International Journal of Mathematics

Some Fano manifolds whose Hilbert polynomial is totally reducible over $\mathbb{Q}$

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Let (X, L) be any Fano manifold polarized by a positive multiple of its fundamental divisor H. The polynomial defining the Hilbert curve of (X, L) boils down to being the Hilbert polynomial of (X, H), hence it is totally reducible over $\mathbb{C}$; moreover, some of the linear factors appearing in the factorization have rational coefficients, e.g. if X has index ≥ 2. It is natural to ask when the same happens for all linear factors. Here the total reducibility over $\mathbb{Q}$ of the Hilbert polynomial is investigated for three special kinds of Fano manifolds: Fano manifolds of large index, toric Fano manifolds of low dimension, and Fano bundles of low coindex.

Keywords: Polarized manifold, Hilbert polynomial; Fano manifold; toric Fano manifold; Fano bundle.

Mathematics Subject Classification 2000: 14C20, 14J45, 14J30, 14J35, 14N30

1. Introduction

Let $P_{-K_X}(m) := \chi(m(-K_X))$ be the Hilbert polynomial of a Fano manifold of dimension n (with respect to the anticanonical polarization). The fact that for $X = \mathbb{P}^n$ one has

$$P_{-K_{\mathbb{P}^n}}(z) = \frac{1}{n!}\prod_{i=1}^{n}\left(z + \frac{i}{n+1}\right)$$

gives rise to various questions concerning the roots of the Hilbert polynomial of a Fano manifold in general. The most celebrated one is the so-called *narrow canonical* *strip hypothesis*, claiming that for any root $\alpha \in \mathbb{C}$ of $P_{-K_X}(z)$ its real part satisfies the conditions

$$-1+\frac{1}{n+1} \leq \operatorname{Re}(\alpha) \leq -\frac{1}{n+1}. \tag{1.1}$$

A large literature is devoted to it and some of its variants, starting with Golyshev's paper [9] (e.g. see [17], [10], and [3], where examples disproving the hypothesis for $n \geq 7$ are provided). Another natural question, which is what we deal with in this paper, is about the rationality of all roots, as it happens in the case of $\mathbb{P}^n$. In other words,

$$\text{for which Fano manifolds is the polynomial } P_{-K_X} \text{ totally reducible over } \mathbb{Q}\,? \tag{1.2}$$

We emphasize that this question makes sense, since for every dimension $n \geq 2$ there are Fano n-folds for which P_{-K_X} is totally reducible over $\mathbb{Q}$ and others for which this is not true. From our results, however, the total reducibility of P_{-K_X} over $\mathbb{Q}$ does not seem to be directly related to any geometric property of the Fano manifold X. Our interest in (1.2) stems from the study of Hilbert curves of polarized manifolds, a notion introduced in [5] and further studied in [14], [15], [16]. The Hilbert curve of a polarized manifold (X, L) is the complex affine plane curve $\Gamma_{(X,L)} \subset \mathbb{A}^2_{\mathbb{C}}$ of degree $n = \dim(X)$, defined by the Hilbert-like polynomial $p_{(X,L)}(x, y) := \chi(xK_X + yL)$, where x and y are regarded as complex variables [5, Section 2]. Specialize the setting to the following case:

($\diamond$) X is a Fano n-fold, $n \geq 2$, of index ι_X, and $L := rH$, where H is the fundamental divisor (i.e. $-K_X = \iota_X H$) and r is a positive integer.

For (X, L) as in ($\diamond$) we simply write p for $p_{(X,L)}$ and Γ for $\Gamma_{(X,L)}$. In [16, Lemma 3.1] the following fact is proven.

Proposition 1.1. *For any pair (X, L) as in ($\diamond$),*

$$p(x, y) = R(x, y) \cdot \prod_{j=1}^{\iota_X - 1} (ry - \iota_X x + j),$$

with $R(x, y) = \sum_{j=0}^{c_X} a_j (ry - \iota_X x)^j$, $c_X = n + 1 - \iota_X$ being the coindex of X, and $(a_0, \ldots, a_{c_X})$ the solution of the linear system

$$U \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{c_X} \end{bmatrix} = \begin{bmatrix} \frac{h^0(\mathcal{O}_X)}{\delta(0)} \\ \frac{h^0(H)}{\delta(1)} \\ \vdots \\ \frac{h^0(c_X H)}{\delta(c_X)} \end{bmatrix}, \tag{1.3}$$

where U is the Vandermonde matrix

$$U := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & \cdots & 2^{c_X-1} & 2^{c_X} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & c_X & \cdots & (c_X)^{c_X-1} & (c_X)^{c_X} \end{pmatrix}, \tag{1.4}$$

and δ is the function defined by $\delta(t) = \frac{(t+\iota_X-1)!}{t!}$ for every $t \in \mathbb{Z}_{\geq 0}$.

In particular, the special setting in ($\diamond$) allows to convert $p(x,y)$ into a polynomial in a single complex variable, which is, of course, totally reducible over $\mathbb{C}$ (hence $p(x,y)$ is the product of n polynomials of degree 1 in $\mathbb{C}[x,y]$). In fact,

$$p(x,y) = P(z) := \left(\sum_{j=0}^{c_X} a_j z^j\right) \cdot \prod_{j=1}^{\iota_X-1} (z+j) = a\prod_{i=1}^{n}(z-\alpha_i), \tag{1.5}$$

where $z := ry - \iota_X x$, $a := a_{c_X}$ and $\alpha_i \in \mathbb{C}$. As a consequence, Γ is always reducible into n lines of $\mathbb{A}^2_{\mathbb{C}}$, which are parallel each other, with slope $\frac{\iota_X}{r}$. Moreover, $\iota_X - 1$ of them are defined over $\mathbb{Q}$ and evenly spaced. As to the polynomial P appearing in (1.5) note that

$$P(z) = P_H(z) = \chi(zH) \tag{1.6}$$

is precisely the Hilbert polynomial of (X,H). In particular it is a numerical polynomial. Because $H = \frac{1}{\iota_X}(-K_X)$, we have $P_H(z) = P_{-K_X}\left(\frac{z}{\iota_X}\right)$. So, while looking at H forces to rescaling the bounds in (1.1), in dealing with (1.2) it is equivalent to work with P instead of P_{-K_X}, as we will do in the following.

Of course, from the point of view of the Hilbert curve, (1.2) is equivalent to asking when the points of Γ defined over $\mathbb{Q}$ (or $\mathbb{R}$), denoted by $\Gamma_{\mathbb{Q}}$ (or $\Gamma_{\mathbb{R}}$), constitute n lines of $\mathbb{A}^2_{\mathbb{Q}}$ (or $\mathbb{A}^2_{\mathbb{R}}$). However, the role of Γ is clearly less relevant in the setting ($\diamond$) than in the case where K_X and L are linearly independent; hence we will confine ourselves to consider Γ in a few points, just to complete the picture.

In this paper we address question (1.2) when X is either:

a) a Fano n-fold of index $\iota_X \geq n-2$,
b) a toric Fano manifold of dimension $n \leq 4$, or
c) a Fano bundle of sufficiently high index.

With regard to a), we first recall that P is totally reducible over $\mathbb{Q}$ for both $\mathbb{P}^n$ and $\mathbb{Q}^n$ (e.g. see [16, Section 3] or Proposition 3.1 (i) and (ii)). So, the first nontrivial situation to be investigated is that occurring for $\iota_X = n-1$, i.e. essentially for pairs (X,H) which are del Pezzo manifolds [7, pp. 44–45]. The answer is the following.

Proposition 1.2. *Let (X,L) be as in ($\diamond$) with $\iota_X \geq n-1 \geq 1$ and let $d := H^n$. Then P is totally reducible over $\mathbb{Q}$, except in the following cases:*

i) *X is a del Pezzo surface with $K_X^2 \leq 7$;*
ii) *(X, H) is a del Pezzo threefold or fourfold with $d \leq 5$;*
iii) *(X, H) is a del Pezzo manifold of dimension $n \geq 5$ with $d \leq 4$ (in particular, any del Pezzo manifold of dimension ≥ 7);*
iv) *(X, H) is a del Pezzo threefold of degree 7, namely, $X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)\big)$, H being the tautological line bundle.*

As to Γ, in cases i)-iii) $\Gamma_{\mathbb{R}}$ (as well as $\Gamma_{\mathbb{Q}}$) reduces to $n-2$ parallel lines; in case iv), $\Gamma_{\mathbb{R}}$ consists of three lines, while $\Gamma_{\mathbb{Q}}$ reduces to the single line of equation $2x - ry - 1 = 0$.

The above result follows from a complete description of $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{Q}}$ we provide in Section 3 for del Pezzo manifolds (Theorem 3.3 and Proposition 3.4), relying on their classification [7] and on Proposition 3.1 (iii). Furthermore we consider the case $\iota_X = n-2$. Here, relying on several results on the classification of Mukai n-folds [12], we obtain a precise answer to (1.2) for any $n \geq 3$ (Theorem 4.1 for $n = 3$ and Theorem 3.8 for $n \geq 4$).

Concerning toric Fano manifolds, we have to mention that the Hilbert polynomial of the anticanonical line bundle can be regarded as the Ehrhart polynomial of the associate lattice polytope. Many properties of these numerical polynomials are known, e. g. concerning the location of the roots and their distribution (see [2, Sec. 4] and [10]), although not related to problem (1.2). So, in relation to b), in Section 4 we provide the precise lists of pairs (X, L) as in ($\diamond$) with X a toric Fano n-fold ($n \leq 4$) for which the Hilbert polynomial P is totally reducible over $\mathbb{Q}$ (Corollary 4.2 for $n = 3$ and Theorem 4.3 for $n = 4$; see also Proposition 2.2 for $n = 2$). This is done by relying on [1]. A similar result could also be obtained for $n = 5, 6, 7$, but the lists that can be obtained in these cases are too long to be included in the paper (see Remark 4.4).

Finally, case c) looks the most interesting. In Section 5, assuming that X is a Fano bundle of index $\iota_X \geq \frac{n+1}{2}$, we show that for a pair (X, L) as in ($\diamond$) the polynomial P is totally reducible over $\mathbb{Q}$ unless $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^m}(1)^{\oplus(m-1)} \oplus \mathcal{O}_{\mathbb{P}^m}(2)$ (Theorem 5.2). In fact to investigate this exceptional case we use a program in Magma to deal with the system (1.3). This allows us to show that the factor of $P(z)$ corresponding to the polynomial $R(x, y)$ appearing in Proposition 1.1 is not totally reducible over $\mathbb{Q}$ only for a limited number of values of m (we tested it only for $m \leq 150$), and we conjecture that the same is true for every $m \geq 2$. There is a good reson for regarding $\frac{n+1}{2}$ as the appropriate threshold for answering positively our problem in case c). For, let $X = Y \times \mathbb{P}^{m-2}$, where (Y, h) is a del Pezzo manifold of dimension $m \geq 3$. Then $n := \dim X = 2(m-1)$, and $-K_X = (m-1)H$, where $H = h \boxtimes \mathcal{O}_{\mathbb{P}^{m-2}}(1)$. Therefore $X = \mathbb{P}_Y(\mathcal{E})$, where $\mathcal{E} = h^{\oplus(m-1)}$, is a Fano bundle of index $\iota_X = \frac{n}{2}$. Since $\chi(kH) = \chi(kh)\chi\big(\mathcal{O}_{\mathbb{P}^{m-2}}(k)\big)$ for every k, $P(z)$ is the product of the Hilbert polynomials of Y and $\mathbb{P}^{m-2}$. The latter is totally reducible over $\mathbb{Q}$, while the former could not be totally reducible over $\mathbb{Q}$, in view of Proposition 1.2. Therefore, for any (Y, h) appearing among the exceptions in Proposition 1.2, not even P is totally reducible over $\mathbb{Q}$. This allows one to produce plenty of Fano bundles X of dimension n and index $\iota_X = \frac{n}{2}$, whose Hilbert polynomials are not totally reducible over $\mathbb{Q}$. Clearly, a similar conclusion can be obtained by working with the exceptions arising in case b).

To conclude, consider the Fano threefold $X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)\big)$, which is isomorphic to $\mathbb{P}^3$ blown-up at one point (sometimes denoted by V_7 in the literature). We would like to stress that this Fano threefold appears in each of the three contexts a), b), c) that we have considered. Moreover, it contributes to underline a significant role that arithmetic plays, in addition to geography, in the study of Hilbert curves of polarized manifolds (see Remark 3.5) Indeed this paper grew out of this observation.

2. Background material

Varieties considered in this paper are defined over the field $\mathbb{C}$ of complex numbers. We use the standard notation and terminology from algebraic geometry. A manifold is any smooth projective variety. Tensor products of line bundles are denoted additively. The pullback of a vector bundle $\mathcal{F}$ on a manifold X by an embedding $Y \hookrightarrow X$ is simply denoted by $\mathcal{F}_Y$. We denote by K_X the canonical bundle of a manifold X. A *Fano manifold* is a manifold X such that $-K_X$ is an ample line bundle (X is also called a *del Pezzo surface* if $\dim X = 2$). The *index* ι_X of X is defined as the greatest positive integer which divides $-K_X$ in the Picard group $\mathrm{Pic}(X)$ of X. We recall that $\iota_X \leq \dim X + 1$, equality holding if and only if $X = \mathbb{P}^n$, by the Kobayashi–Ochiai theorem. By the *coindex* of X we simply mean the non-negative integer $c_X := \dim X + 1 - \iota_X$. Moreover, we say that a polarized manifold (X, H) of dimension n is a *del Pezzo manifold* (respectively a *Mukai manifold*) if $K_X + (n-1)H = \mathcal{O}_X$ (respectively $K_X + (n-2)H = \mathcal{O}_X$). We denote by $d := H^n$ its degree. Note that if X is a Fano manifold of dimension n and index $n-1$, then $\big(X, \frac{1}{n-1}(-K_X)\big)$ is a del Pezzo manifold. The converse is true except for the following pairs: $\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)\big)$, $\big(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(2)\big)$ for $n = 2$ and $\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)\big)$ for $n = 3$.

Let X be any projective manifold of dimension n. For any line bundle D on X, consider the expression of the Euler–Poincaré characteristic $\chi(D)$ provided by the Riemann–Roch–Hirzebruch theorem

$$\chi(D) = \frac{1}{n!} D^n - \frac{1}{2(n-1)!} K_X \cdot D^{n-1} + \text{terms of lower degree} \tag{2.1}$$

(a polynomial of degree n in the Chern class of D, whose coefficients are polynomials with rational coefficients in the Chern classes of X [11, Theorem 20.3.2]). In particular, for (X, L) as in ($\diamond$), letting $D = xK_X + yL = zH$, and recalling that $P(z) = p(x, y)$, (2.1) says that

$$P(z) = H^n \left(\frac{1}{n!} z^n + \frac{\iota_X}{2(n-1)!} z^{n-1} + \text{terms of lower degree} \right).$$

Note that for any root $\alpha \in \mathbb{C}$ of P, also its conjugate $\overline{\alpha}$ is a root, since $P \in \mathbb{Q}[z]$. Moreover, taking into account the Serre involution $D \mapsto K_X - D$ acting on $\mathrm{Num}(X)$, we have

$$P(z) = \chi(zH) = (-1)^n \chi(K_X - zH) = (-1)^n P(-z - \iota_X). \tag{2.2}$$

Hence for any root $\alpha \in \mathbb{C}$ of P, also $-\iota_X - \alpha$ is a root. In particular, $P(\frac{-\iota_X}{2}) = 0$ if n is odd.

This discussion, combined with Proposition 1.1, leads to the following result.

Proposition 2.1. *Let (X, L) be as in ($\diamond$) and consider the polynomial $P(z)$ as in (1.5), where $z = ry - \iota_X x$. Then the following properties hold:*

1) *the leading coefficient of $P(z)$ appearing in (1.5) is $a = \frac{H^n}{n!} = \frac{1}{n!}\frac{(-K_X)^n}{(\iota_X)^n} = \frac{(-1)^n}{\Pi_{i=1}^n \alpha_i}$;*
2) *the subset $A := \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{C}$ of the roots of P is symmetric with respect to the two orthogonal lines $Im(z) = 0$ and $Re(z) = -\frac{\iota_X}{2}$, hence with respect to the point $-\frac{\iota_X}{2}$; moreover, the integers in A, if any, are exactly $\{1 - \iota_X, \ldots, -2, -1\}$ (which implies $\iota_X \geq 2$);*
3) *if either n or c_X is odd, then $z + \frac{\iota_X}{2}$ is a factor of $P(z)$; moreover, if c_X is odd and n is even, then $\left(z + \frac{\iota_X}{2}\right)^2$ is a factor of $P(z)$.*

Proof. 1) Recalling that $-K_X = \iota_X H$, we have $a := a_{c_X} = \frac{(-1)^n}{\Pi_{i=1}^n \alpha_i} = \frac{H^n}{n!} = \frac{1}{n!}\frac{(-K_X)^n}{(\iota_X)^n}$, because $P(0) = \chi(\mathcal{O}_X) = 1$ and

$$P(z) = \chi(zH) = \frac{H^n}{n!} z^n + \ldots ,$$

by equation (2.1), where dots stand for terms of lower degree in z.

2) As we said, $P(\alpha_i) = 0$ if and only if $P(\overline{\alpha_i}) = 0$, since the coefficients of P are real numbers. On the other hand, $P(\alpha_i) = 0$ if and only if $P(-\alpha_i - \iota_X) = 0$, due to (2.2). These facts imply the claimed symmetries of A. Finally, recall that $P(z) = \chi(zH)$. Let $\alpha_i \in A$ be an integer. Since $\chi(\alpha_i H) = 0$ and

$$\chi(\alpha_i H) = \begin{cases} h^0(\alpha_i H) & \text{for } \alpha_i \geq 0 \ , \\ (-1)^n h^0\big((-\iota_X - \alpha_i)H\big) & \text{for } \alpha_i < 0 \ , \end{cases}$$

we deduce that $\alpha_i < 0$ and $\iota_X + \alpha_i > 0$, that is, $\alpha_i \in \{1 - \iota_X, \ldots, -2, -1\}$, which implies $\iota_X \geq 2$. On the other hand, from Proposition 1.1 we know that $\{1 - \iota_X, \ldots, -2, -1\} \subseteq A$.

3) By 2) the set A is symmetric with respect to $-\frac{\iota_X}{2}$, hence, if n is odd, there exists $\alpha_i \in A$ such that $\alpha_i = -\alpha_i - \iota_X$, i.e. $\alpha_i = -\frac{\iota_X}{2}$. Thus $z + \frac{\iota_X}{2}$ is a factor of $P(z)$. Suppose now that c_X is odd. Since both $\{1 - \iota_X, \ldots, -2, -1\}$ and A are symmetric with respect to $-\frac{\iota_X}{2}$ and $R(x, y)$ in Proposition 1.1 has degree c_X, we deduce that the factor of P corresponding to $R(x, y)$ has a root α_j such that $\alpha_j = -\alpha_j - \iota_X$, i.e. $\alpha_j = -\frac{\iota_X}{2}$. Therefore, even in this case, $z + \frac{\iota_X}{2}$ is a factor of $P(z)$. Finally, if c_X is odd and n is even, then also $\iota_X = n + 1 - c_X =: 2p$ is even, hence $\iota_X - 1 = 2p - 1$. Thus the

second polynomial factoring $P(z)$ in (1.5) has the form $\prod_{j=1}^{\iota_x-1}(z+j)=\prod_{j=1}^{2p-1}(z+j)$ and $z+p=z+\frac{\iota_X}{2}$ is one of its factors. □

Sometimes, letting $w := \frac{\iota_X}{2}+z$, i.e. $z=\frac{-\iota_X}{2}+w$, it is useful to look at P as a polynomial in w, namely $Q(w) := P\left(w-\frac{\iota_X}{2}\right)$; in this case, (2.2) becomes $Q(w) = (-1)^n Q(-w)$. As a consequence, the polynomial Q contains only terms of degrees with the same parity as n. This is the advantage of looking at Q instead of P.

As an example, let us discuss here the case of surfaces, by using Q. Let (X, L) be as in ($\diamond$); since X is a del Pezzo surface we have $-K_X = \iota_X H$ and $\chi(\mathcal{O}_X) = 1$. Then the Riemann–Roch theorem gives

$$P(z)=\chi(zH)=1+\frac{1}{2}zH(zH-K_X)=\frac{1}{2}\left(\left[\left(z+\frac{\iota_X}{2}\right)H\right]^2+\;2-\frac{1}{4}K_X^2\right).$$

Hence, replacing $z+\frac{\iota_X}{2}$ with w, we get

$$Q(w)=\frac{1}{2}\left(\frac{K_X^2}{\iota_X^2}w^2-\frac{K_X^2-8}{4}\right). \tag{2.3}$$

Thus Q, hence P, is totally reducible over $\mathbb{Q}$ if and only if $\frac{K_X^2-8}{K_X^2}$ is the square of a rational number. Clearly, this implies $K_X^2 \geq 8$ (total reducibility over $\mathbb{R}$), and then K_X^2 is either 8 or 9, in view of the well-known classification of del Pezzo surfaces (e.g. see [7, (8.1)]). Moreover, from (2.3) we obtain that

$$Q(w)=\begin{cases}4w^2 & \text{if } K_X^2=8 \text{ and } X=\mathbb{F}_1,\\ w^2 & \text{if } K_X^2=8 \text{ and } X=\mathbb{P}^1\times\mathbb{P}^1,\\ \frac{1}{2}\left(w-\frac{1}{2}\right)\left(w+\frac{1}{2}\right) & \text{if } K_X^2=9, \text{ in which case } X=\mathbb{P}^2,\end{cases}$$

recalling that in the three cases above $\iota_X = 1, 2$ or 3 respectively. This, in turn, makes evident the total reducibility over $\mathbb{Q}$. Coming back to $p(x, y)$ to include Γ into the picture, this discussion can be summarized by the following result, which explains i) in Proposition 1.2.

Proposition 2.2. *Let (X, L) be as in ($\diamond$) with $n = 2$. Then the following are equivalent:*

1) *P is totally reducible over $\mathbb{Q}$;*
2) *P is totally reducible over $\mathbb{R}$;*
3) *$K_X^2 = 8$ or 9;*
4) *(X, L) is one of the following polarized surfaces:*
   (a) *$\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r)\right)$, $\iota_X = 3$ and $p_{(X,L)}(x, y) = \frac{1}{2}(ry-3x+1)(ry-3x+2)$;*
   (b) *$\left(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(r,r)\right)$, $\iota_X = 2$ and $p_{(X,L)}(x, y) = (ry-2x+1)^2$;*
   (c) *$(\mathbb{F}_1, -rK_{\mathbb{F}_1})$, $\iota_X = 1$ and $p_{(X,L)}(x, y) = 4\left(ry-x+\frac{1}{2}\right)^2$.*

3. Fano manifolds of large index

Given a Fano manifold X of dimension n and index $\iota_X \geq n-2$, it is known that there exists a smooth element $Y \in |H|$. This is obvious for $\iota_X = n+1$ and n; it follows from Fujita's theory of del Pezzo manifolds [7, §8] for $\iota_X = n-1$, and from a result of Mella [18] for $\iota_X = n-2$. Note that $-K_Y = (\iota_X - 1)H_Y$ by adjunction. In particular, if $n \geq 3$ and (X, H) is a del Pezzo manifold, then (Y, H_Y) is also a del Pezzo manifold, and similarly, if $n \geq 4$ and (X, H) is a Mukai manifold, then (Y, H_Y) is a Mukai manifold too. A consequence of this fact is that for $\iota_X \geq n-2$ we can always apply an inductive argument up to the surface case to compute $h^0(tH)$ for $t = 1, \dots, c_X \leq 3$. So, for (X, L) as in $(\diamond)$, this allows to make the polynomial $p(x, y)$ explicit. On the other hand, this polynomial allows to recover (X, L) provided that the polarization satisfies a mild arithmetic assumption. In fact, translating [16, Theorem 3.3 and Remark 3.5] in terms of $z = ry - \iota_X x$, we can easily obtain the following characterizations of pairs (X, L) as in $(\diamond)$ via Hilbert polynomials for $\iota_X \geq n-1$. From now on we can assume $n \geq 3$, in view of Proposition 2.2.

Proposition 3.1. *Let (X, L) be a polarized manifold of dimension $n \geq 3$.*

(i) $(X, L) = \big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)\big)$ *for a positive integer r coprime with $n+1$ if and only if $H = \frac{1}{r}L$ is an ample line bundle and the Hilbert polynomial of (X, H) is*

$$P(z) = \frac{1}{n!}\prod_{i=1}^{n}(z+i).$$

(ii) $(X, L) = \big(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(r)\big)$ *for a positive integer r coprime with n if and only if $H = \frac{1}{r}L$ is an ample line bundle and the Hilbert polynomial of (X, H) is*

$$P(z) = \frac{2}{n!}\left(z + \frac{n}{2}\right)\prod_{i=1}^{n-1}(z+i).$$

(iii) *X is a Fano manifold of index $\iota_X = n-1$ and $L := \frac{r}{n-1}(-K_X)$ for a positive integer r coprime with $n-1$ if and only if $H = \frac{1}{r}L$ is an ample line bundle such that the Hilbert polynomial of (X, H) is*

$$P(z) = \left(\frac{d}{n!}z^2 + \frac{(n-1)d}{n!}z + \frac{1}{(n-2)!}\right)\prod_{i=1}^{n-2}(z+i).$$

Remark 3.2. Notice that in Proposition 3.1, the coprimality condition is needed just to prove the "if" part, in all the three cases (i)–(iii). However, in the following we do not care it, since we will use only the converse. As to (iii), recall that the equality $\iota_X = n-1$ implies that $\big(X, \frac{1}{\iota_X}(-K_X)\big)$ is a del Pezzo manifold, hence d is its degree. Moreover, for $n = 3$, letting $d = 8$ we see that the equation in (iii) coincides with that provided in (i) by replacing z with $2z$, which corresponds to taking $(X, L) = \big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2r)\big)$. As a consequence, in (iii) we can suppose that $d \leq 7$ if $n = 3$. For the classification of del Pezzo manifolds we refer to [7, (8.11)].

Let (X, L) be as in $(\diamond)$. Clearly the polynomial P is totally reducible over $\mathbb{Q}$ in cases (i) and (ii) of Proposition 3.1. Note that in case (ii), P has a double root, namely $-\frac{n}{2}$, if and only if n is even. In case (iii) of Proposition 3.1, $P(z)$ factors into $(n-2)$ linear factors and the degree 2 factor

$$\frac{d}{n!}\left(z^2 + (n-1)z + \frac{n(n-1)}{d}\right). \tag{3.1}$$

Then P is totally reducible over $\mathbb{Q}$ if and only if the same happens for this factor. Leaving out the coefficient $\frac{d}{n!}$, the discriminant of the above trinomial is

$$\Delta = \frac{n-1}{d}\big((n-1)d - 4n\big). \tag{3.2}$$

Thus P is totally reducible over $\mathbb{Q}$ if and only if $\Delta = k^2$, for some $k \in \mathbb{Q}$. On the other hand, the total reducibility of P over $\mathbb{R}$ is expressed by the condition $\Delta \geq 0$, i.e.

$$d \geq \frac{4n}{n-1}.$$

This means that $d \geq 6$ if $n = 3$ or 4 and $d \geq 5$ if $n \geq 5$. Look at the del Pezzo manifold (X, H). Fujita's classification [7, (8.11)] implies that $d \leq 4$ if $n \geq 7$, hence $\Delta < 0$ for $n \geq 7$, and therefore P cannot be totally reducible over $\mathbb{R}$, and a fortiori over $\mathbb{Q}$, in this case. Consider also that $d \leq 8$ for $n = 3$, $d \leq 6$ for $n = 4$, and $d \leq 5$ if $n = 5, 6$ [7, (8.11)]. In particular, case $d = 8$, which corresponds to $(X, H) = \big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)\big)$, fits into (i), as already observed in Remark 3.2. In terms of coordinates x and y, the factor in (3.1) defines the conic G of equation $\big((n-1)\left(x - \frac{1}{2}\right) - ry\big)^2 - \frac{\Delta}{4} = 0$ and $\Gamma = \ell_1 + \cdots + \ell_{n-2} + G$ consists of G plus $(n-2)$ paralell lines ℓ_i whose equations are $ry - (n-1)x + i = 0$ $(i = 1, \ldots, n-2)$.

In conclusion, a case-by-case analysis leads to the following result, which includes a view on the "geography" of Γ, with obvious meaning of the symbols.

Theorem 3.3. *Let (X, L) be as in case* (iii) *of* Proposition 3.1 *with $d \leq 7$. Then $G_{\mathbb{R}} = \emptyset$, hence $\Gamma_{\mathbb{R}} = \Gamma_{\mathbb{Q}} = \ell_1 + \cdots + \ell_{n-2}$, except in the following cases, in which the description of $G_{\mathbb{R}}$ is provided.*

- $n = 3$ *and there are two possibilities:*
  - *a) $G_{\mathbb{R}} = \lambda + \lambda'$ is the union of two distinct lines λ, λ', both distinct from ℓ_1; this happens for $d = 7$, in which case $X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)\big)$ and H is the tautological line bundle;*
  - *b) $G_{\mathbb{R}} = 2\lambda$ is a double line where $\lambda = \ell_1$, hence $\Gamma_{\mathbb{R}} = 3\ell_1$ is a triple line; this happens for $d = 6$ and either $(X, H) = \big(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1,1)\big)$, or $X = \mathbb{P}(T_{\mathbb{P}^2})$ and H is the tautological line bundle.*
- *$n = 4$, and $G_{\mathbb{R}} = \lambda + \lambda'$ is the union of two distinct lines λ, λ', where, up to renaming, $\lambda = \ell_1$, $\lambda' = \ell_2$, so that $\Gamma_{\mathbb{R}} = 2(\ell_1 + \ell_2)$. This happens for $d = 6$ and it corresponds to $(X, H) = \big(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,1)\big)$.*

- *$n = 5$, and $G_{\mathbb{R}} = 2\lambda$ with $\lambda = \ell_2$, so that $\Gamma_{\mathbb{R}} = \ell_1 + 3\ell_2 + \ell_3$. In this case, $d = 5$, and (X, H) is the general hyperplane section of the Grassmannian $\mathbb{G}(1, 4)$ embedded in $\mathbb{P}^9$ via the Plücker embedding.*
- *$n = 6$, and $G_{\mathbb{R}} = \lambda + \lambda'$ is the union of two distinct lines λ, λ', where, up to renamimg, $\lambda = \ell_2$ and $\lambda' = \ell_3$. So $\Gamma_{\mathbb{R}} = \ell_1 + 2(\ell_2 + \ell_3) + \ell_4$. In this case, $d = 5$ and X is the Grassmannian $\mathbb{G}(1, 4)$ embedded by H in $\mathbb{P}^9$ via the Plücker embedding.*

As to the situation for $G_{\mathbb{Q}}$ (when $G_{\mathbb{R}} \neq \emptyset$), we note the following fact. First of all, for the term in (3.2), we get $\Delta = 0$ when $(n, d) = (3, 6), (5, 5)$. Moreover, $\Delta = k^2$ for some $k \in \mathbb{Q}$ when: $(n, d) = (3, 8), (4, 6)$ and $(6, 5)$ (in which cases $\Delta = 1$). On the other hand, $\Delta = k^2$ with $k \notin \mathbb{Q}$ if and only if $(n, d) = (3, 7)$ (here $\Delta = 4/7$). Therefore, we have

Proposition 3.4. *Let (X, L) be as in case* (iii) *of* Proposition 3.1. *The description of $\Gamma_{\mathbb{Q}}$ is the same as that given for $\Gamma_{\mathbb{R}}$ in* Theorem 3.3, *up to regarding λ, λ' and the ℓ_i's as lines in $\mathbb{A}^2_{\mathbb{Q}}$, except when $(n, d) = (3, 7)$, in which case $G_{\mathbb{Q}} = \emptyset$, so that $\Gamma_{\mathbb{Q}} = \ell_1$, the line of equation $ry - 2x + 1 = 0$.*

The following table summarizes the above results about the Hilbert curves in case (iii) of Proposition 3.1.

Table 1. Hilbert curves of (X, L) as in $(\diamond)$, with $\iota_X = n - 1$, $d \leq 7$.

n	(X, L)	$\Gamma := \Gamma_{(X,L)}$
≥ 3	(X, H) del Pezzo manifold of degree $d := H^n$, $L = rH$ for $r \geq 1$	$\Gamma = \ell_1 + \cdots + \ell_{n-2} + G$ where $\ell_i :\ ry - (n-1)x + i = 0$ for $i = 1, ..., n-2$ and $G :\ \big((n-1)(x - \frac{1}{2}) - ry\big)^2 - \frac{\Delta}{4} = 0$, $\Delta = \frac{n-1}{d}\big((n-1)d - 4n\big)$
	Further information on $G_{\mathbb{R}}$ and $G_{\mathbb{Q}}$:	
≥ 7		$G_{\mathbb{R}} = G_{\mathbb{Q}} = \emptyset$; $\Delta < 0$, $d \leq 4$
6	$X = \mathbb{G}(1, 4) \subset \mathbb{P}^9$	$G_{\mathbb{R}} = G_{\mathbb{Q}} = \ell_2 + \ell_3$; $\Delta = 1$, $d = 5$
5	$X = \mathbb{G}(1, 4) \cap \mathbb{P}^8$	$G_{\mathbb{R}} = G_{\mathbb{Q}} = 2\ell_2$; $\Delta = 0$, $d = 5$
4	$X = \mathbb{P}^2 \times \mathbb{P}^2$	$G_{\mathbb{R}} = G_{\mathbb{Q}} = \ell_1 + \ell_2$; $\Delta = 1$, $d = 6$
3	$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}(T_{\mathbb{P}^2})$	$G_{\mathbb{R}} = G_{\mathbb{Q}} = 2\ell_1$; $\Delta = 0$, $d = 6$
3	$X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$	$G_{\mathbb{R}} = \lambda + \lambda', \lambda \neq \lambda'$ and both $\neq \ell_1$ $G_{\mathbb{Q}} = \emptyset$; $\Delta = 4/7$, $d = 7$

This offers the opportunity to point out the role of arithmetic in the study of

Hilbert curves.

Remark 3.5. Consider the following polarized threefolds: $(X_1, L_1) = \big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)\big)$, $(X_2, L_2) = \big(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1)\big)$, and the del Pezzo threefold (X_2, L_2) of degree 7. Let $p_i = p_{(X_i,L_i)}$ $(i = 1, 2, 3)$ be the polynomials defining the corresponding Hilbert curves. Then

$$p_1(x, y) = \frac{1}{6}\left(y - 2x + \frac{1}{2}\right)(y - 2x + 1)\left(y - 2x + \frac{3}{2}\right) = 0,$$

$$p_2(x, y) = \frac{1}{3}(y - 3x + 1)\left(y - 3x + \frac{3}{2}\right)(y - 3x + 2) = 0,$$

$$p_3(x, y) = \frac{7}{6}(y - 2x + 1)\left((y - 2x + 1)^2 - \frac{1}{7}\right) = 0.$$

Look at them from the real point of view: $(\Gamma_1)_{\mathbb{R}}$ consists of three parallel lines (symmetric with respect to the point $(\frac{1}{2}, 0)$, with slope 2, evenly spaced and with step $\frac{1}{2}$ on the y-axis. The shape is the same for $(\Gamma_2)_{\mathbb{R}}$ except for the slope, which is 3, and also for $(\Gamma_3)_{\mathbb{R}}$, in which case the slope is 2 again, but here the step on the y-axis is $\frac{1}{\sqrt{7}}$, an irrational number. Clearly, these three curves are equivalent each other from the real affine point of view, in particular, Γ_1 and Γ_3 are even similar from the Euclidean point of view. However, they are different in terms of their "geography" (having either different slopes, or different steps on the y-axis). Moreover, the difference between Γ_1 and Γ_3 becomes even more evident if we consider their arithmetic, looking at $(\Gamma_1)_{\mathbb{Q}}$ and $(\Gamma_3)_{\mathbb{Q}}$: indeed, while the former consists of three lines, the latter consist of the single line $y - 2x + 1 = 0$, since the second factor of p_3 is irreducible over $\mathbb{Q}$.

In line with Proposition 3.1, we can also obtain a characterization of pairs (X, L) as in $(\diamond)$ with $\iota_X = n - 2$ and $\mathrm{rk}\langle K_X, L\rangle = 1$ in terms of Hilbert polynomials, provided that $n \geq 6$ (see [16, Theorem 3.3]). The output is the following result.

Proposition 3.6. *Let* (X, L) *be a polarized manifold of dimension* $n \geq 6$*. Then* X *is a Fano manifold of index* $\iota_X = n - 2$ *and* $L := \frac{r}{n-2}(-K_X)$ *with* r *coprime with* $n - 2$ *if and only if* $H = \frac{1}{r}L$ *is an ample line bundle and the Hilbert polynomial of* (X, H) *is as in* (1.5) *with*

$$a_0 = \frac{1}{(n-3)!},\ a_1 = \frac{1}{n!}\left[\left(\frac{d}{2} + 1\right)n^2 - (2d + 1)n + 2d\right],\ a_2 = \frac{3d}{2\ n!}(n-2),\ a_3 = \frac{d}{n!}\ ,$$

where $d := \left(\frac{-K_X}{n-2}\right)^n$.

Remark 3.7. In Proposition 3.6 conditions $n \geq 6$ and coprimality are required only to prove the "if part". In fact, a direct check shows that the above 4-tuple

(a_0, a_1, a_2, a_3) is the solution of (1.3), regardless of the value of n, the column vector on the right hand of (1.3) being the transpose of

$$\left[\tfrac{1}{(n-3)!}, \tfrac{1}{(n-2)!}\left(n-1+\tfrac{d}{2}\right), \tfrac{2}{(n-1)!}\left(\tbinom{n}{2}+(n+2)\tfrac{d}{2}\right), \tfrac{6}{n!}\left(\tbinom{n+1}{3}+\tfrac{(n+1)(n+4)d}{4}\right)\right] .$$

Let us note that, in principle, Algorithm 1 in [16] allows one to express $P(z)$ for any pair as in $(\diamond)$, provided that $h^0(tH)$ is known for every $t = 1, \ldots, c_X$.

Finally, as to the total reducibility of P over $\mathbb{Q}$ (or $\mathbb{R}$), we have the following result.

Theorem 3.8. *Let (X, L) be as in $(\diamond)$ with $\iota_X = n-2 > 0$. Let $\Delta := 1 - \frac{8n(n-1)(n-2)^{n-2}}{(-K_X)^n}$. Then*

$$P(z) = \frac{1}{n!}\frac{(-K_X)^n}{(n-2)^n}\left[z^2 + (n-2)z + (n-2)^2\frac{(1-\Delta)}{4}\right]\left(z + \frac{n-2}{2}\right)\prod_{j=1}^{n-3}(z+j) ,$$

and the following are equivalent:

1) *P is totally reducible over $\mathbb{Q}$ (or $\mathbb{R}$);*
2) *Δ is the square of a rational number (or $\Delta \geq 0$).*

Moreover, suppose that $n \geq 4$. Then the following are equivalent:

(I) *P is totally reducible over $\mathbb{Q}$;*
(II) *either $b_2(X) = 1$ and*

$$\big(n, (-K_X)^n\big) \in \{(5, 18{\cdot}3^5), (6, 16{\cdot}4^6), (7, 14{\cdot}5^7), (8, 14{\cdot}6^8), (9, 12{\cdot}7^9), (10, 12{\cdot}8^{10})\},$$

or $b_2(X) \geq 2$ and X is one of the following Mukai manifolds (where $T_{\mathbb{P}^m}$ stands for the tangent bundle to $\mathbb{P}^m$):

(a) $\mathbb{P}^3 \times \mathbb{P}^3$;
(b) $\mathbb{P}^2 \times \mathbb{Q}^3$;
(c) $\mathbb{P}(T_{\mathbb{P}^3})$;
(d) $\mathbb{P}^1 \times \mathbb{P}^3$;
(e) $\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^2})$;
(f) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;
(g) $\mathbb{P}(\mathcal{N})$, *where $\mathcal{N}$ is the null-correlation bundle on $\mathbb{P}^3$;*

For the description of X in the case $b_2(X) = 1$ we refer to [12, Theorem 5.2.3 and Examples 5.2.2, (ii)–(v)].

Proof. From Proposition 1.1 and Proposition 2.1, it follows that

$$P(z) = \frac{1}{n!}\frac{(-K_X)^n}{\iota_X^n}(z^2 + \alpha z + \beta)\left(z + \frac{n-2}{2}\right)\prod_{j=1}^{n-3}(z+j) ,$$

where $\alpha = (n-2)$ and $\beta = (n-2)^2\frac{(1-\Delta)}{4}$ in view of Proposition 3.6 and Remark 3.7. The equivalence between 1) and 2) follows immediately from the fact that the discriminant of $z^2 + \alpha z + \beta$ is equal to $(n-2)^2\Delta$. Assume now that $n \geq 4$ and let $b_2(X) \geq 2$. Then from [12, Theorem 7.2.1] and [26] we know that $n-2 = \iota_X \leq \frac{n}{2}+1$, i.e. $n \leq 6$. If $n = 6$, then $\iota_X = \frac{n}{2} + 1$ and by [12, Theorem 7.2.2 (i)] and [26, 27] we deduce that $X \cong \mathbb{P}^3 \times \mathbb{P}^3$, hence P is totally reducible over $\mathbb{Q}$ by [5, p. 466]. If $n = 5$, then $\iota_X = \frac{n+1}{2}$ and from [12, Theorem 7.2.2 (ii)] and [26, 27] it follows that either $(-K_X)^5 = 20 \cdot 3^5$ and X is $\mathbb{Q}^3 \times \mathbb{P}^2$ or $\mathbb{P}(T_{\mathbb{P}^3})$, and in these cases P is totally reducible over $\mathbb{Q}$, since condition 2) is fulfilled, or $(-K_X)^5 = 26 \cdot 3^5$, $X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(2)\big)$, and in this case P is not totally reducible over $\mathbb{Q}$ by 2) again. Suppose that $n = 4$, hence $\iota_X = 2$. Then $\Delta = 1 - \frac{384}{(-K_X)^4} = \frac{H^4-24}{H^4}$ and by a close inspection of [12, Theorem 7.2.15 and table 12.7 at p. 225] we see that X is as in (d)–(g). Now let $b_2(X) = 1$. If P is totally reducible over $\mathbb{Q}$, by the equivalence between 1) and 2) it follows that $\Delta \geq 0$, i.e. $(-K_X)^n \geq 8n(n-1)(n-2)^{n-2}$. Moreover, by the genus formula we have $(-K_X)^n = (2g-2)(n-2)^n$, where $g := g\left(X, \frac{1}{n-2}(-K_X)\right)$ is the sectional genus of the polarized manifold $\left(X, \frac{1}{n-2}(-K_X)\right)$. This implies that

$$g \geq \frac{4n(n-1)}{(n-2)^2} + 1 = 5 + \frac{12n-16}{(n-2)^2} \ .$$

Since $n \geq 4$, we get $g \geq 6$ and by the above inequality for g and [20] and [21], combined with [18] (see also [12, Theorem 5.2.3]), we obtain that $(g, n) \in \{(10, 5), (9, 6), (8, 7), (8, 8), (7, 9), (7, 10)\}$. This gives the pairs $(n, (-K_X)^n)$ as in (II) for the case $b_2(X) = 1$. Therefore (I) implies (II). To prove the converse, it is enough to compute Δ and check condition 2), in view of the first part of the statement. □

4. Low-dimensional toric Fano manifolds

In this section we analyze the total reducibility of P over $\mathbb{Q}$ (and $\mathbb{R}$) for pairs (X, L) as in ($\diamond$) when X is a toric Fano n-fold with $n \leq 4$. First of all recall that for $n = 2$ the toric Fano manifolds are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown-up at $s \leq 3$ fixed points of the torus action. Hence those with P totally reducible over $\mathbb{Q}$ are exactly the surfaces listed in Proposition 2.2. We can thus assume that $n \geq 3$.

Assume that (X, L) is as in ($\diamond$). If $\iota_X \geq n$, then by the Kobayashi–Ochiai Theorem [12, Corollary 3.1.15] we know that $(X, H) \cong \big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)\big)$, $\big(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)\big)$ with $L = rH$ and $\iota_X = n+1, n$, respectively. Note that $\mathbb{Q}^n$ is not toric for $n \geq 3$. Furthermore, if $\iota_X = n - 1$ then we can rely on Theorem 3.3. In fact, the only toric del Pezzo n-folds are those with $n \leq 4$ in Theorem 3.3, because $\mathbb{P}^n$ is the only toric (Fano) n-fold with Picard number 1.

Now we address cases $n = 3$ and $n = 4$.

First, let X be any n-fold and write a divisor D on X as $D = \frac{1}{2}K_X + E$. If $n = 3$, then the Riemann–Roch–Hirzebruch formula can be written in the following form (see [4, (7)]):

$$\chi(D) = \frac{1}{6}E^3 + \frac{1}{24}\big(2c_2(X) - K_X^2\big) \cdot E \ . \tag{4.1}$$

Similarly, if $n = 4$, we can write (see [4, (8)]):

$$\chi(D) = \frac{1}{24}E^4 + \frac{1}{48}\big(2c_2(X) - K_X^2\big) \cdot E^2 + \frac{1}{384}\big(K_X^2 - 4c_2(X)\big) \cdot K_X^2 + \chi(\mathcal{O}_X) \ . \tag{4.2}$$

Next, let (X, L) be as in $(\diamond)$ and set $D = xK_X + yL$. Note that writing $z := ry - \iota_X x$ and $w := z + \frac{\iota_X}{2}$, as in Section 2, we have

$$E = \left(x - \frac{1}{2}\right) K_X + yL = \left(ry - \iota_X x + \frac{\iota_X}{2}\right) H = \left(z + \frac{\iota_X}{2}\right) H = wH = \frac{w}{\iota_X}(-K_X). \tag{4.3}$$

Thus, for $n = 3$ we have the following result.

Theorem 4.1. *Let (X, L) be as in $(\diamond)$ with $n = 3$. Then P is totally reducible over $\mathbb{R}$ if and only if $48 \leq (-K_X)^3 \leq 64$. Moreover, the following are equivalent:*

1) *P is totally reducible over $\mathbb{Q}$;*
2) *$(-K_X)^3 \in \{48, 50, 54, 64\}$;*
3) *X is one of the following Fano threefolds:*
   - *(a)* $\mathbb{P}^3$*;*
   - *(b)* $\mathbb{Q}^3$*;*
   - *(c)* $\mathbb{P}(T_{\mathbb{P}^2})$*, where $T_{\mathbb{P}^2}$ is the tangent bundle to $\mathbb{P}^2$;*
   - *(d)* $\mathbb{P}^1 \times \mathbb{P}^2$*;*
   - *(e)* $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$*;*
   - *(f)* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$*;*
   - *(g)* $\mathbb{P}^1 \times \mathbb{F}_1$*;*
   - *(h) X is the blow-up of $V_7 \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$ along a line lying on the exceptional divisor of the blow-up $V_7 \to \mathbb{P}^3$;*
   - *(i) X is the blow-up of $V_7 \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$ along the proper transform of a line passing through the center of the blow-up $V_7 \to \mathbb{P}^3$.*

Proof. Recall that $c_2(X) \cdot K_X = -24\chi(\mathcal{O}_X) = -24$ and $(-K_X)^3 \leq 64$ for any Fano 3-fold X. So, from (4.1) and (4.3) we get

$$Q(w) = \frac{1}{24\ \iota_X^3}\ w\Big(4(-K_X)^3 w^2 - \iota_X^2\big((-K_X)^3 - 48\big)\Big).$$

Therefore, we deduce that P is totally reducible over $\mathbb{R}$ if and only if $48 \leq (-K_X)^3 \leq 64$, which gives the first part of the statement. Furthermore, let us note that

$$P \ \text{ is totally reducible over } \mathbb{Q} \ \text{ if and only if } \ 1 - \frac{48}{(-K_X)^3} \ \text{ is}$$
$$\text{the square of a rational number} \ . \tag{4.4}$$

Then the equivalence between 1), 2) and 3) follows from (4.4) and the classification of Fano 3-folds (see [12, pp. 214–224] and [19]). □

Comparing Theorem 4.1 with the table in [1, Remark 2.5.10], we obtain the following immediate consequence for toric Fano 3-folds.

Corollary 4.2. *Let* (X, L) *be as in* $(\diamond)$ *and assume that* X *is a toric Fano* 3*-fold. Then* P *is totally reducible over* $\mathbb{Q}$ *if and only if* X *is one of the Fano* 3*-folds listed in* 3) *of Theorem 4.1, except cases (b) and (c).*

Finally, we obtain the following result for toric Fano 4-folds.

Theorem 4.3. *Let* (X, L) *be as in* $(\diamond)$ *and assume that* X *is a toric Fano* 4*-fold. Let* $k := K_X^4$ *and* $h := c_2(X) \cdot K_X^2$*. Then* P *is totally reducible over* $\mathbb{R}$ *if and only if* X *is as in [1, Table in §4] with* $h^2 \geq 96k$ *and* $k \geq 2h + 2\sqrt{h^2 - 96k}$*. Moreover, the following are equivalent:*

1) P *is totally reducible over* $\mathbb{Q}$*;*
2) *one of the cases listed in Table* 2 *occurs.*

Proof. Noting that $\chi(\mathcal{O}_X) = 1$ since X is Fano, and recalling that $k = K_X^4$ and $h = c_2(X) \cdot K_X^2$, from (4.2) and (4.3) we obtain that

$$Q(w) = \frac{1}{384\ \iota_X^4}\ \left(16kw^4 + 8\iota_X^2(2h-k)w^2 + \iota_X^4(k - 4h + 384)\right)\ ,$$

where $w := z + \frac{\iota_X}{2}$. Let Δ be the discriminant of the above biquadratic trinomial in brakets. Then

$$\frac{\Delta}{4} := 16\iota_X^4\big((2h-k)^2 - k(k-4h+384)\big) = 64\iota_X^4(h^2 - 96k).$$

Therefore,

$$P \ \text{ is totally reducible over } \mathbb{Q}\ (\mathbb{R}) \ \text{ if and only if } \ \exists\ \alpha, \beta, \gamma \in \mathbb{Q}\ (\mathbb{R}) \text{ such that}$$

$$h^2 - 96k = \alpha^2, \qquad \frac{k - 2h + 2\alpha}{k} = \beta^2 \qquad \text{and} \qquad \frac{k - 2h - 2\alpha}{k} = \gamma^2\ . \tag{4.5}$$

Observe that the first part of the statement simply follows from the first and the third conditions in (4.5) (assuming that $\alpha \geq 0$). Finally, the equivalence between 1) and 2) follows directly from [1, Table in §4], keeping in mind all the three conditions in (4.5). To simplify the check, observe that for some particular Fano 4-folds X in the table we already know that P is totally reducible over $\mathbb{Q}$: this is true for N. 1, by Proposition 3.1 (i), and for N. 2, 4, 7, and 17 because in these cases P is the product of polynomials which are totally reducible over $\mathbb{Q}$ [5, p. 466]. Note also that cases corresponding to the same values of k and h in the table are grouped by horizontal lines and they have the same α. It thus follows that, for all X belonging to a group containing one of the above cases, the three conditions in (4.5) are trivially satisfied, hence the corresponding P is totally reducible over $\mathbb{Q}$. Therefore the only

Table 2. List of the Xs as in [1, Table in §4] with $h^2 \geq 96k, k \geq 2h + 2\sqrt{h^2 - 96k}$.

$N.$	$N.$ in [1]	k	h	Description of X in [1, §4]
1	1	625	250	$\mathbb{P}^4$
2	5	512	224	$\mathbb{P}^3 \times \mathbb{P}^1$
3	6			$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1))$
4	10	486	216	$\mathbb{P}^2 \times \mathbb{P}^2$
5	20			$\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1,1))$
6	21	432	204	$\mathbb{P}^1$-bundle over $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$
7	26			$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$
8	27			$\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1))$
9	28			$\mathbb{P}^2 \times \mathbb{F}_1$
10	29			$\mathbb{P}^1$-bundle over $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$
11	52			$\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
12	31	400	196	$\mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(2,-1))$
13	32			$\mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1,-1))$
14	54			$\mathbb{P}_{\mathbb{P}^1 \times \mathbb{F}_1}(\mathcal{O} \oplus \mathcal{O}(l))$, l is a curve of index 1 on $\mathbb{F}_1$
15	68			$\mathbb{P}^1 \times W$, W is the blow-up of $\mathbb{P}^1$ on $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$
16	55	384	192	$\mathbb{F}_1 \times \mathbb{F}_1$
17	56			$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
18	57			$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{F}_1$
19	58			$\mathbb{P}^1$-bundle over the blow-up of $\mathbb{P}^1$ on $\mathbb{P}^2 \times \mathbb{P}^1$
20	69			the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ on $N.$ 6
21	72			$\mathbb{P}^1$-bundle over the blow-up of $\mathbb{P}^1$ on $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$

group for which one needs to check (4.5) is that containing $N.$ 12–15. In this case we have $h^2 - 96k = 4^2$, hence $\alpha = 4$; on the other hand $\beta^2 = \frac{1}{400}4\alpha = \left(\frac{1}{5}\right)^2$ and $k - 2h = 8 = 2\alpha$, hence $\beta = \frac{1}{5}, \gamma = 0$. □

Remark 4.4. With a case-by-case analysis, by using computer algebra programs (for instance, Magma [6] with its package "Toric Varieties" by Jarosaw Buczyski and Alexander Kasprzyk at `http://magma.maths.usyd.edu.au/magma/handbook/toric_varieties`), and relying on the database at

`http://www.grdb.co.uk/forms/toricsmooth` ,

and on the Belmans' repository

`https://github.com/pbelmans/ehrhart-polynomials-toric-fanos` containing the Ehrhart polynomials of the moment polytopes of smooth toric Fano varieties, up to dimension 7, one could study also the cases $n = 5, 6, 7$; the lists, however, become very long. In particular, for $n = 5$ one can obtain 36 cases in which P is totally reducible over $\mathbb{Q}$.

5. Fano bundles of large index

Let $X = \mathbb{P}_Y(\mathcal{E})$, where $\mathcal{E}$ is an ample vector bundle on a projective manifold Y of dimension $m \geq 2$, such that $K_Y + \det \mathcal{E} = \mathcal{O}_Y$. Bundles like X are called Fano bundles or ruled Fano manifolds (or equivalently $(Y, \mathcal{E})$ is said to be a Mukai pair, according to [13]), since X is Fano; actually, by the canonical bundle formula $-K_X = \mathrm{rk}(\mathcal{E})\xi$, ξ being the tautological line bundle, which is ample, so being $\mathcal{E}$. Moreover, since $\xi \cdot \ell = 1$ for any line ℓ contained in a fiber of the bundle projection $X \to Y$, we get $\iota_X = \mathrm{rk}(\mathcal{E})$, hence $H = \xi$. For $\mathrm{rk}(\mathcal{E}) \geq m - 2$, Fano bundles are (almost) completely classified. This comes from work of Fujita, Ye and Zhang, and Peternell for $\mathrm{rk}(\mathcal{E}) \geq m$ (see references cited in [24]); for $\mathrm{rk}(\mathcal{E}) = m - 1$, we refer to [25] for $m = 3$, [24, Theorem 0.3] for $m \geq 5$ and [24, Proposition 7.4] combined with [23] (for the elimination of a doubtful case in [24]) for $m = 4$. Case $\mathrm{rk}(\mathcal{E}) = m - 2$ has been recently studied by Kanemitsu [13] for $m \geq 5$; for $m = 4$, see [22]. In connection with our problem, here we study the Hilbert polynomial of these Fano bundles, when $\mathrm{rk}(\mathcal{E}) \geq m$, in which case their classification is complete. Notice that, according to results of Wiśniewski [26], [27], they exhaust the class of Fano manifolds X of index $\iota_X \geq (\dim X + 1)/2$ and second Betti number $b_2(X) \geq 2$.

To begin our analysis, we recall that if $\mathrm{rk}(\mathcal{E}) > m$ there is only one possible pair as above, namely $\big(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)^{\oplus(m+1)}\big)$ [8, Main Theorem]. In this case $n = \dim X = 2m$, and $(X, H) = \big(\mathbb{P}^m \times \mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^m}(1, 1)\big)$, in particular $\iota_X = m + 1$ and $L = \big(\mathcal{O}_{\mathbb{P}^m}(r)\big)^{\boxtimes 2}$. Since $\chi\big(\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^m}(k, k)\big) = \big(\chi\big(\mathcal{O}_{\mathbb{P}^m}(k)\big)\big)^2$ for every k, we have

$$P(z) = \left(\frac{1}{m!}\prod_{j=1}^{m}(z + j)\right)^2,$$

by Proposition 3.1 (i). Hence P is totally reducible over $\mathbb{Q}$.

Next suppose that $\mathrm{rk}(\mathcal{E}) = m$, so that $n = \dim(X) = 2m - 1$, and $\iota_X = c_X = m$. In this case, $(Y, \mathcal{E})$ is one of the following pairs (see [24, Theorem 0.1]):

1) $(\mathbb{P}^m, T_{\mathbb{P}^m})$, where $T_{\mathbb{P}^m}$ denotes the tangent bundle,
2) $\big(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)^{\oplus(m-1)} \oplus \mathcal{O}_{\mathbb{P}^m}(2)\big)$,
3) $\big(\mathbb{Q}^m, \mathcal{O}_{\mathbb{Q}^m}(1)^{\oplus m}\big)$.

For any pair in the list above and for any positive integer r, consider the polarized manifold $(X = \mathbb{P}_Y(\mathcal{E}), L = rH)$, where $H = \xi$, the tautological line bundle on X, mentioned above. Clearly (X, L) is as in ($\diamond$). To make the polynomial P explicit in the above cases we should determine the coefficients a_j in the factor $R(x, y)$ appearing in Proposition 1.1. This requires to compute, for any integer $t \leq m$,

$h^0(tH)$, which is equal to $h^0(S^t\mathcal{E})$, S^t standing for the t-th symmetric power, and $\delta(t) = (1+t)(2+t)\dots(t+m-1)$.

First of all, let us consider case 2). In this case we already know that P is not totally reducible over $\mathbb{Q}$ for $m = 2$ (Proposition 1.2 iv); see also Remark 3.5) and $m = 3$ (Theorem 3.8). On the other hand, we can check that this fact is true at least for small values of m by a direct computation along the line described above.

As a first thing let us compute the vector appearing on the right hand of (1.3) in general. To do that we need to determine $h^0(tH)$ for any $t = 0, 1, \dots, m = c_X$. Write $\mathcal{E} = \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^m}(2)$, where $\mathcal{F} = \mathcal{O}_{\mathbb{P}^m}(1)^{\oplus(m-1)}$. Note that

$$S^k\mathcal{F} = \mathcal{O}_{\mathbb{P}^m}(k)^{\oplus\binom{k+m-2}{m-2}}, \qquad \text{for every } k \geq 0.$$

Then

$$S^t\mathcal{E} = \bigoplus_{j=0}^{t} \left(S^j\left(\mathcal{O}_{\mathbb{P}^m}(2)\right) \otimes S^{t-j}\mathcal{F}\right) = \bigoplus_{j=0}^{t} \left(\mathcal{O}_{\mathbb{P}^m}(t+j)^{\oplus\binom{t-j+m-2}{m-2}}\right) .$$

Therefore,

$$h^0(tH) = h^0(S^t\mathcal{E}) = \textstyle\sum_{j=0}^{t} \binom{t-j+m-2}{m-2} h^0\left(\mathcal{O}_{\mathbb{P}^m}(t+j)\right) = \sum_{j=0}^{t} \binom{t-j+m-2}{m-2}\binom{t+j+m}{m}.$$

In conclusion, recalling the expression of $\delta(t)$ in Proposition 1.1, we see that the $(t+1)$-th component of the column vector on the right hand of (1.3) is

$$\frac{t!}{(t+m-1)!}\sum_{j=0}^{t}\binom{t-j+m-2}{m-2}\binom{t+j+m}{m}$$

for $t = 0, 1, \dots, m$.

For example, let $m = 2$; then $\delta(t) = t+1$, moreover, $h^0(H) = 6+3 = 9$ and $h^0(2H) = 15+10+6 = 31$. Thus the column vector on the right hand of (1.3) is the transpose of $[1 \quad \frac{9}{2} \quad \frac{31}{3}]$. Since in the present case the inverse of the matrix in (1.4) is

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} , \tag{5.1}$$

we get $[a_0 \quad a_1 \quad a_2] = \begin{bmatrix} 1 & \frac{7}{3} & \frac{7}{6} \end{bmatrix}$, hence the first factor of $P(z)$ in (1.5) is $\frac{1}{6}(7z^2 + 14z + 6)$. Note that this trinomial is exactly that appearing in Proposition 3.1 (iii), before the product, in the situation corresponding to the case at hand. Moreover, it is clearly not totally reducible over $\mathbb{Q}$. The same happens for $m = 3$, as shown in the proof of Theorem 3.8. In this case the first factor of $P(z)$ in (1.5) is $\frac{1}{120}(26z^3 + 117z^2 + 157z + 60)$, which is totally reducible over $\mathbb{R}$ but not over $\mathbb{Q}$. Furthermore, for $4 \leq m \leq 150$, one can use the following Magma Program [6]:

```
K:=Rationals();
P<x>:=PolynomialRing(K,1);
V:=function(m);
```

```
L:=[];
 for k in [1..m+1] do
  L:=L cat [1];
 end for;
 for j in [1..m] do
  L:=L cat [i^j : i in [0..m]];
 end for;
return Matrix(K,m+1,m+1,L);
end function;
v:=function(m);
L:=[];
 for t in [0..m] do
  L:= L cat [(Factorial(t)/Factorial(t+m-1))*
  (&+[Binomial(t-j+m-2,m-2)*Binomial(t+j+m,m): j in [0..t]])];
 end for;
return Matrix(K,1,m+1,L);
end function;
Test:=function(N,NN);
M:={};
 for m in [N..NN] do
  m; a:=v(m)*V(m)^(-1); p:=P!&+[a[1,j+1]*x^(j) : j in [0..m]];
  b:=&+[Factorization(p)[k][2]: k in [1..#Factorization(p)]];
  if b eq m then
   M:=M join {<m, p>};
  end if;
 end for;
return M;
end function;
```

Typing

```
Test(4,150);
```

in the Magma calculator, one can check that for $4 \leq m \leq 150$ the factor of $P(z)$ corresponding to the polynomial $R(x, y)$ is always not totally reducible over $\mathbb{Q}$.

Next, let us settle case 3). Since $(Y, \mathcal{E}) = \big(\mathbb{Q}^m, \mathcal{O}_{\mathbb{Q}^m}(1)^{\oplus m}\big)$, we have that $(X, H) = \big(\mathbb{Q}^m \times \mathbb{P}^{m-1}, \mathcal{O}_{\mathbb{Q}^m \times \mathbb{P}^{m-1}}(1,1)\big)$, hence $L = \mathcal{O}_{\mathbb{Q}^m}(r) \boxtimes \mathcal{O}_{\mathbb{P}^{m-1}}(r)$. Then $\chi\big(\mathcal{O}_{\mathbb{Q}^m \times \mathbb{P}^{m-1}}(k,k)\big) = \chi\big(\mathcal{O}_{\mathbb{Q}^m}(k)\big)\chi\big(\mathcal{O}_{\mathbb{P}^{m-1}}(k)\big)$ for every k. So, according to Proposition 3.1 (i) and (ii), we get

$$P(z) = \frac{2}{m!(m-1)!} \left(z + \frac{m}{2}\right) \left(\prod_{j=1}^{m-1} (z+j)\right)^2 .$$

Therefore, in case 3) P is totally reducible over $\mathbb{Q}$.

In case 1) we have already seen that P is totally reducible over $\mathbb{Q}$ for $m = 2$ (Theorem 3.3 and Proposition 3.4; see also Table 3) and $m = 3$ (Theorem 3.8). To deal with case 1) in general, note that our pair $(X, H) = \big(\mathbb{P}(T_{\mathbb{P}^m}), \xi\big)$ (ξ being the tautological line bundle) is the general hyperplane section of $\big(\mathbb{P}^m \times \mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^m}(1, 1)\big)$. So denoting this last pair by $(\mathcal{X}, \mathcal{H})$, we have that $X \in |\mathcal{H}|$ and $H = \mathcal{H}_X$. Hence we can use the following

Lemma 5.1. *Let $(\mathcal{X}, \mathcal{H})$ be a polarized manifold such that $|\mathcal{H}|$ contains a smooth element X and let $H = \mathcal{H}_X$. Set $L = rH$, and $\mathcal{L} = r\mathcal{H}$ for some positive integer r. Then*

$$p_{(X,L)}(x, y) = p_{(\mathcal{X},\mathcal{L})}\left(x, \frac{x}{r} + y\right) - p_{(\mathcal{X},\mathcal{L})}\left(x, \frac{x-1}{r} + y\right).$$

Proof. We have $xK_X + yL = \big(x(K_{\mathcal{X}} + \mathcal{H}) + ry\mathcal{H}\big)_X = \big(xK_{\mathcal{X}} + (x + ry)\mathcal{H}\big)_X$, by adjunction. Tensoring the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(-\mathcal{H}) \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_X \to 0$$

by $xK_{\mathcal{X}} + (\frac{x}{r} + y)\mathcal{L}$ we thus get

$$0 \to \mathcal{O}_{\mathcal{X}}\left(xK_{\mathcal{X}} + (\frac{x}{r} + y - \frac{1}{r})\mathcal{L}\right) \to \mathcal{O}_{\mathcal{X}}\left(xK_{\mathcal{X}} + (\frac{x}{r} + y)\mathcal{L}\right) \to \mathcal{O}_X(xK_X + yL) \to 0.$$

Therefore,

$$\chi(xK_X + yL) = \chi\left(xK_{\mathcal{X}} + (\frac{x}{r} + y)\mathcal{L}\right) - \chi\left(xK_{\mathcal{X}} + (\frac{x}{r} + y - \frac{1}{r})\mathcal{L}\right),$$

which proves the assertion. □

Now apply Lemma 5.1 to case 1). We have

$$p_{(\mathcal{X},\mathcal{L})}\left(x, \frac{x}{r} + y\right) = \left(\frac{1}{m!}\right)^2 \left(\prod_{j=1}^{m}(ry - mx + j)\right)^2$$

and

$$p_{(\mathcal{X},\mathcal{L})}\left(x, \frac{x-1}{r} + y\right) = \left(\frac{1}{m!}\right)^2 \left(\prod_{j=1}^{m}(ry - mx + j - 1)\right)^2.$$

Set $z := ry - mx$. Then

$$\begin{aligned}
P(z) &= \left(\frac{1}{m!}\right)^2 \left[\left(\prod_{j=1}^{m}(z+j)\right)^2 - \left(\prod_{j=1}^{m}(z+j-1)\right)^2\right] \\
&= \left(\frac{1}{m!}\right)^2 \left[\prod_{j=1}^{m}(z+j) - \prod_{j=1}^{m}(z+j-1)\right] \cdot \left[\prod_{j=1}^{m}(z+j) + \prod_{j=1}^{m}(z+j-1)\right] \\
&= \left(\frac{1}{m!}\right)^2 \left[m \prod_{j=1}^{m-1}(z+j)\right] \cdot \left[(2z+m) \prod_{j=1}^{m-1}(z+j)\right] \\
&= \left(\frac{1}{m!}\right)^2 m\ (2z+m) \left(\prod_{j=1}^{m-1}(z+j)\right)^2 .
\end{aligned}$$

In conclusion, $P(z)$ has the same expression as in case 3). Therefore P is totally reducible over $\mathbb{Q}$ even in case 1) of the list.

The above discussion proves the following results.

Theorem 5.2. *Let (X, L) be as in $(\diamond)$ with X being a Fano bundle of index $\iota_X \geq \frac{n+1}{2}$. Suppose that either $n \geq 4$ is even, or $n = 2m - 1 \geq 3$ and $X \not\cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^m}(1)^{\oplus m-1} \oplus \mathcal{O}_{\mathbb{P}^m}(2)\right)$. Then the polynomial P is totally reducible over $\mathbb{Q}$.*

Proposition 5.3. *Let (X, L) be as in $(\diamond)$ with $n \leq 299$ and X being a Fano bundle of index $\iota_X \geq \frac{n+1}{2}$. Then the following are equivalent:*

(i) the polynomial P is totally reducible over $\mathbb{Q}$;
(ii) $X \not\cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^{(n+1)/2}}(1)^{\oplus(n-1)/2} \oplus \mathcal{O}_{\mathbb{P}^{(n+1)/2}}(2)\right)$.

Finally, in line with Proposition 5.3, let us state here the following conjecture.

Conjecture 1. Let (X, L) be as in $(\diamond)$, $X = \mathbb{P}(\mathcal{E})$ being an n-dimensional Fano bundle over Y, with $\mathrm{rk}(\mathcal{E}) \geq \dim Y$ (or equivalently, $\iota_X \geq \frac{n+1}{2}$). Then the polynomial P is totally reducible over $\mathbb{Q}$ if and only if $(Y, \mathcal{E})$ is not as in case 2).

One more remark concerning case 2). Relying on computational experiments done with Magma for low values of m we formulate also the following conjecture.

Conjecture 2. For (X, L) as in case 2), the factor of $P(z)$ corresponding to $R(x, y)$ in Proposition 1.1 is totally reducible over $\mathbb{R}$ for any m, but it has either no rational zero or a single rational zero, namely $-\frac{m}{2}$, according to whether m is either even or odd, respectively.

Acknowledgments

The authors are indebted to Prof. Pieter Belmans for calling to their attention the repository quoted in Remark 4.4, containing the Hilbert polynomials of toric Fano n-folds with $n \leq 7$. The first author is a member of G.N.S.A.G.A. of the Italian INdAM. He would like to thank the PRIN 2015 Geometry of Algebraic Varieties and the University of Milano for partial support. During the preparation of this paper, the second author was partially supported by the Proyecto VRID N.219.015.023-INV of the University of Concepción.

References

[1] V.V. Batyrev, *On the classification of toric Fano 4-folds*, J. Math. Sc. **94** (1999) 1021–1050.

[2] M. Beck, J.A. De Loera, M. Develin, J. Pfeifle, and R. Stanley, *Coefficients and roots of Ehrhart polynomials*, In Integer points in polyhedra – geometry, number theory, algebra, optimization, Contemporary Mathematics, vol. 374, pp. 15–36, Amer. Math. Soc., 2005.

[3] P. Belmans, S, Galkin, and S. Mukhopadhyay, *Examples violating Golishev's strip hypothesis*, Exp. Math. DOI:10.1080/10586458,2019.1602571.

[4] M.C. Beltrametti. A. Lanteri, and M. Lavaggi, *Hilbert surfaces of bipolarized varieties*, Rev. Roum. Math. Pures Appl. **60**(3) (2015) 281–319.

[5] M.C. Beltrametti, A. Lanteri, and A.J. Sommese, *Hilbert curves of polarized varieties*, J. Pure Appl. Algebra **214** (2010) 461–479.

[6] W. Bosna, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997) 235–265.

[7] T. Fujita, *Classification Theories of Polarized Varieties*, London Mathematical Society Lecture Note Series, vol. 155 (Cambridge University Press, 1990).

[8] T. Fujita, *On adjoint bundles of ample vector bundles*, in Complex Algebraic Varieties, Bayreuth, 1990, pp. 105–112, Lecture Notes in Math. vol. 1507 (Springer, 1992).

[9] V. V. Golyshev, *On the canonical strip*, Uspekhi Mat. Nauk **64**, no. 1 (385), 139–140; transl in Russian Math. Surveys **64** (2009) 145–147.

[10] G. Hegedüs, A. Higashitani, and A. M. Kasprzyk, *Ehrhart polynomial roots of reflexive polytopes*, Electron. J. Combin. **26** (2019)(1) Paper No. 1.38, 27 pp.

[11] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, 3^{rd} Ed. (Springer, 1966).

[12] V.A. Iskovskikh and Yu. G. Prokhorov, *Fano Varieties*, Algebraic Geometry V, Encyclopaedia of Mathematical Sciences, vol. 47, Springer, 1999.

[13] A. Kanemitsu, *Fano n-folds with ample vector bundles of rank $n-2$ whose adjoint bundles are trivial*, Ann. Inst. Fourier **69**(1) (2019) 231–282.

[14] A. Lanteri, *Characterizing scrolls via the Hilbert curve*, Internat. J. Math. **25**(11) (2014) 1450101 [17 pages].

[15] A. Lanteri, *Hilbert curves of quadric fibrations*, Internat. J. Math. **29**(10) (2018) 1850067 [20 pages].

[16] A. Lanteri and A.L. Tironi, *Characterizing some polarized Fano fibrations via Hilbert curves*, J. Algebra Appl. (2022), to appear.

[17] L. Manivel, *The canonical strip phenomenon for complete intersections in homogeneous spaces*, arXiv:0904.2470v1.

[18] M. Mella, *Existence of good divisors on Mukai varieties*, J. Algebraic Geom. **8**(2) (1999) 197–206.
[19] S. Mori and S. Mukai, *Erratum to "Classification of Fano* 3*-folds with* $B_2 \geq 2$*, Manuscripta Math.* ***36*** (1981), 147–162*"*, Ibidem **110** (2003) 407.
[20] S. Mukai, *Biregular classification of Fano threefolds and Fano manifolds of coindex* 3, Proc. Nat. Acad. Sci. U.S.A. **86** (1989) 3000–3002.
[21] S. Mukai, *Fano* 3*-folds*, London Mathematical Society Lecture Note Series, vol. 179, Cambridge Univ. Press, 1992, pp. 255–263.
[22] C. Novelli and G. Occhetta, *Ruled Fano fivefolds of index two*, Indiana Univ. Math. J. **56**(1) (2007) 207–241.
[23] G. Occhetta, *A note on the classifcation of Fano manifolds of middle index*, Manuscripta Math. **117**(1) (2005) 43–49.
[24] Th. Peternell, M. Szurek and J.A. Wiśniewski, *Fano manifolds and vector bundles*, Math. Ann. **294** (1992) 151–165.
[25] J.A. Wiśniewski, *Ruled Fano* 4*-folds of index* 2, Proc. Amer. Math. Soc. **105** (1989) 55–61.
[26] J.A. Wiśniewski, *On a conjecture of Muka*i. Manuscripta Math. **68**(2) (1990) 135–141.
[27] J.A. Wiśniewski, *On Fano manifolds of large index.* Manuscripta Math. **70**(2) (1991) 145–152.