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Some notes on possibilistic randomisation with t-norm based joint distributions in strategic-form games

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ABSTRACT

This article continues the investigation started in [18] on the role of possibilistic mixed strategies in strategic-form games. In this earlier work we assumed, as standard in possibility theory, that joint possibility distributions were computed by combining possibilistic mixed strategies with the minimum t-norm. In this paper, we investigate the consequences of defining joint possibility distributions by using any continuous t-norm, with players' expected utilities based on the Choquet integral. We characterise under which conditions a pair of possibilistic mixed strategies is an equilibrium, generalising the results first presented in [18], and also show that the set of equilibria in possibilistic mixed strategies depends on the set of idempotent elements of a t-norm and not just on the chosen t-norm.

1. Introduction and motivation

The research reported in this note contributes to the investigation on the formal properties of possibilistic randomisation [1,15–18]. In [18], we asked what kind of game-theoretic equilibria would arise by considering the possibilistic counterpart of the classical notion of a probabilistic mixed strategy. Two notions of possibilistic expected utility, an ordinal one based on the Sugeno integral [28] and a cardinal one based on the Choquet integral [4], were then investigated in the context of strategic-form games and applied to the analysis of a coordination game known as the Weak-link game [29]. Since the publication of [18], further research has addressed this topic, notably [2,26], indicating a continuing interest in understanding the features of game-theoretic models in conjunction with possibility theory (see also [5,25]).

In [18], we defined joint possibility distributions by combining possibility distributions with the minimum t-norm, as this is the natural choice in the standard qualitative setting of possibility theory. In this article, however, we focus on a quantitative decision-theoretic approach based on the Choquet integral: in this setting, any continuous t-norm [19] provides a suitable way of defining a joint possibility distribution. This approach does not simply have a mathematical justification. In fact, depending on how the concept of possibilistic randomisation is interpreted, choosing continuous t-norms other than the minimum might allow a more accurate representation of the meaning of a joint possibility distribution, according to the context (more on this in the last section).

In this paper, we then significantly extend the scope of [18] by investigating the consequences of randomising with distinct continuous t-norms. We give a full characterisation (Theorem 26) of when a pair of possibilistic mixed strategies forms an equilibrium with respect to any continuous t-norm and show that this notion of equilibrium properly generalises the classical one (Proposition 29).

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Our main result (Theorem 30) shows that, modulo the set of its idempotents, the specific choice of a t-norm is irrelevant to the set of equilibria. In particular, Theorem 30 proves that whenever the set of idempotents of a continuous t-norm is a subset of the set of idempotents of another, the set of equilibria with respect to the former t-norm is a subset of the set of equilibria with respect to the latter. In addition, the set of equilibria with respect to any continuous t-norm is always a subset of the set of equilibria with respect to the minimum t-norm.

This article is organised as follows. Section 2 provides the basic background concepts on strategic-form games, possibility measures, and t-norms at the core of this work. Section 3 investigates the notion of possibilistic Choquet equilibria and characterises their main game-theoretic properties. Section 4 studies how the notions introduced in the previous section apply to well-known strategic-form games such as the Prisoner's Dilemma, Matching Pennies, and the Stag-Hunt game. Section 5 illustrates the extent to which defining joint possibility distributions through distinct continuous t-norms makes a difference for the set of possibilistic Choquet equilibria. Finally, Section 6 puts forward concluding remarks and directions for future work.

2. Background

2.1. Strategic-form games

For present purposes it is sufficient to recall a selection of key definitions for static, non-cooperative games in strategic form. All the games in this article are assumed to be *two-person games*. Interested readers can refer to classic introductions such as [24], the more recent [21], or the freely available [3].

Definition 1 *(Strategic-form Game).* A *strategic-form game* is a tuple

$$\mathbf{G} = \langle N, S_1, S_2, u_1, u_2 \rangle$$

where:

1. $N = \{1, 2\}$ is the set of *players* of the game.
2. S_i is a finite set of *strategies* for each player $i \in \{1, 2\}$.
3. $u_i : S_1 \times S_2 \to \mathbb{R}^+$, for each player $i \in \{1, 2\}$, is a non-negative real-valued function (different from the identically zero function) called *utility function* (or *payoff function*).

Given a player $i \in \{1, 2\}$, we sometimes refer to the other player as $-i$. The elements of each S_i are often referred to as *pure strategies*. We usually denote by s_i an arbitrary strategy for player i. Given $s_i \in S_i$, $s_{-i} \in S_{-i}$ is used to denote a strategy for the other player $-i$. A *strategy combination* is any pair $(s_1, s_2) \in S_1 \times S_2$.

Definition 2 *(Best response).* Let $\mathbf{G}$ be a strategic-form game and $(s_1, s_2) \in S_1 \times S_2$ be a strategy combination. Player 1's strategy s_1 is called a *best response* to s_2 if, for all $s_1' \in S_1$,

$$u_1(s_1, s_2) \geq u_1(s_1', s_2).$$

The definition for player 2 is analogous.

Definition 3 *(Pure Strategy Nash Equilibrium).* Let $\mathbf{G}$ be a strategic-form game. We call a pair of pure strategies $(s_1, s_2) \in S_1 \times S_2$ a *pure strategy Nash equilibrium* if each player's strategy is a best response to the other player's strategy.

It is well known that not all strategic-form games admit a pure strategy Nash equilibrium. This situation is obviated by allowing players to not simply choose one among their pure strategies, but among all possible mixed strategies, i.e. all probability distributions over their strategy set. More formally:

Definition 4 *(Mixed Strategy).* In a strategic-form game $\mathbf{G}$, a *mixed strategy* σ_i for player $i \in \{1, 2\}$ is a probability distribution over the set of strategies S_i, i.e. a function $\sigma_i : S_i \to [0, 1]$ such that

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

Similar to pure strategies, any pair of mixed strategies (σ_1, σ_2) is called a *mixed strategy combination.*

Definition 5 *(Mixed Extension).* Let $\mathbf{G}$ be a strategic-form game. The *mixed extension* of $\mathbf{G}$ is the game

$$\mathfrak{G} = \langle N, MS_1, MS_2, eu_1, eu_2 \rangle$$

where, for $i \in \{1, 2\}$:

1. Each MS_i is the set of all mixed strategies of player i over S_i.
2. Each $eu_i : MS_1 \times MS_2 \to \mathbb{R}$ is a function that associates with each mixed strategy combination (σ_1, σ_2) the expected utility

$$eu_i(\sigma_1,\sigma_2) = \sum_{(s_1,s_2)\in S_1\times S_2} ((\sigma_1(s_1)\cdot\sigma_2(s_2))\cdot u_i(s_1,s_2)).$$

Both the concept of best response and Nash equilibrium in pure strategies are easily generalised to mixed strategies.

Definition 6 *(Best Response: Mixed Strategies)*. Let $\mathbf{G}$ be a strategic-form game, $\mathfrak{G}$ be its mixed extension and $(\sigma_1,\sigma_2) \in MS_1 \times MS_2$ be a mixed strategy combination. Player 1's mixed strategy σ_1 is called a *best response* to σ_2 if, for all $\sigma_1' \in MS_1$,

$$eu_1(\sigma_1,\sigma_2) \geq eu_1(\sigma_1',\sigma_2).$$

The definition for player 2 is analogous.

Definition 7 *(Mixed Strategy Nash Equilibrium)*. Let $\mathbf{G}$ be a strategic-form game and let $\mathfrak{G}$ be its mixed extension. We call a pair of mixed strategies $(\sigma_1,\sigma_2) \in MS_1 \times MS_2$ a *mixed strategy Nash equilibrium* for $\mathbf{G}$ if each player's mixed strategy is a best response to the other player's mixed strategy.

The following is the celebrated theorem by John Nash proving the existence of equilibria in mixed strategies.

Theorem 8 *([23]). Every strategic-form game has a mixed strategy Nash equilibrium.*

2.2. Possibility measures and Choquet integration

In this subsection, we provide the background notions concerning possibility measures and distributions, and Choquet integration for finite functions with respect to possibility measures. The reader can find a full account of these topics in [6–10,12,14].

Definition 9 *(Possibility Measure)*. Let X be a finite non-empty set. A *possibility measure* is a function $\Pi : 2^X \to [0,1]$ such that, for all $A, B \in 2^X$:

1. $\Pi\left(\emptyset\right) = 0$;
2. $\Pi(X) = 1$;
3. if $A \subseteq B$, then $\Pi(A) \leq \Pi(B)$;
4. $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$.

Definition 10 *(Possibility Distribution)*. Let X be a finite non-empty set. A *possibility distribution* is a function $\pi : X \to [0,1]$ such that

$$\sup_{x\in X} \pi(x) = 1.$$

Given a possibility distribution π on a finite non-empty set X, the function $\Pi : 2^X \to [0,1]$ such that, for all $A \in 2^X$,

$$\Pi(A) = \sup_{x\in A} \pi(x)$$

is a possibility measure called the *possibility measure generated from* π. Given a possibility measure $\Pi : 2^X \to [0,1]$, the function $\pi : X \to [0,1]$ defined by $\pi(x) = \Pi(\{x\})$, for all $x \in X$, is a possibility distribution.

We now introduce the notion of Choquet integration for non-negative finite functions with respect to possibility measures.

Definition 11 *(Choquet Integral for Possibility Measures)*. Let $X = \{x_1, \dots, x_n\}$ be a finite non-empty set, $f : X \to \mathbb{R}_+$ be a non-negative finite function and $\Pi : 2^X \to [0,1]$ be a possibility measure. Let α be a permutation over X such that

$$f\left(x_{\alpha(1)}\right) \leq f\left(x_{\alpha(2)}\right) \leq \dots \leq f\left(x_{\alpha(n)}\right),$$

and let, for each $1 \leq j \leq n$,

$$A_{\alpha(j)} = \left\{x_{\alpha(j)}, \dots, x_{\alpha(n)}\right\}.$$

The *Choquet integral* of f with respect to Π is defined as:

$$\int^{Ch} f\,\mathrm{d}\Pi = \sum_{j=1}^{n} \left(f\left(x_{\alpha(j)}\right) - f\left(x_{\alpha(j-1)}\right)\right)\cdot \Pi\left(A_{\alpha(j)}\right),$$

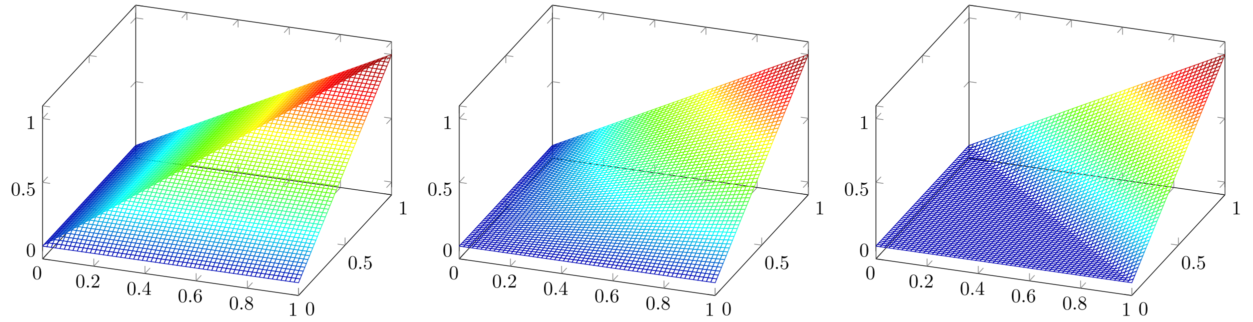

Fig. 1. 3D plot of the minimum, product, and Łukasiewicz t-norms.

with $f\left(x_{\alpha(0)}\right)=0$ by convention.

Possibility measures were originally introduced by Zadeh in [32]. Since then, they have been studied as measures of uncertainty capable of representing incomplete information and as models of partial belief offering both a qualitative and a numerical alternative to probabilities [11,12]. Similar to probabilities, possibility measures lend themselves to several different interpretations. The epistemic interpretation is one of the most prominent ones and sees possibilities as a measure of *plausibility* of the occurrence of an event. In this context, a possibility distribution over a set of alternatives can be seen as a way to rank how plausible or likely each option is. Possibility can also be seen as a measure of logical *consistency*. The possibility of a proposition is a measure of how consistent it is with the available information. Another interpretation of possibilities is that of measures of *feasibility*: they rank how easy to achieve different options are. Finally, possibility measures can be cast in a deontic framework and be seen as a way to measure *permissibility*, to evaluate the degree to which an action is allowed or permitted.

In this work, similar to [18], our view of possibility remains neutral and we do not take any stance concerning its interpretation: we simply see a possibility distribution as a way to formalise a different notion of randomisation.

2.3. Triangular norms

Here we introduce several basic notions and results about triangular norms that we will make extensive use of throughout the paper. A full account of the basic properties of these functions can be found in [19].

Definition 12 *(Triangular Norm* [19]*).* A *triangular norm* $*: [0,1]^2 \to [0,1]$ (*t-norm*, for short) is a binary function such that for all $x, y, z \in [0,1]$:

1. $x * y = y * x$;
2. $x * (y * z) = (x * y) * z$;
3. $x * y \leq x * z$ whenever $y \leq z$;
4. $x * 1 = x$.

According to the above definition, t-norms are then binary, commutative, associative, and monotone functions, having 1 as a neutral element.

A t-norm $*$ is *continuous* if for all convergent sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$,

$$\left(\lim_{n\to\infty} x_n\right) * \left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} \left(x_n * y_n\right).$$

The minimum, product, and Łukasiewicz t-norms (see Fig. 1) are the most prominent examples of continuous t-norms and play a fundamental role in the representation of all continuous t-norms (Theorem 18):

1. Minimum t-norm: $x *_{\min} y = \min(x, y)$;
2. Product t-norm: $x *_{\text{prod}} y = x \cdot y$;
3. Łukasiewicz t-norm: $x *_{\text{Łuk}} y = \max(x + y - 1, 0)$.

A t-norm $*$ is *strictly monotone* if, whenever $x > 0$ and $y < z$, $x * y < x * z$. A t-norm satisfies the *cancellation law* if $x * y = x * z$ implies $x = 0$ or $y = z$. A t-norm $*$ is *Archimedean* if for each $(x, y) \in]0,1[^2$ there is an $n \in \mathbb{N}$ such that $x^n < y$, with

$$x^n = \underbrace{x * \cdots * x}_{n}.$$

A t-norm is strictly monotone if and only if it satisfies the cancellation law. A t-norm is called *strict* if it is continuous and strictly monotone, and it is called *nilpotent* if it is continuous and for any $x \in]0,1[$ there is an $n \in \mathbb{N}$ such that $x^n = 0$. Every continuous Archimedean t-norm is either strict, and isomorphic to product t-norm, or nilpotent, and isomorphic to the Łukasiewicz t-norm.

Proposition 13 *([19]). Let $*_1$ and $*_2$ be two continuous Archimedean t-norms. The following are equivalent:*

1. *$*_1$ and $*_2$ are isomorphic.*
2. *Either both $*_1$ and $*_2$ are strict or both $*_1$ and $*_2$ are nilpotent.*

Proposition 14 *([19]). Let $*$ be any t-norm:*

1. *$*$ is strict if and only if it is isomorphic to the product t-norm $*_{\text{prod}}$.*
2. *$*$ is nilpotent if and only if it is isomorphic to the Łukasiewicz t-norm $*_{\text{Łuk}}$.*

Idempotent elements of continuous t-norms will play a central role in several of the results we present in Section 3 and Section 5.

Definition 15 *(Idempotent Element).* For a t-norm $*$, an element $x \in [0,1]$ is called *idempotent* if $x * x = x$.

For all t-norms, 0 and 1 are trivial idempotent elements. Each $x \in [0,1]$ is an idempotent element of the minimum t-norm, which is the only t-norm whose elements are all idempotent. The Łukasiewicz and product t-norms (and so all continuous Archimedean t-norms in general) have no idempotent elements with the exception of the trivial ones.

Proposition 16. *Let $*$ be a continuous t-norm. Then:*

1. *Any $x \in [0,1]$ is an idempotent element if and only if, for all $y \in [0,1]$, $x * y = \min(x,y)$.*
2. *For all $(x,y) \in [0,1]^2$, $x * y \leq \min(x,y)$.*
3. *If $*$ is Archimedean, then, for all $(x,y) \in]0,1[^2$, $x * y < \min(x,y)$.*

Proof. Proofs for (1) and (2) can be found in [19]. (3) can be derived from results in [19] as follows. Suppose $*$ is Archimedean but there exists a pair of elements $(a,b) \in]0,1[^2$, such that $a * b = \min(a,b)$. Without any loss of generality suppose that $a < b$. Since $*$ is Archimedean, there exists an $n \in \mathbb{N}$ such that $b^n < a$, which, as t-norms are monotone functions, implies that $b^n * a \leq a * a$. As $a * b = a$, it is easy to see that $b^n * a = a$ and so we have that $a \leq a * a$, which, together with the fact that $a * a \leq a$ (statement (2) in this proposition), implies that a is idempotent. If $*$ is nilpotent and $b^n = 0$, then $a = 0$, contradicting the assumption that $a \in]0,1[$. If $b^n > 0$, then a is a non-trivial idempotent element of $*$, contradicting the assumption that $*$ is Archimedean. □

The ordinal sum construction allows the generation of new t-norms.

Theorem 17 *(Ordinal Sum [19]). Let A be a countable set, $\{*_i\}_{i\in A}$ be a family of t-norms and $\left\{]a_i,b_i[\right\}_{i\in A}$ be a family of non-empty, pairwise disjoint, open subintervals of $[0,1]$. Then the function $*\colon [0,1]^2 \to [0,1]$, defined by*

$$x * y = \begin{cases} a_i + \left(b_i - a_i\right) \cdot \left(\frac{x-a_i}{b_i-a_i} *_i \frac{y-a_i}{b_i-a_i}\right) & (x,y) \in [a_i,b_i]^2 \\ \min(x,y) & \text{otherwise} \end{cases},$$

is a t-norm.

We refer to each $*_i$ in the ordinal sum construction as a *component* of the ordinal sum. Notice that ordinal sums preserve continuity. Some examples of t-norms obtained as ordinal sums can be found in Section 5.

As shown by the Mostert-Shields theorem [22], every continuous t-norm can be represented as an ordinal sum of continuous Archimedean t-norms, i.e. of copies of the Łukasiewicz and the product t-norm.

Theorem 18 *(Mostert-Shields Theorem [19,22]). For a function $*\colon [0,1]^2 \to [0,1]$ the following are equivalent:*

1. *$*$ is a continuous t-norm.*
2. *$*$ is uniquely representable as an ordinal sum of continuous Archimedean t-norms.*

3. Possibilistic Choquet equilibria

In this section, we introduce a concept of equilibrium where players' expectations are given by the Choquet integral of their utility function with respect to a possibility measure. For each game, this possibility measure will be defined with respect to some given continuous t-norm. As mentioned above, the games we consider are assumed to be two-person games. The results presented here can be easily generalised for strategic-form games with any finite number of players.

Definition 19 *(Possibilistic Mixed Strategy)*. Given a strategic-form game $\mathbf{G}$, a *possibilistic mixed strategy* π_i for a player $i \in \{1,2\}$ is a possibility distribution $\pi_i : S_i \to [0,1]$.

We denote by Σ_i the set of all possibilistic mixed strategies of player i.

The possibilistic mixed strategies of the players will have an effect on the game and on the players' expectations concerning the outcome. A player's expectation will depend not only on their own possibilistic mixed strategy but also on the other player's. For this, we need a notion of joint possibility distribution.

Definition 20 *(Joint Possibility Distribution)*. Let $\mathbf{G}$ be a strategic-form game and $(\pi_1, \pi_2) \in \Sigma_1 \times \Sigma_2$ be a pair of possibilistic mixed strategies. Given a continuous t-norm $*$, the *joint possibility distribution* π_* of π_1 and π_2 is the function $\pi_* : S_1 \times S_2 \to [0,1]$ such that, for all $(s_1, s_2) \in S_1 \times S_2$,

$$\pi_*(s_1, s_2) = \pi_1(s_1) * \pi_2(s_2).$$

We now define a notion of expected utility based on the Choquet integral with respect to an arbitrary continuous t-norm $*$. In what follows, take a strategic-form game $\mathbf{G}$ with set of strategy combinations

$$S_1 \times S_2 = \left\{ \left(s_1, s_2\right)_1, \dots, \left(s_1, s_2\right)_m \right\}.$$

Note that we will always use m to denote the number of pure strategy combinations in a game. Now, for each $i \in \{1,2\}$, take a permutation α_i on $S_1 \times S_2$ such that

$$u_i\left(\left(s_1, s_2\right)_{\alpha_i(1)}\right) \le u_i\left(\left(s_1, s_2\right)_{\alpha_i(2)}\right) \le \dots \le u_i\left(\left(s_1, s_2\right)_{\alpha_i(m)}\right).$$

Let $A_{\alpha_i(j)} \subseteq S_1 \times S_2$ be defined as

$$A_{\alpha_i(j)} = \left\{ \left(s_1, s_2\right)_{\alpha_i(j)}, \left(s_1, s_2\right)_{\alpha_i(j+1)}, \dots, \left(s_1, s_2\right)_{\alpha_i(m)} \right\},$$

with $i \in \{1,2\}$ and $1 \le j \le m$.

Definition 21 *(Possibilistic Choquet Expected Utility)*. Let $\mathbf{G}$ be a strategic-form game, $(\pi_1, \pi_2) \in \Sigma_1 \times \Sigma_2$ be a pair of possibilistic mixed strategies, $*$ be a continuous t-norm, π_* be the joint possibility distribution of π_1 and π_2, and $\Pi_* : 2^{S_1 \times S_2} \to [0,1]$ be the possibility measure generated from π_*. The *possibilistic Choquet expected utility* of player i is the Choquet integral of the utility function u_i with respect to Π_* and is defined as

$$E_i^{Ch_*}(\pi_1, \pi_2) = \int^{Ch} u_i \mathrm{d}\Pi_* = \sum_{j=1}^{m} \left(u_i\left(\left(s_1, s_2\right)_{\alpha_i(j)}\right) - u_i\left(\left(s_1, s_2\right)_{\alpha_i(j-1)}\right)\right) \cdot \Pi_*\left(A_{\alpha_i(j)}\right),$$

where

$$\Pi_*\left(A_{\alpha_i(j)}\right) = \max_{(s_1, s_2) \in A_{\alpha_i(j)}} (\pi_1(s_1) * \pi_2(s_2)),$$

and $u_i\left(\left(s_1, s_2\right)_{\alpha_i(0)}\right) = 0$, by convention.

We now introduce the possibilistic counterpart of Definition 5 as well as specific notions of best response and equilibrium for Choquet integrals w.r.t. possibility measures.

Definition 22 *(Possibilistic Choquet Mixed Extension)*. Let $\mathbf{G}$ be a strategic-form game and $*$ be a continuous t-norm. The *possibilistic Choquet mixed extension* of $\mathbf{G}$ w.r.t. $*$ is the game

$$\mathfrak{G}^{\Pi_*} = \langle N, \Sigma_1, \Sigma_2, eu_1, eu_2 \rangle$$

where, for $i \in \{1,2\}$, $eu_i : \Sigma_1 \times \Sigma_2 \to \mathbb{R}$ is a function the that associates with each possibilistic mixed strategy combination $(\pi_1, \pi_2) \in \Sigma_1 \times \Sigma_2$ player i's Choquet expected utility w.r.t. the possibility measure generated by the joint possibility distribution π_* of π_1 and π_2:

$$eu_i(\pi_1, \pi_2) = E_i^{Ch_*}(\pi_1, \pi_2).$$

Definition 23 *(Possibilistic Best Response).* Let $\mathbf{G}$ be a strategic-form game and $\mathfrak{G}^{\Pi_*}$ be its possibilistic Choquet mixed extension w.r.t. some continuous t-norm $*$. Let $(\pi_1,\pi_2) \in \Sigma_1 \times \Sigma_2$ be a possibilistic mixed strategy combination. Player 1's possibilistic mixed strategy π_1 is called a *possibilistic best response* to π_2 if, for all $\pi_1' \in \Sigma_1$:

$$E_1^{Ch_*}(\pi_1,\pi_2) \geq E_1^{Ch_*}(\pi_1',\pi_2).$$

The definition for player 2 is analogous.

Definition 24 *(Possibilistic Choquet Equilibrium).* Let $\mathbf{G}$ be a strategic-form game and $\mathfrak{G}^{\Pi_*}$ be its possibilistic Choquet mixed extension w.r.t. some continuous t-norm $*$. We call a pair of possibilistic mixed strategies $(\pi_1,\pi_2) \in \Sigma_1 \times \Sigma_2$ a *possibilistic Choquet equilibrium* for $\mathbf{G}$ with respect to $*$, if each player's possibilistic mixed strategy is a possibilistic best response to the other player's possibilistic mixed strategy.

We are now going to offer a full characterisation of when a pair of possibilistic mixed strategies forms a possibilistic Choquet equilibrium w.r.t. some continuous t-norm $*$.

Definition 25. Let $\mathbf{G}$ be a strategic-form game and $\mathfrak{G}^{\Pi_*}$ be its possibilistic Choquet mixed extension w.r.t. some continuous t-norm $*$. Let $(\pi_1,\pi_2) \in \Sigma_1 \times \Sigma_2$ be a possibilistic mixed strategy combination. For each player $i = \{1,2\}$ we define:

1. $U_i = \left\{1 \leq j \leq m \mid u_i\left(\left(s_1,s_2\right)_{\alpha_i(j)}\right) - u_i\left(\left(s_1,s_2\right)_{\alpha_i(j-1)}\right) > 0\right\}$, and,
2. for each $1 \leq j \leq m$,

$$A^{\max}_{\alpha_i(j)} = \left\{ (s_1,s_2) \mid (s_1,s_2) \in A_{\alpha_i(j)},\ \text{and } s_{-i} \in \underset{s'_{-i}\in S_{-i},\ (s'_1,s'_2)\in A_{\alpha_i(j)}}{\operatorname{argmax}} \pi_{-i}(s'_{-i}) \right\}.$$

For each player i, U_i is the set of indices $1 \leq j \leq m$ such that $u_i\left(\left(s_1,s_2\right)_{\alpha_i(j)}\right)$ is strictly greater that $u_i\left(\left(s_1,s_2\right)_{\alpha_i(j-1)}\right)$. For each player i and index j, $A^{\max}_{\alpha_i(j)}$ is the set of pairs of pure strategies (s_1,s_2) in $A_{\alpha_i(j)}$ such that $\pi_{-i}(s_{-i})$ has the highest value.

The following theorem generalises Theorem 13 from [18].

Theorem 26. *Let $\mathbf{G}$ be a strategic-form game and $\mathfrak{G}^{\Pi_*}$ be its possibilistic Choquet mixed extension w.r.t. some continuous t-norm $*$. Let $(\pi_1,\pi_2) \in \Sigma_1 \times \Sigma_2$ be a possibilistic mixed strategy combination. Then, the following statements are equivalent:*

1. *(π_1,π_2) is a possibilistic Choquet equilibrium for $\mathbf{G}$ w.r.t. $*$.*
2. *For each player $i \in \{1,2\}$ and for every $j \in U_i$, there exists $(s_1,s_2) \in A^{\max}_{\alpha_i(j)}$ such that $\pi_i(s_i) \geq \pi_{-i}(s_{-i})$ and there is an idempotent element $a \in [\pi_{-i}(s_{-i}), \pi_i(s_i)]$.*

Proof. (1) ⇒ (2): Given a pair $(\pi_1,\pi_2) \in \Sigma_1 \times \Sigma_2$, suppose that (2) does not hold. Without any loss of generality we can assume that for player 1 there exists $j \in U_1$ so that, for all $(s_1,s_2) \in A^{\max}_{\alpha_1(j)}$, either $\pi_1(s_1) < \pi_2(s_2)$ or $\pi_1(s_1) \geq \pi_2(s_2)$ and there is no idempotent element $a \in [\pi_2(s_2), \pi_1(s_1)]$. We are going to show that player 1 can unilaterally change their possibilistic mixed strategy and improve their possibilistic Choquet expected utility.

Take then any $(s_1,s_2) \in A^{\max}_{\alpha_1(j)}$ and take a new possibility distribution $\pi_1' : S_1 \to [0,1]$ such that $\pi_1'(s_1) = 1$ and $\pi_1'(s_1') = \pi_1(s_1)$ for all $s_1' \neq s_1$. Let π_*' be the joint possibility distribution of π_1' and π_2 and let Π_*' be the possibility measure generated from π_*'. We first show that, for j and for each $(s_1'',s_2'') \in A_{\alpha_1(j)}$, we have that

$$\pi_1(s_1'') * \pi_2(s_2'') < \pi_2(s_2).$$

In fact:

1. If $(s_1'',s_2'') \in A^{\max}_{\alpha_1(j)}$ we have the following cases:
   (a) $\pi_1(s_1'') < \pi_2(s_2'')$. Then by Proposition 16(2),
   $$\pi_1(s_1'') * \pi_2(s_2'') \quad \leq \quad \pi_1(s_1'') \quad < \quad \pi_2(s_2'').$$
   (b) $\pi_1(s_1'') \geq \pi_2(s_2'')$ and there is no idempotent $a \in [\pi_2(s_2''), \pi_1(s_1'')]$. Then by Theorem 17 and Theorem 18, $\pi_1(s_1'')$ and $\pi_2(s_2'')$ must belong to the same interval $]a,b[$ in the ordinal sum representation of $*$, where the value of $(\pi_1(s_1'') * \pi_2(s_2''))$ is given by an isomorphic copy of a continuous Archimedean t-norm over $[a,b]^2$. By applying Proposition 16(3) we then obtain that
   $$\pi_1(s_1'') * \pi_2(s_2'') \quad < \quad \min(\pi_1(s_1''), \pi_2(s_2'')) \quad = \quad \pi_2(s_2'').$$

Both in case (a) and (b), since for all $(s_1'', s_2'') \in A^{\max}_{\alpha_1(j)}$, $\pi_2(s_2'') = \pi_2(s_2)$, we have:

$$\pi_1(s_1'') * \pi_2(s_2'') \quad < \quad \pi_2(s_2'') \quad = \quad \pi_2(s_2).$$

2. If $(s_1'', s_2'') \notin A^{\max}_{\alpha_1(j)}$ then, by Proposition 16(2) and by definition of $A^{\max}_{\alpha_1(j)}$:

$$\pi_1(s_1'') * \pi_2(s_2'') \quad \leq \quad \pi_2(s_2'') \quad < \quad \pi_2(s_2).$$

We can now show that player 1 can improve their possibilistic Choquet expected utility by changing their possibilistic mixed strategy from π_1 to π_1'. In fact, for j, we have that

$$\max_{(s_1'', s_2'') \in A_{\alpha_1(j)}} (\pi_1(s_1'') * \pi_2(s_2'')) \quad < \quad \pi_2(s_2) \quad = \quad \max_{(s_1'', s_2'') \in A_{\alpha_1(j)}} (\pi_1'(s_1'') * \pi_2(s_2'')).$$

This follows from the fact that for each $(s_1'', s_2'') \in A_{\alpha_1(j)}$

$$\pi_1(s_1'') * \pi_2(s_2'') < \pi_2(s_2),$$

(as shown above) and the fact that $\pi_1'(s_1) = 1$ and

$$\pi_1'(s_1) * \pi_2(s_2) = \pi_2(s_2).$$

For each $j' \in U_1$, with $j' \neq j$:

$$\max_{(s_1'', s_2'') \in A_{\alpha_1(j')}} (\pi_1(s_1'') * \pi_2(s_2'')) \quad \leq \quad \max_{(s_1'', s_2'') \in A_{\alpha_1(j')}} (\pi_1'(s_1'') * \pi_2(s_2'')),$$

since π_1 and π_1' differ only in the value assigned to s_1, and $\pi_1(s_1) < \pi_1'(s_1) = 1$.

From the above we have that

$$E_1^{Ch_*}(\pi_1, \pi_2) < E_1^{Ch_*}(\pi_1', \pi_2),$$

and (π_1, π_2) is not a possibilistic Choquet equilibrium.

(2) $\Rightarrow$ (1): Given a pair $(\pi_1, \pi_2) \in \Sigma_1 \times \Sigma_2$, suppose that for each player i and for every $j \in U_i$, there exists $(s_1, s_2) \in A^{\max}_{\alpha_i(j)}$ such $\pi_i(s_i) \geq \pi_{-i}(s_{-i})$ and there is an idempotent element $a \in [\pi_{-i}(s_{-i}), \pi_i(s_i)]$.

Then, for player 1, for each $j \in U_1$ there exists $(s_1, s_2) \in A^{\max}_{\alpha_1(j)}$ such $\pi_1(s_1) \geq \pi_2(s_2)$ and there is an idempotent $a \in [\pi_2(s_2), \pi_1(s_1)]$. From $a \leq \pi_1(s_1)$, by monotonicity of $*$, Proposition 16(1) and the fact that a is idempotent, we have that

$$\pi_2(s_2) \quad = \quad a * \pi_2(s_2) \quad \leq \quad \pi_1(s_1) * \pi_2(s_2),$$

which, together with the fact that $\pi_1(s_1) * \pi_2(s_2) \leq \pi_2(s_2)$ (Proposition 16(2)), implies that

$$\pi_1(s_1) * \pi_2(s_2) = \pi_2(s_2).$$

Then we have that

$$\Pi_* \left(A_{\alpha_1(j)}\right) = \max_{(s_1'', s_2'') \in A_{\alpha_1(j)}} (\pi_1(s_1'') * \pi_2(s_2'')) = \pi_2(s_2),$$

where Π_* is the possibility measure generated by the joint possibility distribution π_* of π_1 and π_2 w.r.t. $*$.

It is easy to see that for all $\pi_1' \in \Sigma_1$, for each $j \in U_1$ and any $(s_1, s_2) \in A^{\max}_{\alpha_1(j)}$:

$$\Pi_*' \left(A_{\alpha_1(j)}\right) = \max_{(s_1'', s_2'') \in A_{\alpha_1(j)}} (\pi_1'(s_1'') * \pi_2(s_2'')) \leq \pi_2(s_2),$$

where Π_*' is the possibility measure generated by the joint possibility distribution π_*' of π_1' and π_2 w.r.t. $*$. This is a consequence of the definition of $A^{\max}_{\alpha_1(j)}$ and (Proposition 16(2)). In fact, for any $(s_1, s_2) \in A^{\max}_{\alpha_1(j)}$, $\pi_2(s_2)$ is the highest value player 2 assigns to their strategies across all the strategy combinations in $A_{\alpha_1(j)}$ and is the maximum possible value of $\Pi_*'\left(A_{\alpha_1(j)}\right)$ for any possibility distribution π_1' chosen by player 1.

As a consequence, we have that, for all $\pi_1' \in \Sigma_1$, for each $j \in U_1$ and any $(s_1, s_2) \in A^{\max}_{\alpha_1(j)}$

$$\Pi_*'\left(A_{\alpha_1(j)}\right) \quad \leq \quad \pi_2(s_2) \quad = \quad \Pi_*\left(A_{\alpha_1(j)}\right),$$

which means that

$$E_1^{Ch_*}(\pi_1', \pi_2) \leq E_1^{Ch_*}(\pi_1, \pi_2).$$

A similar argument holds for player 2, and so (π_1, π_2) is a possibilistic Choquet equilibrium. □

The following corollary shows that every strategic-form game has a (trivial) possibilistic Choquet equilibrium, independent of the chosen t-norm.

Corollary 27. *Given any continuous t-norm $*$, every strategic-form game* $\mathbf{G}$ *has a possibilistic Choquet equilibrium w.r.t.* $*$.

Proof. Given any strategic-form game $\mathbf{G}$ and any continuous t-norm $*$, let π_1 and π_2 be possibility distributions such that $\pi_1(s_1) = 1$ for all $s_1 \in S_1$ and $\pi_2(s_2) = 1$ for all $s_2 \in S_2$. It is easy to see that (π_1, π_2) is a possibilistic Choquet equilibrium for $\mathbf{G}$ w.r.t. $*$ as each player maximises the value of their possibilistic Choquet expected utility. □

It is clear that every pure strategy $s_i \in S_i$ can be seen as a possibilistic mixed strategy by taking the distribution $\pi_i : S_i \to [0,1]$ such that $\pi_i(s_i) = 1$ and $\pi_i(s_i') = 0$ for all $s_i' \neq s_i$. Given any pure strategy, we refer to the possibilistic mixed strategy built as above as its corresponding *degenerate distribution*.

Lemma 28. *For any strategic-form game* $\mathbf{G}$*, let* $(s_1, s_2) \in S_1 \times S_2$ *be a pair of pure strategies,* $(\pi_1, \pi_2) \in \Sigma_1 \times \Sigma_2$ *be their corresponding pair of degenerate distributions, and* $*$ *be any continuous t-norm. Then, for each player* $i \in \{1, 2\}$*:*

$$u_i(s_1, s_2) = E_i^{Ch_*}(\pi_1, \pi_2).$$

Proof. Given any strategic-form game $\mathbf{G}$ and any continuous t-norm $*$, let $(s_1, s_2) \in S_1 \times S_2$ be a pair of pure strategies and $(\pi_1, \pi_2) \in \Sigma_1 \times \Sigma_2$ be their corresponding pair of degenerate distributions. Let Π_* be the possibility measure generated by the joint possibility distribution π_* of π_1 and π_2 w.r.t. $*$. For each $i \in \{1, 2\}$, it is easy to see that there is an index $1 \leq k \leq m$ such that, for all $1 \leq k' \leq k$, $(s_1, s_2) \in A_{\alpha_i(k')}$, and for all $k < k'' \leq m$, $(s_1, s_2) \notin A_{\alpha_i(k'')}$. Then:

1. For all k' such that $1 \leq k' \leq k$,
$$\Pi_* \left(A_{\alpha_i(k')} \right) = 1,$$
since $(s_1, s_2) \in A_{\alpha_i(k')}$ and $\pi_1(s_1) * \pi_2(s_2) = 1$.
2. For all k'' such that $k < k'' \leq m$,
$$\Pi_* \left(A_{\alpha_i(k'')} \right) = 0,$$
since $(s_1, s_2) \notin A_{\alpha_i(k'')}$ and, for all $(s_1'', s_2'') \in A_{\alpha_i(k'')}$, either $\pi_1(s_1'') = 0$ or $\pi_2(s_2'') = 0$.

As a consequence:

$$\begin{aligned} E_i^{Ch_*}(\pi_1, \pi_2) &= \sum_{k'=1}^{k} \left(u_i\left((s_1, s_2)_{\alpha_i(k')} \right) - u_i\left((s_1, s_2)_{\alpha_i(k'-1)} \right) \right) \cdot \Pi_*\left(A_{\alpha_i(k')} \right) \quad + \\ & \sum_{k''=k+1}^{m} \left(u_i\left((s_1, s_2)_{\alpha_i(k'')} \right) - u_i\left((s_1, s_2)_{\alpha_i(k''-1)} \right) \right) \cdot \Pi_*\left(A_{\alpha_i(k'')} \right) \\ &= \sum_{k'=1}^{k} \left(u_i\left((s_1, s_2)_{\alpha_i(k')} \right) - u_i\left((s_1, s_2)_{\alpha_i(k'-1)} \right) \right) \cdot \Pi_*\left(A_{\alpha_i(k')} \right), \end{aligned}$$

and so

$$E_i^{Ch_*}(\pi_1, \pi_2) \quad = \quad u_i\left((s_1, s_2)_{\alpha_i(k)} \right) \quad = \quad u_i(s_1, s_2). \quad \square$$

The next proposition shows that the concept of possibilistic Choquet equilibrium is a proper generalisation of the notion of Nash equilibrium in pure strategies.

Proposition 29. *Let* $\mathbf{G}$ *be a strategic-form game and* $\mathfrak{G}^{\Pi_*}$ *be its possibilistic Choquet mixed extension w.r.t. some continuous t-norm* $*$*. Let* $(s_1, s_2) \in S_1 \times S_2$ *be a pair of pure strategies and* $(\pi_1, \pi_2) \in \Sigma_1 \times \Sigma_2$ *be their corresponding pair of degenerate distributions. Then the following statements are equivalent:*

1. (s_1, s_2) *is a pure strategy Nash equilibrium.*
2. (π_1, π_2) *is a possibilistic Choquet equilibrium.*

Proof. Given any strategic-form game $\mathbf{G}$ and any continuous t-norm $*$, let $(s_1, s_2) \in S_1 \times S_2$ be a pair of pure strategies and $(\pi_1, \pi_2) \in \Sigma_1 \times \Sigma_2$ be their corresponding pair of degenerate distributions. Let Π_* be the possibility measure generated by the joint possibility distribution π_* of π_1 and π_2 w.r.t. $*$.

Suppose that (π_1, π_2) is not a possibilistic Choquet equilibrium. By Theorem 26, without any loss of generality, we can assume that for player 1 there exists $j \in U_1$ so that, for all $(s_1', s_2') \in A^{\max}_{\alpha_1(j)}$, either $\pi_1(s_1') < \pi_2(s_2')$ or $\pi_1(s_1') \geq \pi_2(s_2')$ and there is no idempotent element $a \in [\pi_2(s_2'), \pi_1(s_1')]$. Since π_1 and π_2 are degenerate distributions, this means that $A^{\max}_{\alpha_1(j)}$ contains strategy combinations (s_1', s_2), such that $\pi_2(s_2) = 1$ and $\pi_1(s_1') = 0$. Notice that, for all j' such that $j < j' \leq m$, $A_{\alpha_1(j')} \subset A_{\alpha_1(j)}$, which also means that $(s_1, s_2) \notin A_{\alpha_1(j')}$. So there must exist some $1 \leq k < j$ such that, for all $1 \leq k' \leq k$, $(s_1, s_2) \in A_{\alpha_1(k')}$, and for all $k < k'' \leq m$, $(s_1, s_2) \notin A_{\alpha_1(k'')}$. Then,

1. for all k' such that $1 \leq k' \leq k$,
$$\Pi_* \left(A_{\alpha_1(k')} \right) = 1,$$
and
2. for all k'' such that $k < k'' \leq m$,
$$\Pi_* \left(A_{\alpha_1(k'')} \right) = 0$$

(see Lemma 28).

Now, take any $(s_1', s_2) \in A^{\max}_{\alpha_1(j)}$ and let π_1' be a degenerate possibility distribution such that $\pi_1'(s_1') = 1$ but $\pi_1'(s_1'') = 0$ for all $s_1'' \neq s_1'$. (s_1', s_2) belongs to all $A_{\alpha_1(j'')}$ with $1 \leq j'' \leq j$, and, clearly, for all these sets, $\Pi'_* \left(A_{\alpha_1(j'')} \right) = 1$, where Π'_* is the possibility measure generated by the joint possibility distribution π'_* of π_1' and π_2 w.r.t. $*$. By the fact that $j \in U_1$ and $k < j$, and given that

$$\begin{aligned} E_1^{Ch_*}(\pi_1, \pi_2) &= \sum_{k'=1}^{k} \left(u_1\left((s_1, s_2)_{\alpha_1(k')} \right) - u_1\left((s_1, s_2)_{\alpha_1(k'-1)} \right) \right) \cdot \Pi_* \left(A_{\alpha_1(k')} \right) \quad + \\ & \sum_{k''=k+1}^{m} \left(u_1\left((s_1, s_2)_{\alpha_1(k'')} \right) - u_1\left((s_1, s_2)_{\alpha_1(k''-1)} \right) \right) \cdot \Pi_* \left(A_{\alpha_1(k'')} \right), \end{aligned}$$

and

$$\begin{aligned} E_1^{Ch_*}(\pi_1', \pi_2) &= \sum_{k'=1}^{k} \left(u_1\left((s_1, s_2)_{\alpha_1(k')} \right) - u_1\left((s_1, s_2)_{\alpha_1(k'-1)} \right) \right) \cdot \Pi'_* \left(A_{\alpha_1(k')} \right) \quad + \\ & \sum_{k''=k+1}^{j} \left(u_1\left((s_1, s_2)_{\alpha_1(k'')} \right) - u_1\left((s_1, s_2)_{\alpha_1(k''-1)} \right) \right) \cdot \Pi'_* \left(A_{\alpha_1(k'')} \right) \quad + \\ & \sum_{j'=j+1}^{m} \left(u_1\left((s_1, s_2)_{\alpha_1(j')} \right) - u_1\left((s_1, s_2)_{\alpha_1(j'-1)} \right) \right) \cdot \Pi'_* \left(A_{\alpha_1(j')} \right), \end{aligned}$$

it is easy to see that

$$E_1^{Ch_*}(\pi_1, \pi_2) < E_1^{Ch_*}(\pi_1', \pi_2)$$

and, by Lemma 28,

$$u_1(s_1, s_2) < u_1(s_1', s_2)$$

(where (s_1', s_2) is the pure strategy combination corresponding to the pair (π_1', π_2) of degenerate distributions), which means that (s_1, s_2) is not a pure strategy Nash equilibrium.

Conversely, suppose that (s_1, s_2) is not a pure strategy Nash equilibrium. Then, without any loss of generality, we can suppose that for player 1, there exists a pure strategy s_1' such that

$$u_1(s_1, s_2) < u_1(s_1', s_2).$$

By Lemma 28, given the degenerate distributions π_1, π_1', π_2 corresponding to s_1, s_1', s_2, respectively, we have that

$$E_1^{Ch_*}(\pi_1, \pi_2) < E_1^{Ch_*}(\pi_1', \pi_2),$$

which means that (π_1, π_2) is not a possibilistic Choquet equilibrium. □

4. Examples

We look now at some well-known examples of strategic-form games. Please note that in these examples we will slightly change the notation used in Section 3, where we described a more general approach.

Table 1
The Prisoner's Dilemma.

		Player 2 S	Player 2 C
Player 1	S	2 2	3 0
	C	0 3	1 1

4.1. The Prisoner's Dilemma

Recall that in the Prisoner's Dilemma two criminals are apprehended and questioned separately and in isolation. Each criminal can make the choice of cooperating by remaining silent (S) or defecting by betraying their fellow criminal and confessing (C). If both confess, each prisoner will face two years in prison. If only one confesses, they will be freed while the other prisoner will face three years in prison. If neither of them confesses, both will be charged with a sentence of one year in prison. Table 1 shows the Prisoner's Dilemma in matrix form using a convenient payoff representation of the prisoners' preferences.

The game has the following set of pure strategy combinations

$$\{(S,S),(S,C),(C,S),(C,C)\},$$

where (C,C) is the only pure strategy Nash equilibrium, corresponding to the situation where both criminals choose to confess. We compute the set of possibilistic Choquet equilibria of the game.

We begin by taking an ordering of the payoffs of each player $i \in \{1,2\}$ as follows:

$$\begin{array}{ccccccc} u_1(S,C) & \leq & u_1(C,C) & \leq & u_1(S,S) & \leq & u_1(C,S) \\ u_2(C,S) & \leq & u_2(C,C) & \leq & u_2(S,S) & \leq & u_2(S,C) \end{array}.$$

Given the above ordering, we define the sets

$$\begin{array}{lllllll} A_{1(1)} & = & \{(S,C),(C,C),(S,S),(C,S)\} & \quad & A_{2(1)} & = & \{(C,S),(C,C),(S,S),(S,C)\} \\ A_{1(2)} & = & \{(C,C),(S,S),(C,S)\} & & A_{2(2)} & = & \{(C,C),(S,S),(S,C)\} \\ A_{1(3)} & = & \{(S,S),(C,S)\} & & A_{2(3)} & = & \{(S,S),(S,C)\} \\ A_{1(4)} & = & \{(C,S)\} & & A_{2(4)} & = & \{(S,C)\} \end{array},$$

i.e.: $A_{i(1)}$, $A_{i(2)}$, $A_{i(3)}$ etc. are the sets of strategy combinations for player i that have the lowest, second lowest, third lowest etc. payoff according to the above ordering.

For each player i, U_i is the set of indices $1 \leq j \leq 4$ such that the difference between the lowest utility of the strategy combinations in $A_{i(j)}$ and the lowest utility of the strategy combinations in $A_{i(j-1)}$ is strictly greater that 0 (cfr. Definition 25). Since

$$\begin{array}{lllll} u_1(S,C) & - & 0 & = & 0 \\ u_1(C,C) & - & u_1(S,C) & = & 1 \\ u_1(S,S) & - & u_1(C,C) & = & 1 \\ u_1(C,S) & - & u_1(S,S) & = & 1 \end{array} \qquad \begin{array}{lllll} u_2(C,S) & - & 0 & = & 0 \\ u_2(C,C) & - & u_2(C,S) & = & 1 \\ u_2(S,S) & - & u_2(C,C) & = & 1 \\ u_2(S,C) & - & u_2(S,S) & = & 1 \end{array},$$

we have $U_1 = U_2 = \{2,3,4\}$.

To compute the equilibria of the game, we now make use of Theorem 26 and look at all possible pairs (π_1, π_2) of possibilistic mixed strategies. In what follows, given any (π_1, π_2), $A^{\max}_{1(j)}$ will denote the set of strategy combinations (s_1, s_2) in $A_{1(j)}$, with $1 \leq j \leq 4$, where $\pi_2(s_2)$ has the highest value (cfr. Definition 25). $A^{\max}_{2(j)}$ is defined in a similar way.

By Corollary 27, we know that the pair (π_1, π_2) where

$$\begin{array}{ll} \pi_1(S) = 1 & \pi_2(S) = 1 \\ \pi_1(C) = 1 & \pi_2(C) = 1 \end{array}$$

trivially is a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(S) = 1 & \pi_2(S) = 1 \\ \pi_1(C) < 1 & \pi_2(C) = 1 \end{array}, \quad \begin{array}{ll} \pi_1(S) = 1 & \pi_2(S) = 1 \\ \pi_1(C) = 1 & \pi_2(C) < 1 \end{array}, \text{ or } \quad \begin{array}{ll} \pi_1(S) = 1 & \pi_2(S) = 1 \\ \pi_1(C) < 1 & \pi_2(C) < 1 \end{array}.$$

In the first and third case, we have $\pi_1(C) < \pi_2(S)$ for $A_{1(4)}$, while in the second and third we have $\pi_1(S) > \pi_2(C)$ for $A_{2(4)}$. This means that none of the above pairs is a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(S) < 1 & \pi_2(S) = 1 \\ \pi_1(C) = 1 & \pi_2(C) = 1 \end{array}, \quad \begin{array}{ll} \pi_1(S) = 1 & \pi_2(S) < 1 \\ \pi_1(C) = 1 & \pi_2(C) = 1 \end{array}, \text{ or } \quad \begin{array}{ll} \pi_1(S) < 1 & \pi_2(S) < 1 \\ \pi_1(C) = 1 & \pi_2(C) = 1 \end{array}.$$

Table 2
Matching Pennies with rescaled payoffs.

		Player 2: H	Player 2: T
Player 1	H	2, 0	0, 2
	T	0, 2	2, 0

In all these cases, (C,C) belongs to $A^{\max}_{1(2)}$ and $A^{\max}_{2(2)}$, while (C,S) belongs to $A^{\max}_{1(3)}$ and $A^{\max}_{1(4)}$ and $\pi_1(C) \geq \pi_2(S)$, and (S,C) belongs to $A^{\max}_{2(3)}$ and $A^{\max}_{2(4)}$ and $\pi_2(C) \geq \pi_1(S)$. Then, the above pairs are all possibilistic Choquet equilibria.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(S)=1 & \pi_2(S)<1 \\ \pi_1(C)<1 & \pi_2(C)=1 \end{array} \quad \text{or} \quad \begin{array}{ll} \pi_1(S)<1 & \pi_2(S)=1 \\ \pi_1(C)=1 & \pi_2(C)<1 \end{array}.$$

In the first case, (S,C) belongs to each $A_{2(j)}$, with $1 \leq j \leq 4$, so player 2 cannot improve their possibilistic Choquet expected utility. For player 1, we have that $A^{\max}_{1(2)} = \{(C,C)\}$ and $\pi_1(C) < \pi_2(C)$, and so (π_1,π_2) is not a possibilistic Choquet equilibrium. For the second case, (C,S) belongs to each $A_{1(j)}$, with $1 \leq j \leq 4$, so player 1 cannot improve their possibilistic Choquet expected utility. For player 2, we have that $A^{\max}_{2(2)} = \{(C,C)\}$ and $\pi_2(C) < \pi_1(C)$, and so (π_1,π_2) is not a possibilistic Choquet equilibrium.

This covers all the possible combinations of possibilistic mixed strategies and the set of possibilistic Choquet equilibria is given by

$$\left\{(\pi_1,\pi_2) \mid \pi_1(S) \leq 1, \pi_1(C)=1 \text{ and } \pi_2(S) \leq 1, \pi_2(C)=1\right\}.$$

Notice that the above results for the Prisoner's Dilemma are independent of the choice of a continuous t-norm.

4.2. Matching Pennies

The two-person game of *Matching Pennies* is an example of a game with no pure strategies Nash equilibria. In this game, each player has a penny and decides to show either heads (H) or tails (T). If both players make the same choice (i.e. they both choose H or T), then player 1 keeps both coins. If the players' choices are different, then player 2 keeps both coins. The game can be presented in the matrix form displayed in Table 2, where the payoffs have been rescaled from the original version to non-negative values.

The game has the following set of pure strategy combinations

$$\{(H,H),(H,T),(T,H),(T,T)\}.$$

We compute the set of possibilistic Choquet equilibria of the game.

We begin by taking an ordering of the payoffs of each player $i \in \{1,2\}$ as follows:

$$\begin{array}{ccccccc} u_1(T,H) & \leq & u_1(H,T) & \leq & u_1(H,H) & \leq & u_1(T,T) \\ u_2(H,H) & \leq & u_2(T,T) & \leq & u_2(T,H) & \leq & u_2(H,T) \end{array}.$$

From the above, we obtain the sets

$$\begin{array}{lcl} A_{1(1)} & = & \{(T,H),(H,T),(H,H),(T,T)\} \\ A_{1(2)} & = & \{(H,T),(H,H),(T,T)\} \\ A_{1(3)} & = & \{(H,H),(T,T)\} \\ A_{1(4)} & = & \{(T,T)\} \end{array} \qquad \begin{array}{lcl} A_{2(1)} & = & \{(H,H),(T,T),(T,H),(H,T)\} \\ A_{2(2)} & = & \{(T,T),(T,H),(H,T)\} \\ A_{2(3)} & = & \{(T,H),(H,T)\} \\ A_{2(4)} & = & \{(H,T)\} \end{array}.$$

Since

$$\begin{array}{lclcl} u_1(T,H) & - & 0 & = & 0 \\ u_1(H,T) & - & u_1(T,H) & = & 0 \\ u_1(H,H) & - & u_1(H,T) & = & 2 \\ u_1(T,T) & - & u_1(H,H) & = & 0 \end{array} \qquad \begin{array}{lclcl} u_2(H,H) & - & 0 & = & 0 \\ u_2(T,T) & - & u_2(H,H) & = & 0 \\ u_2(T,H) & - & u_2(T,T) & = & 2 \\ u_2(H,T) & - & u_2(T,H) & = & 0 \end{array},$$

we have $U_1 = U_2 = \{3\}$.

To compute the equilibria of the game, we now make use of Theorem 26 and look at all possible pairs (π_1,π_2) of possibilistic mixed strategies.

By Corollary 27, we know that the pair (π_1,π_2), where

$$\begin{array}{ll} \pi_1(H)=1 & \pi_2(H)=1 \\ \pi_1(T)=1 & \pi_2(T)=1 \end{array},$$

trivially is a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(H) = 1 & \pi_2(H) < 1 \\ \pi_1(T) = 1 & \pi_2(T) = 1 \end{array}.$$

For player 1 we have $A^{\max}_{1(3)} = \{(T,T)\}$ and $\pi_1(T) \geq \pi_2(T) = 1$. For player 2 we have $A^{\max}_{2(3)} = \{(T,H),(H,T)\}$ and $\pi_2(T) \geq \pi_1(H) = 1$. Then (π_1, π_2) is a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(H) < 1 & \pi_2(H) = 1 \\ \pi_1(T) = 1 & \pi_2(T) = 1 \end{array}.$$

For player 1 we have $A^{\max}_{1(3)} = \{(H,H),(T,T)\}$ and $\pi_1(T) \geq \pi_2(T) = 1$. For player 2 we have $A^{\max}_{2(3)} = \{(T,H)\}$ and $\pi_2(H) \geq \pi_1(T) = 1$. Then (π_1, π_2) is a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(H) = 1 & \pi_2(H) = 1 \\ \pi_1(T) < 1 & \pi_2(T) = 1 \end{array}.$$

For player 1 we have $A^{\max}_{1(3)} = \{(H,H),(T,T)\}$ and $\pi_1(H) \geq \pi_2(H) = 1$. For player 2 we have $A^{\max}_{2(3)} = \{(H,T)\}$ and $\pi_2(T) \geq \pi_1(H) = 1$. Then (π_1, π_2) is a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(H) = 1 & \pi_2(H) = 1 \\ \pi_1(T) = 1 & \pi_2(T) < 1 \end{array}.$$

For player 1 we have $A^{\max}_{1(3)} = \{(H,H)\}$ and $\pi_1(H) \geq \pi_2(H) = 1$. For player 2 we have $A^{\max}_{2(3)} = \{(T,H),(H,T)\}$ and $\pi_2(H) \geq \pi_1(T) = 1$. Then (π_1, π_2) is a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(H) < 1 & \pi_2(H) < 1 \\ \pi_1(T) = 1 & \pi_2(T) = 1 \end{array}.$$

For player 2 we have $A^{\max}_{2(3)} = \{(T,H)\}$ and $\pi_2(H) < \pi_1(T)$. Then (π_1, π_2) is not a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(H) = 1 & \pi_2(H) = 1 \\ \pi_1(T) < 1 & \pi_2(T) < 1 \end{array}.$$

For player 2 we have $A^{\max}_{2(3)} = \{(H,T)\}$ and $\pi_2(T) < \pi_1(H)$. Then (π_1, π_2) is not a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(H) < 1 & \pi_2(H) = 1 \\ \pi_1(T) = 1 & \pi_2(T) < 1 \end{array}.$$

For player 1 we have $A^{\max}_{1(3)} = \{(H,H)\}$ and $\pi_1(H) < \pi_2(H)$. Then (π_1, π_2) is not a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(H) = 1 & \pi_2(H) < 1 \\ \pi_1(T) < 1 & \pi_2(T) = 1 \end{array}.$$

For player 1 we have $A^{\max}_{1(3)} = \{(T,T)\}$ and $\pi_1(T) < \pi_2(T)$. Then (π_1, π_2) is not a possibilistic Choquet equilibrium.

This covers all the possible combinations of possibilistic mixed strategies and the set of possibilistic Choquet equilibria is given by

$$\begin{array}{lll} \{ \ (\pi_1, \pi_2) & | & \pi_1(H) \leq 1, \pi_1(T) = 1 \text{ and } \pi_2(H) = 1, \pi_2(T) = 1, \text{or} \\ & & \pi_1(H) = 1, \pi_1(T) \leq 1 \text{ and } \pi_2(H) = 1, \pi_2(T) = 1, \text{or} \\ & & \pi_1(H) = 1, \pi_1(T) = 1 \text{ and } \pi_2(H) \leq 1, \pi_2(T) = 1, \text{or} \\ & & \pi_1(H) = 1, \pi_1(T) = 1 \text{ and } \pi_2(H) = 1, \pi_2(T) \leq 1 \ \}. \end{array}$$

Similar to the Prisoner's Dilemma, the set of possibilistic Choquet equilibria for Matching Pennies is independent of the choice of a continuous t-norm. Not surprisingly, this is not true for all strategic-form games, and even for 2×2 games in particular, as shown in the next subsection.

4.3. Stag Hunt game

The Stag Hunt game is an example of a two-player coordination game. Two hunters can choose to hunt a stag or a hare. Hunting the hare requires a small effort and offers a small reward, but if only one hunter makes this choice, they will receive a much higher

Table 3
The Stag Hunt game.

		Player 2	
		S	H
Player 1	S	10 10	8 1
	H	1 8	5 5

payoff than the other as they won't have to share their catch. In this case, the other hunter will have to hunt the stag alone, which requires a much higher effort for little chance of success. If, however, the hunters make the same choice, they will catch their prey, with the higher payoff obtained when both hunt the stag. The game is well known to have two pure strategy Nash equilibria, corresponding to the players coordinating on the same choice, and it can be presented in the matrix form displayed in Table 3.

The game has the following set of pure strategy combinations

$$\{(S,S),(S,H),(H,S),(H,H)\}.$$

We compute the set of possibilistic Choquet equilibria of the game.

We begin by taking an ordering of the payoffs of each player $i \in \{1,2\}$ as follows:

$$\begin{array}{ccccccc} u_1(S,H) & \leq & u_1(H,H) & \leq & u_1(H,S) & \leq & u_1(S,S) \\ u_2(H,S) & \leq & u_2(H,H) & \leq & u_2(S,H) & \leq & u_2(S,S) \end{array}.$$

From the above, we obtain the sets

$$\begin{array}{lcllcl} A_{1(1)} & = & \{(S,H),(H,H),(H,S),(S,S)\} & A_{2(1)} & = & \{(H,S),(H,H),(S,H),(S,S)\} \\ A_{1(2)} & = & \{(H,H),(H,S),(S,S)\} & A_{2(2)} & = & \{(H,H),(S,H),(S,S)\} \\ A_{1(3)} & = & \{(H,S),(S,S)\} & A_{2(3)} & = & \{(S,H),(S,S)\} \\ A_{1(4)} & = & \{(S,S)\} & A_{2(4)} & = & \{(S,S)\} \end{array}.$$

Since

$$\begin{array}{lclcllclcl} u_1(S,H) & - & 0 & = & 1 & u_2(H,S) & - & 0 & = & 1 \\ u_1(H,H) & - & u_1(S,H) & = & 4 & u_2(H,H) & - & u_2(H,S) & = & 4 \\ u_1(H,S) & - & u_1(H,H) & = & 3 & u_2(S,H) & - & u_2(H,H) & = & 3 \\ u_1(S,S) & - & u_1(H,S) & = & 2 & u_2(S,S) & - & u_2(S,H) & = & 2 \end{array},$$

we have $U_1 = U_2 = \{1,2,3,4\}$.

To compute the equilibria of the game, we now make use of Theorem 26 and look at all possible pairs (π_1, π_2) of possibilistic mixed strategies.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(S) = 1 & \pi_2(S) = 1 \\ \pi_1(H) \leq 1 & \pi_2(H) \leq 1 \end{array}.$$

Since $(S,S) \in A^{\max}_{i(j)}$ for each player i and each $1 \leq j \leq 4$, any of the above pairs trivially is a possibilistic Choquet equilibrium.

Consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(S) < 1 & \pi_2(S) = 1 \\ \pi_1(H) = 1 & \pi_2(H) \leq 1 \end{array} \quad \text{or} \quad \begin{array}{ll} \pi_1(S) = 1 & \pi_2(S) < 1 \\ \pi_1(H) \leq 1 & \pi_2(H) = 1 \end{array}.$$

In the first case, we have that $\{(S,S)\} = A^{\max}_{1(4)}$ and $\pi_1(S) < \pi_2(S)$. In the second, we have that $\{(S,S)\} = A^{\max}_{2(4)}$ and $\pi_2(S) < \pi_1(S)$. In both cases then, (π_1, π_2) is not a possibilistic Choquet equilibrium.

Finally, consider any pair (π_1, π_2) such that

$$\begin{array}{ll} \pi_1(S) < 1 & \pi_2(S) < 1 \\ \pi_1(H) = 1 & \pi_2(H) = 1 \end{array}.$$

Again, we have that $\{(S,S)\} = A^{\max}_{1(4)} = A^{\max}_{2(4)}$. If $\pi_1(S) \neq \pi_2(S)$, then (π_1, π_2) is obviously not an equilibrium. If $\pi_1(S) = \pi_2(S)$, then (π_1, π_2) is a possibilistic Choquet equilibrium as long as both $\pi_1(S)$ and $\pi_2(S)$ are idempotent. This, of course, depends on the specific choice of a t-norm and shows that the sets of equilibria associated to different t-norms for the same game are not necessarily the same.

To make this more explicit, consider the pair (π_1, π_2) of possibility distributions such that

$$\begin{array}{ll} \pi_1(S) = 0.4 & \pi_2(S) = 0.4 \\ \pi_1(H) = 1 & \pi_2(H) = 1 \end{array}.$$

It is easy to check that (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t. the minimum t-norm: this is a consequence of both Theorem 26 in Section 3 and Theorem 13 from [18]. In fact, for player 1,

$$\Pi_{*_{\min}}\left(A_{1(1)}\right) = \Pi_{*_{\min}}\left(A_{1(2)}\right) = 1$$

and

$$\begin{aligned}\Pi_{*_{\min}}\left(A_{1(3)}\right) &= \max(\min(\underset{=1}{\pi_1(H)}, \underset{=0.4}{\pi_2(S)}), \min(\underset{=0.4}{\pi_1(S)}, \underset{=0.4}{\pi_2(S)})) = 0.4,\\ \Pi_{*_{\min}}\left(A_{1(4)}\right) &= \min(\underset{=0.4}{\pi_1(S)}, \underset{=0.4}{\pi_2(S)}) = 0.4\end{aligned}$$

(where $\Pi_{*_{\min}}$ is the possibility measure generated by the joint possibility distribution $\pi_{*_{\min}}$ of π_1 and π_2 w.r.t. the minimum t-norm). It is clear that there is no possible distribution for player 1 that would unilaterally increase the value of $\Pi_{*_{\min}}\left(A_{1(3)}\right)$ and $\Pi_{*_{\min}}\left(A_{1(4)}\right)$. A very similar argument can be made for player 2, showing that (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t. the minimum t-norm.

The same pair of distributions is not, however, a possibilistic Choquet equilibrium w.r.t. the product t-norm $*_{\text{prod}}$. As an example, for any pair (π_1', π_2) of possibility distributions such that

$$\begin{array}{ll}\pi_1'(S) = a & \pi_2(S) = 0.4\\ \pi_1'(H) = 1 & \pi_2(H) = 1\end{array},$$

with $a > 0.4$, player 1 increases their possibilistic Choquet expected utility. In fact, for the pair (π_1, π_2) of possibilistic mixed strategies such that

$$\begin{array}{ll}\pi_1(S) = 0.4 & \pi_2(S) = 0.4\\ \pi_1(H) = 1 & \pi_2(H) = 1\end{array},$$

we have, for player 1,

$$\Pi_{*_{\text{prod}}}\left(A_{1(1)}\right) = \Pi_{*_{\text{prod}}}\left(A_{1(2)}\right) = 1,$$

and

$$\begin{aligned}\Pi_{*_{\text{prod}}}\left(A_{1(3)}\right) &= \max((\underset{=1}{\pi_1(H)} \cdot \underset{=0.4}{\pi_2(S)}), (\underset{=0.4}{\pi_1(S)} \cdot \underset{=0.4}{\pi_2(S)})) = 0.4,\\ \Pi_{*_{\text{prod}}}\left(A_{1(4)}\right) &= \underset{=0.4}{\pi_1(S)} \cdot \underset{=0.4}{\pi_2(S)} = 0.16\end{aligned}$$

(where $\Pi_{*_{\text{prod}}}$ is the possibility measure generated by the joint possibility distribution $\pi_{*_{\text{prod}}}$ of π_1 and π_2 w.r.t. the product t-norm). For any pair (π_1', π_2) of possibilistic mixed strategies such that

$$\begin{array}{ll}\pi_1'(S) = a & \pi_2(S) = 0.4\\ \pi_1'(H) = 1 & \pi_2(H) = 1\end{array},$$

with $a > 0.4$, we have, for player 1,

$$\Pi'_{*_{\text{prod}}}\left(A_{1(1)}\right) = \Pi'_{*_{\text{prod}}}\left(A_{1(2)}\right) = 1,$$

and

$$\begin{aligned}\Pi'_{*_{\text{prod}}}\left(A_{1(3)}\right) &= \max((\underset{=1}{\pi_1'(H)} \cdot \underset{=0.4}{\pi_2(S)}), (\underset{=a}{\pi_1'(S)} \cdot \underset{=0.4}{\pi_2(S)})) = 0.4,\\ \Pi'_{*_{\text{prod}}}\left(A_{1(4)}\right) &= \underset{=a}{\pi_1'(S)} \cdot \underset{=0.4}{\pi_2(S)} > 0.16\end{aligned}$$

(where $\Pi'_{*_{\text{prod}}}$ is the possibility measure generated by the joint possibility distribution $\pi'_{*_{\text{prod}}}$ of π_1' and π_2 w.r.t. the product t-norm). Then (π_1, π_2) is not a possibilistic Choquet equilibrium w.r.t. the product t-norm. A similar argument can be made for the Łukasiewicz t-norm and any continuous Archimedean t-norm, as a consequence of Corollary 31 below.

5. Relationship between equilibria of different T-norms

As mentioned above, any continuous t-norm $*$ is isomorphic to an ordinal sum of isomorphic copies of continuous Archimedean t-norms (Theorem 18). In any ordinal sum, the set of idempotents can be seen as a sort of skeleton supporting and separating the components. In this section, we show that idempotent elements also play a pivotal role in determining the set of possibilistic Choquet equilibria.

In what follows, for any continuous t-norm $*$, we denote by $I(*) \subseteq [0,1]$ its set of idempotent elements.

Theorem 30. *Let $*$ and $*'$ be any two continuous t-norms, and $I(*)$ and $I(*')$ be their sets of idempotents. Let* $\mathbf{G}$ *be a strategic-form game, and $\mathfrak{G}^{\Pi_*}$ and $\mathfrak{G}^{\Pi_{*'}}$ be the possibilistic Choquet mixed extensions of* $\mathbf{G}$ *w.r.t. $*$ and $*'$. Let $h : [0,1] \to [0,1]$ be an order-preserving bijection. Then the following hold:*

1. *If h is an order-embedding of $I(*)$ into $I(*')$, then if (π_1, π_2) is a possibilistic Choquet equilibrium for* $\mathbf{G}$ *w.r.t.* $*$*, so is* $(h(\pi_1), h(\pi_2))$ *w.r.t.* $*'$.
2. *If $I(*)$ and $I(*')$ are order-isomorphic under h, then (π_1, π_2) is a possibilistic Choquet equilibrium for* $\mathbf{G}$ *w.r.t.* $*$ *if and only if* $(h(\pi_1), h(\pi_2))$ *is a possibilistic Choquet equilibrium for* $\mathbf{G}$ *w.r.t.* $*'$.

Proof. Given any strategic-form game $\mathbf{G}$, consider any two continuous t-norms $*$ and $*'$ along with their corresponding possibilistic Choquet mixed extensions $\mathfrak{G}^{\Pi_*}$ and $\mathfrak{G}^{\Pi_{*'}}$. Let $h : [0,1] \to [0,1]$ be an order-preserving bijection that is an order embedding of $I(*)$ into $I(*')$. Suppose that (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t. $*$. We show that $(h(\pi_1), h(\pi_2))$ is a possibilistic Choquet equilibrium w.r.t. $*'$.

First, notice that, since h is an order preserving bijection, for each j,

$$\begin{aligned} A^{\max}_{\alpha_1(j)} &= \left\{ (s'_1, s'_2) \mid (s'_1, s'_2) \in A_{\alpha_1(j)} \text{ and } s'_2 \in \underset{s''_2 \in S_2,\ (s''_1, s''_2) \in A_{\alpha_1(j)}}{\operatorname{argmax}} \pi_2(s''_2) \right\} \\ &= \left\{ (s'_1, s'_2) \mid (s'_1, s'_2) \in A_{\alpha_1(j)} \text{ and } s'_2 \in \underset{s''_2 \in S_2,\ (s''_1, s''_2) \in A_{\alpha_1(j)}}{\operatorname{argmax}} h(\pi_2(s''_2)) \right\}. \end{aligned}$$

Now, by assumption (π_1, π_2) is a possibilistic Choquet equilibrium for $\mathbf{G}$ w.r.t. $*$. Then by Theorem 26, for player 1 and for every $j \in U_1$, there exists $(s_1, s_2) \in A^{\max}_{\alpha_1(j)}$ such that $\pi_1(s_1) \geq \pi_2(s_2)$ and there is an idempotent element $a \in [\pi_2(s_2), \pi_1(s_1)]$. Since h is an order-preserving bijection that maps idempotents into idempotents, it is easy to see that

$$h(\pi_1(s_1)) \geq h(a) \geq h(\pi_2(s_2)),$$

where $h(a)$ is an idempotent element. The same argument can be made for player 2. Then, by Theorem 26, $(h(\pi_1), h(\pi_2))$ is a possibilistic Choquet equilibrium for $\mathbf{G}$ w.r.t. to $*'$.

This proves (1). (2) can be shown in a similar way. □

From the above theorem we can derive some interesting results concerning equilibria of any continuous t-norm and the minimum t-norm, and equilibria of continuous Archimedean t-norms.

Corollary 31. *Let* $\mathbf{G}$ *be a strategic-form game, let $*$ and $*'$ be any two continuous t-norms and let $\mathfrak{G}^{\Pi_*}$ and $\mathfrak{G}^{\Pi_{*'}}$ be the possibilistic Choquet mixed extensions of* $\mathbf{G}$ *w.r.t.* $*$ *and* $*'$.

1. *If (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t. $*$, then (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t. the minimum t-norm.*
2. *If $*$ and $*'$ are Archimedean, then the sets of possibilistic Choquet equilibria w.r.t. $*$ and $*'$ coincide.*
3. *If $*$ is Archimedean and (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t. $*$, then (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t.* $*'$.

Proof. (1) follows from Theorem 30(1) along with the fact that every element of the minimum t-norm is idempotent, and that the identity mapping $id : [0,1] \to [0,1]$ is a bijection that trivially maps the set of idempotents of $*$ into the set of idempotents of the minimum.

(2) is a consequence of Theorem 30(2), as every continuous Archimedean t-norm has only two idempotent elements, i.e. 0 and 1, and of taking the identity mapping $id : [0,1] \to [0,1]$.

(3) follows from Theorem 30(1) by an argument similar to (2). □

The previous corollary shows that, for any continuous t-norm $*$, a possibilistic Choquet equilibrium w.r.t. $*$ will also be a possibilistic Choquet equilibrium w.r.t. the minimum t-norm. However, we know from Section 4.3 that not every possibilistic Choquet equilibrium w.r.t. the minimum t-norm will necessarily be a possibilistic Choquet equilibrium w.r.t. $*$. While there are strategic-form games for which the set of possibilistic Choquet equilibria does not change no matter the choice of a continuous t-norm (sections 4.1 and 4.2), in general, games with mixed extensions based on different t-norms will have different sets of equilibria.

In what follows, we see some examples of ordinal sums of continuous t-norms and the relationship between their sets of idempotents.

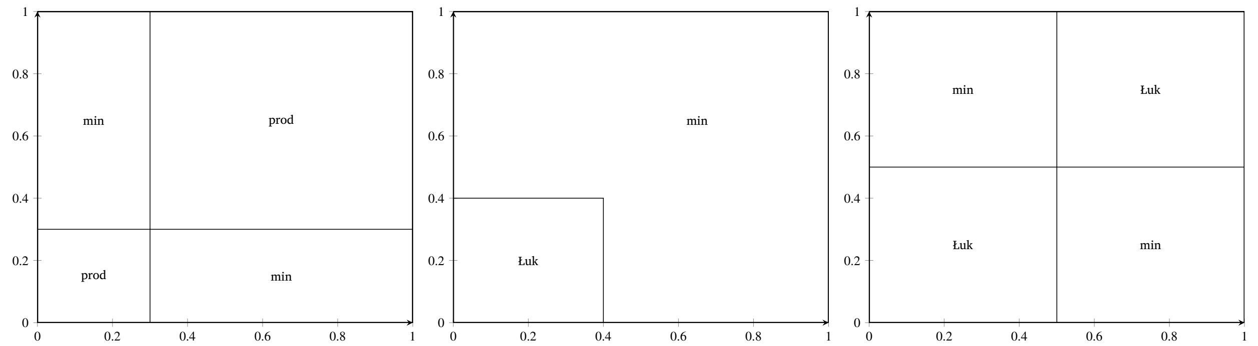

Fig. 2. Domains of the t-norms $*_1$, $*_2$, and $*_3$ with their components (Example 32).

Example 32. Consider the following continuous t-norms (see Fig. 2):

$$x *_1 y = \begin{cases} 0.3 \cdot \left(\frac{x}{0.3} \cdot \frac{y}{0.3}\right) & (x,y) \in [0,0.3]^2 \\ 0.3 + 0.7 \cdot \left(\frac{x-0.3}{0.7} \cdot \frac{y-0.3}{0.7}\right) & (x,y) \in [0.3,1]^2 \\ \min(x,y) & \text{otherwise} \end{cases},$$

$$x *_2 y = \begin{cases} 0.4 \cdot \left(\max\left(0, \frac{x}{0.4} + \frac{y}{0.4} - 1\right)\right) & (x,y) \in [0,0.4]^2 \\ \min(x,y) & \text{otherwise} \end{cases},$$

$$x *_3 y = \begin{cases} 0.5 \cdot \left(\max\left(0, \frac{x}{0.5} + \frac{y}{0.5} - 1\right)\right) & (x,y) \in [0,0.5]^2 \\ 0.5 + 0.5 \cdot \left(\max\left(0, \frac{x-0.5}{0.5} + \frac{y-0.5}{0.5} - 1\right)\right) & (x,y) \in [0.5,1]^2 \\ \min(x,y) & \text{otherwise} \end{cases}.$$

$*_1$ is a continuous t-norm that is the ordinal sum of two isomorphic copies of the product t-norm: the first over $[0,0.3]^2$ and the second over $[0.3,1]^2$, with set of idempotents $I(*_1) = \{0, 0.3, 1\}$. $*_2$ is a continuous t-norm obtained as an ordinal sum built from an isomorphic copy of the Łukasiewicz t-norm over $[0,0.4]^2$, with $I(*_2) = \{0\} \cup [0.4,1]$. $*_3$ is a continuous t-norm that is the ordinal sum of two isomorphic copies of the Łukasiewicz t-norm: the first over $[0,0.5]^2$ and the second over $[0.5,1]^2$, with $I(*_3) = \{0, 0.5, 1\}$.

Let

$$h_1(x) = \begin{cases} \frac{4}{3}x & x \in [0,0.3] \\ \frac{6}{7}x + \frac{1}{7} & x \in (0.3,1] \end{cases}.$$

h_1 is an order-preserving bijection that is an order-embedding of $I(*_1)$ into $I(*_2)$. By Theorem 30, we have that, for any strategic-form game $\mathbf{G}$, if (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t. $*_1$, then so is $(h_1(\pi_1), h_1(\pi_2))$ w.r.t. $*_2$.

Let

$$h_2(x) = \begin{cases} \frac{5}{3}x & x \in [0,0.5] \\ \frac{5}{7}x + \frac{2}{7} & x \in (0.5,1] \end{cases}.$$

h_2 is an order-preserving bijection that is an order-isomorphism between $I(*_1)$ into $I(*_3)$. Again, by Theorem 30, we have that, for any strategic-form game $\mathbf{G}$, (π_1, π_2) is a possibilistic Choquet equilibrium w.r.t. $*_1$, if and only if so is $(h_2(\pi_1), h_2(\pi_2))$ w.r.t. $*_3$.

We conclude this section by showing how, given any strategic-form game $\mathbf{G}$, for any pair of possibility distributions satisfying certain conditions, we can construct a family of continuous t-norms from a finite set of elements so that each resulting t-norm $*$ makes (π_1, π_2) into a possibilistic Choquet equilibrium for $\mathbf{G}$ w.r.t. $*$. We describe this construction here.

Let $\mathbf{G}$ be any strategic-form game and (π_1, π_2) be any pair of possibilistic mixed strategies for $\mathbf{G}$ such that for each player i and for every $j \in U_i$, there exists $(s_1, s_2) \in A^{\max}_{\alpha_i(j)}$ such $\pi_i(s_i) \geq \pi_{-i}(s_{-i})$. For each player i and each $j \in U_i$, take a real number $a_{ij} \in [\pi_{-i}(s_{-i}), \pi_i(s_i)]$. Take the set

$$X = \{0, 1\} \cup \bigcup_{j \in U_1} \{a_{1j}\} \cup \bigcup_{j' \in U_2} \{a_{2j'}\}.$$

$X \subset [0,1]$ can be presented as an ordered set of elements

$$X = \{b_1, b_2, \dots, b_{k-1}, b_k\}$$

where $k' < k''$ if and if only $b_{k'} < b_{k''}$. Then, for each $1 \le l \le k-1$, let $*_l$ be either a continuous Archimedean t-norm or the minimum t-norm and $]b_l, b_{l+1}[$ be an open subinterval of $[0,1]$. Let $*$ be the t-norm defined by

$$x * y = \begin{cases} b_l + (b_{l+1} - b_l) \cdot \left(\frac{x-b_l}{b_{l+1}-b_l} *_l \frac{y-b_l}{b_{l+1}-b_l} \right) & (x,y) \in [b_l, b_{l+1}]^2 \\ \min(x,y) & \text{otherwise} \end{cases}.$$

$*$ is a continuous t-norm built from X where the elements of X are taken as idempotents bounding the components of the ordinal sum, each being isomorphic to either the minimum, product or Łukasiewicz t-norm. It is easy to check that for any choice of $*_l$, (π_1, π_2) is a possibilistic Choquet equilibrium for $\mathbf{G}$ w.r.t. $*$.

6. Final remarks

The above results suggest that the set of possibilistic Choquet equilibria of a strategic-form game essentially depends on the set of idempotents of a t-norm and not strictly on the t-norm itself. By taking the minimum, we have the guarantee of obtaining the set of all possible possibilistic Choquet equilibria, as any other choice will not return possibilistic mixed strategy combinations that are not in this set. The question then is: why would anyone choose to model joint possibility distributions with the Łukasiewicz or product t-norms (or any other continuous t-norm) over the minimum and would this really matter? One could argue that the choice of a t-norm is still relevant depending on the interpretation given to a possibilistic mixed strategy and a joint possibility distribution. Suppose that, for instance, we were to interpret a possibilistic mixed strategy as a measure of the commitment of a player to make a specific choice (as we do with respect to the Weak-link game in [18]). The choice of the Łukasiewicz t-norm can model the situation in which to obtain a positive aggregated value of the commitment of different players requires the combined commitment to pass a certain threshold. In that case, for a strategy combination (s_1, s_2), we need $\pi_1(s_1) + \pi_2(s_2) > 1$ to have $\pi_1(s_1) *_{Łuk} \pi_2(s_2) > 0$. Other situations that require a different threshold might be more adequately represented by taking other nilpotent t-norms or taking ordinal sums where the Łukasiewicz t-norm (or another nilpotent t-norm) is the first component. The choice of the minimum t-norm can instead model the situation in which the aggregated value is rather seen as the minimum commitment level among the players. It is then clear that the choice of a particular t-norm is relevant in the context of what possibilistic mixed strategies are used to model in strategic interactions.

The above suggests that to gain a better understanding of the role the possibilistic approach plays in game theory, further research should explore a behavioural interpretation of the notion of possibilistic mixed strategies. Possibility measures are a special kind of imprecise probabilities [30] and can be seen, in behavioural terms, as coherent upper probabilities [31], i.e. as upper bounds of some set of probability measures. Whilst the game theoretic literature abounds with behavioural generalisations of Nash equilibria aimed at overcoming the limitations of the additive representations of uncertainty [13,20], this has not quite been the motivating question for our research. As discussed in full detail in [18], our goal has been rather that of investigating the mathematical consequences of randomising with possibility instead of probability distributions. However, the framework of Ellsberg games [27] may provide an interesting bridge between our analysis and the approach to game theory aimed at rationalising ambiguity aversion. The reason is that in Ellsberg games players can be seen to make a strategic use of ambiguity by concealing their intentions of playing one probability distribution and rather playing a set of distributions. Seeing possibility measures as upper bounds of sets of probabilities could offer then an interesting link to Ellsberg games. We plan to explore this connection in our future work.

CRediT authorship contribution statement

Esther Anna Corsi: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. **Hykel Hosni:** Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. **Enrico Marchioni:** Conceptualization, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Methodology, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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