Journal of Mathematical Physics

RESEARCH ARTICLE | APRIL 24 2014

Pseudo-Z symmetric space-times

Carlo Alberto Mantica; Young Jin Suh

J. Math. Phys. 55, 042502 (2014)
https://doi.org/10.1063/1.4871442

JOURNAL OF MATHEMATICAL PHYSICS 55, 042502 (2014)

Pseudo-Z symmetric space-times

Carlo Alberto Mantica[1,a)] and Young Jin Suh[2,b)]
[1]Physics Department, Università degli Studi di Milano, Via Celoria 16, 20133 Milano, Italy
[2]Department of Mathematics, Kyungpook National University, Taegu 702-701, South Korea

(Received 13 December 2012; accepted 3 April 2014; published online 24 April 2014)

In this paper, we investigate Pseudo-Z symmetric space-time manifolds. First, we deal with elementary properties showing that the associated form A_k is closed: in the case the Ricci tensor results to be Weyl compatible. This notion was recently introduced by one of the present authors. The consequences of the Weyl compatibility on the magnetic part of the Weyl tensor are pointed out. This determines the Petrov types of such space times. Finally, we investigate some interesting properties of $(PZS)_4$ space-time; in particular, we take into consideration perfect fluid and scalar field space-time, and interesting properties are pointed out, including the Petrov classification. In the case of scalar field space-time, it is shown that the scalar field satisfies a generalized *eikonal* equation. Further, it is shown that the integral curves of the gradient field are geodesics. A classical method to find a general integral is presented. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4871442]

I. INTRODUCTION

Recently, the present authors[27] (see also Ref. 25) defined a generalized (0,2) symmetric Z tensor given by

$$Z_{kl} = R_{kl} + \varphi g_{kl}, \tag{1.1}$$

where φ is an arbitrary scalar function. In Refs. 27 and 25, various properties of the Z tensor were pointed out; it was used to introduce the new differential structures of *Pseudo-Z symmetric* and *weakly Z symmetric* Riemannian manifolds. The first one is defined by the condition[27]

$$\nabla_k Z_{jl} = 2A_k Z_{jl} + A_j Z_{kl} + A_l Z_{jk}. \tag{1.2}$$

If $\varphi = 0$, we recover a Pseudo-Ricci symmetric manifold introduced by Chaki.[5] This notion of Pseudo-Ricci symmetric is different from that of Deszcz.[10] The fundamental properties of such manifolds were investigated in Ref. 27. The second is defined by the condition[25]

$$\nabla_k Z_{jl} = A_k Z_{jl} + B_j Z_{kl} + D_l Z_{jk}. \tag{1.3}$$

A complete study of (1.3) was pursued in Ref. 25. Finally, in Ref. 28 manifolds on which a Z form is recurrent were studied. This embraces both *Pseudo-Z symmetric* and *weakly Z symmetric* Riemannian manifolds.

From the results in Refs. 27 and 25, the Z tensor may be used to write the Einstein field equations of general relativity (Refs. 8, 20, and 36). In fact, the equation $Z_{kl} = kT_{kl}$ being $k = \frac{8\pi G}{c^4}$ the Einstein gravitational constant (see Ref. 8) and the condition $\nabla^l Z_{kl} = 0$ coming from the stress energy tensor give $\nabla_k(\frac{R}{2} + \varphi) = 0$ that is $\varphi = -\frac{R}{2} + \Lambda$. The term Λ is thus the cosmological constant and Einstein's equations take the form $R_{kl} - \frac{R}{2} g_{kl} + \Lambda g_{kl} = kT_{kl}$. Here, we have defined the Ricci tensor to be $R_{kl} = -R^m_{\ mkl}$[37] and the scalar curvature $R = g^{ij} R_{ij}$.

a)E-mail: carloalberto.mantica@libero.it
b)E-mail: yjsuh@knu.ac.kr

In the present paper, we investigate the fundamental properties of $(PZS)_4$ space-times. In Sec. II, we deal with elementary properties showing that the associated form A_k is closed. In Sec. III, we will show that in the case of closed associated covector, the Ricci tensor results to be *Weyl compatible*. This notion was recently introduced by one of the present authors in Refs. 24 and 26. The consequences (recently obtained in Ref. 26) of the Weyl compatibility on the electric and magnetic part of the Weyl tensor are pointed out. In Sec. IV, we investigate some interesting properties of $(PZS)_4$ space-time manifolds: Weyl compatibility ensures that the space-time is of Petrov D or I;[34] moreover, we take into consideration perfect fluid space times with cosmological constant (see Refs. 27, 34, and 36) and provide a state equation. Finally, in Sec. V a $(PZS)_4$ scalar field space-time is considered, and interesting properties are pointed out. In this case, it is shown that the scalar field satisfies a generalized *eikonal* equation.[18,19] Further, it is shown that the integral curves of the field gradient are geodesics. A classical method to find a general solution is presented.

Throughout the paper, all manifolds under consideration are assumed to be smooth connected Hausdorff manifolds endowed with a Lorentz metric[20] (i.e., a metric of signature $+2$).

II. ELEMENTARY PROPERTIES OF $(PZS)_4$ SPACE-TIMES

In Ref. 27, elementary properties of a $(PZS)_4$ space-times are shown. In this section, we collect them for successive use. Transvecting Eq. (1.2) with g^{jl} gives immediately

$$\nabla_k Z = 2A_k Z + 2A^l Z_{kl}, \tag{2.1}$$

where $Z = g^{il} Z_{jl}$ and $A^l = g^{jl} A_j$.

In the same manner, transvecting Eq. (1.2) with g^{kl} one obtains

$$\nabla^l Z_{kl} = A_k Z + 3A^l Z_{kl}, \tag{2.2}$$

where $\nabla^l Z_{kl} = g^{lj}\nabla_j Z_{kl}$. Because of the condition $\nabla^l Z_{kl} = 0$ coming from the stress energy tensor and using Eqs. (2.1) and (2.2), we get

$$\begin{aligned} A^l Z_{kl} &= -\frac{A_k}{3} Z, \\ \nabla_k Z &= \frac{4}{3} A_k Z. \end{aligned} \tag{2.3}$$

We can thus state the following.

Theorem 2.1. (Ref. 27) *For a $(PZS)_4$ space-time manifold A_k is a closed one form and it is an eigenvector of the Z tensor with eigenvalue $-\frac{Z}{3}$.*

III. WEYL COMPATIBLE $(PZS)_4$ SPACE-TIMES

In this section, we consider four-dimensional Pseudo-Z symmetric manifold with closed associated covector. We will show that this condition implies that the Ricci tensor is *Weyl compatible*. This notion was recently introduced in Refs. 24 and 26. As a consequence, strong restrictions on the structure of the Weyl tensor are imposed, with geometric and topological implications (see Refs. 24 and 26).

From Eq. (1.2), we infer

$$\nabla_k Z_{jl} - \nabla_j Z_{kl} = A_k Z_{jl} - A_j Z_{kl}. \tag{3.1}$$

Now from (6.3) in Ref. 27, we introduce the following:

$$\begin{aligned} (\nabla_l \nabla_k - \nabla_k \nabla_l) Z_{ij} = \; & 2(\nabla_l A_k - \nabla_k A_l) Z_{ij} \\ & + (\nabla_l A_i - A_i A_l) - (\nabla_k A_i - A_i A_k) Z_{jl} \\ & + (\nabla_l A_j - A_j A_l) Z_{ik} - (\nabla_k A_j - A_j A_k) Z_{il}, \end{aligned} \tag{3.2}$$

which by the Ricci identity turns into

$$-g^{rs}(Z_{rj}R_{sikl} + Z_{ir}R_{sjkl}) = 2(\nabla_l A_k - \nabla_k A_l)Z_{ij}$$
$$+ (\nabla_l A_i - A_i A_l) - (\nabla_k A_i - A_i A_k)Z_{jl}$$
$$+ (\nabla_l A_j - A_j A_l)Z_{ik} - (\nabla_k A_j - A_j A_k)Z_{il}.$$

Applying this to the condition $\nabla_l A_k = \nabla_k A_l$ and summing cyclically the resulting equation in the indices j, k, l, we get

$$R_{ri}R^r_{jkl} + R_{rj}R^r_{kil} + R_{rk}R^r_{ijl} = 0, \tag{3.3}$$

where R^l_{ijk} denotes the component of the curvature tensor R defined by

$$R_{ijkm} = g(R(\partial_i, \partial_j)\partial_k, \partial_m) = g_{lm}R^l_{ijk}.$$

Here, we recall that the component of (0, 4) type curvature tensor R_{ijkl} satisfies the following properties:

$$R_{jklm} + R_{kljm} + R_{ljkm} = 0, \quad R_{jklm} = -R_{kjlm} = -R_{jkml}, \quad R_{jklm} = R_{lmkj}.$$

Any semi-Riemannian manifold satisfying (3.3) is called a *Riemann compatible* manifold.[24,26] Thus, we have

Theorem 3.1. *Every space-time* $(PZS)_4$ *is a Riemann compatible manifold.*

Remark. We mention that there are results on manifolds satisfying (3.3) published earlier than Refs. 23 and 24 and not cited in those papers, see, e.g., Lemma 3.3 in Ref. 1 and Ref. 13, and Proposition 3.1 (iv) in Ref. 12.

Geometric and topological consequences of this condition were extensively studied in Ref. 26. If we insert in the previous relation the local form of the Weyl tensor[31] is defined by

$$C^m_{jkl} = R^m_{jkl} + \frac{1}{n-2}(\delta^m_j R_{kl} - \delta^m_k R_{jl} + R^m_j g_{kl} - R^m_k g_{jl})$$
$$- \frac{R}{(n-1)(n-2)}(\delta^m_j g_{kl} - \delta^m_k g_{jl}),$$

where the component C^l_{ijk} of type (1, 3) of the Weyl tensor C is defined by

$$C_{ijkm} = g(C(e_i, e_j)e_k, e_m) = g_{lm}C^l_{ijk}.$$

Of course, the component of (0, 4) type Weyl tensor C_{ijkl} satisfies the following properties:

$$C_{jklm} + C_{kljm} + C_{ljkm} = 0, \quad C_{jklm} = -C_{kjlm} = -C_{jkml}, \quad C_{jklm} = C_{lmkj}.$$

Any semi-Riemannian manifold satisfying

$$R_{ri}C^r_{jkl} + R_{rj}C^r_{kil} + R_{rk}C^r_{ijl} = 0 \tag{3.4}$$

is called *Weyl-compatible*. In recent works, Weyl compatibility has been extensively investigated in the Riemannian case.[26] It is known that both conditions (3.3) and (3.4) are equivalent. If we use Einstein's equations in (3.4), we get

$$T_{rj}C^r_{ikl} + T_{rk}C^r_{ilj} + T_{rl}C^r_{ijk} = 0. \tag{3.5}$$

From the above discussion, we may state the following:

Theorem 3.2. *Let M be a* $(PZS)_4$ *space-time manifold: then the relation (3.5) is fulfilled for any stress-energy tensor.*

In General Relativity, it is customary to define the *electric* and *magnetic* part of the Weyl tensor (see Refs. 2 and 34). Precisely given a normalized velocity vector u^i (*i.e* $u_i u^i = -1$) the following (0,2) tensors are defined:

$$E_{kl} = u^j u^m C_{jklm},$$
$$H_{kl} = \frac{1}{4} u^j u^m (\varepsilon_{\alpha\beta jk} C^{\alpha\beta}_{lm} + \varepsilon_{\alpha\beta jl} C^{\alpha\beta}_{km}), \tag{3.6}$$

where the component $C^{\alpha\beta}_{lm}$ of type (2, 2) of the Weyl tensor can be defined by

$$C_{jklm} = g_{\alpha m} C^{\alpha}_{jkl} = g_{\alpha m} g_{\beta l} C^{\alpha\beta}_{jk}$$

and ε_{ijkl} denotes the completely skew-symmetric Levi-Civita symbol.[22,29] The tensor E_{kl} is named *electric part* of the Weyl tensor, while the tensor H_{kl} is named *magnetic part* of the Weyl tensor; elementary properties are found to be[2]

$$g^{kl} E_{kl} = g^{kl} H_{kl} = 0,$$
$$u^k E_{kl} = u^k H_{kl} = 0.$$

Moreover, the Weyl tensor is uniquely decomposed in its electric and magnetic parts. In Ref. 2, it was specified that Eq. (3.6) is also valid in the case $u_i u^i = +1$. A fundamental property of the magnetic part of the Weyl tensor satisfying condition (3.5) was stated in Ref. 26: we reproduce it here for completeness. We focus on stress-energy tensors of the form $T_{kl} = \alpha u_k u_l + \beta g_{kl}$ with normalized covector u_j: in this case such covector permits the decomposition of the Weyl tensor. Equation (3.5) takes the form

$$u^i u^m C^{jk}_{lm} + u^j u^m C^{ki}_{lm} + u^k u^m C^{ij}_{lm} = 0.$$

The previous equation is thus multiplied by ε_{ijkp} to get

$$\varepsilon_{ijkp} u^i u^m C^{jk}_{lm} + \varepsilon_{ijkp} u^j u^m C^{ki}_{lm} + \varepsilon_{ijkp} u^k u^m C^{ij}_{lm} = 0.$$

Recalling the skew-symmetric properties of the Levi-Civita symbol, we simply have

$$\varepsilon_{ijkp} u^i u^m C^{jk}_{lm} = \varepsilon_{kijp} u^k u^m C^{ij}_{lm} = \varepsilon_{ijkp} u^k u^m C^{ij}_{lm},$$
$$\varepsilon_{ijkp} u^j u^m C^{ki}_{lm} = \varepsilon_{jkip} u^j u^m C^{ki}_{lm} = \varepsilon_{ijkp} u^k u^m C^{ij}_{lm}.$$

Thus, we infer that $3\varepsilon_{ijkp} u^k u^m C^{ij}_{lm} = 0$ and so the magnetic part of the Weyl tensor vanishes.

Theorem 3.3. (Ref. 26) *Let M be any space-time manifold having a Weyl compatible stress energy tensor of the form $T_{kl} = \alpha u_k u_l + \beta g_{kl}$: then the magnetic part of the Weyl tensor vanishes.*

Space-times in which $H_{ki} = 0$ are named *purely electric* space-times, while the condition $E_{kl} = 0$ defines *purely magnetic* space-times.[34] In general relativity, a deep comprehension of gravitational fields structure is obtained from the algebraic classification of the Weyl tensor in terms of its eigenvalues and eigenvectors (see Refs. 30 and 34). This is known as the *Petrov classification*.[30] It turns out that the eigenvalues of the Weyl tensor satisfy a fourth order equation. The eigenvalue multiplicity classifies five different types of space-times. Thus, for Petrov type I space time the quartic roots are all distinct, for type II one double root is present, for type D there are two double roots, for type III one triple root is found, and finally for type N there is a fourfold root. The completely degenerate case of conformally flat space-time forms the sixth type (named O). It is well known that purely electric space-times are of Petrov type I, D, or O (conformally flat).[34] We have thus:

Theorem 3.4. *Let M be a non-conformally flat Pseudo-Z symmetric space-time with stress energy tensor of the form $T_{kl} = \alpha u_k u_l + \beta g_{kl}$: then $H_{kl} = 0$ and the Petrov types are I or D.*

Let us now consider a $(PZS)_4$ space-time with an associated covector A of the concircular form

$$\nabla_j A_i = \gamma g_{jl} + \delta A_j A_i, \tag{3.7}$$

where γ and δ are constant. As, for example, satisfying (3.7), we introduce that in the class of four-dimensional warped product manifolds with one-dimensional base it will be possible to find suitable examples of non-conformally $(PZS)_4$ space-times with an associated covector A satisfying (3.7). Evidently, the fibre of investigated warped products is a three-dimensional manifold. Note that the metric of such fibre can be presented in a diagonal form (see Ref. 21).

The integrability conditions of the previous equation (3.7) read

$$A_m R^m_{jkl} = \gamma\delta(A_j g_{kl} - A_k g_{jl}). \tag{3.8}$$

After a straightforward calculation we infer easily

$$A_i A_m R^m_{jkl} + A_j A_m R^m_{kil} + A_k A_m R^m_{ijl} = 0.$$

Thus, the tensor $b_{kl} = A_k A_l$ turns out to be Riemann compatible.[23] Consequently, it is also Weyl compatible as it satisfies

$$A_i A_m C^m_{jkl} + A_j A_m C^m_{kil} + A_k A_m C^m_{ijl} = 0. \tag{3.9}$$

If $A_i A^i = -1$, this again allows the decomposition of the Weyl tensor into an electric and magnetic part. Thus, again it is inferred that $\epsilon_{ijkp} A^k A^m C^{ij}_{lm} = 0$ and the magnetic part H_{kl} vanishes. From the above discussion, the following theorem may be stated as follows:

Theorem 3.5. *Let M be a non-conformally flat Pseudo-Z symmetric space-time with associated time-like covector A, $A_i A^i = -1$, of the form $\nabla_j A_l = \gamma g_{jl} + \delta A_j A_l$ being γ, δ constants, then $H_{kl} = 0$ and the Petrov types are I or D.*

In the sequel, we present an example of $(PZS)_4$ space-time as follows:

Example 3.6. Let us define a semi-Riemannian metric on the four-dimensional vector space as follows:

$$ds^2 = g_{kl} dx^k dx^l = f(dx^1)^2 + 4dx^1 dx^2 + (dx^3)^2 - (kx^1)^2 (dx^4)^2,$$

where $f = \alpha_0 + \alpha_1 x^3 + \alpha_2 (x_3)^2$, $\alpha_0, \alpha_1, \alpha_2$ are non-constant scalar functions of x^1 only and k is a non-null arbitrary constant. So the metric tensor may be written in the form

$$(g_{ij}) = \begin{pmatrix} f & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(kx^1)^2 \end{pmatrix}.$$

Then the inverse matrix of $g = (g_{ji})$ can be given as follows:

$$(g^{ij}) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{f}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{(kx^1)^2} \end{pmatrix}.$$

We also note that the metric g considered in this example if defined on an non-empty open (and connected) subset U of $\mathbb{R}^4$ such that at every point of U we have $x^1 > 0$ or at every point of U we have $x^1 < 0$. Then the Christoffel symbols may be calculated with the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

Then it can be given as follows:

$$\Gamma^2_{11} = \frac{1}{4}\nabla_1 f, \quad \Gamma^2_{13} = \frac{1}{4}\nabla_3 f, \quad \Gamma^3_{11} = -\frac{1}{2}\nabla_3 f, \quad \Gamma^4_{14} = \frac{1}{x^1}, \quad \Gamma^2_{44} = \frac{k^2}{2} x^1.$$

Now let us verify that

$$R_{1331} = \frac{1}{2}\nabla_3\nabla_3 f, \quad R_{11} = \frac{1}{2}\nabla_3\nabla_3 f = \alpha_2 \neq 0, \quad \nabla_1 R_{11} = \nabla_1 \alpha_2 \neq 0.$$

But $R_{kl} = -R^m_{mkl} = -R^1_{1kl} - R^2_{2kl} - R^3_{3kl} - R^4_{4kl}$ so we evaluate

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}.$$

Then by using the expressions of the Christoffel symbol Γ^k_{ij} and the function f given above, explicitly we calculate the component of the curvature tensor R as follows:

$$R^2_{211} = 0, \quad R^3_{311} = \frac{\partial \Gamma^3_{11}}{\partial x^3} = -\frac{1}{2}\nabla_3\nabla_3 f,$$

and

$$R^4_{411} = -\frac{\partial \Gamma^4_{41}}{\partial x^1} - \Gamma^4_{14}\Gamma^4_{14} = \frac{1}{(x^1)^2} - \frac{1}{(x^1)^2} = 0.$$

We have shown that $R_{11} = \frac{1}{2}\nabla_3\nabla_3 f = \alpha_2 \neq 0$ and thus that $\nabla_1 R_{11} = \nabla_1 \alpha_2 \neq 0$. It is easily found that the scalar curvature of this metric is zero. In fact, $R = g^{11}R_{11} = 0$. Thus, since we have $\varphi = -\frac{R}{2} + \Lambda$ we may set $\varphi = 0$ choosing the cosmological constant $\Lambda = 0$. From the definitions (1.1) of the Z tensor, (1.2) of Pseudo-Z symmetric manifold, and considering the previous relations concerning the Ricci tensor and its covariant derivatives we have to satisfy

$$\nabla_1 R_{11} = 4A_1 R_{11}$$

for some one-form A_1. We get $\nabla_1\alpha_2 = 4A_1\alpha_2$, from which it follows that

$$A_1 = \frac{1}{4}\nabla_1 \log|\alpha_2|.$$

The covector A is thus a closed one form. Moreover, from the definition of conformal curvature tensor we have $C_{1331} = R_{1331} + \frac{g_{33}}{2} R_{11} = \frac{3}{2}\alpha_2 \neq 0$ and the Pseudo-Z-symmetric manifold is not conformally flat. The electric component of the Weyl tensor is thus $E_{33} = C_{1331}u^l u^l \neq 0$ and the magnetic component vanishes. Thus, we have an example on non-conformally flat Pseudo-Z symmetric space-time. Since the Ricci tensor is of rank one, it is a quasi-Einstein manifold. We recall that a pseudo-Riemannian manifold (M, g), $n \geq 3$, is said to be a quasi-Einstein manifold if at every point of M we have $\mathrm{rank}(S - \alpha g) \leq 1$, for some $\alpha \in \mathbb{R}$.

Moreover, since $R_{11} = kT_{11} = k(\mu + p)u_1u_1 + kpg_{11}$, we know that $k(\mu + p) = b$ and $kp = 0$. This satisfies the assumption of Theorem III.4. So the magnetic part of the Weyl tensor identically vanishes. Also this becomes a perfect fluid space-time in Sec. IV.

Remark. As it was stated above, R_{1331} is non-zero and other local components of the curvature tensor R vanish. Now we can easily check that the tensor R satisfies

$$\omega_h R_{ijkl} + \omega_i R_{jhkl} + \omega_j R_{hikl} = 0, \tag{3.10}$$

where ω_h are the local components of the 1-form ω defined by $\omega = (\beta_1, 0, \beta_3, 0)$ and β_1 and β_3 are some non-zero smooth functions. We note that (3.10) leads to some curvature condition of pseudo-symmetry type (see Ref. 14, Theorem 1). Moreover, we can also check that the metric g is a semi-symmetric metric, that is, the condition $R \cdot R = 0$ is satisfied (see Ref. 35).

Consider the metric of example 1 of Ref. 14 (also see Ref. 33) defined on $M = R^n$, $n \geq 4$ by the formula

$$g_{rs}dx^r dx^s = A(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{3.11}$$

where $(k_{\alpha\beta})$ is a symmetric and non-singular matrix consisting of constants, A is a function independent of x^n, and $\alpha, \beta, \gamma, \delta \in \{2, 3, \ldots, n-1\}$. The only components of the Riemann and Weyl tensors not identically zero are those related to

$$R_{1\alpha\beta 1} = \frac{1}{2}\nabla_\alpha\nabla_\beta A$$

and

$$C_{1\alpha\beta 1} = \frac{1}{2}\nabla_\alpha \nabla_\beta A - \frac{1}{2(n-1)} k_{\alpha\beta} k^{\gamma\delta} \nabla_\gamma \nabla_\delta A.$$

Note that in Ref. 14 it was shown that the curvature tensor, as well as the Weyl tensor of the metric g, defined by (3.11), satisfies (3.10) for some 1-form ω. We refer to Ref. 9 for further results on manifolds satisfying (3.10).

IV. PERFECT FLUID PSEUDO-Z SYMMETRIC SPACE-TIME

In this section, we consider perfect fluid $(PZS)_4$ space-times. Perfect fluid $(PZS)_4$ were investigated in Ref. 27. They are characterized by a non-null stress-energy tensor given by the following equation (see Refs. 20, 34, and 36):

$$T_{kl} = (\mu + p) u_k u_l + p g_{kl}, \tag{4.1}$$

where μ is the energy density, p is the isotropic pressure, and u_i the fluid flow velocity with the condition $u_i u^i = -1$. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity.[20] In addition, p and μ are related by an *equation of state* governing the particular sort of perfect fluid under consideration. In general, this is an equation of the form $p = p(\mu, T)$ where T is the absolute temperature. However, we shall only be concerned with situations in which T is effectively constant so that the equation of state reduces to $p = p(\mu)$. In this case, the perfect fluid is called *isentropic*.[20] From Theorem 3.4, we immediately have

Theorem 4.1. *Let M be a non-conformally flat $(PZS)_4$ perfect fluid space-time: then $H_{kl} = 0$ and the Petrov types are I or D.*

The most important features of perfect fluid Pseudo-Z symmetric space-times were studied in Ref. 27. The following argument was stated. The condition $A^l Z_{kl} = -\frac{A_k}{3} Z$ is applied on Einstein's equation $Z_{kl} = kT_{kl}$ giving

$$k(\mu + p) u_k A^l u_l + kpA_k = -\frac{Z}{3} A_k. \tag{4.2}$$

Transvecting with u^k, we infer

$$(k\mu - \frac{Z}{3}) A_k u^k = 0. \tag{4.3}$$

If the condition $A_k u^k \neq 0$ is fulfilled, we have $k\mu = \frac{Z}{3}$. Now $Z_{kl} = kT_{kl}$ give rise to $Z = kT$ and so from $T = k(3p - \mu)$, to $Z = k(3p - \mu)$. It follows immediately that $kp = \frac{4}{9}Z$ and that $\mu = \frac{3}{4}p$: this is the equation of state of this kind of space-times. Inserting $k\mu = \frac{Z}{3}$ in Eq. (4.2) one easily obtain $k(\mu + p)u_k A^l u_l = -k(\mu + p)A_k$ and thus $A_k = -u_k A^l u_l$. Now from $\frac{R}{2} + \varphi = \Lambda$ and $Z = R + 4\varphi$ it follows that $Z = 4\Lambda - R$. Inserting these relations in Einstein's equations it follows after a straightforward calculation that

$$R_{kl} = \frac{7}{9}(4\Lambda - R) u_k u_l + \frac{1}{9}(7\Lambda - \frac{R}{2}) g_{kl}. \tag{4.4}$$

Thus, the manifold satisfying (4.4) is a quasi-Einstein manifold. For instance, in the Riemannian case they were investigated in Refs. 6 and 11; in the pseudo-Riemannian case they arose during the study of exact solutions of Einstein equations and during the investigations of quasi-umbilical hypersurfaces of pseudo-Euclidean spaces.[12,15] For example, the Robertson-Walker space-times are quasi-Einstein.[12] We refer to Refs. 7 and 16 for recent results on quasi-Einstein manifolds. Thus, the following theorem holds.

Theorem 4.2. (Ref. 27) *Let $(PZS)_4$ be a perfect fluid space-time manifold: if the condition $A_k u^k \neq 0$ is fulfilled, then the space is quasi-Einstein and the one form A_k is proportional to the fluid flow velocity.*

The fluid flow velocity is thus irrotational. This is a generalization of the result due to Ray-Guha:[32] a perfect fluid pseudo-Ricci Symmetric space-time is a quasi-Einstein manifold with each of its associated scalars equal to $\frac{R}{3}$.

If the condition $A_k u^k = 0$ is fulfilled, we have from (4.2) $(kp + \frac{Z}{3})A_k = 0$ and thus $kp = -\frac{Z}{3}$ (being $A_k \neq 0$). Again from $Z = k(3p - \mu)$ we infer $k\mu = -2Z$ and the state equation $p = \frac{\mu}{6}$. Inserting again in Einstein equations, we get

$$R_{kl} = \frac{7}{3}(R - 4\Lambda)u_k u_l + \frac{1}{6}(5R - 14\Lambda)g_{kl}. \tag{4.5}$$

Thus, the following theorem may be stated.

Theorem 4.3. *Let $(PZS)_4$ be a perfect fluid space-time manifold: if the condition $A_k u^k = 0$ is fulfilled, then the space is quasi-Einstein.*

V. SCALAR FIELD PSEUDO-Z SYMMETRIC SPACE-TIMES

In the sequel, we consider a $(PZS)_4$ scalar field space-time and study its properties. The Lagrangian (density) of a real *spin-0* field ψ is defined as[20,3,4]

$$L = -\frac{1}{2}(\nabla_k \psi)(\nabla_l \psi)g^{kl} - V(\psi). \tag{5.1}$$

In the previous expression, $V(\psi)$ is a potential that models the self-interaction between particles. The Euler-Lagrange equations are

$$\nabla^l \nabla_l \psi - \frac{dV(\psi)}{d\psi} = 0, \tag{5.2}$$

where $\Box^2 \psi = \nabla^j \nabla_j \psi = g^{jl}\nabla_j \nabla_l \psi$ is the covariant d' Alembertian operator in curved space.[20,36] In the case $V(\psi) = \frac{m^2}{2\hbar^2}\psi^2$ (m is the particle mass and $\hbar$ is the Planck constant divided by 2π), these are known as the Klein-Gordon equation.[20] The stress-energy tensor of a scalar field ψ space-time is written as[3,4,20]

$$T_{kl} = (\nabla_k \psi)(\nabla_l \psi) - \frac{1}{2}g_{kl}[(\nabla_j \psi)(\nabla^j \psi) + V(\psi)]. \tag{5.3}$$

This defines a scalar field minimally coupled with matter. Upon quantisation, this field theory describes collection of neutral particles subjected to their mutual interaction and gravitational attraction. In this case of a $(PZS)_4$ scalar field space-time, we have a differential structure that may modify the mutual interaction. Moreover, the coupling determines constrictions on the scalar field and on the potential $V(\psi)$. Scalar field space-times were investigated in Refs. 3 and 4.

If we define $u_j = \frac{\nabla_j \psi}{\sqrt{|(\nabla_l \psi)(\nabla^l \psi)|}}$ (with the condition $(\nabla_j \psi)(\nabla^j \psi) \neq 0$), we obviously have $u_j u^j = -1$ if $(\nabla_j \psi)(\nabla^j \psi) < 0$ and $u_j u^j = +1$ if $(\nabla_j \psi)(\nabla^j \psi) > 0$, and the stress energy tensor may be written in the form $T_{kl} = \alpha u_k u_l + \beta g_{kl}$ being $\alpha = |(\nabla_l \psi)(\nabla^l \psi)|$ and $\beta = -\frac{1}{2}[(\nabla_l \psi)(\nabla^l \psi) + V(\psi)]$. From Theorem 3.4, we immediately have the following.

Theorem 5.1. *Let M be a non-conformally flat $(PZS)_4$ scalar field space-time: if $(\nabla_j \psi)(\nabla^j \psi) < 0$, then $H_{kl} = 0$ and the Petrov types are I or D.*

A possible example of a non-conformally flat $(PZS)_4$ of the Petrov types *I* or *D* satisfying the condition $(\nabla_j \psi)(\nabla^j \psi) < 0$ is possibly that described in example 3.6 with u_j defined as before.

Again if the condition $A^l Z_{kl} = -\frac{A_k}{3}Z$ is applied on Einstein's equation $Z_{kl} = kT_{kl}$, one can obtain the following relation:

$$k[(\nabla_k \psi)A^l(\nabla_l \psi) + \beta A_k] = -\frac{A_k}{3}Z. \tag{5.4}$$

The previous equation is then transvected with $\nabla^k\psi$ to obtain easily

$$\left(k[(\nabla^k\psi)(\nabla_k\psi)+\beta]+\frac{Z}{3}\right)A^l(\nabla_l\psi)=0. \tag{5.5}$$

If we suppose that $A^l(\nabla_l\psi)\neq 0$, then we have

$$k[(\nabla^k\psi)(\nabla_k\psi)+\beta]=-\frac{Z}{3}. \tag{5.6}$$

Inserting back in Eq. (5.4) we obtain $A_k=\nabla_k\psi\left[\frac{A^l(\nabla_l\psi)}{(\nabla_j\phi)(\nabla^j\psi)}\right]$. We thus state the following.

Theorem 5.2. *Let $(PZS)_4$ be a scalar field space-time: if the condition $A^l(\nabla_l\psi)\neq 0$ is fulfilled, then the one form A_k is proportional to the field gradient.*

Again $Z_{kl}=kT_{kl}$ gives rise to $Z=kT$ and thus from $T=(\nabla_k\psi)(\nabla^k\psi)+4\beta$ we easily have $Z=k[(\nabla_k\psi)(\nabla^k\psi)+4\beta]$. Combining with (5.6) after straightforward calculations we infer

$$Z=-18kV(\phi),\quad k(\nabla_l\psi)(\nabla^l\psi)=-\frac{7}{9}Z. \tag{5.7}$$

We thus get $(\nabla_l\psi)(\nabla^l\psi)=14V(\psi)$. From the previous results inserting back in the Lagrangian, it is easily seen that $L=-\frac{4}{7}(\nabla_k\psi)(\nabla_l\psi)g^{kl}=-8V(\psi)$. Thus, the Euler-Lagrange equations become

$$\frac{dV(\psi)}{d\psi}=0,\quad \nabla^l\nabla_l\psi=0. \tag{5.8}$$

From the relation $\nabla_kV(\psi)=\frac{dV(\psi)}{d\psi}\nabla_k\psi$, we easily infer $\nabla_kV(\psi)=0$. Thus, the relevant conditions imposed on the scalar field may be written as

$$g^{jl}(\nabla_j\psi)(\nabla_l\psi)=K,\quad g^{jl}\nabla_j\nabla_l\psi=0, \tag{5.9}$$

being $K=14V(\psi)$ a constant. If the condition $A^l(\nabla_l\psi)=0$ is fulfilled from (5.4), we have $A_k(\beta k+\frac{Z}{3})=0$ and thus $\beta k=-\frac{Z}{3}$ (being $A_k\neq 0$). Combining with $Z=k[(\nabla_k\psi)(\nabla^k\psi)+4\beta]$ we infer

$$Z=-\frac{6}{5}kV(\psi),\quad k(\nabla_l\psi)(\nabla^l\psi)=\frac{7}{3}Z, \tag{5.10}$$

and finally $(\nabla_l\psi)(\nabla^l\psi)=-\frac{14}{5}V(\psi)$. These results give the same equations (5.9) (with a constant K'). We may state the following.

Theorem 5.3. *Let $(PZS)_4$ be a scalar field space-time. Then the field ψ satisfies Eq. (5.9).*

The first equation is a generalization of the well-known *eikonal equation* from geometrical optics (see Refs. 17–20). The *eikonal* equation arises during the study of Maxwell's equations for the vector potential Φ_k (see Ref. 36)

$$\nabla^l\nabla_l\Phi_k-R_k^l\Phi_l=-4\pi j_k$$

(being j_k the current density) in terms of waves oscillating with nearly constant amplitude, i.e., with the ansatz $\Phi_k=C_ke^{iS}$, being C_k a constant vector field and S a scalar function called *phase* of the wave. Neglecting source terms, small derivatives terms and the curvature term, it is inferred that $g^{kl}(\nabla_kS)(\nabla_lS)=0$. The second equation is simply the wave equation for the scalar field.

We focus on some consequences of the first of equations (5.9) $(\nabla^j\psi)(\nabla_j\psi)=K$. As it is well known[8,36] the vector $k_j=\nabla_j\psi$ is orthogonal to the surface of constant ψ. Moreover, it is obviously a closed form, i.e., $\nabla_nk_l=\nabla_lk_n$. We infer thus

$$k_m(\nabla_nk_l-\nabla_lk_n)+k_n(\nabla_lk_m-\nabla_mk_l)=0.$$

This last relation may be written in the form[8]

$$k_{[m}\nabla_{n]}k_l + \nabla_l k_{[m}k_{n]} = 0,$$

where $A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$ (see Ref. 8). Transvecting with k^l the previous relation takes the final form

$$(k_{[m}\nabla_{n]}k_l)k^l + k^l\nabla_l k_{[m}k_{n]} = 0. \tag{5.11}$$

Now from $k^l k_l = K$ being K constant we have simply $k_m\nabla_n(k^l k_l) = 0$ and thus $k_m(\nabla_n k_l)k^l = 0$ and finally $(k_{[m}\nabla_{n]}k_l)k^l = 0$. Considering (5.11) this implies that[8]

$$k^l\nabla_l k_{[m}k_{n]} = 0. \tag{5.12}$$

We have just proved that the integral curves of the vector k_j are geodesics (see Ref. 8, p. 86 or Ref. 36). Thus, we state the following.

Theorem 5.4. *Let $(PZS)_4$ be a scalar field space-time manifold. Then the integral curves of field $\nabla_j\psi$ are geodesics. If $K = 0$, they are null geodesics.*

Now we consider a possible solution of Eq. (5.9): they may be evaluated in a local inertial (flat) frame for which[34]

$$g_{kl} = \eta_{kl} = \mathrm{diag}(-1, 1, 1, 1).$$

We have thus

$$\begin{aligned} &(\psi_x)^2 + (\psi_y)^2 + (\psi_z)^2 - (\psi_t)^2 = K, \\ &\psi_{xx} + \psi_{yy} + \psi_{zz} - \psi_{tt} = 0, \end{aligned} \tag{5.13}$$

being $\psi_x = \frac{\partial\psi}{\partial x}$, $\psi_{xx} = \frac{\partial^2\psi}{\partial x^2}$ and so on. The first of the previous equations admits a complete integral of the form (see Refs. 18 and 36)

$$\psi_0 = \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 t, \tag{5.14}$$

where a_i are constants subjected to the conditions

$$a_4 = \sqrt{\sum_{i=1}^{3} a_i^2 - K} \quad \text{and} \quad \sum_{i=1}^{3} a_i^2 - K \geq 0.$$

Equation (5.14) obviously satisfies the second condition of (5.13) (the wave equation). We have thus obtained a particular solution of the system (5.13). In an arbitrary Lorentzian metric, the equation

$$g^{jl}(\nabla_j\psi)(\nabla_l\psi) = 0 \tag{5.15}$$

was extensively studied in the literature[18,19] and it is of great importance in General Relativity.[8,36]

Following a similar method employed in Refs. 18 and 19, we are able to produce an arbitrary solution of the generalized *eikonal* equation

$$g^{jl}(\nabla_j\psi)(\nabla_l\psi) = K \tag{5.16}$$

if a special class of solution is known. A particular solution is given by

$$\psi_0 = \psi_0(q^i a_1, \ldots, a_r), \tag{5.17}$$

where a_i are r parameters and q^i are the coordinates on a local chart. A general integral can thus be written as

$$\psi(q^i, a_1, \ldots, a_r) = \psi_0(q^i, a_1, \ldots, a_r) - h(a_1, \ldots, a_r), \tag{5.18}$$

where $h(a_1, \ldots, a_r)$ is an arbitrary function (see Refs. 18 and 19). In Ref. 18, three parameters are considered for the solution in flat space-time (see Eq. (5.14)); in Ref. 19, the parameters are the complex stereographic coordinates $\zeta, \bar{\zeta}$ on the sphere S^2. Now the field $\psi(q^i, a_1, \ldots, a_r)$ (5.18) is extremized with respect to the parameters

$$\frac{\partial \psi}{\partial a_i} = \frac{\partial \psi_0}{\partial a_i} - \frac{\partial h}{\partial a_i} = 0, \quad i = 1, 2, \ldots, r. \tag{5.19}$$

As stated in Ref. 18, if the condition

$$\left| \frac{\partial^2 \psi_0}{\partial a_i \partial a_j} \right| \neq 0$$

is fulfilled, Eq. (5.19) can be solved for $a_i = a_i(q^j)$, $i = 1, 2, \ldots, r$ and inserting back in (5.18) a solution of the generalized *eikonal* equation is obtained. In fact, being $\psi = \psi(q^i, a_1(q^j), a_2(q^j), \ldots, a_r(q^j))$ we infer

$$\nabla_j \psi = \frac{\partial \psi}{\partial q^j} + \sum_{i=1}^{r} \frac{\partial \psi}{\partial a_i} \frac{\partial a_i}{\partial q^j} = \frac{\partial \psi_0}{\partial q^j} + \sum_{i=1}^{r} \frac{\partial \psi}{\partial a_i} \nabla_j a_i = \nabla_j \psi_0. \tag{5.20}$$

In Refs. 18 and 19, the authors stated that given arbitrary Cauchy data for the *eikonal* equation, then the function $h(a_1, \ldots, a_r)$ is determined, and thus by construction, the general solution of such equation. Moreover, as pointed out in Refs. 18 and 19, the regions where Eq. (5.19) cannot be solved for $a_i = a_i(q^j)$, $i = 1, 2, \ldots, r$ are named the *caustics* of the solution.

This method allows us to find the general solution of $g^{jl}(\nabla_j \psi)(\nabla_l \psi) = K$ in a flat space-time, being ψ_0 defined in Eq. (5.14) with three independent parameters. Finally, we underline that the filed ψ_0 in general non-flat metric should solve the wave equation $g^{kl}\nabla_k \nabla_l \psi_0 = 0$ and, by construction, the general solution too.

The condition $g^{jl}(\nabla_j \psi)(\nabla_l \psi) = K$ has a further peculiar geometric picture. In Theorem 1 in Ref. 4, the scalar field space time manifold was partitioned in three components named T, S, and F. In the T region, it is $(\nabla_j \psi)(\nabla^j \psi) < 0$ (time-like), in the S region $(\nabla_j \psi)(\nabla^j \psi) > 0$ (space-like) and finally in the F region $(\nabla_j \psi)(\nabla^j \psi) = 0$. Moreover, $\nabla_k \psi$ is always an eigenvector of the energy momentum tensor with corresponding eigenvalue $\lambda = \frac{1}{2}(\nabla_k \psi)(\nabla^k \psi) - V(\psi)$ (see Ref. 4). Further in the T region, the eigenvalues are λ and $\sigma = -\frac{1}{2}(\nabla_k \psi)(\nabla^k \psi) - V(\psi)$ (degenerate) and in the S region they are σ and λ (degenerate). From the condition $(\nabla_l \psi)(\nabla^l \psi) = 14V(\psi)$ and from Theorem 1 in Ref. 4 we easily infer.

Theorem 5.5. *Let $(PZS)_4$ be a scalar field space-time. If the condition $A^l(\nabla_l \psi) \neq 0$ is fulfilled, then the space-time may be decomposed in the following sets:*

(1) $V(\psi) < 0$ *with eigenvalues* $\lambda = 6V(\psi)$ *(degenerate) and* $\sigma = -8V(\psi)$,
(2) $V(\psi) < 0$ *with eigenvalues* $\lambda = 6V(\psi)$ *and* $\sigma = -8V(\psi)$ *(degenerate).*

From the above discussion, an analogous result is valid if the condition $A^l(\nabla_l \psi) = 0$ is fulfilled.

ACKNOWLEDGMENTS

The authors would like to express their deep gratitude to the referees for careful comments and valuable suggestions for the improvement of the first version of our paper. This work was supported by Grant Project Nos. NRF-2011-220-C00002 and NRF-2012-R1A2A2A-01043023 from National Research Foundation and the second by KNU Research Grant 2013.

[1] K. Arslan, Y. Çelik, R. Deszcz, and R. Ezentaș, "On the equivalence of the Ricci-semisymmetry and semisymmetry," Colloq. Math. **76**, 279–294 (1998).
[2] E. Bertschinger and A. J. S. Hamilton, "Lagrangian evolution of the Weyl tensor," Astrophys. J. **435**, 1–7 (1994).
[3] J. Carot and M. M. Collinge, "Scalar field spacetimes," Class. Quant. Grav. **18**(24), 5441–5454 (2001).
[4] J. Carot and M. M. Collinge, "Scalar field cosmologies: A dynamical system study," Class. Quant. Grav. **20**(4), 707–728 (2003).
[5] M. C. Chaki, "On pseudo Ricci symmetric manifolds," Bulgar. J. Phys. **15**(6), 526–531 (1988).
[6] M. C. Chaki and R. K. Maity, "On quasi Einstein manifolds," Publ. Math. Debrecen **57**(3–4), 297–306 (2000).

[7] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska, and M. Prvanović, "On warped product manifolds satisfying some curvature conditions," J. Geom. Phys. **74**, 328–341 (2013).
[8] F. De Felice and C. J. S. Clarke, *Relativity on Curved Manifolds*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1990).
[9] F. Defever, R. Deszcz, M. Głogowska, V. V. Goldberg, and L. Verstraelen, "A class of four-dimensional warped products," Demonstratio Math. **35**(4), 853–864 (2002).
[10] R. Deszcz, "On pseudo symmetric spaces," Bull. Soc. Math. Belg. Sér. A **44**(1), 1–34 (1992).
[11] R. Deszcz, F. Dillen, L. Verstraelen, and L. Vrancken, "Quasi-Einstein totally real submanifolds of the nearly Kähler 6-sphere," Tohoku Math. J. **51**(4), 461–478 (1999).
[12] R. Deszcz, M. Głogowska, M. Hotloś, and Z. Şentürk, "On certain quasi-Einstein semisymmetric hypersurfaces," Annu. Univ. Sci. Budapest. Eötvös Sect. Math. **41**, 151–164 (1998).
[13] R. Deszcz, M. Głogowska, J. Jełowicki, M. Petrović-Torgašev, and G. Zafindratafa, "On Riemann and Weyl compatible tensors," Publ. Inst. Math. (Beograd)(N.S.) **94**(108), 111–124 (2013).
[14] R. Deszcz and W. Grycak, "On certain curvature conditions on Riemannian manifolds," Colloq. Math. **58**(2), 259–268 (1990).
[15] R. Deszcz, M. Hotloś, and Z. Şentürk, "Quasi-Einstein hypersurfaces in semi-Riemannian space forms," Colloq. Math. **89**(1), 81–97 (2001).
[16] R. Deszcz, M. Hotloś, J. Jełowicki, H. Kundu, and A. A. Shaikh, "Curvature properties of Gödel metric," Int. J. Geom. Methods Mod. Phys. **11**, 1450025 (2014).
[17] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics Vol. **19** (American Mathematical Society, Providence, RI, 1998).
[18] S. Frittelli, E. T. Newman, and G. Silva-Ortigoza, "The eikonal equation in flat space: Null surfaces and their singularities I," J. Math. Phys. **40**(1), 383–407 (1999).
[19] S. Frittelli, E. T. Newman, and G. Silva-Ortigoza, "The eikonal equation in asymptotically flat space-times," J. Math. Phys. **40**(2), 1041–1056 (1999).
[20] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space Time*, Cambridge Monographs on Mathematical Physics Vol. 1 (Cambridge University Press, London, 1973).
[21] O. Kowalski and M. Sekizawa, "Diagonalization of three-dimensional pseudo-Riemannian metrics," J. Geom. Phys. **74**, 251–255 (2013).
[22] D. Lovelock and H. Rund, *Tensors, Differential Forms and Variational Principles*, Pure and Applied Mathematics (Wiley-Interscience, New York, 1975).
[23] C. A. Mantica and L. G. Molinari, "A second order identity for the Riemann tensor and applications," Colloq. Math. **122**(1), 69–82 (2011).
[24] C. A. Mantica and L. G. Molinari, "Extended Derdzinski-Shen theorem for curvature tensors," Colloq. Math. **128**(1), 1–6 (2012).
[25] C. A. Mantica and L. G. Molinari, "Weakly Z symmetric manifolds," Acta Math. Hungar. **135**(1–2), 80–96 (2012).
[26] C. A. Mantica and L. G. Molinari, "Riemann compatible tensors," Colloq. Math. **128**(2), 197–210 (2012).
[27] C. A. Mantica and Y. J. Suh, "Pseudo-Z symmetric Riemannian manifolds with harmonic curvature tensors," Int. J. Geom. Methods Mod. Phys. **9**(1), 1250004 (2012).
[28] C. A. Mantica and Y. J. Suh, "Recurrent Z forms on Riemannian and Kaehler manifolds," Int. J. Geom. Methods Mod. Phys. **9**(7), 1250059 (2012).
[29] C. A. Mantica and Y. J. Suh, "Pseudo Q-symmetric Riemannian manifolds," Int. J. Geom. Methods Mod. Phys. **10**(5), 1350013 (2013).
[30] A. Z. Petrov, "Classification of spaces defining gravitational fields," Kazan. Gos. Univ. Uč. Zap. **114**(8), 55–69 (1954); "English translation: The classification of spaces defining gravitational fields," Gen. Relat. Grav. **32**(8), 1665–1685 (2000).
[31] M. M. Postnikov, *Geometry VI, Riemannian Geometry*, Encyclopaedia of Mathematical Sciences Vol. 91 (Springer-Verlag, Berlin, 2001) (Translated from the 1998 Russian edition by S. A. Vakhrameev).
[32] S. Ray-Guha, "On perfect fluid pseudo Ricci symmetric space time," Tensor (N.S.) **67**, 101–107 (2006).
[33] W. Roter, "On conformally symmetric Ricci-recurrent spaces," Colloq. Math. **31**, 87–96 (1974).
[34] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Hertl, *Exact Solutions of Einstein's Field Equations*, 2nd ed., Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2003).
[35] Z. I. Szabó, "Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$ I. The local version," J. Differ. Geom. **17**, 531–582 (1982).
[36] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, IL, 1984).
[37] R. C. Wrede, *Introduction to Vector and Tensor Analysis* (John Wiley & Sons, Inc., New York, 1963).