ON EXTENSION OF UNIFORMLY CONTINUOUS QUASICONVEX FUNCTIONS

CARLO ALBERTO DE BERNARDI AND LIBOR VESELÝ

Abstract. We show that each uniformly continuous quasiconvex function defined on a subspace of a normed space X admits a uniformly continuous quasiconvex extension to the whole X with the same "invertible modulus of continuity". This implies an analogous extension result for Lipschitz quasiconvex functions, preserving the Lipschitz constant.

We also show that each uniformly continuous quasiconvex function defined on a uniformly convex set $A \subset X$ admits a uniformly continuous quasiconvex extension to the whole X. However, our extension need not preserve moduli of continuity in this case, and a Lipschitz quasiconvex function on A may admit no Lipschitz quasiconvex extension to X at all.

1. Introduction

A real-valued function f defined on a convex set A is said to be *quasiconvex* if all its sub-level sets are convex (see Definition 2.3). Quasiconvex functions represent a natural generalization of convex functions and play a crucial role in Optimization, in Mathematical programming, in Mathematical economics, and in many other areas of mathematical analysis (see [1, 2, 6, 9] and the references therein).

Let X be a normed space of dimension at least two. Here we are interested in extending uniformly continuous quasiconvex functions either from a subspace $Y \subset X$ or from an open convex set $A \subset X$ to the whole X. In both cases, we make advantage of the relation between continuity properties of a quasiconvex function and some particular strict-monotonicity properties of its sub-level sets, formulated in Proposition 2.5. In this way, extension of functions can be reduced to extension of convex sets, which is an easier task.

After a section of notations and preliminaries, the subsequent Section 3 deals with extensions from a subspace Y. The main result, Theorem 3.4, says that each uniformly continuous quasiconvex function f on Y admits a uniformly continuous quasiconvex extension defined on the whole X. Since

---

2010 *Mathematics Subject Classification.* Primary 26B25, 52A05; Secondary 46A55, 52A99.

Key words and phrases. Quasiconvex function, extension, uniformly convex set, normed space.

our method preserves moduli of uniform continuity, this also implies that if f is even Lipschitz on Y then it admits a quasiconvex extension on X which is Lipschitz with the same Lipschitz constant.

Then we consider extendability from an open convex set $A \subset X$. We have shown elsewhere (see [5]) that this is not always possible: if X is a Banach space and A is not "locally uniformly convex" then there exists a Lipschitz quasiconvex function on A admitting no continuous quasiconvex extension to X whatsoever. In the present paper, we provide a positive result concerning "uniformly convex sets". Such sets were defined and studied by Balashov and Repovš [3]. In Section 4 we give a similar, maybe more natural, definition and we prove that the two definitions are equivalent. As a corollary we obtain that each proper, uniformly convex subset of X is bounded. The main result in the subsequent Section 5, Theorem 5.5, states that if A is uniformly convex then every uniformly continuous quasiconvex function on A admits a uniformly continuous quasiconvex extension to the whole X. However, the modulus of continuity need not be preserved in this case. We have shown in [5] that there exists a Lipschitz quasiconvex function on a disc in the Euclidean plane admitting no Lipschitz quasiconvex extension to any larger disc.

Finally, our results immediately imply the following corollary in the spirit of our paper [4]: if $A \subset X$ is an open convex set whose intersection with a subspace Y is a nonempty, proper, uniformly convex subset of Y, then each uniformly continuous quasiconvex function on $A \cap Y$ admits a uniformly quasiconvex extension to A (even to the whole X).

2. Preliminaries

2.1. **Notation.** By a *normed space* we mean a real normed linear space of dimension at least two. If not specified otherwise, X denotes such a space. By $U(x;r) \equiv U_X(x;r)$ and $B(x;r) \equiv B_X(x;r)$ we denote the open and the closed ball centered in $x \in X$ with radius $r \geq 0$, respectively. In particular, $U(x;0) = \emptyset$ and $B(x;0) = \{x\}$. Moreover we put $U(x;\infty) := X$. We also denote $U_X := U(0;1)$ and $B_X := B(0;1)$, the open and closed unit ball of X, respectively. The unit sphere of the dual Banach space X^* of X will be denoted by S_{X^*}.

The closed segment with endpoints $x, y \in X$ is denoted by $[x,y]$, and $[x,y) := [x,y] \setminus \{y\}$, $(x,y) := [x,y] \setminus \{x,y\}$.

If $W \subset Z \subset X$ then $\mathrm{int}_Z W$, $\overline{W}^Z$ and $\partial_Z W$ denote the relative interior, relative closure and relative boundary, respectively, of W in Z. In the case of $Z = X$, we simply write $\mathrm{int}\, W$, $\overline{W}$ and ∂W.

The distance of two sets $B, C \subset X$ is defined as

$$d(B,C) := \inf\{\|b - c\| : b \in B,\, c \in C\}$$

with the usual convention that $\inf \emptyset := \infty$. We also put $d(x, B) := d(\{x\}, B)$, $x \in X$. By $\operatorname{diam}(B)$ we denote the diameter of B; by $\operatorname{span}(B)$, $\operatorname{aff}(B)$ and $\operatorname{conv}(B)$ we mean the linear, affine and convex hull of B, respectively.

Let A be an open convex set in X. Then it is well known that

$$x \in \overline{A},\ a \in A \ \Rightarrow\ [a, x) \subset A. \tag{1}$$

If $C \subset X$ is a convex set with nonempty interior and $x \in \partial C$, we denote by $\Sigma(x, C)$ the (nonempty) set of elements of X^* that support C at x, that is,

$$\Sigma(x, C) = \{f \in X^* \setminus \{0\} : \sup f(C) = f(x)\}.$$

Given a nonempty set E, a function $f\colon E \to \mathbb{R}$, and $\alpha \in \mathbb{R}$, we shall often use the following simplified notation:

$$[f \le \alpha] := \{x \in E : f \le \alpha\}\,;$$

the sets $[f < \alpha]$, $[f = \alpha]$, $[f \ge \alpha]$ and $[f > \alpha]$ are defined in an obvious analogous way. If E is a metric space and f is Lipschitz with a Lipschitz constant $L > 0$ on E, we shall briefly say that f is *L-Lipschitz.*

2.2. **Invertible modulus of continuity.** The following lemma introduces a terminology and a tool which will be useful for our purposes. Its proof is quite standard.

Lemma 2.1 (Invertible modulus of continuity)**.** *Let E be a (nonempty) convex set in a normed space X, and $f\colon E \to \mathbb{R}$ a function. Then f is uniformly continuous on E if and only if there exists a function $\omega\colon [0, \infty) \to [0, \infty)$ such that:*

(a) ω is an increasing homeomorphism of $[0, \infty)$ onto itself;
(b) $|f(x) - f(y)| \le \omega(\|x - y\|)$ whenever $x, y \in E$.

Such function ω will be called an invertible modulus of continuity *of f.*

Sketch of proof. It is well-known that every uniformly continuous function f has a modulus of continuity, that is, a function $\omega\colon [0, \infty) \to [0, \infty)$ which is nondecreasing, right-continuous at 0 with $\omega(0) = 0$, and satisfies (b). Then $\widetilde{\omega}(t) := (1/t) \int_t^{2t} \omega(s)\, ds$ $(t > 0)$ defines a continuous modulus of continuity, and $t + \widetilde{\omega}(t)$ is an invertible modulus of continuity. The other implication is obvious. □

The following well-known fact is very easy.

Fact 2.2. *Let X be a normed space, $E \subset X$ a bounded convex set, and $f\colon E \to \mathbb{R}$ a uniformly continuous function. Then f is bounded on E.*

2.3. **Quasiconvex functions and their sub-level sets.** Main results of the present paper are based on the following simple idea: we transform the problem of extending quasiconvex functions into a problem of extending monotone families of convex sets, which we find to be an easier task.

First, let us recall the definition of a quasiconvex function. Notice that each convex function is quasiconvex, but not vice-versa.

Definition 2.3. Let E be a convex set in a vector space. A function $f\colon E \to \mathbb{R}$ is called *quasiconvex* if $f((1-t)x+ty) \le \max\{f(x), f(y)\}$ whenever $x, y \in E$ and $t \in [0,1]$. Thus f is quasiconvex if and only if all its strict sub-level sets $[f < \beta]$ ($\beta \in \mathbb{R}$) are convex, if and only if all its sub-level sets $[f \le \beta]$ ($\beta \in \mathbb{R}$) are convex.

For brevity of formulations, we introduce the following definition from [4].

Definition 2.4. Given a nonempty set $E \subset X$, an $\Omega(E)$-*family* is a family $\{D_\alpha\}_{\alpha\in\mathbb{R}}$ of sets in X such that

$$\bigcap_{\alpha\in\mathbb{R}} D_\alpha = \emptyset, \quad \bigcup_{\alpha\in\mathbb{R}} D_\alpha = E, \quad \overline{D}_\alpha^E \subset D_\beta \text{ whenever } \alpha, \beta \in \mathbb{R},\ \alpha < \beta.$$

The following simple, but important proposition, which is a variant of [4, Proposition 2.4], provides a basic connection between regularity of a quasiconvex function and certain monotonicity properties of the family of its sub-level sets.

Proposition 2.5. *Let $E \subset X$ be a convex set.*

(a) If $\{D_\alpha\}_{\alpha\in\mathbb{R}}$ is an $\Omega(E)$-family of relatively open convex subsets of E, then the function

$$f(x) := \sup\{\alpha \in \mathbb{R} : x \notin D_\alpha\} \qquad (x \in E)$$

is a continuous quasiconvex function on E such that for each $\alpha \in \mathbb{R}$

$$[f < \alpha] = \bigcup_{\beta<\alpha} D_\beta \quad \text{and} \quad [f \le \alpha] = \bigcap_{\gamma>\alpha} D_\gamma\,. \tag{2}$$

(b) If g is a continuous quasiconvex function on E, then the sets

$$D_\alpha := [g < \alpha] \qquad (\alpha \in \mathbb{R})$$

form an $\Omega(E)$-family of relatively open convex subsets of E. Moreover, the function f associated to $\{D_\alpha\}_{\alpha\in\mathbb{R}}$ by (a) coincides with g.

(c) Let $\{D_\alpha\}_{\alpha\in\mathbb{R}}$ and f be as in (a). Let $\omega\colon [0,\infty) \to [0,\infty)$ be an increasing homeomorphism of $[0,\infty)$ onto itself. Then the following two assertions are equivalent:

(c1) f is uniformly continuous with invertible modulus of continuity ω;

(c2) $d(D_\alpha, E \setminus D_\beta) \ge \omega^{-1}(\beta-\alpha)$ whenever $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$.

In particular, f is Lipschitz with a Lipschitz constant $L > 0$ (L-Lipschitz, for short) if and only if $d(D_\alpha, E \setminus D_\beta) \geq (1/L)(\beta - \alpha)$ whenever $\alpha < \beta$.

Proof.

(a) It is clear that f is well-defined, and it is an elementary exercise to show that it satisfies (2). Since all the sets $[f < \alpha]$ are convex and relatively open in E, f is quasiconvex and upper semicontiuous on E. Moreover, the definition of an $\Omega(E)$-family implies that $[f \leq \alpha] = \bigcap_{\gamma > \alpha} \overline{D}_\gamma^E$ for each $\alpha \in \mathbb{R}$, and hence f is also lower semicontinuous on E.

(b) The first part is clear. For the second part, if $x \in E$ then we have $f(x) = \sup\{\alpha \in \mathbb{R};\, g(x) \geq \alpha\} = g(x)$.

(c) Let f be uniformly continuous with invertible modulus of continuity ω. Let $\alpha < \beta$ be real numbers and $x \in D_\alpha$, $y \in E \setminus D_\beta$. Then $f(x) \leq \alpha$ and $f(y) \geq \beta$, and hence $\beta - \alpha \leq f(y) - f(x) \leq \omega(\|y - x\|)$. It follows that $\|y - x\| \geq \omega^{-1}(\beta - \alpha)$, and (c2) follows. To show the other implication, assume (c2) and consider $x, y \in E$ such that $f(x) < f(y)$. For any two reals α, β with $f(x) < \alpha < \beta < f(y)$, we have $x \in D_\alpha$ and $y \notin D_\beta$, and hence $\|y - x\| \geq \omega^{-1}(\beta - \alpha)$. By letting $\alpha \to f(x)^+$ and $\beta \to f(y)^-$ we obtain $\|y - x\| \geq \omega^{-1}(f(y) - f(x))$. This implies (c1). The last part is now obvious since f is L-Lipschitz if and only if it is uniformly continuous with invertible modulus of continuity $\omega(t) = Lt$ ($t \geq 0$). □

3. Extension from a subspace

3.1. **Extension of convex sets from a subspace.** We shall need the following known fact which is an easy consequence of the identity

$$U_X\big((1-t)x_1 + tx_2; (1-t)r_1 + tr_2\big) = (1-t)U_X(x_1; r_1) + tU_X(x_2; r_2),$$

$x_1, x_2 \in X$, $r_1, r_2 > 0$, $t \in (0,1)$.

Fact 3.1. *Let X be a normed space, and $C \subset X$ an open convex set. Then the distance function $d(\cdot, X \setminus C)$ is concave on C.*

The following definition provides a method of extension of relatively open convex sets in a subspace to open convex sets in the whole space, and the subsequent lemma shows that this extension method preserves certain distances of sets.

Definition 3.2. Let Y be a subspace of a normed space X, and $C \subset Y$ an open convex set in Y. We define the "natural extension" of C as the set

$$\widehat{C} := \bigcup_{y \in Y} U_X\big(y; d(y, Y \setminus C)\big).$$

Lemma 3.3. *Let Y be a subspace of a normed space X, and let $C, C_1, C_2 \subset Y$ be open convex sets in Y.*
(a) $\widehat{C}$ is an open convex set in X, and $\widehat{C} \cap Y = C$.
(b) If $C_1 \subset C_2$ then $\mathrm{dist}(\widehat{C}_1, X \setminus \widehat{C}_2) = \mathrm{dist}(C_1, Y \setminus C_2)$.

Proof. (a) It is clear that $\widehat{C}$ is open, and $\widehat{C} \cap Y = \bigcup_{y \in C} U_Y(y; d(y, Y \setminus C)) = C$. It remains to show that $\widehat{C}$ is convex. Let $x_1, x_2 \in \widehat{C}$, $t \in (0,1)$ and $x = (1-t)x_1 + tx_2$. For each $i \in \{1,2\}$ there is $y_i \in C$ such that $\|x_i - y_i\| < d(y_i, Y \setminus C)$. Consider $y := (1-t)y_1 + ty_2 \in C$. Then, by Fact 3.1, $\|x - y\| \leq (1-t)\|x_1 - y_1\| + t\|x_2 - y_2\| < (1-t)d(y_1, Y \setminus C) + td(y_2, Y \setminus C) \leq d(y, Y \setminus C)$. Thus $x \in \widehat{C}$, and (a) is proved.

(b) By (a), $\widehat{C}_1 \supset C_1$ and $X \setminus \widehat{C}_2 \supset Y \setminus C_2$, and this implies the inequality $\leq$ in (b). To show the inequality $\geq$, assume that $\delta := \mathrm{dist}(C_1, Y \setminus C_2) > 0$. Then

$$\widehat{C}_1 + \delta U_X = \bigcup_{y \in C_1} U_X\big(y; d(y, Y \setminus C_1) + \delta\big) \subset \bigcup_{y \in C_1} U_X\big(y; d(y, Y \setminus C_2)\big) \subset \widehat{C}_2,$$

and this means that $\mathrm{dist}(\widehat{C}_1, X \setminus \widehat{C}_2) \geq \delta$. We are done. □

3.2. **Extension of quasiconvex functions from a subspace.** Now we are ready for the main result of the present section.

Theorem 3.4. *Let Y be a subspace of a normed space X, and $f\colon Y \to \mathbb{R}$ a quasiconvex function. If f is uniformly continuous with an invertible modulus of continuity, then it admits a quasiconvex extension $F\colon X \to \mathbb{R}$ which is uniformly continuous with the same invertible modulus of continuity.*

In particular, every L-Lipschitz quasiconvex function on Y admits an L-Lipschitz quasiconvex extension to the whole X.

Proof. Assume that f is uniformly continuous with an invertible modulus of continuity ω, and define $C_\alpha := [f < \alpha]$ ($\alpha \in \mathbb{R}$). By Proposition 2.5, $\{C_\alpha\}_{\alpha \in \mathbb{R}}$ is an $\Omega(Y)$-family of open convex sets in Y such that $\mathrm{dist}(C_\alpha, Y \setminus C_\beta) \geq \omega^{-1}(\beta - \alpha)$ whenever $\alpha < \beta$. By Lemma 3.3, the "naturally extended sets" $D_\alpha := \widehat{C}_\alpha$ ($\alpha \in \mathbb{R}$) also satisfy the inequality $\mathrm{dist}(D_\alpha, X \setminus D_\beta) \geq \omega^{-1}(\beta - \alpha)$ whenever $\alpha < \beta$. Notice that this also gives that $\bigcup_{\alpha \in \mathbb{R}} D_\alpha = X$. Now Proposition 2.5 easily implies that the formula

$$F(x) := \sup\{\alpha \in \mathbb{R};\, x \notin D_\alpha\} \qquad (x \in X)$$

defines a quasiconvex extension of f to the whole X which is uniformly continuous with modulus ω. Moreover, F is L-Lipschitz on X whenever f is L-Lipschitz on Y. □

4. Uniformly convex sets

Our second task is now to study extendibility of uniformly continuous quasiconvex functions from an open convex subset A of a normed space X to the whole X. We show elsewhere (see [5]) that this is not always possible. However, we shall show in the next section that uniformly continuous quasiconvex extensions do exist provided A is uniformly convex. The present section is devoted to introducing uniform convexity of sets. Let us remark that uniformly convex sets have already been defined by Polyak in [10] (see also [3]), though in a slightly different way (called by us "$\widetilde{\delta}$-uniformly convex") which turns out to be equivalent to our definition (see Proposition 4.3).

We can define uniform convexity of the unit ball to be equivalent to uniform convexity of the space in question, which is a well-known notion from the Banach space theory (see e.g. [7]). The notion of a uniformly convex set represents a direct, natural generalization.

Definition 4.1. Let C be a convex set in X. We shall say that

- C *is nontrivial* if $\emptyset \neq C \neq X$ (equivalently, if $\partial C \neq \emptyset$);
- C *is strictly convex* if it satisfies the implication:

$$x, y \in \partial C,\ x \neq y \quad \Rightarrow \quad \tfrac{x+y}{2} \in \operatorname{int} C\,;$$

- C *is uniformly convex* if for each $\varepsilon \in (0, \operatorname{diam}(C))$ there exists $\delta > 0$ such that

$$x, y \in \partial C,\ \|x-y\| \geq \varepsilon \quad \Rightarrow \quad d(\tfrac{x+y}{2}, \partial C) \geq \delta\,.$$

- C *is $\widetilde{\delta}$-uniformly convex* if for each $\varepsilon \in (0, \operatorname{diam}(C))$ there exists $\delta > 0$ such that

$$x, y \in \overline{C},\ \|x-y\| \geq \varepsilon \quad \Rightarrow \quad d(\tfrac{x+y}{2}, \partial C) \geq \delta\,.$$

- Moreover, for $0 \leq \varepsilon < \operatorname{diam}(C)$ we define the following two *moduli of convexity*:

$$\delta_C(\varepsilon) := \inf\Big\{ d(\tfrac{x+y}{2}, \partial C) :\ x, y \in \partial C,\ \|x-y\| \geq \varepsilon \Big\}\,,$$

$$\widetilde{\delta}_C(\varepsilon) := \inf\Big\{ d(\tfrac{x+y}{2}, \partial C) :\ x, y \in \overline{C},\ \|x-y\| \geq \varepsilon \Big\}\,.$$

Let us collect some immediate properties.

Observation 4.2. *Let $C \subset X$ be a nontrivial convex set.*

(a) The following assertions are equivalent:

(a1) C is uniformly convex;

(a2) $\|x_n - y_n\| \to 0$ whenever $\{x_n\}, \{y_n\} \subset \partial C$, $d(\frac{x_n+y_n}{2}, \partial C) \to 0$;

(a3) $\delta_C(\varepsilon) > 0$ for every $\varepsilon \in (0, \operatorname{diam}(C))$.

(b) Likewise, the following assertions are equivalent:

(b1) C *is* $\widetilde{\delta}$*-uniformly convex;*

(b2) $\|x_n - y_n\| \to 0$ *whenever* $\{x_n\}, \{y_n\} \subset \overline{C}$, $d(\frac{x_n+y_n}{2}, \partial C) \to 0$;

(b3) $\widetilde{\delta}_C(\varepsilon) > 0$ *for every* $\varepsilon \in (0, \operatorname{diam}(C))$.

(c) Moreover, for each $0 \leq \varepsilon < \operatorname{diam}(C)$ *one has*

$$\delta_C(\varepsilon) \geq \widetilde{\delta}_C(\varepsilon) = \inf\left\{d(\tfrac{x+y}{2}, \partial C) : \ x, y \in \overline{C}, \ \|x-y\| = \varepsilon\right\}.$$

Proof. To see the equality in (c), it suffices to observe that if $x, y \in \overline{C}$ are such that $\|x-y\| \geq \varepsilon$, there exist $x', y' \in [x, y]$ such that $\|x'-y'\| = \varepsilon$ and $\frac{x'+y'}{2} = \frac{x+y}{2}$. The rest is very easy. □

In our opinion, the above notion of uniform convexity (and the modulus δ_C) is more natural that the notion of $\widetilde{\delta}$-uniform convexity (and the modulus $\widetilde{\delta}_C$), but it turns out to be less comfortable to work with. It is natural to ask whether the two notions are really different. As we shall see in Proposition 4.3, the two notions reveal to be equivalent.

Moreover, Observation 4.2(c) implies that the modulus of convexity $\widetilde{\delta}_C$ coincides with the modulus used by Balashov and Repovš in [3] to define "uniformly convex sets". This enables us to take advantage of some of their results.

So it is time to prove the promised equivalence, which is the main result of the present section.

Proposition 4.3. *Let* C *be a nontrivial convex set in a normed space* X*. Then* C *is uniformly convex if and only if* C *is* $\widetilde{\delta}$*-uniformly convex.*

Proof. One implication is clear by the inequality in Observation 4.2(c). To show the opposite implication, let us proceed by contradiction. Assume that $\delta_C(\cdot) > 0$ on $(0, \operatorname{diam}(C))$ and there exists $\varepsilon_0 \in (0, \operatorname{diam}(C))$ such that $\widetilde{\delta}_C(\varepsilon_0) = 0$. Denote $\delta_0 := \delta_C(\frac{\varepsilon_0}{4})$ (>0). Let $\theta > 0$ be such that:

(I) $\frac{\varepsilon_0}{2} - 2\theta \geq \frac{\varepsilon_0}{4}$;

(II) $\frac{2}{3}(\delta_0 - \theta) \geq \theta$.

Since $\widetilde{\delta}_C(\varepsilon_0) = 0$, there exist $x, y \in C$ such that $\|x-y\| \geq \varepsilon_0$ and, for $z = \frac{x+y}{2}$, there exists $z' \in \partial C$ satisfying $\|z - z'\| < \theta$. Without any loss of generality, we can suppose that $d(x, \partial C) < \theta$ (indeed, if $d(x, \partial C) \geq \theta$ and $d(y, \partial C) \geq \theta$ then also $d(z, \partial C) \geq \theta$). Let $x' \in \partial C$ be such that $\|x - x'\| < \theta$ and observe that

$$\|x'-z'\| \geq \|x-z\| - \|x-x'\| - \|z-z'\| \geq \tfrac{\varepsilon_0}{2} - 2\theta \geq \tfrac{\varepsilon_0}{4},$$

where the last inequality holds by (I). Then, by definition of δ_C and since $x', z' \in \partial C$, we have that $d(\frac{x'+z'}{2}, \partial C) \geq \delta_0$. Moreover, $\|\frac{x'+z'}{2} - \frac{x+z}{2}\| < \theta$. By (II), we have that $\delta_0 - \theta > 0$ and that $d\big(\frac{x+z}{2}, \partial C\big) \geq \delta_0 - \theta > 0$. Since $z = \frac{2}{3}\frac{x+z}{2} + \frac{1}{3}y$, we have

$$z + \tfrac{2}{3}(\delta_0 - \theta)B_X \subset z + \tfrac{2}{3}d\big(\tfrac{x+z}{2}, \partial C\big)B_X \subset \overline{C}.$$

By applying (II) again, we have that $z + \theta B_X \subset \overline{C}$, which is a contradiction since $\|z - z'\| < \theta$. □

Now, a result in [3] about $\widetilde{\delta}$-uniformly convex sets in Banach spaces will provide us with the following corollary.

Corollary 4.4. *Each nontrivial uniformly convex set in a normed space is bounded, strictly convex, and has nonempty interior.*

Proof. Strict convexity and nonempty interior are obvious from the the definition. To show boundedness, we claim that we can assume that the normed space in question is a Banach space.

Indeed, let X and X_1 be the given normed space and its completion, respectively, and $C \subset X$ a nontrivial uniformly convex set. We can (and do) assume that C is closed in X. Let C_1 be the closure of C in X_1. Then $\mathrm{int}_X C$ is dense in $\mathrm{int}_{X_1} C_1$, as well as $\partial_X C$ is dense in $\partial_{X_1} C_1$. And this implies that C_1 is uniformly convex in X_1.

Boundedness now follows by [3, Theorem 2.1] which implies that each (nontrivial) $\widetilde{\delta}$-uniformly convex set in a Banach space is bounded. □

5. Extension from uniformly convex sets

Convention. *Unless specified otherwise, throughout the present section A denotes a nontrivial open convex set in a normed space X.*

Let us remark that we restrict ourselves to *open* convex sets just for the sake of simplicity. Indeed, the nonempty interior of a convex set is always dense in the convex set; and every uniformly continuous quasiconvex function on a set has a unique continuous extension to the closure of the set and this extension is uniformly continuous and quasiconvex.

5.1. Extension of convex sets from a uniformly convex set.

Like in Subsection 3.1, we shall first define a method of extension of convex sets from a uniformly convex set.

First, let us recall that if $x_0 \in \partial A$ then $\Sigma(x_0, A)$ denotes the set of (nonzero) supporting functionals to A at x_0. Let us also define

$$K(x_0, A) := \bigcap_{f \in \Sigma(x_0, A)} [f \leq f(x_0)]\,.$$

It is an easy exercise, based on the Hahn-Banach theorem, to show that

$$\overline{A} = \bigcap_{x \in \partial A} K(x, A)$$

and

$$K(x_0, A) = x_0 + \overline{\bigcup_{t>0} t(A - x_0)}\,.$$

Thus $K(x_0, A)$ is the closed convex cone with vertex at x_0, determined by A.

Now we are ready for defining our extension method.

Definition 5.1. For an open convex set $C \subset A$, let us define its *extension* $\widetilde{C}$ as follows. If $\emptyset \neq C \neq A$ we define

$$\widetilde{C} := \bigcap_{y \in \overline{A \cap \partial C}} K(y, C)\,.$$

Moreover, for $C = \emptyset$ we put $\widetilde{C} := \emptyset$, and for $C = A$ we put $\widetilde{C} := X$.

We shall need the following two, quite technical lemmas.

Lemma 5.2. *Let $C \subset A$ be an open convex set.*

(a) $\widetilde{C}$ is a closed convex set containing $\overline{C}$.
(b) $\widetilde{C} \cap \overline{A} = \overline{C}$.
(c) If $x \in \widetilde{C} \setminus \overline{A}$, $c \in C$ and $[c, x] \cap \partial A = \{x'\}$, then $x' \in \mathrm{int}_{\partial A}(\partial C \cap \partial A)$.
(d) $\partial_{\partial A}(\partial C \cap \partial A) \subset \overline{A \cap \partial C}$.

Proof. (a) follows easily from definitions and the text at the beginning of the present section.

(b) The inclusion "$\supset$" follows from (a). To show the other inclusion let us proceed by contradiction. Assume there exists $x \in \widetilde{C} \cap \overline{A}$ such that $x \notin \overline{C}$. Take an arbitrary $y \in C$. Then there exists a unique z such that $\{z\} = (x, y) \cap \partial C$. Obviously, $z \in A \cap \partial C$. Separating C from $[x, z]$ we find $f \in X^*$ such that $f(u) < f(v)$ whenever $u \in C$ and $v \in [x, z]$; in particular, $f(y) < f(z)$. It follows that $f(x) > f(z)$. But $f \in \Sigma(z, C)$ so $x \notin K(z, C)$. This contradicts $x \in \widetilde{C}$.

(c) The set $B := \bigcup_{t \in (0,1]} \big((1 - t)x + tC\big) = \mathrm{conv}(C \cup \{x\}) \setminus \{x\}$ is an open convex set that contains x'. Moreover, since $B \subset \widetilde{C}$, (b) implies that $B \cap \partial A = B \cap \partial C$ and this set is contained in $\partial C \cap \partial A$. Thus $x' \in \mathrm{int}_{\partial A}(\partial C \cap \partial A)$ as needed.

(d) Without any loss of generality, we can suppose that $0 \in C$. Let $x \in \partial_{\partial A}(\partial C \cap \partial A)$. Then $x \in \partial C \cap \partial A$ and there exists a sequence $\{y_n\}_n \subset \partial A \setminus \overline{C}$ such that $y_n \to x$. Then the points $y'_n := \frac{y_n}{\mu_C(y_n)}$ $(n \in \mathbb{N})$ belong to $\partial C \cap A$ and converge to $\frac{x}{\mu_A(x)} = x$. Consequently $x \in \overline{\partial C \cap A}$. □

Lemma 5.3. *Let A be bounded and let $C \subset A$ be an open convex set. Let $x \in \widetilde{C} \setminus \overline{C}\, (= \widetilde{C} \setminus \overline{A})$ and $y \in \overline{A} \setminus \overline{C}$. Then $[x, y] \cap \overline{C} \neq \emptyset$.*

Proof. If $y \in A$ and the unique point $y' \in [x,y] \cap \partial A$ belongs to $\overline{C}$, we are done. Otherwise we can consider y' instead of y. So we can (and do) assume that $y \in \partial A$.

We can (and do) assume that $0 \in C \setminus \mathrm{aff}\{x,y\}$, and then x, y are linearly independent. Let $[0,x] \cap \partial C = \{x'\}$, and notice that $x' \in \mathrm{int}_{\partial A}(\partial C \cap \partial A)$ by Lemma 5.2. Consider the two-dimensional subspace $Y := \mathrm{span}\{x,y\}$. Let $H \subset Y$ be the closed half-plane such that

$$\partial_Y H = \mathbb{R}(x' - y) \quad \text{and} \quad x, y \in H.$$

Since $\gamma := H \cap \partial A$ is a simple arc, there exists $u \in \partial_{\partial A}(\partial C \cap \partial A) \cap \gamma$ that separates x' and y in γ, that is, each of the two components of $\gamma \setminus \{u\}$ contains just one of the points x', y. The half-line $(0,\infty)u$ intersects $[x,y]$ at a point u'. So we have

$$tu = u' = (1-\lambda)x + \lambda y \quad \text{for some } t > 0,\ \lambda \in (0,1).$$

Now consider $f \in \Sigma(u, A)$ such that $f(u) = 1$. Since $f \in \Sigma(u, C)$ and $u \in \overline{A \cap \partial C}$ (by Lemma 5.2(iv)), we must have $f(x) \le 1$ by the definition of $\widetilde{C}$. Therefore, $t = tf(u) = (1-\lambda)f(x) + \lambda f(y) \le 1$ and hence $u' \in [0,u] \cap [x,y] \subset \overline{C} \cap [x,y]$. We are done. □

The next proposition shows that our extension method gives possibility to control distance properties of sets constructed by our extension method.

Proposition 5.4. *Let $C_1 \subset C_2 \subset A$ be open convex sets. If A is uniformly convex and $d(C_1, A \setminus C_2) =: \varepsilon > 0$, then*

$$d(\widetilde{C}_1, X \setminus \widetilde{C}_2) \ge \widetilde{\delta}_A(\varepsilon),$$

where the modulus $\widetilde{\delta}_A$ was defined in Definition 4.1.

Proof. Fix arbitrary $x \in \widetilde{C}_1$ and $y \in \overline{A \cap \partial C_2}$. By Lemma 5.3, there exists $z \in \overline{C}_1 \cap [x,y]$. Since $\|z - y\| \ge \varepsilon$, there exists a unique $v \in [z,y]$ with $\|z - v\| = \varepsilon$. Denoting $w := \frac{z+v}{2}$, we know that the open ball $U(w; \widetilde{\delta}_A(\varepsilon))$ is contained in A. Moreover, since $d(w, A \cap \partial C_2) \ge d(z, A \cap \partial C_2) - \|w - z\| \ge \varepsilon - (\varepsilon/2) = \varepsilon/2 \ge \widetilde{\delta}_A(\varepsilon)$, the ball $U(w; \widetilde{\delta}_A(\varepsilon))$ does not intersect $\overline{A \cap \partial C_2}$. Since $z \in \overline{C_1} \subset \overline{C_2}$, we have that $w \in \overline{C_2}^A$. All this implies that $U(w, \widetilde{\delta}_A(\varepsilon))$ is contained in C_2. Since the cone $K(y, C_2)$ contains $U(w, \widetilde{\delta}_A(\varepsilon))$, it contains also the cone $y + \bigcup_{t>0}[U(w; \widetilde{\delta}_A(\varepsilon)) - y]$. Since $x = y + \tau(w - y)$ for some $\tau > 1$, the last cone certainly contains $U(x; \widetilde{\delta}_A(\varepsilon))$. By the definition of $\widetilde{C}_2$ and by arbitrariness of $x \in \widetilde{C}_1$, we conclude that $\widetilde{C}_1 + \widetilde{\delta}_A(\varepsilon)U_X \subset \widetilde{C}_2$ as needed. □

5.2. **Extension of quasiconvex functions from uniformly convex sets.** The main result of the present section is now quite easy to obtain, after the work done before.

Theorem 5.5. *Let X be a normed space, $A \subset X$ a nontrivial uniformly convex set, and $f\colon A \to \mathbb{R}$ a uniformly continuous quasiconvex function. Then there exists a uniformly continuous quasiconvex function $F\colon X \to \mathbb{R}$ that extends f and satisfies $F(X) \subset \overline{f(A)}$.*

Proof. As remarked at the beginning of the present section, we can (and do) assume that A is open. Recall that A is bounded by Corollary 4.4. Assume that f is not constant, and let $\omega\colon [0,\infty) \to [0,\infty)$ be an invertible modulus of continuity for f (see Lemma 2.1). The modulus of convexity $\widetilde{\delta}_A$ is an increasing bounded continuous function on $[0, \mathrm{diam}(A))$ (see [3, Corollary 2.1]), and hence it admits an extension, denoted again by $\widetilde{\delta}_A$, which is an increasing homeomorphism of $[0,\infty)$ onto itself. Since f is bounded (see Fact 2.2),

$$\iota := \inf f(A) \quad \text{and} \quad \sigma := \sup f(A)$$

are real numbers. For $\alpha \in \mathbb{R}$, the sets $C_\alpha := [f < \alpha]$ are open convex subsets of A. Moreover, $C_\alpha = \emptyset$ for $\alpha \le \iota$, and $C_\alpha = A$ for $\alpha > \sigma$. Let us define

$$D_\alpha := \mathrm{int}\, \widetilde{C}_\alpha \quad \text{for } \alpha \ne \sigma, \text{ and} \quad D_\sigma := X,$$

where $\widetilde{C}_\alpha$ is the extension of C_α defined in Definition 5.1. So $D_\alpha = \emptyset$ whenever $\alpha \le \iota$, and $D_\alpha = X$ whenever $\alpha \ge \sigma$. For $\iota < \alpha < \beta < \sigma$, we have $d(C_\alpha, A \setminus C_\beta) \ge \omega^{-1}(\beta - \alpha)$ by Proposition 2.5, and hence

$$d(D_\alpha, X \setminus D_\beta) = d(\widetilde{C}_\alpha, X \setminus \widetilde{C}_\beta) \ge \widetilde{\delta}_A(\omega^{-1}(\beta - \alpha))$$

by Proposition 5.4. Notice that this implies that $\{D_\alpha\}_{\alpha \in \mathbb{R}}$ is an $\Omega(X)$-family of open convex sets. By Proposition 2.5, the formula

$$F(x) := \sup\{\alpha \in \mathbb{R} : x \notin D_\alpha\} \quad (x \in X)$$

defines a quasiconvex extension of f such that F is uniformly continuous with invertible modulus of continuity

$$\omega_F(t) = \omega(\widetilde{\delta}_A^{-1}(t)) \quad (t \ge 0).$$

Finally, it is easy to see that $\iota \le F(x) \le \sigma$ for each $x \in X$. We are done. □

The rest of this section contain some complementary results that follow easily from our method.

Remark 5.6. We say that a uniformly convex set $C \subset X$ has $\widetilde{\delta}$-modulus of convexity of *power type* $p > 0$ if there exists a constant $k > 0$ for which $\widetilde{\delta}_C(\varepsilon) \ge k\varepsilon^p$ for each $\varepsilon \in (0, \mathrm{diam}(C))$.

By [3, Corollary 2.3], *a power type p of $\widetilde{\delta}_C$ cannot be smaller than* 2 (for balls this is due to Nordlander [8]).

Let $E \subset X$, $f\colon E \to \mathbb{R}$ and $a \in (0,1]$. Recall that f is a-Hőlder on E if it is uniformly continuous with an invertible modulus of continuity of the form $\omega(t) = Lt^a$ $(t \geq 0)$, where $L > 0$ is a constant. Notice that 1-Hőlder functions are just Lipschitz functions.

The above Theorem 5.5 and its proof enable us to get the following quantitative result about extension of quasiconvex hőlderian functions.

Theorem 5.7. *Let A be a nontrivial uniformly convex set with modulus of convexity $\widetilde{\delta}_A$ of power type $p \geq 2$ (see Remark 5.6), and $f\colon A \to \mathbb{R}$ an a-Hőlder quasiconvex function for some $a \in (0,1]$. Then f admits an (a/p)-Hőlder quasiconvex extension to the whole X, such that $F(X) \subset \overline{f(A)}$.*

Proof. The function f is uniformly continuous with an invertible modulus of continuity of the form $\omega(t) = Lt^a$, and we can (and do) assume that $\widetilde{\delta}_A$ is an increasing homeomorphism of $[0,\infty)$ onto itself such that $\widetilde{\delta}_A(\varepsilon) \geq k\varepsilon^p$ for each $\varepsilon \geq 0$. Let us follow the proof of Theorem 5.5. The sets D_α, D_β $(\iota < \alpha < \beta < \sigma)$ satisfy $d(D_\alpha, X \setminus D_\beta) \geq \widetilde{\delta}_A(\omega^{-1}(\beta-\alpha)) \geq k[\omega^{-1}(\beta-\alpha)]^p = (k/L^{p/a})(\beta-\alpha)^{p/a} =: \widetilde{\omega}_F^{-1}(\beta-\alpha)$. Now it is easy to verify from Proposition 2.5 that the (quasiconvex) extension F from the proof of Theorem 5.5 is uniformly continuous with the invertible modulus of continuity

$$\widetilde{\omega}_F(t) = (L/k^{a/p})t^{a/p} \quad (t \geq 0).$$

So F is (a/p)-Hőlder, and we are done. □

Notice that if f is Lipschitz on a nontrivial uniformly convex set $A \subset X$, the last theorem assures only existence of a $(1/p)$-Hőlder extension of f, where necessarily $1/p \leq 1/2$ (see Remark 5.6). This means that our results in the present section are not as "good" as the extension result from Subsection 3.2. The following two-dimensional example, constructed in [5], shows that we cannot hope for an extension result in the class of quasiconvex Lipschitz functions from a uniformly convex set to the whole space.

Example 5.8 ([5])**.** There exists a Lipschitz quasiconvex function f on the open circle $A := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ such that f admits no Lipschitz quasiconvex extension defined on an open convex set containing $\overline{A}$.

Using Theorem 3.4, we immediately obtain the following corollary in the spirit of [4]. Let us stress that our extension does not necessarily preserve invertible moduli of continuity, in this case.

Corollary 5.9. *Let Y be a subspace of a normed space X, and $A \subset X$ an open convex set intersecting Y. Assume that the set $A \cap Y$ is nontrivial and uniformly convex in Y. Then every uniformly continuous quasiconvex function on $A \cap Y$ admits a uniformly continuous quasiconvex extension defined on X, in particular, on A.*

Proof. Theorem 5.5 gives a uniformly continuous quasiconvex extension to Y. Then apply Theorem 3.4. □

Acknowledgement. The research of the first author has been partially supported by the GNAMPA (INdAM – Istituto Nazionale di Alta Matematica) Research Project 2020 and by the Ministry for Science and Innovation, Spanish State Research Agency (Spain), under project PID2020-112491GB-I00. The research of the second author has been partially supported by the GNAMPA (INdAM – Istituto Nazionale di Alta Matematica) Research Project 2020 and by the University of Milan, Research Support Plan PSR 2020.

References

[1] K.J. Arrow and A.C. Enthoven, *Quasi-concave programming*, Econometrica **29** (1961), 779–800.

[2] M. Avriel, W.E. Diewert, S. Schaible, and I. Zang, *Generalized concavity*, Classics in Applied Mathematics, No. 63, reprint of the 1988 original, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2010.

[3] M.V. Balashov and D. Repovš, *Uniform convexity and the splitting problem for selections*, J. Math. Anal. Appl. **360** (2009), 307–316.

[4] C.A. De Bernardi and L. Veselý, *Extendability of continuous quasiconvex functions from subspaces*, preprint, 2022.

[5] C.A. De Bernardi and L. Veselý, *Rotundity properties, and non-extendability of Lipschitz quasiconvex functions*, preprint, 2022.

[6] H. Greenberg and W. Pierskalla, *A Review of Quasi-Convex Functions*, Operations Research **19** (1971), 1553–1570.

[7] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II. Function Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 97, Springer-Verlag, Berlin-New York, 1979.

[8] G. Nordlander, *The modulus of convexity in normed linear spaces*, Ark. Mat. **4** (1960), 15–17.

[9] J.-P. Penot, *What is quasiconvex analysis?*, Optimization **47** (2000), 35–110.

[10] B.T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. **7** (1966), 72–75.

Dipartimento di Matematica per le Scienze economiche, finanziarie ed attuariali, Università Cattolica del Sacro Cuore, 20123 Milano,Italy

E-mail address: carloalberto.debernardi@unicatt.it

E-mail address: carloalberto.debernardi@gmail.com

Dipartimento di Matematica, Università degli Studi, Via C. Saldini 50, 20133 Milano, Italy

E-mail address: libor.vesely@unimi.it