A NOTE ON A RESIDUAL SUBSET OF LIPSCHITZ FUNCTIONS ON METRIC SPACES

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ABSTRACT. Let (X,d) be a quasi-convex, complete and separable metric space with reference probability measure m. We prove that the set of of real valued Lipschitz function with non zero point-wise Lipschitz constant m-almost everywhere is residual, and hence dense, in the Banach space of Lipschitz and bounded functions. The result is the metric analogous of a result proved for real valued Lipschitz maps defined on \mathbb{R}^2 by Alberti, Bianchini and Crippa in [1].

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1. Introduction

In the context of metric spaces, say (X, d), it is possible to look at the point-wise variation of a real valued map considering

(1.1)
$$\operatorname{Lip} f(x) := \limsup_{y \to x, y \neq x} \frac{|f(x) - f(y)|}{d(x, y)},$$

that is called *point-wise Lipschitz constant*. In the smooth framework Lip f corresponds to the modulus of ∇f : if (X,d) is an open subset of \mathbb{R}^d endowed with the euclidean norm and f is locally Lipschitz, then Lip $f=|\nabla f|$ almost everywhere with respect to the Lebesgue measure. Or more in general if (X,d,m) is a metric measure space admitting a differentiable structure in the sense of Cheeger, see [4], [5] for the definitions, and f is Lipschitz, then Lip f=|df| m-a.e. where df is the Cheeger's differential of f.

Once a point-wise information is given we are interested at looking at those points where the "differential" vanishes: define the singular set of f as follows

$$S(f) := \{ x \in X : \text{Lip } f(x) = 0 \}.$$

The classical Sard's Theorem states that if $f: \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth then the Lebesgue measure of f(S(f)) is 0. As soon as the regularity assumption on f is dropped, the conclusion of Sard's Theorem does not hold anymore and one may look for weaker property to hold.

The question is if it is possible to approximate any Lipschitz function with functions having negligible S(f) with respect to a given reference measure.

For real valued Lipschitz functions defined on \mathbb{R}^2 , with Lebesgue measure playing the role of the reference measure, a positive answer is contained in [1], see Proposition 4.10. We prove the following.

Theorem 1.1. Assume (X,d) is a quasi-convex, complete and separable metric space and let m be a Borel probability measure over it. The set of those $f \in D^{\infty}(X)$ so that m(S(f)) = 0 is residual, and therefore dense, in $D^{\infty}(X)$.

The Banach space $D^{\infty}(X)$ will be the space of bounded functions with bounded point-wise Lipschitz constant, endowed with the uniform norm. See below for a precise definition. Recall that a set in a topological space is residual if it contains a countable intersection of open dense set. By Baire Theorem, a residual set in a complete metric space is dense.

2. Setting

Let (X,d) be a metric space and m is a Borel probability measure over X so that X coincides with its support. For $f: X \to \mathbb{R}$ the Lipschitz constant of f is defined as usual by

$$LIP(f) := \sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)},$$

and we say that f is Lipschitz if LIP(f) is a finite number. Accordingly denote by $LIP^{\infty}(X)$ the space of bounded Lipschitz functions. The natural norm on $LIP^{\infty}(X)$ is given by

$$||f||_{\operatorname{LIP}^{\infty}(X)} = ||f||_{\infty} + \operatorname{LIP}(f),$$

where $\|\cdot\|_{\infty}$ is the uniform norm. The space of bounded Lipschitz functions endowed with $\|f\|_{\mathrm{LIP}^{\infty}(X)}$ turns out to be a Banach space. The point-wise version of $\mathrm{LIP}(f)$ is given by the point-wise Lipschitz constant as defined in 1.1. The corresponding space of bounded functions with bounded point-wise Lipschitz constant can be considered:

$$D^{\infty}(X) := \{ f : X \to \mathbb{R} : ||f||_{\infty} + ||\text{Lip } f||_{\infty} < \infty \}.$$

A study of $D^{\infty}(X)$ and $LIP^{\infty}(X)$ can be found in [3]. The following results are taken from [3].

It is straightforward to note that $LIP^{\infty}(X) \subset D^{\infty}(X)$ and for a general metric space this is the only valid inclusion. Examples of metric spaces and functions in $D^{\infty}(X)$ not satisfying a global Lipschitz bound can be constructed, see [3]. If (X, d) is quasi-convex also the other inclusion holds and $LIP^{\infty}(X) = D^{\infty}(X)$ and the two semi-norms are comparable: there exists $C \geq 1$ so that

$$\|\operatorname{Lip} f\|_{\infty} \le \operatorname{LIP}(f) \le C \|\operatorname{Lip} f\|_{\infty}.$$

Hence $D^{\infty}(X)$, or equivalently $\mathrm{LIP}^{\infty}(X)$, endowed with the norm $\|\cdot\|_{\infty} + \|\mathrm{Lip}(\cdot)\|_{\infty}$ is a Banach space. We will denote this norm with $\|\cdot\|_{D^{\infty}}(X)$.

Recall that a metric space (X,d) is quasi-convex if there exists a constant $C \geq 1$ such that for each pair of points $x,y \in X$ there exists a curve γ connecting the two points such that $l(\gamma) \leq Cd(x,y)$, where $l(\gamma)$ denotes the length of γ defined with the usual "affine" approximation: for $\gamma:[a,b] \to X$ its length $l(\gamma)$ is defined by

$$l(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(x_i, x_{i+1}) : a = x_1 < x_2 < \dots < x_{n+1} = b, n \in \mathbb{N} \right\}.$$

Associated to the length $l(\gamma)$ there is the distance obtained minimizing it:

$$d_L(x, y) = \inf\{l(\gamma) : \gamma_0 = x, \gamma_1 = y\}.$$

The function d_L is indeed a distance on each component of accessibility by rectifiable paths, i. e. those paths having finite l. By quasi-convexity it follows that

$$d(x,y) < d_L(x,y) < Cd(x,y),$$

with C > 1. Hence (X, d_L) is a complete and separable metric space that is also a length space. Clearly (X, d_L) has the same open sets of (X, d). For a more detailed discussion on length spaces see [2].

We will use the following notation. For r > 0 and $z \in X$, we will denote with $B_r(z)$ the ball of radius r centered in z. The complement in X of a set A will be denoted by A^c and ∂A denotes the topological boundary of A. The closure of A is cl(A) and the interior part int(A). Associated to a set we can consider the distance from it: for $x \in X$ and $A \subset X$

$$d(x,A) := \inf_{w \in A} d(x,w).$$

3. The Result

Lemma 3.1. For any Borel function $f: X \to \mathbb{R}$, the function $\text{Lip } f: X \to \overline{\mathbb{R}}$ is universally measurable.

Proof. In order to prove the claim we just have to show that the set $\{x \in X : \text{Lip } f(x) \geq a\}$ is Souslin for any $a \in \mathbb{R}$. Since f is a Borel map then

$$\bigcap_{n \in \mathbb{N}} \left\{ (x, y) \in X \times X : 0 < d(x, y) \le \frac{1}{n}, \ \frac{|f(x) - f(y)|}{d(x, y)} \ge a \right\}$$

is a Borel set. Note that

$$\{x \in X : \text{Lip } f(x) \ge a\} = P_1 \left(\bigcap_{x \in \mathbb{N}} \left\{ (x, y) \in X \times X : 0 < d(x, y) \le \frac{1}{n}, \ \frac{|f(x) - f(y)|}{d(x, y)} \ge a \right\} \right),$$

where $P_1: X \times X \to X$ denotes the projection on the first element. It follows from the definition of Souslin set that $\{x \in X : \text{Lip } f(x) \geq a\}$ is Souslin and the claim follows.

Then after Lemma 3.1 it makes sense to look at those functions f so that m(S(f)) = 0. We will need the following

Lemma 3.2. Let $K \subset X$ be a closed set and consider the length distance function from K that is $g(x) := d_L(x, K)$. Then

$$1 \le \text{Lip } g(x) \le C, \quad for \ x \in K^c,$$

Proof. Step 1. Assume that $d = d_L$ so that (X, d) is also a length space and g = d(x, K). Then fix $x \in K^c$: for any $z \in K$ and $y \in K^c$ it holds

$$d(x,z) - d(y,z) \le d(x,y)$$

hence trivially Lip $g(x) \leq 1$.

Consider now a minimizing sequence $z_n \in K$ for x, that is that $g(x) \geq d(x, z_n) - 1/n$. From the length structure it follows that for any n there exists $\gamma^n : [0, 1] \to X$ rectifiable curve starting in x and arriving in z_n so that $d(x, z_n) \geq l(\gamma^n) - 1/n$. So for any y_n in the image of γ^n

$$\frac{g(x)-g(y_n)}{d(x,y_n)} \ge \frac{l(\gamma_n)-d(y_n,z_n)-2/n}{d(x,y_n)}.$$

Since $l(\gamma^n) \ge d(x, y_n) + d(y_n, z_n)$ it follows that

$$\frac{g(x)-g(y_n)}{d(x,y_n)} \geq \frac{d(x,y_n)-2/n}{d(x,y_n)}.$$

Since the only constrain on y_n was to belong to the image of γ^n , we can choose y_n so that the previous ratio converges to 1. Hence Lip g(x) = 1.

Step 2. We now drop the assumption on the length structure of the space. Let (X, d) be quasi-convex and $g(x) = d_L(x, K)$. Since (X, d_L) is a length space for any $x \in K^c$

$$\limsup_{y \to x, y \neq x} \frac{|g(x) - g(y)|}{d_L(x, y)} = 1.$$

Having (X, d_L) and (X, d) the same open set, K^c does not depend on the metric. Since $d \le d_L \le Cd$ the claim follows.

We can now prove Theorem 1.1. The proof uses now the ideas contained in Proposition 4.10 in [1].

Theorem 3.3. Assume (X,d) is a quasi-convex, complete and separable space and let m be a Borel probability measure over it. The set of those $f \in D^{\infty}(X)$ so that m(S(f)) = 0 is residual in $D^{\infty}(X)$ and therefore dense.

Proof. Consider the following sets

$$G := \{ f \in D^{\infty}(X) : m(S(f)) = 0 \}, \qquad G_r := \{ f \in D^{\infty}(X) : m(S(f)) < r \}.$$

The claim is then to prove that G is a residual set. Since $G = \cap G_r$, where the intersection runs over a sequence of r converging to 0, the claim is proved once it is proved that each G_r is open and dense in $D^{\infty}(X)$.

Step 1. The set G_r is open in $D^{\infty}(X)$. Fix $f \in G_r$. Then there exists $\delta > 0$ so that

$$m(\{x \in X : \text{Lip } f(x) \le \delta\}) < r.$$

Since for any $g \in D^{\infty}(X)$ it holds that

$$\operatorname{Lip} f(x) \le \operatorname{Lip} g(x) + \operatorname{Lip} (f - g)(x),$$

for any $g \in D^{\infty}(X)$ so that $||g - f||_{D^{\infty}}(X) \le \delta$ it holds that

$$S(g) \subset \{x \in X : \text{Lip } f(x) \le \delta\},\$$

and therefore m(S(g)) < r and consequently $g \in G_r$.

Step 2. The set G_r is dense in $D^{\infty}(X)$. Given $f \in D^{\infty}(X)$ and $\delta > 0$ we have to find $g \in G_r$ so that $||f - g||_{D^{\infty}(X)} \leq \delta$. Without loss of generality we can assume $m(S(f)) \geq r$.

For every $\varepsilon > 0$ denote with $S(f)^{\varepsilon}$ the ε -neighborhood of the set of singular points of f, i.e.

$$S(f)^{\varepsilon} = \{ z \in X : d(z, S(f)) < \varepsilon \}.$$

The set $S(f)^{\varepsilon}$ is open and denote by K its complementary in X. Associated to K we consider the distance function \hat{g} as defined in Lemma 3.2 that is $\hat{g}(x) := d_L(x, K)$. A rough bound on $\hat{g}(x)$ can be given in terms of the "diameter" of S(f):

$$\hat{g}(x) \leq C \sup\{d(x,z) : cl(S(f)^{\varepsilon})\},$$

where $cl(S(f)^{\varepsilon})$ stands for the closure of $S(f)^{\varepsilon}$. Since to approximate with functions in G_r we can make an error in measure strictly less than r and since m is a probability measure, we can assume S(f) to have finite diameter and by inner regularity we can even assume it to be closed. Therefore

$$\|\hat{g}\|_{\infty} \le M, \qquad M > 0.$$

From Lemma 3.2 we have $\operatorname{Lip} \hat{g}(x) > 0$ for $x \in S(f)^{\varepsilon}$ and clearly $\operatorname{Lip} \hat{g}(x) = 0$ for $x \in \operatorname{int}(K)$, where $\operatorname{int}(K)$ stands for the interior part of K.

Note that the boundary of $S(f)^{\varepsilon}$ is contained in the set $\{z:d(z,S(f))=\varepsilon\}$. Indeed $z\in\partial S(f)^{\varepsilon}$ if and only if $d(z,S(f))\geq\varepsilon$ and for every $\eta>0$ there exists a point $w\in X$ so that

$$d(z, w) \le \eta, \qquad d(w, S(f)) < \varepsilon.$$

Let η_n be a sequence converging to 0 and w_n the corresponding sequence converging to z. To each w_n associate $x_n \in S(f)$ so that $d(w_n, x_n) < \varepsilon$. Then

$$d(z, x_n) \le d(z, w_n) + d(w_n, x_n) < \eta_n + \varepsilon.$$

Passing to the limit $d(z, S(f)) \leq \varepsilon$ and therefore necessarily $d(z, S(f)) = \varepsilon$.

Moreover for $\varepsilon \neq \varepsilon'$

$$\{z: d(z, S(f)) = \varepsilon\} \cap \{z: d(z, S(f)) = \varepsilon'\} = \emptyset,$$

hence there exists at most countably many ε so that $m(\{z:d(z,S(f))=\varepsilon\})>0$. Hence for any r>0 there exists $\varepsilon>0$ so that

$$m(\lbrace z : d(z, S(f)) = \varepsilon \rbrace) = 0, \qquad m(S(f)^{\varepsilon} \setminus S(f)) < r,$$

where the second expression holds because S(f) is closed. From what said so far, denoting $g:=f+(\delta/2M)\hat{g}$ is such that

$$||f - g||_{D^{\infty}(X)} \le \delta.$$

To conclude the proof observe that $S(g) \subset S(f)^{\varepsilon} \setminus S(f)$, hence by construction $g \in G_r$.

References

- [1] G. Alberti, S. Bianchini, and G. Crippa. Structure of level sets and sard-type properties of lipschitz maps. to appear on Ann. Sc. Norm. Super. Pisa, Cl. Sci., 2011.
- [2] D. Burago, Y. Burago, and S. Ivanov. A course in metric geometry. Graduate studies in mathematics. AMS, 2001.
- [3] E. Durand Cartegna and J. A. Jaramillo. Pointwise lipschitz functions on metric spaces. J. Math. Anal. Appl., 363:525
 –548, 2010.
- [4] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal., 9:428-517, 1999.
- [5] B. Kleiner and J. Mackay. Differentiable structures on metric measure spaces: a primer. preprint arXiv:1108.1324, 2011.

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