta_{jk}}\right) + 3\text{Li}_3(1-\eta_{ik}) \\
& - 3\text{Li}_3(1-\eta_{jk}) + \text{Li}_2\left(\frac{1-\eta_{jk}}{1-\eta_{ik}}\right)\log(\eta_{ik}\eta_{jk}) + \text{Li}_2\left(\frac{\eta_{ik}}{\eta_{jk}}\right)\log(\eta_{ik}\,\eta_{jk}) \\
& - \log(\eta_{jk})\text{Li}_2\left(-\frac{\eta_{jk}-\eta_{ik}}{\eta_{ik}}\right) - \log(\eta_{jk})\text{Li}_2\left(-\frac{\eta_{jk}-\eta_{ik}}{1-\eta_{jk}}\right) + \text{Li}_3(\eta_{ik}) \\
& - \text{Li}_3(\eta_{jk}) + \log^2(\eta_{ik})\left[\frac{1}{2}\log\left(\frac{1-\eta_{jk}}{\eta_{ij}}\right) + \log\left(\frac{\eta_{jk}-\eta_{ik}}{\eta_{jk}}\right)\right] \\
& + \log(\eta_{ik})\left[-\frac{1}{2}\log^2(\eta_{ij}) + \log(1-\eta_{ij})\log(\eta_{ij}) + \frac{1}{2}\log^2\left(\frac{1-\eta_{jk}}{1-\eta_{ik}}\right)\right. \\
& \left. + \frac{1}{2}\log^2\left(\frac{\eta_{ik}}{\eta_{jk}}\right) + \log(1-\eta_{jk})\log(\eta_{jk}(\eta_{jk}-\eta_{ik})) + \log^2(\eta_{jk}) - \frac{13\pi^2}{6}\right] \\
& + \log(1-\eta_{jk})\left[-\log(\eta_{jk})\log\left(\frac{\eta_{ij}}{\eta_{jk}-\eta_{ik}}\right) - \log^2(\eta_{jk}) - \frac{\pi^2}{6}\right] \\
& + \log(1-\eta_{ik})\left[\log(\eta_{ik})\left[\log\left(\frac{\eta_{ij}}{\eta_{jk}-\eta_{ik}}\right) - \log(1-\eta_{jk})\right] + \log^2(\eta_{ik})\right. \\
& \left. - \log(\eta_{jk})\log(\eta_{jk}-\eta_{ik}) - \frac{\log^2(\eta_{jk})}{2} + \frac{\pi^2}{6}\right] + \frac{1}{2}\log(\eta_{ij})\log^2(\eta_{jk}) \\
& + \frac{1}{2}\log^2(\eta_{ij})\log(\eta_{jk}) - \log(1-\eta_{ij})\log(\eta_{ij})\log(\eta_{jk}) - \frac{1}{3}\log^3\left(\frac{\eta_{ik}}{\eta_{jk}}\right) \\
& - \frac{1}{2}\log\left(\frac{1-\eta_{jk}}{1-\eta_{ik}}\right)\log^2\left(\frac{\eta_{ik}}{\eta_{jk}}\right) - \frac{1}{2}\log^2\left(\frac{1-\eta_{jk}}{1-\eta_{ik}}\right)\log\left(\frac{\eta_{ik}}{\eta_{jk}}\right) \\
& - \log^2(\eta_{jk})\log(\eta_{jk}-\eta_{ik}) + \frac{2}{3}\pi^2\log\left(\frac{\eta_{ik}}{\eta_{jk}}\right) + \log\left(\frac{\eta_{ik}}{1-\eta_{ik}}\right) \\
& \times\left[\frac{\pi^2}{6} - \log(1-\eta_{jk})\log(\eta_{jk})\right] - \frac{1}{2}\log^3(\eta_{ik}) + \log^2(1-\eta_{ik})\log(\eta_{ik}) \\
& + \frac{\log^3(\eta_{jk})}{2} - \log^2(1-\eta_{jk})\log(\eta_{jk}) + \frac{3}{2}\pi^2\log(\eta_{jk}) - \frac{1}{6}\pi^2\log\left(\frac{\eta_{jk}}{1-\eta_{jk}}\right) \\
& + \log(\eta_{ij})\Big[G(\tilde\eta_{ik},w^+,1) - G(\tilde\eta_{jk},w^+,1) + G(\tilde\eta_{ik},w^-,1) - G(\tilde\eta_{jk},w^-,1)\Big] \\
& - G(\tilde\eta_{ik},w^+,\tilde\eta_{jk},1) + G(\tilde\eta_{jk},w^+,\tilde\eta_{ik},1) + G(\tilde\eta_{ik},w^+,1,1) - G(\tilde\eta_{ik},w^+,\tilde\eta_{ik},1) \\
& - G(\tilde\eta_{jk},w^+,1,1) + G(\tilde\eta_{jk},w^+,\tilde\eta_{jk},1) - G(\tilde\eta_{ik},w^-,\tilde\eta_{jk},1) + G(\tilde\eta_{jk},w^-,\tilde\eta_{ik},1) \\
& + G(\tilde\eta_{ik},w^-,1,1) - G(\tilde\eta_{ik},w^-,\tilde\eta_{ik},1) - G(\tilde\eta_{jk},w^-,1,1) + G(\tilde\eta_{jk},w^-,\tilde\eta_{jk},1)\,,
\end{aligned} \tag{H.16}$$

where we defined

$$w^{\pm} = \frac{2-\eta_{ij}-\eta_{ik}-\eta_{jk} \pm \sqrt{(\eta_{ij}-\eta_{ik}-\eta_{jk})^2 - 4\eta_{ik}\eta_{jk}(1-\eta_{ij})}}{2\,(\eta_{ik}\eta_{jk}-\eta_{ik}-\eta_{jk}+1)}\,, \tag{H.17}$$

and $\tilde\eta_{ab} = 1/(1-\eta_{ab})$.

The expression in eq. (H.16) is well defined in the region $\eta_{ik} < \eta_{jk}$. However, this is sufficient to cover the entire phase space since the other region can be obtained by swapping indices i and j. Thanks to the antisymmetry of the result under such an exchange, this only amounts to an overall sign change.

I Collection of functions used in the final result

In this appendix we collect all the functions that are necessary to write the final result for the NNLO QCD contribution to the partonic cross section of the process $q\bar{q} \to X + Ng$ given in section 6. For the reader's convenience, we attempt to make this appendix as self-contained as possible.

We use the following notations

$$\bar{z} = 1 - z\,, \qquad \mathcal{D}_n(z) = \left[\frac{\log^n(1-z)}{1-z}\right]_+\,, \tag{I.1}$$

$$\widetilde{L}_i = \log\left(\frac{2E_i}{\mu}\right)\,, \quad L_i = \log\left(\frac{E_{\max}}{E_i}\right)\,, \quad L_{\max} = \log\left(\frac{2E_{\max}}{\mu}\right)\,. \tag{I.2}$$

To present the double-boosted contribution in eq. (6.3) we have used the following splitting function

$$\mathcal{P}_{qq}^{\rm NLO}(z,E_i) = C_F\Big[\bar{z} + 4\mathcal{D}_1(z) + \big[4\mathcal{D}_0(z) + 3\delta(\bar{z})\big]\widetilde{L}_i - 2(1+z)\big[\widetilde{L}_i + \log(\bar{z})\big]\Big]\,. \tag{I.3}$$

The single-boosted contribution in eq. (6.4) depends on the function $\mathcal{W}_i^{i\|\mathfrak{n},\rm fin}$, defined in eq. (G.12), and an operator $I_{\rm T}^{(0)}$, reported in eq. (A.66). We have also introduced the function

$$\begin{aligned}\mathcal{P}_{qq}^{\mathcal{W}}(z,E_i) &= -\frac{1}{2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{2E_i}{\mu}\right)^{-4\epsilon}\Big[\mathcal{P}_{qq}^{(4)}(z,L_i) - e^{-2\epsilon L_i}\mathcal{P}_{qq}^{(2)}(z,L_i)\Big]\\ &= C_F\Big[\big[1+z-2\mathcal{D}_0(z)\big]L_i + 2\mathcal{D}_1(z) + \delta(\bar{z})L_i^2 - (1+z)\log(\bar{z})\Big]\,,\end{aligned} \tag{I.4}$$

where in the second line we have taken the $\epsilon \to 0$ limit. Furthermore, we use

$$\mathcal{P}_{qq}^{\rm NNLO}(z,E_i) = C_F^2 P_{qq}^{\rm NNLO,a}(z,E_i) + C_F C_A P_{qq}^{\rm NNLO,na}(z,E_i)\,, \tag{I.5}$$

with

$$\begin{aligned}P_{qq}^{\rm NNLO,a}(z,E_i) = 2\widetilde{L}_i^2\Big[&8\mathcal{D}_1(z)+6\mathcal{D}_0(z)-\frac{(3z^2+1)\log(z)}{\bar{z}}-4(z+1)\log(\bar{z})\\ &-z-5\Big]+\widetilde{L}_i\Big[24\mathcal{D}_2(z)+12\mathcal{D}_1(z)-\frac{8\pi^2}{3}\mathcal{D}_0(z)+\frac{8{\rm Li}_2(\bar{z})}{\bar{z}}-\left(1+3z^2\right)\frac{\log^2(z)}{\bar{z}}\\ &+4\left(1+z+z^2\right)\frac{\log(z)}{\bar{z}}-\log(\bar{z})\left(\frac{8z^2\log(z)}{\bar{z}}+2(5+z)\right)-12(1+z)\log^2(\bar{z})\\ &+\frac{4\pi^2}{3}(z+1)+9-7z\Big]+8\mathcal{D}_3(z)-\frac{8\pi^2}{3}\mathcal{D}_1(z)+16\zeta_3\mathcal{D}_0(z)-2\left(5+3z^2\right)\frac{{\rm Li}_3(z)}{\bar{z}}\\ &-\left(5-3z^2\right)\frac{{\rm Li}_3(\bar{z})}{\bar{z}}+\frac{\log(\bar{z})}{\bar{z}}\Big[\left(7-z^2\right){\rm Li}_2(\bar{z})-6\left(1+z^2\right)\log^2(z)+\frac{4\pi^2}{3}(1-z^2)\end{aligned}$$

$$
+\left(7-2z+7z^2\right)\log(z)+\bar z\left(6-\frac{9}{2}z\right)-4(z+1)\bar z\log^2(\bar z)\bigg]+\frac{\log(z)}{\bar z}\left[\left(\frac{5}{2}-\frac{9z}{2}\right)\bar z\right.
$$

$$
-2\left(1+z^2\right)\left(\mathrm{Li}_2(\bar z)-\frac{5\pi^2}{6}\right)\bigg]+3\bar z\left(\mathrm{Li}_2(\bar z)+\frac{2\pi^2}{9}\right)+\left(\frac{5}{4}+\frac{13}{12}z^2\right)\frac{\log^3(z)}{\bar z}
$$

$$
+\frac{2\zeta_3(1+7z^2)}{\bar z}-\log^2(\bar z)\left[2\bar z-\left(\frac{5}{2}-\frac{3z^2}{2}\right)\frac{\log(z)}{\bar z}\right]+8z+\frac{z}{2}\log^2(z)-9
$$

$$
+\delta(\bar z)\left[\left(\frac{9}{2}-\frac{4\pi^2}{3}\right)\widetilde{L}_i^2+\left(16\zeta_3+\frac{9}{2}\log 2\right)\widetilde{L}_i+\frac{3\pi^2}{16}-\frac{\pi^4}{45}-\frac{9}{16}\log^2 2\right], \tag{I.6}
$$

and

$$
P_{qq}^{\mathrm{NNLO,na}}(z,E_i)=-\frac{11}{3}\widetilde{L}_i^2\Big[2\mathcal{D}_0(z)-1-z\Big]+\widetilde{L}_i\left[\left(\frac{134}{9}-\frac{2\pi^2}{3}\right)\mathcal{D}_0(z)-\frac{44}{3}\mathcal{D}_1(z)\right.
$$

$$
+2\left(1+z^2\right)\frac{\mathrm{Li}_2(z)}{\bar z}+\left[\frac{2}{3}+\frac{11}{3}z^2+2(1+z^2)\log(\bar z)\right]\frac{\log(z)}{\bar z}+\frac{22}{3}(z+1)\log(\bar z)
$$

$$
\left.+\frac{2\pi^2}{3}-\frac{52}{9}-\frac{91z}{9}\right]-\frac{22}{3}\mathcal{D}_2(z)+\left(\frac{134}{9}-\frac{2\pi^2}{3}\right)\mathcal{D}_1(z)+\left[9\zeta_3-\frac{208}{27}+\frac{11\pi^2}{6}\right.
$$

$$
\left.-\frac{2\log 2}{3}\right]\mathcal{D}_0(z)-\frac{(1+6z+19z^2)}{6}\frac{\mathrm{Li}_2(z)}{\bar z}+\frac{2\log 2}{3}+\frac{(1+z^2)}{4\bar z}\Big[2\mathrm{Li}_3(z)-8\mathrm{Li}_3(\bar z)
$$

$$
-2\big(\log(z)-2\log(\bar z)\big)\mathrm{Li}_2(z)-\log^2(z)\log(\bar z)+4\log(z)\log^2(\bar z)\Big]+\frac{2+11z^2}{8\bar z}\log^2(z)
$$

$$
+\frac{11}{3}(1+z)\log^2(\bar z)+\frac{20-57z-49z^2}{36\bar z}\log(z)-\log\bar z\left[\frac{52}{9}+\frac{173}{18}z-\frac{1}{2}(1-z)\log(z)\right.
$$

$$
\left.-\frac{\pi^2}{6}\frac{1+3z^2}{\bar z}\right]-\frac{5-4z^2}{\bar z}\zeta_3+\frac{\pi^2}{36}\frac{12z+49z^2-35}{\bar z}+\frac{563}{108}+\frac{197}{108}z+\delta(\bar z)\left\{L_{\max}^2\left[\frac{64}{9}\right.\right.
$$

$$
\left.-\frac{\pi^2}{3}+\frac{22\log 2}{3}\right]+\widetilde{L}_i^2\left(\frac{\pi^2}{3}-\frac{227}{18}-\frac{22}{3}\log 2\right)+L_{\max}\left[\frac{11\zeta_3}{2}-\frac{22\pi^2}{9}+\frac{383}{54}\right.
$$

$$
\left.-\frac{77}{3}\log^2 2-\frac{125\log 2}{9}\right]+\widetilde{L}_i\left[\frac{263}{6}-7\zeta_3-\frac{7\pi^2}{9}+\frac{11\log^2 2}{3}+\left(\frac{224}{9}-\frac{4\pi^2}{3}\right)\log 2\right]
$$

$$
-2\mathrm{Li}_4(1/2)+\frac{22\log^3 2}{9}+\zeta_3\left(\frac{217}{8}+\frac{25\log 2}{4}\right)+\frac{211\pi^4}{1440}-\frac{1561}{36}-\frac{103\pi^2}{432}-\frac{\log^4 2}{12}
$$

$$
\left.+\left(\frac{15\pi^2}{4}-\frac{284}{9}\right)\log 2+\left(\frac{5\pi^2}{12}-\frac{415}{36}\right)\log^2 2\right\}. \tag{I.7}
$$

It remains to discuss functions that contribute to $\mathrm{d}\hat\sigma_{\mathrm{el}}^{\mathrm{NNLO}}$, see eq. (6.5). The quantity $I_{\mathrm{cc}}^{\mathrm{fin}}$ collects color-correlated contributions and reads

$$
I_{\mathrm{cc}}^{\mathrm{fin}}=\frac{1}{2}(I_{\mathrm{T}}^{(0)})^2+KI_{\mathrm{T}}^{(0)}+C_A\left[\frac{11}{6}\left(I_{\mathrm{T}}^{(1)}+\widetilde{I}_{\mathrm{S}}^{(1)}-2I_{\mathrm{S}}^{(1)}+\frac{\pi^2}{24}I_{\mathrm{V}}^{(-1)}\right)\right.
$$

$$
+I_{\mathrm{S}}^{(-1)}\left[\left(\frac{2\pi^2}{3}-\frac{131}{18}+\frac{22}{3}\log 2\right)L_{\max}-\frac{17\zeta_3}{4}+\frac{1975}{108}-\frac{11}{12}\pi^2\right. \tag{I.8}
$$

$$
\left.\left.-11\log^2 2-\frac{2}{3}\pi^2\log 2\right]+I_{\mathrm{S}}^{(0)}\left(\frac{\pi^2}{3}-\frac{131}{36}+\frac{11\log 2}{3}\right)\right],
$$

where K is a constant given in eq. (A.7) and $I_{\mathrm{S}}^{(n)}$, $\widetilde{I}_{\mathrm{S}}^{(n)}$, $I_{\mathrm{V}}^{(n)}$, $I_{\mathrm{T}}^{(n)}$ are the coefficients of the n-th power in the ϵ-expansion of the corresponding operators reported in appendix A.5. The finite part of the triple color-correlated operator is given by

$$I_{\rm tri}^{\rm fin} = I_{\rm tri}^{\rm (cc),fin} + \sum_{(ijk)}^{N_p} \kappa_{ij}\, \overline{G}_{r,\rm fin}^{kij}\, F^{(kij)}\,, \tag{I.9}$$

where $F^{(kij)} = f_{abc}\, T_k^a\, T_i^b\, T_j^c$. We note that $I_{\rm tri}^{\rm (cc),fin}$ corresponds to the $\mathcal{O}(\epsilon^0)$ contributions of the operator $I_{\rm tri}^{\rm (cc)}$ in eq. (5.15) and reads

$$I_{\rm tri}^{\rm (cc),fin} = \frac{\pi}{2} \sum_{(ijk)}^{N_p} F^{(kij)} \left(\delta_{ij}^- + \delta_{ji}^-\right) \left(\delta_{kj}^+ + \delta_{jk}^+ - 2\phi_{jk}\right)\,, \tag{I.10}$$

where

$$\begin{aligned}
\delta_{ij}^+ &= \frac{1}{2} L_{ij}^2 + \frac{\gamma_i}{\boldsymbol{T}_i^2}\, L_{ij} - \frac{1}{2}\pi^2 \lambda_{ij}^2\,, \\
\delta_{ij}^- &= \frac{\gamma_i}{\boldsymbol{T}_i^2}\, \lambda_{ij} + L_{ij}\, \lambda_{ij}\,, \\
\phi_{ij} &= -2 L_{\rm max} \log(\eta_{ij}) - \frac{1}{2} \log^2(\eta_{ij}) - {\rm Li}_2(1-\eta_{ij})\,.
\end{aligned} \tag{I.11}$$

Furthermore, the term $\overline{G}_{r,\rm fin}^{kij}$ can be found in eq. (H.16).

The operator $I_{\rm unc}^{\rm fin}$ in eq. (6.5) collects color-uncorrelated contributions. It reads

$$\begin{aligned}
I_{\rm unc}^{\rm fin} = \sum_{i=1}^{N_p} D_{\rm c}(E_i) + I_{\rm S}^{(-2)} C_A \Bigg\{ &\left[\frac{2\pi^2}{3} - \frac{131}{18} + \frac{22\log 2}{3}\right] L_{\rm max}^2 - \frac{935\zeta_3}{72} + \frac{9607}{324} \\
&+ \left[-8\zeta_3 - \frac{11\pi^2}{6} + \frac{1433}{108}\right] \log 2 - \frac{\pi^2\,(945 + 199\pi^2)}{1440} - \frac{11}{3}\log^3 2 \\
&+ \left(\frac{143}{36} - \frac{\pi^2}{3}\right) \log^2 2 \Bigg\}\,,
\end{aligned} \tag{I.12}$$

where we have introduced

$$\begin{aligned}
D_{\rm c}(E_i) = C_A C_F \Bigg\{ & L_i \bigg[\frac{2}{9}\left(3\pi^2 - 64 - 66\log 2\right) \widetilde{L}_i - 16\zeta_3 \\
&+ \frac{1}{27}\left(802 - 36\pi^2 \log 2 + 3(131 + 33\log 2)\log 2\right) \bigg] \\
&+ \frac{1}{9}\left(3\pi^2 - 64 - 66\log 2\right) L_i^2 + \frac{1}{6}\left(9\pi^2 - 64 - 66\log 2\right) \widetilde{L}_i \\
&- 12\zeta_3 + \frac{1}{36}\left(802 - 36\pi^2\log 2 + 3(131 + 33\log 2)\log 2\right) \Bigg\} \\
&+ C_F^2 \left(-\frac{3}{16}\left(\pi^2 - 3\log^2 2\right) - \frac{9}{2}\widetilde{L}_i \log 2 \right)\,,
\end{aligned} \tag{I.13}$$

if $i = 1, 2,$ and

$$\begin{aligned}
D_{\rm c}(E_i) = C_A^2 \Bigg\{ & \left[-\frac{15\zeta_3}{2} + \frac{1010}{27} - 22\log^2 2 - \frac{1}{6}\pi^2 (11 + 8\log 2) \right] \widetilde{L}_i \\
&+ \left[-\frac{21\zeta_3}{2} + \frac{1987}{54} - 22\log^2 2 + \frac{2\log 2}{3} - \frac{2}{9}\pi^2(11 + 6\log 2) \right] L_i
\end{aligned}$$

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$$
\begin{aligned}
&-2\mathrm{Li}_4(1/2)+\zeta_3\left(\frac{583}{24}+\frac{25\log 2}{4}\right)+\frac{47\pi^4}{160}-\frac{40201}{648} \\
&-\frac{\log 2}{36}\Big[713+\log 2\Big(316+\log 2(3\log 2-88)\Big)\Big] \\
&+\frac{\pi^2}{432}\Big[259+36\log 2(33+5\log 2)\Big]\Bigg\}\,,
\end{aligned} \tag{I.14}
$$

if $i \in [3, N_p]$.

The function $\gamma^{\mathcal{W}}$ in eq. (6.5) is a combination of anomalous dimensions. It is given by

$$
\begin{aligned}
\gamma^{\mathcal{W}}(L_i) &= \frac{1}{2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{2E_i}{\mu}\right)^{-4\epsilon}\Big[\gamma^{24}_{z,g\to gg}(\epsilon,L_i)-e^{-2\epsilon L_i}\gamma^{22}_{z,g\to gg}(\epsilon,L_i)\Big] \\
&= C_A\left[\frac{203}{72}+L_i\left(\frac{11}{6}+L_i\right)\right]\,,
\end{aligned} \tag{I.15}
$$

where in the second line we have taken $\epsilon \to 0$. Furthermore, the functions $\mathcal{W}_i^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}}$, $\mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}}$ and $\mathcal{W}_r^{(i)}$ are given in eqs. (G.10), (G.12) and (F.41), respectively. The quantities $\delta_g^{(0)}$ and $\delta_g^{\perp}$ correspond to

$$
\delta_g^{(0)} = C_A\Big(-\frac{131}{72}+\frac{\pi^2}{6}+\frac{11}{6}\log 2\Big)\,, \qquad \delta_g^{\perp} = C_A\Big(\frac{13}{36}-\frac{\log 2}{3}\Big)\,. \tag{I.16}
$$

The finite remainder of the double-soft integrated subtraction term is given by

$$
\begin{aligned}
&\langle S_{\mathfrak{m}\mathfrak{n}}\Theta_{\mathfrak{m}\mathfrak{n}}F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n})\rangle^{\mathrm{fin}}_{T^2} = \\
&=\left[\frac{\alpha_s(\mu)}{2\pi}\right]^2\sum_{(ij)}^{N_p}C_A\Bigg\langle\Bigg\{-\frac{\mathrm{Si}_2(2\delta_{ij})}{6\tan(\delta_{ij})}-\frac{11}{3}\mathrm{Ci}_3(2\delta_{ij})-2G_{-1,0,0,1}(\eta_{ij}) \\
&+\frac{7}{2}G_{0,1,0,1}(\eta_{ij})-\frac{5}{24}\log^4(\eta_{ij})-\frac{1}{12}\log^4(1+\eta_{ij})+\frac{1}{2}\log(1-\eta_{ij})\log^3(\eta_{ij})-\Bigg[\frac{5\pi^2}{12} \\
&+\frac{11}{12}\log(1-\eta_{ij})+\frac{7}{4}\log^2(1-\eta_{ij})\Bigg]\log^2(\eta_{ij})+\frac{\pi^2}{12}\log^2(1+\eta_{ij})-\frac{7}{4}\mathrm{Li}_2(\eta_{ij})^2 \\
&+3\mathrm{Li}_4(\eta_{ij})-5\mathrm{Li}_4\left(1-\frac{1}{\eta_{ij}}\right)-5\mathrm{Li}_4(1-\eta_{ij})-2\mathrm{Li}_4\left(\frac{1}{1+\eta_{ij}}\right)+\mathrm{Li}_4\left(\frac{1-\eta_{ij}}{1+\eta_{ij}}\right) \\
&-\mathrm{Li}_4\left(-\frac{1-\eta_{ij}}{1+\eta_{ij}}\right)-\frac{1}{2}\mathrm{Li}_4\left(1-\eta_{ij}^2\right)-\mathrm{Li}_2(\eta_{ij})\Bigg[\log^2(\eta_{ij})+\frac{11}{6}\log(\eta_{ij})+\frac{1+2\pi^2}{12} \\
&+\frac{11\log 2}{3}\Bigg]+\mathrm{Li}_2(-\eta_{ij})\Bigg[2\log(1-\eta_{ij})\log(\eta_{ij})+2\mathrm{Li}_2(1-\eta_{ij})-\frac{\pi^2}{3}\Bigg] \\
&-2\log(1-\eta_{ij})\mathrm{Li}_3(-\eta_{ij})+2\mathrm{Li}_3(1-\eta_{ij})\Big(\log(1+\eta_{ij})-\log(\eta_{ij})\Big)+\mathrm{Li}_3(\eta_{ij})\Bigg[\frac{11}{6} \\
&+2\log(\eta_{ij})-2\log(\eta_{ij}+1)-7\log(1-\eta_{ij})\Bigg]+\log 2\Bigg[-\frac{11}{3}\log(1-\eta_{ij})\log(\eta_{ij}) \\
&+\frac{21}{4}\zeta_3+\frac{33\pi^2-868}{108}\Bigg]+\frac{11}{2}\zeta_3\log(1-\eta_{ij})-\frac{7}{4}\zeta_3\log(1+\eta_{ij})+6\mathrm{Li}_4\left(\frac{1}{2}\right) \\
&+\log(\eta_{ij})\Bigg[\left(\pi^2-\frac{1}{12}\right)\log(1-\eta_{ij})+2\zeta_3-\frac{1}{6}\Bigg]-\frac{11}{24}\zeta_3+\frac{137\pi^2}{432}-\frac{17\pi^4}{160}+\frac{649}{162} \\
&+\frac{\log^4 2}{4}-\frac{11\log^3 2}{9}-\frac{137+9\pi^2}{36}\log^2 2\Bigg\}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\mathrm{LM}}\Bigg\rangle\,.
\end{aligned} \tag{I.17}
$$

In the above equation we used $\delta_{ij} = \theta_{ij}/2$, where θ_{ij} is the opening angle between momenta of partons i and j. The Clausen functions are defined as

$$\mathrm{Ci}_n(z) = \frac{\mathrm{Li}_n(e^{iz}) + \mathrm{Li}_n(e^{-iz})}{2}, \qquad \mathrm{Si}_n(z) = \frac{\mathrm{Li}_n(e^{iz}) - \mathrm{Li}_n(e^{-iz})}{2i}, \tag{I.18}$$

and $G_{a_1,a_2,\ldots,a_m}(x)$ are the standard Goncharov polylogarithms.

The last two functions in eq. (6.5) are $F^{\mathrm{fin}}_{\mathrm{LV}^2}$ and $F^{\mathrm{fin}}_{\mathrm{VV}}$, which refer to the infrared-finite components of the one-loop squared amplitude and the two-loop amplitude interfered with tree level, respectively.

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