# THE POLAR CONE OF THE SET OF MONOTONE MAPS 

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#### Abstract

We prove that every element of the polar cone to the closed convex cone of monotone transport maps can be represented as the divergence of a measure field taking values in the positive definite matrices.


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## 1. Introduction

The one-dimensional pressureless gas dynamics equations

$$
\left.\begin{array}{rl}
\partial_{t} \varrho+\nabla \cdot(\varrho \boldsymbol{u}) & =0  \tag{1.1}\\
\partial_{t}(\varrho \boldsymbol{u})+\nabla \cdot(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) & =0
\end{array}\right\} \quad \text { in }[0, \infty) \times \mathbb{R}
$$

has recently been shown equivalent (in the regime of sticky particles) to a first-order differential inclusion on the space of monotone transport maps from the reference measure space $\left([0,1],\left.\mathcal{L}^{1}\right|_{[0,1] \mid}\right)=:(\Omega, m)$ (where $\mathcal{L}^{1}$ is the one-dimensional Lebesgue measure) to $\mathbb{R}$; see [7. More precisely, to every density/velocity ( $\varrho, \boldsymbol{u}$ ) solving (1.1) one can associate a unique map $X \in \mathscr{L}^{2}(\Omega, \mathrm{~m})$ with $X$ monotone such that

$$
\begin{equation*}
\varrho(t, \cdot)=X(t, \cdot) \# \mathrm{~m} \quad \text { for all } t \in[0, \infty) \tag{1.2}
\end{equation*}
$$

Here \# indicates the push-forward of measures. Then $X$ satisfies

$$
\begin{equation*}
\dot{X}+\partial I_{\mathscr{K}}(X) \ni \bar{V} \quad \text { for all } t \in[0, \infty) \tag{1.3}
\end{equation*}
$$

where $\mathscr{K}$ denotes the closed convex cone of all transport maps $X \in \mathscr{L}^{2}(\Omega, \mathrm{~m})$ that are monotone, and where $\partial I_{\mathscr{K}}$ is the subdifferential of the indicator function of $\mathscr{K}$. If $X$ satisfies (1.3) and is related to $\varrho$ through (1.2), then the Eulerian velocity $\boldsymbol{u}$ can be recovered from the Lagrangian velocity $V:=\dot{X}$ through

$$
\begin{equation*}
V(t, \cdot)=\boldsymbol{u}(t, X(t, \cdot)) \quad \text { for all } t \in[0, \infty) \tag{1.4}
\end{equation*}
$$

Assuming finite kinetic energy, it is natural to require that

$$
V(t, \cdot) \in \mathscr{L}^{2}(\Omega, \mathrm{~m}), \quad \boldsymbol{u}(t, \cdot) \in \mathscr{L}^{2}(\mathbb{R}, \varrho(t, \cdot))
$$

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The relation (1.4) in particular determines the initial Lagrangian velocity $\bar{V}$ in (1.3) in terms of the initial data $(\varrho, \boldsymbol{u})(0, \cdot)=:(\bar{\varrho}, \overline{\boldsymbol{u}})$ of the system (1.1).

It is shown in [7] that the solution of (1.3) can be written explicitly as

$$
\begin{equation*}
X(t, \cdot)=P_{\mathscr{K}}(\bar{X}+t \bar{V}) \quad \text { for all } t \in[0, \infty) \tag{1.5}
\end{equation*}
$$

with $\bar{X}:=X(0, \cdot) \in \mathscr{K}$ given by (1.2). Here $P_{\mathscr{K}}$ denotes the metric projection onto the cone $\mathscr{K}$. The connection between (1.1) and (1.3) makes it possible to apply classical results from the theory of first-order differential inclusions in Hilbert spaces to study the pressureless gas dynamics equations, which form a system of hyperbolic conservation laws. We refer the reader to [3, 7] for further information.

It is known that if $X$ satisfies (1.5), then the difference $(\bar{X}+t \bar{V})-X(t, \cdot)$ must be an element of the polar cone $N_{\mathscr{K}}(X(t, \cdot))$ of $\mathscr{K}$, which is defined as

$$
\begin{equation*}
N_{\mathscr{K}}(X):=\left\{Y \in \mathscr{L}^{2}(\Omega, \mathrm{~m}): \int_{\Omega} Y\left(X^{\prime}-X\right) \leqslant 0 \quad \text { for all } X^{\prime} \in \mathscr{K}\right\} \tag{1.6}
\end{equation*}
$$

for all $X \in \mathscr{K}$. We observe that $N_{\mathscr{K}}(X)$ coincides with the subdifferential $\partial I_{\mathscr{K}}(X)$. Since $\mathscr{K}$ is a cone, one can choose $X^{\prime}=2 X, X^{\prime}=0$ in (1.6) to obtain that

$$
\begin{equation*}
Y \in N_{\mathscr{K}}(X) \quad \Longleftrightarrow \quad \int_{\Omega} Y X=0, \int_{\Omega} Y X^{\prime} \leqslant 0 \quad \text { for all } X^{\prime} \in \mathscr{K} . \tag{1.7}
\end{equation*}
$$

One is therefore naturally led to the problem of characterizing the polar cone of the set of monotone transport maps, beyond the basic definition (1.6). It is shown in [7] that if $Y \in \mathscr{L}^{2}(\Omega, \mathrm{~m})$ is an element of the polar cone $N_{\mathscr{K}}(X)$, then $Y$ coincides with the derivative of a nonnegative function. We refer the reader to 7 for more details, and to [4, 6, 9] for similar results.

In this paper, we will give a generalization of this result to the multi-dimensional case. We are interested in the following setting: We assume that a Borel probability measure $\varrho$ on $\mathbb{R}^{d}$ is given with finite second moments, so that $\int_{\mathbb{R}^{d}}|x|^{2} \varrho(d x)<\infty$. We consider the closed convex cone of monotone transport maps

$$
\mathscr{K}_{\varrho}:=\left\{f \in \mathscr{L}^{2}\left(\mathbb{R}^{d}, \varrho\right): f \text { is monotone }\right\} .
$$

Here we call any Borel map $f: \mathbb{R}^{d} \longrightarrow \mathbb{R}^{d}$ monotone if the support of the induced transport plan $\gamma_{\boldsymbol{f}}:=(\mathrm{id}, \boldsymbol{f}) \# \varrho$, which is a Borel probability measure on the product space $\mathbb{R}^{d} \times \mathbb{R}^{d}$, is a monotone set. Recall that $\Gamma \subset \mathbb{R}^{d} \times \mathbb{R}^{d}$ is monotone if

$$
\left(y_{1}-y_{2}\right) \cdot\left(x_{1}-x_{2}\right) \geqslant 0 \quad \text { for all }\left(x_{i}, y_{i}\right) \in \Gamma \text { with } i=1 . .2,
$$

where • denotes the Euclidean inner product on $\mathbb{R}^{d}$. Our goal is to find a representation of elements of the polar cone $\mathscr{K}_{\varrho}^{\perp}$ (at the zero map), defined as

$$
\mathscr{K}_{\varrho}^{\perp}:=\left\{\boldsymbol{g} \in \mathscr{L}^{2}\left(\mathbb{R}^{d}, \varrho\right): \int_{\mathbb{R}^{d}} \boldsymbol{g}(x) \cdot \boldsymbol{f}^{\prime}(x) \varrho(d x) \leqslant 0 \quad \text { for all } \boldsymbol{f}^{\prime} \in \mathscr{K}_{\varrho}\right\}
$$

Notice that since $\varrho$ has finite second moments, any smooth monotone function with at most linear growth at infinity (see details below) is an element of $\mathscr{K} \varrho$. Moreover, whenever $\boldsymbol{g} \in \mathscr{K}_{\varrho}^{\perp}$ is given, then the product $\boldsymbol{g} \varrho$ is in fact an $\mathbb{R}^{d}$-valued finite Borel measure, because of Cauchy-Schwarz inequality. We will show below in Theorem 2.1 that for any $\boldsymbol{g} \in \mathscr{K}_{\varrho}^{\perp}$ the measure $\boldsymbol{g} \varrho$ can be written as the divergence of a finite Borel measure taking values in the symmetric, positive semidefinite matrices. In the one-dimensional case, we therefore obtain the derivative of a nonnegative function (measure) as in [7. Our proof relies on an application of the Hahn-Banach theorem and is inspired by a similar argument in [2] for the construction of Michell trusses.

It is possible to prove a representation of the polar cone $\mathscr{K}_{\varrho}^{\perp}$ similar to ours by using a characterization of polar cones from [10] and subharmonic functions; see [6, 9 for instance. Compared to these presentations, our proof is shorter and simpler.

## 2. The Main Result

We will denote by $x \cdot y$ the Euclidean inner product of $x, y \in \mathbb{R}^{k}$, and by $|x|$ the induced norm. We write $\mathbb{R}^{l \times l}$ for the space of real matrices. For any $A, B \in \mathbb{R}^{l \times l}$ with components $A=\left(a_{i j}\right)$ and $B=\left(b_{i j}\right)$ we define an inner product

$$
\langle A, B\rangle:=\operatorname{tr}\left(A B^{\mathrm{T}}\right)=\sum_{i, j=1}^{l} a_{i, j} b_{i, j}
$$

(with $B^{\mathrm{T}}$ the transpose of $B$ ), which induces the Frobenius norm

$$
\|A\|:=\sqrt{\operatorname{tr}\left(A A^{\mathrm{T}}\right)}=\sum_{i, j=1}^{l} a_{i, j}^{2}
$$

We denote by $\mathcal{S}^{l}$ the space of symmetric real matrices and by $\mathcal{S}_{+}^{l}$ the subset of positive semidefinite symmetric matrices. The space of all positive definite, but not necessarily symmetric matrices will be denoted by $\mathbb{R}_{+}^{l \times l}$. Recall that

$$
A \in \mathbb{R}_{+}^{l \times l} \Longleftrightarrow v \cdot(A v) \geqslant 0 \quad \text { for all } v \in \mathbb{R}^{l}
$$

Equivalently, we have $A \in \mathbb{R}_{+}^{l \times l}$ if and only if $A^{\text {sym }}:=\left(A+A^{\mathrm{T}}\right) / 2 \in \mathcal{S}_{+}^{l}$.
Let $\mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathbb{R}^{l \times l}\right)$ be the space of all continuous functions $w: \mathbb{R}^{d} \longrightarrow \mathbb{R}^{l \times l}$ with the property that $\lim _{|x| \rightarrow \infty} w(x) \in \mathbb{R}^{l \times l}$ exists. Note that we can write

$$
\mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathbb{R}^{l \times l}\right)=\mathbb{R}^{l \times l}+\mathscr{C}_{0}\left(\mathbb{R}^{d} ; \mathbb{R}^{l \times l}\right)
$$

where $\mathscr{C}_{0}\left(\mathbb{R}^{d} ; \mathbb{R}^{l \times l}\right)$ is the closure of the space of all compactly supported continuous $\mathbb{R}^{l \times l}$-valued maps, w.r.t. the sup-norm. In an analogous way, we define $\mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathcal{S}^{l}\right)$ and $\mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathcal{S}_{+}^{l}\right)$. For any map $u \in \mathscr{C}^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right)$ we denote by

$$
e(u(x)):=D u(x)^{\text {sym }} \quad \text { for all } x \in \mathbb{R}^{d}
$$

its deformation tensor, which is an element of $\mathscr{C}\left(\mathbb{R}^{d} ; \mathcal{S}^{d}\right)$. Let

$$
\begin{gathered}
\mathscr{C}_{*}^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right):=\left\{u \in \mathscr{C}^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right): D u \in \mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathbb{R}^{d \times d}\right)\right\}, \\
\operatorname{MON}\left(\mathbb{R}^{d}\right):=\left\{u \in \mathscr{C}_{*}^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right): u \text { is monotone }\right\}
\end{gathered}
$$

so that $e(u) \in \mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathcal{S}_{+}^{d}\right)$ if $u \in \operatorname{MON}\left(\mathbb{R}^{d}\right)$. The cone $\operatorname{MON}\left(\mathbb{R}^{d}\right)$ contains all linear maps $u(x):=A x$ for $x \in \mathbb{R}^{d}$, where $A \in \mathbb{R}_{+}^{d \times d}$. See [1] for more details.

We will denote by $\mathscr{M}\left(\mathbb{R}^{d} ; \mathbb{R}^{k}\right)$ the space of finite $\mathbb{R}^{k}$-valued Borel measures. In an analogous way, we define $\mathscr{M}\left(\mathbb{R}^{d} ; \mathcal{S}^{l}\right)$ and $\mathscr{M}\left(\mathbb{R}^{d} ; \mathcal{S}_{+}^{l}\right)$. If $f_{i}, i=1 \ldots k$, are the components of $\mathbf{F} \in \mathscr{M}\left(\mathbb{R}^{d} ; \mathbb{R}^{k}\right)$ and $u \in \mathscr{C}_{b}\left(\mathbb{R}^{d} ; \mathbb{R}^{k}\right)$ we write

$$
\int_{\mathbb{R}^{d}} u(x) \cdot \mathbf{F}(d x)=\sum_{i=1}^{k} u_{i}(x) f_{i}(d x) .
$$

We will say that $\mathbf{F}$ has finite first moment if $\sum_{i=1}^{k} \int_{\mathbb{R}^{d}}|x|\left|f_{i}\right|(d x)<\infty$. If $\mu_{i, j}=\mu_{j, i}$, $i, j=1 \ldots l$, are the components of $\mathbf{M} \in \mathscr{M}\left(\mathbb{R}^{d} ; \mathcal{S}^{l}\right)$ and $v \in \mathscr{C}_{b}\left(\mathbb{R}^{d} ; \mathcal{S}^{l}\right)$, then

$$
\int_{\mathbb{R}^{d}}\langle v(x), \mathbf{M}(d x)\rangle=\sum_{i, j=1}^{l} v_{i, j}(x) \mu_{i, j}(d x)
$$

For any $\mathbf{M}=\left(\mu_{i, j}\right) \in \mathscr{M}\left(\mathbb{R}^{d} ; \mathcal{S}^{l}\right)$ we have $\mathbf{M} \in \mathscr{M}\left(\mathbb{R}^{d} ; \mathcal{S}_{+}^{l}\right)$ if and only if

$$
\sum_{i, j=1}^{l} \mu_{i, j} v_{i} v_{j} \quad \text { is a positive measure for all } v \in \mathbb{R}^{l}
$$

We can now state our representation result.
Theorem 2.1 (Stress Tensor). Assume that there exist a measure $\mathbf{F} \in \mathscr{M}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right)$ with finite first moment and a matrix-valued field $\mathbf{H} \in \mathscr{M}\left(\mathbb{R}^{d} ; \mathcal{S}_{+}^{d}\right)$ with

$$
\begin{equation*}
G(u):=-\int_{\mathbb{R}^{d}} u(x) \cdot \mathbf{F}(d x)-\int_{\mathbb{R}^{d}}\langle e(u(x)), \mathbf{H}(d x)\rangle \geqslant 0 \tag{2.1}
\end{equation*}
$$

for all $u \in \operatorname{MON}\left(\mathbb{R}^{d}\right)$. Then there exists $\mathbf{M} \in \mathscr{M}\left(\mathbb{R}^{d} ; \mathcal{S}_{+}^{d}\right)$ such that

$$
\begin{align*}
G(u) & =\int_{\mathbb{R}^{d}}\langle e(u(x)), \mathbf{M}(d x)\rangle \quad \text { for all } u \in \mathscr{D}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right)  \tag{2.2}\\
\int_{\mathbb{R}^{d}} \operatorname{tr}(\mathbf{M}(d x)) & \leqslant-\int_{\mathbb{R}^{d}} x \cdot \mathbf{F}(d x)-\int_{\mathbb{R}^{d}} \operatorname{tr}(\mathbf{H}(d x)) \tag{2.3}
\end{align*}
$$

Notice that the integrals in (2.1) are finite for any choice of $u \in \mathscr{C}_{*}^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right)$, by our assumptions on $\mathbf{F}$ and $\mathbf{H}$. Recall that the trace of a symmetric matrix is equal to the sum of its eigenvalues, which in the case of a positive semidefinite matrix are all nonnegative. Therefore (2.3) controls the size of the measure $\mathbf{M}$.

For $\mathbf{H} \equiv 0$ we obtain the representation announced in the introduction:

$$
\int_{\mathbb{R}^{d}} u(x) \cdot \mathbf{F}(d x)=-\int_{\mathbb{R}^{d}}\langle D u(x), \mathbf{M}(d x)\rangle
$$

for all test functions $u$. Recall that $\mathbf{M}$ takes values in the symmetric matrices. The more general form of (2.1) is motivated by a variational time discretization for the compressible Euler equations, for which a minimization problem of the form

$$
\begin{equation*}
\inf _{\boldsymbol{f} \in \mathscr{\mathscr { C }}}\left\{\frac{1}{2} \int_{\mathbb{R}^{d}}|\boldsymbol{h}(x)-\boldsymbol{f}(x)|^{2} \varrho(d x)+\int_{\mathbb{R}^{d}} e(x) \operatorname{det}\left(D \boldsymbol{f}(x)^{\text {sym }}\right)^{1-\gamma} d x\right\} \tag{2.4}
\end{equation*}
$$

for suitable $\boldsymbol{h} \in \mathscr{L}^{2}\left(\mathbb{R}^{d}, \varrho\right)$ and nonnegative $e \in \mathscr{L}^{1}\left(\mathbb{R}^{d}\right)$ must be solved, with $\gamma>1$ some constant. Denoting by $\boldsymbol{f} \in \mathscr{K} \varrho$ the minimizer of (2.4) and letting $\boldsymbol{g}:=\boldsymbol{h}-\boldsymbol{f}$, we can write the corresponding first-order optimality condition (formally) as

$$
\begin{aligned}
& -\int_{\mathbb{R}^{d}} \boldsymbol{g}(x) \cdot \boldsymbol{f}^{\prime}(x) \varrho(d x) \\
& \quad-(\gamma-1) \int_{\mathbb{R}^{d}} e(x) \operatorname{det}\left(D \boldsymbol{f}(x)^{\mathrm{sym}}\right)^{-\gamma} \operatorname{tr}\left(\operatorname{cof}\left(D \boldsymbol{f}(x)^{\mathrm{sym}}\right)^{\mathrm{T}} D \boldsymbol{f}^{\prime}(x)\right) d x \geqslant 0
\end{aligned}
$$

for all $\boldsymbol{f}^{\prime} \in \mathscr{K}_{\varrho}$. From this, assumption (2.1) follows if we define

$$
\mathbf{F}:=\boldsymbol{g} \varrho \quad \text { and } \quad \mathbf{H}:=(\gamma-1) e \operatorname{det}\left(D \boldsymbol{f}^{\mathrm{sym}}\right)^{-\gamma} \operatorname{cof}\left(D \boldsymbol{f}^{\mathrm{sym}}\right)^{\mathrm{T}} .
$$

One can then check that $\mathbf{F}$ has finite first moments and that $\mathbf{H} \in \mathscr{M}\left(\mathbb{R}^{d} ; \mathcal{S}_{+}^{d}\right)$. This application will be discussed in more detail in an upcoming publication.
2.1. Positive Functionals. In this section, we will discuss a general result about extensions of positive functionals, which is due to Riedl [8]. Let us start with some notation: In the following, we denote by $E$ a normed vector space. We call positive cone any subset $C \subset E$ with $C \neq E$ with the following properties:

$$
\begin{equation*}
C+C \subset C, \quad \lambda C \subset C \quad \text { for all } \lambda>0, \quad C \cap(-C)=\{0\} \tag{2.5}
\end{equation*}
$$

The positive cone $C$ induces a partial ordering $\geqslant$ on the space $E$ by

$$
y \geqslant x \quad \Longleftrightarrow \quad y-x \in C
$$

A linear map $F: L \longrightarrow \mathbb{R}$ defined on a subspace $L \subset E$ is called positive if

$$
\begin{equation*}
F(x) \geqslant 0 \quad \text { for all } x \in L \cap C \tag{2.6}
\end{equation*}
$$

A linear map $F: E \longrightarrow \mathbb{R}$ is called functional if it is continuous.
Proposition 2.2. Let $E$ be a Banach space, partially ordered by a positive cone $C$. If some subspace $L \subset E$ contains an interior point of $C$, then every positive linear $\operatorname{map} F_{0}: L \longrightarrow \mathbb{R}$ can be extended to a positive functional $F: E \longrightarrow \mathbb{R}$.

Proof. See Theorem 10.10 of [8]. We include a proof for the reader's convenience.
Step 1. We first observe that $E=L-C$. Indeed if $x_{0} \in L$ is an inner point of $C$, then there exists a $\delta>0$ with $B_{\delta}\left(x_{0}\right) \subset C$. Moreover, for all $x \in E$ there exists $\lambda>0$ (choose $\lambda:=2\|x\| / \delta$, for example) with the property that

$$
x / \lambda \subset B_{\delta}(0)=x_{0}-B_{\delta}\left(x_{0}\right) \subset x_{0}-C
$$

Since $L$ is a subspace we obtain, using $\lambda C \subset C$ for all $\lambda>0$, that

$$
E \subset \bigcup_{\lambda>0} \lambda\left(x_{0}-C\right) \subset \mathbb{R} x_{0}-C \subset L-C
$$

Step 2. Since $E=L-C$, for every $x \in E$ there exist $y_{ \pm} \in L$ and $z_{ \pm} \in C$ such that $\pm x=y_{ \pm}-z_{ \pm}$, which implies that $y_{+} \geqslant x \geqslant-y_{-}$. We now define

$$
\begin{equation*}
p(x):=\inf \left\{F_{0}(y): y \in L, y \geqslant x\right\} \quad \text { for all } x \in E \tag{2.7}
\end{equation*}
$$

Then $p(x) \leqslant F_{0}\left(y_{+}\right)<\infty$. On the other hand, for every $y \in L$ with $y \geqslant x$ we have $y \geqslant-y_{-}$. Since $y+y_{-} \in L \cap C$, we have $F_{0}\left(y+y_{-}\right) \geqslant 0$, by positivity of $F_{0}$. This implies that $F_{0}(y) \geqslant-F_{0}\left(y_{-}\right)>-\infty$. We conclude that $p(x)$ is finite for all $x \in E$. It is easy to check that for all $x_{1}, x_{2} \in E$ and for all $\lambda>0$ we have

$$
p\left(x_{1}+x_{2}\right) \leqslant p\left(x_{1}\right)+p\left(x_{2}\right), \quad p\left(\lambda x_{1}\right)=\lambda p\left(x_{1}\right)
$$

For every $x \in L$ and $z \in E$ with $z \geqslant x$, we have $F_{0}(x) \leqslant p(z)$ (in particular, we may choose $z=x$ ). Indeed for every $y \in L$ with $y \geqslant z$, we have $y \geqslant x$, thus $y-x \in L \cap C$. Hence $F_{0}(y-x) \geqslant 0$, by positivity, which yields $F_{0}(y)=F_{0}(x)+F_{0}(y-x) \geqslant F_{0}(x)$. Taking the inf over all $y \in L$ with $y \geqslant z$, we obtain the estimate.

Step 3. We can now apply the Hahn-Banach theorem and obtain a linear map $F: E \longrightarrow \mathbb{R}$ with $F(x) \leqslant p(x)$ for all $x \in E$. In order to show that $F$ is positive, let $x \in C$. Then $0 \geqslant-x$ and $0 \in L$, so we may choose $y=0$ in the definition of $p(-x)$ (see (2.7)) to obtain $p(-x) \leqslant 0$. Therefore $F(-x) \leqslant p(-x) \leqslant 0$, and so $F(x) \geqslant 0$ for all $x \in C$. To prove that $F$ is an extension of $F_{0}$, let $x \in L$. Then we may choose $y=-x$ in (2.7) to obtain $p(-x) \leqslant F_{0}(-x)$ for all $x \in L$. Then

$$
-F(x)=F(-x) \leqslant p(-x) \leqslant F_{0}(-x)=-F_{0}(x)
$$

hence $F_{0}(x) \leqslant F(x)$. Applying the same argument to $-x \in L$, we get $F_{0}(x) \geqslant F(x)$. It follows that $F_{0}(x)=F(x)$ for all $x \in L$. Therefore $F$ is an extension of $F_{0}$.

Step 4. It remains to prove that $F$ is continuous. Let $x_{0}$ be the interior point of $C$ from Step 1, for which $B_{\delta}\left(x_{0}\right) \subset C$. Then $B_{\delta}(0) \subset \pm\left(x_{0}-C\right)$. Let $\lambda:=F\left(x_{0}\right) \geqslant 0$ (recall that $F(x) \geqslant 0$ for all $x \in C$ ). Then for all $x \in B_{\delta}(0)$ we have $x_{0}-x \in C$, thus $F\left(x_{0}-x\right) \geqslant 0$. It follows that $F\left(x_{0}\right) \geqslant F(x)$. Similarly, we obtain $F(x) \geqslant-F\left(x_{0}\right)$. Then either $F$ vanishes (if $\lambda=0$ ), or the preimage of the nonempty interval $(-\lambda, \lambda)$ contains a neighborhood of 0 , and so $F$ (being linear) is continuous.
2.2. Proof of Theorem 2.1. We apply Proposition 2.2 with

$$
E:=\mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathcal{S}^{d}\right), \quad C:=\mathscr{C}_{*}\left(\mathbb{R}^{d}, \mathcal{S}_{+}^{d}\right), \quad L:=\left\{e(u): u \in \mathscr{C}_{*}^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right)\right\}
$$

Clearly $C$ satisfies conditions (2.5). The identity map id is an element of $\operatorname{MON}\left(\mathbb{R}^{d}\right)$, with constant deformation tensor $e(i d)$ equal to the identity matrix $1 \in \mathcal{S}_{+}^{d}$. Since the eigenvalues of a symmetric matrix depend continuously on the matrix entries, we have that $e(\mathrm{id})=\mathbf{1}$ is an interior point of $C$ : For all $\| v$-id $\|_{E}$ sufficiently small, the eigenvalues of $v(x)$ are bigger than $1 / 2$ for all $x \in \mathbb{R}^{d}$ and $v \in E$.

On the subspace $L \subset E$, we define the functional $F_{0}$ as

$$
F_{0}(v):=-\int_{\mathbb{R}^{d}} u(x) \cdot \mathbf{F}(d x)-\int_{\mathbb{R}^{d}}\langle v(x), \mathbf{H}(d x)\rangle \quad \text { where } v=e(u)
$$

Note that $F_{0}$ is well-defined: If there exists another map $\tilde{u} \in \mathscr{C}_{*}^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right)$ such that $e(\tilde{u}(x))=v(x)$ for all $x \in \mathbb{R}^{d}$, then we have $e(u-\tilde{u}) \equiv 0$, by linearity. Consequently, there exist an antisymmetric matrix $B \in \mathbb{R}^{d \times d}$ and $c \in \mathbb{R}^{d}$ such that

$$
\bar{u}(x):=u(x)-\tilde{u}(x)=B x+c \quad \text { for all } x \in \mathbb{R}^{d} .
$$

Indeed assume that $e(\bar{u}(x))=0$ and define

$$
W \bar{u}(x):=\frac{D \bar{u}(x)-D \bar{u}(x)^{\mathrm{T}}}{2} \quad \text { for all } x \in \mathbb{R}^{d}
$$

Then $\partial_{k}(W \bar{u})_{i, j} \equiv 0$ for all indices $i, j, k$. Since $D \bar{u}=e(\bar{u})+W \bar{u}$ it follows that $D \bar{u}$ is a constant matrix with vanishing symmetric part, so $\bar{u}$ is a rigid deformation. We now observe that both $\pm \bar{u} \in \operatorname{MON}\left(\mathbb{R}^{d}\right)$, which implies $F_{0}(e(\bar{u}))=0$ because of (2.1). As $F_{0}$ is linear, we conclude that $F_{0}$ is well-defined. Similarly, one can check that $F_{0}(v) \geqslant 0$ for all $v \in L \cap C$, so the linear map $F_{0}: L \longrightarrow \mathbb{R}$ is positive.

Applying Proposition 2.2, we obtain that $F_{0}$ can be extended to a continuous linear $\operatorname{map} F: E \longrightarrow \mathbb{R}$. Notice that $\mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathbb{R}\right)$ is a separable and closed subalgebra of the space $\mathscr{C}_{b}\left(\mathbb{R}^{d} ; \mathbb{R}\right)$ of bounded continuous $\mathcal{S}^{d}$-functions. As is well-known, to any closed subalgebra of a space of bounded continuous functions, there corresponds a compactification of the domain. In our case, we obtain the one-point (also called Alexandroff) compactification of $\mathbb{R}^{d}$, which we will denote by $\beta \mathbb{R}^{d}$. Then $\mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathcal{S}^{d}\right)$ is isomorphic to $\mathscr{C}\left(\beta \mathbb{R}^{d} ; \mathcal{S}^{d}\right)$. We refer the reader to [5] Section 4.8 for more details. By the Riesz representation theorem, there therefore exists a finite Radon measure $\mathbf{M} \in \mathscr{M}\left(\beta \mathbb{R}^{d} ; \mathcal{S}^{d}\right)$ that represents the functional $F$ in the sense that

$$
F(v)=\int_{\beta \mathbb{R}^{d}}\langle v(x), \mathbf{M}(d x)\rangle \quad \text { for all } v \in \mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathcal{S}^{d}\right)
$$

Since $F(v) \geqslant 0$ for all $v \in \mathscr{C}_{*}\left(\mathbb{R}^{d} ; \mathcal{S}_{+}^{d}\right)$ we obtain that $\mathbf{M}$ takes in fact values in $\mathcal{S}_{+}^{d}$. Moreover, as $F$ is an extension of $F_{0}$, the following identity holds:

$$
F_{0}(v)=-\int_{\mathbb{R}^{d}} u(x) \cdot \mathbf{F}(d x)-\int_{\mathbb{R}^{d}}\langle v(x), \mathbf{H}(d x)\rangle=\int_{\beta \mathbb{R}^{d}}\langle v(x), \mathbf{M}(d x)\rangle
$$

for any $v=e(u)$ and $u \in \mathscr{C}_{*}^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right)$; see (2.2). In particular, we may choose $u=\mathrm{id}$ (with $e(\mathrm{id})=\mathbf{1}$ ) to obtain the control (recall that $\mathbf{M}$ is $\mathcal{S}_{+}^{d}$-valued)

$$
\int_{\beta \mathbb{R}^{d}} \operatorname{tr}(\mathbf{M}(d x))=-\int_{\mathbb{R}^{d}} x \cdot \mathbf{F}(d x)-\int_{\mathbb{R}^{d}} \operatorname{tr}(\mathbf{H}(d x))
$$

Restricting the representation from $\beta \mathbb{R}^{d}$ to $\mathbb{R}^{d}$, we obtain the result.

## References

[1] G. Alberti and L. Ambrosio, A geometrical approach to monotone functions in $\mathbf{R}^{n}$, Math. Z. 230 (1999), no. 2, 259-316.
[2] G. Bouchitté, W. Gangbo, and P. Seppecher, Michell trusses and lines of principal action, Math. Models Methods Appl. Sci. 18 (2008), no. 9, 1571-1603.
[3] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).
[4] G. Carlier and T. Lachand-Robert, Representation of the polar cone of convex functions and applications, J. Convex Anal. 15 (2008), no. 3, 535-546.
[5] G. B. Folland, Real analysis, Pure and Applied Mathematics (New York), John Wiley \& Sons Inc., New York, 1984. Modern techniques and their applications, A Wiley-Interscience Publication.
[6] P.-L. Lions, Identification du cône dual des fonctions convexes et applications, C. R. Acad. Sci. Paris Sér. I Math. 326 (1998), no. 12, 1385-1390.
[7] L. Natile and G. Savaré, A Wasserstein approach to the one-dimensional sticky particle system, SIAM J. Math. Anal. 41 (2009), no. 4, 1340-1365.
[8] J. Riedl, Partially ordered locally convex vector spaces and extensions of positive continuous linear mappings, Math. Ann. 157 (1964), 95-124.
[9] M. Westdickenberg, Projections onto the cone of optimal transport maps and compressible fluid flows, J. Hyperbolic Differ. Equ. 7 (2010), no. 4, 605-649.
[10] E. H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory. I. Projections on convex sets, Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Madison, WI, 1971), Academic Press, New York, 1971, pp. 237-341.

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