HILBERT CURVES OF SCROLLS OVER THREEFOLDS

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Abstract. Let (X, L) be a complex polarized n-fold with the structure of a classical scroll over a smooth projective threefold Y. The Hilbert curve of such a pair (X, L) is a complex affine plane curve of degree n, consisting of $n-3$ evenly spaced parallel lines plus a cubic. This paper is devoted to a detailed study of this cubic. In particular, existence of triple points, behavior with respect to the line at infinity, and non-reducedness, are analyzed in connection with the structure of (X, L). Special attention is reserved to the case $n = 4$, where various examples are presented and the possibility that the cubic is itself the Hilbert curve of the base threefold Y for a suitable polarization is discussed.

Introduction

The Hilbert curve of a polarized manifold (X, L) with $\dim(X) = n \geq 2$ is the complex affine plane curve $\Gamma = \Gamma_{(X,L)}$, of degree n, defined by the Hilbert-like polynomial $\chi(xK_X + yL)$, where K_X is the canonical bundle of X and x and y are regarded as complex variables. This notion was introduced in [4] and extensively studied in [5]–[10], for varieties which are special from the adjunction theoretic point of view. The natural expectation is that several properties of the polarized manifold that one considers are encoded by its Hilbert curve, as suggested by [4, Theorem 6.1]. In particular, if X is endowed with a fibration $\varphi : X \to Y$ over a normal variety Y of dimension $< n$ and $K_X + aL = \varphi^* A$, for some positive integer a and some $\mathbb{Q}$-line bundle A on Y, then Γ contains $a-1$ parallel lines of prescribed equations as components, and therefore it becomes important to understand the properties of the residual curve of the union of such lines in Γ, which is a plane curve of degree $n - a + 1$. Up to now the study of such residual curve has been done in some particular cases, like for scrolls over a curve (where $a = n$) [8], or scrolls over a surface ($a = n - 1$) [9], in these cases the residual curve is a line and a conic, respectively. The other class of varieties considered are quadric fibrations over a surface ($a = n - 2$), where the residual curve is a cubic for which some of the geometric properties can be described in terms of the base surface and of an appropriate vector bundle, [6]. Our interest on scrolls over a 3-fold ($a = n - 2$) derives from the evident analogy with the quadric fibration case, due to the fact that the nef value is the same for both. Hence in this paper, inspired by [6] and [9], we investigate the Hilbert curves of n-dimensional pairs (X, L) with $n \geq 4$, which are scrolls over a smooth threefold Y.

In this setting $\Gamma = \ell_1 + \cdots + \ell_{n-3} + C$, where the ℓ_i's are certain $n-3$ evenly spaced parallel lines and C is the residual cubic; moreover, both Γ and C are Serre-invariant, that is they are invariant under the involution induced on the affine plane containing Γ by Serre duality on X. As a first thing we determine the explicit equation of C in terms of the numerical invariants of (X, L) (Proposition 1.2). If we let $\pi : X \to Y$ denote the scroll projection, then $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} := \pi_* L$ is an ample vector bundle on Y of rank $n-2$, and L is the

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tautological line bundle on X. Our purpose is achieved through computations involving the Chern classes $c_i = c_i(\mathcal{E})$ $(i = 1, 2, 3)$ and appropriate intersection numbers on Y, by taking advantage of the special feature of the equation of Γ, which derives from [4, Theorem 6.1].

As to the first properties of C, we make explicit the conditions for the existence of a triple point as well as that of a singular point at infinity (Proposition 3.1), and putting them together we characterize when C is non-reduced (Corollary 3.2). In Section 2 we focus on the subclass of scrolls for which $K_Y + \det \mathcal{E}$ is not an ample line bundle (namely, scrolls which fail to be adjunction theoretic one's). The precise list of them is given in Proposition 2.1, where for the convenience of the reader we have collected results due to several authors ([2], [7], [20], [13], [19]). In particular, every scroll over a smooth projective threefold is an adjunction theoretic one for $n \geq 7$. We also characterize those pairs in Proposition 2.1 for which $(K_Y + c_1)^2 c_1 = 0$ and $(K_Y + c_1)^3 = 0$ (Lemma 2.2), since such intersection numbers come up in studying the properties of C. The corresponding pairs (X, L) are characterized in Section 3, by the property that the projective closure $\overline{C}$ of the residual cubic C of their Hilbert curve contains the point at infinity of the remaining $n-3$ lines constituting Γ (Corollary 3.4). This follows from a precise analysis of the intersection multiplicity of $\overline{C}$ with the line at infinity at that point in terms of the properties of $K_Y + \det \mathcal{E}$ (Theorem 3.3).

In the case $n = 4$ the equation of C simplifies considerably and therefore one can provide a more detailed specification of the above result. This is done in Section 4, where, in particular, we obtain the precise list of pairs $(Y, \mathcal{E})$ for which $C = 3\ell_0$, where ℓ_0 is the line through the point representing $\frac{1}{2}K_X$ and parallel to the ℓ_i's $(i = 1, \dots, n-3)$ (Corollary 4.2). Moreover, letting T denote the 1-cycle of Y given by $K_Y^2 - 2c_2(Y) + c_1^2 - 4c_2$, we see that the vanishing of T implies that C has a triple point. Looking for special situations in which this condition is satisfied leads to nontrivial examples.

In Section 5, still remaining with $n = 4$, we address for our scrolls (X, L) the problem of when the cubic C itself is the Hilbert curve of the base threefold Y with the average polarization $\frac{1}{2} \det \mathcal{E}$, [4, Problem 6.6]. If this happens, we show that either $c_1(c_1^2 - 4c_2) = K_Y(c_1^2 - 4c_2) = 0$, or $\chi(\mathcal{O}_Y) = 0$ and $K_Y(K_Y + c_1)^2 = 0$ (Proposition 5.2). This fact highlights pairs $(Y, \mathcal{E})$ for which $c_1^2 - 4c_2 = 0$. We emphasize the analogy with the condition of Bogomolov proper semistability of the vector bundle $\mathcal{E}$, which was found in [9] while discussing the same problem for scrolls over surfaces. In some instances we show that the condition $c_1^2 - 4c_2 = 0$ implies that $\mathcal{E} = M^{\oplus 2}$ for some ample line bundle M on Y, hence $X \cong Y \times \mathbb{P}^1$. This happens in particular, when Y is either $\mathbb{P}^3$, the quadric $\mathbb{Q}^3$, a del Pezzo 3-fold of degree $d = 3, 4, 5, 8$, or a prime Fano 3-fold with Picard number one (Proposition 5.4). The paper ends with several examples, including the discussion of the cases in which Y is a del Pezzo 3-fold of degree 6 or 7 and the vector bundle $\mathcal{E}$ is decomposable.

1. Hilbert curve and residual cubic

Let (X, L) be a scroll (in the classical sense) with $\dim(X) = n \geq 4$ over a smooth projective threefold Y, with scroll projection $\pi : X \to Y$. Then $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} := \pi_* L$ is an ample vector bundle of rank $n-2$ on Y, L is the tautological line bundle of $\mathcal{E}$ on X, and $K_X + (n-2)L = \pi^*(K_Y + \det \mathcal{E})$, by the canonical bundle formula. Moreover, if (X, L) is a scroll also in the adjunction theoretic sense, then, according to the definition [3, p. 81], $A := K_Y + \det \mathcal{E}$ is ample.

In the following we denote by c_i the i-th Chern class of $\mathcal{E}$, hence $c_1 = \det \mathcal{E}$.

Let $p(x,y)=0$ be the equation of the Hilbert curve of (X,L). Recall that $p(x,y)=\chi(xK_X+yL)$ is the polynomial expressing the Euler–Poincaré characteristic of xK_X+yL, when x and y are regarded as complex variables. According to [4, Theorem 6.1], we have that

$$p(x,y)=\prod_{i=1}^{n-3}\big((n-2)x-y-i\big)R(x,y), \tag{1}$$

where $R(x,y)$ is a polynomial of degree 3. From the qualitative point of view, this means that the Hilbert curve Γ of (X,L) can be written as

$$\Gamma=\ell_1+\cdots+\ell_{n-3}+C, \tag{2}$$

i.e., it consists of $n-3$ evenly spaced parallel lines with slope $(n-2)$ (the nef value of (X,L)) plus a cubic C, which we call the *residual cubic*.

We call Serre involution the map $s:\mathbb{A}^2\to\mathbb{A}^2$ sending (x,y) to $(1-x,-y)$, induced by Serre duality. Note that Γ is *Serre-invariant*, i.e., invariant under s. Moreover, s exchanges the line ℓ_i of equation $(n-2)x-y-i=0$ with ℓ_{n-2-i} $(i=1,\dots,n-3)$, hence the set consisting of the $n-3$ lines $\ell_1,\dots,\ell_{n-3}$ is globally Serre-invariant. It thus follows that the cubic C itself is also Serre-invariant. We use coordinates (u,v) in place of $(x=\frac{1}{2}+u, y=v)$ in order to make this invariance more evident. Since the degree of C is odd, its equation in u and v does not contain terms of even degree, hence $R(\frac{1}{2}+u,v)$ is the sum of two homogeneous polynomials in u and v of degrees 3 and 1 respectively [4, Claim 7.1]. Thus we can write

$$R\left(\frac{1}{2}+u,v\right)=R_3(u,v)+R_1(u,v). \tag{3}$$

where

$$R_3(u,v)=\alpha u^3+\beta u^2v+\gamma uv^2+\delta v^3 \tag{4}$$

with $(\alpha,\beta,\gamma,\delta)\neq(0,0,0,0)$, because $\deg C=3$, and

$$R_1(u,v)=\sigma u+\tau v. \tag{5}$$

Note that the property of having an equation of this type characterizes any Serre-invariant plane cubic, which is not necessarily the residual cubic of a Hilbert curve.

Our aim it to obtain the explicit expression of $R\left(\frac{1}{2}+u,v\right)$ in our specific case, which in particular describes our cubic C. To do that, first recall that for any divisor D on X,

$$\chi(D)=\frac{1}{n!}D^n+\dots,$$

where the dots stand for lower degree terms. So, by using homogeneous coordinates $(x:y:z)$, where z is the homogenizing coordinate, and letting $p_0(x,y,z)$ denote the homogeneous polynomial associated to p, we have:

$$\begin{aligned} p_0(x,1,0) &= \frac{1}{n!}(xK_X+L)^n \\ &= \frac{1}{n!}\Big[d_nx^n+\binom{n}{1}d_{n-1}x^{n-1}+\binom{n}{2}d_{n-2}x^{n-2}+\dots \\ &\qquad \dots+\binom{n}{n-3}d_3x^3+\binom{n}{n-2}d_2x^2+\binom{n}{n-1}d_1x+d\Big], \end{aligned} \tag{6}$$

where $d_i := K_X^i L^{n-i}$ for $i = 0, 1, \dots, n$ ($d_0 = d$ being the degree of (X, L)). On the other hand, from (1) and (3) we see that $p_0(x, y, 0) = R_3(x, y)\big((n-2)x - y\big)^{n-3}$. Hence (4) gives

$$p_0(x, 1, 0) = (\alpha x^3 + \beta x^2 + \gamma x + \delta)\big((n-2)x - 1\big)^{n-3}. \tag{7}$$

By comparing (6) with (7), easy manipulations lead to the following expressions:

$$\alpha = \frac{1}{n!(n-2)^{(n-3)}}\ d_n, \tag{8}$$

$$\beta = (-1)^{n-1}\frac{1}{2(n-1)!}\left(\frac{(n-2)^3(n-3)}{n}\ d + 2(n-2)(n-3)d_1 + (n-1)d_2\right), \tag{9}$$

$$\gamma = (-1)^{n-1}\frac{1}{n!}\Big((n-2)(n-3)d + nd_1\Big), \tag{10}$$

$$\delta = (-1)^{n-1}\frac{1}{n!}\ d. \tag{11}$$

Lemma 1.1. *We have:*

$$d = c_1^3 - 2c_1c_2 + c_3, \tag{12}$$

$$d_1 = (3-n)c_1^3 + (2n-5)c_1c_2 + (2-n)c_3 + K_Yc_1^2 - K_Yc_2, \tag{13}$$

$$\begin{aligned} d_2 \;&=\; (n-3)^2c_1^3 - 2(n-2)(n-3)c_1c_2 + (n-2)^2c_3 \\ &\quad -2(n-3)K_Yc_1^2 + 2(n-2)K_Yc_2 + K_Y^2c_1, \end{aligned} \tag{14}$$

$$\begin{aligned} d_n \;&=\; (-1)^n(n-2)^{n-2}\Big[\frac{1}{3}(n-3)(n-4)c_1^3 - (n-2)(n-4)c_1c_2 + (n-2)^2c_3 \\ &\quad -\frac{1}{2}n(n-3)K_Yc_1^2 + n(n-2)K_Yc_2 - \frac{1}{6}n(n-1)K_Y^3\Big]. \end{aligned} \tag{15}$$

Proof. Clearly $L^{n-3}\pi^*D_1\pi^*D_2\pi^*D_3 = D_1D_2D_3$ for any divisors D_1, D_2, D_3 on Y; moreover, recalling the Chern–Wu relation

$$L^{n-2} = \pi^*c_1L^{n-3} - \pi^*c_2L^{n-4} + \pi^*c_3L^{n-5}, \tag{16}$$

we see that $L^{n-2}\pi^*D_1\pi^*D_2 = c_1D_1D_2$ and $L^{n-1}\pi^*D_1 = D_1(c_1^2 - c_2)$. Thus, recalling that $d = L^n$ and $K_X = -(n-2)L + \pi^*(K_Y + c_1)$, by an iterated application of (16) we get the above expressions of d, d_1, d_2 and d_n. $\square$

Lemma 1.1 along with the relations (8)–(11) allows us to get the explicit expressions of α, β, γ and δ in terms of the natural invariants of (X, L). Next, recalling that $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and that the higher direct images are zero, we get $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_Y)$ for every i, by the Leray spectral sequence, hence

$$p(0,0) = \chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y). \tag{17}$$

On the other hand, from (1) and (3) we get

$$p(0,0) = \prod_{i=1}^{n-3}(-i)\Big(-\frac{\alpha}{8} - \frac{\sigma}{2}\Big) = (-1)^n\frac{(n-3)!}{8}\ (\alpha + 4\sigma). \tag{18}$$

Thus the equation obtained by comparing (17) with (18) allows us to get the expression of σ. To determine τ, recall that $K_X + (n-2)L = \pi^*A = \pi^*(K_Y + c_1)$. It thus follows that

$\pi_*\big(K_X+(n-2)L\big) = K_Y + c_1$ by projection formula, the higher direct images are zero, hence $h^i(K_X+(n-2)L) = h^i(K_Y+c_1)$ for every i. Therefore

$$p(1,n-2) = \chi\big(K_X+(n-2)L\big) = \chi(K_Y+c_1), \tag{19}$$

and thus it can be computed by using the Riemann–Roch theorem on Y. On the other hand, from (1) and (3) we get

$$\begin{aligned} p(1,n-2) &= \prod_{i=1}^{n-3}\big(n-2-(n-2)-i\big)R(1,n-2) \\ &= (-1)^{n-3}(n-3)!\,\Big(\frac{\alpha}{8}+\frac{\beta}{4}(n-2)+\frac{\gamma}{2}(n-2)^2 \\ &\qquad +\delta(n-2)^3+\frac{\sigma}{2}+\tau(n-2)\Big). \end{aligned} \tag{20}$$

So (19) and (20) give another equation, which, added to the previous ones, allows us to determine τ.

With the help of the Maple package we were able to compute the coefficients of the residual cubic of the Hilbert curve of (X,L).

$$\begin{aligned} \alpha &= \frac{(-1)^{n-1}(n-2)}{6n!}\Big\{n(n-1)K_Y^3+3nK_Y[(n-3)c_1^2-2(n-2)c_2] \\ &\quad -2(n-3)(n-4)c_1^3+6(n-2)(n-4)c_1c_2-6(n-2)^2c_3\Big\} \end{aligned} \tag{21}$$

$$\begin{aligned} \beta &= \frac{(-1)^{n-1}}{2n!}\Big\{n(n-1)K_Y^2c_1-2nK_Y[(n-3)c_1^2-2(n-2)c_2] \\ &\quad +(3n-8)(n-3)c_1^3-8(n-2)(n-3)c_1c_2+6(n-2)^2c_3\Big\} \end{aligned} \tag{22}$$

$$\gamma = \frac{(-1)^{n-1}}{n!}\Big[nK_Y(c_1^2-c_2)-2(n-3)c_1^3+(5n-12)c_1c_2-3(n-2)c_3\Big\} \tag{23}$$

$$\delta = \frac{(-1)^{n-1}}{n!}(c_1^3-2c_1c_2+c_3) \tag{24}$$

$$\begin{aligned} \sigma &= \frac{(-1)^{n}(n-2)}{24n!}\Big\{n(n-1)K_Y^3+nK_Y\Big[3(n-3)c_1^2-6(n-2)c_2-2(n-1)c_2(Y)\Big] \\ &\quad -2(n-3)(n-4)c_1^3+6(n-2)(n-4)c_1c_2-6(n-2)^2c_3\Big\} \end{aligned} \tag{25}$$

$$\begin{aligned} \tau &= \frac{(-1)^{n}}{24n!}\Big[n(n-1)K_Y^2c_1+(5n-8)(n-3)c_1^3-12(n-2)^2c_1c_2 \\ &\quad -2n(n-1)c_1c_2(Y)+6(n-2)^2c_3\Big]. \end{aligned} \tag{26}$$

Proposition 1.2. *Let (X, L) be a scroll over a smooth threefold Y, as in Section 1. Then the residual cubic of its Hilbert curve is defined by (3), where the homogeneous part of degree 3 is*

$$
\begin{aligned}
(27)\quad R_3(u,v) \;=\; & \frac{(-1)^{n-1}(n-2)}{6n!}\Big\{n(n-1)K_Y^3 + 3nK_Y[(n-3)c_1^2 - 2(n-2)c_2] \\
& -2(n-3)(n-4)c_1^3 + 6(n-2)(n-4)c_1c_2 - 6(n-2)^2c_3\Big\}\, u^3 \\
& \frac{(-1)^{n-1}}{2n!}\Big\{n(n-1)K_Y^2c_1 - 2nK_Y[(n-3)c_1^2 - 2(n-2)c_2] \\
& +(3n-8)(n-3)c_1^3 - 8(n-2)(n-3)c_1c_2 + 6(n-2)^2c_3\Big\}\, u^2v \\
& +\frac{(-1)^{n-1}}{n!}\Big[nK_Y(c_1^2 - c_2) - 2(n-3)c_1^3 + (5n-12)c_1c_2 - 3(n-2)c_3\Big]\, uv^2 \\
& +\frac{(-1)^{n-1}}{n!}(c_1^3 - 2c_1c_2 + c_3)\, v^3,
\end{aligned}
$$

while the homogenous part of degree 1 is

$$
\begin{aligned}
(28)\quad R_1(u,v) \;=\; & \frac{(-1)^{n}(n-2)}{24n!}\Big\{n(n-1)K_Y^3 + nK_Y\Big[3(n-3)c_1^2 - 6(n-2)c_2 \\
& -2(n-1)c_2(Y)\Big] - 2(n-3)(n-4)c_1^3 + 6(n-2)(n-4)c_1c_2 \\
& -6(n-2)^2c_3\Big\}\, u + \\
& \frac{(-1)^{n}}{24n!}\Big[n(n-1)K_Y^2c_1 + (5n-8)(n-3)c_1^3 - 12(n-2)^2c_1c_2 \\
& -2n(n-1)c_1c_2(Y) + 6(n-2)^2c_3\Big]\, v.
\end{aligned}
$$

Here are some examples which show various possibilities for the residual cubic C, for different values of n.

Example 1.1. Let $(Y, \mathcal{E}) = (\mathbb{P}^3, T_{\mathbb{P}^3} \oplus \mathcal{N}(2))$, where $T_{\mathbb{P}^3}$ and $\mathcal{N}$ are the tangent bundle and the null correlation bundle, respectively. Let $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$. In this case $K_Y = -4H$ and $Hc_2(Y) = 6$ $c_1 = 8H, c_2 = 27H^2, c_3 = 48H^3 = 48$. Plugging these values and $n = 7$ in (27) and (28) we get that C is defined by

$$-\frac{65}{252}u^3 + \frac{241}{630}u^2v - \frac{221}{1260}uv^2 + \frac{8}{315}v^3 - \frac{19}{1008}u + \frac{1}{126}v = 0.$$

Here C is an irreducible smooth cubic.

Example 1.2. Let $(Y, \mathcal{E}) = (\mathbb{P}^3, \mathcal{N}(2)^{\oplus 2})$, and let $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$. In this case $K_Y = -4H$ and $Hc_2(Y) = 6$ again; moreover, $c_1 = 8H, c_2 = 26H^2, c_3 = 40$. Plugging these values and $n = 6$ in (27) and (28) we get that C is defined by

$$\frac{1}{90}(2u - v)(32u^2 - 56uv + 17v^2 + 7) = 0.$$

Thus C consists of a line and an irreducible conic.

Example 1.3. Let $(Y,\mathcal{E}) = (\mathbb{Q}^3, \mathcal{S}(2)^{\oplus 2})$, where $\mathcal{S}$ is the spinor bundle [14]. Let $H \in |\mathcal{O}_{\mathbb{Q}^3}(1)|$. In this case $K_Y = -3H, Hc_2(Y) = 8, c_1 = 6H, c_2 = 14H^2, c_3 = 27$, and thus, plugging these values and $n = 6$ in (27) and (28) we get that C is defined by

$$\frac{5}{6}u^3 - \frac{33}{20}u^2v + \frac{19}{20}uv^2 - \frac{41}{240}v^3 + \frac{1}{8}u - \frac{11}{240}v = 0.$$

Here C is an irreducible smooth cubic.

Example 1.4. Let $(Y,\mathcal{E}) = (\mathbb{P}^3, T_{\mathbb{P}^3})$, and let $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$. In this case $K_Y = -4H, Hc_2(Y) = 6, c_1 = 4H, c_2 = 6H^2, c_3 = 4$, and thus, plugging these values and $n = 5$ in (27) and (28) one can see that C is defined by

$$-\frac{1}{24}(3u - v)(6u - 2v - 1)(6u - 2v + 1) = 0.$$

Here C consists of three parallel lines.

Example 1.5. Let $(Y,\mathcal{E}) = (\mathbb{P}^3, \mathcal{N}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$, let $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$. We have $K_Y = -4H$ and $Hc_2(Y) = 6$ again; moreover, $c_1 = 5H, c_2 = 9H^2, c_3 = 5$, and thus, plugging these values and $n = 5$ in (27) and (28) we get that C is defined by

$$-\frac{1}{48}(2u - v)(10u - 4v - 1)(10u - 4v + 1) = 0.$$

Here C consists of two parallel lines plus a third line transverse to them.

Example 1.6. Let $(Y,\mathcal{E}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(3))$, let $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$. In this case $c_1 = 6H, c_2 = 11H^2, c_3 = 6$, and thus, plugging these values and $n = 5$ in (27) and (28) we get that C is defined by

$$-\frac{1}{48}(2u - v)(164u^2 - 152uv + 36v^2 - 17) = 0.$$

Here C consists of a line and an irreducible conic.

Example 1.7. Let $(Y,\mathcal{E}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(3))$, let $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$. In this case $c_1 = 5H, c_2 = 7H^2, c_3 = 3$, and thus, plugging these values and $n = 5$ in (27) and (28) we get that C is defined by

$$-\frac{112}{15}u^3 + \frac{269}{30}u^2v - \frac{18}{5}uv^2 + \frac{29}{60}v^3 + \frac{13}{15}u - \frac{43}{120}v = 0.$$

Here C is a smooth irreducible cubic.

For an example with C irreducible and its projective closure singular at a point at infinity, see Example 5.2.

2. Classical and adjunction theoretic scrolls

Let (X, L) be as in Section 1. It is useful to recall that (X, L) is also a scroll in the adjunction theoretic sense unless $K_Y + \det \mathcal{E}$ is not ample. By combining [2, Corollary 2.5 and Theorem 3.1], with [7], [20, Theorem 3], and [19] (see also [15, Theorem 0.4], [13, Theorem 1.3], and [18, Proposition 3.1]), we can state the following result.

Proposition 2.1. *Let (X, L) be a classical scroll over a smooth threefold Y, with $\dim(X) = n \geq 4$ and let $\mathcal{E}$ be the ample vector bundle on Y of rank $n - 2$ defined by $\pi_* L$, where $\pi : X \to Y$ is the scroll projection. Then $K_Y + \det \mathcal{E}$ fails to be ample exactly in the following cases:*

A) *$n = 6$ and $(Y, \mathcal{E}) = \left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}\right)$;*
B) *$n = 5$ and $(Y, \mathcal{E})$ is one of the following pairs:*
(B1) *$Y = \mathbb{P}^3$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3}$, $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$, or the tangent bundle $T_{\mathbb{P}^3}$;*
(B2) *$\left(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 3}\right)$;*
(B3) *Y is a $\mathbb{P}^2$-bundle over a smooth curve B and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$ for every fiber $F = \mathbb{P}^2$ of the bundle projection $Y \to B$;*
C) *$n = 4$ and $(Y, \mathcal{E})$ is one of the following pairs:*
(C1) *$Y = \mathbb{P}^3$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$, $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$, $\mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$, or $\mathcal{N}(2)$, where $\mathcal{N}$ is a null correlation bundle;*
(C2) *$Y = \mathbb{Q}^3$ and $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 2}$, $\mathcal{O}_{\mathbb{Q}^3}(2) \oplus \mathcal{O}_{\mathbb{Q}^3}(1)$, or $\mathcal{S}(2)$, where $\mathcal{S}$ is the spinor bundle;*
(C3) *(Y, H) is a del Pezzo threefold and $\mathcal{E} = H^{\oplus 2}$ (this includes $\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2}\right)$;*
(C4) *Y is a quadric fibration over a smooth curve B and $\mathcal{E}_F = \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2}$ for the general fiber $F = \mathbb{Q}^2$ of the fibration $Y \to B$;*
(C5) *Y is a $\mathbb{P}^2$-bundle over a smooth curve B and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, or the tangent bundle $T_{\mathbb{P}^2}$ for every fiber $F = \mathbb{P}^2$ of the bundle projection $Y \to B$ (the former case includes the possibility that $Y = \mathbb{P}^2 \times \mathbb{P}^1$ with $\mathcal{E} = \mathcal{O}(2,1) \oplus \mathcal{O}(1,1)$ or $p_1^* T_{\mathbb{P}^2} \otimes \mathcal{O}(0,1)$, where p_1 stands for the first projection);*
(C6) *Y is a $\mathbb{P}^1$-fibration over a smooth surface S and $\mathcal{E}_f = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ for the general fiber $f = \mathbb{P}^1$ of the fibration $Y \to S$;*
(C7) *there exist a birational morphism $\eta : Y \to Y'$ expressing Y as a smooth threefold Y' blown-up at a finite set and an ample vector bundle $\mathcal{E}'$ of rank 2 on Y' such that $\mathcal{E} = \eta^* \mathcal{E}' \otimes \mathcal{O}_Y(-E)$, where E is the exceptional divisor of η; moreover, either $K_{Y'} + \det \mathcal{E}'$ is ample, or $(Y', \mathcal{E}') = \left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2}\right)$.*

In particular, $K_Y + \det \mathcal{E}$ is always ample if $n \geq 7$.

We recall that the two pairs $\left(\mathbb{P}^3, \mathcal{N}(2)\right)$ and $\left(\mathbb{Q}^3, \mathcal{S}(2)\right)$ define the same scroll (X, L) (with respect to two distinct projections) [17, Proposition 2.6 and Proposition 3.4]. The intersection properties of the adjoint bundle $K_Y + \det \mathcal{E}$ will be relevant in Theorem 3.3, especially when it is not ample. So let us look in some detail at the exceptional pairs $(Y, \mathcal{E})$ listed in Proposition 2.1.

Remark 2.1. i) In case A), $K_Y + \det \mathcal{E}$ is trivial.

ii) For $(Y, \mathcal{E})$ as in B), it follows from [20, Theorem 2] that $K_Y + \det \mathcal{E}$ is nef except in the case of (B1) with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3}$ (for which $K_Y + \det \mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(-1)$): moreover it is trivial in the other cases of (B1) and in (B2).

iii) Finally, for pairs $(Y, \mathcal{E})$ as in C), $K_Y + \det \mathcal{E}$ is nef except in the following situations.

E_{-1}) $K_Y + \det \mathcal{E}$ is not nef in cases: (C1) with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ and $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$, (C2) with $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 2}$, and (C5) with $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$ (see [20, Theorem 3]); in all these cases, except the last one, $K_Y + \det \mathcal{E}$ is the opposite of an ample line bundle, hence $(K_Y + c_1)^3 < 0$. The remaining case is settled by Lemma 2.2 below.

E_0) $K_Y + \det \mathcal{E} = \mathcal{O}_Y$ in cases: (C1) with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ and $\mathcal{N}(2)$, (C2) with $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^3}(2) \oplus \mathcal{O}_{\mathbb{Q}^3}(1)$, or $\mathcal{S}(2)$, (C3), and (C5) with $(Y, \mathcal{E})$ being one of the two possibilities mentioned in the brackets (see [19]);

E_1) the morphism defined by a multiple of $K_Y + \det \mathcal{E}$ has a 1-dimensional image, hence $(K_Y + c_1)^2 = 0$ in $H^4(Y)$, in case (C4) and in case (C5) with $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, except the subcases mentioned in E_0;

E_2) the morphism defined by a multiple of $K_Y + \det \mathcal{E}$ has a 2-dimensional image, hence $(K_Y + c_1)^3 = 0$, in case (C6).

Lemma 2.2. *Let* $(Y, \mathcal{E})$ *be as in case* (C5) *of Proposition 2.1, with* $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ *for any fiber* F *of* Y*. Then*

(i) *the equality* $(K_Y + c_1)^2 c_1 = 0$ *occurs if and only if* $Y = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$*, i.e.* Y *is isomorphic to* $\mathbb{P}^3$ *blown-up along a line, and* $\mathcal{E} = [\zeta + F]^{\oplus 2}$*, where* ζ *is the tautological line bundle on* Y*;*

(ii) *we have* $(K_Y + c_1)^3 = 0$ *if and only if* $(Y, \mathcal{E}) = \big(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(1,1)^{\oplus 2}\big)$.

Proof. Write $Y = \mathbb{P}(\mathcal{V})$, where $\mathcal{V}$ is an ample vector bundle of rank 3 on the base curve B. Let ξ denote the tautological line bundle of $\mathcal{V}$ and $q : Y \to B$ the bundle projection; then $\mathcal{E} = \xi \otimes q^*\mathcal{G}$, where $\mathcal{G}$ is a rank-2 vector bundle on B. So $c_1 = \det \mathcal{E} = 2\xi + q^* \det \mathcal{G}$ and $c_2 = c_2(\mathcal{E}) = \xi^2 + \xi q^* \det \mathcal{G}$. Moreover, $K_Y = -3\xi + q^*(K_B + \det \mathcal{V})$. Then $K_Y + c_1 = -\xi + q^*(K_B + \det \mathcal{V} + \det \mathcal{G}) \equiv -\xi + (2g - 2 + \deg \mathcal{V} + \deg \mathcal{G})F$, where g is the genus of B. Recall that $\xi^3 = \xi^2 q^* \det \mathcal{V} = \deg \mathcal{V}$, due to the Chern–Wu relation. Recalling (12) we get

$$
\begin{aligned}
(29) \qquad 0 < d = c_1^3 - 2c_1c_2 &= (2\xi + \deg \mathcal{G}\ F)^3 - 2(2\xi + q^* \det \mathcal{G})(\xi^2 + \xi q^* \det \mathcal{G}) \\
&= 2(2 \deg \mathcal{V} + 3 \deg \mathcal{G}).
\end{aligned}
$$

This shows that $2 \deg \mathcal{V} + 3 \deg \mathcal{G} \geq 1$. Now, suppose that $(K_Y + c_1)^2 c_1 = 0$. Then we get

$$
(30) \qquad 0 = (K_Y + c_1)^2 c_1 = -4(2g - 2) - (2 \deg \mathcal{V} + 3 \deg \mathcal{G}).
$$

Recalling (29) this implies that $0 \leq 7 - 8g$, hence $g = 0$, i.e. $B = \mathbb{P}^1$. Rewrite Y as $Y = \mathbb{P}(\mathcal{U})$ where $\mathcal{U}$ is normalized as in [3, Lemma 3.2.4, p. 74]. So, $\mathcal{U} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$, where $0 \leq a_1 \leq a_2$. In particular, $\deg \mathcal{U} = a_1 + a_2 \geq 0$. Denote by ζ the tautological line bundle of $\mathcal{U}$. Then $\zeta^3 = \deg \mathcal{U} \geq 0$, by Chern–Wu. Moreover, $\mathcal{E} = \zeta \otimes q^*\mathcal{F}$, where $\mathcal{F}$ is a vector bundle of rank 2 on $\mathbb{P}^1$, hence of the form $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)$. Thus $\mathcal{E} = [\zeta + b_1 F] \oplus [\zeta + b_2 F]$, and the ampleness of $\mathcal{E}$ implies that of the two summands $[\zeta + b_i\ F]$, which is expressed by the condition $b_i > 0$ for $i = 1, 2$, since $\mathcal{U}$ is normalized. Thus $\deg \mathcal{F} = b_1 + b_2 \geq 2$. In this setting $c_1 = 2\zeta + \deg \mathcal{F}\ F$ and $K_Y = -3\zeta + (\deg \mathcal{U} - 2)\ F$, hence $K_Y + c_1 = -\zeta + (\deg \mathcal{U} + \deg \mathcal{F} - 2)\ F$. So (30) becomes

$$
(31) \qquad 0 = (K_Y + c_1)^2 c_1 = 8 - 2 \deg \mathcal{U} - 3 \deg \mathcal{F} \leq 2(1 - \deg \mathcal{U}).
$$

Thus $\deg \mathcal{U} \leq 1$; on the other hand, if $\deg \mathcal{U} = 0$ then (31) gives $3 \deg \mathcal{F} = 8$, but this is clearly impossible. Therefore $\deg \mathcal{U} = 1$, hence $\mathcal{U} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and $\deg \mathcal{F} = 2$, which in turn implies that $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. In particular, $Y = \mathbb{P}(\mathcal{U})$ is isomorphic to $\mathbb{P}^3$ blown-up along a line. This proves (i). To prove (ii), come back to the description of $(Y, \mathcal{E})$ in terms of $\mathcal{V}$ and $\mathcal{G}$ and suppose that

$$
(32) \qquad 0 = (K_Y + c_1)^3 = 3(2g - 2) + 2 \deg \mathcal{V} + 3 \deg \mathcal{G}.
$$

The same argument used in the proof of (i) shows that $B = \mathbb{P}^1$. So we can use the description of $(Y, \mathcal{E})$ in terms of the vector bundles $\mathcal{U}$ and $\mathcal{F}$ once more. In these terms the expression in (32) can be rewritten in the form

$$
(33) \qquad (K_Y + c_1)^3 = (-\zeta)^3 + 3(\deg \mathcal{U} + \deg \mathcal{F} - 2)\zeta\ F = 2 \deg \mathcal{U} + 3 \deg \mathcal{F} - 6.
$$

Recalling that $\deg \mathcal{F} \geq 2$, we thus see that $2 \deg \mathcal{U} \leq 0$, hence $\deg \mathcal{U} = 0$ and thus $\mathcal{U}$ is the trivial bundle and $\mathcal{E} = [\zeta + F]^{\oplus 2}$. This means exactly that $Y = \mathbb{P}^2 \times \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}(1,1)^{\oplus 2}$. $\square$

By the way, note that for the pair $(Y, \mathcal{E})$ in (i) we have $(K_Y + c_1)^3 = 2$.

3. First properties of the residual cubic

In this Section we discuss some properties of the residual cubic C of the Hilbert curve of a scroll over a threefold Y as in Section 1 and of its projective closure $\overline{C}$. As we will see, the behavior of C with respect to the line at infinity of the (u,v)-plane is strictly related to the properties of the adjoint bundle $K_Y + \det \mathcal{E}$.

As a first thing, note that when C is irreducible, then it cannot have a singular point in $\mathbb{A}^2$. In fact, if it did, due to the symmetry of C, the origin itself would be a singular point and then, necessarily, a triple point, in view of the equation (3). But then C would be reducible, a contradiction. On the other hand, it can happen that the projective closure of an irreducible C is singular at a point at infinity, as Example 5.2 shows. We have the following basic result.

Proposition 3.1. *Let (X,L) be a scroll over a smooth threefold Y as in Section 1, and let C be the residual cubic of its Hilbert curve. Then*

(i) *C has a triple point if and only if $\sigma = \tau = 0$,*

(ii) *$\overline{C}$ has a singular point at infinity if and only if*

$$27\alpha^2\delta^2 - 18\alpha\beta\gamma\delta + 4\alpha\gamma^3 + 4\beta^3\delta - \beta^2\gamma^2 = 0. \tag{34}$$

In this case the singular point at infinity is $Q_\infty = (1 : m : 0)$, where either $m = 0$ or

$$m = \frac{9\alpha\delta - \beta\gamma}{2(\gamma^2 - 3\beta\delta)}\ . \tag{35}$$

Proof. To prove (i) note that C has a triple point, if and only the origin O of the (u,v)-plane is a triple point of C, and this happens if and only if $\sigma = \tau = 0$. To prove (ii), consider the homogeneous equation of $\overline{C}$:

$$\alpha u^3 + \beta u^2 v + \gamma u v^2 + \delta v^3 + \sigma u w^2 + \tau v w^2 = 0,$$

where $w = 0$ represents the line at infinity. Recall that $\delta \neq 0$, since δ is the degree of (X,L) up to a non zero factor, by (11). Therefore $\overline{C}$ cannot be singular at $(0:1:0)$. So if it is singular at some point at infinity, this point must be $Q_\infty = (1 : m : 0)$, for some $m \in \mathbb{C}$. In particular, $m = 0$ if and only if $\alpha = \beta = 0$. Suppose that this is not the case. Looking for the singular points via the Jacobian criterion and imposing that one of them is at $Q_\infty = (1 : m : 0)$ we obtain

$$\left\{ \begin{array}{rcl} \gamma\, m^2 + 2\beta\, m + 3\alpha & = & 0 \\ 3\delta\, m^2 + 2\gamma\, m + \beta & = & 0. \end{array} \right. \tag{36}$$

Thus the assertion follows from the fact that the resultant of the two polynomials in (36) is the expression appearing in (34), up to a constant factor. Finally, by equating the expressions of m^2 deriving from the two above equations we immediately get (35). $\square$

Clearly the two conditions in (i) and (ii) of Proposition 3.1 are independent. For example, for the curve C in Example 5.2 condition (34) is satisfied and $m = 0$, whereas $(\sigma, \tau) = (-2, 2 + \frac{d}{2})$. However, by combining the two conditions together we obtain the following characterization.

Corollary 3.2. *Let (X,L) be a scroll over a smooth threefold Y as in Section 1, and let C be the residual cubic of its Hilbert curve. Then C is non-reduced if and only if $\sigma = \tau = 0$ and (34) is satisfied.*

Proof. Claiming that C is non-reduced is equivalent to writing that $C = 2\ell_1 + \ell_2$, where both ℓ_1 and ℓ_2 are lines (possibly coinciding) passing through O. This, in turn, means that O is a triple point of C and that the point at infinity of ℓ_1 is a singular point of $\overline{C}$. Then the assertion follows from Proposition 3.1 $\square$

Remark 3.1. (i) Let $C = 2\ell_1 + \ell_2$ be as above. As already observed in the proof of Proposition 3.1, $\delta \neq 0$, hence we can describe both ℓ_1 and ℓ_2 by equations of the form $mu - v = 0$ and $m'u - v = 0$, for some $m, m' \in \mathbb{C}$, respectively. Then letting $t = v/u$ we see that $R_3(u, v) = u^3\ f(t)$, where $f(t) = \delta t^3 + \gamma t^2 + \beta t + \alpha$. By imposing that $f(t) = \delta(t - m)^2(t - m')$, we thus get

$$\alpha = -\delta m^2 m', \quad \beta = \delta m(m + 2m'), \quad \gamma = -\delta(2m + m').$$

By the way, note that eliminating m and m' from these equations we directly obtain the expression in (34).

(ii) Clearly, if $C = 3\ell_1$, the above equations specialize to

$$\alpha = -\delta m^3, \quad \beta = 3\delta m^2, \quad \gamma = -3\delta m,$$

which are equivalent to $\gamma^2 - 3\beta\delta = \beta^2 - 3\alpha\gamma = 0$.

Now, let ℓ_∞ be the line at infinity of the (u, v)-plane. We denote by ℓ_0 the line of equation $(n-2)u - v = 0$, whose point at infinity is $P_\infty := (1 : n-2 : 0)$. Observe that $\ell_0 =< O, P_\infty >$, where O is the origin.

Theorem 3.3. *Let (X, L) be a scroll over a smooth threefold Y, as in Section 1, and let C be the residual cubic of its Hilbert curve. Then $\overline{C}$ intersects ℓ_∞ at P_∞ with multiplicity ν, where,*

j) *$\nu \geq 1$ if and only if $(K_Y + c_1)^3 = 0$;*
jj) *$\nu \geq 2$ if and only if, in addition, $c_1(K_Y + c_1)^2 = 0$ (in view of* j) *this is equivalent to the two single conditions $(K_Y + c_1)^2 c_1 = (K_Y + c_1)^2 K_Y = 0$);*
jjj) *$\nu = 3$ if and only if, in addition to the above, $(K_Y + c_1)(c_1^2 - c_2) = 0$ (in view of* jj) *this is equivalent to the two conditions $(K_Y + c_1)(c_1^2 - c_2) = (K_Y + c_1)(c_1^2 - K_Y^2) = 0$, the latter summarizing* jj)*).*

Proof. From (11) and (12) we know that $\delta \neq 0$. So, letting $t = v/u$ we can consider again the degree 3 polynomial $f(t) = \delta t^3 + \gamma t^2 + \beta t + \alpha$ and then ν turns out to be the multiplicity of $n - 2$ as a root of f. In particular, $\overline{C} \cap \ell_\infty \ni P_\infty$ if and only if $n - 2$ is a root of f. Plugging the values of $\alpha, \beta, \gamma, \delta$ given by (21), (22), (23), and (24) respectively in $f(n - 2)$ we get

$$f(n-2) = \alpha + (n-2)\beta + (n-2)^2\gamma + (n-2)^3\delta = \frac{(-1)^{n-1}(K_Y + c_1)^3}{6(n-3)!}.$$

This proves j). We have $\nu \geq 2$ if and only if $n - 2$ is a root of f and of its first derivative $f'(t) = 3\delta t^2 + 2\gamma t + \beta$. Taking into account (24), (23) and (22) we see that

$$f'(n-2) = 3(n-2)^2\delta + 2(n-2)\gamma + \beta = \frac{(-1)^{n-1}}{2(n-2)!} c_1(K_Y + c_1)^2. \tag{37}$$

This proves jj). Finally, condition $\nu = 3$ is equivalent to imposing that $n-2$ is a triple root of f. To do that, in addition to the previous conditions of being a root of f and of f', we have to require that $n-2$ is also a root of the second derivative $f''(t) = 6\delta t + 2\gamma$. A computation taking into account (24) and (23) gives

$$2\gamma + 6\delta(n-2) = \frac{2(-1)^{n-1}}{(n-1)!}(K_Y + c_1)(c_1^2 - c_2), \tag{38}$$

and this proves jjj). $\square$

Clearly, we have the following consequence.

Corollary 3.4. *Let (X, L) and C be as in Theorem 3.3. If (X, L) is also an adjunction theoretic scroll, then $\overline{C}$ does not contain P_∞. In particular, this happens for every scroll, if $n \geq 7$.*

More generally, the same is true if $(Y, \mathcal{E})$ is as in (C7) of Proposition 2.1, in which case $K_Y + \det \mathcal{E}$ is nef and big. Now consider the line ℓ_0. Since C is defined by $R_3(u, v) + R_1(u, v) = 0$, with R_3 and R_1 given by Proposition 1.2, imposing that C contains ℓ_0 is equivalent to requiring that $R_3(u, (n-2)u) = R_1(u, (n-2)u) = 0$, identically. The first of these two conditions is equivalent to $f(n-2) = 0$, hence to $(K_Y + c_1)^3 = 0$, as shown in the proof of Theorem 3.3. On the other hand, the second condition, namely $\sigma + (n-2)\tau = 0$, becomes

$$\frac{(-1)^n(n-2)(K_Y + c_1)\big[3(n-3)c_1^2 + (n-1)K_Y^2 - 6(n-2)c_2 - 2(n-1)c_2(Y)\big]}{24(n-1)!} = 0,$$

after replacing σ and τ with the values appearing in (28). In particular, this shows that if $K_Y + c_1 \equiv 0$, then C contains ℓ_0. In fact, a slightly weaker condition is enough.

Corollary 3.5. *Let (X, L) and C be as in Theorem 3.3. If $(K_Y + c_1)^3 = (K_Y + c_1)^2 c_1 = (K_Y + c_1)(c_1^2 - c_2) = 0$, then C contains ℓ_0.*

Proof. The three conditions above are equivalent to $\nu = 3$ by Theorem 3.3. This implies that C consists of three parallel lines with slope $n-2$ and then, due to the symmetry of C, necessarily one of them has to be ℓ_0. $\square$

We finally note that if $\sigma = \tau = 0$ in addition to the three conditions above, then O is a triple point of C, as well as P_∞. Therefore $C = 3\ell_0$, a situation fitting with Remark 3.1 (ii).

4. The case $n = 4$

In this Section we specialize the situation to the case $n = 4$. So, let (X, L) be a 4-dimensional scroll over a smooth threefold Y, as in Section 1, and let C be the residual cubic of its Hilbert curve. First of all, from Proposition 1.2, letting $n = 4$, we see that the polynomial $R_3(u,v)+R_1(u,v)$ defining C becomes

$$
\begin{aligned}
(39)\qquad R_3(u,v)+R_1(u,v) \;=\; & -\frac{1}{6}\big(K_Y^3+K_Y(c_1^2-4c_2)\big)u^3 \\
& -\frac{1}{12}\big(c_1(c_1^2-4c_2)-2K_Y(c_1^2-4c_2)+3K_Y^2c_1\big)u^2v \\
& +\frac{1}{48}\big(4c_1(c_1^2-4c_2)-8K_Y(c_1^2-c_2)\big)uv^2 \\
& -\frac{1}{24}\; c_1(c_1^2-2c_2)v^3 \\
& +\frac{1}{48}\; 2K_Y\big(K_Y^2-2c_2(Y)+c_1^2-4c_2\big)u \\
& +\frac{1}{48}\; c_1\big(K_Y^2-2c_2(Y)+c_1^2-4c_2\big)v.
\end{aligned}
$$

Here is a non trivial example.

Example 4.1. Let Y be either an abelian 3-fold or a Calabi-Yau 3-fold of Type A, according to [12] (we recall that such varieties arise as quotients of an abelian threefold by a finite group acting freely in codimension one, e.g. see [16]). Then $K_Y=0$ and $c_2(Y)=0$ in both cases. Thus, letting $h:=c_1(c_1^2-4c_2)$, according to the above expression, the equation of C takes the form

$$(40)\qquad -\frac{1}{48}\big(4hu^2-4huv+2(h+2c_1c_2)v^2-h\big)v=0.$$

Therefore C is reducible, consisting of the u-axis plus a conic G whose matrix is

$$(41)\qquad A=\begin{bmatrix} 4h & -2h & 0 \\ -2h & 2(h+2c_1c_2) & 0 \\ 0 & 0 & -h \end{bmatrix}.$$

In particular, we see that $\det A=-4h^2c_1^3$, hence G is irreducible if and only if $h\neq 0$.

Again with regard to the equation of C, the expression of the last two coefficients allows us to explore the condition $\sigma=\tau=0$ (or equivalently that C has a triple point at O) at least in the special case $n=4$. Examples 5.2–5.5 fit in this case. Actually, if we consider the 1-cycle $T:=K_Y^2-2c_2(Y)+c_1^2-4c_2$, then we can write

$$(42)\qquad \sigma=2K_YT \qquad \text{and} \qquad \tau=c_1T.$$

In particular this shows that if $T=0$ then C has a triple point. Certainly this happens when each of the two 1-cycles $K_Y^2-2c_2(Y)$ and $c_1^2-4c_2$, summands of T, is trivial (e.g. the first of these two conditions, namely $K_Y^2-2c_2(Y)=0$, holds if $K_Y\equiv 0$ and $c_2(Y)=0$, for example if Y is an abelian 3-fold or a Calabi-Yau 3-fold of Type A, as observed in Example 4.1). On the other hand, if $(\sigma,\tau)\neq(0,0)$, then O is a smooth point of C, and then (42) allows us to express the equation of the tangent line to C at O in the form: $(2K_YT)u+(c_1T)v=0$.

Now we revisit the three assertions in Theorem 3.3 in connection with Proposition 2.1 C). By Corollary 3.4, condition $\nu\geq 1$ implies that (X,L) cannot be an adjunction theoretic scroll. In fact we can add that it cannot even be as in the first two cases of (C1), in the first case of (C2), and as in (C7). As to the first case in (C5) Lemma 2.2 (ii) says that $(K_Y+c_1)^3=0$ occurs only for the pair $(\mathbb{P}^2\times\mathbb{P}^1,\mathcal{O}(1,1)^{\oplus 2})$.

For instance we have

Example 4.2. Let $(Y, \mathcal{E}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2})$. In this case (40) easily shows that C is defined by

$$\frac{1}{6}(4u-v)(4u-v-1)(4u-v+1) = 0.$$

Here $\overline{C}$ does not contain P_∞, since $(K_Y + c_1)^3 \neq 0$. However, it intersects ℓ_∞ at Q_∞ with multiplicity 3, where $Q_\infty = (1 : 4 : 0)$. Moreover, the Hilbert curve of the corresponding scroll (X, L) has equation

$$p_{(X,L)}\left(\frac{1}{2}+u, v\right) = \frac{1}{6}(2u-v)(4u-v)(4u-v-1)(4u-v+1) = 0.$$

This is in accordance with [9, Theorem 4.1], since $(X, L) = (\mathbb{P}^3 \times \mathbb{P}^1, \mathcal{O}(1,1))$ can also be regarded as a scroll over $\mathbb{P}^1$.

As to condition $\nu = 2$, note that pairs $(Y, \mathcal{E})$ as in case (C5) of Proposition 2.1, obviously satisfy the equality $(K_Y + c_1)^2 c_1 = 0$, since $(K_Y + c_1)^2$ is the trivial 1-cycle. In case (C6) we have that $(K_Y + c_1)^2 c_1 \neq 0$. Moreover, this is the only possibility when $K_Y + c_1$ is nef. On the other hand, when $K_Y + c_1$ is not nef, Lemma 2.2 (i) shows that the equality $(K_Y + c_1)^2 c_1 = 0$ occurs only in a special subcase of the first case of (C5), in which, however, $(K_Y + c_1)^3 \neq 0$. So we have

Corollary 4.1. *Let (X, L) be a scroll over a smooth threefold Y, as in Section 1, with $n = 4$, and let C be the residual cubic of its Hilbert curve. Then*

1) *$\overline{C}$ intersects ℓ_∞ at P_∞ transversally if and only if $(K_Y + c_1)^3 = 0$ and $(K_Y + c_1)^2 c_1 \neq 0$. This happens exactly in the following cases:*
   1-1) *$K_Y + \det \mathcal{E}$ is nef and $(Y, \mathcal{E})$ is as in* (C6) *of Proposition 2.1;*
   1-2) *$K_Y + \det \mathcal{E}$ is not nef and $(Y, \mathcal{E})$ is as in Lemma 2.2,* (ii).
2) *Suppose that (X, L) is not an adjunction theoretic scroll. Then $\overline{C}$ intersects ℓ_∞ at P_∞ with multiplicity $\nu = 2$ if and only if $(K_Y + c_1)^3 = (K_Y + c_1)^2 c_1 = 0$ and $(K_Y + c_1)(c_1^2 - c_2) \neq 0$. This happens exactly when $K_Y + \det \mathcal{E}$ is nef and $(Y, \mathcal{E})$ is as in* (C5) *of Proposition 2.1.*

Finally, condition $\nu = 3$ is certainly satisfied when $K_Y + c_1 \equiv 0$, which means that (X, L) is a "Fano-bundle" of index 2 (recalling that $\mathrm{rk}(\mathcal{E}) = n - 2 = 2$). In particular Y is a Fano threefold, and $c_1 = -K_Y$. In this case the equation of C can be made explicit since the coefficients displayed in (39) simplify considerably. Set $Z := K_Y^2 - 2c_2$. Recalling that the pairs $(Y, \mathcal{E})$ of this type are listed in E_0 of Remark 2.1 iii), a direct check shows that $K_Y Z \neq 0$ for all pairs in the list. Recalling also that $K_Y c_2(Y) = -24$, since Y is Fano, a straightforward check shows that the equation $R_3(u, v) + R_1(u, v) = 0$ becomes

$$-\frac{1}{24}\, K_Y Z\, (2u-v)\left((2u-v)^2 - \left(1 + \frac{24}{K_Y Z}\right)\right) = 0. \tag{43}$$

In particular, we see that $C = 3\ell_0$ if and only if $K_Y Z = -24$ and a quick inspection shows when this is the case. So we have

Corollary 4.2. *Suppose that $K_Y + c_1 \equiv 0$. Then $C = 3\ell_0$ if and only if $K_Y^3 = 2(K_Y c_2 - 12)$; this occurs exactly for the following pairs $(Y, \mathcal{E})$: $\big(\mathbb{P}^3, \mathcal{N}(2)\big)$, $\big(\mathbb{Q}^3, \mathcal{S}(2)\big)$, $(Y, H^{\oplus 2})$, where (Y, H) is any of the two del Pezzo threefolds of degree 6, and $\big(\mathbb{P}^2 \times \mathbb{P}^1, p_1^* T_{\mathbb{P}^2} \otimes \mathcal{O}(0,1)\big)$.*

5. The case $n = 4$: C as a Hilbert curve

Let (X, L) be a 4-dimensional scroll over a smooth threefold Y, as in Section 1 and let C be the residual cubic of its Hilbert curve. Referring to [4, Problem 6.6], and stimulated by what [8, Remark 4.1] and [9, Theorem 4.1] suggest, we ask under what conditions the cubic C will be the Hilbert curve of Y polarized by the ample $\mathbb{Q}$-line bundle $\frac{1}{2}\det\mathcal{E}$, induced by $\mathcal{E}$, which we call "average" polarization, since $\mathrm{rk}(\mathcal{E}) = n-2 = 2$. To motivate this question we recall that if $(V, \mathcal{M})$ and $(W, \mathcal{N})$ are two polarized manifolds, then $p_{(V\times W, \mathcal{M}\boxtimes\mathcal{N})} = p_{(V,\mathcal{M})}\, p_{(W,\mathcal{N})}$ [4, 2.5]. In particular, if $n = 4$ and $(X \cong Y \times \mathbb{P}^1, L)$ is the product scroll, namely $(\mathbb{P}_Y(\mathcal{E}), M \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$, where $\mathcal{E} = M^{\oplus 2}$ for some ample line bundle M on Y, then

$$p_{(X,L)} = -p_{(Y,M)}\ (2u - v), \tag{44}$$

hence the residual cubic of the Hilbert curve of (X, L) is itself the Hilbert curve of Y polarized by the average polarization $M = (1/2)\det\mathcal{E}$. First of all, as a consequence of this fact we can produce explicit examples of scrolls (X, L) as in Section 1, with $n = 4$, for which the residual cubic C is either irreducible and smooth or irreducible with projective closure having a singular point.

Example 5.1. Choose as Y a general element in $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(2, 3)|$, and let $M = \big(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1, 1)\big)_Y$. According to [4, Example 4.11], the projective closure of $\Gamma_{(Y,M)}$ is a smooth cubic. Then $X := \mathbb{P}(M^{\oplus 2})$, polarized by the tautological line bundle, is a scroll over Y and, according to (44), its Hilbert curve consists of $\Gamma_{(Y,M)}$ plus the line $2u - v = 0$. Thus $C = \Gamma_{(Y,M)}$ is a smooth cubic. The same conclusion holds, if we take as (Y, M) a 3-dimensional geometric conic fibration in $\mathbb{P}^6$ as in [5, Remark 6.4].

Example 5.2. Let (Y, M) be a geometric conic fibration over a del Pezzo surface of degree $d \geq 3$, constructed as in [5, p. 559], and consider $X := \mathbb{P}(M^{\oplus 2}) \cong Y \times \mathbb{P}^1$. Here the tautological line bundle is $L = M \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. Then, according to (44), the residual cubic C of the Hilbert curve of the 4 dimensional scroll (X, L) is the Hilbert curve $\Gamma_{(Y,M)}$, hence C is defined by

$$dv^2(v - u) - 2u + \left(2 + \frac{d}{2}\right) v = 0$$

[5, Example 6.3]. Therefore C is irreducible and its projective closure is singular at the point at infinity of the u axis.

Now consider the base Y of our scroll (X, L), endowed with the average polarization. We compute $p_{(Y,\frac{1}{2}c_1)}(\frac{1}{2} + u, v)$ using Riemann–Roch theorem and we get that

$$\begin{aligned} p_{(Y,\frac{1}{2}c_1)}\left(\frac{1}{2} + u, v\right) &= \frac{K_Y^3}{6}\ u^3 + \frac{K_Y^2 c_1}{4}\ u^2 v + \frac{K_Y c_1^2}{8}\ uv^2 + \frac{c_1^3}{48}\ v^3 \\ &\quad + \frac{2K_Y c_2(Y) - K_Y^3}{24}\ u + \frac{-K_Y^2 c_1 + 2c_1 c_2(Y)}{48}\ v. \end{aligned} \tag{45}$$

In particular, if $K_Y + c_1 \equiv 0$, then $c_1 = -K_Y$ and $K_Y c_2(Y) = -24$ since Y is Fano. Then (45) takes the form

$$p_{(Y,\frac{1}{2}c_1)}\left(\frac{1}{2} + u, v\right) = \frac{1}{48} K_Y^3 (2u - v)\left((2u - v)^2 - \left(1 + \frac{48}{K_Y^3}\right)\right) = 0. \tag{46}$$

This says that the Hilbert curve of (X, L) consists of three parallel lines one of which is ℓ_0, and they collapse to ℓ_0 exactly in case $K_Y^3 = -48$. Moreover, comparing (46) with the equation of C given by (43), we see that $\Gamma_{(Y,\frac{1}{2}c_1)} = C$ if and only if the two equations are proportional, hence if and only if $K_Y^3 = 2K_Y Z$. Since $Z = K_Y^2 - 2c_2$, this is in turn equivalent to $K_Y^3 = 4K_Y c_2$. A direct check for all pairs $(Y, \mathcal{E})$ in E_0 of Remark 2.1 iii) shows that this condition if fulfilled only in case (C3) of Proposition 2.1, i.e. when Y is a del Pezzo threefold and $\mathcal{E} = H^{\oplus 2}$. So, we have

Proposition 5.1. *Let (X, L) be a scroll over a smooth threefold Y, as in Section 1, with $n = 4$, and let C be the residual cubic of its Hilbert curve. Suppose that $K_Y + c_1 \equiv 0$. Then $\Gamma_{(Y,\frac{1}{2}c_1)} = C$ if and only if $(Y, H = -\frac{1}{2}K_Y)$ is a del Pezzo threefold of degree d. In particular, $\Gamma_{(Y,\frac{1}{2}c_1)} = 3\ell_0$ if and only if $d = 6$.*

Comparing Proposition 5.1 with Corollary 4.2, we deduce that for the three pairs $(Y, \mathcal{E}) = (\mathbb{P}^3, \mathcal{N}(2)), (\mathbb{Q}^3, \mathcal{S}(2))$, and $(\mathbb{P}^2 \times \mathbb{P}^1, p_1^* T_{\mathbb{P}^2} \otimes \mathcal{O}(0,1))$, C is not the Hilbert curve of $(Y, \frac{1}{2}c_1)$. In fact a straightforward calculation shows that $\Gamma_{(Y,\frac{1}{2}c_1)}$ has equation

$$p_{(Y,\frac{1}{2}c_1)}\left(\frac{1}{2} + u, v\right) = -\frac{1}{3}(2u - v)(4u - 2v + 1)(4u - 2v - 1) = 0$$

in the first of the above three cases, and

$$p_{(Y,\frac{1}{2}c_1)}\left(\frac{1}{2} + u, v\right) = -\frac{1}{8}(2u - v)(6u - 3v + 1)(6u - 3v - 1) = 0,$$

in the last two cases, respectively.

Although this situation is outside the range of adjunction theoretic scrolls, this shows that the question whether the residual cubic C is itself the Hilbert curve of $(Y, \frac{1}{2}c_1)$ is nontrivial.

Next, we come back to the general case. The polynomial defining C, displayed in (39), is proportional to that in (45) if its coefficients are the same as the corresponding ones up to a non-zero constant, say λ.

For simplicity we call $K_Y^3 = x_0$, $K_Y c_1^2 = x_1$, $K_Y c_2 = x_2$, $c_1^3 = x_3$, $K_Y^2 c_1 = x_4$, $c_1 c_2 = x_5$, $K_Y c_2(Y) = x_6$, $c_1 c_2(Y) = x_7$. Hence we get a system of six homogeneous linear equations in the unknown $x_0, \dots, x_7$, regarding λ as a parameter. Solving such system of equations we get that either

i) $\lambda = -1$, $x_2 = \frac{1}{4}x_1$ and $x_5 = \frac{1}{4}x_3$, that is $\lambda = -1$, $K_Y(c_1^2 - 4c_2) = 0$ and $c_1(c_1^2 - 4c_2) = 0$, or

ii) $\chi(\mathcal{O}_Y) = 0$, $x_0 = -x_3 - 3x_7$, $x_1 = -x_3 - x_7$, $x_2 = -(\frac{1}{4}\lambda + \frac{1}{2})x_3 - (\frac{3}{4}\lambda - 1)x_7$, $x_4 = x_3 + 2x_7$, $x_5 = (\frac{1}{4}\lambda + \frac{1}{2})x_3$.

A special case corresponding to i) is when $c_1^2 - 4c_2 = 0$. Some information on this case is provided by Proposition 5.4 and the examples below when Y is a Fano threefold.

In case ii), we can assume that $\lambda \neq -1$, and then, solving the system gives in particular

$$c_1^3 + 3c_1c_2(Y) = -K_Y^3,$$
$$c_1^3 + c_1c_2(Y) = -K_Y c_1^2,$$
$$c_1^3 + 2c_1c_2(Y) = K_Y^2 c_1.$$

The second and third relations show that $c_1c_2(Y) = K_Y c_1(K_Y + c_1)$, while the first and the third one imply $c_1c_2(Y) = -K_Y^2(K_Y + c_1)$. Combining these two relations gives $K_Y(K_Y + c_1)^2 = 0$. Now, if (X, L) is an adjunction theoretic scroll, from the ampleness of $K_Y + c_1$ we conclude that no positive multiple of K_Y can be effective, hence Y has negative Kodaira dimension. On the other hand, if (X, L) is not an adjunction theoretic scroll, then condition $\chi(\mathcal{O}_Y) = 0$ implies that (X, L) can only be as in cases (C5)–(C7) of Proposition 2.1. Moreover, in case (C5) the base curve of Y turns out to be an elliptic curve. Summing up the above discussion, we have

Proposition 5.2. *Let (X, L) be a 4-dimensional scroll over a smooth threefold Y and suppose that the residual cubic C of its Hilbert curve is the Hilbert curve of $(Y, \frac{1}{2}\det\mathcal{E})$, precisely*

$$p_{(X,L)}\left(\frac{1}{2} + u, v\right) = \lambda\ (2u - v)\ p_{(Y,\frac{1}{2}\det\mathcal{E})}\left(\frac{1}{2} + u, v\right) \tag{47}$$

for some proportionality factor $\lambda \neq 0$. Then either

i) $\lambda = -1$, *and*
$c_1(c_1^2 - 4c_2) = K_Y(c_1^2 - 4c_2) = 0$, *or*

ii) $\lambda \neq -1$, $\chi(\mathcal{O}_Y) = 0$, *and* $K_Y(K_Y + c_1)^2 = 0$. *In particular, if (X, L) is also an adjunction theoretic scroll, then Y has negative Kodaira dimension. If (X, L) is not an adjunction theoretic scroll, then (X, L) is as in one of the cases* (C5)–(C7) *of Proposition 2.1.*

Case $c_1^2 - 4c_2 = 0$ provides an interesting situation for i). We stress that this condition is the one characterizing the Bogomolov proper semistability of the rank two vector bundle, which arises in the study of the same problem for scrolls over surfaces [9]. Note that this condition is clearly fulfilled when $(Y, \mathcal{E})$ is as in case (C3) of Proposition 2.1 (see also Proposition 5.1). More generally, let us observe the following fact. Let (X, L) be a 4-dimensional scroll over a smooth threefold Y and suppose that $X \cong Y \times \mathbb{P}^1$. Then $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = M^{\oplus 2}$ for some ample line bundle M on Y. In this case, $c_1 = 2M$ and $c_2 = M^2$, hence $c_1^2 - 4c_2 = 0$. A natural question is the converse: when does condition $c_1^2 - 4c_2 = 0$ imply that $X \cong Y \times \mathbb{P}^1$? This is not always true as Example 5.5 will show. Here is a partial answer.

Proposition 5.3. *Let $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is an ample vector bundle of rank 2 on a smooth threefold Y with Picard number $\rho(Y) = 1$. If $\mathcal{E}$ is decomposable and $c_1^2 - 4c_2 = 0$, then necessarily $X \cong Y \times \mathbb{P}^1$.*

Proof. Actually, letting M denote the ample generator of $\mathrm{NS}(Y)$ we can write $\mathcal{E} = [aM] \oplus [bM]$ for some positive integers a and b. Hence $c_1 = (a+b)M$ and $c_2 = abM^2$, and then the condition $0 = c_1^2 - 4c_2 = (a-b)^2M^2$ shows that $a = b$. Therefore $\mathcal{E} = [aM]^{\oplus 2}$, hence $X \cong Y \times \mathbb{P}^1$. □

For instance the cases in which Y is either a complete intersection or a general abelian threefold fall into the above Proposition since for both the Picard group has rank 1. The following result gives another partial answer to the above question.

Proposition 5.4. *Let (X, L) be a 4-dimensional scroll over a smooth threefold Y. Assume that Y is either $\mathbb{P}^3$, $\mathbb{Q}^3$, V_d, the del Pezzo threefold of degree $d = 3, 4, 5, 8$, or a prime Fano threefold with Picard number $\rho(Y) = 1$. Let $\mathcal{E}$ be a rank two vector bundle on Y such that $c_1^2 - 4c_2 = 0$. Then $X \cong \mathbb{P}^1 \times Y$.*

Proof. Because Y has Picard number $\rho(Y) = 1$, let $\mathcal{O}_Y(H)$ be the ample generator of Pic(Y). Let $\mathcal{E}' = \mathcal{E}(-aH)$ be the normalized rank two vector bundle on Y, that is, whose determinant

$c_1(\mathcal{E}')$ equals $\mathcal{O}_Y(\varepsilon H)$, with $\varepsilon = 0$ or -1. Since $\mathcal{E}$ is a rank two vector bundle on Y such that $c_1^2 - 4c_2 = 0$, then also $c_1(\mathcal{E}')^2 - 4c_2(\mathcal{E}') = 0$. By [1, Corollary 1] and [11, Lemma 2.5] the vector bundle $\mathcal{E}'$ is not semistable unless it is the trivial bundle. If $\mathcal{E}' = \mathcal{O}_Y^{\oplus 2}$ then $\mathcal{E} = \mathcal{O}_Y(aH)^{\oplus 2}$, which implies the assertion. On the other hand, if $\mathcal{E}'$ is not semistable then there exists a destabilizing line bundle $A \subset \mathcal{E}'$ and hence a short exact sequence

$$0 \to A \to \mathcal{E}' \to B \to 0\ .$$

Since Pic(Y) $\cong \mathbb{Z}$ it follows that $A = \mathcal{O}_Y(pH)$, $B = \mathcal{O}_Y(qH)$, for some $p, q \in \mathbb{Z}$. From $c_1(\mathcal{E}')^2 - 4c_2(\mathcal{E}') = 0$, dotting with H we get $((p+q)^2 - 4pq)H^3 = 0$, which implies that $p = q$. If $c_1(\mathcal{E}') = 0$ then $p = q = 0$ and moreover because $Ext^1(A, B) \cong H^1(\mathcal{O}_Y) = 0$ it follows that $\mathcal{E}'$ is the trivial bundle, a contradiction. On the other hand, it cannot be that $c_1(\mathcal{E}') = -1$, otherwise $-1 = 2p$, which is impossible because $p \in \mathbb{Z}$. In conclusion, $\mathcal{E}'$ has to be the trivial bundle and then our claim that $X = \mathbb{P}_Y(\mathcal{E}) \cong \mathbb{P}^1 \times Y$ follows. $\square$

Example 5.3. Consider over $\mathbb{P}^3$ the rank two bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2}$ and let $X = \mathbb{P}(\mathcal{E}) \cong \mathbb{P}^1 \times \mathbb{P}^3$. Because $c_1(\mathcal{E}) = 4 = c_2(\mathcal{E})$, it follows that $c_1^2 - 4c_2 = 0$. The equation of the Hilbert curve of (X, L) is

$$p_{(X,L)}\left(\frac{1}{2} + u, v\right) = \frac{1}{6}\ (2u - v)^2(4u - 2v + 1)(4u - 2v - 1) = 0,$$

and (47) holds with $\lambda = -1$. We like to point out that $X = \mathbb{P}(\mathcal{E})$ is a Fano bundle, [17, Theorem 2.1].

Example 5.4. Consider over $\mathbb{Q}^3$ the rank two bundle $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^3}(2)^{\oplus 2}$ and let $X = \mathbb{P}(\mathcal{E}) \cong \mathbb{P}^1 \times \mathbb{Q}^3$. Because $c_1(\mathcal{E}) = 4H$ and $c_2(\mathcal{E}) = 4H^2$, where H is the hyperplane bundle, it follows that $c_1^2 - 4c_2 = 0$. In this case the equation of the Hilbert curve of (X, L) is

$$p_{(X,L)}\left(\frac{1}{2} + u, v\right) = \frac{1}{12}(2u - v)(6u - 4v + 1)(3u - 2v)(6u - 4v - 1) = 0$$

and (47) holds with $\lambda = -1$.

We like to point out that if in Proposition 5.4 we drop the assumption $\rho(Y) = 1$ it is not anymore true that we always have $X \cong \mathbb{P}^1 \times Y$. The next two examples deal with the two del Pezzo threefolds of degree six.

Example 5.5. Let $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, let $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, for $i = 1, 2, 3$, be the projections onto each factor and let h_i denote the pull back of the class of a point in the i-th copy of $\mathbb{P}^1$. Consider over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ the rank two bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(h_1 + ph_2 + h_3) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(h_1 + h_2 + h_3)$, for an integer $p \geq 2$. In this case $c_1(\mathcal{E}) = 2h_1 + (1 + p)h_2 + 2h_3$, $c_2 = (\mathcal{E}) = (1 + p)h_2h_3 + 2h_1h_3 + (1 + p)h_1h_2$ and hence $c_1^2 - 4c_2 = 0$. On the other hand, for $X = \mathbb{P}(\mathcal{E})$ we have that $X \ncong \mathbb{P}^1 \times Y$.

The equation of the Hilbert curve of the polarized 4-fold (X, L) is

$$p_{(X,L)}\left(\frac{1}{2} + u, v\right) = \frac{1}{2}\ (2u - v)^3\big(4u - (1 + p)v\big) = 0,$$

and (47) holds with $\lambda = -1$. Note that $K_Y + c_1 = 0$ only when $p = 1$ and in this case we see that $C = 3\ell_0$, in accordance with Corollary 4.2.

In some instances, however, even droppping the assumption $\rho(Y) = 1$ but assuming that the rank two vector bundle $\mathcal{E}$ is decomposable, the condition $c_1^2 - 4c_2 = 0$ allows to prove that $\mathcal{E} = M^{\oplus 2}$ for some ample line bundle M on Y, which in turn implies that $X = \mathbb{P}(\mathcal{E}) \cong \mathbb{P}^1 \times Y$. This is shown by the following example.

Example 5.6. Let $Y = \mathbb{P}(T_{\mathbb{P}^2})$ and let $q : \mathbb{P}(T_{\mathbb{P}^2}) \to \mathbb{P}^2$ be the projection morphism. Denote by ξ the tautological line bundle of $\mathbb{P}(T_{\mathbb{P}^2})$ and let h_2 denote the pull back of the class of $h \in |\mathcal{O}_{\mathbb{P}^2}(1)|$, $h_2 = q^*h$. Let $h_1 = H - h_2$, with $H \in |\xi|$ and consider h_1, h_2 as generators of Pic(Y). One has $h_1^3 = 0$, $h_2^3 = 0$; moreover, by using the Chern–Wu relation we see that $h_1^2 + h_2^2 - h_1h_2 = 0$. On Y we consider the rank two vector bundle $\mathcal{E} = \mathcal{O}_Y(a\,h_1 + b\,h_2) \oplus \mathcal{O}_Y(c\,h_1 + d\,h_2)$, with a, b, c, d positive integers. In this case $c_1(\mathcal{E}) = \mathcal{O}_Y\big((a+c)\,h_1 + (b+d)\,h_2\big)$, $c_2(\mathcal{E}) = (ac)h_1^2 + (ad+bc)h_1h_2 + (bd)h_2^2$. If we require that $c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = 0$ then

$$(a+c)^2h_1^2 + (b+d)^2h_2^2 + 2(a+c)(b+d)h_1h_2 - 4\big[(ac)h_1^2 + (ad+bc)h_1h_2 + (bd)h_2^2\big] = 0.$$

Easy computations along with the fact that $h_1h_2 = h_1^2 + h_2^2$ give

$$(a-c)(a-c+2b-2d)h_1^2 + (b-d)(b-d+2a-2c)h_2^2 = 0.$$

Thus

$$(a-c)(a-c+2b-2d) = 0 \tag{48}$$

and

$$(b-d)(b-d+2a-2c) = 0 \quad . \tag{49}$$

If $a - c = 0$ then plugging this information in equation (49) we get $(b-d)^2 = 0$ and thus $d = b$ and $a = c$, therefore $\mathcal{E} = \mathcal{O}_Y(a\,h_1 + b\,h_2)^{\oplus 2}$. Same conclusion holds if $b - d = 0$.
If $a - c + 2b - 2d = 0$ then $a - c = 2d - 2b$ and plugging this in equation (48) we get $-3(b-d)^2 = 0$ and thus $b = d$ from which it follows $a = c$ and therefore $\mathcal{E} = \mathcal{O}_Y(a\,h_1 + b\,h_2)^{\oplus 2}$. Same conclusion holds if $b - d + 2a - 2c = 0$. Thus the fourfold $X = \mathbb{P}(\mathcal{E}) = \mathbb{P}\big(\mathcal{O}_Y(a\,h_1 + b\,h_2)^{\oplus 2}\big) \cong \mathbb{P}^1 \times Y$.

In this case, the Hilbert curve of the polarized 4-fold (X, L) has equation

$$p_{(X,L)}\left(\frac{1}{2} + u, v\right) = \frac{1}{2}(2u - v)(2u - bv)(2u - av)\big(4u - (a+b)v\big) = 0,$$

and (47) holds with $\lambda = -1$. Note that here $K_Y + c_1 = 0$ only when $a = b = 1$, and in this case the above equation shows that $C = 3\ell_0$, in accordance with Corollary 4.2.

Arguing in a similar way, the conclusion that $X = Y \times \mathbb{P}^1$ can be obtained also for the del Pezzo threefold Y of degree 7, recalling that $Y = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$. In this case, as in Example 5.6, we can see that if $\mathcal{E}$ is a rank two decomposable vector bundle on Y with $c_1^2 - 4c_2 = 0$ then $\mathcal{E} = M^{\oplus 2}$, where $M = \mathcal{O}_Y(ah_1 + bh_2)$, h_1 is the tautological line bundle on Y and $h_2 = q^*\mathcal{O}_{\mathbb{P}^2}(1)$, $q : Y \to \mathbb{P}^2$ being the bundle projection. Then for the Hilbert curve of (X, L) we get

$$p_{(X,L)}\left(\frac{1}{2} + u, v\right) = -\frac{1}{6}(2u - v)(2u - av)\big(28u^2 - (10a + 18b)uv + (a^2 + 3ab + 3b^2)v^2 - 1\big),$$

and (47) holds with $\lambda = -1$. Note that here the residual cubic C has a conic as a component, which is reducible if and only if $a = b$.

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