TEMPORAL INTERPRETATION OF MONADIC INTUITIONISTIC QUANTIFIERS

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Abstract. We show that monadic intuitionistic quantifiers admit the following temporal interpretation: "always in the future" (for $\forall$) and "sometime in the past" (for $\exists$). It is well known that Prior's intuitionistic modal logic MIPC axiomatizes the monadic fragment of the intuitionistic predicate logic, and that MIPC is translated fully and faithfully into the monadic fragment MS4 of the predicate S4 via the Gödel translation. To realize the temporal interpretation mentioned above, we introduce a new tense extension TS4 of S4 and provide a full and faithful translation of MIPC into TS4. We compare this new translation of MIPC with the Gödel translation by showing that both TS4 and MS4 can be translated fully and faithfully into a tense extension of MS4, which we denote by MS4.t. This is done by utilizing the relational semantics for these logics. As a result, we arrive at the diagram of full and faithful translations shown in Figure 1 which is commutative up to logical equivalence. We prove the finite model property (fmp) for MS4.t using algebraic semantics, and show that the fmp for the other logics involved can be derived as a consequence of the fullness and faithfulness of the translations considered.

Contents

§1. Introduction. It is well known that, unlike classical quantifiers, the interpretation of intuitionistic quantifiers is non-symmetric in that $\forall xA$ is true at a world w iff A is true at every object a in the domain D_v of every world v accessible from w, while $\exists xA$ is true at w iff A is true at some object a in the domain D_w of w. This non-symmetry is also evident in the Gödel translation of the intuitionistic predicate calculus IQC into the predicate S4, denoted QS4,

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since $\forall x A$ is translated as $\Box\forall x A^t$ while $\exists x A$ as $\exists x A^t$, where A^t is the translation of A.

Our aim is to provide a more symmetric temporal interpretation of intuitionistic quantifiers as "always in the future" (for $\forall$) and "sometime in the past" (for $\exists$). In this paper we restrict our attention to monadic quantifiers (that quantify over one fixed variable), but in Section 7 we discuss the connection to the full predicate case, which is treated in detail in [4]. One of the main reasons to treat the monadic case separately is because it gives rise to a new interesting temporal system $\mathsf{TS4}$ (see below).

It is well known that the monadic fragment of IQC is axiomatized by Prior's monadic intuitionistic propositional calculus MIPC [8, 25]. The monadic fragment of $\mathsf{QS4}$ was studied by Fischer-Servi [14] who showed that the Gödel translation of IQC into $\mathsf{QS4}$ restricts to the monadic case. We denote the monadic fragment of $\mathsf{QS4}$ by $\mathsf{MS4}$. One of our main contributions is to introduce a tense counterpart of $\mathsf{MS4}$, denoted by $\mathsf{TS4}$, and prove that a modification of the Gödel translation embeds MIPC into $\mathsf{TS4}$ fully and faithfully. This allows us to give the desired temporal interpretation of intuitionistic monadic quantifiers as "always in the future" (for $\forall$) and "sometime in the past" (for $\exists$).

While $\mathsf{MS4}$ and $\mathsf{TS4}$ are not comparable, we introduce a common extension, which we denote by $\mathsf{MS4.t}$. The system $\mathsf{MS4.t}$ can be thought of as a tense extension of $\mathsf{MS4}$. We prove that there exist full and faithful translations of MIPC, $\mathsf{MS4}$, and $\mathsf{TS4}$ into $\mathsf{MS4.t}$, yielding the diagram in Figure 1. The Gödel translation is denoted by $(\ )^t$, our new translation by $(\ )^\natural$, and the three translations into $\mathsf{MS4.t}$ by $(\ )^\flat$, $(\ )^\#$, and $(\ )^\dagger$, respectively.

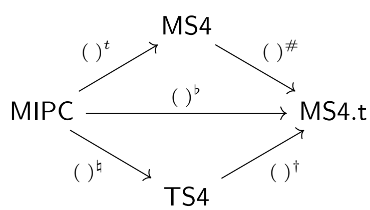


FIGURE 1. Diagram of translations.

We prove these results by employing relational semantics. In addition, we utilize algebraic semantics to prove that $\mathsf{MS4.t}$ has the fmp. It is then an easy consequence of the fullness and faithfulness of the translations considered that the other systems also have the fmp. That MIPC has the fmp was proved in [7], but the proof contained a gap, which was corrected in [15, 23]. The fmp for $\mathsf{MS4}$ follows from the results in [17, Sec. 12]. An advantage of our approach is in that it provides a uniform means for proving the fmp for all four systems in Figure 1.

In [4] we extended the translation of MIPC into $\mathsf{MS4.t}$ to the predicate setting.[1] We showed that the same interpretation of intuitionistic quantifiers can

[1]While [4] is a sequel to this paper, as it often happens, it appeared in print before this paper.

be realized via a full and faithful translation of IQC into a version of predicate S4.t that can be thought of as a predicate analogue of MS4.t. We conclude this paper by comparing the translations investigated here with those studied in [4].

§2. Logics of interest. In this section we present our four main logics of interest (see Figure 1). We start by recalling the monadic intuitionistic propositional calculus MIPC. This system was introduced by Prior [26, p. 38] and it was shown by Bull [8] that MIPC axiomatizes the monadic fragment of the intuitionistic predicate calculus IQC (that is, the fragment of IQC consisting of all predicate formulas containing one fixed variable). For this reason we denote the modalities of MIPC by $\forall$ and $\exists$. Let $\mathcal{L}$ be a propositional language and let $\mathcal{L}_{\forall\exists}$ be the extension of $\mathcal{L}$ with two modalities $\forall$ and $\exists$.

Definition 2.1. The *monadic intuitionistic propositional calculus* MIPC is the intuitionistic modal logic in the propositional modal language $\mathcal{L}_{\forall\exists}$ containing

1. all theorems of the intuitionistic propositional calculus IPC;
2. the S4-axioms for $\forall$:
   (a) $\forall(p \wedge q) \leftrightarrow (\forall p \wedge \forall q)$,
   (b) $\forall p \to p$,
   (c) $\forall p \to \forall\forall p$;
3. the S5-axioms for $\exists$:
   (a) $\exists(p \vee q) \leftrightarrow (\exists p \vee \exists q)$,
   (b) $p \to \exists p$,
   (c) $\exists\exists p \to \exists p$,
   (d) $(\exists p \wedge \exists q) \to \exists(\exists p \wedge q)$;
4. the axioms connecting $\forall$ and $\exists$:
   (a) $\exists\forall p \leftrightarrow \forall p$,
   (b) $\exists p \leftrightarrow \forall\exists p$;

and closed under the rules of modus ponens, substitution, and necessitation ($\varphi/\forall\varphi$).

Remark 2.2. There are several equivalent axiomatizations of MIPC (see, e.g., [2, Lem. 2]).

We next recall the monadic extension of S4 studied by Fischer-Servi [14] who showed that it axiomatizes the monadic fragment of the predicate S4. Let $\mathcal{L}_{\Box\forall}$ be a propositional bimodal language with two modal operators $\Box$ and $\forall$. As usual, $\Diamond$ is an abbreviation for $\neg\Box\neg$ and $\exists$ is an abbreviation for $\neg\forall\neg$.

Definition 2.3. The *monadic* S4, denoted MS4, is the smallest bimodal logic containing all theorems of the classical propositional calculus CPC, the S4-axioms for $\Box$, the S5-axioms for $\forall$, the left commutativity axiom

$$\Box\forall p \to \forall\Box p,$$

and closed under modus ponens, substitution, $\Box$-necessitation, and $\forall$-necessitation.

Remark 2.4. Recalling the definition of fusion of two logics (see [16]), MS4 is obtained from the fusion $\mathsf{S4}\otimes\mathsf{S5}$ by adding the left commutativity axiom $\Box\forall p \to$

$\forall\Box p$ which is the monadic version of the converse Barcan formula. The monadic version of the Barcan formula is the right commutativity axiom $\forall\Box p \to \Box\forall p$. Adding it to MS4 yields the product logic S4 $\times$ S5; see [16, Ch. 5] for details.

The following lemma will be useful in Section 3.

LEMMA 2.5. *An equivalent axiomatization of* MS4 *is obtained by replacing the left commutativity axiom* $\Box\forall p \to \forall\Box p$ *by* $\exists\Box p \to \Box\exists p$.

PROOF. We show that MS4 $\vdash \exists\Box p \to \Box\exists p$. That $\exists\Box p \to \Box\exists p$ together with the other axioms of MS4 implies $\Box\forall p \to \forall\Box p$ is proved similarly. Since $\forall$ is an S5-modality, $\Box\exists p \to \Box\forall\exists p$ is a theorem of MS4. By the left commutativity axiom, $\Box\forall\exists p \to \forall\Box\exists p$ is also a theorem of MS4. Therefore, MS4 $\vdash \Box\exists p \to \forall\Box\exists p$, and hence MS4 $\vdash \exists\Box\exists p \to \exists\forall\Box\exists p$. But $\exists\Box p \to \exists\Box\exists p$, $\exists\forall\Box\exists p \to \forall\Box\exists p$, and $\forall\Box\exists p \to \Box\exists p$ are all theorems of MS4 because $\forall$ is an S5-modality. Thus, MS4 $\vdash \exists\Box p \to \Box\exists p$, concluding the proof. ⊣

To introduce our main tense logic TS4, we first need to recall the tense logic S4.t. Let $\mathcal{L}_T$ be the propositional tense language with two modalities $[F]$ and $[P]$. As usual, $[F]$ is interpreted as "always in the future" and $[P]$ as "always in the past." We use the following standard abbreviations: $\langle F\rangle$ for $\neg[F]\neg$ and $\langle P\rangle$ for $\neg[P]\neg$. Then $\langle F\rangle$ is interpreted as "sometime in the future" and $\langle P\rangle$ as "sometime in the past."

DEFINITION 2.6. Let S4.t be the smallest bimodal logic containing all theorems of the classical propositional calculus CPC, the S4-axioms for $[F]$ and $[P]$, the tense axioms

$$p \to [P]\langle F\rangle p$$
$$p \to [F]\langle P\rangle p$$

and closed under modus ponens, substitution, $[F]$-necessitation, and $[P]$-necessitation.

REMARK 2.7. The system S4.t is the extension of the least tense logic K.t in which both tense modalities satisfy the S4-axioms. It was studied by several authors. Esakia [10] showed that the Gödel translation can be extended to embed the Heyting-Brouwer logic HB of Rauszer [28] into S4.t fully and faithfully. The language of HB is obtained by enriching the language of IPC by an additional connective of *coimplication*, and the logic HB is the extension of IPC by the axioms for coimplication, which are dual to the axioms for implication. Wolter [31] extended the celebrated Blok-Esakia Theorem to this setting.

We are ready to define TS4 by combining S4 and S4.t. In Section 4 we will translate MIPC into TS4 fully and faithfully. We will use S4 to translate intuitionistic connectives and S4.t to translate monadic intuitionistic quantifiers. Let $\mathcal{ML}$ be the multimodal propositional language with three modalities $\Box$, $[F]$, and $[P]$. We use $\Diamond$, $\langle F\rangle$, and $\langle P\rangle$ as usual abbreviations.

DEFINITION 2.8. The logic TS4 is the least multimodal logic containing all theorems of the classical propositional calculus CPC, the S4-axioms for $\Box$, $[F]$, and $[P]$, the tense axioms for $[F]$ and $[P]$, the connecting axioms

$$\Diamond p \to \langle F\rangle p$$
$$\langle F\rangle p \to \Diamond(\langle F\rangle p \wedge \langle P\rangle p)$$

and closed under modus ponens, substitution, and three necessitation rules (for $\Box$, $[F]$, and $[P]$).

Our final logic of interest is the monadic tense logic $\mathsf{MS4.t}$ which is obtained by combining $\mathsf{MS4}$ and $\mathsf{S4.t}$. As we will see, both $\mathsf{MS4}$ and $\mathsf{TS4}$ translate fully and faithfully into $\mathsf{MS4.t}$. In order to avoid confusion between the tense modalities of $\mathsf{TS4}$ and $\mathsf{MS4.t}$, we denote the tense modalities of $\mathsf{MS4.t}$ by $\Box_F$ and $\Box_P$. Let $\mathcal{L}_{T\forall}$ be the propositional language with the tense modalities $\Box_F$ and $\Box_P$, and the monadic modality $\forall$.

Definition 2.9. The *tense* $\mathsf{MS4}$, denoted $\mathsf{MS4.t}$, is the least multimodal logic containing all theorems of the classical propositional calculus CPC, the $\mathsf{S4.t}$-axioms for $\Box_F$ and $\Box_P$, the $\mathsf{S5}$-axioms for $\forall$, the left commutativity axiom

$$\Box_F\forall p \to \forall\Box_F p,$$

and closed under modus ponens, substitution, and the necessitation rules (for $\Box_F$, $\Box_P$, and $\forall$).

Remark 2.10. We can think of $\mathsf{MS4.t}$ as a tense extension of $\mathsf{MS4}$. It is worth stressing that $\mathsf{MS4.t}$ is not the monadic fragment of the standard predicate extension $\mathsf{QS4.t}$ of $\mathsf{S4.t}$. To see this, it is well known that the Barcan formula $\forall x\Box_F\varphi \to \Box_F\forall x\varphi$ and the converse Barcan formula $\Box_F\forall x\varphi \to \forall x\Box_F\varphi$ are both theorems of any tense predicate logic containing the standard axioms of first order logic. Therefore, both are theorems of $\mathsf{QS4.t}$. Thus, the monadic fragment of $\mathsf{QS4.t}$ contains both the left commutativity axiom $\Box_F\forall p \to \forall\Box_F p$ and the right commutativity axiom $\forall\Box_F p \to \Box_F\forall p$. On the other hand, it is easy to see (e.g., using the Kripke semantics for $\mathsf{MS4.t}$ which we will define in the next section) that, while $\mathsf{MS4.t}$ contains the left commutativity axiom, the right commutativity axiom is not provable in $\mathsf{MS4.t}$.

§3. Relational semantics. In this section we investigate the relational semantics for MIPC, $\mathsf{MS4}$, $\mathsf{TS4}$, and $\mathsf{MS4.t}$. The relational semantics for MIPC and $\mathsf{MS4}$ have already been studied in the literature, and the relational semantics for $\mathsf{TS4}$ and $\mathsf{MS4.t}$ are obtained by a straightforward adaptation.

Definition 3.1. Let R be a quasi-order (reflexive and transitive relation) on a set X. As usual, for $x \in X$, we write

$$R[x] = \{y \in X \mid xRy\} \text{ and } R^{-1}[x] = \{y \in X \mid yRx\},$$

and for $A \subseteq X$, we write

$$R[A] = \bigcup\{R[x] \mid x \in A\} \text{ and } R^{-1}[A] = \bigcup\{R^{-1}[x] \mid x \in A\}.$$

We say that $A \subseteq X$ is an R-upset if $R[A] = A$ and that it is an R-downset if $R^{-1}[A] = A$.

We first describe the relational semantics for MIPC. There are several such (see, e.g., [3]), but we concentrate on the one introduced by Ono [23].

Definition 3.2. An MIPC-*frame* is a triple $\mathfrak{F} = (X, R, Q)$ where X is a set, R is a partial order, Q is a quasi-order, and the following two conditions are satisfied:

(O1) $R \subseteq Q$,
(O2) $xQy \Rightarrow (\exists z)(xRz \ \& \ zE_Q y)$.

Here E_Q is the equivalence relation defined by $xE_Q y$ iff xQy and yQx.

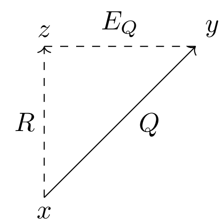


Figure 2. Condition (O2).

Remark 3.3. If U is an R-upset of an MIPC-frame $\mathfrak{F}$, then Condition (O2) implies that $E_Q[U] = Q[U]$. This motivates our interpretation of $\exists$ as "sometime in the past." Indeed, taking $Q[U]$ is the standard way to associate an operator on $\wp(X)$ to the tense modality "sometime in the past" (see, e.g., [30, p. 151]).

Definition 3.4. A *valuation* on an MIPC-frame $\mathfrak{F} = (X, R, Q)$ is a map v associating an R-upset of $\mathfrak{F}$ to any propositional letter of $\mathcal{L}_{\forall\exists}$. The connectives $\wedge, \vee, \to, \neg$ are then interpreted as in intuitionistic Kripke frames, and $\forall, \exists$ are interpreted as

$$\begin{array}{lll} x \vDash_v \forall\varphi & \text{iff} & (\forall y \in X)(xQy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v \exists\varphi & \text{iff} & (\exists y \in X)(xE_Q y \ \& \ y \vDash_v \varphi). \end{array}$$

As usual, we say that φ is *valid* in $\mathfrak{F}$, and write $\mathfrak{F} \vDash \varphi$, if $x \vDash_v \varphi$ for every valuation v and every $x \in X$.

It is well known that MIPC is a canonical logic (see, e.g., [3]). Thus, we have:

Theorem 3.5. MIPC *is a canonical logic, hence it is sound and complete with respect to its relational semantics. Therefore,*

$$\mathsf{MIPC} \vdash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi \textit{ for every } \mathsf{MIPC}\textit{-frame } \mathfrak{F}.$$

Remark 3.6. In addition, MIPC has the fmp [7, 15, 23] and hence is decidable. As we will see in Section 6, the fmp of MIPC can be derived from the fmp of MS4.t.

The relational semantics for MS4 was introduced by Esakia [12].

Definition 3.7. An MS4-*frame* is a triple $\mathfrak{F} = (X, R, E)$ where X is a set, R is a quasi-order, E is an equivalence relation, and the following commutativity condition is satisfied:

(E) $\qquad (\forall x, y, z \in X)(xEy \ \& \ yRz) \Rightarrow (\exists u \in X)(xRu \ \& \ uEz)$.

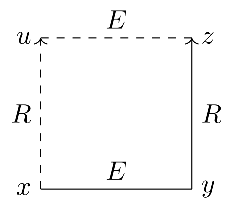


FIGURE 3. Condition (E).

DEFINITION 3.8. A *valuation* on an MS4-frame $\mathfrak{F} = (X, R, E)$ is a map v associating a subset of X to each propositional letter of $\mathcal{L}_{\Box\forall}$. The boolean connectives are interpreted as usual, and

$$\begin{array}{lll} x \vDash_v \Box\varphi & \text{iff} & (\forall y \in X)(xRy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v \forall\varphi & \text{iff} & (\forall y \in X)(xEy \Rightarrow y \vDash_v \varphi). \end{array}$$

By Lemma 2.5, in the axiomatization of MS4, the left commutativity axiom $\Box\forall p \to \forall\Box p$ can be replaced by $\exists\Box a \to \Box\exists a$. Therefore, MS4 can be axiomatized by Sahlqvist formulas (see, e.g, [5, Sec. 3.6]). Thus, by the Sahlqvist completeness theorem (see, e.g., [5, Thm. 4.42]), it is sound and complete with respect to its relational semantics:

THEOREM 3.9. MS4 *is a Sahlqvist logic, hence it is sound and complete with respect to its relational semantics. Therefore,*

$$\mathsf{MS4} \vdash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi \textit{ for every } \mathsf{MS4}\textit{-frame } \mathfrak{F}.$$

REMARK 3.10. In addition, MS4 has the fmp and is decidable. This can be derived from the results in [17, Sec. 12] (see also [16, Thms. 6.52, 9.12]). As we will see in Section 6, the fmp of MS4 can also be derived from the fmp of MS4.t.

We next recall the relational semantics for S4.t.

DEFINITION 3.11. An S4.t-frame is a pair $\mathfrak{F} = (X, Q)$ where X is a set and Q is a quasi-order on X.

REMARK 3.12. While S4.t-frames coincide with S4-frames, the difference is in the interpretation of the modalities as we use Q to interpret $[F]$ and its inverse relation $Q^{\smile}$ to interpret $[P]$.

DEFINITION 3.13. A *valuation* on an S4.t-frame $\mathfrak{F} = (X, Q)$ is a map v associating a subset of X to each propositional letter of $\mathcal{L}_T$. The boolean connectives are interpreted as usual, and the tense modalities are interpreted as

$$\begin{array}{lll} x \vDash_v [F]\varphi & \text{iff} & (\forall y \in X)(xQy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v [P]\varphi & \text{iff} & (\forall y \in X)(yQx \Rightarrow y \vDash_v \varphi). \end{array}$$

REMARK 3.14. It is straightforward to see that all the axioms of S4.t are Sahlqvist formulas. Therefore, S4.t is a Sahlqvist logic, and hence it is sound and complete with respect to its relational semantics. That S4.t has the fmp follows from [29, pp. 313–314] (see also [18, p. 44] and Remark 6.18(3)).

We now introduce the relational semantics for TS4.

Definition 3.15. A TS4-frame is a triple $\mathfrak{F} = (X, R, Q)$ where X is a set and R, Q are quasi-orders on X satisfying Conditions (O1) and (O2).

It follows that TS4-frames are a version of MIPC-frames, where the relation R is a quasi-order. We interpret $\Box$ using R, and $[F]$, $[P]$ using Q and its inverse $Q^{\smile}$.

Definition 3.16. A *valuation* on a TS4-frame $\mathfrak{F} = (X, R, Q)$ is a map associating a subset of X to each propositional letter of $\mathcal{ML}$. The classical connectives are interpreted as usual, and the modalities $\Box$, $[F]$, and $[P]$ are interpreted as

$$\begin{array}{lll} x \vDash_v \Box\varphi & \text{iff} & (\forall y \in X)(xRy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v [F]\varphi & \text{iff} & (\forall y \in X)(xQy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v [P]\varphi & \text{iff} & (\forall y \in X)(yQx \Rightarrow y \vDash_v \varphi). \end{array}$$

Consequently,

$$\begin{array}{lll} x \vDash_v \Diamond\varphi & \text{iff} & (\exists y \in X)(xRy \;\&\; y \vDash_v \varphi), \\ x \vDash_v \langle F\rangle\varphi & \text{iff} & (\exists y \in X)(xQy \;\&\; y \vDash_v \varphi), \\ x \vDash_v \langle P\rangle\varphi & \text{iff} & (\exists y \in X)(yQx \;\&\; y \vDash_v \varphi). \end{array}$$

Remark 3.17. It is straightforward to check that if (X, R, Q) is a TS4-frame, then (X, R, E_Q) is an MS4-frame, and that if (X, R, E) is an MS4-frame, then (X, R, Q_E) is a TS4-frame, where Q_E is defined by

$$xQ_Ey \text{ iff } (\exists z \in X)(xRz \;\&\; zEy).$$

If (X, R, Q) is a TS4-frame, by definition we have that

$$xQy \text{ iff } (\exists z \in X)(xRz \;\&\; zE_Qy).$$

Thus, $Q = Q_{E_Q}$. On the other hand, there exist MS4-frames (X, R, E) such that $E \neq E_{Q_E}$ (see [3, p. 24]). Therefore, this correspondence is not a bijection.

Since all TS4-axioms are Sahlqvist formulas, we have:

Theorem 3.18. TS4 *is a Sahlqvist logic, hence it is sound and complete with respect to its relational semantics. Therefore,*

$$\mathsf{TS4} \vdash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi \textit{ for every } \mathsf{TS4}\textit{-frame } \mathfrak{F}.$$

Remark 3.19. In Section 6 we will see that TS4 has the fmp and hence is decidable.

Finally, we introduce the relational semantics for MS4.t. As with S4 and S4.t, we have that MS4.t-frames are simply MS4-frames, the difference is in interpreting tense modalities.

Definition 3.20. A *valuation* on an MS4.t-frame $\mathfrak{F} = (X, R, E)$ is a map v associating a subset of X to each propositional letter of $\mathcal{L}_{T\forall}$. The boolean connectives are interpreted as usual, and

$$\begin{array}{lll} \mathfrak{F}, x \vDash_v \Box_F\varphi & \text{iff} & (\forall y \in X)(xRy \Rightarrow y \vDash_v \varphi), \\ \mathfrak{F}, x \vDash_v \Box_P\varphi & \text{iff} & (\forall y \in X)(yRx \Rightarrow y \vDash_v \varphi), \\ \mathfrak{F}, x \vDash_v \forall\varphi & \text{iff} & (\forall y \in X)(xEy \Rightarrow y \vDash_v \varphi). \end{array}$$

Since both MS4 and S4.t are Sahlqvist logics, the same is true for MS4.t. Thus, we have:

THEOREM 3.21. $\mathsf{MS4.t}$ *is a Sahlqvist logic, hence it is sound and complete with respect to its relational semantics. Therefore,*

$$\mathsf{MS4.t} \vdash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi \textit{ for every } \mathsf{MS4.t}\textit{-frame } \mathfrak{F}.$$

In Section 6 we will prove that $\mathsf{MS4.t}$ has the fmp and hence is decidable.

§4. The four translations. In this section we define the translations of Figure 1 and show that they are full and faithful by using relational semantics. We start by recalling that the Gödel translation of MIPC into $\mathsf{MS4}$ is defined by

$$\begin{aligned}
\bot^t &= \bot \\
p^t &= \Box p \quad \text{for each propositional letter } p \\
(\varphi \wedge \psi)^t &= \varphi^t \wedge \psi^t \\
(\varphi \vee \psi)^t &= \varphi^t \vee \psi^t \\
(\varphi \to \psi)^t &= \Box(\neg\varphi^t \vee \psi^t) \\
(\forall\varphi)^t &= \Box\forall\varphi^t \\
(\exists\varphi)^t &= \exists\varphi^t.
\end{aligned}$$

DEFINITION 4.1. The translation $(-)^\natural : \mathsf{MIPC} \to \mathsf{TS4}$ is defined as $(-)^t$ on propositional letters, $\bot$, $\wedge$, $\vee$, and $\to$; and for $\forall$ and $\exists$ we set:

$$\begin{aligned}
(\forall\varphi)^\natural &= [F]\varphi^\natural \\
(\exists\varphi)^\natural &= \langle P\rangle\varphi^\natural.
\end{aligned}$$

REMARK 4.2. Thus, $(-)^\natural$ realizes the desired temporal interpretation of the intuitionistic monadic quantifiers: $\forall$ as "always in the future" and $\exists$ as "sometime in the past."

DEFINITION 4.3. The translation $(-)^\dagger : \mathsf{TS4} \to \mathsf{MS4.t}$ is defined by

$$\begin{aligned}
p^\dagger &= p \quad \text{for each propositional letter } p \\
(\varphi \circ \psi)^\dagger &= \varphi^\dagger \circ \psi^\dagger \quad \text{for } \circ = \wedge, \vee \\
(\neg\varphi)^\dagger &= \neg\varphi^\dagger \\
(\Box\varphi)^\dagger &= \Box_F\varphi^\dagger \\
([F]\varphi)^\dagger &= \Box_F\forall\varphi^\dagger \\
([P]\varphi)^\dagger &= \forall\Box_P\varphi^\dagger.
\end{aligned}$$

REMARK 4.4.

1. The translation $(-)^\dagger : \mathsf{TS4} \to \mathsf{MS4.t}$ is suggested by the correspondence between $\mathsf{TS4}$-frames and $\mathsf{MS4}$-frames described in Remark 3.17. Each $\mathsf{MS4.t}$-frame $\mathfrak{F} = (X, R, E)$ is an $\mathsf{MS4}$-frame, the relation Q_E on the corresponding $\mathsf{TS4}$-frame is the composition of R and E, and the inverse relation $Q_E^\smile$ is the composition of E and $R^\smile$. Therefore, the modalities $[F]$ and $[P]$ are translated as $\Box_F\forall$ and $\forall\Box_P$, respectively. Notice that, since $\mathsf{MS4}$ lacks a modality corresponding to the relation $R^\smile$, we are not able to translate $\mathsf{TS4}$ into $\mathsf{MS4}$.

2. It is natural to consider a modification of $(-)^\dagger : \mathsf{TS4} \to \mathsf{MS4.t}$ where $[P]$ is translated as $\Box_P\forall$. However, such a modification does not result in a faithful translation. Nevertheless, as we will see in Section 5, its composition with $(-)^\natural : \mathsf{MIPC} \to \mathsf{TS4}$ is full and faithful.

The translation of MS4 into MS4.t reflects that MS4.t is a tense extension of MS4.

Definition 4.5. The translation $(-)^\# : \mathsf{MS4} \to \mathsf{MS4.t}$ replaces in each formula φ of $\mathcal{L}_{\Box\forall}$ every occurrence of $\Box$ with $\Box_F$.

We show that these four translations are full and faithful by using relational semantics. For this we first need to generalize the well-known notion of the skeleton of an S4-frame (see, e.g., [9, Sec. 3.9]).

Definition 4.6.

1. If R is a quasi-order on a set X, we define $\sim$ to be the equivalence relation on X given by
$$x \sim y \text{ iff } xRy \text{ and } yRx.$$
We let X' be the set of equivalence classes of $\sim$, and define R' on X' by
$$[x]R'[y] \text{ iff } xRy.$$
2. Let $\mathfrak{F} = (X, R, E)$ be an MS4-frame. Recall that Q_E is defined on X by xQ_Ey iff $(\exists z \in X)(xRz \;\&\; zEy)$ (see Remark 3.17). Define Q' on X' by
$$[x]Q'[y] \text{ iff } xQ_Ey.$$
We call $\mathfrak{F}^t = (X', R', Q')$ the *skeleton* of the MS4-frame $\mathfrak{F}$.
3. Let $\mathfrak{F} = (X, R, Q)$ be a TS4-frame. Define Q' on X' by
$$[x]Q'[y] \text{ iff } xQy.$$
We call $\mathfrak{F}^\natural = (X', R', Q')$ the *skeleton* of the TS4-frame $\mathfrak{F}$.
4. For an MS4.t-frame $\mathfrak{F} = (X, R, E)$ let $\mathfrak{F}^\dagger = (X, R, Q_E)$ where Q_E is defined as in (2).

Remark 4.7. If a TS4-frame $\mathfrak{F} = (X, R, Q)$ is such that R is a partial order, then $\mathfrak{F}$ is isomorphic to its skeleton $\mathfrak{F}^\natural$. However, we cannot always recover an MS4-frame $\mathfrak{F} = (X, R, E)$ from its skeleton $\mathfrak{F}^t$ even if R is a partial order. Indeed, it is not always the case that $E = E_{Q_E}$. Nonetheless, if $\mathfrak{F}$ is canonical (and in particular finite) and R is a partial order, then $E = E_{Q_E}$; see [3, Sec. 2] for details.

Proposition 4.8.

1. *If $\mathfrak{F}$ is an* MS4*-frame, then $\mathfrak{F}^t$ is an* MIPC*-frame.*
2. *If $\mathfrak{F}$ is a* TS4*-frame, then $\mathfrak{F}^\natural$ is an* MIPC*-frame.*
3. *If $\mathfrak{F}$ is an* MS4.t*-frame, then $\mathfrak{F}^\dagger$ is a* TS4*-frame.*

Proof. (1). It is well known that (X', R') is an intuitionistic Kripke frame. That Q' is well defined follows from Condition (E). Showing that Q' is a quasi-order and that (O1), (O2) hold in $\mathfrak{F}^t$ is straightforward.

(2). We have that Q' is well defined on X' because $R \subseteq Q$ in $\mathfrak{F}$. Showing that Q' is a quasi-order and that (O1), (O2) hold in $\mathfrak{F}^\natural$ is straightforward.

(3). Since MS4.t-frames coincide with MS4-frames, it follows from Remark 3.17 that $\mathfrak{F}^\dagger$ is a TS4-frame. ⊣

It is well known that an S4-frame validates the Gödel translation of an intuitionistic formula φ iff its skeleton validates φ (see, e.g., [9, Cor. 3.82]). We will prove in Proposition 4.11 that analogous results hold for the four translations defined in this section. For this we need the following technical lemma.

LEMMA 4.9. *For any formula χ of $\mathcal{L}_{\forall\exists}$, we have*

$$\mathsf{MS4} \vdash \chi^t \to \Box\chi^t.$$

Consequently, if $\mathfrak{F} = (X, R, E)$ is an MS4*-frame and v a valuation on $\mathfrak{F}$, then the set of points $x \in X$ such that $\mathfrak{F}, x \vDash_v \chi^t$ forms an R-upset of $\mathfrak{F}$.*

PROOF. We prove that $\mathsf{MS4} \vdash \chi^t \to \Box\chi^t$ by induction on the complexity of χ. This is obvious when $\chi = \bot$. The cases when χ is p, $\varphi \to \psi$, or $\forall\varphi$ follow from the axiom $\Box\varphi \to \Box\Box\varphi$. We next consider the cases when χ is $\varphi \wedge \psi$ or $\varphi \vee \psi$. Suppose that the claim is true for φ and ψ, so $\varphi^t \to \Box\varphi^t$ and $\psi^t \to \Box\psi^t$ are theorems of MS4. Then $\varphi^t \wedge \psi^t \to \Box(\varphi^t \wedge \psi^t)$ and $\varphi^t \vee \psi^t \to \Box(\varphi^t \vee \psi^t)$ are also theorems of MS4. Finally, if χ is $\exists\varphi$ and $\mathsf{MS4} \vdash \varphi^t \to \Box\varphi^t$, then $\mathsf{MS4} \vdash \exists\varphi^t \to \exists\Box\varphi^t$. Therefore, since $\mathsf{MS4} \vdash \exists\Box\varphi^t \to \Box\exists\varphi^t$ by Lemma 2.5, we conclude that $\mathsf{MS4} \vdash \exists\varphi^t \to \Box\exists\varphi^t$.

Let $\mathfrak{F} = (X, R, E)$ be an MS4-frame, v a valuation of $\mathfrak{F}$, and $x \in X$. Since $\mathsf{MS4} \vdash \chi^t \to \Box\chi^t$, if $\mathfrak{F}, x \vDash_v \chi^t$, then $\mathfrak{F}, x \vDash_v \Box\chi^t$. Therefore, for each y such that xRy we have $\mathfrak{F}, y \vDash_v \chi^t$. Thus, $\{x \in X \mid \mathfrak{F}, x \vDash_v \chi^t\}$ is an R-upset. ⊣

The next result generalizes to our setting a well-known correspondence result [9, Lem. 3.81] between IPC-models and S4-models.

PROPOSITION 4.10.

1. *For each valuation v on an* MS4*-frame $\mathfrak{F}$ there is a valuation v' on $\mathfrak{F}^t$ such that for each $x \in \mathfrak{F}$ and $\mathcal{L}_{\forall\exists}$-formula φ, we have*

$$\mathfrak{F}^t, [x] \vDash_{v'} \varphi \quad \textit{iff} \quad \mathfrak{F}, x \vDash_v \varphi^t.$$

2. *For each valuation v on a* TS4*-frame $\mathfrak{F}$ there is a valuation v' on $\mathfrak{F}^\natural$ such that for each $x \in \mathfrak{F}$ and $\mathcal{L}_{\forall\exists}$-formula φ, we have*

$$\mathfrak{F}^\natural, [x] \vDash_{v'} \varphi \quad \textit{iff} \quad \mathfrak{F}, x \vDash_v \varphi^\natural.$$

3. *Each valuation v on an* MS4.t*-frame $\mathfrak{F}$ is also a valuation on $\mathfrak{F}^\dagger$ and for each $x \in \mathfrak{F}$ and $\mathcal{ML}$-formula φ, we have*

$$\mathfrak{F}^\dagger, x \vDash_v \varphi \quad \textit{iff} \quad \mathfrak{F}, x \vDash_v \varphi^\dagger.$$

4. *Each valuation v on an* MS4.t*-frame $\mathfrak{F}$ is also a valuation on $\mathfrak{F}$ as an* MS4*-frame and for each $x \in \mathfrak{F}$ and $\mathcal{L}_{\Box\forall}$-formula φ, we have*

$$\mathfrak{F}, x \vDash_v \varphi \quad \textit{iff} \quad \mathfrak{F}, x \vDash_v \varphi^\#.$$

PROOF. (1). Define v' on $\mathfrak{F}^t$ by $v'(p) = \{[x] \in X' \mid R[x] \subseteq v(p)\}$. We show that $\mathfrak{F}^t, [x] \vDash_{v'} \varphi$ iff $\mathfrak{F}, x \vDash_v \varphi^t$ by induction on the complexity of φ. Since $v'(p) = \{[x] \mid \mathfrak{F}, x \vDash_v \Box p\}$, the claim is obvious when φ is a propositional letter. We prove the claim for φ of the form $\forall\psi$ and $\exists\psi$ since the other cases are well known (see, e.g., [9, Lem. 3.81]). Suppose $\varphi = \forall\psi$. By the definition of Q' and induction hypothesis, we have

$$\begin{aligned}\mathfrak{F}^t, [x] \vDash_{v'} \forall\psi &\text{ iff } (\forall [y] \in X')([x]Q'[y] \Rightarrow \mathfrak{F}^t, [y] \vDash_{v'} \psi)\\ &\text{ iff } (\forall y \in X)(xQ_E y \Rightarrow \mathfrak{F}^t, [y] \vDash_{v'} \psi)\\ &\text{ iff } (\forall y \in X)(xQ_E y \Rightarrow \mathfrak{F}, y \vDash_v \psi^t).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathfrak{F}, x \vDash_v (\forall\psi)^t &\text{ iff } \mathfrak{F}, x \vDash_v \Box\forall\psi^t\\ &\text{ iff } (\forall z \in X)(xRz \Rightarrow (\forall y \in X)(zEy \Rightarrow \mathfrak{F}, y \vDash_v \psi^t))\\ &\text{ iff } (\forall y \in X)(xQ_E y \Rightarrow \mathfrak{F}, y \vDash_v \psi^t).\end{aligned}$$

Thus, $\mathfrak{F}^t, [x] \vDash_{v'} \forall\psi$ iff $\mathfrak{F}, x \vDash_v (\forall\psi)^t$.

Suppose $\varphi = \exists\psi$. As noted in Remark 3.3, Q' and $E_{Q'}$ coincide on R'-upsets, and it is straightforward to see by induction that the set $\{[y] \mid \mathfrak{F}^t, [y] \vDash_{v'} \psi\}$ is an R'-upset. Therefore, by the induction hypothesis,

$$\begin{aligned}\mathfrak{F}^t, [x] \vDash_{v'} \exists\psi &\text{ iff } (\exists [y] \in X')([x]E_{Q'}[y] \;\&\; \mathfrak{F}^t, [y] \vDash_{v'} \psi)\\ &\text{ iff } [x] \in E_{Q'}[\{[y] \mid \mathfrak{F}^t, [y] \vDash_{v'} \psi\}]\\ &\text{ iff } [x] \in Q'[\{[y] \mid \mathfrak{F}^t, [y] \vDash_{v'} \psi\}]\\ &\text{ iff } x \in Q_E[\{y \mid \mathfrak{F}^t, [y] \vDash_{v'} \psi\}]\\ &\text{ iff } x \in Q_E[\{y \mid \mathfrak{F}, y \vDash_v \psi^t\}].\end{aligned}$$

On the other hand, since $\{y \mid \mathfrak{F}, y \vDash_v \psi^t\}$ is an R-upset (see Lemma 4.9), and E and Q_E coincide on R-upsets,

$$\begin{aligned}\mathfrak{F}, x \vDash_v (\exists\psi)^t &\text{ iff } \mathfrak{F}, x \vDash_v \exists\psi^t\\ &\text{ iff } (\exists y \in X)(xEy \;\&\; \mathfrak{F}, y \vDash_v \psi^t)\\ &\text{ iff } x \in E[\{y \mid \mathfrak{F}, y \vDash_v \psi^t\}]\\ &\text{ iff } x \in Q_E[\{y \mid \mathfrak{F}, y \vDash_v \psi^t\}].\end{aligned}$$

Thus, $\mathfrak{F}^t, [x] \vDash_{v'} \exists\psi$ iff $\mathfrak{F}, x \vDash_v (\exists\psi)^t$.

(2). As in (1) we define v' by $v'(p) = \{[x] \in X' \mid R[x] \subseteq v(p)\}$. We show that $\mathfrak{F}^\natural, [x] \vDash_{v'} \varphi$ iff $\mathfrak{F}, x \vDash_v \varphi^\natural$ by induction on the complexity of φ. It is sufficient to only consider the cases when φ is of the form $\forall\psi$ or $\exists\psi$. Suppose $\varphi = \forall\psi$. Then by the definition of Q' and induction hypothesis,

$$\begin{aligned}\mathfrak{F}^\natural, [x] \vDash_{v'} \forall\psi &\text{ iff } (\forall [y] \in X')([x]Q'[y] \Rightarrow \mathfrak{F}^\natural, [y] \vDash_{v'} \psi)\\ &\text{ iff } (\forall y \in X)(xQy \Rightarrow \mathfrak{F}^\natural, [y] \vDash_{v'} \psi)\\ &\text{ iff } (\forall y \in X)(xQy \Rightarrow \mathfrak{F}, y \vDash_v \psi^\natural)\\ &\text{ iff } \mathfrak{F}, x \vDash_v [F]\psi^\natural\\ &\text{ iff } \mathfrak{F}, x \vDash_v (\forall\psi)^\natural.\end{aligned}$$

Suppose $\varphi = \exists\psi$. As noted in Remark 3.3, Q' and $E_{Q'}$ coincide on R'-upsets. Since the set $\{[y] \mid \mathfrak{F}^\natural, [y] \vDash_{v'} \psi\}$ is an R'-upset, by the induction hypothesis, we have

$$\begin{aligned}
\mathfrak{F}^\natural, [x] \vDash_{v'} \exists\psi &\text{ iff } (\exists [y] \in X')([x] E_{Q'} [y] \ \&\ \mathfrak{F}^\natural, [y] \vDash_{v'} \psi) \\
&\text{ iff } [x] \in E_{Q'}[\{[y] \mid \mathfrak{F}^\natural, [y] \vDash_{v'} \psi\}] \\
&\text{ iff } [x] \in Q'[\{[y] \mid \mathfrak{F}^\natural, [y] \vDash_{v'} \psi\}] \\
&\text{ iff } x \in Q[\{y \mid \mathfrak{F}^\natural, [y] \vDash_{v'} \psi\}] \\
&\text{ iff } x \in Q[\{y \mid \mathfrak{F}, y \vDash_{v} \psi^\natural\}] \\
&\text{ iff } (\exists y \in X)(yQx \ \&\ \mathfrak{F}, y \vDash_v \psi^\natural) \\
&\text{ iff } \mathfrak{F}, x \vDash_v \langle P \rangle \psi^\natural \\
&\text{ iff } \mathfrak{F}, x \vDash_v (\exists\psi)^\natural.
\end{aligned}$$

(3). It is clear that if v is a valuation on $\mathfrak{F}$, then v is also a valuation on $\mathfrak{F}^\dagger$. We show that $\mathfrak{F}^\dagger, x \vDash_v \varphi$ iff $\mathfrak{F}, x \vDash_v \varphi^\dagger$ by induction on the complexity of φ. The only nontrivial cases are when φ is of the form $\Box\psi$, $[F]\psi$, and $[P]\psi$. Suppose $\varphi = \Box\psi$. Then, by the induction hypothesis,

$$\begin{aligned}
\mathfrak{F}^\dagger, x \vDash_v \Box\psi &\text{ iff } (\forall y \in X)(xRy \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi) \\
&\text{ iff } (\forall y \in X)(xRy \Rightarrow \mathfrak{F}, y \vDash_v \psi^\dagger) \\
&\text{ iff } \mathfrak{F}, x \vDash_v \Box_F \psi^\dagger \\
&\text{ iff } \mathfrak{F}, x \vDash_v (\Box\psi)^\dagger.
\end{aligned}$$

Suppose $\varphi = [F]\psi$. Then, by the induction hypothesis,

$$\begin{aligned}
\mathfrak{F}^\dagger, x \vDash_v [F]\psi &\text{ iff } (\forall y \in X)(xQ_E y \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi) \\
&\text{ iff } (\forall z \in X)(xRz \Rightarrow (\forall y \in X)(zEy \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi)) \\
&\text{ iff } (\forall z \in X)(xRz \Rightarrow (\forall y \in X)(zEy \Rightarrow \mathfrak{F}, y \vDash_v \psi^\dagger)) \\
&\text{ iff } (\forall z \in X)(xRz \Rightarrow \mathfrak{F}, z \vDash \forall\psi^\dagger) \\
&\text{ iff } \mathfrak{F}, x \vDash_v \Box_F \forall\psi^\dagger \\
&\text{ iff } \mathfrak{F}, x \vDash_v ([F]\psi)^\dagger.
\end{aligned}$$

Suppose $\varphi = [P]\psi$. Then, by the induction hypothesis,

$$\begin{aligned}
\mathfrak{F}^\dagger, x \vDash_v [P]\psi &\text{ iff } (\forall y \in X)(yQ_E x \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi) \\
&\text{ iff } (\forall y, z \in X)(yRz \ \&\ zEx \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi) \\
&\text{ iff } (\forall z \in X)(zEx \Rightarrow (\forall y \in X)(yRz \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi)) \\
&\text{ iff } (\forall z \in X)(zEx \Rightarrow (\forall y \in X)(yRz \Rightarrow \mathfrak{F}, y \vDash_v \psi^\dagger)) \\
&\text{ iff } (\forall z \in X)(zEx \Rightarrow \mathfrak{F}, z \vDash \Box_P \psi^\dagger) \\
&\text{ iff } (\forall z \in X)(xEz \Rightarrow \mathfrak{F}, z \vDash \Box_P \psi^\dagger) \\
&\text{ iff } \mathfrak{F}, x \vDash_v \forall\Box_P \psi^\dagger \\
&\text{ iff } \mathfrak{F}, x \vDash_v ([P]\psi)^\dagger.
\end{aligned}$$

(4). This is an immediate consequence of the definition of $(-)^{\#}$ and the relational semantics for MS4 and MS4.t. ⊣

Proposition 4.11.

1. *For each* MS4*-frame* $\mathfrak{F}$ *and* $\mathcal{L}_{\forall\exists}$*-formula* φ*, we have*
$$\mathfrak{F}^t \vDash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi^t.$$
2. *For each* TS4*-frame* $\mathfrak{F}$ *and* $\mathcal{L}_{\forall\exists}$*-formula* φ*, we have*
$$\mathfrak{F}^\natural \vDash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi^\natural.$$
3. *For each* MS4.t*-frame* $\mathfrak{F}$ *and* $\mathcal{ML}$*-formula* φ*, we have*
$$\mathfrak{F}^\dagger \vDash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi^\dagger.$$
4. *For each* MS4.t*-frame* $\mathfrak{F}$ *and* $\mathcal{L}_{\Box\forall}$*-formula* φ*, we have*
$$\mathfrak{F} \vDash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi^{\#}.$$

Proof. We only prove (2) since the proofs of (1), (3), and (4) are similar. If $\mathfrak{F} \nvDash \varphi^\natural$, then there is a valuation v on $\mathfrak{F}$ such that $\mathfrak{F}, x \nvDash_v \varphi^\natural$ for some $x \in X$. By Proposition 4.10(2), v' is a valuation on $\mathfrak{F}^\natural$ such that $\mathfrak{F}^\natural, [x] \nvDash_{v'} \varphi$. Therefore, $\mathfrak{F}^\natural \nvDash \varphi$. If $\mathfrak{F}^\natural \nvDash \varphi$, then there is a valuation w on $\mathfrak{F}^\natural$ and $[x] \in X'$ such that $\mathfrak{F}^\natural, [x] \nvDash_w \varphi$. Let v be the valuation on $\mathfrak{F}$ given by $v(p) = \{x \mid [x] \in w(p)\}$. Since $\mathfrak{F}^\natural$ is an MIPC-frame, $w(p)$ is an R'-upset of $\mathfrak{F}^\natural$ for each p. So $v(p)$ is an R-upset of $\mathfrak{F}$ for each p. Therefore, $w = v'$ because
$$v'(p) = \{[x] \in X' \mid R[x] \subseteq v(p)\} = \{[x] \in X' \mid x \in v(p)\} = w(p).$$
Thus, $\mathfrak{F}^\natural, [x] \nvDash_{v'} \varphi$. By Proposition 4.10(2), $\mathfrak{F}, x \nvDash_v \varphi^\natural$. Consequently, $\mathfrak{F} \nvDash \varphi^\natural$. ⊣

In order to show that the translations are full, we also need the following result, which generalizes to our setting a well-known fact that each IPC-frame is the skeleton of an S4-frame.

Proposition 4.12.

1. *For each* MIPC*-frame* $\mathfrak{G}$ *there is an* MS4*-frame* $\mathfrak{F}$ *such that* $\mathfrak{G}$ *is isomorphic to* $\mathfrak{F}^t$*.*
2. *Each* MIPC*-frame* $\mathfrak{G}$ *is also a* TS4*-frame and* $\mathfrak{G}^\natural$ *is isomorphic to* $\mathfrak{G}$*.*
3. *For each* TS4*-frame* $\mathfrak{G}$ *there is an* MS4.t*-frame* $\mathfrak{F}$ *such that* $\mathfrak{G} = \mathfrak{F}^\dagger$*.*

Proof. (1). Let $\mathfrak{G} = (X, R, Q)$ be an MIPC-frame. We show that $\mathfrak{F} = (X, R, E_Q)$ is an MS4-frame. If xE_Qy and yRz, then by definition of E_Q and Condition (O1), xQy and yQz. Since Q is transitive, xQz. Condition (O2) then implies that there is $u \in X$ with xRu and uE_Qz. Thus, $\mathfrak{F}$ is an MS4-frame. Since R is a partial order, it is an immediate consequence of Definition 4.6(1) that $\sim$ is the identity relation. It then follows from condition (O2) that $Q = Q_{E_Q}$, and hence $\mathfrak{G}$ is isomorphic to $\mathfrak{F}^t$.

(2). Let $\mathfrak{G} = (X, R, Q)$ be an MIPC-frame. It is clear from the definition of TS4-frames that $\mathfrak{G}$ is also a TS4-frame. Since R is a partial order, $\sim$ is the identity relation. Therefore, $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^\natural$.

(3). Let $\mathfrak{G} = (X, R, Q)$ be a TS4-frame. As we observed in Remark 3.17, $\mathfrak{F} = (X, R, E_Q)$ is an MS4-frame, and so an MS4.t-frame. By definition of TS4-frames we have that $Q = Q_{E_Q}$, and hence $\mathfrak{G} = \mathfrak{F}^\dagger$. ⊣

We are ready to prove the main result of this section that the four translations are full and faithful. That the Gödel translation of MIPC into MS4 is full and faithful was first observed by Fischer-Servi [14] using the translations of MIPC and MS4 into IQC and QS4 respectively, and the predicate version of the Gödel translation. In [15] she gave a different proof of this result, using that MIPC has the fmp. Our proof utilizes the relational semantics and generalizes the semantic proof that the Gödel translation of IPC into S4 is full and faithful (see, e.g., [9, Sec. 3.9]).

THEOREM 4.13.

1. *The Gödel translation* $(-)^t$ *of* MIPC *into* MS4 *is full and faithful; that is,*

$$\mathsf{MIPC} \vdash \varphi \quad \textit{iff} \quad \mathsf{MS4} \vdash \varphi^t.$$

2. *The translation* $(-)^\natural$ *of* MIPC *into* TS4 *is full and faithful; that is,*

$$\mathsf{MIPC} \vdash \varphi \quad \textit{iff} \quad \mathsf{TS4} \vdash \varphi^\natural.$$

3. *The translation* $(-)^\dagger$ *of* TS4 *into* MS4.t *is full and faithful; that is,*

$$\mathsf{TS4} \vdash \varphi \quad \textit{iff} \quad \mathsf{MS4.t} \vdash \varphi^\dagger.$$

4. *The translation* $(-)^\#$ *of* MS4 *into* MS4.t *is full and faithful; that is,*

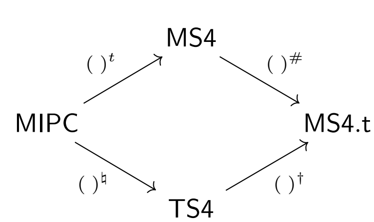
$$\mathsf{MS4} \vdash \varphi \quad \textit{iff} \quad \mathsf{MS4.t} \vdash \varphi^\#.$$

PROOF. We prove (2). For faithfulness, suppose that TS4 $\nvdash \varphi^\natural$. By Theorem 3.18, there is a TS4-frame $\mathfrak{F}$ such that $\mathfrak{F} \nvDash \varphi^\natural$. By Propositions 4.8(2) and 4.11(2), $\mathfrak{F}^\natural$ is an MIPC-frame and $\mathfrak{F}^\natural \nvDash \varphi$. Thus, by Theorem 3.5, MIPC $\nvdash \varphi$. For fullness, let MIPC $\nvdash \varphi$. Then there is an MIPC-frame $\mathfrak{G}$ such that $\mathfrak{G} \nvDash \varphi$. By Proposition 4.12(2), there is a TS4-frame such that $\mathfrak{G}$ is isomorphic to $\mathfrak{F}^\natural$. Therefore, $\mathfrak{F}^\natural \nvDash \varphi$. Proposition 4.11(2) implies that $\mathfrak{F} \nvDash \varphi^\natural$. Thus, TS4 $\nvdash \varphi^\natural$.

The proofs of (1), (3), and (4) are obtained analogously using Theorems 3.5, 3.9 3.18, and 3.21, and Propositions 4.8, 4.11, and 4.12 ⊣

§5. Compositions of the translations. In this section we show that the translations introduced in the previous section form a commutative diagram up to logical equivalence.

We denote the composition of $(-)^\#$ and $(-)^t$ by $(-)^{t\#}$, and the composition of $(-)^\dagger$ and $(-)^\natural$ by $(-)^{\natural\dagger}$. Since we proved that all these four translations are full and faithful, we also have that $(-)^{t\#}$ and $(-)^{\natural\dagger}$ are full and faithful translations of MIPC into MS4.t. We have thus obtained the following diagram of full and faithful translations. We next show that this diagram is commutative up to logical equivalence in MS4.t.



LEMMA 5.1. *For each formula φ of $\mathcal{L}_{\forall\exists}$, we have*

$$\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \Diamond_P \varphi^{t\#}.$$

PROOF. By Lemma 4.9 and Theorem 4.13(4), $\mathsf{MS4.t} \vdash \varphi^{t\#} \to \Box_F \varphi^{t\#}$. Therefore, $\mathsf{MS4.t} \vdash \Diamond_P \varphi^{t\#} \to \Diamond_P \Box_F \varphi^{t\#}$. The tense axiom then gives $\mathsf{MS4.t} \vdash \Diamond_P \varphi^{t\#} \to \varphi^{t\#}$. Thus, $\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \Diamond_P \varphi^{t\#}$. ⊣

THEOREM 5.2. *For each $\mathcal{L}_{\forall\exists}$-formula χ we have*

$$\mathsf{MS4.t} \vdash \chi^{t\#} \leftrightarrow \chi^{\natural\dagger}.$$

PROOF. The two compositions compare as follows:

$\bot^{t\#} = \bot$	$\bot^{\natural\dagger} = \bot$
$p^{t\#} = \Box_F p$	$p^{\natural\dagger} = \Box_F p$
$(\varphi \wedge \psi)^{t\#} = \varphi^{t\#} \wedge \psi^{t\#}$	$(\varphi \wedge \psi)^{\natural\dagger} = \varphi^{\natural\dagger} \wedge \psi^{\natural\dagger}$
$(\varphi \vee \psi)^{t\#} = \varphi^{t\#} \vee \psi^{t\#}$	$(\varphi \vee \psi)^{\natural\dagger} = \varphi^{\natural\dagger} \vee \psi^{\natural\dagger}$
$(\varphi \to \psi)^{t\#} = \Box_F(\neg\varphi^{t\#} \vee \psi^{t\#})$	$(\varphi \to \psi)^{\natural\dagger} = \Box_F(\neg\varphi^{\natural\dagger} \vee \psi^{\natural\dagger})$
$(\forall\varphi)^{t\#} = \Box_F \forall \varphi^{t\#}$	$(\forall\varphi)^{\natural\dagger} = \Box_F \forall \varphi^{\natural\dagger}$
$(\exists\varphi)^{t\#} = \exists\varphi^{t\#}$	$(\exists\varphi)^{\natural\dagger} = (\langle P\rangle\varphi^{\natural})^{\dagger} = (\neg[P]\neg\varphi^{\natural})^{\dagger}$ $= \neg\forall\Box_P\neg\varphi^{\natural\dagger}$

Thus, they are identical except the $\exists$-clause. Therefore, to prove that $\mathsf{MS4.t} \vdash \chi^{t\#} \leftrightarrow \chi^{\natural\dagger}$ it is sufficient to prove that $\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \varphi^{\natural\dagger}$ implies $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \neg\forall\Box_P\neg\varphi^{\natural\dagger}$. Since $\mathsf{MS4.t} \vdash \neg\forall\Box_P\neg\varphi^{\natural\dagger} \leftrightarrow \exists\Diamond_P\varphi^{\natural\dagger}$, it is enough to prove that $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \exists\Diamond_P\varphi^{\natural\dagger}$. From the assumption $\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \varphi^{\natural\dagger}$ it follows that $\mathsf{MS4.t} \vdash \exists\Diamond_P\varphi^{t\#} \leftrightarrow \exists\Diamond_P\varphi^{\natural\dagger}$. By Lemma 5.1, $\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \Diamond_P\varphi^{t\#}$ and hence $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \exists\Diamond_P\varphi^{t\#}$. Thus, $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \exists\Diamond_P\varphi^{\natural\dagger}$. ⊣

As we pointed out in Remark 4.4(2), there is another natural translation of MIPC into MS4.t.

DEFINITION 5.3. Let $(-)^\flat : \mathsf{MIPC} \to \mathsf{MS4.t}$ be the translation that differs from $(-)^{t\#}$ and $(-)^{\natural\dagger}$ only in the $\exists$-clause:

$$(\exists\varphi)^\flat = \Diamond_P \exists\varphi^\flat.$$

The translation $(-)^\flat$ provides a temporal interpretation of intuitionistic monadic quantifiers that is similar to the translation $(-)^\natural$ (see also Section 7).

THEOREM 5.4. *For each $\mathcal{L}_{\forall\exists}$-formula χ we have*

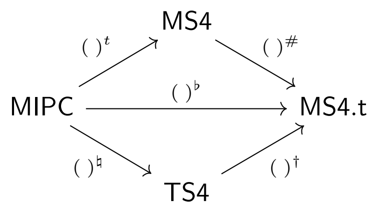
$$\mathsf{MS4.t} \vdash \chi^\flat \leftrightarrow \chi^{t\#}.$$

Consequently, the translation $(-)^\flat$ of MIPC *into* MS4.t *is full and faithful.*

PROOF. The translations $(-)^\flat$ and $(-)^{t\#}$ are identical except the $\exists$-clause. Therefore, to prove that $\mathsf{MS4.t} \vdash \chi^\flat \leftrightarrow \chi^{t\#}$ it is sufficient to prove that $\mathsf{MS4.t} \vdash \varphi^\flat \leftrightarrow \varphi^{t\#}$ implies $\mathsf{MS4.t} \vdash \Diamond_P\exists\varphi^\flat \leftrightarrow \exists\varphi^{t\#}$. By Lemma 5.1, $\mathsf{MS4.t} \vdash (\exists\varphi)^{t\#} \leftrightarrow \Diamond_P(\exists\varphi)^{t\#}$ which means $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \Diamond_P\exists\varphi^{t\#}$. From the assumption $\mathsf{MS4.t} \vdash \varphi^\flat \leftrightarrow \varphi^{t\#}$ it follows that $\mathsf{MS4.t} \vdash \Diamond_P\exists\varphi^\flat \leftrightarrow \Diamond_P\exists\varphi^{t\#}$. Thus, $\mathsf{MS4.t} \vdash$

$\Diamond_P \exists \varphi^\flat \leftrightarrow \exists \varphi^{t\#}$. Since $(-)^{t\#}$ is full and faithful, it follows that $(-)^\flat$ is full and faithful as well. ⊣

As a result, we obtain the following diagram of full and faithful translations that is commutative up to logical equivalence in MS4.t.



The Gödel translation of IPC into S4 extends to a translation of extensions of IPC into normal extensions of S4. It has been investigated in detail what logical properties are preserved by this translation (see, e.g., [9, p. 328]). A landmark result in this direction is the Blok-Esakia theorem stating that the lattice of extensions of IPC is isomorphic to the lattice of normal extensions of the Grzegorczyk logic Grz (see, e.g., [9, p. 325]). It is natural to investigate whether the Blok-Esakia theorem generalizes to MIPC.

§6. Finite model property. In this section we give a uniform proof of the fmp for the four logics studied in this paper. Our strategy is to first establish the fmp for MS4.t via algebraic methods, and then use the full and faithful translations to conclude that the other three logics also have the fmp.

The algebraic semantics for MS4.t is given by MS4.t-algebras. To define these algebras, we first recall the definition of S4-algebras, S5-algebras, and S4.t-algebras, which provide algebraic semantics for S4, S5, and S4.t, respectively. S4-algebras are known under various names. They were first introduced by McKinsey and Tarski [22] under the name of *closure algebras*. Rasiowa and Sikorski [27] call them *topological boolean algebras* and Blok [6] calls them *interior algebras*. S5-algebras were first introduced by Halmos [20] under the name of *monadic algebras*, and S4.t-algebras by Esakia [11] under the name of S4^2-*algebras*.

Definition 6.1. Let B be a boolean algebra.

1. A unary function $\Box : B \to B$ is an *interior operator* on B if
$$\Box(a \wedge b) = \Box a \wedge \Box b, \qquad \Box 1 = 1, \qquad \Box a \leq a, \qquad \Box a \leq \Box\Box a$$
for all $a, b \in B$.
2. An S4-*algebra* is a pair $\mathfrak{B} = (B, \Box)$ where B is a boolean algebra and $\Box$ is an interior operator on B.
3. An S5-*algebra* is an S4-algebra $\mathfrak{B} = (B, \forall)$ that in addition satisfies $a \leq \forall\exists a$ for all $a \in B$, where $\exists$ denotes the dual operator $\neg\forall\neg$.
4. An S4.t-*algebra* is a triple $\mathfrak{B} = (B, \Box_F, \Box_P)$ where B is a boolean algebra and $\Box_F, \Box_P$ are interior operators on B such that

(PF) $$a \leq \Box_P \Diamond_F a$$

(FP) $$a \leq \Box_F \Diamond_P a$$

for all $a \in B$, where $\Diamond_F = \neg\Box_F\neg$ and $\Diamond_P = \neg\Box_P\neg$.

MS4.t-algebras are obtained by combining S4.t-algebras and S5-algebras.

DEFINITION 6.2. An MS4.t-*algebra* is a tuple $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ where

1. $(B, \Box_F, \Box_P)$ is an S4.t-algebra,
2. $(B, \forall)$ is an S5-algebra,
3. $\Box_F \forall a \leq \forall \Box_F a$ for each $a \in B$.

Validity of $\mathcal{L}_{T\forall}$-formulas in MS4.t-algebras is defined in the usual way (see, e.g., [9, 27]). If a formula φ is valid in an MS4.t-algebra $\mathfrak{B}$, we write $\mathfrak{B} \models \varphi$. The standard Lindenbaum-Tarski construction (see, e.g., [27, Ch. VI]) yields the following:

THEOREM 6.3. MS4.t *is sound and complete with respect to its algebraic semantics. Therefore,*

$$\mathsf{MS4.t} \vdash \varphi \quad \textit{iff} \quad \mathfrak{B} \models \varphi \textit{ for every } \mathsf{MS4.t}\textit{-algebra } \mathfrak{B}.$$

DEFINITION 6.4. Let $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ be an MS4.t-algebra. We define

1. $H_F := \{a \in B \mid \Box_F a = a\}$ the set of $\Box_F$-fixpoints,
2. $H_P := \{a \in B \mid \Box_P a = a\}$ the set of $\Box_P$-fixpoints,
3. $B_0 := \{a \in B \mid \forall a = a\}$ the set of $\forall$-fixpoints.

REMARK 6.5.

1. It is well known (see, e.g., [13, Prop. 2.2.4]) that H_F and H_P with the restricted order from B are both Heyting algebras that are bounded sublattices of B. Moreover, it follows from Definition 6.1(4) that H_F coincides with the set of $\Diamond_P$-fixpoints and H_P with the set of $\Diamond_F$-fixpoints. Furthermore, $\neg$ maps H_F to H_P and vice versa. Indeed, if $a \in H_F$, then $a = \Box_F a$. By (PF), $\Diamond_P a = \Diamond_P \Box_F a \leq a$, so $\Diamond_P a = a$, and hence $\Box_P \neg a = \neg \Diamond_P a = \neg a$. Therefore, $\neg a \in H_P$. Similarly, if $a \in H_P$, then $\neg a \in H_F$. Thus, $\neg$ is a dual isomorphism between H_F and H_P.
2. It is easy to see that B_0 is an S4-subalgebra of $(B, \Box_F)$ because the inequality $\Box_F \forall a \leq \forall \Box_F a$, which corresponds to the left commutativity axiom, is equivalent to the equality $\forall \Box_F \forall a = \Box_F \forall a$.

We now prove that MS4.t has the fmp. For this we must show that if MS4.t $\nvdash \varphi$, then φ is refuted on a finite MS4.t-algebra.

DEFINITION 6.6. Let $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ be an MS4.t-algebra and $S \subseteq B$ a finite subset. Then $(B, \forall)$ is an S5-algebra. Let $(B', \forall')$ be the S5-subalgebra of $(B, \forall)$ generated by S. It is well known (see [1]) that $(B', \forall')$ is finite. Define $\Box'_F$ and $\Box'_P$ on B' by

$$\Box'_F a = \bigvee \{b \in B' \cap H_F \mid b \leq a\}$$
$$\Box'_P a = \bigvee \{b \in B' \cap H_P \mid b \leq a\}.$$

We denote $(B', \Box'_F, \Box'_P, \forall')$ by $\mathfrak{B}_S$.

LEMMA 6.7. $\mathfrak{B}_S$ *is an* MS4.t-*algebra.*

PROOF. $(B', \forall')$ is an S5-algebra by definition. Since $(B, \Box_F)$ and $(B, \Box_P)$ are both S4-algebras, a standard argument (see [22, Lem. 4.14]) shows that $(B', \Box'_F)$ and $(B', \Box'_P)$ are also S4-algebras. We show that $(B', \Box'_F, \Box'_P)$ is an S4.t-algebra. As noted in Remark 6.5(1), $\neg$ is a dual isomorphism between the algebras H_F and H_P of $\Box_F$-fixpoints and $\Box_P$-fixpoints of $\mathfrak{B}$. Therefore,

$$\begin{aligned}\Diamond'_F a := \neg\Box'_F\neg a &= \neg \bigvee\{b \in B' \cap H_F \mid b \le \neg a\} \\ &= \neg \bigvee\{b \in B' \cap H_F \mid a \le \neg b\} \\ &= \bigwedge\{\neg b \mid b \in B' \cap H_F,\ a \le \neg b\} \\ &= \bigwedge\{c \in B' \cap H_P \mid a \le c\}.\end{aligned}$$

Since this meet is finite and $\Box_P$ commutes with finite meets, we obtain

$$\begin{aligned}\Box_P \Diamond'_F a &= \Box_P \left(\bigwedge\{c \in B' \cap H_P \mid a \le c\}\right) \\ &= \bigwedge\{\Box_P c \mid c \in B' \cap H_P,\ a \le c\} \\ &= \bigwedge\{c \in B' \cap H_P \mid a \le c\} \\ &= \Diamond'_F a.\end{aligned}$$

Thus, $\Diamond'_F a \in B' \cap H_P$ which yields

$$\Box'_P \Diamond'_F a = \bigvee\{b \in B' \cap H_P \mid b \le \Diamond'_F a\} = \Diamond'_F a.$$

Similarly, we have that $\Diamond'_P a = \bigwedge\{c \in B' \cap H_F \mid a \le c\}$ from which we deduce that $\Box'_F \Diamond'_P a = \Diamond'_P a$. This implies that $a \le \Box'_P \Diamond'_F a$ and $a \le \Box'_F \Diamond'_P a$. Consequently, $(B, \Box'_F, \Box'_P)$ is an S4.t-algebra.

It remains to show that $\Box'_F \forall' a \le \forall' \Box'_F a$ holds in $\mathfrak{B}_S$. For this it is sufficient to show that the set $B'_0 := B' \cap B_0$ of the $\forall'$-fixpoints of B' is an S4-subalgebra of $(B', \Box'_F)$ because then $\Box'_F \forall' a = \forall' \Box'_F \forall' a \le \forall' \Box'_F a$. Suppose that $d \in B'_0$. Then $\Box'_F d = \bigvee\{b \in B' \cap H_F \mid b \le d\}$. Let $b \in B' \cap H_F$. By Lemma 2.5, $\exists b = \exists \Box_F b = \Box_F \exists \Box_F b = \Box_F \exists b$. Therefore, $\exists b \in B' \cap H_F$. Moreover, $b \le \exists b$ and $b \le d$ imply $\exists b \le \exists d = d$. Thus, $\Box'_F d = \bigvee\{\exists b \mid b \in B' \cap H_F,\ b \le d\}$. Since $(B', \forall')$ is an S5-algebra, B'_0 is the set of $\exists'$-fixpoints of B' and is closed under finite joins. Consequently, $\Box'_F d \in B'_0$. ⊣

THEOREM 6.8. MS4.t *has the fmp.*

PROOF. It is sufficient to prove that each $\mathcal{L}_{T\forall}$-formula φ refuted on some MS4.t-algebra is also refuted on a finite MS4.t-algebra. Let $t(x_1, \dots, x_n)$ be the term in the language of MS4.t-algebras that corresponds to φ, and suppose there is an MS4.t-algebra $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ and $a_1, \dots, a_n \in B$ such that $t(a_1, \dots, a_n) \ne 1$ in $\mathfrak{B}$. Let

$$S = \{t'(a_1, \dots, a_n) \mid t' \text{ is a subterm of } t\}.$$

Then S is a finite subset of B. Therefore, by Lemma 6.7, $\mathfrak{B}_S = (B', \Box'_F, \Box'_P, \forall)$ is a finite MS4.t-algebra. It follows from the definition of $\Box'_F$ that, for each $b \in B'$, if $\Box_F b \in B'$, then $\Box'_F b = \Box_F b$. Similarly, if $\Box_P b \in B$, then $\Box'_P b = \Box_P b$. Thus, for each subterm t' of t, the computation of t' in $\mathfrak{B}_S$ is the same as that in $\mathfrak{B}$.

Consequently, $t(a_1, \dots, a_n) \neq 1$ in $\mathfrak{B}_S$, and we have found a finite MS4.t-algebra refuting φ. ⊣

We conclude this section by showing that the fmp for TS4, MS4, and MIPC can be obtained as a consequence of Theorem 6.8 via the full and faithful translations into MS4.t described in Section 4. In order to do so, we state the fmp of MS4.t in terms of MS4.t-frames thanks to the correspondence between finite MS4.t-algebras and finite MS4.t-frames. In fact, we will obtain such a correspondence as a consequence of a representation result for MS4.t-algebras.

Definition 6.9. Let R be a quasi-order on a set X and $A \subseteq X$. We define

$$\Box_R(A) = X \setminus R^{-1}[X \setminus A].$$

If $R^{\smile}$ is the inverse relation of R, we have

$$\Box_{R^{\smile}}(A) = X \setminus R[X \setminus A].$$

If E is an equivalence relation on X, we use the notation

$$\forall_E(A) = X \setminus E^{-1}[X \setminus A] = X \setminus E[X \setminus A].$$

Proposition 6.10. *For each* MS4.t-*frame* $\mathfrak{F} = (X, R, E)$ *we have that* $\mathfrak{F}^+ := (\wp(X), \Box_R, \Box_{R^{\smile}}, \forall_E)$ *is an* MS4.t-*algebra.*

Proof. Since R is a quasi-order, so is $R^{\smile}$, hence $(\wp(X), \Box_R)$ and $(\wp(X), \Box_{R^{\smile}})$ are S4-algebras; and since E is an equivalence relation, $(\wp(X), \forall_E)$ is an S5-algebra (see [21, Thm. 3.5]). In addition, the commutativity condition yields that $\Box_R \forall_E(A) \leq \forall_E \Box_R(A)$ for each $A \in \wp(X)$. A standard argument (see [21, Thm. 3.6]) gives that $\Box_R$ and $\Box_{R^{\smile}}$ satisfy (PF) and (FP). Therefore, $\mathfrak{F}^+$ is an MS4.t-algebra. ⊣

Remark 6.11. If $\mathfrak{B} = \mathfrak{F}^+$, then the elements of H_F and H_P are respectively the R-upsets and R-downsets of $\mathfrak{F}$, and the elements of B_0 are the E-saturated subsets of $\mathfrak{F}$ (that is, unions of E-equivalence classes).

We next prove that each MS4.t-algebra is represented as a subalgebra of $\mathfrak{F}^+$ for some MS4.t-frame $\mathfrak{F}$.

Definition 6.12. Let $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ be an MS4.t-algebra. The *canonical frame* of $\mathfrak{B}$ is the frame $\mathfrak{B}_+ = (X_\mathfrak{B}, R_\mathfrak{B}, E_\mathfrak{B})$ where $X_\mathfrak{B}$ is the set of ultrafilters of B, $x R_\mathfrak{B} y$ iff $x \cap H_F \subseteq y$ iff $y \cap H_P \subseteq x$, and $x E_\mathfrak{B} y$ iff $x \cap B_0 = y \cap B_0$.

Lemma 6.13. *If* $\mathfrak{B}$ *is an* MS4.t-*algebra, then* $\mathfrak{B}_+$ *is an* MS4.t-*frame.*

Proof. Since $(B, \Box_F)$ is an S4-algebra, we have that $R_\mathfrak{B}$ is a quasi-order (see [21, Thm. 3.14]); and since $(B, \forall)$ is an S5-algebra, $E_\mathfrak{B}$ is an equivalence relation (see [21, Thm. 3.18]). It remains to show that Definition 3.7(E) is satisfied. Let $x, y, z \in X_\mathfrak{B}$ be such that $x E_\mathfrak{B} y$ and $y R_\mathfrak{B} z$. This means that $x \cap B_0 = y \cap B_0$ and $y \cap H_F \subseteq z$. Let F be the filter of $\mathfrak{B}$ generated by $(x \cap H_F) \cup (z \cap B_0)$. We show that F is proper. Otherwise, since $x \cap H_F$ and $z \cap B_0$ are closed under finite meets, there are $a \in x \cap H_F$ and $b \in z \cap B_0$ such that $a \wedge b = 0$. Therefore, $a \leq \neg b$. Thus, $a = \Box_F a \leq \Box_F \neg b$, so $\Box_F \neg b \in x$. Since B_0 is an S4-subalgebra of $(B, \Box_F)$ (see Remark 6.5(2)) and $b \in B_0$, we have $\Box_F\neg b \in B_0$. This yields $\Box_F\neg b \in x \cap B_0 = y \cap B_0$, which implies $\Box_F\neg b \in y \cap H_F \subseteq z$. Therefore, $\neg b \in z$ which contradicts $b \in z$. Thus, F is proper, and so there is an ultrafilter u of B such that $F \subseteq u$. Consequently, $x \cap H_F \subseteq u$ and $z \cap B_0 \subseteq u \cap B_0$. Since $z \cap B_0$ and $u \cap B_0$ are both ultrafilters of B_0, we conclude that $z \cap B_0 = u \cap B_0$. Thus, there is $u \in X_{\mathfrak{B}}$ with $xR_{\mathfrak{B}}u$ and $uE_{\mathfrak{B}}z$. ⊣

DEFINITION 6.14. Let $\mathfrak{B}$ be an MS4.t-algebra. The *Stone map* $\beta : \mathfrak{B} \to (\mathfrak{B}_+)^+$ is defined by

$$\beta(a) = \{x \in X_{\mathfrak{B}} \mid a \in x\}.$$

It is straightforward to see that β is a homomorphism of MS4.t-algebras, that the ultrafilter lemma for boolean algebras yields that β is an embedding, and that the embedding is an isomorphism in the finite case. Thus, we obtain the following representation theorem for MS4.t-algebras.

THEOREM 6.15 (Representation theorem). *Let $\mathfrak{B}$ be an* MS4.t-*algebra.*

1. *$\mathfrak{B}$ is isomorphic to a subalgebra of $(\mathfrak{B}_+)^+$.*
2. *When $\mathfrak{B}$ is finite, its embedding into $(\mathfrak{B}_+)^+$ is an isomorphism, and hence the categories of finite* MS4.t-*algebras and finite* MS4.t-*frames are dually equivalent.*

REMARK 6.16. As usual, to recover the image of the embedding of $\mathfrak{B}$ into $(\mathfrak{B}_+)^+$ we need to endow $\mathfrak{B}_+$ with a Stone topology (see, e.g., [13, Def. 3.3.3]). This leads to the notion of perfect MS4.t-frames and a duality between the categories of MS4.t-algebras and perfect MS4.t-frames.

Thanks to the representation theorem, the fmp of MS4.t can be equivalently stated as follows: if φ is not a theorem of MS4.t, then it is refuted in a finite MS4.t-frame. We now obtain the fmp of TS4, MS4, and MIPC as a consequence of the fmp of MS4.t.

THEOREM 6.17.

1. TS4 *has the fmp.*
2. MS4 *has the fmp.*
3. MIPC *has the fmp.*

PROOF. (1). Suppose that TS4 $\nvdash \varphi$. By Theorem 4.13(3), MS4.t $\nvdash \varphi^\dagger$. Since MS4.t has the fmp, there is a finite MS4.t-frame $\mathfrak{F}$ such that $\mathfrak{F} \nvDash \varphi^\dagger$. By Proposition 4.11(3), $\mathfrak{F}^\dagger \nvDash \varphi$. We have thus obtained a finite TS4-frame $\mathfrak{F}^\dagger$ refuting φ.

(2). Similar to the proof of (1) but uses the translation $(-)^\#$: MS4 $\to$ MS4.t instead of $(-)^\dagger$.

(3). Similar to the proof of (1) but uses the composition $(-)^{t\#}$: MIPC $\to$ MS4.t instead of $(-)^\dagger$. Alternatively, we can use the other translations $(-)^{\natural\dagger}$ and $(-)^\flat$ of MIPC into MS4.t. ⊣

REMARK 6.18.

1. That MIPC has the fmp was first established by Bull [8] using algebraic semantics. His proof contained a gap, which was corrected independently by Fischer-Servi [15] and Ono [23]. A semantic proof is given in [16], which is based on a technique developed by Grefe [19].

2. The fmp for MS4 can be derived from the results in [17, Sec. 12] (see also [16, Thms. 6.52, 9.12]). The proof given above is more direct.
3. The proof of the fmp for MS4.t contains the proof of the fmp for S4.t, but the latter is known (see [29, pp. 313–314] or [18, p. 44]). In fact, MS4.t is a conservative extension of S4.t.

§7. Connection with the full predicate case. In [4] we extended the full and faithful translation of MIPC into MS4.t to the full predicate case. We proved that this translation embeds IQC fully and faithfully into a weakening of the tense predicate logic QS4.t. This weakening is necessary since QS4.t proves the Barcan formula for both $\Box_F$ and $\Box_P$, so Kripke frames of QS4.t have constant domains, and hence they validate the translation of the constant domain axiom $\forall x(A \vee B) \to (A \vee \forall x B)$, where x is not free in A. Since this is not provable in IQC, the translation cannot be full. Instead we considered the tense predicate logic Q°S4.t in which the universal instantiation axiom $\forall x A \to A(y/x)$ is replaced by its weakened version $\forall y(\forall x A \to A(y/x))$. The main result of [4] proves that IQC translates fully and faithfully into Q°S4.t (provided the translation is restricted to sentences).

It is natural to investigate the relationship between MS4.t and predicate extensions of S4.t. As we already pointed out in Remark 2.10, MS4.t is not the monadic fragment of QS4.t. In addition, MS4.t cannot be the monadic fragment of Q°S4.t either since the formula $\forall x A \to A$ is not in general provable in Q°S4.t, whereas $\forall \varphi \to \varphi$ is provable in MS4.t. On the other hand, call a formula φ (in the language of MS4.t) *bounded* if each occurrence of a propositional letter in φ is under the scope of $\forall$. Bounded formulas play the same role as sentences of Q°S4.t containing only one fixed variable. It is quite plausible that for a bounded formula φ we have MS4.t $\vdash \varphi$ iff Q°S4.t proves the translation of φ where each occurrence of a propositional letter p is replaced with the unary predicate $P(x)$ and $\forall$ is replaced with $\forall x$ (for a similar translation of MIPC and its extensions into IQC and its extensions, see [24]). If true, this would yield that the monadic sentences provable in Q°S4.t are exactly the bounded formulas φ provable in MS4.t. It would also yield that restricting the translation IQC $\to$ Q°S4.t of [4] to the monadic setting gives the translation $(-)^\flat$: MIPC $\to$ MS4.t for bounded formulas.

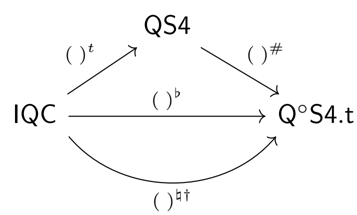
It is natural to seek an axiomatization of the full monadic fragment of Q°S4.t. Note that in this fragment $\forall$ does not behave like an S5-modality. For example, $\forall \varphi \to \varphi$ is not in general a theorem of this fragment.

Finally, the translation $(-)^\#$: MS4 $\to$ MS4.t suggests a translation of QS4 into Q°S4.t which replaces each occurrence of $\Box$ with $\Box_F$. It is easy to see that for sentences this translation is full and faithful. Composing it with the standard Gödel translation of IQC into QS4 yields a translation IQC $\to$ Q°S4.t which is different from the translation of [4]. This translation restricts to the translation $(-)^{t\#}$: MIPC $\to$ MS4.t for bounded formulas. Thus, the upper part of the diagram of Section 4 extends to the predicate case.

On the other hand, we do not see a natural way to interpret the tense modalities of TS4 as monadic quantifiers, and hence we cannot think of a natural predicate logic which could take the role of TS4 in the diagram of Section 4.

Thus, the lower part of the diagram does not seem to have a natural extension to the predicate case. Nevertheless, we can consider the predicate analogue of the translation $(-)^{\natural\dagger} : \mathsf{MIPC} \to \mathsf{MS4.t}$. Arguing as in Theorems 5.2 and 5.4 yields a translation of IQC into $\mathsf{Q^{\circ}S4.t}$ that is full and faithful on sentences and coincides, up to logical equivalence in $\mathsf{Q^{\circ}S4.t}$, with the other two predicate translations described in this section.

We thus obtain the following diagram in the predicate case which is commutative up to logical equivalence in $\mathsf{Q^{\circ}S4.t}$.



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