SOME NOTES ON FUNCTIONS OF LEAST $W^{s,1}$-FRACTIONAL SEMINORM
ALCUNE NOTE SULLE FUNZIONI DI SEMINORMA FRAZIONARIA $W^{s,1}$ MINIMA

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Abstract. In this survey we discuss some existence and asymptotic results, originally obtained in [4, 3], for functions of least $W^{s,1}$-fractional seminorm. We present the connection between these functions and nonlocal minimal surfaces, leveraging this relation to build a function of least fractional seminorm. We further prove that a function of least fractional seminorm is the limit for $p \to 1$ of the sequence of minimizers of the $W^{s,p}$-energy. Additionally, we consider the fractional 1-Laplace operator and study the equivalence between weak solutions and functions of least fractional seminorm.

Sunto. In questa nota discutiamo alcuni risultati di esistenza e asintotici, originariamente ottenuti in [4, 3], per le funzioni di seminorma frazionaria $W^{s,1}$ minima. Presentiamo la connessione tra queste funzioni e le superfici minime nonlocali, e ricorriamo a tale relazione per costruire una funzione di seminorma frazionaria minima. Otteniamo inoltre una funzione di seminorma frazionario minima come limite per $p \to 1$ del minimo dell'energia frazionaria $W^{s,p}$. Consideriamo in più l'1-Laplaciano frazionario e mostriamo l'equivalenza tra le soluzioni deboli e le funzioni di seminorma frazionaria $W^{s,1}$ minima.

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1. Introduction

The problem of *functions of least* $W^{s,1}$*-seminorm* is the fractional counterpart of a renowned problem in the calculus of variations, that of *functions of least gradient.* In this survey we present some results regarding functions of least $W^{s,1}$-seminorm, originally published in [4, 3]. In particular, we will discuss an existence result based on the relation between these minimizers and nonlocal minimal surfaces, the asymptotics as $p \to 1$ of the p-problem, and the equivalence between minimizers and weak solutions, all with an eye of regard towards the results obtained for functions of least gradient in [2, 21, 20, 11, 13].

Functions of least gradient were first introduced in [15, 2] because of their connection to minimal surfaces, but have turned out to be useful in numerous applications such as in image processing, TV denoising, conductivity problems or free material design. A mathematical model for denoising an image (the so-called ROF model, after the names of its authors [17]) is based on functions of least gradient. To present it in a nutshell, let Ω be a domain where a blurred image is represented, and let $f\colon \Omega \to \mathbb{R}$ be this observed image – in terms of pixels. One can then decompose f in a regularized component representing the true image, a function $u\colon \Omega \to \mathbb{R}$ of bounded variation, and be left with the noise, $f - u$. Denoising the image requires then to minimize the energy

$$\mathcal{E}(u) = \int_\Omega |Du| + \lambda(u - f)^2 \, dx,$$

with λ a positive tuning parameter. Historically, in numerical models, a quadratic term was first used in the kinetic energy, replaced in sequel by a linear term, due to better numerical results. In particular, the BV space allows to recover the discontinuities of the image, thus keeping edges and contours in the picture. Later, refinements of this model were introduced, see e.g. [16]. In more recent years in [12], a fractional model using the $W^{s,1}$-fractional seminorm, replacing the BV seminorm, was proposed. Due to its nonlocal nature, such a fractional energy maintains low frequency texture details in the smooth area, thus is better preserving fine structures.

1.1. **Setting of the problem and results in the classical case.** Here and in the rest of this note, let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set with Lipschitz boundary. Let $s \in (0,1)$ be a fractional parameter.

The problem of functions of least gradient is set in the space of functions of bounded variation. Let $u \in L^1(\Omega)$, we say that $u \in BV(\Omega)$ if the distributional derivative of $u \colon \Omega \to \mathbb{R}$ can be represented by a signed vector valued Radon measure Du, whose total variation $|Du|$ in Ω,

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u \operatorname{div} \varphi \,|\, \varphi \in C_c^1(\Omega, \mathbb{R}^n), \|\varphi\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq 1 \right\},$$

is finite. Furthermore, $BV(\Omega)$ is a Banach space with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + [u]_{BV(\Omega)},$$

where we denote $[u]_{BV(\Omega)} := |Du|(\Omega)$.

To impose a boundary condition, let $g \in L^1(\partial\Omega)$ be given. For all $u \in BV(\Omega)$, we say that $u = g$ on $\partial\Omega$ in the sense of traces if the equality

$$\lim_{\rho \to 0} \rho^{-n} \int_{\Omega \cap B_\rho(x)} |u(y) - g(x)| \, dy = 0$$

holds for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$. We then set

$$BV_g(\Omega) := \{u \in BV(\Omega) \,|\, u = g \text{ in the sense of traces on } \partial\Omega\}.$$

We say that u is of least gradient if it realizes

$$\min \left\{ \int_\Omega |Du| \,\middle|\, u \in BV_g(\Omega) \right\}. \tag{1}$$

We point out that the energy $|Du|(\Omega)$ is lower semicontinuous and that the space $BV(\Omega)$ is compact in $L^1(\Omega)$. However, a minimizer of the energy may not attain the boundary data $g \in L^1(\Omega)$ in the sense of traces, and this makes the problem of existence of least gradient functions, i.e. a solution of (1), quite delicate. Moreover, even considering a more regular boundary data, say $g \in C(\partial\Omega)$, and trying to solve the problem

$$\min \left\{ \int_\Omega |Du| \,\middle|\, u \in BV(\Omega) \cap C(\overline{\Omega}), \ u = g \text{ on } \partial\Omega \right\}, \tag{2}$$

a solution may not exist, unless one imposes further assumptions on the domain Ω.

The problem of the existence of a continuous (and unique) least gradient function, which reaches it boundary data $g \in C(\partial\Omega)$ in the pointwise sense, i.e. of a minimizer of (2), has been treated with two different approaches in [21, 11]. Provided $\partial\Omega$ has non-negative mean curvature in a weak sense, and that it is not locally a minimal set,

- in [21], the authors use a level set method based on a result from [2], that is that
  - level sets of functions of least gradient are minimal surfaces,

  to prove the existence of a unique minimizer of (2);
- while in [11], the author shows that the solution obtained in [21] is the uniform limit as $p \to 1$ of the sequence of continuous minimizers of the $W^{1,p}$-energy.

We point out that the assumptions on Ω say that, if $\partial\Omega$ is smooth, then the mean curvature should be positive on a dense set of $\partial\Omega$, while if $\Omega \subset \mathbb{R}^2$, then it should be strictly convex. We underline that in the absence of any of the two additional geometric assumptions on $\partial\Omega$, the existence of a minimizer of (2) is not guaranteed (see [21, Theorem 3.8]).

What is more, for $g \in C(\partial\Omega)$, a minimizer of (2) is also a minimizer of (1), see [21, Theorem 3.7]. The problem of uniqueness in the larger class of BV functions was addressed in [20] and answered positively, provided that Ω additionally satisfies an interior ball condition. Under these three hypothesis on Ω (i.e., $\partial\Omega$ has non-negative mean curvature in a weak sense, it is not locally a minimal set, and Ω satisfies an interior ball condition), the problem (1) has a unique, continuous, minimizer.

In our papers [4, 3], we studied whether we could obtain similar results in the fractional setting, focusing in particular on existence results. We set the fractional problem in the space $W^{s,1}(\Omega)$, which is a proper counterpart of the classical case. Indeed, it is known that as $s \to 1$, the $W^{s,1}$ seminorm of a BV function, renormalized by $(1-s)$, converges to the BV seminorm of the function (see [8]).

With an abuse of notation, we define

$$W^{s,1}(\Omega) := \{u \colon \mathbb{R}^n \to \mathbb{R} \,|\, u|_\Omega \in W^{s,1}(\Omega)\},$$

and denote the fractional $W^{s,1}$-seminorm on Ω as

$$[u]_{W^{s,1}(\Omega)} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|^{n+s}}\, dxdy.$$

As customary in nonlocal problems, the "boundary data" is now an exterior data, given on the whole complement on Ω. For a function $g\colon \mathcal{C}\Omega \to \mathbb{R}$ we then define

$$W^{s,1}_g(\Omega) := \{u \in W^{s,1}_g(\Omega) \,|\, u = g \text{ on } \mathcal{C}\Omega\}.$$

Finally, the nonlocal energy to minimize is

$$\mathcal{E}_s(u,\Omega) = \frac{1}{2}\iint_{Q(\Omega)} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dxdy,$$

with $Q(\Omega) = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$. We remark that this energy is the sum of a local contribution $\Omega \times \Omega$ – precisely the seminorm $[u]_{W^{s,1}(\Omega)}$–, and a nonlocal one, given by the interaction $\Omega \times \mathcal{C}\Omega$, that accounts for the exterior given data. The interaction $\mathcal{C}\Omega \times \mathcal{C}\Omega$ is disregarded, since it does not contribute to the minimization problem – and might as well be infinite.

We say that a function is of least $W^{s,1}$-seminorm if it realizes

$$\min\left\{\mathcal{E}_s(u,\Omega) \,|\, u \in W^{s,1}_g(\Omega)\right\}. \tag{3}$$

In Section 2, the two main results take their inspiration from [2, 21]. We prove that

- level sets of functions of least $W^{s,1}$-fractional seminorm[1] are nonlocal minimal surfaces,
- a least $W^{s,1}$-fractional seminorm function exists, for a given, general, exterior data.

In Section 3, we follow the approach in [11] and study the asymptotics for $p \to 1$ of the minimizers of the $W^{s,p}$-energy, also in the Gamma-convergence sense.

We turn now to the Euler-Lagrange equation associated to problems (1), respectively (3).

We note that when the datum $g \in L^1(\partial\Omega)$ (and it is not continuous), a least gradient function $u \in BV_g(\Omega)$ may not exist, check [19, 9] for examples in $\mathbb{R}^2$. However, there exists (see [13, Theorem 1.1 (2)]) a weak solution, i.e. a 1-harmonic function. If furthermore, the boundary data is achieved in the sense of traces, then the weak solution is also a

[1] We avoid using the term "fractional gradient" – which would resemble the nomenclature of the classical problem – to avoid confusion with [18] and related work. As a matter of fact, there is no apparent connection between the results presented in this notes and the fractional gradient therein introduced.

least gradient problem (and viceversa), without any further restrictions on $\partial\Omega$ (besides assuming it Lipschitz).

The definition of weak solution/1-harmonic function (hence, the representation of the 1-Laplace operator) is quite articulate, and we postpone it to Section 4. There, we handle the fractional counterpart of the equivalence between mimimizers and weak solutions. To complete the fractional picture, we underline that the definition of weak solution of an equation involving the fractional $(s,1)$-Laplacian, with given right hand side data $f \in L^2(\Omega)$ and zero exterior condition, was given in [14]. The authors use there the asymptotics as $p \to 1$ of the weak solution of the (s,p)-Laplace operator (along the lines of [13]). In [3], we adapted the definition given in [14] to $(s,1)$-harmonic functions with given exterior data. See also [10], where the existence of a weak solution for a fractional 1-Laplacian evolution equation is obtained differently with respect to [14].

In the last brief Section 5, we make some notes on regularity and uniqueness of minimizers for the fractional problem, which are still open problems.

2. Existence of minimizers by the level sets method

A first existence result can be obtained in the following context.

Definition 2.1. *We say that* $u\colon \mathbb{R}^n \to \mathbb{R}$ *is an* $(s,1)$*-minimizer of the energy* $\mathcal{E}_s$ *if* $\mathcal{E}_s(u,\Omega) < +\infty$ *and*

$$\mathcal{E}_s(u,\Omega) \leq \mathcal{E}_s(v,\Omega) \quad \textit{for all } v = u \textit{ on } \mathcal{C}\Omega.$$

The energy $\mathcal{E}_s$ is the sum of a local contribution and a nonlocal one, precisely

$$\mathcal{E}_s(u,\Omega) = \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dxdy + \int_\Omega\left(\int_{\mathcal{C}\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dy\right)dx.$$

Notice that if the energy is finite, then $u \in W^{s,1}(\Omega)$. Moreover, we must impose some condition on the nonlocal contribution $\Omega \times \mathcal{C}\Omega$. Applying the fractional Hardy inequality (see [7, Proposition A.2]) and [4, (1.4)]), we have that

(4)
$$\begin{aligned}\int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\, dy \right) dx \leq& \int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(x)|}{|x-y|^{n+s}}\, dy \right) dx + \int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(y)|}{|x-y|^{n+s}}\, dy \right) dx \\ \leq& \|u\|_{W^{s,1}(\Omega)} + \int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(y)|}{|x-y|^{n+s}}\, dy \right) dx.\end{aligned}$$

If we consider $u \in W^{s,1}(\Omega)$, it is thus enough to require the latter integral to be finite to have the energy finite. The assumptions on the exterior data should therefore be given in terms of the so-called *nonlocal tail* of g,

$$\mathrm{Tail}_s(g, \mathcal{O}, x) = \int_{\mathcal{O}} \frac{|g(y)|}{|x-y|^{n+s}}\, dy,$$

defined for any $\mathcal{O} \subset \mathcal{C}\Omega$.

We have the following preliminary result, which is easily proved by direct methods, see [4, Theorem A.1].

Proposition 2.1. *Let $g \colon \mathcal{C}\Omega \to \mathbb{R}$ be such that*

(5)
$$\mathrm{Tail}_s(g, \mathcal{C}\Omega, \cdot) \in L^1(\Omega).$$

Then there exists an $(s,1)$-minimizer $u \in W^{s,1}_g(\Omega)$ of the energy $\mathcal{E}_s(\cdot, \Omega)$.

Notice that in this proposition, we impose a global condition, that is we take $\mathcal{O}$ in the definition of the tail to be the whole $\mathcal{C}\Omega$. One hopes to be able to ask less, but in order to do so, a less restrictive definition of minimizer should be in place.

Definition 2.2. *We say that $u \in W^{s,1}(\Omega)$ is an s-minimal function if and only if*

$$\iint_{Q(\Omega)} \frac{|u(x)-u(y)| - |v(x)-v(y)|}{|x-y|^{n+s}}\, dxdy \leq 0,$$

for all $v \in W^{s,1}(\Omega)$ such that $v = u$ on $\mathcal{C}\Omega$.

We remark that this definition is well posed without assuming, a priori, any condition on the function in $\mathcal{C}\Omega$. This is due to the possibility of applying the fractional Hardy inequality, see [4, (1.5)]. Observe furthermore that if (5) holds, then $u \in W^{s,1}_g(\Omega)$ is an s-minimal function according to Definition 2.2 if and only if u is an $(s,1)$-minimizer, in agreement with Definition 2.1.

Let us denote, furthermore, for some $\eta > 0$,

$$\Omega_\eta := \{x \in \mathcal{C}\Omega \,|\, \mathrm{dist}(x, \partial\Omega) < \eta\},$$

and use the familiar notation

$$\mathrm{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y|.$$

We have the following existence theorem.

Theorem 2.1. *There exists $\Theta = \Theta(n, s) > 1$ such that the following statement holds true. If $g \colon \mathcal{C}\Omega \to \mathbb{R}$ is such that*

$$\int_\Omega \left[\int_{\Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega} \frac{|g(y)|}{|x - y|^{n+s}}\, dy \right] dx < +\infty, \tag{6}$$

then there exists an s-minimal function $u \in W^{s,1}_g(\Omega)$ in Ω.

The proof, as in the classical case, is based on the connection between s-minimal functions and s-minimal sets. We briefly recall here some notions on s-minimal sets, useful for our purposes.

The classical perimeter of a set $E \subset \mathbb{R}^n$ in Ω is

$$P(E, \Omega) = [\chi_E]_{BV(\Omega)},$$

where χ_E is the characteristic function of E.

The fractional perimeter is defined as

$$\begin{aligned} \mathrm{Per}_s(E, \Omega) &:= \frac{1}{2} \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+s}} dx\, dy \\ &= \mathcal{E}_s(\chi_E, \Omega). \end{aligned}$$

We say that $E \subset \mathbb{R}^n$ is an s-minimal set in Ω if

$$\mathrm{Per}_s(E, \Omega) \le \mathrm{Per}_s(F, \Omega) \qquad \forall\, F \text{ such that } F \setminus \Omega = E \setminus \Omega.$$

The boundary of E is a *nonlocal minimal surface* (see [6] for more details on this argument).

The relation between sets of minimal perimeter and functions of least gradient can be observed immediately by comparing their definitions: if $E \subset \mathbb{R}^n$ is a minimal set then χ_E

is of least gradient. The same holds in the fractional case, see [4, Theorem 1.4]. Moreover, according to the result in [2, Theorem 1], the level sets of functions of least gradient are minimal sets. The fractional counterpart of that result is the content of the following lemma, see [4, Theorem 1.3].

Lemma 2.1. *If $u \in W^{s,1}(\Omega)$ is an s-minimal function in Ω, then for all $\lambda \in \mathbb{R}$, the set $\{u \geq \lambda\}$ is an s-minimal set in Ω.*
If $u \in W^{s,1}(\Omega)$ and for almost all $\lambda \in \mathbb{R}$, the set $\{u \geq \lambda\}$ is an s-minimal set in Ω, then u is an s-minimal function in Ω.

We give a brief sketch of the proof and invite the reader to consult [4] for all the details.

Sketch of the proof of Lemma 2.1. The proof that "an s-minimal set implies an s-minimal function" readily follows from the coarea formula (see [22], or e.g. [5, Theorem 3.2.2])

$$\mathcal{E}_s(u,\Omega) = \int_{-\infty}^{\infty} \operatorname{Per}_s(\{u \geq \lambda\}, \Omega)\, d\lambda,$$

observing that

$$\begin{aligned}\iint_{Q(\Omega)} & \frac{|u(x)-u(y)| - |v(x)-v(y)|}{|x-y|^{n+s}}\, dxdy \\ &= \int_{-\infty}^{\infty} \Big(\operatorname{Per}_s(\{u \geq \lambda\}, \Omega) - \operatorname{Per}_s(\{v \geq \lambda\}, \Omega)\Big) d\lambda,\end{aligned}$$

for all v such that $v = u$ on $\mathcal{C}\Omega$.

To prove that an "s-minimal function implies an s-minimal set", we build a sequence $\varphi_{\lambda,\varepsilon}$ of s-minimal functions, such that as $\varepsilon \to 0$, almost everywhere in $\mathbb{R}^n$,

$$\varphi_{\lambda,\varepsilon} \to \chi_{\{u \geq \lambda\}}.$$

More precisely, let u be an s-minimal function. For a fixed arbitrary $\varepsilon > 0$ we let

$$\varphi_{\lambda,\varepsilon} := \frac{\min\{\varepsilon, \max\{u - \lambda + \sqrt{\varepsilon}, 0\}\}}{\varepsilon},$$

and show that

- $0 \leq \varphi_{\lambda,\varepsilon} \leq 1$,
- the positive and negative parts, the translation and multiplication by constant of an s-minimal function is still s-minimal, thus $\varphi_{\lambda,\varepsilon}$ are s-minimal functions,

- for ε small enough, if $u < \lambda$ then $\varphi_{\lambda,\varepsilon} = 0$, when $u \geq \lambda$, then
$$\varphi_{\lambda,\varepsilon} = \frac{\min\{\varepsilon, u - \lambda + \sqrt{\varepsilon}\}}{\varepsilon} \geq \min\{1, \varepsilon^{-1/2}\} \geq 1,$$
so we obtain that
$$\varphi_{\lambda,\varepsilon} \to \chi_{\{u\geq\lambda\}} \quad \text{a.e. in } \mathbb{R}^n \quad \text{as } \varepsilon \to 0,$$
- the limit, $\chi_{\{u\geq\lambda\}}$ is an s-minimal function, hence
$$\mathrm{Per}_s(\{u \geq \lambda\}, \Omega) = \mathcal{E}_s(\chi_{\{u\geq\lambda\}}, \Omega) \leq \mathcal{E}_s(\chi_F, \Omega) = \mathrm{Per}_s(F, \Omega)$$
for all competitors F such that $\{u \geq \lambda\} = F$ on $\mathcal{C}\Omega$, hence $\{u \geq \lambda\}$ is an s-minimal set.

This concludes the sketch of the proof of the lemma. $\square$

Having set this lemma, we give a sketch of the proof of the existence theorem, leaving the details to [4].

Sketch of the proof of Theorem 2.1. We observe that if E, F are s-minimal, then so are $E \cup F$, $E \cap F$ and furthermore,

$$\mathrm{Per}_s(E \cap F, \Omega) + \mathrm{Per}_s(E \cup F, \Omega) \leq \mathrm{Per}_s(E, \Omega) + \mathrm{Per}_s(F, \Omega). \tag{7}$$

This observation allows to prove the existence and uniqueness of an s-minimal set of maximum volume, precisely
$$E = \bigcup_{F\in\mathcal{F}} F,$$
where $\mathcal{F}$ is the set of all s-minimal sets with a fixed exterior data. We fix the exterior data
$$\tilde{E}_t := \{g \geq t\} \subset \mathcal{C}\Omega,$$
and let E_t be the s-minimal set of maximum volume with exterior data $\tilde{E}_t$. It is obvious that $\tilde{E}_\tau \supset \tilde{E}_t$ if $\tau < t$, and moreover
$$(E_t \cap E_\tau) \setminus \Omega = \tilde{E}_t \cap \tilde{E}_\tau = \tilde{E}_t, \qquad (E_t \cup E_\tau) \setminus \Omega = \tilde{E}_t \cup \tilde{E}_\tau = \tilde{E}_\tau,$$
hence by the minimality of E_t and E_τ,
$$\mathrm{Per}_s(E_t, \Omega) \leq \mathrm{Per}_s(E_t \cap E_\tau, \Omega), \qquad \mathrm{Per}_s(E_\tau, \Omega) \leq \mathrm{Per}_s(E_t \cup E_\tau, \Omega).$$

We then have, employing (7), that

$$\begin{aligned}\operatorname{Per}_s(E_t,\Omega)+\operatorname{Per}_s(E_t\cup E_\tau,\Omega)\leq&\ \operatorname{Per}_s(E_t\cap E_\tau,\Omega)+\operatorname{Per}_s(E_t\cup E_\tau,\Omega)\\ \leq&\ \operatorname{Per}_s(E_t,\Omega)+\operatorname{Per}_s(E_\tau,\Omega).\end{aligned}$$

Hence $\operatorname{Per}_s(E_\tau,\Omega) = \operatorname{Per}_s(E_t\cup E_\tau,\Omega)$, and since E_τ is the s-minimal set of maximal volume, $|E_t\cup E_\tau| = |E_\tau|$, therefore

$$E_t \subset E_\tau \qquad \text{if } \tau < t.$$

We now define

$$u(x) := \begin{cases} g(x), & \text{for } x\in\mathcal{C}\Omega,\\ \sup\left\{t \mid x\in \overline{E_t}\right\}, & \text{for } x\in\Omega, \end{cases}$$

and we claim that up to null sets, $E_t=\{u\geq t\}$, thus $\{u\geq t\}$ is an s-minimal set. This, from Lemma 2.1 gives that u is an s-minimal function.

We notice that $\{u\geq t\}\cap\mathcal{C}\Omega = \{g\geq t\}\cap\mathcal{C}\Omega=\tilde{E}_t$. Moreover, by the definition of u in Ω, $\overline{E}_t\cap\Omega\subset\{u\geq t\}\cap\Omega$. Thus

$$E_t\triangle\{u\geq t\}\subset\left(\{u\geq t\}\setminus\overline{E_t}\right)\cap\Omega.$$

Let now p be a point belonging to the right-hand side set $\left(\{u\geq t\}\setminus\overline{E_t}\right)\cap\Omega$. First we notice that since E_t is s-minimal, then it follows from [6, Corollary 4.4 (i)] that $|\partial E_t\cap\Omega|=0$. Then, if $p\in\left(\{u\geq t\}\setminus E_t\right)\cap\Omega$ we have that $u(p)\geq t$ and that $u(p)=\sup\{\theta\,|\,p\in E_\theta\}$. Further on, for all $\theta\geq t$, $E_\theta\subset E_t$, hence if $p\notin E_t$ then surely $p\notin E_\theta$. Then if we want θ such that $p\in E_\theta$ we need $\theta<t$, and taking the supremum for all θ such that $p\in E_\theta$, we obtain that $u(p)\leq t$. This implies that $u(p)=t$ and

$$\{u\geq t\}\triangle E_t\subset\{u=t\}\cap\Omega.$$

But we are able to prove that $u\in L^1(\Omega)$, and this leads to

$$|E_t\triangle\{u\geq t\}|\leq|\{u=t\}|=0.$$

We point out that using the condition on the exterior data g, we get that

$$\|u\|_{W^{s,1}(\Omega)}\leq C\|\operatorname{Tail}_s(f,\Omega_{\Theta\operatorname{diam}(\Omega)}\setminus\Omega;\cdot)\|_{L^1(\Omega)}$$

for some positive constant C, hence $u\in W^{s,1}(\Omega)$. □

3. Asymptotics as $p \to 1$ of the (s,p)-problem

In the classical case, in [11] the author proves that if $\partial\Omega$ is smooth with positive mean curvature and $g \in C(\partial\Omega)$, then the sequence of unique p-minimizers $u_p \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $u_p = g$ on $\partial\Omega$ converges uniformly, as $p \to 1$, to a function $u \in BV(\Omega) \cap C(\overline{\Omega})$, the unique function of least gradient with boundary data g.

In the fractional case, we consider

$$1 < p < \frac{1}{s}$$

and with an abuse of notation, the fractional Sobolev space

$$W^{s,p}(\Omega) = \{u \colon \mathbb{R}^n \to \mathbb{R} \mid u|_\Omega \in W^{s,p}(\Omega)\}.$$

We denote the $W^{s,p}$-seminorm by

$$[u]_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dxdy,$$

and for $g \colon \mathcal{C}\Omega \to \mathbb{R}$, we define the space

$$W^{s,p}_g(\Omega) = \{u \in W^{s,p}(\Omega) \mid u = g \text{ on } \mathcal{C}\Omega\}.$$

The (s,p)-fractional energy is

$$\mathcal{E}^p_s(u,\Omega) = \frac{1}{2p} \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dx\, dy.$$

We say that $u \colon \mathbb{R}^n \to \mathbb{R}$, $u \in W^{s,p}(\Omega)$ is an (s,p)-minimizer if $\mathcal{E}^p_s(u,\Omega) < \infty$ and

$$\mathcal{E}^p_s(u,\Omega) \le \mathcal{E}^p_s(v,\Omega) \quad \text{for all } \ v \in W^{s,p}(\Omega) \ \text{ such that } \ v = u \text{ in } \mathcal{C}\Omega.$$

We discuss now the assumption we take on exterior data $g \colon \mathcal{C}\Omega \to \mathbb{R}$. Again, as already mentioned in the case $p = 1$ (see the computations in (4)), a fractional Hardy inequality still holds in the regime of $sp < 1$, and the condition $\mathcal{E}^p_s(u,\Omega) < \infty$ can be replaced by $u \in W^{s,p}(\Omega)$ and some bound on the nonlocal contribution $\Omega \times \mathcal{C}\Omega$. Precisely, defining the global nonlocal (s,p)-tail of $g \colon \mathcal{C}\Omega \to \mathbb{R}$, for all $x \in \Omega$, as

$$\mathrm{Tail}^p_s(g,x) = \int_{\mathcal{C}\Omega} \frac{|g(y)|^p}{|x-y|^{n+sp}} \, dxdy,$$

we remark that $\mathcal{E}_s^p(u,\Omega) < \infty$ implies that $u \in W^{s,p}(\Omega)$, and that $u \in W^{s,p}(\Omega)$ together with $\mathrm{Tail}_s^p(u,\cdot) \in L^1(\Omega)$ gives that $\mathcal{E}_s^p(u,\Omega) < \infty$. Notice also that according to our Proposition 2.1, if $\mathrm{Tail}_s(u,\mathcal{C}\Omega,\cdot) = \mathrm{Tail}_s^1(u,\cdot) \in L^1(\Omega)$, we know that an $(s,1)$-minimizers exists (and that it is also an s-minimal function). The goal is to prove the convergence of (s,p)-minimizers to an $(s,1)$-minimizer.

To obtain such a convergence result, we ask that there exists $q \in (1, 1/s)$ such that

$$\sup_{p\in(1,q)} \mathrm{Tail}_s^p(g,\cdot) \in L^1(\Omega). \tag{8}$$

See [3, Theorem 1.6] for a more general statement (there, we let the exterior data g vary with p) and for the complete proof.

Theorem 3.1. *Let $g\colon \mathcal{C}\Omega \to \mathbb{R}$ be such that there exists $q \in (1, 1/s)$ for which (8) holds. Let $\{u_p\} \in W_g^{s,p}(\Omega)$ be a sequence of (s,p)-minimizers. Then, there exists $u_1 \in W_g^{s,1}(\Omega)$ such that, up to a subsequence,*

$$u_p \longrightarrow u_1 \quad \text{in } L^1(\Omega) \text{ and a.e. in } \mathbb{R}^n.$$

Furthermore, u_1 is an $(s,1)$-minimizer.

We remark that there exists $q \in (1, 1/s)$ such that (8) holds if and only if

$$\| \sup_{p\in(1,q)} \mathrm{Tail}_s^p(g,\cdot)\|_{L^1(\Omega)} < \infty$$

and if and only if

$$\mathrm{Tail}_s^1(g,\cdot), \mathrm{Tail}_s^q(g,\cdot) \in L^1(\Omega).$$

Notice that $g \in W^{s,q}(\mathcal{C}\Omega)$ implies

$$\| \sup_{p\in(1,q)} \mathrm{Tail}_s^p(g,\cdot)\|_{L^1(\Omega)} < \infty.$$

Sketch of the proof of Theorem 3.1. We notice that, by comparing with a competitor that vanishes in Ω, from (8) we get that the (s,p)-energy is uniformly bounded, hence

$$\sup_{p\in(1,q)} \|u_p\|_{W^{s,p}(\Omega)} < \infty.$$

We use the continuous embedding $W^{s,p}(\Omega) \hookrightarrow W^{\sigma,1}(\Omega)$, which holds for some $\sigma \in (0,s)$ and for every $p \in [1,\infty)$ (with constant independent of p). It follows that

$$\sup_{p\in(1,q)} \|u_p\|_{W^{\sigma,1}(\Omega)} < \infty.$$

By compactness and Fatou's lemma, there exists $u_1 \in W^{s,1}_g(\Omega)$ such that $u_p \to u_1$ in $L^1(\Omega)$ norm, as $p \to 1$. We let $v \in W^{s,1}_g(\Omega)$ be any competitor for u_1. We claim that

$$\mathcal{E}^1_s(u_1,\Omega) \le \mathcal{E}^1_s(v,\Omega),$$

and by density of $C^\infty_c(\Omega)$ in $W^{s,1}(\Omega)$, we reduce the claim to proving that

$$\mathcal{E}^1_s(u_1,\Omega) \le \mathcal{E}^1_s(\psi,\Omega),$$

for all $\psi \in C^\infty_c(\Omega)$, with $\psi = \varphi$ in $\mathcal{C}\Omega$. We obtain this by showing the line of inequalities

$$\mathcal{E}^1_s(u_1,\Omega) \le \liminf_{p\to 1} \mathcal{E}^p_s(u_p,\Omega) \le \liminf_{p\to 1} \mathcal{E}^p_s(\psi,\Omega) = \mathcal{E}^1_s(\psi,\Omega),$$

which follows using, in order, Fatou's lemma, that u_p are (s,p)-minimizers, and carefully employing the dominated convergence theorem. $\square$

We further obtain the Gamma-convergence of the $\mathcal{E}^p_s$ energy to the $\mathcal{E}^1_s$ energy in the following setting.

Let for any $q \in [1,\infty)$,

$$\mathcal{X}^q(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \;\middle|\; \iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^q}{|x-y|^{n+sq}} dx\, dy < \infty \right\}.$$

We introduce the (extended) functional on the space $L^1_{\mathrm{loc}}(\mathbb{R}^n)$

$$\tilde{\mathcal{E}}^q_s(u,\Omega) := \begin{cases} \dfrac{1}{2q} \displaystyle\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^q}{|x-y|^{n+sq}} dx\, dy & \text{if } \ u \in \mathcal{X}^q(\Omega), \\ +\infty & \text{if } \ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \setminus \mathcal{X}^q(\Omega). \end{cases}$$

We then have the following result.

Theorem 3.2. *It holds that*

$$\Gamma\text{-}\lim_{p\searrow 1} \tilde{\mathcal{E}}^p_s = \tilde{\mathcal{E}}^1_s,$$

in the $L^1_{\mathrm{loc}}(\mathbb{R}^n)$*-topology.*

The reader can check Section 4 of [3] for the proof and other Gamma-convergence results.

4. Existence of a weak solution and equivalence with minimizers

The main result of this section is the equivalence between minimizers and weak solutions. We will first introduce the Euler-Lagrange equation – thus the fractional 1-Laplacian – and the definition of weak solution.

We begin by discussing some results from the classical theory on the 1-Laplacian. For starters, we recall that for $p > 1$, the p-Laplacian is defined as

$$(-\Delta)_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

We say that the function $u \in W^{1,p}_g(\Omega)$ is p-harmonic, i.e. that it solves in a weak sense $(-\Delta)_p u = 0$ in Ω, if and only if it is a p-minimizer for the $W^{1,p}$ energy. Formally then, the 1-Laplacian is

$$(-\Delta)_1 u(x) = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right),$$

whenever $\nabla u \neq 0$. The rigorous definition in [13, Section 2] is the following.

Definition 4.1. *Let $g\colon \partial\Omega \to \mathbb{R}$ be given. The function $u \in BV(\Omega)$ is said to be 1-harmonic with the given boundary data g, if there exists the vector field $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^n)$ with $\|\mathbf{z}\|_\infty \leq 1$ and*

$$\begin{cases} -\operatorname{div}\mathbf{z} = 0 \text{ in the sense of distributions,} \\ (\mathbf{z}, Du) = |Du| \text{ as (Radon) measures,} \\ [\mathbf{z}, \nu] \in sgn(g-u)\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega,\ \nu \text{ the normal exterior to } \partial\Omega. \end{cases}$$

We recall that $\operatorname{sgn}(x)$ denotes a generalized sign function, satisfying $\operatorname{sgn}(x)\, x = |x|$ for $x \neq 0$ and $\operatorname{sgn}(0) = [-1, 1]$.
To obtain such an equation, heuristically, one would like to take the first variation of the relaxed functional

$$\mathcal{F}(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u-g| d\mathcal{H}^{n-1}.$$

If u were smooth with a non-vanishing gradient, for any test function φ, one would get

$$\begin{aligned} 0 &= \int_\Omega \frac{Du}{|Du|} \cdot \nabla\varphi + \int_{\partial\Omega} \operatorname{sgn}(u-g)\varphi \\ &= -\int_\Omega \varphi \operatorname{div}\left(\frac{Du}{|Du|}\right) + \int_{\partial\Omega} \left(\frac{Du}{|Du|} \cdot \nu + \operatorname{sgn}(u-g)\right)\varphi. \end{aligned}$$

Here, the vector field $\mathbf{z}$ would take the place of $Du/|Du|$. To make the definition rigorous, one uses the trace theory of BV-functions (see [1]) and the generalized Gauss-Green theorem for BV functions. Note that the pairing $(\mathbf{z}, Du)$ is a Radon measure with finite total variation, whereas $[\mathbf{z}, \gamma]$ represents the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$.

We reiterate some differences with respect to [21, 11]. In [13], the authors work with a $L^1(\partial\Omega)$ boundary data and avoid any geometric requirements on $\partial\Omega$ (except for it being Lipschitz). We recall that for an L^1 boundary data, a least gradient function may not exist, however, a weak solution does, in the sense of Definition 4.1. If further the boundary data is assumed in the sense of traces, then the weak solution is also a least gradient function, see [13].

The Euler-Lagrange equation for functions of least $W^{s,1}$-seminorm was defined in [14]. Therein, among other things, the authors introduce the fractional 1-Laplacian and the definition of weak solution, while dealing with the problem

$$\begin{cases} (-\Delta)^s_1 u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega, \end{cases}$$

for $f \in L^2(\Omega)$. We adapt their definition to our Dirichlet problem with exterior data as follows.

Definition 4.2. *Let $g\colon \mathcal{C}\Omega \to \mathbb{R}$ be given. We say that $u \in W^{s,1}_g(\Omega)$ is a weak solution of the $(s,1)$-problem*

$$\begin{cases} (-\Delta)^s_1 u = 0 & \textit{in } \Omega \\ u = g & \textit{in } \mathcal{C}\Omega \end{cases}$$

if there exists

$$\mathbf{z} \in L^\infty(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2), \qquad \|\mathbf{z}\|_\infty \le 1, \qquad \mathbf{z}(x,y) = -\mathbf{z}(y,x),$$

$$\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{\mathbf{z}(x,y)(w(x)-w(y))}{|x-y|^{n+s}}dx\,dy = 0 \qquad \textit{for all } w \in W_0^{s,1}(\Omega)$$

and

$$\mathbf{z}(x,y) \in sgn(u(x)-u(y)) \quad \textit{for almost all } (x,y) \in \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2.$$

As in [13], to prove existence of a weak solution, we start with a weak solution of the fractional p-Laplacian, and by sending $p \to 1$, we obtain our desiderata. We recall that the fractional p-Laplacian is defined as

$$(-\Delta)_p^s u(x) := P.V. \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}dy.$$

Let $g\colon \mathcal{C}\Omega \to \mathbb{R}$ be such that $\mathrm{Tail}_s^p(g,\cdot) \in L^1(\Omega)$.
Then $u \in W_g^{s,p}(\Omega)$ is a weak solution to the (s,p)-problem $(-\Delta)_p^s u = 0$ in Ω if

$$\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+sp}}dx\,dy = 0,$$

for every $w \in W_0^{s,p}(\Omega)$. We point out that a weak solution is unique and that it is also the minimizer of the (s,p)-energy.

Since we will adopt the proof of [13, 14], we will use the asymptotics as $p \to 1$ of the weak (s,p)-solution. In order to do this however, we have to pay a price in terms of assumptions on the exterior data. We need to consider a suitable fractional parameter s_p, depending on p, and some $g\colon \mathcal{C}\Omega \to \mathbb{R}$ for which, as p approaches 1, all (s_p,p)-weak solutions exist. In [3, Theorem 1.11 (iii)] we prove the following result.

Theorem 4.1. *Let $q \in (1, c_{n,s})$, with $c_{n,s} > 0$ such that*

$$s_q := s + n - \frac{n}{q} \in (s,1) \quad \textit{and} \quad s_q q < 1.$$

Let $g\colon \mathcal{C}\Omega \to \mathbb{R}$ be such that

$$\mathrm{Tail}_s^1(g,\cdot),\ \mathrm{Tail}_{s_q}^q(g,\cdot) \in L^1(\Omega).$$

Then, there exists a weak solution $u \in W_g^{s,1}(\Omega)$ to the $(s,1)$-problem.

Notice that

$$(n+s)p = n + s_p p,$$

and let $u_p \in W^{s_p,p}(\Omega)$ be weak solutions of the (s_p, p)-problem.
The core of the proof is to show that there exists $\mathbf{z} \in L^\infty(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2)$ such that, roughly speaking,

$$\frac{|u_p(x) - u_p(y)|^{p-2}(u_p(x) - u_p(y))}{|x-y|^{(n+s)(p-1)}} \frac{1}{|x-y|^{n+s}} \rightharpoonup \frac{\mathbf{z}(x,y)}{|x-y|^{n+s}}$$
$$\text{weakly}^* \text{ in } L^\infty(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2), \text{ as } p \to 1.$$

Recall that

$$\mathrm{Tail}^p_{s_p}(g, \cdot) \in L^1(\Omega) \qquad \Longrightarrow \qquad \mathcal{E}^p_{s_p}(u, \Omega) < \infty.$$

We also point out that the two conditions on g are equivalent to

$$\sup_{p \in (1,q]} \mathrm{Tail}^q_{s_p}(g, \cdot) \in L^1(\Omega),$$

so the assumption on g guarantees that all the (s_p, p)-minimizers/weak solutions u_p exist at all steps, as $p \to 1$. All details of the proof can be consulted in [3].

The equivalence between minimizers and weak solutions is given in the following result.

Theorem 4.2. *Let $u \in W^{s,1}(\Omega)$. The following holds:*

(i) *If u is a weak solution to the $(s,1)$-problem, then u is an $(s,1)$-minimizer for the energy $\mathcal{E}^1_s(u, \Omega)$.*

(ii) *Assume that there exists a weak solution $\overline{u} \in W^{s,1}(\Omega)$ of the $(s,1)$-problem. Then any $(s,1)$-minimizer u in Ω such that $\overline{u} = u$ almost everywhere in $\mathcal{C}\Omega$, is a weak solution of the $(s,1)$-problem.*

Sketch of the proof. (i) If u is a weak solution to the $(s,1)$-problem, let $v \in W^{s,1}(\Omega)$ be a competitor and use $w := v - u \in W^{s,1}_0(\Omega)$ as a test function.
(ii) Assume that there exists a weak solution $\overline{u} \in W^{s,1}(\Omega)$ of the $(s,1)$-problem. We use $w = u - \overline{u} \in W^{s,1}_0(\Omega)$ as a test function.
The claims easily follow. □

5. A comment on continuity and uniqueness

We recall that the results obtained in the classical case in [21, 20, 11] establish, in presence of a continuous boundary data and with some geometric assumptions on the domain Ω, that a least gradient function is continuous up to the boundary and unique.

We underline moreover that the lack of continuity of the boundary data can result in lack of uniqueness. There are examples, see [13, Example 2.7], where uniqueness is not expected if the boundary data $g \in L^1(\partial\Omega) \setminus C(\partial\Omega)$.

The regularity of s-minimal functions is still an open problem. In the classical case, to obtain the continuity in [21], the authors use a comparison principle for minimal surfaces, see [21, Theorem 2.2]. Such a result is not available in the fractional setting. This is an important impediment to obtain a similar result in the fractional case. In [11], the uniform continuity of the p-minimizers follows from a comparison principle and the construction of some barriers, instruments that do not appear to translate without effort to the fractional setting.

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