RICCI SOLITONS ON SINGLY WARPED PRODUCT MANIFOLDS AND APPLICATIONS

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ABSTRACT. The purpose of this article is to study implications of a Ricci soliton warped product manifold to its base and fiber manifolds. First, it is proved that if a warped product manifold is Ricci soliton then its factors are Ricci soliton. Then we study Ricci soliton on warped product manifolds admitting either a conformal vector field or a concurrent vector field. Finally, we study Ricci soliton on some warped product space-times.

1. INTRODUCTION

A Riemannian manifold (M, g) is said to admit a Ricci soliton structure, denoted by (M, g, ζ, λ) , if there exists a vector field $\zeta \in \mathfrak{X}(M)$ and a scalar λ satisfying

(1.1)
$$\frac{1}{2}\mathcal{L}_{\zeta}g + \operatorname{Ric} = \lambda g$$

where Ric denotes the Ricci tensor of M and \mathcal{L}_{ζ} denotes the Lie derivative in the direction of ζ . A Ricci soliton is said to be shrinking, steady or expanding if the scalar λ is positive, zero or negative respectively If $\zeta = \operatorname{grad} u$, for a smooth function u, the Ricci soliton $(M, g, \zeta, \lambda) \equiv (M, g, u, \lambda)$ is called a gradient Ricci soliton and the function u is called the potential function. Gradient Ricci solitons are natural generalizations of Einstein manifolds [4]. The study of Ricci solitons was first introduced by Hamilton as fixed or stationary points of the Ricci flow in the space of the metrics on M modulo diffeomorphisms and scaling. Since then, Ricci solitons have been extensively studied for different reasons and in different spaces [6, 8, 10, 12, 13, 20–23]. A large and growing body of research has continued to study Ricci soliton after Pereleman used Ricci soliton to solve the Poincare conjecture posed in 1904.

Generally speaking, it is possible to categorize the research problems on this topic under the perspective of warped product manifolds into two ways:

1. Under what conditions does the warped product become a Ricci soliton?

2. What does a factor of a warped product Ricci soliton inherit?

There are many partial answers for these questions. For example, if a gradient soliton splits $(M, g, f, \lambda) = (M_1 \times M_2, g_1 \oplus f^2 g_2, u, \lambda)$ as a Riemannian product, then $u(x_1, x_2) = u_1(x_1) + u_2(x_2)$ also splits in such a way that each (M_i, g_i, u_i, λ) is a soliton [24]. In [15, Sections 3 and 4], the authors obtain a criteria that the Riemannian manifold M is Einstein or a gradient Ricci soliton using of the second derivative of warping function f in the warped and Lorentzian warped product

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space of the form $\mathbb{R} \times_f M$ with gradient Ricci solitons. In [11], it is shown that a non-shrinking gradient Ricci soliton warped product whose warping function attains its extremes is a Riemannian product. Moreover, existence conditions for a warped product gradient Ricci soliton are derived on the warping function, a gradient vector field and on the fiber. The authors of [17, Section 3] derived conditions on the gradient warped product Ricci soliton to have Einstein base manifold or to have Einstein fiber manifold. They also considered warped product Ricci soliton either with one dimensional Euclidean base or with one dimensional circle base in [18]. Both necessary and sufficient conditions for multiply warped product manifolds to be gradient Ricci solitons are obtained in [14]. Special doubly warped gradient Ricci soliton with harmonic Weyl tensor are considered in [16] where a doubly warped product means a multiply warped product with two fibers. In [7], it is shown that locally conformally flat Lorentzian gradient Ricci soliton is locally isometric to a Robertson–Walker warped product, if the gradient of the potential function is nonnull. In [29], the concept of gradient Ricci solitons on a semi-Riemannian warped product is studied and it is proved that the potential function depends only on the base factor of the underlying warped product and its fiber is an Einstein manifold.

It is clear that the work is restricted either to gradient warped Ricci solitons or to special cases of warped Ricci solitons. In the current study, we intend to fill this gap in the literature by providing a complete study of warped product Ricci solitons. Moreover, we study warped product Ricci solitons admitting conformal, concurrent or Killing vector fields. Finally, we apply our results on generalized Robertson-Walker space-times and standard static space-times.

2. Preliminaries

Let (M_i, g_i, D^i) , i = 1, 2 be two C^{∞} pseudo-Riemannian manifolds equipped with metric tensors g_i where D^i is the Levi-Civita connection of the metric g_i for i = 1, 2. Let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the natural projection maps of the Cartesian product $M_1 \times M_2$ onto M_1 and M_2 respectively. Also, let $f : M_1 \to (0, \infty)$ be a positive real-valued smooth function. The warped product manifold $M_1 \times_f M_2$ is the the product manifold $M_1 \times M_2$ equipped with the metric tensor $g = g_1 \oplus f^2 g_2$ defined by

$$g = \pi_1^* (g_1) \oplus (f \circ \pi_1)^2 \pi_2^* (g_2)$$

where * denotes the pull-back operator on tensors [5,25,26]. The function f is called the warping function of the warped product manifold $M_1 \times_f M_2$. In particular, if f = 1, then $M_1 \times_1 M_2 = M_1 \times M_2$ is the usual Cartesian product manifold. It is clear that the submanifold $M_1 \times \{q\}$ is isometric to M_1 for every $q \in M_2$. Also, $\{p\} \times M_2$ is homothetic to M_2 for every $p \in M_1$. Throughout this article, we use the same notation for a tensor field and for its lift to the product manifold. Let D be the Levi-Civita connection of the metric tensor g. In [5,25,26], the Ricci curvature Ric of the warped product manifold in terms of the lift of Ricci curvatures, Ricⁱ, on M_i , for i = 1, 2 and H^f , the Hessian tensor of f on M_1 is described.

3. WARPED PRODUCT RICCI SOLITONS

This section presents a study of warped product Ricci soliton. The following result considers the first inheritance property.

Theorem 1. Let (M, g, ζ, λ) be a Ricci soliton where $(M, g) = (M_1 \times_f M_2, g_1 \oplus$ f^2g_2). Then,

- (1) $(M_1, g_1, \zeta_1 \eta, \lambda)$ is a Ricci soliton provided that $\frac{n_2}{f} D^1_{X_1} \nabla^1 f = D^1_{X_1} \eta_1$, and (2) $(M_2, g_2, f^2 \zeta_2, \lambda f^2 + f^{\sharp} f \zeta_1(f))$ is a Ricci soliton when $\lambda f^2 + f^{\sharp} f \zeta_1(f)$ is constant.

Proof. It is well-known that

(3.1)
$$\mathcal{L}_{\zeta}g = \mathcal{L}_{\zeta_1}^1 g_1 + f^2 \mathcal{L}_{\zeta_2}^2 g_2 + 2f\zeta_1(f) g_2,$$

(3.2)
$$\operatorname{Ric} = \operatorname{Ric}^{1} - \frac{n_{2}}{f} \operatorname{H}^{f} + \operatorname{Ric}^{2} - f^{\sharp} g_{2},$$

where $f^{\sharp} = f\Delta f + (n_2 - 1) \|\nabla f\|_1^2$. Thus Equation (1.1) becomes

(3.3)
$$\lambda g_1 + \lambda f^2 g_2 = \frac{1}{2} \mathcal{L}^1_{\zeta_1} g_1 + \frac{1}{2} f^2 \mathcal{L}^2_{\zeta_2} g_2 + f \zeta_1 (f) g_2 + \operatorname{Ric}^1 - \frac{n_2}{f} \operatorname{H}^f + \operatorname{Ric}^2 - f^\sharp g_2.$$

It is noted that

(3.4)
$$\begin{pmatrix} \frac{1}{2}\mathcal{L}_{\zeta_{1}}^{1}g_{1} - \frac{n_{2}}{f}\mathrm{H}^{f} \end{pmatrix} (X_{1}, Y_{1}) = \frac{1}{2}g_{1}\left(D_{X_{1}}^{1}\zeta_{1}, Y_{1}\right) + \frac{1}{2}g_{1}\left(X_{1}, D_{Y_{1}}^{1}\zeta_{1}\right) \\ - \frac{n_{2}}{f}g_{1}\left(D_{X_{1}}^{1}\nabla^{1}f, Y_{1}\right).$$

However, $g_1\left(D_{X_1}^1\nabla^1 f, Y_1\right) = g_1\left(D_{Y_1}^1\nabla^1 f, X_1\right)$. Thus,

$$\frac{n_2}{f}g_1\left(D_{X_1}^1\nabla^1 f, Y_1\right) = \frac{n_2}{2f}g_1\left(D_{X_1}^1\nabla^1 f, Y_1\right) + \frac{n_2}{2f}g_1\left(D_{X_1}^1\nabla^1 f, Y_1\right) \\ = \frac{n_2}{2f}g_1\left(D_{X_1}^1\nabla^1 f, Y_1\right) + \frac{n_2}{2f}g_1\left(D_{Y_1}^1\nabla^1 f, X_1\right)$$

Now, assume that there is a vector field η_1 on the base manifold such that $\frac{n_2}{f}D_{X_1}^1\nabla^1 f =$ $D_{X_1}^1 \eta_1$, then Equation (3.4) reads as

$$\begin{pmatrix} \frac{1}{2}\mathcal{L}_{\zeta_{1}}^{1}g_{1} - \frac{n_{2}}{f}\mathbf{H}^{f} \end{pmatrix} (X_{1}, Y_{1}) = \frac{1}{2}g_{1} \left(D_{X_{1}}^{1}\zeta_{1}, Y_{1} \right) - \frac{n_{2}}{2f}g_{1} \left(D_{X_{1}}^{1}\nabla^{1}f, Y_{1} \right) \\ + \frac{1}{2}g_{1} \left(X_{1}, D_{Y_{1}}^{1}\zeta_{1} \right) - \frac{n_{2}}{2f}g_{1} \left(X_{1}, D_{Y_{1}}^{1}\nabla^{1}f \right) \\ = \frac{1}{2}g_{1} \left(D_{X_{1}}^{1} \left(\zeta_{1} - \eta_{1} \right), Y_{1} \right) + \frac{1}{2}g_{1} \left(X_{1}, D_{Y_{1}}^{1} \left(\zeta_{1} - \eta_{1} \right) \right) \\ = \frac{1}{2} \left(\mathcal{L}_{\zeta_{1} - \eta_{1}}^{1}g_{1} \right) \left(X_{1}, Y_{1} \right).$$

Thus, Equation (3.3) may be rewritten as

$$\lambda g_1 + \left(\lambda f^2 + f^{\sharp} - f\zeta_1(f)\right) g_2 = \frac{1}{2} \mathcal{L}^1_{\zeta_1 - \eta_1} g_1 + \operatorname{Ric}^1 + \frac{1}{2} f^2 \mathcal{L}^2_{\zeta_2} g_2 + \operatorname{Ric}^2.$$

Since f is a function on the base factor, it implies that $f^2 \mathcal{L}^2_{\zeta_2} g_2 = \mathcal{L}^2_{f^2 \zeta_2} g_2$ and hence

$$\lambda g_1 + \left(\lambda f^2 + f^{\sharp} - f\zeta_1(f)\right) g_2 = \frac{1}{2} \mathcal{L}^1_{\zeta_1 - \eta_1} g_1 + \operatorname{Ric}^1 + \frac{1}{2} \mathcal{L}^2_{f^2 \zeta_2} g_2 + \operatorname{Ric}^2.$$

Now, when the arguments are restricted to the factor manifolds, one can obtain

$$\lambda g_1 = \frac{1}{2} \mathcal{L}^1_{\zeta_1 - \eta_1} g_1 + \operatorname{Ric}^1,$$
$$\left(\lambda f^2 + f^{\sharp} - f\zeta_1(f)\right) g_2 = \frac{1}{2} \mathcal{L}^2_{f^2 \zeta_2} g_2 + \operatorname{Ric}^2,$$

which completes the proof.

Remark 1. In the preceding theorem, the properties of the vector field η_1 depend on the geometry of the base manifold M_1 and the warping function f. For that, we consider the following cases:

(1) Assume that M_1 is compact and η_1 is a gradient vector field. Then, there is a smooth function ϕ on M_1 , such that $\eta_1 = \nabla^1 \phi$ and so $D_{Y_1}^1 \nabla^1 \phi = \frac{1}{f} D_{Y_1}^1 \nabla^1 f$. Hence, $\Delta_1 (\phi - \log f) = |\nabla^1 f|^2 / f^2$. The integration of both sides over M_1 implies that $\nabla^1 f = 0$, i.e., f is constant, say f = 1, and consequently M is a product manifold. The soliton equations on the factor manifolds become

$$\lambda g_1 = \frac{1}{2} \mathcal{L}_{\zeta_1}^1 g_1 + \operatorname{Ric}^1,$$

$$\lambda g_2 = \frac{1}{2} \mathcal{L}_{\zeta_2}^2 g_2 + \operatorname{Ric}^2.$$

- (2) The warped product manifold $M = I \times_{e^t} M_2$, where $M_1 = I$ is an open connected subinterval of \mathbb{R} , admits a vector field $\eta = t\partial_t$ where $\frac{1}{f}D^1_{\partial_t}\nabla^1 f = D^1_{\partial_t}\eta_1$.
- (3) Assume that M_1 has a constant curvature κ_1 . In [19], it is proved that a parallel symmetric (0,2) tensor in a manifold of constant curvature is a constant multiple of the metric tensor. Let us define the tensor T as

$$T(X_1, Y_1) = g_1\left(D_{Y_1}^1 \eta_1, X_1\right) = \frac{1}{f}g_1\left(D_{Y_1}^1 \nabla^1 f, X_1\right).$$

It is clear that T is symmetric and so the parallelism of T is sufficient for $D_{Y_1}^1\eta = cY_1$ for some constant c and $\mathcal{L}_{\eta_1}^1g_1 = 2cg_1$. Thus, the soliton equation reduces to

$$\begin{aligned} (\lambda + c) g_1 &= \frac{1}{2} \mathcal{L}^1_{\zeta_1} g_1 + \operatorname{Ric}^1, \\ (\lambda + c) g_1 &= \frac{1}{2} \mathcal{L}^1_{\zeta_1} g_1 + (n_1 - 1) \kappa_1 g_1 \\ [\lambda + c - (n_1 - 1) \kappa_1] g_1 &= \frac{1}{2} \mathcal{L}^1_{\zeta_1} g_1, \end{aligned}$$

that is, ζ_1 is also a conformal vector field on M_1 .

Let ω be a one-form metrically equivalent to η_1 . It is clear that $g_1(X_1, D_{Y_1}^1 \eta_1) = g_1(Y_1, D_{X_1}^1 \eta_1)$ and consequently ω is closed. In simply connected manifolds, every closed one-form is exact. A direct consequence of the above theorem is the following corollary.

Corollary 1. Let (M, g, ζ, λ) be a Ricci soliton where $(M, g) = (M_1 \times_f M_2, g_1 \oplus f^2g_2)$ and ζ_1 is homothetic on M_1 , that is $\mathcal{L}^1_{\zeta_1}g_1 = 2ag_1$ for some constant a. Then, $(M_1, g_1, -\eta_1, \lambda - a)$ is a gradient Ricci soliton whenever M_1 is simply connected.

It is clear that a Ricci soliton (M, g, ζ, λ) is Einstein with factor $\lambda - \rho$ if and only if ζ is conformal with factor 2ρ .

Theorem 2. Let (M, g, ζ, λ) be a Ricci soliton where $(M, g) = (M_1 \times_f M_2, g_1 \oplus$ f^2q_2). Then (M,q) is Einstein if one of the following conditions holds

- (1) ζ_i is conformal on M_i with factor $2\rho_i$ for any i = 1, 2 and $\rho_1 = \rho_2 + \zeta_1 (\ln f)$
- (2) $\zeta = \zeta_1$ and ζ_1 is a Killing vector field on M_1 .
- (3) $\zeta = \zeta_2$ and ζ_2 is a Killing vector field on M_2 .
- (4) ζ_i is a Killing vector field on M_i for i = 1, 2 and $\zeta_1(f) = 0$.

The following theorem considers the converse.

Theorem 3. Let $(M_1, g_1, \zeta_1, \lambda)$ be a Ricci soliton and (M_2, g_2) be an Einstein manifold with factor μ . Then (M, g, ζ, λ) is a Ricci soliton where (M, g) is a warped product manifold of the form $(M_1 \times_f M_2, g_1 \oplus f^2 g_2)$ if

- (1) ζ_2 is conformal with factor 2ρ ,
- (2) $H^f = 0$ and,
- (3) $(\lambda \rho) f^2 = f\zeta_1(f) + \mu + (n_2 1) c^2$ where $g_1(\nabla f, \nabla f) = c^2$ for some $c \in \mathbb{R}$.

A vector field ζ on a Riemannian manifold M which satisfies

$$\nabla_X \zeta = X$$

for any vector field $X \in \mathfrak{X}(M)$ is called a concurrent vector field [9]. It is clear that a concurrent vector field is a homothetic one with factor $\rho = 2$ since $(\mathcal{L}_{\mathcal{C}}g)(X,Y) = 2g(X,Y)$. Furthermore, a constant vector field is not concurrent. ζ is called gradient if there is a smooth function u defined on M such that $\zeta = \nabla u$. In this case $(\mathcal{L}_{\zeta}g)(X,Y) = 2\mathrm{H}^{u}(X,Y)$ where H^{u} is the Hessian tensor of u defined on M. Let ζ be a concurrent vector field and let $u = \frac{1}{2}g(\zeta,\zeta)$, then $\zeta = \nabla u$. The converse is also true. A vector field ζ on a manifold M is concurrent with respect to a Riemannian metric g if and only if $\zeta = \nabla u$, and $\mathcal{L}_{\zeta}g = 2g$ where $u = \frac{1}{2}g(\zeta, \zeta)$. Hence, a concurrent vector field is a gradient vector field.

Proposition 1. Let $\zeta = \zeta_1 + \zeta_2$ be a vector field on $M = (M_1 \times_f M_2, g)$. ζ is concurrent on M if and only if ζ_1 is a concurrent vector field on M_1 and one of the following conditions hold.

- (1) ζ_2 is a concurrent vector field on M_2 and f is constant. (2) $\zeta_2 = 0$ and $\zeta_1(f) = f$.

Proof. Suppose that ζ is a concurrent vector field on M. Then

(3.5)
$$D_{\partial_i}\zeta = \partial_i, i = 1, 2, ..., n_1, n_1 + 1, ..., n_1 + n_2$$

The first n_1 equations of (3.5) imply that

$$D_{\partial_i} \left(\zeta_1 + \zeta_2 \right) = \partial_i$$

$$D^1_{\partial_i} \zeta_1 + \partial_i \left(\ln f \right) \zeta_2 = \partial_i$$

The tangential and normal parts of the last equation are

$$(3.6) D^1_{\partial_i}\zeta_1 = \partial_i$$

 $D_{\partial_i}\zeta_1 = O_i$ $\partial_i (\ln f) \zeta_2 = 0$ (3.7)

Equation (3.6) implies that ζ_1 is concurrent vector field on M_1 . The second equation implies that

$$\partial_i (\ln f) = 0 \quad \text{or} \quad \zeta_2 = 0$$

Now, we have two cases:

Case 1: $\partial_i (\ln f) = 0$ for any $i = 1, 2, ..., n_1$: This equation implies that f is constant. Now, we use the second n_2 equations of (3.5)

$$D_{\partial_i} \left(\zeta_1 + \zeta_2 \right) = \partial_i \quad i = n_1 + 1, n_1 + 2, \dots, n_1 + n_2$$

$$\zeta_1 \left(\ln f \right) \partial_i + D_{\partial_i}^2 \zeta_2 - fg_2 \left(\partial_i, \zeta_2 \right) \operatorname{grad} f = \partial_i$$

Since f is constant,

$$D_{\partial_i}^2 \zeta_2 = \partial$$

i.e, ζ_2 is a concurrent vector field on M_2 . Thus the first condition holds.

Case 2: $\zeta_2 = 0$: Now, we use the second n_2 equations of (3.5)

$$D_{\partial_i} (\zeta_1 + \zeta_2) = \partial_i \text{ for any } i = n_1 + 1, n_1 + 2, \dots, n_1 + n_2$$

$$\zeta_1 (\ln f) \partial_i = \partial_i$$

This equation implies that $\zeta_1(\ln f) = 1$ and hence $\zeta_1(f) = f$. This is the second condition.

Conversely, suppose that the first condition holds. Then for $i = 1, 2, ..., n_1$ we have

$$D_{\partial_i}\zeta = D^1_{\partial_i}\zeta_1\zeta + \partial_i\left(\ln f\right)\zeta_2 = D^1_{\partial_i}\zeta_1 = \partial_i$$

and for $i = n_1 + 1, n_1 + 2, ..., n_1 + n_2$ we have

$$D_{\partial_i}\zeta = \zeta_1 \left(\ln f\right)\partial_i + D^2_{\partial_i}\zeta_2 - fg_2\left(\zeta_2, \partial_i\right)\operatorname{grad} f = D^2_{\partial_i}\zeta_2 = \partial_i$$

Therefore, ζ is a concurrent vector field. Now suppose that the second condition holds. Then for $i = 1, 2, ..., n_1$ we have

$$D_{\partial_i}\zeta = D_{\partial_i}\zeta_1 = D^1_{\partial_i}\zeta_1 = \partial_i$$

and for $i = n_1 + 1, n_1 + 2, ..., n_1 + n_2$ we have

Remark 2. The above result ensures that

$$D_{\partial_i}\zeta = \zeta_1 \left(\ln f\right)\partial_i = \partial_i$$

Therefore ζ is concurrent in this case also and the proof is complete.

(1) There is no concurrent vector field on $M_1 \times_f M_2$ of the form $\zeta = \zeta_2$. Thus, there is no space-like concurrent vector field on $I \times_f M$ and there is no time-like concurrent vector field on standard static space-time $I_f \times M$.

- (2) The only time-like concurrent vector field on $I \times_f M$ is given by $\zeta_1 = (t+c) \partial_t$ where f(t) = a (t+c) and a > 0.
- (3) The only concurrent vector field of the form $\zeta = \zeta_1$ on $M_1 \times_f M_2$ exists if ζ_1 is concurrent on M_1 and $\zeta_1(f) = f$.

Let $\overline{M} = I \times_f M$ be a generalized Robertson-Walker space-times equipped with the metric $\overline{g} = -dt^2 \oplus f^2 g$ where (M, g) is a Riemannian manifold and I is an open connected interval with the usual flat metric $-dt^2$. A vector field of the form $\zeta = u\partial_t$ is a concurrent vector field on \overline{M} if u = (t + c) and f = au where a > 0and t + c > 0. However, the vector field $\zeta = \coth t\partial_t$ on $I \times_{\cosh t} M$ satisfies that $\zeta(f) = f$. But ζ is not concurrent on $I \times_{\cosh t} M$ since $\zeta_1 = \coth t\partial_t$ is not concurrent on I. The rigorous of generalized Robertson-Walker space-times will be given in Section 4. **Theorem 4.** Let (M, g, ζ, λ) be a Ricci soliton and ζ be a concurrent vector field on M where (M, g) is a warped product of the form $(M_1 \times_f M_2, g_1 \oplus f^2 g_2)$. If $\zeta_2 \neq 0$, then M, M_1 and M_2 are Ricci flat, gradient Ricci soliton with $\lambda = 1$.

Proof. Let $(M_1 \times_f M_2, g, \zeta, \lambda)$ be a Ricci soliton and ζ be a concurrent vector field on $M_1 \times_f M_2$. Then

(3.8)
$$\operatorname{Ric}(X,Y) = (\lambda - 1) g(X,Y)$$

Suppose that $X = X_2$ and $Y = Y_2$, then

$$\operatorname{Ric}^{2}(X_{2}, Y_{2}) = f^{\sharp}g_{2}(X_{2}, Y_{2}) + (\lambda - 1) f^{2}g_{2}(X_{2}, Y_{2})$$

where $f^{\sharp} = f \Delta f + (n_2 - 1) \|\nabla f\|_1^2$. Since ζ is concurrent and $\zeta_2 \neq 0$, ζ_2 is concurrent and f = c is constant. This implies that $f^{\sharp} = 0$ and so

(3.9)
$$\operatorname{Ric}^{2}(X_{2}, Y_{2}) = (\lambda - 1) c^{2} g_{2}(X_{2}, Y_{2})$$

i.e. M_2 is Einstein with factor $\mu = (\lambda - 1) f^2$. This equation is true for any vector field in $\mathfrak{X}(M_2)$ and so

(3.10)
$$\operatorname{Ric}^{2}(\zeta_{2},\zeta_{2}) = (\lambda - 1) c^{2} \|\zeta_{2}\|_{2}^{2}$$

Let $\{\zeta_2, e_1, e_2, ..., e_{n_2-1}\}$ be an orthogonal basis of $\mathfrak{X}(M_2)$, then the curvature tensor is given by

$$R^{2}(\zeta_{2}, e_{i}, \zeta_{2}, e_{i}) = g_{2}(R^{2}(\zeta_{2}, e_{i})\zeta_{2}, e_{i})$$

$$= g_{2}(D_{\zeta_{2}}D_{e_{i}}\zeta_{2} - D_{e_{i}}D_{\zeta_{2}}\zeta_{2} - D_{[\zeta_{2}, e_{i}]}\zeta_{2}, e_{i})$$

$$= g_{2}(D_{\zeta_{2}}e_{i} - D_{e_{i}}\zeta_{2} - [\zeta_{2}, e_{i}], e_{i})$$

$$= 0$$

Thus $\operatorname{Ric}^2(\zeta_2, \zeta_2) = 0$. By substitution in equation (3.10) we get that $\lambda = 1$ and so equations 3.8 and 3.9 imply that $\operatorname{Ric}(X, Y) = \operatorname{Ric}^2(X_2, Y_2) = 0$. Now suppose that $X = X_1$ and $Y = Y_1$, then we get that

Suppose that
$$A = A_1$$
 and $I = I_1$, then we get that

$$\operatorname{Ric}(X_{1}, Y_{1}) = 0$$
$$\operatorname{Ric}^{1}(X_{1}, Y_{1}) = \frac{n_{2}}{f} \operatorname{H}^{f}(X_{1}, Y_{1}) = 0$$

It is easy to show that all of them are shrinking Ricci soliton with the same factor $\lambda = 1$. Moreover, ζ and ζ_i are gradient vector fields with potential functions $u = \frac{1}{2}g(\zeta, \zeta)$ and $u_i = \frac{1}{2}g(\zeta_i, \zeta_i)$ where i = 1, 2 since

$$g(X, \nabla u) = X(u) = g(D_X \zeta, \zeta)$$
$$= g(X, \zeta)$$

i.e. $\zeta = \nabla u$ and similarly $\zeta_i = \nabla_i u_i$.

Theorem 5. Let (M, g, u, λ) be a gradient Ricci soliton where (M, g) is a warped product of the form $(M_1 \times_f M_2, g_1 \oplus f^2 g_2)$. Then

(1) $(M_1, g_1, \phi_1, \lambda)$ is a gradient Ricci soliton with $\phi_1 = u_1 - n_2 \ln f$ and $u_1 = u$ at some fixed point of M_2 .

(2) $(M_2, g_2, \phi_2, \lambda f^2)$ is a gradient Ricci soliton with $\phi_2 = u$ at some fixed point of M_1 if f is constant.

Proof. Suppose that $(M_1 \times_f M_2, g, u, \lambda)$ is a gradient Ricci soliton, then

$$\mathrm{H}^{u}(X,Y) + \mathrm{Ric}(X,Y) = \lambda g(X,Y)$$

for any vector fields $X, Y \in \mathfrak{X}$ $(M_1 \times_f M_2)$. Let $X = X_1$ and $Y = Y_1$, then

$$H^{u}(X_{1}, Y_{1}) + \operatorname{Ric}(X_{1}, Y_{1}) = \lambda g(X_{1}, Y_{1})$$
$$H^{u_{1}}_{1}(X_{1}, Y_{1}) + \operatorname{Ric}^{1}(X_{1}, Y_{1}) - \frac{n_{2}}{f} H^{f}_{1}(X_{1}, Y_{1}) = \lambda g_{1}(X_{1}, Y_{1})$$
$$H^{\phi_{1}}_{1}(X_{1}, Y_{1}) + \operatorname{Ric}^{1}(X_{1}, Y_{1}) = \lambda g_{1}(X_{1}, Y_{1})$$

where $\phi_1 = u_1 - n_2 \ln f$ and $u_1 = u$ at a fixed point of M_2 . Thus $(M_1, g_1, \phi_1, \lambda)$ is a gradient Ricci soliton. Now let $X = X_2$ and $Y = Y_2$, then

$$H^{u}(X_{2}, Y_{2}) + \operatorname{Ric}(X_{2}, Y_{2}) = \lambda g(X_{2}, Y_{2})$$

$$H^{\phi_{2}}_{2}(X_{2}, Y_{2}) + \operatorname{Ric}^{2}(X_{2}, Y_{2}) - f^{\sharp}g_{2}(X_{2}, Y_{2}) = \lambda f^{2}g_{2}(X_{2}, Y_{2})$$

$$H^{\phi_{2}}_{2}(X_{2}, Y_{2}) + \operatorname{Ric}^{2}(X_{2}, Y_{2}) = (\lambda f^{2} + f^{\sharp})g_{2}(X_{2}, Y_{2})$$

$$H^{\phi_{2}}_{2}(X_{2}, Y_{2}) + \operatorname{Ric}^{2}(X_{2}, Y_{2}) = \lambda_{2}g_{2}(X_{2}, Y_{2})$$

where $u_2 = u$ at a fixed point of M_1 and $\lambda_2 = \lambda f^2 + f^{\sharp}$ and $f^{\sharp} = f\Delta f + (n_2 - 1) \|\nabla f\|_1^2$. If f is constant, then $(M_1, g_2, u_2, \lambda f^2)$ is a gradient Ricci soliton.

4. RICCI SOLITONS ON WARPED SPACE-TIMES

In this section we will consider Ricci soliton on two well-known space-time models, namely generalized Robertson-Walker space-times and standard static spacetimes. More explicitly, we state some necessary conditions for these models to be Ricci soliton. Then we also explore implications of that on the components of the underlying models.

We begin by the following straightforward result.

Remark 3. Let I be an open and connected subinterval of \mathbb{R} furnished with $-dt^2$. Suppose that $u\partial_t \in \mathfrak{X}(I)$ is a vector field on I where $u: I \to \mathbb{R}$ is smooth. Then

- (1) $u\partial_t$ is concurrent on $(I, -dt^2)$ if and only if u(t) = t + a for some a.
- (2) $u\partial_t$ is conformal on $(I, -dt^2)$ with the conformal factor $\mu = 2u'$.

4.1. Ricci Solitons on Generalized Robertson-Walker Space-times. We first define generalized Robertson-Walker space-times. Let (M, g) be an *n*-dimensional Riemannian manifold and $f : I \to (0, \infty)$ be a smooth function. Then (n + 1)-dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -\mathrm{d}t^2 \oplus f^2 g$$

is called a generalized Robertson-Walker space-time and is denoted by $\overline{M} = I \times_f M$ where I is an open, connected subinterval of \mathbb{R} and dt^2 is the Euclidean metric tensor on I. This structure was introduced to the literature to extend Robertson-Walker space-times [27, 28]. From now on, we will denote $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ by ∂_t to state our results in simpler forms.

Let $\overline{\zeta} = u\partial_t + \zeta$ be a concurrent vector field on a generalized Robertson-Walker space-time $\overline{M} = I \times_f M$ furnished with the metric $\overline{g} = -\mathrm{d}t^2 \oplus f^2 g$, then

$$D_{\bar{X}}\bar{\zeta} = X$$

for any $\bar{X} = x\partial_t + X \in \mathfrak{X}(\bar{M})$. This equation implies that

$$\bar{D}_{\partial_t} \bar{\zeta} = \partial_t \bar{D}_{\partial_i} \bar{\zeta} = \partial_i$$

where $\{\partial_i | i = 1, 2, ..., n\}$ is an orthonormal set of vector fields on M. The first equation yields

(4.1)
$$\dot{u}\partial_t + \partial_t \left(\ln f\right)\zeta = \partial_t$$

and the second equation yields

(4.2)
$$u\partial_t \left(\ln f\right)\partial_i + D_{\partial_i}\zeta - ff\zeta^i\partial_t = \partial_i$$

where $\zeta^i = g(\zeta, \partial_i)$. Equation (4.1) implies that $\dot{u} = 1$ and $\dot{f}\zeta = 0$. Therefore, u = t + a where $a \in \mathbb{R}$. Now, we have two cases, namely, $\dot{f} = 0$ or $\zeta = 0$.

The first case implies that $D_X \zeta = X$. That is, ζ is concurrent on M. The second case implies that $u\dot{f} = f$ and so f(t) = b(t+a) where b > 0 and t+a > 0.

Theorem 6. Let $\overline{\zeta} = u\partial_t + \zeta$ be a field on a generalized Robertson-Walker spacetime $\overline{M} = I \times_f M$ furnished with the metric $\overline{g} = -dt^2 \oplus f^2 g$. Then $\overline{\zeta}$ is a concurrent vector field on \overline{M} if and only if u = t + a and one of the following conditions hold

- (1) ζ is concurrent on M and $\dot{f} = 0$.
- (2) $\zeta = 0$ and f = b(t+a) where b > 0 and t+a > 0.

Theorem 7. Let $\overline{M} = I \times_f M$ be a generalized Robertson-Walker space-time equipped with the metric $\overline{g} = -dt^2 \oplus f^2 g$. If $(\overline{M}, \overline{g}, u, \lambda)$ is a Ricci soliton where

$$u = \int_{a}^{t} f(r) \, \mathrm{d}r, \quad for \ some \ constant \quad a \in I$$

then

(4.3)
$$\operatorname{Ric} = \left(\lambda - \dot{f}\right)g$$

Proof. Let $\zeta = \operatorname{grad} u$, then $\zeta = f(t) \partial_t$. It is clear that the vector field is perpendicular to M. Suppose that $\{\partial_t, \partial_1, \partial_2, ..., \partial_m\}$ is an orthogonal basis for $\mathfrak{X}(\overline{M})$, then the Hessian tensor of u is given by

$$\mathrm{H}^{u}\left(\partial_{t},\partial_{t}\right)=\bar{g}\left(D_{X}\mathrm{grad}u,Y\right)$$

Now we have the following cases. The first case when $X = Y = \partial_t$. In this case we have

$$\begin{aligned} H^u(\partial_t, \partial_t) &= \bar{g}\left(D_{\partial_t} \operatorname{grad} u, \partial_t\right) \\ &= \dot{f} \bar{g}\left(\partial_t, \partial_t\right) \end{aligned}$$

The second case when $X = \partial_t$ and $Y = \partial_i$ for any i = 1, 2, ..., m. In this case

$$H^{u}(\partial_{t}, \partial_{i}) = \bar{g}(D_{\partial_{t}} \operatorname{grad} u, \partial_{i})$$
$$= f\bar{g}(\partial_{t}, \partial_{i})$$

Finally, $X = \partial_i$ and $Y = \partial_j$ for any i, j = 1, 2, ..., m. In this case

$$H^{u}(\partial_{i},\partial_{j}) = \bar{g}(D_{\partial_{i}}\operatorname{grad} u,\partial_{j})$$

$$= f\bar{g}(D_{\partial_{i}}\partial_{t},\partial_{j})$$

$$= f\bar{g}\left(\frac{\dot{f}}{f}\partial_{i},\partial_{j}\right) = \dot{f}\bar{g}(\partial_{i},\partial_{j})$$

Thus $\mathrm{H}^{u}(X,Y) = \dot{f}\bar{g}(X,Y)$ and hence

$$(\mathcal{L}_{\zeta}\bar{g})(X,Y) = \bar{g}(D_X \operatorname{grad} u, Y) + \bar{g}(D_Y \operatorname{grad} u, X)$$

= $2\operatorname{H}^u(X,Y) = 2\dot{f}\bar{g}(X,Y)$

Suppose that $(\overline{M}, \overline{g}, u, \lambda)$ is a Ricci soliton, then

$$\frac{1}{2}\mathcal{L}_{\zeta}\bar{g} + \operatorname{Ric} = \lambda \bar{g}$$
$$\dot{f}\bar{g} + \operatorname{Ric} = \lambda \bar{g}$$
$$\operatorname{Ric} = \left(\lambda - \dot{f}\right)\bar{g}$$

Corollary 2. Let $\overline{M} = I \times_f M$ be a generalized Robertson-Walker space-time equipped with the metric $\overline{g} = -dt^2 \oplus f^2 g$. Suppose that $(\overline{M}, \overline{g}, u, \lambda)$ is a gradient Ricci soliton where

$$u = \int_{a}^{t} f(r) \, \mathrm{d}r, \quad for \ some \ constant \quad a \in I.$$

Then

(1) $(\overline{M}, \overline{g})$ is Einstein if \dot{f} is constant

(2) $(\overline{M}, \overline{g})$ is Ricci flat if $\lambda = \dot{f}$.

Theorem 8. Let $\overline{M} = I \times_f M$ be a generalized Robertson-Walker space-time equipped with the metric $\overline{g} = -dt^2 \oplus f^2 g$. If $(\overline{M}, \overline{g}, \zeta, \lambda)$ is a Ricci soliton with concurrent vector field ζ , then (M, g) is Einstein with factor $(n-1)c^2$ where $c = \|\text{grad}f\|_1$ is a constant.

4.2. Ricci Solitons on Standard Static Space-times. We begin by defining standard static space-times. Let (M, g) be an *n*-dimensional Riemannian manifold and $f: M \to (0, \infty)$ be a smooth function. Then (n + 1)-dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -f^2 \mathrm{d}t^2 \oplus g$$

is called a standard static space-time and is denoted by $\overline{M} =_f I \times M$ where I is an open, connected subinterval of \mathbb{R} and dt^2 is the Euclidean metric tensor on I.

Note that standard static space-times can be considered as a generalization of the Einstein static universe [1-4]. The following propositions are well-known and so the proofs are omitted.

By using *Theorem 1*, one can obtain the following result for standard static space-times.

Theorem 9. Let $\overline{\zeta} = u\partial_t + \zeta$ be a vector field on a standard static space-time of the form $\overline{M} =_f I \times M$ furnished with the metric $\overline{g} = -f^2 dt^2 \oplus g$. Then $\overline{\zeta}$ is a concurrent vector field on \overline{M} if and only if $\zeta \in (M)$ is concurrent on M and one of the following conditions hold

- (1) u = t + a (i.e, $u\partial t$ is concurrent on I) and f is constant,
- (2) $u = 0 \text{ and } \zeta(f) = f$

The next result can be considered as a consequence of *Theorem 2*.

Theorem 10. Let $\overline{M} =_f I \times M$ be a standard static space-time with the metric tensor $\overline{g} = -f^2 dt^2 \oplus g$. Suppose that $(\overline{M}, \overline{g}, \overline{\zeta}, \lambda)$ is a Ricci soliton where $\overline{\zeta} = u\partial_t + \zeta$. Then $(\overline{M}, \overline{g})$ is Einstein if

- (1) $\overline{\zeta} = \zeta$ is Killing on M
- (2) $\overline{\zeta} = u\partial_t$ and u = t + a (i.e, $u\partial_t$ is Killing on I),
- (3) $u\partial_t$ and ζ are Killing vector fields on I and M, respectively and $\zeta(f) = 0$.

Now an application of *Theorem* 3 yields that:

Theorem 11. Let $\overline{M} =_f I \times M$ be a standard static space-time with the metric tensor $\overline{g} = -f^2 dt^2 \oplus g$. Suppose that (M, g, ζ, λ) is a Ricci soliton. Then $(\overline{M}, \overline{g}, \overline{\zeta}, \lambda)$ is a Ricci soliton where $\overline{\zeta} = u\partial_t + \zeta$ if

- (1) $H^f = 0$ and,
- (2) $\zeta(f) = (\lambda u')f$

The preceding corollary due to *Theorem 5* is about the implications of a gradient Ricci soliton standard static space-time.

Corollary 3. Let $\overline{M} =_f I \times M$ be a standard static space-time furnished with the metric $\overline{g} = -f^2 dt^2 \oplus g$. Suppose that $(\overline{M}, \overline{g}, u, \lambda)$ is a gradient Ricci soliton. Then

- (1) (M, g, ϕ_2, λ) is a gradient Ricci soliton with $\phi_2 = u_1 \ln f$ where $u_1 = u$ at some fixed point of I,
- (2) $(I, -dt^2, \phi_1, \lambda_1)$ is a gradient Ricci soliton with $\lambda_1 = \lambda f^2$ and $\phi_1 = u = t^2/2 + at + b$ for some a and b and also $\phi_1 = u$ at some fixed point of M when f is constant.

Now, we will obtain some results by computing the following fundamental Ricci soliton equation on a standard static space-time of the form $\overline{M} =_f I \times M$ with the metric $\overline{g} = -f^2 dt^2 \oplus g$. Suppose that $\overline{\zeta} = u\partial_t + \zeta$ is a vector field on \overline{M} where ζ is a vector field on M and $u: I \to \mathbb{R}$ is smooth.

(4.4)
$$\frac{1}{2}\mathcal{L}_{\bar{\zeta}}\bar{g} + \operatorname{Ric} = \lambda_{\bar{\zeta}}$$

Evaluating both sides of Equation 4.4 at (∂_t, ∂_t) , we have that

$$\Delta_M(f) = (u' - \lambda) f + \zeta(f)$$

It is clear that $\lambda = u'$ if f is constant. Equation 4.4 at (X, Y) where $X, Y \in \mathfrak{X}(M)$, yields

$$\frac{1}{2}\mathcal{L}_{\bar{\zeta}}^{M}g(X,Y) + \operatorname{Ric}^{M}(X,Y) = \lambda g(X,Y) + \frac{1}{f}\operatorname{H}^{f}(X,Y)$$

Moreover if ζ is conformal on M with factor 2ρ , then

$$\operatorname{Ric}^{M}(X,Y) = (\lambda - \rho) g(X,Y) + \frac{1}{f} \operatorname{H}^{f}(X,Y)$$

Thus M is Einstein if $H^f = 0$. By taking the trace of both sides one gets that

$$S = n\left(\lambda - \rho\right) + \frac{1}{f}\Delta_M(f)$$

Theorem 12. Let $\overline{M} =_f I \times M$ be a standard static space-time furnished with the metric $\overline{g} = -f^2 dt^2 \oplus g$. Suppose that $(\overline{M}, \overline{g}, \overline{\zeta}, \lambda)$ is a Ricci soliton. Then

$$\Delta_M(f) = (u' - \lambda) f + \zeta(f)$$

Moreover, if ζ is a conformal vector field on M with factor 2ρ , then the scalar curvature S of M is given by

$$S = n\left(\lambda - \rho\right) + \frac{1}{f}\Delta_M(f)$$

Corollary 4. Let $\overline{M} =_f I \times M$ be a standard static space-time furnished with the metric $\overline{g} = -f^2 dt^2 \oplus g$. Suppose that $(\overline{M}, \overline{g}, \overline{\zeta}, \lambda)$ is a Ricci soliton. Then M is Einstein if ζ is conformal on M and $H^f = 0$.

References

- D.E. Allison, Energy conditions in standard static space-times, General Relativity and Gravitation, 20 (1988) 115-122. 4.2
- [2] D.E. Allison, Geodesic Completeness in Static Space-times, Geometriae Dedicata 26 (1988) 85-97. 4.2
- [3] D.E. Allison and B. Ünal, Geodesic Structure of Standard Static Space-times, Journal of Geometry and Physics 46 (2003) 193-200. 4.2
- [4] A. L. Besse, *Einstein Manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, (2008). 1, 4.2
- [5] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. 145 (1969) 1-49. 2
- [6] Simon Brendle, Rotational symmetry of Ricci solitons in higher dimensions, Journal of Differential Geometry 97 (2014) 191-214.
- [7] M. Brozos-Vazquez, E. Garcia-Rio and S. Gavino-Fernandez, Locally Conformally Flat Lorentzian Gradient Ricci Solitons, J Geom Anal, 23 (2013) 1196-1212. 1
- Huai-Dong Cao and Detang Zhou, On complete gradient shrinking Ricci solitons, Journal of Differential Geometry, 85 (2010) 175-186. 1
- Bang-Yen Chen, Some results on concircular vector fields and their applications to Ricci solitons, Bull. Korean Math. Soc. 52 (2015), No. 5, pp. 1535–1547.
- [10] Bang-Yen Chen, Classification of torqued vector fields and its applications to Ricci solitons, Kragujevac Journal of Mathematics, Volume 41(2) (2017), Pages 239–250. 1
- [11] F. E. S. Feitosa, A. A. Freitas Filho, J. N. V. Gomes, On the construction of gradient Ricci soliton warped product, Nonlinear Analysis 161 (2017) 30–43. 1
- [12] Manuel Fernández-López, Eduardo García-Río, Rigidity of shrinking Ricci solitons, Mathematische Zeitschrift, 269 (2011) 461-466. 1
- [13] M. Brozos-Vázquez, E. Garcia-Rio and S. Gavino-Fernández, Locally conformally flat Lorentzian gradient Ricci solitons, J. Geom. Anal. 23 (2013) 1196-1212. 1
- [14] Fatma Karaca, Cihan Özgur, Gradient Ricci Solitons on Multiply Warped Product Manifolds, Filomat 32:12 (2018), 4221–4228 1
- [15] Byung Hak Kim, Sang Deok Lee, Jin Hyuk Choi, and Young Ok Lee, On warped product spaces with a certain Ricci condition, Bull. Korean Math. Soc. 50 (2013) 683-1691.
- [16] Jongsu Kim, Some doubly-warped product gradient Ricci solitons, Commun. Korean Math. Soc. 31 (2016), No. 3, pp. 625–635 1
- [17] Sang Deok Lee, Byung Hak Kim, and Jin Hyuk Choi, On a classification of warped product spaces with gradient Ricci solitons, Korean J. Math. 24 (2016), No. 4, pp. 627-636. 1
- [18] Sang Deok Lee, Byung Hak Kim, and Jin Hyuk Choi, Warped product spaces with Ricci conditions, Turk J. Math (2017) 41: 1365-1375. 1
- [19] H. Levy, Symmetric Tensors of the second order whose covariant derivatives vanish, Annals of Mathematics, Second Series, Vol. 27, no. 2, pp. 91-98, (1925). 3
- [20] Mancho Manev, Ricci-Like Solitons with Vertical Potential on Sasaki-Like Almost Contact B-Metric Manifolds, Results Math. 75, 136 (2020). 1
- [21] Ovidiu Munteanu, Natasa Sesum, On Gradient Ricci Solitons, Journal of Geometric Analysis 23 (2013) 539-561. 1
- [22] Peter Petersen and William Wylie, Rigidity of gradient Ricci solitons, Pacific Journal of Mathematics, 241 (2009) 329-345. 1
- [23] P. Petersen and W. Wylie, On the classification of gradient Ricci solitons, Geom. Topol. 14 (2010) 2277-2300. 1

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- [24] P. Petersen and W. Wylie, On gradient Ricci solitons with symmetry, Proc. Amer. Math. Soc., 137 (2009) 2085-2092. 1
- [25] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press Limited, London, 1983. 2
- [26] S. Shenawy and B. Ünal, 2-Killing vector fields on warped product manifolds, International Journal of Mathematics, 26 (2015) 1550065 2
- [27] M. Sánchez, On the Geometry of Generalized Robertson-Walker Spacetimes: geodesics, Gen. Relativ. Gravitation 30 (1998) 915-932. 4.1
- [28] M. Sánchez, On the Geometry of Generalized Robertson-Walker Spacetimes: Curvature and Killing fields, J. Geom. Phys. 31 (1999) 1-15. 4.1
- [29] Lemes de Sousa, M., Pina, R. Gradient Ricci Solitons with Structure of Warped Product, Results Math 71, (2017), 825-840.

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