# PERIMETER INEQUALITY UNDER CIRCULAR AND STEINER SYMMETRISATION: GEOMETRIC CHARACTERISATION OF EXTREMALS 

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#### Abstract

We study the perimeter inequality under circular symmetrisation, and we provide a full geometric characterisation of equality cases. A careful inspection of the proof shows that a similar characterisation holds true also for the perimeter inequality under Steiner symmetrisation. Our result is based on a new short proof of the perimeter inequality under symmetrisation.


## 1. Introduction

In this paper we give a geometric characterisation of the extremals of circular and Steiner perimeter inequalities.
1.1. Overview. Symmetrisation procedures have been proved to be important tools in mathematical analysis, and indeed they have been widely used to deduce geometric properties of minimizers of variational problems, and of solutions to PDEs. For instance, Steiner symmetrisation is a fundamental instrument in the proof by Ennio De Giorgi of the isoperimetric inequality (see [9, 10], and [13, Chapter 14]), while Schwarz symmetrisation was used to prove the classic Faber-Krahn inequality (see [12, Chapter II.8]).

Despite these techniques have been used for many decades, the detailed study of the equality cases for the perimeter inequalities under symmetrisation, is a relatively recent topic of investigation. One of the first results in this direction is due to De Giorgi: in his proof of the isoperimetric inequality he showed that if a set satisfies equality in Steiner's inequality, then it must be convex along the direction in which one performs the symmetrisation. After that, the problem of characterizing the equality cases for Steiner's inequality was resumed, and intensively investigated by Chlebík, Cianchi and Fusco. In their seminal work [7], the authors gave necessary conditions that a set must satisfy in order to be an extremal [7, Theorem 1.1], and provided sufficient conditions under which rigidity of equality cases holds true. Here, by rigidity we mean the case where the only sets achieving equality are those that are already symmetric (w.r.t. the symmetrisation procedure under consideration). The results obtained in [7] were successfully extended to the Steiner symmetrisation in any codimension in [2] (thus including also the Schwarz symmetrisation), but still no full characterisation of the cases of equality was proved.

Finally in [4], Cagnetti, Colombo, De Philippis and Maggi gave a full analytic characterisation of equality cases for the Steiner's inequality in terms of the properties of the barycenter function (see [4, Theorem 1.9]). Thanks to new tools introduced by the same authors in [5], they were able to further push the study of the rigidity, obtaining new important results. Still in the framework of Steiner symmetrisation, inspired by [4] and employing some general notions of convex analysis, the author was able to extend the analytic characterisation of equality cases to the anisotropic setting (see [14, Theorem 1.8]).

Despite a full characterisation of equality cases was successfully achieved for the Steiner's inequality, for other types of symmetrisation procedures such result is still missing. In particular, in the aforementioned work presented in [5], the authors were able to fully characterize the rigidity of equality cases for the Gaussian perimeter inequality under Ehrhard's symmetrization, but they only showed useful necessary conditions (not sufficient) for equality cases (see [5, Theorem A]). Lastly, a similar situation to the one just described for the Gaussian perimeter was obtained but in the setting of the perimeter inequality under spherical symmetrisation. Indeed in [6] the author together with Cagnetti and Stöger were also able to provide the full characterisation of the rigidity problem, but regarding the characterisation of extremals nothing more than a result that can be considered as the spherical counterpart of [7, Theorem 1.1] was achieved (see [6, Theorem 1.1]).

The analytic characterisations given in [4, Theorem 1.9] and [14, Theorem 1.8] have proven to be quite helpful in the study of rigidity. However, they can be quite difficult to use in specific situations, since they are expressed in terms of fine properties of the barycenter function of the one dimensional slices of the sets.

In this paper we present a geometric characterisation of extremals for the perimeter inequality under both circular and Steiner symmetrisation. Such characterisation is written in terms of geometric properties of the (measure-theoretic) inner unit normal $\nu^{E}$ to the set $E$ to which the symmetrisation is applied (see Theorem 1.3). In the Steiner setting, these properties appear easier to check than the analytic conditions given in [4, Theorem 1.9]. In the framework of circular symmetrisation, to the best of our knowledge this is the first characterisation result for the extremals of the perimeter inequality.

We will provide a detailed proof of our result for circular symmetrisation, and we will then show how this can be adapted to the Steiner setting. Inspired by [11, Section 4.1.5], we introduce a measure associated to the distribution function of the set under consideration (see (3.8)). This allows us to give a short and direct proof of the perimeter inequality and, in turn, to describe the extremals.

As far as we know, the circular symmetrisation for sets, and its application to rearrangements of functions, was firstly introduced by Pólya in [15] (see also [16, A.7-A8], and [12, Chapter II.9]). Let us now precisely introduce the circular symmetrisation for sets (see also Figure 1.1).
1.2. Circular symmetrisation for sets. Let us start presenting some of the notation we will use in this paper. Let $k \in \mathbb{N}$, with $k \geq 2$. We will decompose $\mathbb{R}^{k}$ as $\mathbb{R}^{2} \times \mathbb{R}^{k-2}$, and we will write $(x, z) \in \mathbb{R}^{k}$, with $x \in \mathbb{R}^{2}$ and $z \in \mathbb{R}^{k-2}$. We will denote by $|\cdot|$ the Euclidean norm of $\mathbb{R}, \mathbb{R}^{2}, \mathbb{R}^{k-2}$, $\mathbb{R}^{k}$, or the total variation of a Radon measure, depending on the context. For $d \in \mathbb{N}$, with $1 \leq d \leq k$ we denote by $\mathcal{H}^{d}$, and $\mathcal{L}^{d}$ the $d$-dimensional Hausdorff and Lebesgue measure in $\mathbb{R}^{k}$, respectively. We set $\mathbb{R}_{0}^{2}:=\mathbb{R}^{2} \backslash\{(0,0)\}, \mathbb{S}^{1}=\left\{x \in \mathbb{R}_{0}^{2}:|x|=1\right\}$, and $\mathbb{S}^{k-1}=\left\{(x, z) \in \mathbb{R}^{k}:|(x, z)|=1\right\}$. Moreover, given $r>0$ we write $\partial B(r)=\left\{x \in \mathbb{R}^{2}:|x|=r\right\}$ to denote the boundary of the 2 -dimensional ball centered at the origin with radius $r$. Lastly, for every $x \in \mathbb{R}_{0}^{2}$ we set $\hat{x}=x /|x|$.

We are now going to define the circular symmetral of a Borel set in $\mathbb{R}^{k}$ with respect to the half-hyperplane $\left\{\left(x_{1}, x_{2}, z_{1}, \ldots, z_{k-2}\right) \in \mathbb{R}^{k}: x_{1}>0, x_{2}=0\right\}=(0, \infty) \times\{0\} \times \mathbb{R}^{k-2}$. For every Borel set $E \subset \mathbb{R}^{k}$ we define

$$
\begin{equation*}
E_{(r, z)}:=\left\{x \in \mathbb{R}_{0}^{2}:|x|=r \text { and }(x, z) \in E\right\} \subset \partial B(r) \quad \text { for every }(r, z) \in(0, \infty) \times \mathbb{R}^{k-2} \tag{1.1}
\end{equation*}
$$

Note that, by definition, we have

$$
0 \leq \mathcal{H}^{1}\left(E_{(r, z)}\right) \leq 2 \pi r, \quad \text { for every }(r, z) \in(0, \infty) \times \mathbb{R}^{k-2}
$$

Let now $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying

$$
\begin{equation*}
0 \leq \mu(r, z) \leq 2 \pi r, \quad \text { for } \mathcal{L}^{k-1} \text {-a.e. }(r, z) \in(0, \infty) \times \mathbb{R}^{k-2} \tag{1.2}
\end{equation*}
$$

We will say that $E$ is $\mu$-distributed if

$$
\mu(r, z)=\mathcal{H}^{1}\left(E_{(r, z)}\right), \quad \text { for } \mathcal{L}^{k-1} \text {-a.e. }(r, z) \in(0, \infty) \times \mathbb{R}^{k-2}
$$

Given a Lebesgue measurable function $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ satisfying (1.2), we define the set $F_{\mu} \subset \mathbb{R}^{k}$ as

$$
\begin{equation*}
F_{\mu}:=\left\{(x, z) \in \mathbb{R}_{0}^{2} \times \mathbb{R}^{k-2}: 2|x| \arccos \left(\hat{x} \cdot e_{1}\right)<\mu(|x|, z)\right\} \tag{1.3}
\end{equation*}
$$

where $e_{1} \in \mathbb{R}^{2}$ is defined as $e_{1}=(1,0)$.
Remark 1.1. Note that by definition of $F_{\mu}$, we have

$$
(x, z) \in F_{\mu} \Longrightarrow(w, z) \in F_{\mu} \quad \forall w \text { with }|w|=|x| \operatorname{and} \arccos \left(\hat{w} \cdot e_{1}\right) \leq \arccos \left(\hat{x} \cdot e_{1}\right)
$$

If $E \subset \mathbb{R}^{k}$ is a $\mu$-distributed Borel set, we say that $F_{\mu}$ is the circular symmetral of $E$ with respect to the half-hyperplane $\left\{\left(x_{1}, x_{2}, z_{1}, \ldots, z_{k-2}\right) \in \mathbb{R}^{k}: x_{1}>0, x_{2}=0\right\}$.
There is a particular bond between circular symmetrisation and Steiner symmetrisation. Firstly, both symmetrisation techniques act by slicing sets with lines of dimension 1 , and second, as observed by Pólya and Szegö themselves, the limit as $c \rightarrow-\infty$ of the circular symmetrisation of a set $E$ w.r.t. the half-hyperplane $\left\{\left(x_{1}, x_{2}, z_{1}, \ldots, z_{k-2}\right) \in \mathbb{R}^{k}: x_{1}>c, x_{2}=0\right\}$ "tends" to the Steiner symmetrisation of $E$ w.r.t. the full hyperplane $\left\{\left(x_{1}, x_{2}, z_{1}, \ldots, z_{k-2}\right) \in \mathbb{R}^{k}: x_{2}=0\right\}$.


Figure 1.1. A pictorial representation in $\mathbb{R}^{3}$ of a $\mu$-distributed set $E$ and of its circular symmetral $F_{\mu}$.
1.3. Main results. Let us now present the main results of this work. The first result consists in a precise description of the geometric properties of the symmetral set $F_{\mu}$ defined in (1.3). In the following, given any set $E \subset \mathbb{R}^{k}$ of locally finite perimeter we denote with $\partial^{*} E$ and with $\nu^{E}(x, z)$ the reduced boundary of $E$, and the (measure-theoretic) inner unit normal to $\partial^{*} E$ at $(x, z)$, respectively (see Section 2 for more details). Given any vector $\nu \in \mathbb{R}^{k}$ we set

$$
\nu_{x}=\left(\nu_{1}, \nu_{2}\right), \quad \nu_{z}=\left(\nu_{3}, \nu_{4}, \ldots, \nu_{k}\right)
$$

In particular, given any vector field $\nu: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$ we use the following notation:

$$
\begin{equation*}
\nu_{x \perp}(x, z)=\left(\hat{x} \cdot \nu_{x}(x, z)\right) \hat{x}, \quad \nu_{x \|}(x, z)=\nu_{x}(x, z)-\nu_{x \perp}(x, z) \quad \forall(x, z) \in \mathbb{R}_{0}^{2} \times \mathbb{R}^{k-2} \tag{1.4}
\end{equation*}
$$

Given $E \subset \mathbb{R}^{k}$ set of locally finite perimeter, we set

$$
\begin{equation*}
\nu_{\mathrm{c}}^{E}(x, z):=\left(\hat{x} \cdot \nu_{x}^{E}(x, z),\left|\nu_{x \|}^{E}(x, z)\right|, \nu_{z}^{E}(x, z)\right), \quad \text { for } \mathcal{H}^{k} \text {-a.e. }(x, z) \in \partial^{*} E \cap\left(\mathbb{R}_{0}^{2} \times \mathbb{R}^{k-2}\right) \tag{1.5}
\end{equation*}
$$

First of all, we show some useful symmetry properties of the (measure-theoretic) inner unit normal $\nu^{F_{\mu}}$ of $F_{\mu}$.
Proposition 1.2. Let $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying (1.2) such that $F_{\mu}$ is a set of locally finite perimeter. Then, for every $(r, z) \in(0, \infty) \times \mathbb{R}^{k-2}$ such that $\left(\partial^{*} F_{\mu}\right)_{(r, z)} \neq \emptyset$, the functions

$$
\begin{equation*}
x \mapsto \hat{x} \cdot \nu_{x}^{F_{\mu}}(x, z), \quad x \mapsto\left|\nu_{x \|}^{F_{\mu}}(x, z)\right|, \quad x \mapsto \nu_{z}^{F_{\mu}}(x, z), \tag{1.6}
\end{equation*}
$$

are constant in $\left(\partial^{*} F_{\mu}\right)_{(r, z)}$, that is $x \mapsto \nu_{\mathrm{c}}^{F_{\mu}}(x, z)$ is constant in $\left(\partial^{*} F_{\mu}\right)_{(r, z)}$.
We observe that a weaker version of the above result in the Steiner setting was already known (see [2, Remark 2.5]). Let us now introduce some further notation that we will need in order to state the next theorem. Thanks to Proposition 1.2 we can define the Borel vector field $\bar{\nu}_{\mathrm{c}}^{F_{\mu}}$ : $(0, \infty) \times \mathbb{R}^{k-2} \rightarrow \mathbb{R}^{k}$ as

$$
\bar{\nu}_{\mathrm{c}}^{F_{\mu}}(r, z)= \begin{cases}\nu_{\mathrm{c}}^{F_{\mu}}(x, z) & \text { if }\left(\partial^{*} F_{\mu}\right)_{(r, z)} \neq \emptyset, \text { and } x \in\left(\partial^{*} F_{\mu}\right)_{(r, z)}  \tag{1.7}\\ 0 & \text { otherwise }\end{cases}
$$

Proposition 1.2 is the new ingredient for the characterisation of equality cases for the perimeter inequality under circular symmetrisation. In the following, we define the diffeomorphism $\Phi$ : $(0, \infty) \times \mathbb{R}^{k-2} \times \mathbb{S}^{1} \rightarrow \mathbb{R}_{0}^{2} \times \mathbb{R}^{k-2}$ as:

$$
\Phi(r, z, \omega):=(r \omega, z) \quad \text { for every }(r, z, \omega) \in(0, \infty) \times \mathbb{R}^{k-2} \times \mathbb{S}^{1}
$$

Thus, more in general, for every Borel set $B \subset(0, \infty) \times \mathbb{R}^{k-2}$, we set

$$
\Phi\left(B \times \mathbb{S}^{1}\right):=\left\{(x, z) \in \mathbb{R}^{k}:(|x|, z) \in B\right\}
$$

We can now state our main result.
Theorem 1.3. Let $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying (1.2), let $U \subset(0, \infty) \times \mathbb{R}^{k-2}$ be an open set, and let $E \subset \mathbb{R}^{k}$ be a $\mu$-distributed set such that $E$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. Then, $F_{\mu}$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$ and

$$
\begin{equation*}
P\left(F_{\mu} ; \Phi\left(B \times \mathbb{S}^{1}\right)\right) \leq P\left(E ; \Phi\left(B \times \mathbb{S}^{1}\right)\right), \quad \forall B \subset U \text { Borel. } \tag{1.8}
\end{equation*}
$$

Moreover, equality holds in (1.8) for some Borel set $B \subset U$ if and only if both the following two conditions are satisfied.
a) For $\mathcal{L}^{k-1}$-a.e. $(r, z) \in B$ we have that $(E)_{(r, z)}$ is $\mathcal{H}^{1}$-equivalent to a connected arc in $\mathbb{R}^{2}$.
b) There exists $N \subset \partial^{*} E$ with $\mathcal{H}^{k-1}(N)=0$, with the property that for every $(r, z) \in B$ such that $\left(\partial^{*} E \backslash N\right)_{(r, z)} \neq \emptyset$, and $\left(\partial^{*} F_{\mu}\right)_{(r, z)} \neq \emptyset$, we have that

$$
\begin{equation*}
\nu_{\mathrm{c}}^{E}(x, z)=\bar{\nu}_{\mathrm{c}}^{F_{\mu}}(r, z) \quad \forall x \in\left(\partial^{*} E \backslash N\right)_{(r, z)} . \tag{1.9}
\end{equation*}
$$

Remark 1.4. By definition of $\nu_{\mathrm{c}}^{E}$, condition b) of the above result implies that for every $(r, z) \in B$ such that $\left(\partial^{*} E \backslash N\right)_{(r, z)} \neq \emptyset$, and $\left(\partial^{*} F_{\mu}\right)_{(r, z)} \neq \emptyset$ the functions

$$
x \mapsto \hat{x} \cdot \nu_{x}^{E}(x, z), \quad x \mapsto\left|\nu_{x \|}^{E}(x, z)\right|, \quad x \mapsto \nu_{z}^{E}(x, z),
$$

are constant in $\left(\partial^{*} E \backslash N\right)_{(r, z)}$.
Roughly speaking, we can say that condition b) of Theorem 1.3 holds true if and only if the symmetric properties of $\nu^{F_{\mu}}$ described by Proposition 1.2 holds true also for $\nu^{E}$. Let us point out that in [6, Theorem 1.4] condition a) and a weaker version of condition b) were shown to be necessary condition for a set $E$ to be an extremal of (1.8). In particular, condition b) of [6, Theorem 1.4] (see also condition b) of [2, Theorem 1.1]) was only discussed for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in B$, and no information was given on the $\mathcal{L}^{k-1}$-negligible subset of $B$ where coarea formula cannot be used. In order to clarify the meaning of condition b) of Theorem 1.3, let us give some examples.


Figure 1.2. A pictorial representation of a $\mu$-distributed set $E \subset \mathbb{R}^{3}$ that satisfies both conditions a) and b) of Theorem 1.3, thus being an equality case for (1.8).

Example 1.5 (Case of equality). Let us explain with an example in $\mathbb{R}^{3}$ the meaning of condition b) of Theorem 1.3. In Figure 1.2 we marked in green the sets $\left(\left(\partial^{*} E_{(r, z)}\right), z\right)$, and $\left(\left(\partial^{*} F_{\mu}\right)_{(r, z)}, z\right)$ for some $(r, z) \in(0, \infty) \times \mathbb{R}$ (in the picture, with a little abuse of notation, we simply call


Figure 1.3. A pictorial representation of a $\mu$-distributed set $E_{1} \subset \mathbb{R}^{3}$ that does not satisfy condition $b$ ) of Theorem 1.3.
them $\left(\partial^{*} E\right)_{(r, z)}$, and $\left(\partial^{*} F_{\mu}\right)_{(r, z)}$, respectively). In the picture in the left one can notice the $\mathcal{H}^{2}$ negligible set $N$, which is represented by the blue dashed vertical line. Let us point out that the isolated green dot appearing in the left picture is indeed part of $\left(\left(\partial^{*} E_{(r, z)}\right), z\right)$, and it coincides with $\left(\left(\partial^{*} E_{(r, z)}\right), z\right) \cap N$. It can be shown that $\nu_{c}^{E}$ evaluated at that isolated point differs from $\nu_{\mathrm{c}}^{E}$ evaluated at any other point of $\left(\left(\partial^{*} E_{(r, z)}\right), z\right) \backslash N$. Nonetheless, $\nu_{\mathrm{c}}^{E}$ is constant when restricted to $\left(\left(\partial^{*} E_{(r, z)}\right), z\right) \backslash N$ and it coincides with $\nu_{\mathrm{c}}^{F_{\mu}}(x, z)$ restricted to $\left(\left(\partial^{*} F_{\mu}\right)_{(r, z)}, z\right)$, namely $\nu_{\mathrm{c}}^{E}(x, z)=\bar{\nu}_{\mathrm{c}}^{F_{\mu}}(r, z)$ for all $x \in\left(\partial^{*} E \backslash N\right)_{(r, z)}$. Thus, condition b) of Theorem 1.3 holds true.
Example 1.6 (Non equality case). In Figure 1.3 we show an example of a $\mu$-distributed set $E_{1} \subset \mathbb{R}^{3}$ that does not satisfy condition b) of Theorem 1.3. Indeed, it can be shown that $\nu_{\mathrm{c}}^{E_{1}}(x, z)$ changes depending on weather $(x, z)$ belongs to the green or to the red part of $\left(\left(\partial^{*} E_{1}\right)_{(r, z)}, z\right)$. Note that this phenomenon cannot be avoided by removing an $\mathcal{H}^{2}$-negligible set from $\partial^{*} E_{1}$. Thus, condition b) is not satisfied and therefore $E_{1}$ is not an extremal of (1.8). Let us stress that, despite the set $E_{1}$ does not satisfy condition b) of Theorem 1.3, it does satisfy all the necessary conditions in order to be a case of equality for (1.8) that are listed in [6, Theorem 1.4].

Theorem 1.3 is a refinement of [6, Theorem 1.4], where the inequality (1.8) was already stated without an explicit proof. Let us stress that, apart from some technical intermediate results, the arguments we use to prove Theorem 1.3 differ from the standard ones used while proving perimeter inequalities under symmetrisation (see once more [7, Theorem 1.1], and [6, Theorem 1.1]), and deeply rely on the new information given by Proposition 1.2 about the symmetral set $F_{\mu}$. Indeed, as a consequence of that, our proof of (1.8) is much more direct, and leads quite simply to the characterisation of the equality cases.

Finally, we are able to show that analogous results hold true for the Steiner symmetrisation (see Theorem 5.9). In fact, we believe that our short proof of (1.8) and the techniques we used to show Theorem 1.3 can be adapted to other symmetrisation procedures, and that they can be helpful in simplifying the study of rigidity of perimeter inequality under symmetrisation.
Structure of the paper. The paper is divided as follows. In Section 2 we recall some basic notions of geometric measure theory and functions of bounded variation. In Section 3 for the reader convenience we start off by stating once more the precise notation we will use throughout the paper, and then we focus on proving Proposition 1.2 and other technical results we will need later on. In Section 4 we present the proof of Theorem 1.3. Lastly, in Section 5 we state, without
proofs, the Steiner counterpart of the results we obtained for the circular symmetrisation, thus including a Steiner version of both Proposition 1.2, and Theorem 1.3 (see Proposition 5.5, and Theorem 5.9, respectively).

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## 2. Fundamentals of geometric measure theory

The aim of this section is to introduce some basic concepts of Geometric Measure Theory that will be largely used in the article. For more details the reader can have a look in the monographs $[1,11,13,17]$. For $(x, z) \in \mathbb{R}^{k}$ and $\nu \in \mathbb{S}^{k-1}$, we will denote by $H_{(x, z), \nu}^{+}$and $H_{(x, z), \nu}^{-}$the closed half-spaces whose boundaries are orthogonal to $\nu$ :
$H_{(x, z), \nu}^{+}:=\left\{(\bar{x}, \bar{z}) \in \mathbb{R}^{k}:(x-\bar{x}, z-\bar{z}) \cdot \nu \geq 0\right\}, \quad H_{(x, z), \nu}^{-}:=\left\{(\bar{x}, \bar{z}) \in \mathbb{R}^{k}:(x-\bar{x}, z-\bar{z}) \cdot \nu \leq 0\right\}$.
In the following, given a measurable set $E \subset \mathbb{R}^{k}$ we will denote by $\chi_{E}$ its characteristic function, while the $k$-dimensional ball of $\mathbb{R}^{k}$ of radius $r>0$ and center in $(x, z)$ is denoted with $B_{r}(x, z)$.
2.1. Density points. Let $E \subset \mathbb{R}^{k}$ be a Lebesgue measurable set and let $(x, z) \in \mathbb{R}^{k}$. The upper and lower $k$-dimensional densities of $E$ at $(x, z)$ are defined as

$$
\theta^{*}(E,(x, z)):=\limsup _{\rho \rightarrow 0^{+}} \frac{\mathcal{H}^{k}\left(E \cap B_{\rho}(x, z)\right)}{\omega_{k} \rho^{k}}, \quad \theta_{*}(E,(x, z)):=\liminf _{\rho \rightarrow 0^{+}} \frac{\mathcal{H}^{k}\left(E \cap B_{\rho}(x, z)\right)}{\omega_{k} \rho^{k}}
$$

respectively, where $\omega_{k} \rho^{k}=\mathcal{H}^{k}\left(B_{\rho}(x, z)\right)$. It turns out that $(x, z) \mapsto \theta^{*}(E,(x, z))$ and $(x, z) \mapsto$ $\theta_{*}(E,(x, z))$ are Borel functions that agree $\mathcal{H}^{k}$-a.e. on $\mathbb{R}^{k}$. Therefore, the $k$-dimensional density of $E$ at $(x, z)$

$$
\theta(E,(x, z)):=\lim _{\rho \rightarrow 0^{+}} \frac{\mathcal{H}^{k}\left(E \cap B_{\rho}(x, z)\right)}{\omega_{k} \rho^{k}}
$$

is defined for $\mathcal{H}^{k}$-a.e. $(x, z) \in \mathbb{R}^{k}$, and $(x, z) \mapsto \theta(E,(x, z))$ is a Borel function on $\mathbb{R}^{k}$. Given $t \in[0,1]$, we set

$$
E^{(t)}:=\left\{(x, z) \in \mathbb{R}^{k}: \theta(E,(x, z))=t\right\}
$$

The set $\partial^{e} E:=\mathbb{R}^{n} \backslash\left(E^{(0)} \cup E^{(1)}\right)$ is called the essential boundary of $E$.
2.2. Functions of bounded variation. Let $f:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow \mathbb{R}$ be a Lebesgue measurable function, and let $\Omega \subset(0, \infty) \times \mathbb{R}^{k-2}$ be open, such that $f \in L^{1}(\Omega)$. Then we say that $f$ is of bounded variation in $\Omega$, and we write $f \in B V(\Omega)$ if and only if

$$
\begin{equation*}
\sup \left\{\int_{\Omega} f(r, z) \operatorname{div} T(r, z) d r d z: T \in C_{c}^{1}\left(\Omega ; \mathbb{R}^{k-1}\right),|T| \leq 1\right\}<\infty \tag{2.1}
\end{equation*}
$$

where $C_{c}^{1}\left(\Omega ; \mathbb{R}^{k-1}\right)$ is the set of $C^{1}$ functions from $\Omega$ to $\mathbb{R}^{k-1}$ with compact support. More in general, we say that $f \in B V_{\text {loc }}(\Omega)$ if $f \in B V\left(\Omega^{\prime}\right)$ for every open set $\Omega^{\prime}$ compactly contained in $\Omega$. If $f \in B V_{\text {loc }}(\Omega)$ the distributional derivative $D f$ of $f$ is representable as a $\mathbb{R}^{k-1}$-valued Radon measure defined on $\Omega$, and its total variation $|D f|$ is finite in $\Omega$, and its value $|D f|(\Omega)$ coincides with (2.1). Moreover, for every $T \in C_{c}^{1}\left(\Omega ; \mathbb{R}^{k-1}\right)$ we have

$$
\int_{\Omega} f(r, z) \operatorname{div} T(r, z) d r d z=-\int_{\Omega} T(r, z) \cdot d D f(r, z)
$$

One can write the Radon-Nykodim decomposition of $D f$ with respect to $\mathcal{L}^{k-1}$ as $D f=D^{a} f+D^{s} f$, where $D^{s} f$ and $\mathcal{L}^{k-1}$ are mutually singular, and where $D^{a} f \ll \mathcal{L}^{k-1}$. We denote the density of $D^{a} f$ with respect to $\mathcal{L}^{k-1}$ by $\nabla f$, so that $\nabla f \in L^{1}\left(\Omega ; \mathbb{R}^{k-1}\right)$ with $D^{a} f=\nabla f d \mathcal{L}^{k-1}$. Moreover, for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in \Omega, \nabla f(r, z)$ is the approximate differential of $f$ at $(r, z)$.
2.3. Sets of finite perimeter. Let $E \subset \mathbb{R}^{k}$ be a Lebesgue measurable set, and let $O \subset \mathbb{R}^{k}$ be an open set. We say that $E \subset \mathbb{R}^{k}$ is a set of finite perimeter in $O$ if and only if

$$
\begin{equation*}
\sup \left\{\int_{\mathbb{R}^{k}} \chi_{E}(x, z) \operatorname{div}_{(x, z)} T(x, z) d x d z: T \in C_{c}^{1}\left(O ; \mathbb{R}^{k}\right)\right\}<\infty \tag{2.2}
\end{equation*}
$$

where by $\operatorname{div}_{(x, z)}$ we mean the classical divergence in $\mathbb{R}^{k}$ w.r.t. the variables $(x, z)$. If $E \subset \mathbb{R}^{k}$ is a set of finite perimeter in $O$, we denote with $P(E ; O)$ its relative perimeter in $O$, where $P(E ; O)$ coincides with the quantity in (2.2). If $P(E):=P\left(E ; \mathbb{R}^{k}\right)<\infty$ we say that $E$ is a set of finite perimeter, while more generally if $P(E ; V)<\infty$ for every $V \subset \subset O$, we say that $E$ is a set of locally finite perimeter in $O$. If $E \subset \mathbb{R}^{k}$ is a set of finite perimeter and finite volume in $O$, then we have that $\chi_{E} \in B V(O)$, while in general if $E \subset \mathbb{R}^{k}$ is a set of finite perimeter in $O$ then $\chi_{E} \in B V_{\text {loc }}(O)$. Moreover, if $E \subset \mathbb{R}^{k}$ is a set of locally finite perimeter in $O$ we define the reduced boundary $\partial^{*} E \subset \mathbb{R}^{k}$ of $E$ as the set of those points such that

$$
\nu^{E}(x, z):=\lim _{\rho \rightarrow 0^{+}} \frac{D \chi_{E}\left(B_{\rho}(x, z)\right)}{\left|D \chi_{E}\right|\left(B_{\rho}(x, z)\right)}
$$

exists and belongs to $\mathbb{S}^{k-1}$. The Borel function $\nu^{E}: \partial^{*} E \rightarrow \mathbb{S}^{k-1}$ is called the (measure-theoretic) inner unit normal to $E$. Given $E \subset \mathbb{R}^{k}$ set of locally finite perimeter in $O$, we have that $D \chi_{E}=$ $\nu^{E} \mathcal{H}^{k-1}\left\llcorner\left(\partial^{*} E \cap O\right)\right.$ and,

$$
\int_{\mathbb{R}^{k}} \chi_{E}(x, z) \operatorname{div}_{(x, z)} T(x, z) d x d z=-\int_{\partial^{*} E \cap O} T(x, z) \cdot \nu^{E}(x, z) d \mathcal{H}^{k-1}(x, z), \quad \forall T \in C_{c}^{1}\left(O ; \mathbb{R}^{k}\right)
$$

The relative perimeter of $E$ in $A \subset O$ is then defined by

$$
P(E ; A):=\left|D \chi_{E}\right|(A)=\mathcal{H}^{k-1}\left(\partial^{*} E \cap A\right)
$$

for every Borel set $A \subset O$. If $E$ is a set of locally finite perimeter in $O$, it turns out that

$$
\left(\partial^{*} E \cap O\right) \subset\left(E^{(1 / 2)} \cap O\right) \subset\left(\partial^{\mathrm{e}} E \cap O\right)
$$

Moreover, Federer's theorem holds true (see [1, Theorem 3.61] and [13, Theorem 16.2]):

$$
\mathcal{H}^{n-1}\left(\left(\partial^{e} E \cap O\right) \backslash\left(\partial^{*} E \cap O\right)\right)=0 .
$$

## 3. Properties of $F_{\mu}$ and $\mu$

We start this section stating two important results. The first one, is a special case of Coarea Formula (see [6, Proposition 6.1], and [13, Theorem 18.8]). In the following, given $O \subset \mathbb{R}^{k}$ open set, and given $E \subset \mathbb{R}^{k}$ set of locally finite perimeter in $O$, we denote with $L^{1}\left(\mathbb{R}^{k}, \mathcal{H}^{k-1}\left\llcorner\partial^{*} E \cap O\right)\right.$ the space of integrable functions from $\mathbb{R}^{k}$ to $\mathbb{R}$ w.r.t. the Radon measure $\mathcal{H}^{k-1} L \partial^{*} E \cap O$.

Proposition 3.1. Let $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying (1.2), let $U \subset(0, \infty) \times \mathbb{R}^{k-2}$ be an open set, and let $E \subset \mathbb{R}^{k}$ be a $\mu$-distributed set such that $E$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. Let $g: \mathbb{R}^{k} \rightarrow[-\infty, \infty]$ be a Borel function, such that either $g \geq 0$ on $\partial^{*} E \cap \Phi\left(U \times \mathbb{S}^{1}\right)$, or $g \in L^{1}\left(\mathbb{R}^{k}, \mathcal{H}^{k-1}\left\llcorner\partial^{*} E \cap \Phi\left(U \times \mathbb{S}^{1}\right)\right)\right.$. Then,

$$
\int_{\partial^{*} E \cap \Phi\left(U \times \mathbb{S}^{1}\right)} g(x, z)\left|\nu_{x \|}^{E}(x, z)\right| d \mathcal{H}^{k-1}(x, z)=\int_{U} d r d z \int_{\left(\partial^{*} E\right)_{(r, z)}} g(x, z) d \mathcal{H}^{0}(x)
$$

Next result is about circular one-dimensional slices of sets of finite perimeter (see [6, Theorem $6.2]$ ), and it can be seen as the circular counterpart of a classic result by Vol'pert (see [18], and [8, Theorem D]).
Proposition 3.2 (Vol'pert). Let $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying (1.2), let $U \subset(0, \infty) \times \mathbb{R}^{k-2}$ be an open set, and let $E \subset \mathbb{R}^{k}$ be a $\mu$-distributed set such that $E$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. Then, there exists a Borel set $G_{E} \subset(\{\mu>0\} \cap U)$ with $\mathcal{L}^{k-1}\left((\{\mu>0\} \cap U) \backslash G_{E}\right)=0$ such that the following properties hold true:
(i) for every $(r, z) \in G_{E}$ :
(ia) $E_{(r, z)}$ is a set of finite perimeter in $\partial B(r)$;
(ib) $\partial^{*}\left(E_{(r, z)}\right)=\left(\partial^{*} E\right)_{(r, z)}$;

We are now going to show some useful properties of the symmetric set $F_{\mu}$. Such properties (see Proposition 1.2) are the new fundamental ingredient with which proving the perimeter inequality under circular symmetrisation, and then characterize the cases of equality. In the following, for every $\gamma \in[-2 \pi, 2 \pi]$ we define $R_{\gamma}$ as the the counterclockwise rotation of an angle $\gamma$ in the plane $\left(x_{1}, x_{2}\right)$. Lastly, for every $(r, z) \in(0, \infty) \times \mathbb{R}^{k-2}$ we set

$$
\begin{equation*}
\partial B_{r}^{2}(0, z):=\left\{(x, z) \in \mathbb{R}_{0}^{2} \times \mathbb{R}^{k-2}:|x|=r\right\} \tag{3.1}
\end{equation*}
$$

Roughly speaking, $\partial B_{r}^{2}(0, z)$ stands for the 1-dimensional circle in $\mathbb{R}_{0}^{2} \times\{z\} \subset \mathbb{R}^{k}$ centered in $(0, z)$ and having radius $r$.

Lemma 3.3. Let $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying (1.2). Let $(r, z) \in(0, \infty) \times \mathbb{R}^{k-2}$, and set $x_{r}:=(r, 0)$. Then, the functions

$$
\gamma \mapsto \theta_{*}\left(F_{\mu},\left(R_{\gamma} x_{r}, z\right)\right) \quad \text { and } \quad \gamma \mapsto \theta^{*}\left(F_{\mu},\left(R_{\gamma} x_{r}, z\right)\right)
$$

are even in $[-\pi, \pi]$ and non increasing in $[0, \pi]$.
Proof. The fact that $\gamma \mapsto \theta_{*}\left(F_{\mu},\left(R_{\gamma} x_{r}, z\right)\right)$ and $\gamma \mapsto \theta^{*}\left(F_{\mu},\left(R_{\gamma} x_{r}, z\right)\right)$ are even in $[-\pi, \pi]$ follows directly from Remark 1.1. We now divide the rest of the proof into two steps.
Step 1: We show that, if $0 \leq \gamma_{1}<\gamma_{2} \leq \pi$, and $\rho>0$ is so small that

$$
\begin{equation*}
B_{\rho}\left(\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap B_{\rho}\left(\left(R_{\gamma_{2}} x_{r}, z\right)\right)=\emptyset \tag{3.2}
\end{equation*}
$$

then for every $(\lambda, \bar{z}) \in(0, \infty) \times \mathbb{R}^{k-2}$ one has

$$
\begin{align*}
& \mathcal{H}^{1}\left(F_{\mu} \cap \partial B_{\lambda}^{2}(0, \bar{z}) \cap B_{\rho}\left(\left(R_{\gamma_{2}} x_{r}, z\right)\right) \cap T_{\gamma_{2}} A\right) \\
& \leq \mathcal{H}^{1}\left(F_{\mu} \cap \partial B_{\lambda}^{2}(0, \bar{z}) \cap B_{\rho}\left(\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap T_{\gamma_{1}} A\right) \tag{3.3}
\end{align*}
$$

for every $A \subset \mathbb{R}^{k}$ where, for every $\gamma \in[-\pi, \pi]$, we set

$$
T_{\gamma}(x, z):=\left(R_{\gamma} x, z\right)
$$

If $F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma_{2}} x_{r}, z\right)\right) \cap \partial B_{\lambda}^{2}(0, \bar{z})=\emptyset$, the left hand side of (3.3) equals 0 and therefore the inequality is satisfied. Instead, suppose that

$$
F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma_{2}} x_{r}, z\right)\right) \cap \partial B_{\lambda}^{2}(0, \bar{z}) \neq \emptyset .
$$

Then, from (3.2) and Remark 1.1 we have

$$
F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap \partial B_{\lambda}^{2}(0, \bar{z})=B_{\rho}\left(\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap \partial B_{\lambda}^{2}(0, \bar{z})
$$

Therefore,

$$
\begin{aligned}
& \mathcal{H}^{1}\left(F_{\mu} \cap \partial B_{\lambda}^{2}(0, \bar{z}) \cap B_{\rho}\left(\left(R_{\gamma_{2}} x_{r}, z\right)\right) \cap T_{\gamma_{2}} A\right) \\
& \leq \mathcal{H}^{1}\left(\partial B_{\lambda}^{2}(0, \bar{z}) \cap B_{\rho}\left(\left(R_{\gamma_{2}} x_{r}, z\right)\right) \cap T_{\gamma_{2}} A\right) \\
& =\mathcal{H}^{1}\left(\partial B_{\lambda}^{2}(0, \bar{z}) \cap B_{\rho}\left(\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap T_{\gamma_{1}} A\right) \\
& =\mathcal{H}^{1}\left(F_{\mu} \cap \partial B_{\lambda}^{2}(0, \bar{z}) \cap B_{\rho}\left(\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap T_{\gamma_{1}} A\right)
\end{aligned}
$$

which gives (3.3).
Step 2: We will show that if $0 \leq \gamma_{1}<\gamma_{2} \leq \pi$, then

$$
\theta_{*}\left(F_{\mu},\left(R_{\gamma_{2}} x_{r}, z\right)\right) \leq \theta_{*}\left(F_{\mu},\left(R_{\gamma_{1}} x_{r}, z\right)\right)
$$

and

$$
\theta^{*}\left(F_{\mu},\left(R_{\gamma_{2}} x_{r}, z\right)\right) \leq \theta^{*}\left(F_{\mu},\left(R_{\gamma_{1}} x_{r}, z\right)\right)
$$

Let $\rho>0$ be such that (3.2) is satisfied. Then,

$$
\begin{aligned}
& \left.\mathcal{H}^{k}\left(B_{\rho}\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap F_{\mu}\right)=\int_{B_{\rho}\left(\left(R_{\gamma_{1}} x_{r}, z\right)\right)} \chi_{F_{\mu}}(\bar{x}, \bar{z}) d \mathcal{H}^{k}(\bar{x}, \bar{z}) \\
& =\int_{\mathbb{R}^{k-2}} \int_{r-\rho}^{r+\rho} \mathcal{H}^{1}\left(F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap \partial B_{\lambda}^{2}(0, \bar{z})\right) d \lambda d \bar{z} \\
& \geq \int_{\mathbb{R}^{k-2}} \int_{r-\rho}^{r+\rho} \mathcal{H}^{1}\left(F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma_{2}} x_{r}, z\right)\right) \cap \partial B_{\lambda}^{2}(0, \bar{z})\right) d \lambda d \bar{z} \\
& =\mathcal{H}^{k}\left(B_{\rho}\left(\left(R_{\gamma_{2}} x_{r}, z\right)\right) \cap F_{\mu}\right),
\end{aligned}
$$

where the inequality follows from (3.3) with $A=\mathbb{R}^{k}$. Thus,

$$
\frac{\left.\mathcal{H}^{k}\left(B_{\rho}\left(R_{\gamma_{1}} x_{r}, z\right)\right) \cap F_{\mu}\right)}{\omega_{k} \rho^{k}} \geq \frac{\left.\mathcal{H}^{k}\left(B_{\rho}\left(R_{\gamma_{2}} x_{r}, z\right)\right) \cap F_{\mu}\right)}{\omega_{k} \rho^{k}} .
$$

Passing to the liminf and the limsup as $\rho \rightarrow 0^{+}$, the conclusion follows.
Proposition 3.4. Let $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying (1.2) such that $F_{\mu}$ is a set of locally finite perimeter. Suppose that $(x, z) \in \partial^{*} F_{\mu}$, and let $r \in(0, \infty)$ and $\beta \in(-\pi, \pi]$ be such that $x=r(\cos \beta, \sin \beta)$. Then,

$$
\begin{equation*}
\nu^{F_{\mu}}\left(R_{\gamma} x, z\right)=\left(R_{\gamma} \nu_{x}^{F_{\mu}}(x, z), \nu_{z}^{F_{\mu}}(x, z)\right), \tag{3.4}
\end{equation*}
$$

for every $\gamma \in[\min \{-\beta, 0\}, \max \{-\beta, 0\}]$ such that $\left(R_{\gamma} x, z\right) \in\left(\partial^{*} F_{\mu}\right)_{(r, z)}$.
Roughly speaking, what the above result says is that, given $(x, z) \in \partial^{*} F_{\mu}$ as in the statement, if there exists any other point $(\bar{x}, z) \in \partial^{*} F_{\mu}$ satisfying the following properties, namely $\bar{x} \in$ $\left(\partial^{*} F_{\mu}\right)_{(|x|, z)}, \arccos \left(\hat{\bar{x}} \cdot e_{1}\right) \leq|\beta|$, and $x_{2} \bar{x}_{2} \geq 0$, then there exists an angle $\gamma \in[\min \{-\beta, 0\}, \max \{-\beta, 0\}]$ such that $\bar{x}=R_{\gamma} x$ and the corresponding $\nu^{F_{\mu}}(\bar{x}, z)$ can be written as

$$
\nu^{F_{\mu}}(\bar{x}, z)=\left(R_{\gamma} \nu_{x}^{F_{\mu}}(x, z), \nu_{z}^{F_{\mu}}(x, z)\right) .
$$

Proof of Proposition 3.4. In the following, we set $x_{r}=(r, 0)$. If $\beta=0$ there is nothing to prove, so we can assume $\beta \neq 0$. We will only consider the case $\beta>0$, since for $\beta<0$ the proof is analogous. Also, since $R_{0} x=x$ and the statement is true for $x$, we only need to consider the case $\gamma \neq 0$. Let $\gamma \in[-\beta, 0)$, and $\rho>0$ be such that $r-\rho>0$, and

$$
\emptyset=B_{\rho}((x, z)) \cap B_{\rho}\left(\left(R_{\gamma} x, z\right)\right)=B_{\rho}\left(\left(R_{\beta} x_{r}, z\right)\right) \cap B_{\rho}\left(\left(R_{\gamma+\beta} x_{r}, z\right)\right) .
$$

In the following, to ease the notation, let us set

$$
\nu=\nu^{F_{\mu}}(x, z) \quad \text { and } \quad \nu_{\gamma}=\left(R_{\gamma} \nu_{x}^{F_{\mu}}(x, z), \nu_{z}^{F_{\mu}}(x, z)\right) .
$$

We have

$$
\begin{aligned}
& \mathcal{H}^{k}\left(H_{\left(R_{\gamma} x, z\right), \nu_{\gamma}}^{+} \cap F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma} x, z\right)\right)\right)=\mathcal{H}^{k}\left(H_{\left(R_{\gamma+\beta} x_{r}, z\right), \nu_{\gamma}}^{+} \cap F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma+\beta} x_{r}, z\right)\right)\right) \\
& =\int_{\mathbb{R}^{k-2}} \int_{r-\rho}^{r+\rho} \mathcal{H}^{1}\left(F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma+\beta} x_{r}, z\right)\right) \cap H_{\left(R_{\gamma+\beta} x_{r}, z\right), \nu_{\gamma}}^{+} \cap \partial B_{\lambda}^{2}(0, \bar{z})\right) d \lambda d \bar{z} \\
& =\int_{\mathbb{R}^{k-2}} \int_{r-\rho}^{r+\rho} \mathcal{H}^{1}\left(F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma+\beta} x_{r}, z\right)\right) \cap T_{\gamma}\left(H_{\left(R_{\beta} x_{r}, z\right), \nu}^{+}\right) \cap \partial B_{\lambda}^{2}(0, \bar{z})\right) d \lambda d \bar{z} \\
& \geq \int_{\mathbb{R}^{k-2}} \int_{r-\rho}^{r+\rho} \mathcal{H}^{1}\left(F_{\mu} \cap B_{\rho}\left(\left(R_{\beta} x_{r}, z\right)\right) \cap H_{\left(R_{\beta} x_{r}, z\right), \nu}^{+} \cap \partial B_{\lambda}^{2}(0, \bar{z})\right) d \lambda d \bar{z} \\
& =\int_{\mathbb{R}^{k-2}} \int_{r-\rho}^{r+\rho} \mathcal{H}^{1}\left(F_{\mu} \cap B_{\rho}((x, z)) \cap H_{(x, z), \nu}^{+} \cap \partial B_{\lambda}^{2}(0, \bar{z})\right) d \lambda d \bar{z} \\
& =\mathcal{H}^{k}\left(H_{(x, z), \nu}^{+} \cap F_{\mu} \cap B_{\rho}((x, z))\right),
\end{aligned}
$$

where in the inequality we used (3.3) with $A=H_{(x, z), \nu}^{+}$, and the fact that $\gamma<0$. From the last chain of inequalities we obtain

$$
\begin{aligned}
& \frac{\mathcal{H}^{k}\left(H_{\left(R_{\gamma} x, z\right), \nu_{\gamma}}^{+} \cap B_{\rho}\left(\left(R_{\gamma} x, z\right)\right)\right)}{\omega_{k} \rho^{k}} \geq \frac{\mathcal{H}^{k}\left(H_{\left(R_{\gamma} x, z\right), \nu_{\gamma}}^{+} \cap F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma} x, z\right)\right)\right)}{\omega_{k} \rho^{k}} \\
& \geq \frac{\mathcal{H}^{k}\left(H_{(x, z), \nu}^{+} \cap F_{\mu} \cap B_{\rho}((x, z))\right)}{\omega_{k} \rho^{k}} .
\end{aligned}
$$

Passing to the limit as $\rho \rightarrow 0^{+}$, we have

$$
\begin{aligned}
& \frac{1}{2}=\lim _{\rho \rightarrow 0^{+}} \frac{\mathcal{H}^{k}\left(H_{\left(R_{\gamma} x, z\right), \nu_{\gamma}}^{+} \cap B_{\rho}\left(\left(R_{\gamma} x, z\right)\right)\right)}{\omega_{k} \rho^{k}} \geq \limsup _{\rho \rightarrow 0^{+}} \frac{\mathcal{H}^{k}\left(H_{\left(R_{\gamma} x, z\right), \nu_{\gamma}}^{+} \cap F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma} x, z\right)\right)\right)}{\omega_{k} \rho^{k}} \\
& \geq \liminf _{\rho \rightarrow 0^{+}} \frac{\mathcal{H}^{k}\left(H_{\left(R_{\gamma} x, z\right), \nu_{\gamma}}^{+} \cap F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma} x, z\right)\right)\right)}{\omega_{k} \rho^{k}} \geq \lim _{\rho \rightarrow 0^{+}} \frac{\mathcal{H}^{k}\left(H_{(x, z), \nu}^{+} \cap F_{\mu} \cap B_{\rho}((x, z))\right)}{\omega_{k} \rho^{k}}=\frac{1}{2}
\end{aligned}
$$

where the last equality follows from the fact that $\nu$ is the inner unit normal to $\partial^{*} F_{\mu}$ at $(x, z)$. Therefore,

$$
\frac{1}{2}=\lim _{\rho \rightarrow 0^{+}} \frac{\mathcal{H}^{k}\left(H_{\left(R_{\gamma} x, z\right), \nu_{\gamma}}^{+} \cap F_{\mu} \cap B_{\rho}\left(\left(R_{\gamma} x, z\right)\right)\right)}{\omega_{k} \rho^{k}}
$$

Since by assumption $R_{\gamma} x \in \partial^{*} F_{\mu}$, it has to be

$$
\nu^{F_{\mu}}\left(R_{\gamma} x, z\right)=\left(R_{\gamma} \nu_{x}^{F_{\mu}}(x, z), \nu_{z}^{F_{\mu}}(x, z)\right)
$$

and this allows us to conclude.
Now we state a useful remark. For a similar result in the context of Steiner symmetrisation see [2, Remark 2.5].

Remark 3.5. Let us notice that, by symmetry of the set $F_{\mu}$ w.r.t. the hyperplane $\left\{x_{2}=0\right\} \subset \mathbb{R}^{k}$, the following property holds true. Given any $\nu \in \mathbb{R}^{2}$ we denote with $\operatorname{Ref}(\nu) \in \mathbb{R}^{2}$ the reflection of $\nu$ with respect to $\left\{x_{2}=0\right\} \subset \mathbb{R}^{2}$, namely $\operatorname{Ref}(\nu)=\left(\nu_{1},-\nu_{2}\right)$. Then, for every $(x, z) \in \partial^{*} F_{\mu}$ we have that $(\operatorname{Ref}(x), z) \in \partial^{*} F_{\mu}$ and

$$
\nu^{F_{\mu}}(\operatorname{Ref}(x), z)=\left(\operatorname{Ref}\left(\nu_{x}^{F_{\mu}}(x, z)\right), \nu_{z}^{F_{\mu}}(x, z)\right)
$$

We are now ready to prove Proposition 1.2.
Proof of Proposition 1.2. Let $(r, z) \in(0, \infty) \times \mathbb{R}^{k-2}$ such that the slice $\left(\partial^{*} F_{\mu}\right)_{(r, z)} \neq \emptyset$. We divide the slice in two parts, namely

$$
\left(\partial^{*} F_{\mu}\right)_{(r, z)}=\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+} \cup\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{-}
$$

where we set $\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}=\left(\partial^{*} F_{\mu}\right)_{(r, z)} \cap\left\{x_{2} \geq 0\right\}$, and $\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{-}=\left(\partial^{*} F_{\mu}\right)_{(r, z)} \cap\left\{x_{2}<0\right\}$. We now divide the proof in steps, depending on how many points are contained in the slice.
Step 1a. Let us suppose that $\mathcal{H}^{0}\left(\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}\right)=1$. Let $x \in \mathbb{R}_{0}^{2}$ such that $\{x\}=\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}$, and suppose in addition that $x_{2}=0$. Then, by symmetry properties of $F_{\mu}$, the point $x$ is the only point in the entire slice $\left(\partial^{*} F_{\mu}\right)_{(r, z)}$, and so we conclude.
Step 1b. Let us suppose that $\mathcal{H}^{0}\left(\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}\right)=1$. Let $x \in \mathbb{R}_{0}^{2}$ such that $\{x\}=\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}$, and suppose in addition that $x_{2}>0$. Then, by symmetry properties of $F_{\mu}$, the points $x$, and $\operatorname{Ref}(x)$ namely the reflection of $x$ w.r.t $\left\{x_{2}=0\right\}$ (see Remark 3.5), are the only points in the entire slice $\left(\partial^{*} F_{\mu}\right)_{(r, z)}$. Applying Remark 3.5 we get that the two vectors $\nu^{F_{\mu}}(x, z)$, and $\nu^{F_{\mu}}(\operatorname{Ref}(x), z)$ are symmetric to each other w.r.t. $\left\{x_{2}=0\right\}$, and so by a direct computation we show that the three functions in (1.6) are constant in the slice $\left(\partial^{*} F_{\mu}\right)_{(r, z)}$. This concludes the second part of the first step.
Step 2. Let us suppose that $\mathcal{H}^{0}\left(\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}\right)>1$. Let $x \in\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}$and let $\beta \in(0, \pi]$ be such that $x=r(\cos \beta, \sin \beta)$. Thanks to Proposition 3.4 we get that

$$
\nu^{F_{\mu}}\left(R_{\gamma} x, z\right)=\left(R_{\gamma} \nu_{x}^{F_{\mu}}(x, z), \nu_{z}^{F_{\mu}}(x, z)\right)
$$

for every $\gamma \in[-\beta, 0]$ such that $\left(R_{\gamma} x, z\right) \in\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}$. As a consequence of the fact that the above relation holds true for every $x \in\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}$, we get that the three functions in (1.6) are constant in $\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}$. By symmetry of $F_{\mu}$ w.r.t. $\left\{x_{2}=0\right\}$, the same conclusion holds true when restricting the three functions in (1.6) to $\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{-}$. Finally, the fact that the constant values of those three functions does not change when passing from $\left(\partial^{*} F_{\mu}\right)_{(r, z)}^{+}$to $\left(\partial^{*} F_{\mu}\right)_{(r, z)}$ is a consequence of Remark 3.5. This concludes the proof of the second step. Putting together the informations obtained in all these steps, we conclude.

We now focus our attention on the properties of the function $\mu$. Parts of the following results were already stated without an explicit proof in [6, Section 6]. For completeness, and future references we decide to provide here a detailed proof. In the following, we denote by $C_{c}^{0}\left(\Omega ; \mathbb{R}^{k-1}\right)$ the class of all continuous functions from $\Omega$ to $\mathbb{R}^{k-1}$, while with $C_{b}^{0}\left(\Omega ; \mathbb{R}^{k-1}\right)$ we denote the set of continuous and bounded function from $\Omega$ to $\mathbb{R}^{k-1}$.

In the following, given $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ a Lebesgue measurable function satisfying (1.2), we denote by $\xi:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ the function defined as

$$
\begin{equation*}
\xi(r, z):=\mu(r, z) / r \quad \text { for } \mathcal{L}^{k-1} \text {-a.e. }(r, z) \in(0, \infty) \times \mathbb{R}^{k-2} \tag{3.5}
\end{equation*}
$$

Lemma 3.6. Let $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying (1.2), let $U \subset(0, \infty) \times \mathbb{R}^{k-2}$ be an open set, and let $E \subset \mathbb{R}^{k}$ be a $\mu$-distributed set such that $E$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. Then, both the function $\mu$, and the function $\xi$ defined in (3.5) are in $B V_{\text {loc }}(U)$. In addition, $\left|D_{z} \mu\right|$, and $\left|r D_{r} \xi\right|$ are finite Radon measures on $U$, and for every Borel set $B \subset U$ we have

$$
\begin{align*}
& \int_{B} \varphi(r, z) d D_{z_{i}} \mu(r, z)=\int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z),  \tag{3.6}\\
& \int_{B} \varphi(r, z) r d D_{r} \xi(r, z)=\int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \hat{x} \cdot \nu_{x}^{E}(x, z) d \mathcal{H}^{k-1}(x, z), \tag{3.7}
\end{align*}
$$

for every $i \in\{1, \ldots, k-2\}$, and for every bounded Borel function $\varphi: B \rightarrow \mathbb{R}$. Moreover, let $\sigma_{\mu}$ be the $\mathbb{R}^{k}$-valued Radon measure on $U$ defined as

$$
\begin{equation*}
\sigma_{\mu}(B):=\int_{B} d\left(r D_{r} \xi, 2 \mathcal{L}^{k-1}\left\llcorner(\{\mu>0\} \cap U), D_{z} \mu\right)(r, z), \quad \forall B \subset U\right. \text { Borel. } \tag{3.8}
\end{equation*}
$$

Then, for every Borel set $B \subset U$ we get

$$
\begin{equation*}
\int_{B} \varphi(r, z) \cdot d \sigma_{\mu}(r, z) \leq \int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \cdot \nu_{\mathrm{c}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z) \tag{3.9}
\end{equation*}
$$

for every bounded Borel function $\varphi: B \rightarrow \mathbb{R}^{k}$ with non-negative second component, where $\nu_{c}^{E}$ was defined in (1.5). In particular, equality sign holds true in (3.9) if and only if $(E)_{(r, z)}$ is $\mathcal{H}^{1}$-equivalent to a connected arc for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in B$.

Proof of Lemma 3.6. We divide the proof in several steps.
Step 1. Let us prove that $\mu \in B V_{\text {loc }}(U)$. Let us start by proving that $\mu \in L_{\text {loc }}^{1}(U)$. Let $V \subset \subset U$, then

$$
\|\mu\|_{L^{1}(V)}=\int_{V} \mu(r, z) d r d z=\int_{V} d r d z \int_{E_{(r, z)}} 1 d \mathcal{H}^{1}(x)=\int_{E \cap \Phi\left(V \times \mathbb{S}^{1}\right)} 1 d \mathcal{H}^{k}<\infty .
$$

This proves that $\mu \in L_{\mathrm{loc}}^{1}(U)$. Similarly, we get that $\xi \in L_{\mathrm{loc}}^{1}(U)$. In order to conclude this first step we need to show that for every $V \subset \subset U$ open set, we have

$$
\begin{equation*}
\sup \left\{\int_{V} \mu(r, z) \operatorname{div} T(r, z) d r d z: T \in C_{c}^{1}\left(V ; \mathbb{R}^{k-1}\right),|T| \leq 1\right\}<\infty \tag{3.10}
\end{equation*}
$$

Let $i \in\{1, \ldots, k-2\}$, and let $\varphi \in C_{c}^{1}(V)$ with $|\varphi| \leq 1$. Then,

$$
\begin{aligned}
& \int_{V} \mu(r, z) \frac{\partial \varphi}{\partial z_{i}}(r, z) d r d z=\int_{V} d r d z \int_{E_{(r, z)}} \frac{\partial \varphi}{\partial z_{i}}(|x|, z) d \mathcal{H}^{1}(x) \\
& =\int_{\Phi\left(V \times \mathbb{S}^{1}\right)} \chi_{E}(x, z) \frac{\partial \varphi}{\partial z_{i}}(|x|, z) d \mathcal{H}^{k}(x, z)=-\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z) \\
& \leq P\left(E ; \Phi\left(V \times \mathbb{S}^{1}\right)\right)<\infty
\end{aligned}
$$

Let us now recall that, for any $\varphi \in C^{1}(V)$

$$
\operatorname{div}_{(x, z)}(\varphi(|x|, z) \hat{x})=\frac{\partial \varphi}{\partial r}(|x|, z)+\frac{1}{|x|} \varphi(|x|, z)
$$

where by $\operatorname{div}_{(x, z)}$ we denoted the divergence in $\mathbb{R}^{k}$ with respect to the variables $(x, z)$; we do that to distinguish when we consider the divergence in $(0, \infty) \times \mathbb{R}^{k-2}$ w.r.t. the variables $r$, and $z$. Then, for any $\varphi \in C_{c}^{1}(V)$ with $|\varphi| \leq 1$ we get

$$
\begin{aligned}
& \int_{V} \mu(r, z) \frac{\partial \varphi}{\partial r}(r, z) d r d z=\int_{\Phi\left(V \times \mathbb{S}^{1}\right)} \chi_{E}(x, z) \frac{\partial \varphi}{\partial r}(|x|, z) d x d z \\
& =\int_{\Phi\left(V \times \mathbb{S}^{1}\right)} \chi_{E}(x, z)\left(\operatorname{div}_{(x, z)}(\varphi(|x|, z) \hat{x})-\frac{1}{|x|} \varphi(|x|, z)\right) d x d z \\
& =-\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \hat{x} \cdot \nu^{E}(x, z) d \mathcal{H}^{k-1}(x, z)-\int_{\Phi\left(V \times \mathbb{S}^{1}\right)} \chi_{E}(x, z) \frac{1}{|x|} \varphi(|x|, z) d x d z \\
& =-\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)} \varphi(|x|, y, t) \hat{x} \cdot \nu_{x}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)-\int_{V} \xi(r, y, t) \varphi(r, y, t) d r d z \\
& \leq P\left(E ; \Phi\left(V \times \mathbb{S}^{1}\right)\right)+\|\xi\|_{L^{1}(V)}<\infty
\end{aligned}
$$

where for the last inequality we used that $\xi \in L_{\mathrm{loc}}^{1}(U)$, and $V \subset \subset U$. Putting together the above calculations we get that (3.10) holds true, and this proves that $\mu \in B V_{\text {loc }}(U)$. Since the maps $(r, z) \mapsto 1 / r$ and $(r, z) \mapsto \mu(r, z)$ belong to $B V(V)$, thanks to [1, Example 3.97] we have that $\xi(r, z)=\mu(r, z) / r \in B V(V)$ for every $V \subset \subset U$ open set, and so $\xi(r, z) \in B V_{\text {loc }}(U)$. In particular,

$$
\begin{equation*}
D_{r} \mu=D_{r}(r \xi)=r D_{r} \xi+\xi d r d z \tag{3.11}
\end{equation*}
$$

This concludes the first step.
Step 2a. Let us prove that relations (3.6), and (3.7) holds true for every $\varphi \in C_{c}^{1}(U)$. Let $\varphi \in C_{c}^{1}(U)$ be a test function, and let $V \subset \subset U$ be an open set $\operatorname{such}$ that $\operatorname{supp}(\varphi) \subset V$. Then, by properties of $B V_{\text {loc }}$ functions, together with the calculation we made in the first step, we have

$$
\begin{aligned}
& -\int_{U} \varphi(r, z) d D_{z_{i}} \mu(r, z)=\int_{U} \mu(r, z) \frac{\partial \varphi}{\partial_{z_{i}}}(r, z) d r d z \\
& =\int_{V} \mu(r, z) \frac{\partial \varphi}{\partial_{z_{i}}}(r, z) d r d z=-\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z) \\
= & -\int_{\partial^{*} E \cap \Phi\left(U \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z) .
\end{aligned}
$$

Thus, for all $i \in\{1, \ldots, \kappa-2\}$, and for all $\varphi \in C_{c}^{1}(U)$ we have,

$$
\begin{equation*}
\int_{U} \varphi(r, z) d D_{z_{i}} \mu(r, z)=\int_{\partial^{*} E} \varphi(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z) \tag{3.12}
\end{equation*}
$$

which proves that (3.6) holds true for every $\varphi \in C_{c}^{1}(U)$. Let us now prove that (3.7) holds true for every $\varphi \in C_{c}^{1}(U)$. Let $\varphi \in C_{c}^{1}(U)$ be a test function, and let $V \subset \subset U$ be an open set such that $\operatorname{supp}(\varphi) \subset V$. Then, analogously to what we proved above, by properties of $B V_{\text {loc }}$ functions,
together with the calculation we made in the first step, we have

$$
\begin{aligned}
& -\int_{U} \varphi(r, z) d D_{r} \mu(r, z)=\int_{U} \mu(r, z) \frac{\partial \varphi}{\partial r}(r, z) d r d z=\int_{V} \mu(r, z) \frac{\partial \varphi}{\partial r}(r, z) d r d z \\
& =-\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)} \varphi(|x|, y, t) \hat{x} \cdot \nu_{x}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)-\int_{V} \xi(r, y, t) \varphi(r, y, t) d r d z, \\
& =-\int_{\partial^{*} E \cap \Phi\left(U \times \mathbb{S}^{1}\right)} \varphi(|x|, y, t) \hat{x} \cdot \nu_{x}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)-\int_{U} \xi(r, y, t) \varphi(r, y, t) d r d z,
\end{aligned}
$$

from which we get

$$
\begin{align*}
\int_{U} \varphi(r, z) d D_{r} \mu(r, z)= & \int_{\partial^{*} E \cap \Phi\left(U \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \hat{x} \cdot \nu_{x}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)  \tag{3.13}\\
& +\int_{U} \xi(r, z) \varphi(r, z) d r d z
\end{align*}
$$

Comparing (3.13) with (3.11), we get that (3.7) holds true for every $\varphi \in C_{c}^{1}(U)$. Before concluding this first step, let us observe as a consequence of the previous calculations, we have that

$$
\begin{align*}
& \left|D_{z_{i}} \mu\right|(U) \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right), \quad \text { for } i=1, \ldots, k-2  \tag{3.14}\\
& \left|r D_{r} \xi\right|(U) \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right) \tag{3.15}
\end{align*}
$$

This proves that $\left|D_{z} \mu\right|$ and $\left|r D_{r} \xi\right|$ are finite Radon measures on $U$, and we conclude the first step. Step 2b. We are now ready to prove (3.6), and (3.7) whenever $B \subset \subset U$. We will only show (3.6), since the proof of (3.7) is similar. Let $i \in\{1, \ldots, k-2\}$, let $B \subset \subset U$ be a Borel set, let $\varphi: B \rightarrow \mathbb{R}$ be a bounded Borel function, and let $V \subset \subset U$ open set such that $B \subset V$. We call $\bar{\varphi}: V \rightarrow \mathbb{R}$ the Borel function that coincides with $\varphi$ in $B$, and it is zero in $V \backslash B$. Since every function in $C_{b}^{0}(V)$ can be approximated uniformly on compact subsets of $V$ by functions in $C_{c}^{1}(V)$, and since $D_{z_{i}} \mu$ is a bounded Radon measure on $V$, we have that (3.6) holds true for every function in $C_{b}^{0}(V)$. Let $\lambda$ be the bounded Radon measure on $V$ defined by

$$
\begin{equation*}
\lambda(B):=\left|D_{z_{i}} \mu\right|(B)+\mathcal{H}^{k-1}\left(\partial^{*} E \cap\left(\Phi\left(B \times \mathbb{S}^{1}\right)\right)\right) \tag{3.16}
\end{equation*}
$$

for every Borel set $B \subset V$. By Lusin Theorem, for every $h \in \mathbb{N}$ there exists $\varphi_{h} \in C_{b}^{0}(V)$ such that $\left\|\varphi_{h}\right\|_{L^{\infty}(V)} \leq\|\bar{\varphi}\|_{L^{\infty}(V)}$ and

$$
\lambda\left(\left\{(r, z) \in V: \bar{\varphi}(r, z) \neq \varphi_{h}(r, z)\right\}\right)<\frac{1}{h} .
$$

For each $h \in \mathbb{N}$ we can apply (3.6) to $\varphi_{h}$, obtaining

$$
\int_{V} \varphi_{h}(r, z) d D_{z_{i}} \mu(r, z)=\int_{\partial^{*} E \cap\left(\Phi\left(V \times \mathbb{S}^{1}\right)\right)} \varphi_{h}(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)
$$

Using this identity, we have

$$
\begin{aligned}
& \left|\int_{B} \varphi(r, z) d D_{z_{i}} \mu(r, z)-\int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{n}(x, z)\right| \\
& =\left|\int_{V} \bar{\varphi}(r, z) d D_{z_{i}} \mu(r, z)-\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)} \bar{\varphi}(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{n}(x, z)\right| \\
& \leq\left|\int_{V}\left(\bar{\varphi}(r, z)-\varphi_{h}(r, z)\right) d D_{z_{i}} \mu(r, z)\right| \\
& +\left|\int_{V} \varphi_{h}(r, z) d D_{z_{i}} \mu(r, z)-\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)} \varphi_{h}(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)\right| \\
& +\left|\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)}\left(\bar{\varphi}(|x|, z)-\varphi_{h}(r, z)\right) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)\right| \\
& =\left|\int_{V}\left(\bar{\varphi}(r, z)-\varphi_{h}(r, z)\right) d D_{z_{i}} \mu(r, z)\right| \\
& +\left|\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)}\left(\bar{\varphi}(r, z)-\varphi_{h}(r, z)\right) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)\right| \\
& \leq \int_{V}\left|\bar{\varphi}(r, z)-\varphi_{h}(r, z)\right| d\left|D_{z_{i}} \mu\right|(r, z) \\
& +\int_{\partial^{*} E \cap \Phi\left(V \times \mathbb{S}^{1}\right)}\left|\bar{\varphi}(r, z)-\varphi_{h}(r, z)\right| d \mathcal{H}^{k-1}(x, z) \leq \frac{4}{h}\|\bar{\varphi}\|_{L^{\infty}(V)} .
\end{aligned}
$$

Passing to the limit as $h \rightarrow \infty$ we obtain (3.6) whenever $B \subset \subset U$. This concludes step 2b.
Step 2c. We finally prove (3.6), and (3.7). As done in step 2 b , we will only show (3.6). Fix $i \in\{1, \ldots, k-2\}$ and consider the Radon measure $\lambda$ on $U$ defined as in (3.16). Let $B \subset U$ be a Borel set, and let $\left(B_{h}\right)_{h \in \mathbb{N}} \subset B$ be a sequence of compact sets, with the property that $\lambda\left(B \backslash B_{h}\right)<\epsilon_{h}$, where $\left(\epsilon_{h}\right)_{h \in \mathbb{N}} \subset[0,1]$ and $\lim _{h \rightarrow \infty} \epsilon_{h}=0$. Let $\varphi: B \rightarrow \mathbb{R}$ be a bounded Borel function, and let us set $\varphi_{h}(r, z)=\chi_{B_{h}}(r, z) \varphi(r, z)$ for every $(r, z) \in B$, for every $h \in \mathbb{N}$. By construction, up to pass to a subsequence, we have that $\lim _{h \rightarrow \infty} \varphi_{h}(r, z)=\varphi(r, z)$ for $\lambda$-a.e. $(r, z) \in B$. Thus,

$$
\begin{aligned}
& \left|\int_{B} \varphi(r, z) d D_{z_{i}} \mu(r, z)-\int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)\right| \\
& =\left|\int_{B}\left(\varphi(r, z)-\varphi_{h}(r, z)\right) d D_{z_{i}} \mu(r, z)-\int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)}\left(\varphi(|x|, z)-\varphi_{h}(|x|, z)\right) \nu_{z_{i}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z)\right| \\
& \leq \sup _{(r, z) \in B} \varphi(r, z) \lambda\left(B \backslash B_{h}\right) \leq \epsilon_{h} \sup _{(r, z) \in B} \varphi(r, z) .
\end{aligned}
$$

Passing to the limit in the above relation as $h \rightarrow \infty$ we prove (3.6). Formula (3.7) can be obtained in similar way using the approximation argument we just presented.
Step 3. Let us prove (3.9). Let $B \subset U$ be a Borel set, and let $g: B \rightarrow[0, \infty]$ be a Borel function. Let us denote with $\operatorname{Pr}\left(\partial^{*} E\right)$ the projection in $U$ of the set $\partial^{*} E \cap \Phi\left(U \times \mathbb{S}^{1}\right)$, namely

$$
\operatorname{Pr}\left(\partial^{*} E\right):=\left\{(r, z) \in U:\left(\partial^{*} E\right)_{(r, z)} \neq \emptyset\right\}
$$

By construction, it can be shown that $\mathcal{L}^{k-1}\left(\operatorname{Pr}\left(\partial^{*} E\right) \backslash(\{\mu>0\} \cap U)\right)=0$, while by Proposition 3.2 we have that $\mathcal{L}^{k-1}\left((\{\mu>0\} \cap U) \backslash \operatorname{Pr}\left(\partial^{*} E\right)\right)=0$. Thus, by the Coarea formula (3.1) we get

$$
\begin{aligned}
& \int_{B \cap\{\mu>0\} \cap U} 2 g(r, z) d r d z=\int_{B \cap \operatorname{Pr}\left(\partial^{*} E\right)} 2 g(r, z) d r d z \leq \int_{B \cap \operatorname{Pr}\left(\partial^{*} E\right)} g(r, z) \int_{\left(\partial^{*} E\right)_{(r, z)}} 1 d \mathcal{H}^{0}(x) d r d z \\
& =\int_{\partial^{*} E \cap \Phi\left(\left(B \cap \operatorname{Pr}\left(\partial^{*} E\right)\right) \times \mathbb{S}^{1}\right)} g(|x|, z)\left|\nu_{x \|}^{E}(x, z)\right| d \mathcal{H}^{k-1}(x, z) \\
& =\int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)} g(|x|, z)\left|\nu_{x \|}^{E}(x, z)\right| d \mathcal{H}^{k-1}(x, z),
\end{aligned}
$$

where for the inequality sign we used Proposition 3.2, and the properties of the set $\operatorname{Pr}\left(\partial^{*} E\right)$ to infer that $\mathcal{H}^{0}\left(\left(\partial^{*} E\right)_{(r, z)}\right) \geq 2$ for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in \operatorname{Pr}\left(\partial^{*} E\right)$. The above relation, together with (3.6), and (3.7) proves (3.9). This concludes the third step and the proof of the lemma.

Remark 3.7. Under the assumptions of the above lemma, let $B \subset U$ be a Borel set, and let $E \subset \mathbb{R}^{k}$ be a $\mu$-distributed set of finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$ such that $(E)_{(r, z)}$ is $\mathcal{H}^{1}$-equivalent to a connected arc for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in B$. Then, we get that (3.9) holds true with equality and in addition, as a consequence of Proposition 3.1, we can drop the assumption of the non-negativity of the second component of the vector field appearing the formula, namely

$$
\begin{equation*}
\int_{B} \varphi(r, z) \cdot d \sigma_{\mu}(r, z)=\int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \cdot \nu_{\mathrm{c}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z) \tag{3.17}
\end{equation*}
$$

for every bounded Borel function $\varphi: B \rightarrow \mathbb{R}^{k}$.
Let us mention that the idea of considering the Radon measure $\sigma_{\mu}$ was inspired by [11, Section 4.1.5]. The next result can be seen as a refinement of [ 6 , Proposition 6.8].

Lemma 3.8. Let $\mu:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow[0, \infty)$ be a Lebesgue measurable function satisfying (1.2), let $U \subset(0, \infty) \times \mathbb{R}^{k-2}$ be an open set, and let $E \subset \mathbb{R}^{k}$ be a $\mu$-distributed set such that $E$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. Then, the set $F_{\mu} \subset \mathbb{R}^{k}$ defined in (1.3) is a set of finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. Moreover,

$$
\begin{equation*}
\left|\sigma_{\mu}\right|(B)=P\left(F_{\mu} ; \Phi\left(B \times \mathbb{S}^{1}\right)\right), \quad \forall B \subset U \text { Borel } \tag{3.18}
\end{equation*}
$$

where $\sigma_{\mu}$ is the Radon measure defined in (3.8).
Proof. We divide the proof in several steps. We start by proving that the set $F_{\mu} \subset \mathbb{R}^{k}$ is of finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. The argument we are going to use is standard, but for the seek of completeness and for future references we decided to include it (see [6, Proposition 4.3] for the same argument but in the spherical symmetrisation setting). Let $\Omega \subset \subset U$ be an open set. By Lemma 3.6 $\xi \in B V(\Omega)$. Thus, by standard approximation techniques, let $\left(\xi_{j}\right)_{j \in \mathbb{N}} \subset C_{c}^{1}\left(\Omega ; \mathbb{R}^{k}\right)$ be a sequence of non negative functions such that $\xi_{j} \rightarrow \xi$ for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in \Omega$, and $\left|\nabla \xi_{j}\right| \mathcal{L}^{k-1} \xrightarrow{*}|D \xi|$, where the function $\xi$ was defined in (3.5), and with the symbol $\stackrel{*}{\rightharpoonup}$ we denote the weak star convergence of Radon measures. In the following, denoting with $\mu_{j}(r, z)=r \xi_{j}(r, z)$, we call $F_{\mu_{j}} \subset \mathbb{R}^{k}$ the set defined as in (1.3) w.r.t. the function $\mu_{j}$.
Step 0. In this step we present some circular notation that we will need for the following calculations. Let $\varphi \in C_{c}^{1}\left(\Phi\left(\Omega \times \mathbb{S}^{1}\right), \mathbb{R}^{k}\right)$ with $|\varphi| \leq 1$. A direct calculation shows that

$$
\begin{align*}
\operatorname{div}_{(x, z)} \varphi(x, z) & =\operatorname{div}_{(x)} \varphi_{x}(x, z)+\operatorname{div}_{(z)} \varphi_{z}(x, z) \\
& =\operatorname{div}_{(x) \|} \varphi_{x \|}(x, z)+\nabla_{x} \varphi_{x}(x, z)[\hat{x}] \cdot \hat{x}+\frac{\varphi_{x}(x, z) \cdot \hat{x}}{|x|}+\operatorname{div}_{(z)} \varphi_{z}(x, z) \tag{3.19}
\end{align*}
$$

where $\operatorname{div}_{(x)}$, and $\operatorname{div}_{(z)}$ stand for the classical divergence in $\mathbb{R}^{2}$ w.r.t. the variables $x_{1}$, and $x_{2}$, and the classical divergence in $\mathbb{R}^{k-2}$ w.r.t. the variables $z_{1}, \ldots, z_{k-2}$, respectively, $\operatorname{div}_{(x) \|} \varphi_{x \|}(x, z)$ stands for the tangential divergence in $\mathbb{R}^{2}$ of $\varphi_{x \|}(\cdot, z)$ at $(x, z)$ in $\partial B(|x|)$, and finally $\nabla_{x}$ is the classical gradient in $\mathbb{R}^{2}$ w.r.t. the variables $x_{1}$, and $x_{2}$. Thus,

$$
\begin{equation*}
\int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \operatorname{div}_{(x, z)} \varphi(x, z) d x d z=\mathbf{I}+\mathbf{I I}+\mathbf{I I I} \tag{3.20}
\end{equation*}
$$

where we set

$$
\begin{aligned}
\mathbf{I} & :=\int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \operatorname{div}_{(x) \|} \varphi_{x \|}(x, z) d x d z ; \\
\mathbf{I I} & :=\int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \nabla_{x} \varphi_{x}(x, z)[\hat{x}] \cdot \hat{x} d x d z+\int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \frac{\varphi_{x}(x, z) \cdot \hat{x}}{|x|} d x d z ; \\
\mathbf{I I I} & :=\int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \operatorname{div}_{(z)} \varphi_{z}(x, z) d x d z .
\end{aligned}
$$

Step 1. In this step we study the quantity identified with I. Let us observe that, by construction of $F_{\mu_{j}}$, the slice $\left(F_{\mu_{j}}\right)_{(r, z)}$ is a connected arc in $\partial B(r)$ for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in \Omega$. By the theory of sets of finite perimeter in $\partial B(r)$ (see for instance [ 6 , Section 3.2]), and Proposition 3.2 we get that
$2=\mathcal{H}^{0}\left(\partial^{*}\left(\left(F_{\mu_{j}}\right)_{(r, z)}\right)\right) \leq \mathcal{H}^{0}\left(\partial^{*}\left((E)_{(r, z)}\right)\right)=\mathcal{H}^{0}\left(\left(\partial^{*} E\right)_{(r, z)}\right) \quad$ for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in\left\{\mu_{j}>0\right\} \subset \Omega$, while $\mathcal{H}^{0}\left(\partial^{*}\left(\left(F_{\mu_{j}}\right)_{(r, z)}\right)\right)=0$ for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in \Omega \backslash\left\{\mu_{j}>0\right\}$. Thus, applying the structure theorem for sets of finite perimeter on $\partial B(r)$ (see once more [6, Section 3.2]) we get

$$
\begin{align*}
& \int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \operatorname{div}_{(x) \|} \varphi_{x \|}(x, z) d x d z=\int_{\Omega} d r d z \int_{\left(F_{\mu_{j}}\right)_{(r, z)}} \operatorname{div}_{(x) \|} \varphi_{x \|}(x, z) d \mathcal{H}^{1}(x) \\
& \leq \int_{\Omega} \mathcal{H}^{0}\left(\left(\partial^{*} E\right)_{(r, z)}\right) d r d z=\int_{\partial^{*} E \cap \Phi\left(\Omega \times \mathbb{S}^{1}\right)}\left|\nu_{x \|}^{E}(x, z)\right| d \mathcal{H}^{k-1}(x, z) \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right) \tag{3.21}
\end{align*}
$$

where for the last equality sign we used Proposition 3.1. This concludes the first step.
Step 2. In this step we study the quantity identified with II. Let us introduce the following quantity

$$
V_{j}(r, z):=\int_{\left(F_{\mu_{j}}\right)_{(r, z)}} \varphi_{x}(x, z) \cdot \hat{x} d \mathcal{H}^{1}(x)=r \int_{-\xi_{j}(r, z) / 2}^{\xi_{j}(r, z) / 2} \varphi_{x}(r \omega(\theta), z) \cdot \omega(\theta) d \theta \quad \forall(r, z) \in \Omega
$$

where $\xi(r, z)=\mu_{j}(r, z) / r$, and $\omega(\theta)=(\cos (\theta), \sin (\theta)) \in \mathbb{S}^{1}$. By regularity properties of $\mu_{j}$, and of $\varphi_{x}$ the above quantity is differentiable in the $r$ variable, and a direct computation shows that

$$
\begin{aligned}
& \frac{\partial}{\partial r} V_{j}(r, z)=\int_{-\xi_{j}(r, z) / 2}^{\xi_{j}(r, z) / 2} \varphi_{x}(r \omega(\theta), z) \cdot \omega(\theta) d \theta \\
& +\frac{1}{2} r \frac{\partial}{\partial r} \xi_{j}(r, z)\left(\varphi_{x}\left(r \omega\left(\xi_{j}(r, z) / 2\right), z\right) \cdot \omega\left(\xi_{j}(r, z) / 2\right)+\varphi_{x}\left(r \omega\left(-\xi_{j}(r, z) / 2\right), z\right) \cdot \omega\left(-\xi_{j}(r, z) / 2\right)\right) \\
& +r \int_{-\xi_{j}(r, z) / 2}^{\xi_{j}(r, z) / 2} \nabla_{x} \varphi_{x}(r \omega(\theta), z)[\omega(\theta)] \cdot \omega(\theta) d \theta
\end{aligned}
$$

In order to keep the notation a bit more compact, let us set

$$
\begin{aligned}
& A_{x}(r, z)=\varphi_{x}\left(r \omega\left(\xi_{j}(r, z) / 2\right), z\right) \cdot \omega\left(\xi_{j}(r, z) / 2\right) \\
& B_{x}(r, z)=\varphi_{x}\left(r \omega\left(-\xi_{j}(r, z) / 2\right), z\right) \cdot \omega\left(-\xi_{j}(r, z) / 2\right)
\end{aligned}
$$

Let us observe that, by construction, the function $V_{j}$ has compact support in $\Omega$. Thus, integrating both sides of the above relation over $\Omega$ and applying Fubini theorem together with the fundamental theorem of calculus we get

$$
\begin{aligned}
0 & =\int_{\Omega} \frac{\partial}{\partial r} V_{j}(r, z) d r d z=\int_{\Omega} d r d z \int_{-\xi_{j}(r, z) / 2}^{\xi_{j}(r, z) / 2} \varphi_{x}(r \omega(\theta), z) \cdot \omega(\theta) d \theta \\
& +\frac{1}{2} \int_{\Omega} r \frac{\partial}{\partial r} \xi_{j}(r, z) A_{x}(r, z) d r d z \\
& +\frac{1}{2} \int_{\Omega} r \frac{\partial}{\partial r} \xi_{j}(r, z) B_{x}(r, z) d r d z \\
& +\int_{\Omega} r \int_{-\xi_{j}(r, z) / 2}^{\xi_{j}(r, z) / 2} \nabla_{x} \varphi_{x}(r \omega(\theta), z)[\omega(\theta)] \cdot \omega(\theta) d \theta d r d z
\end{aligned}
$$

Thus, after a changing variables, we get

$$
\begin{align*}
& \int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \frac{\varphi_{x}(x, z) \cdot \hat{x}}{|x|} d x d z+\int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \nabla_{x} \varphi_{x}(x, z)[\hat{x}] \cdot \hat{x} d x d z \\
& =-\frac{1}{2} \int_{\Omega} r \frac{\partial}{\partial r} \xi_{j}(r, z) A_{x}(r, z) d r d z-\frac{1}{2} \int_{\Omega} r \frac{\partial}{\partial r} \xi_{j}(r, z) B_{x}(r, z) d r d z \tag{3.22}
\end{align*}
$$

This concludes the second step.
Step 3. In this step we study the quantity identified with III. Similarly to what we did in the
previous step, we consider the following auxiliary quantity

$$
Z_{j}^{i}(r, z):=\int_{\left(F_{\mu_{j}}\right)_{(r, z)}}\left(\varphi_{z}\right)_{i}(x, z) d \mathcal{H}^{1}(x)=r \int_{-\xi_{j}(r, z) / 2}^{\xi_{j}(r, z) / 2}\left(\varphi_{z}\right)_{i}(x, z) d \theta \quad \forall(r, z) \in \Omega
$$

where by $\left(\varphi_{z}\right)_{i}$ stands for the $i$-th component of the vector $\varphi_{z}$, with $i=1, \ldots, k-2$. Let us set

$$
\nabla_{z} \xi_{j}(r, z)=\left(\frac{\partial}{\partial z_{1}} \xi_{j}(r, z), \ldots, \frac{\partial}{\partial z_{k-2}} \xi_{j}(r, z)\right)
$$

Following verbatim the argument used in the step 2, and calling

$$
A_{z}(r, z)=\varphi_{z}\left(r \omega\left(\xi_{j}(r, z) / 2\right), z\right), \quad B_{z}(r, z)=\varphi_{z}\left(r \omega\left(-\xi_{j}(r, z) / 2\right), z\right)
$$

we get that

$$
\begin{align*}
& \int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \operatorname{div}_{(z)} \varphi_{z}(x, z) d x d z \\
& =-\frac{1}{2} \int_{\Omega} r \nabla_{z} \xi_{j}(r, z) \cdot\left(\varphi_{z}\left(r \omega\left(\xi_{j}(r, z) / 2\right), z\right)+\varphi_{z}\left(r \omega\left(-\xi_{j}(r, z) / 2\right), z\right)\right) d r d z  \tag{3.23}\\
& =-\frac{1}{2} \int_{\Omega} r \nabla_{z} \xi_{j}(r, z) \cdot A_{z}(r, z) d r d z-\frac{1}{2} \int_{\Omega} r \nabla_{z} \xi_{j}(r, z) \cdot B_{z}(r, z) d r d z
\end{align*}
$$

This concludes the third step.
Step 4. In this step we finally prove that $F_{\mu}$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. Indeed, thanks to the previous step, in particular plugging into (3.20) the relations obtained in (3.21), (3.22), and (3.23) we get

$$
\begin{aligned}
& \int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \operatorname{div}_{(x, z)} \varphi(x, z) d x d z \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right) \\
& -\frac{1}{2} \int_{\Omega} r \nabla \xi_{j}(r, z) \cdot\left(A_{x}(r, z), A_{z}(r, z)\right) d r d z-\frac{1}{2} \int_{\Omega} r \nabla \xi_{j}(r, z) \cdot\left(B_{x}(r, z), B_{z}(r, z)\right) d r d z
\end{aligned}
$$

Let us now observe that by construction, we have that both quantities $\left|\left(A_{x}(r, z), A_{z}(r, z)\right)\right|$, and $\left|\left(B_{x}(r, z), B_{z}(r, z)\right)\right|$ are less than 1 . Thus, from the above relation we get that

$$
\int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \operatorname{div}_{(x, z)} \varphi(x, z) d x d z \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right)+\int_{\operatorname{Pr}(\operatorname{supp}(\varphi))} r\left|\nabla \xi_{j}(r, z)\right| d r d z
$$

where $\operatorname{Pr}(\operatorname{supp}(\varphi)) \subset \Omega$ is the projection in $(0, \infty) \times \mathbb{R}^{k-2}$ of the support of $\varphi$, namely

$$
\operatorname{Pr}(\operatorname{supp}(\varphi))=\left\{(r, z) \in \Omega:(\operatorname{supp}(\varphi))_{(r, z)} \neq \emptyset\right\}
$$

Let us also observe that $\operatorname{Pr}(\operatorname{supp}(\varphi))$ is a compact set in $\Omega$. Recalling that $\left|\nabla \xi_{j}\right| \mathcal{L}^{k-1} \xrightarrow{*}|D \xi|$ we immediately get that $r\left|\nabla \xi_{j}\right| \mathcal{L}^{k-1} \xrightarrow{*} r|D \xi|$. Moreover, since $\xi_{j} \rightarrow \xi$ for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in \Omega$, by the definition of $\mu_{j}$ we get $\mu_{j} \rightarrow \mu$ for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in \Omega$, which implies that $\chi_{F_{\mu_{j}}} \rightarrow \chi_{F_{\mu}}$ for $\mathcal{L}^{k}$-a.e. $(x, z) \in \Phi\left(\Omega \times \mathbb{S}^{1}\right)$. Thus,

$$
\begin{align*}
& \int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu}}(x, z) \operatorname{div}_{(x, z)} \varphi(x, z) d x d z=\limsup _{j \rightarrow \infty} \int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu_{j}}}(x, z) \operatorname{div}_{(x, z)} \varphi(x, z) d x d z \\
& \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right)+\limsup _{j \rightarrow \infty} \int_{\operatorname{Pr}(\operatorname{supp}(\varphi))} r\left|\nabla \xi_{j}(r, z)\right| d r d z \\
& \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right)+r|D \xi|(\operatorname{Pr}(\operatorname{supp}(\varphi))) \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right)+r|D \xi|(\Omega) \tag{3.24}
\end{align*}
$$

In order to conclude, let us observe that

$$
\begin{aligned}
& r|D \xi|(\Omega)=\sup \left\{\int_{\Omega} \psi(r, z) \cdot d\left(r D_{r} \xi, r D_{z_{i}} \xi, \ldots, r D_{z_{k-2}} \xi\right)(r, z): \psi \in C_{c}^{0}\left(\Omega ; \mathbb{R}^{k}\right),|\psi| \leq 1\right\} \\
& \stackrel{(3.11)}{=} \sup \left\{\int_{\Omega} \psi(r, z) \cdot d\left(r D_{r} \xi, D_{z_{i}} \mu, \ldots, D_{z_{k-2}} \mu\right)(r, z): \psi \in C_{c}^{0}\left(\Omega ; \mathbb{R}^{k}\right),|\psi| \leq 1\right\} \\
& \leq P\left(E ; \Phi\left(\Omega \times \mathbb{S}^{1}\right)\right) \leq P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right)
\end{aligned}
$$

where for the second last inequality we used (3.6), and (3.7). Combining the above relation with the estimate obtained in (3.24) we get that

$$
\begin{equation*}
\int_{\Phi\left(\Omega \times \mathbb{S}^{1}\right)} \chi_{F_{\mu}}(x, z) \operatorname{div}_{(x, z)} \varphi(x, z) d x d z \leq 2 P\left(E ; \Phi\left(U \times \mathbb{S}^{1}\right)\right)<\infty \tag{3.25}
\end{equation*}
$$

Taking the sup over all test functions $\varphi \in C_{c}^{1}\left(\Phi\left(\Omega \times \mathbb{S}^{1}\right)\right)$ with $|\varphi| \leq 1$ on the left hand side of the above relation we get that $F_{\mu}$ has finite perimeter in $\Phi\left(\Omega \times \mathbb{S}^{1}\right)$ for every $\Omega \subset \subset U$ open set. Since the right hand side of (3.25) does not depend on $\Omega$, by standard arguments we conclude that $F_{\mu}$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$. This concludes step four, and we can now proceed to prove relation (3.18).
Step 5. As we said, we are left to prove (3.18). By standard measure theory, since $\left|\sigma_{\mu}\right|$ is a Radon measures on $U$, it is sufficient to show that (3.18) holds true for every open set $A \subset U$. We start proving that

$$
\begin{equation*}
\left|\sigma_{\mu}\right|(A) \leq P\left(F_{\mu} ; \Phi\left(A \times \mathbb{S}^{1}\right)\right) \quad \forall A \subset U \text { open. } \tag{3.26}
\end{equation*}
$$

Let $\varphi \in C_{c}^{0}\left(A ; \mathbb{R}^{k}\right)$ with $|\varphi| \leq 1$, and let $V \subset \subset U$ be an open set such that $\operatorname{supp}(\varphi) \subset V$. Then, since we proved that $F_{\mu}$ has finite perimeter in $\Phi\left(U \times \mathbb{S}^{1}\right)$, and since by construction $\left(F_{\mu}\right)_{(r, z)}$ is $\mathcal{H}^{1}$-equivalent to a connected arc in $\partial B(r)$ for $\mathcal{L}^{k-1}$-a.e. $(r, z) \in A$, we can apply (3.17) thus obtaining

$$
\begin{aligned}
& \int_{A} \varphi(r, z) \cdot d \sigma_{\mu}(r, z)=\int_{V} \varphi(r, z) \cdot d \sigma_{\mu}(r, z) \\
& \stackrel{(3.17)}{=} \int_{\partial^{*} F_{\mu} \cap \Phi\left(V \times \mathbb{S}^{1}\right)} \varphi(|x|, z) \cdot \nu_{\mathrm{C}}^{F_{\mu}}(x, z) d \mathcal{H}^{k-1}(x, z) \leq P\left(F_{\mu} ; \Phi\left(A \times \mathbb{S}^{1}\right)\right)<\infty,
\end{aligned}
$$

where in the first inequality we used Schwartz inequality. Passing to the sup in the left hand side among all $\varphi \in C_{c}^{0}\left(A ; \mathbb{R}^{k}\right)$ with $|\varphi| \leq 1$, we prove (3.26). Let us now prove the reverse inequality, namely

$$
\begin{equation*}
\left|\sigma_{\mu}\right|(A) \geq P\left(F_{\mu} ; \Phi\left(A \times \mathbb{S}^{1}\right)\right) \tag{3.27}
\end{equation*}
$$

Recall now the definition of the Borel vector field $\bar{\nu}_{\mathrm{c}}^{F_{\mu}}:(0, \infty) \times \mathbb{R}^{k-2} \rightarrow \mathbb{R}^{k}$ that was given in (1.7). Thus, denoting by $d \sigma_{\mu} / d\left|\sigma_{\mu}\right|: U \rightarrow \mathbb{S}^{k-1}$ the polar decomposition of $\sigma_{\mu}$, we get

$$
\begin{aligned}
& P\left(F_{\mu} ; \Phi\left(A \times \mathbb{S}^{1}\right)\right)=\int_{\partial^{*} F_{\mu} \cap \Phi\left(A \times \mathbb{S}^{1}\right)} d \mathcal{H}^{k-1}(x, z)=\int_{\partial^{*} F_{\mu} \cap \Phi\left(A \times \mathbb{S}^{1}\right)} \bar{\nu}_{\mathrm{c}}^{F_{\mu}}(|x|, z) \cdot \nu_{\mathrm{c}}^{F_{\mu}}(x, z) d \mathcal{H}^{k-1}(x, z) \\
& \stackrel{(3.17)}{=} \int_{A} \bar{\nu}_{\mathrm{c}}^{F_{\mu}}(r, z) \cdot d \sigma_{\mu}(r, z)=\int_{A} \bar{\nu}_{\mathrm{c}}^{F_{\mu}}(r, z) \cdot \frac{d \sigma_{\mu}}{d\left|\sigma_{\mu}\right|}(r, z) d\left|\sigma_{\mu}\right|(r, z) \leq \int_{A} 1 d\left|\sigma_{\mu}\right|(r, z)=\left|\sigma_{\mu}\right|(A)
\end{aligned}
$$

where in the last inequality we used the Schwartz inequality. This concludes the proof of (3.27) which together with (3.26) gives (3.18).

Remark 3.9. Let us observe that, as a consequence of (3.18), and thanks to the argument used to prove it, we get that

$$
\begin{equation*}
\frac{d \sigma_{\mu}}{d\left|\sigma_{\mu}\right|}(r, z)=\bar{\nu}_{\mathrm{c}}^{F_{\mu}}(r, z) \quad \text { for }\left|\sigma_{\mu}\right| \text {-a.e. }(r, z) \in U \tag{3.28}
\end{equation*}
$$

where $\bar{\nu}_{c}^{F_{\mu}}$ was defined in (1.7).
Remark 3.10. Another consequence of relation (3.18) is the following formula for the perimeter of $F_{\mu}$, namely for every $B \subset U$ Borel we have that

$$
P\left(F_{\mu} ; \Phi\left(B \times \mathbb{S}^{1}\right)\right)=2 \int_{B} \sqrt{1+\frac{1}{4}\left|r \frac{\partial}{\partial r} \xi(r, z)\right|^{2}+\frac{1}{4}\left|\nabla_{z} \mu(r, z)\right|^{2}} d r d z+\left|\left(D_{r}^{s} \xi, D_{z}^{s} \mu\right)\right|(B)
$$

where by $\frac{\partial}{\partial r} \xi$, and $\nabla_{z} \mu$ we denote the first component of $D^{a} \xi$, and the last $(k-2)$ components of $D^{a} \mu$, respectively.

## 4. Characterisation of equality cases

Proof of Theorem 1.3. Let us prove (1.8). Indeed,

$$
\begin{aligned}
&\left|\sigma_{\mu}\right|(B) \stackrel{(3.28)}{=} \int_{B} \bar{\nu}_{\mathrm{c}}^{F_{\mu}}(r, z) \cdot \frac{d \sigma_{\mu}}{d\left|\sigma_{\mu}\right|}(r, z) d\left|\sigma_{\mu}\right|(r, z)=\int_{B} \bar{\nu}_{\mathrm{c}}^{F_{\mu}}(r, z) \cdot d \sigma_{\mu}(r, z) \\
& \stackrel{(3.9)}{\leq} \int_{\partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)} \bar{\nu}_{\mathrm{c}}^{F_{\mu}}(|x|, z) \cdot \nu_{\mathrm{c}}^{E}(x, z) d \mathcal{H}^{k-1}(x, z) \leq P\left(E ; \Phi\left(B \times \mathbb{S}^{1}\right)\right)
\end{aligned}
$$

where for the last inequality we used Schwartz inequality. This, together with (3.18) proves (1.8). Immediately from the above chain of inequalities we get that conditions a) and b) are sufficient to have $P\left(F_{\mu} ; \Phi\left(B \times \mathbb{S}^{1}\right)\right)=P\left(E ; \Phi\left(B \times \mathbb{S}^{1}\right)\right)$. Indeed, by condition a) we get an equality sign in (3.9), while by condition b) we get the equality sign in the last inequality appearing above. Vice versa, let us assume that $P\left(F_{\mu} ; \Phi\left(B \times \mathbb{S}^{1}\right)\right)=P\left(E ; \Phi\left(B \times \mathbb{S}^{1}\right)\right)$. Then, by the equality sign in (3.9) we get that condition a) is satisfied. Moreover, by imposing the equality sign also in the last inequality appearing in the above relations we get that,

$$
\bar{\nu}_{\mathrm{c}}^{F_{\mu}}(|x|, z) \cdot \nu_{\mathrm{c}}^{E}(x, z)=1 \quad \text { for } \mathcal{H}^{k-1} \text {-a.e. }(x, z) \in \partial^{*} E \cap \Phi\left(B \times \mathbb{S}^{1}\right)
$$

Thus, up to remove a set $N \subset \partial^{*} E$ with $\mathcal{H}^{k-1}(N)=0$ we have that

$$
\bar{\nu}_{\mathrm{c}}^{F_{\mu}}(|x|, z) \cdot \nu_{\mathrm{c}}^{E}(x, z)=1 \quad \text { for every }(x, z) \in\left(\partial^{*} E \backslash N\right) \cap \Phi\left(B \times \mathbb{S}^{1}\right)
$$

which recalling Proposition 1.2, is equivalent to say that for every $(r, z) \in B$ such that both $\left(\partial^{*} F_{\mu}\right)_{(r, z)} \neq \emptyset$, and $\left(\partial^{*} E \backslash N\right)_{(r, z)} \neq \emptyset$, we have that $\nu_{\mathrm{c}}^{E}(x, z)=\bar{\nu}_{\mathrm{c}}^{F_{\mu}}(|x|, z)$ for every $x \in\left(\partial^{*} E \backslash\right.$ $N)_{(r, z)}$. This directly implies condition b), and so we conclude the proof.

Remark 4.1. Let $B \subset U$ be a Borel set such that we are in an equality case for (1.8) w.r.t. the set $B$. Let us stress that calling with $\tilde{B} \subset B$ the set

$$
\tilde{B}:=\left\{(r, z) \in B:\left(\partial^{*} F_{\mu}\right)_{(r, z)} \neq \emptyset, \text { and }\left(\partial^{*} E \backslash N\right)_{(r, z)} \neq \emptyset\right\},
$$

we have that

$$
\begin{equation*}
P\left(E ; \Phi\left(\tilde{B} \times \mathbb{S}^{1}\right)\right)=P\left(E ; \Phi\left(B \times \mathbb{S}^{1}\right)\right)=P\left(F_{\mu} ; \Phi\left(B \times \mathbb{S}^{1}\right)\right)=P\left(F_{\mu} ; \Phi\left(\tilde{B} \times \mathbb{S}^{1}\right)\right) \tag{4.1}
\end{equation*}
$$

Indeed, if we consider the following two sets

$$
\begin{aligned}
& B_{1}:=\left\{(r, z) \in B:\left(\partial^{*} F_{\mu}\right)_{(r, z)}=\emptyset\right\}, \\
& B_{2}:=\left\{(r, z) \in B:\left(\partial^{*} E \backslash N\right)_{(r, z)}=\emptyset\right\},
\end{aligned}
$$

we get that $B \backslash \tilde{B}=B_{1} \cup B_{2}$ and

$$
\begin{gathered}
0=P\left(F_{\mu} ; \Phi\left(B_{1} \times \mathbb{S}^{1}\right)\right)=P\left(E ; \Phi\left(B_{1} \times \mathbb{S}^{1}\right)\right)=\left|\sigma_{\mu}\right|\left(B_{1}\right) \\
0=P\left(E ; \Phi\left(B_{2} \times \mathbb{S}^{1}\right)\right)=P\left(F_{\mu} ; \Phi\left(B_{2} \times \mathbb{S}^{1}\right)\right)=\left|\sigma_{\mu}\right|\left(B_{2}\right)
\end{gathered}
$$

from which we easily deduce (4.1).

## 5. Steiner symmetrisation Setting

In this section we will present the results obtained for the circular symmetrisation, but for the Steiner setting. We will present the results without proofs since they can be obtained by adapting the arguments used in the previous sections.

Let $k \in \mathbb{N}$, with $k \geq 2$. We will decompose $\mathbb{R}^{k}$ as $\mathbb{R}^{k-1} \times \mathbb{R}$, and we will write $\left(x^{\prime}, y\right) \in \mathbb{R}^{k}$, with $x^{\prime} \in \mathbb{R}^{k-1}$ and $y \in \mathbb{R}$. We are now going to define the Steiner symmetral of a Borel set in $\mathbb{R}^{k}$ with respect to the hyperplane $\left\{\left(x^{\prime}, y\right) \in \mathbb{R}^{k}: y=0\right\}=\mathbb{R}^{k-1} \times\{0\}$. For every Borel set $E \subset \mathbb{R}^{k}$ we define

$$
E_{x^{\prime}}:=\left\{y \in \mathbb{R}:\left(x^{\prime}, y\right) \in E\right\} \quad \text { for every } x^{\prime} \in \mathbb{R}^{k-1}
$$

Let now $v: \mathbb{R}^{k-1} \rightarrow[0, \infty)$ be a Lebesgue measurable function. We will say that $E$ is $v$-distributed if

$$
v\left(x^{\prime}\right)=\mathcal{H}^{1}\left(E_{x^{\prime}}\right), \quad \text { for } \mathcal{L}^{k-1} \text {-a.e. } x^{\prime} \in \mathbb{R}^{k-1}
$$

Given a Lebesgue measurable function $v: \mathbb{R}^{k-1} \rightarrow[0, \infty)$ we define the set $F[v] \subset \mathbb{R}^{k}$ as

$$
\begin{equation*}
F[v]:=\left\{\left(x^{\prime}, y\right) \in \mathbb{R}^{k}:|y|<\frac{1}{2} v\left(x^{\prime}\right)\right\} . \tag{5.1}
\end{equation*}
$$

Remark 5.1. Note that by definition of $F[v]$, we have

$$
\left(x^{\prime}, y\right) \in F[v] \Longrightarrow\left(x^{\prime}, z\right) \in F[v] \quad \forall z \in \mathbb{R} \text { such that }|z| \leq|y|
$$

If $E \subset \mathbb{R}^{k}$ is a $v$-distributed Borel set, we say that $F[v]$ is the Steiner symmetral of $E$ with respect to the hyperplane $\left\{\left(x^{\prime}, y\right) \in \mathbb{R}^{k}: y=0\right\}$.

### 5.1. Properties of $F[v]$ and $v$. Next result is the Steiner counterpart of Lemma 3.3.

Lemma 5.2. Let $v: \mathbb{R}^{k-1} \rightarrow[0, \infty)$ be a Lebesgue measurable. Let $x^{\prime} \in \mathbb{R}^{k-1}$. Then, the functions

$$
z \mapsto \theta_{*}\left(F[v],\left(x^{\prime}, z\right)\right) \quad \text { and } \quad z \mapsto \theta^{*}\left(F[v],\left(x^{\prime}, z\right)\right)
$$

are even in $(-\infty, \infty)$ and non increasing in $[0, \infty)$.
The following result is the Steiner counterpart of Proposition 3.4.
Proposition 5.3. Let $v: \mathbb{R}^{k-1} \rightarrow[0, \infty)$ be a Lebesgue measurable function such that $F[v]$ is a set of locally finite perimeter. Suppose that $\left(x^{\prime}, y\right) \in \partial^{*} F[v]$. Then

$$
\begin{equation*}
\nu^{F[v]}\left(x^{\prime}, z\right)=\nu^{F[v]}\left(x^{\prime}, y\right) \tag{5.2}
\end{equation*}
$$

for every $z \in[\min \{y, 0\}, \max \{y, 0\}]$ such that $\left(x^{\prime}, z\right) \in\left(\partial^{*} F[v]\right)_{x^{\prime}}$.
The following remark is the Steiner counterpart of Remark 3.5 (compare it with [2, Remark 2.5]).
Remark 5.4. Let us notice that, by symmetry of the set $F[v]$ w.r.t. the hyperplane $\{y=0\} \subset \mathbb{R}^{k}$, the following property holds true: for every $\left(x^{\prime}, y\right) \in \partial^{*} F[v]$ we have that $\left(x^{\prime},-y\right) \in \partial^{*} F[v]$ and

$$
\nu^{F_{\mu}}\left(x^{\prime},-y\right)=\left(\nu_{1}^{F[v]}\left(x^{\prime}, y\right), \ldots, \nu_{k-1}^{F[v]}\left(x^{\prime}, y\right),-\nu_{k}^{F[v]}\left(x^{\prime}, y\right)\right)
$$

The following result represents the Steiner counterpart of Proposition 1.2.
Proposition 5.5. Let $v: \mathbb{R}^{k-1} \rightarrow[0, \infty)$ be a Lebesgue measurable function such that $F[v]$ is a set of locally finite perimeter. Then, for every $x^{\prime} \in \mathbb{R}^{k-1}$ such that $\left(\partial^{*} F[v]\right)_{x^{\prime}} \neq \emptyset$, the functions

$$
\begin{equation*}
y \mapsto \nu_{i}^{F[v]}\left(x^{\prime}, y\right) \quad \text { for } i=1, \ldots, k-1, \quad y \mapsto\left|\nu_{k}^{F[v]}\left(x^{\prime}, y\right)\right|, \tag{5.3}
\end{equation*}
$$

are constant in $\left(\partial^{*} F[v]\right)_{x^{\prime}}$.
Given $E \subset \mathbb{R}^{k}$ set of locally finite perimeter, we set

$$
\begin{equation*}
\nu_{\mathrm{s}}^{E}\left(x^{\prime}, y\right):=\left(\nu_{1}^{E}\left(x^{\prime}, y\right), \ldots, \nu_{k-1}^{E}\left(x^{\prime}, y\right),\left|\nu_{k}^{E}\left(x^{\prime}, y\right)\right|\right), \quad \text { for } \mathcal{H}^{k} \text {-a.e. }\left(x^{\prime}, y\right) \in \partial^{*} E . \tag{5.4}
\end{equation*}
$$

Thanks to Proposition 5.5, we set

$$
\bar{\nu}_{\mathrm{s}}^{F[v]}\left(x^{\prime}\right):= \begin{cases}\nu_{\mathrm{s}}^{F[v]}\left(x^{\prime}, y\right) & \text { if }\left(\partial^{*} F[v]\right)_{x^{\prime}} \neq \emptyset, \text { and } y \in\left(\partial^{*} F[v]\right)_{x^{\prime}}  \tag{5.5}\\ 0 & \text { otherwise }\end{cases}
$$

Next result is the Steiner counterpart of Lemma 3.6 (compare this result with [7, Lemma 3.1]).
Lemma 5.6. Let $v: \mathbb{R}^{k-1} \rightarrow[0, \infty)$ be a Lebesgue measurable function, and let $E \subset \mathbb{R}^{k}$ be a $v$-distributed set of finite perimeter and finite volume. Then, $v \in B V\left(\mathbb{R}^{k-1}\right)$. In addition, $\left|D_{i} v\right|$ is a finite Radon measure on $\mathbb{R}^{k-1}$ for every $i=1, \ldots, k-1$, and for every Borel set $B \subset \mathbb{R}^{k-1}$ we have

$$
\begin{equation*}
\int_{B} \varphi\left(x^{\prime}\right) d D_{i} v\left(x^{\prime}\right)=\int_{\partial^{*} E \cap(B \times \mathbb{R})} \varphi\left(x^{\prime}\right) \nu_{i}^{E}\left(x^{\prime}, y\right) d \mathcal{H}^{k-1}\left(x^{\prime}, y\right) \tag{5.6}
\end{equation*}
$$

for every $i=1, \ldots, k-1$, and for every bounded Borel function $\varphi: B \rightarrow \mathbb{R}$. Moreover, let $\sigma_{v}$ be the $\mathbb{R}^{k}$-valued Radon measure on $\mathbb{R}^{k-1}$ defined as

$$
\begin{equation*}
\sigma_{v}(B):=\int_{B} d\left(D_{1} v, \ldots, D_{k-1} v, 2 \mathcal{L}^{k-1}\llcorner\{v>0\})\left(x^{\prime}\right), \quad \forall B \subset \mathbb{R}^{k-1}\right. \text { Borel. } \tag{5.7}
\end{equation*}
$$

Then, for every Borel set $B \subset \mathbb{R}^{k-1}$ we get

$$
\begin{equation*}
\int_{B} \varphi\left(x^{\prime}\right) \cdot d \sigma_{v}\left(x^{\prime}\right) \leq \int_{\partial^{*} E \cap(B \times \mathbb{R})} \varphi\left(x^{\prime}\right) \cdot \nu_{\mathrm{s}}^{E}\left(x^{\prime}, y\right) d \mathcal{H}^{k-1}\left(x^{\prime}, y\right) \tag{5.8}
\end{equation*}
$$

for every bounded Borel function $\varphi: B \rightarrow \mathbb{R}^{k}$ with non-negative last component, where $\nu_{\mathrm{s}}^{E}$ was defined in (5.4). In particular, equality sign holds true in (5.8) if and only if $(E)_{x^{\prime}}$ is $\mathcal{H}^{1}$-equivalent to a segment, for $\mathcal{L}^{k-1}$-a.e. $x^{\prime} \in B$.

Remark 5.7. Under the assumptions of the above lemma, let $B \subset \mathbb{R}^{k-1}$ be a Borel set, and let $E \subset \mathbb{R}^{k}$ be a $v$-distributed set of finite perimeter and finite volume such that $(E)_{x^{\prime}}$ is $\mathcal{H}^{1}$-equivalent to a segment for $\mathcal{L}^{k-1}$-a.e. $x^{\prime} \in B$. Then, we get

$$
\begin{equation*}
\int_{B} \varphi\left(x^{\prime}\right) \cdot d \sigma_{v}\left(x^{\prime}\right)=\int_{\partial^{*} E \cap B \times \mathbb{R}} \varphi\left(x^{\prime}\right) \cdot \nu_{\mathrm{s}}^{E}\left(x^{\prime}, y\right) d \mathcal{H}^{k-1}\left(x^{\prime}, y\right) \tag{5.9}
\end{equation*}
$$

for every bounded Borel function $\varphi: B \rightarrow \mathbb{R}^{k}$.
The next result is the Steiner counterpart of Lemma 3.8 (compare this result with [4, Corollary 3.4], and with [7, Lemma 3.5]).

Lemma 5.8. Let $v: \mathbb{R}^{k-1} \rightarrow[0, \infty)$ be a Lebesgue measurable function, and let $E \subset \mathbb{R}^{k}$ be a $v$-distributed set of finite perimeter and finite volume. Then, the set $F[v] \subset \mathbb{R}^{k}$ defined in (5.1) is a set of finite perimeter and finite volume. Moreover,

$$
\begin{equation*}
\left|\sigma_{v}\right|(B)=P(F[v] ; B \times \mathbb{R}), \quad \forall B \subset \mathbb{R}^{k-1} \text { Borel, } \tag{5.10}
\end{equation*}
$$

where $\sigma_{v}$ is the Radon measure defined in (5.7).
5.2. Characterisation of equality cases. Next result is the Steiner counterpart of Theorem 1.3 (compare this result with [7, Theorem 1.1, Lemma 3.4]).

Theorem 5.9. Let $v: \mathbb{R}^{k-1} \rightarrow[0, \infty)$ be a Lebesgue measurable function, and let $E \subset \mathbb{R}^{k}$ be a $v$-distributed set of finite perimeter and finite volume. Then,

$$
\begin{equation*}
P(F[v] ; B \times \mathbb{R}) \leq P(E ; B \times \mathbb{R}), \quad \forall B \subset \mathbb{R}^{k-1} \text { Borel. } \tag{5.11}
\end{equation*}
$$

Moreover, equality holds in (5.11) for some Borel set $B \subset \mathbb{R}^{k-1}$ if and only if both the following two conditions are satisfied.
a) For $\mathcal{L}^{k-1}$-a.e. $x^{\prime} \in B$ we have that $(E)_{x^{\prime}}$ is $\mathcal{H}^{1}$-equivalent to a segment.
b) There exists $N \subset \partial^{*} E$ with $\mathcal{H}^{k-1}(N)=0$, with the property that for every $x^{\prime} \in B$ such that $\left(\partial^{*} E \backslash N\right)_{x^{\prime}} \neq \emptyset$, and $\left(\partial^{*} F[v]\right)_{x^{\prime}} \neq \emptyset$, we have that

$$
\nu_{\mathrm{s}}^{E}\left(x^{\prime}, y\right)=\bar{\nu}_{\mathrm{s}}^{F[v]}\left(x^{\prime}\right) \quad \forall y \in\left(\partial^{*} E \backslash N\right)_{x^{\prime}},
$$

where $\bar{\nu}_{\mathrm{s}}^{F[v]}$ was defined in (5.5).
Remark 5.10. By definition of $\nu_{\mathrm{s}}^{E}$, condition b) of the above result implies that for every $x^{\prime} \in B$ such that $\left(\partial^{*} E \backslash N\right)_{x^{\prime}} \neq \emptyset$, and $\left(\partial^{*} F[v]\right)_{x^{\prime}} \neq \emptyset$ the functions

$$
y \mapsto \nu_{i}^{E}\left(x^{\prime}, y\right) \quad \text { for } i=1, \ldots, k-1, \quad y \mapsto\left|\nu_{k}^{E}\left(x^{\prime}, y\right)\right|,
$$

are constant in $\left(\partial^{*} E \backslash N\right)_{x^{\prime}}$.
Remark 5.11. Let us point out that if $B=\mathbb{R}^{k-1}$, condition a) of the above result coincides with [7, (1.7) of Theorem 1.1], while condition b) is a refinement of [7, (1.8) of Theorem 1.1].

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