RESEARCH ARTICLE

On real Calabi–Yau threefolds twisted by a section

Diego Matessi [ORCID]

Dipartimento di Matematica, Università
degli Studi di Milano, Milan, Italy

Correspondence
Diego Matessi, Dipartimento di
Matematica, Università degli Studi di
Milano, Via Saldini 50, I-20133 Milan,
Italy.
Email: diego.matessi@unimi.it

Abstract

We study the mod 2 cohomology of real Calabi–Yau threefolds given by real structures that preserve the torus fibrations constructed by Gross. We extend the results of Castaño-Bernard–Matessi and Arguz–Prince to the case of real structures twisted by a Lagrangian section. In particular, we find exact sequences linking the cohomology of the real Calabi–Yau with the cohomology of the complex one. Applying Strominger–Yau–Zaslow mirror symmetry, we show that the connecting homomorphism is determined by a "twisted squaring of divisors" in the mirror Calabi–Yau, that is, by $D \mapsto D^2 + DL$ where D is a divisor in the mirror and L is the divisor mirror to the twisting section. We use this to find an example of a connected $(M - 2)$-real quintic threefold.

MSC 2020
14J33, 14P25 (primary), 53D12 (secondary)

1 | INTRODUCTION

A real structure on a complex manifold X is an anti-holomorphic involution $\iota : X \to X$, as, for example, conjugation on an algebraic variety $X \subset \mathbb{CP}^{n+1}$ defined over $\mathbb{R}$. The real part of X is the fixed point set of ι, which we denote by Σ. Understanding the topology of Σ is a notoriously difficult problem, see, for instance, Wilson's classical survey on Hilbert's sixteenth problem [25]. A remarkable method to construct real hypersurfaces in toric varieties with controlled topology is Viro's patchworking technique [19, 23, 24], which laid the foundations for the field of

tropical geometry. Indeed patchworking allowed the construction of many interesting examples and counterexamples, especially in the case of curves and surfaces [15, 16].

One of the questions one may ask is about the relationship between the topologies of Σ and X. For instance, a famous result is the Smith–Thom inequality relating the $\mathbb{Z}_2$ Betti numbers of Σ and X:

$$\sum b_j(\Sigma, \mathbb{Z}_2) \leqslant \sum b_j(X, \mathbb{Z}_2).$$

When the equality is satisfied, then Σ is said to be maximal, or an M-real hypersurface [25, Definition 3.2]. We say it is of type $(M - k)$ if the difference between the two sums of Betti numbers is $2k$. There are many examples of maximal hypersurfaces in the case of curves and surfaces, but little is known in higher dimensions [16]. Another problem is to find bounds on individual Betti numbers. For instance, a sharp bound on individual Betti numbers of real surfaces in $\mathbb{RP}^3$ is unknown in high degrees [17].

One may investigate the same questions for real hypersurfaces constructed via patchworking. In this context, Itenberg [17] conjectured that if Σ is a hypersurface in $\mathbb{RP}^{n+1}$ constructed by primitive patchworking then

$$b_q(\Sigma, \mathbb{Z}_2) \leqslant \begin{cases} h^{q,q}(X) \text{ if } q = n/2, \\ h^{q,n-q}(X) + 1 \text{ otherwise.} \end{cases}$$

This conjecture has been recently proved by Renaudineau and Shaw [20], who actually proved a more general version for real hypersurfaces in toric varieties (see inequalities (24)). In this paper, we investigate similar questions but for real structures arising in a different context.

1.1 | Lagrangian fibrations with real structures

Our goal is to generalize the results of Castaño–Bernard and Matessi [6] and Arguz and Prince [2] on the cohomology of real Calabi–Yau threefolds constructed via Lagrangian torus fibrations. In this context, (X, ω) is a (real) $2n$-dimensional symplectic manifold, with symplectic form ω, together with a Lagrangian torus fibration $f : X \to B$ onto a real n-dimensional manifold B. A compatible real structure is an anti-symplectic involution ι, that is, $\iota^*\omega = -\omega$, which preserves the fibers of the torus fibration. The real variety Σ is the fixed point set of ι. Let $\pi : \Sigma \to B$ be the restriction of f to Σ. The general idea in [2, 6] and in this paper is to relate the cohomology with $\mathbb{Z}_2$ coefficients of X and Σ by comparing the Leray spectral sequences associated to f and π.

The torus fibrations that we consider in this paper are those constructed topologically by Gross in [11] starting from the data of a 3-dimensional affine manifold with singularities B. It follows from [12] that one can construct these fibrations over affine manifolds with singularities associated to Calabi–Yau hypersurfaces or complete intersections in toric Fano varieties. It is expected, although not yet proved, that X is homeomorphic to the corresponding Calabi–Yau. In the case of the quintic threefold in $\mathbb{P}^4$, this has been proved by Gross in [11]. The motivation behind these torus fibrations comes from mirror symmetry, which is a highly nontrivial duality between a pair of Calabi–Yau manifolds X and $\check{X}$. There are various constructions of mirror pairs of Calabi–Yau manifolds, one of the earliest is [5], and the duality can be studied at different levels of complexity [4]. The most basic manifestation of mirror symmetry is the exchange of Hodge numbers $h^{p,q}(X) = h^{n-p,q}(\check{X})$. The Strominger–Yau–Zaslow (SYZ) conjecture [22] suggested that a pair of

mirror Calabi–Yau manifolds should be related by the process of dualizing (special) Lagrangian torus fibrations. The main feature of Gross' fibrations is that they are built to naturally incorporate SYZ mirror symmetry at a topological level (higher levels should include dualizing higher structures). In fact there is a standard procedure to dualize the torus fibrations to obtain the mirror Calabi–Yau $\check{X}$ together with a fibration $\check{f} : \check{X} \to B$ in such a way to reproduce the exchange of Hodge numbers (see Section 2 for more details).

In [7], it was shown that Gross' fibrations could be made into Lagrangian fibrations with respect to a symplectic form extending the natural one existing on the union of smooth fibers. These fibrations also come with a Lagrangian zero section $\sigma_0 : B \to M$.

A family of fiber preserving real structures on X was constructed in [8]. We have the "standard real structure" that fixes the zero section. Denote the corresponding real Calabi–Yau by Σ. Then Σ has at least two connected components, one of them being the zero section, isomorphic to B. Given a Lagrangian section $\tau : B \to X$, one can "twist" the standard real structure to get another real structure ι_τ (formula (7)). If τ is not the square of another section, then ι_τ does not fix a section and therefore it is not standard (see Subsection 3.2). Let us denote by Σ_τ the corresponding real Calabi–Yau and by $\pi_\tau : \Sigma_\tau \to B$ the restriction of f to Σ_τ. We expect, although this has to be proved, that such real structures on X are isomorphic to algebraic ones, possibly arising from some patchworking. The results in [6] and [2] concern the topology of the standard real Calabi–Yau. In this paper, we generalize to the twisted case.

1.2 │ The Leray spectral sequence and mirror symmetry

The Leray spectral sequence of a Gross fibration was investigated in [9–11]. Given some group of coefficients G, we have the sheaves $R^p f_* G$ on B, whose stalk at a point b is the cohomology of the fiber F_b, that is, $H^p(F_b, G)$. The second page of the Leray spectral sequence is given by $E_2^{q,p} = H^q(B, R^p f_* G)$. It is a feature of mirror symmetry, that is, of the duality between the torus fibrations $f : X \to B$ and $\check{f} : \check{X} \to B$, that we have the following isomorphism (see Subsection 2.7)

$$H^q(B, R^p f_* G) \cong H^q(B, R^{n-p} \check{f}_* G).$$

Gross shows that for various choices of G (e.g., $G = \mathbb{Q}, \mathbb{Z}$ or $\mathbb{Z}_p$) and with some assumptions on B, X and $\check{X}$, the spectral sequence degenerates at the E_2 page. In this case, the cohomology of X can be read off from the E_2 page. In particular, the Hodge numbers of X satisfy

$$h^{p,q}(X) = \dim H^q(B, R^p f_* \mathbb{Q}).$$

This equality holds in higher dimensions and it has been proved in more generality in [14]. Notice that together with the above mirror symmetry isomorphism, this implies the famous relationship between the Hodge numbers of mirror Calabi–Yau manifolds $h^{p,q}(X) = h^{n-p,q}(\check{X})$.

1.3 │ Main results

Let B_0 be the set of regular values of f, so that for every $b \in B_0$, the fiber $F_b = f^{-1}(b)$ is a smooth n-dimensional torus. By the Arnold–Liouville theorem, F_b is of the type V/Λ^* where V is an

affine space modeled on $T^*_b B_0$ and $\Lambda^* \cong \mathbb{Z}^n$ is an n-dimensional lattice in $T^*_b B_0$. It follows that a compatible real structure ι on X, restricted to the fiber F_b, acts as reflection with respect to some point on V (i.e., as the affine map $y \mapsto y + 2(o - y)$ for some point $o \in V$). In particular, $\pi^{-1}(b) = \Sigma \cap F_b$ consists of 2^n points that have the structure of an n-dimensional affine space defined over $\mathbb{Z}_2$. In the case of Gross' fibrations, $\pi^{-1}(b)$ is finite for all $b \in B$. In particular, the Leray spectral sequence of π is trivial: the cohomology of Σ satisfies

$$H^q(\Sigma, \mathbb{Z}_2) \cong H^q(B, \pi_* \mathbb{Z}_2).$$

Our results consider the case when X is a Calabi–Yau threefold, that is, $n = 3$. The first result is the following.

Theorem 1. *Let τ be a Lagrangian section of $f : X \to B$ and ι_τ the associated real structure. There exist sheaves $\mathcal{L}^1_\tau$ and $\mathcal{L}^2_\tau$ over B and a short exact sequence*

$$0 \longrightarrow \mathcal{L}^1_\tau \longrightarrow \pi_{\tau_*} \mathbb{Z}_2 \longrightarrow \mathcal{L}^2_\tau \longrightarrow 0,$$

such that $\mathcal{L}^1_\tau$ and $\mathcal{L}^2_\tau$ are related to the topology of X by the following short exact sequences

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{L}^1_\tau \longrightarrow R^1 f_* \mathbb{Z}_2 \longrightarrow 0,$$

$$0 \longrightarrow R^2 f_* \mathbb{Z}_2 \longrightarrow \mathcal{L}^2_\tau \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Notice that at a regular value $b \in B_0$, we have

$$(\pi_{\tau_*} \mathbb{Z}_2)_b = H^0(\pi_\tau^{-1}(b), \mathbb{Z}_2) = \text{Maps}(\pi_\tau^{-1}(b), \mathbb{Z}_2).$$

The sheaf $\mathcal{L}^1_\tau$ in b coincides with the affine maps. This also explains the second sequence, which is the usual splitting of affine functions as the sum of a constant function and a linear function. In the case of the standard real structure, the first sequence coincides with the one found in [6]. Indeed in this case $\pi^{-1}(b)$ is naturally a vector space, as the zero section defines an origin. This implies that the second and third sequence are both split.

The first sequence gives a long exact sequence in cohomology that computes the $\mathbb{Z}_2$ cohomology of Σ_τ. In particular, we have the connecting homomorphism

$$\beta : H^1(B, \mathcal{L}^2_\tau) \to H^2(B, \mathcal{L}^1_\tau).$$

By composing β with the morphisms from the second and third sequence (see diagram (18)) we get the homomorphism

$$\beta' : H^1(B, R^2 f_* \mathbb{Z}_2) \to H^2(B, R^1 f_* \mathbb{Z}_2).$$

It was shown in [6] that in the case B is a homology $\mathbb{Z}_2$-sphere and $H^1(X, \mathbb{Z}_2) = H^1(\check{X}, \mathbb{Z}_2) = 0$, then the untwisted Σ has exactly two connected components. Under the same hypothesis, we prove here that in the twisted case, Σ_τ is connected. In both cases the cohomology of Σ_τ is uniquely determined by β'. As a corollary of this construction we also get that if in addition the integral cohomologies of X and $\check{X}$ have no torsion, then the Betti numbers of Σ_τ satisfy the same bounds as those proved by Renaudineau–Shaw (inequalities (24)). Indeed, in the twisted case the bound

is stronger: a twisted Σ_τ can be at most of type $(M-2)$ and this happens if and only if β' is the zero map.

If we apply the mirror symmetry isomorphism we can view β' as a map $\beta' : H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \to H^2(B, R^2 \check{f}_* \mathbb{Z}_2)$ on the cohomology of the mirror $\check{X}$. We now have that a Lagrangian section τ can be naturally viewed as a class $\tau \in H^1(B, R^2 f_* \mathbb{Z}_2)$. If we apply mirror symmetry to τ, we get an element $L_\tau \in H^1(B, R^1 \check{f}_* \mathbb{Z}_2)$. Notice that the latter group can be interpreted as a $\mathbb{Z}_2$ version of the Picard group of $\check{X}$. It is expected that Lagrangian sections on X should be mirror to line bundles on $\check{X}$. Although we do not have in this context an explicit correspondence, we view L_τ as the $\mathbb{Z}_2$-reduction of the first Chern class of the conjectured line bundle that is mirror to τ. By abuse of notation we will refer to L_τ as the mirror line bundle.

We can now state the main theorem of this paper.

Theorem 2. *The map* $\beta' : H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \to H^2(B, R^2 \check{f}_* \mathbb{Z}_2)$ *coincides with the map*

$$S_\tau : H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \longrightarrow H^2(B, R^2 \check{f}_* \mathbb{Z}_2)$$

$$D \longmapsto D^2 + DL_\tau.$$

This theorem generalizes the main result in [2], which proves the untwisted case, that is, when $L_\tau = 0$.

As an application we find a connected $(M-2)$-real quintic. We use the torus fibration on a quintic in $\mathbb{P}^4$ constructed by Gross in [11]. Then, on the mirror quintic $\check{X}$, we have that $H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \cong H^2(\check{X}, \mathbb{Z}_2)$, which coincides with the Picard group mod 2. To find an $(M-2)$-real quintic it is then enough to find an $L \in H^2(\check{X}, \mathbb{Z}_2)$ such that $D^2 + DL = 0$ for all $D \in H^2(\check{X}, \mathbb{Z}_2)$. The real quintic will then be Σ_τ, such that $L = L_\tau$. We find such an L by using the explicit description of the triple cup product form $(\alpha, \beta, \gamma) \mapsto \alpha \smile \beta \smile \gamma$ on $H^2(\check{X}, \mathbb{Z})$ given in [11]. Arguz and Prince have computed the Betti numbers of the untwisted real quintic Σ, obtaining $b_1(\Sigma) = 29$. In particular, Σ is far from being maximal, therefore none of the real quintics constructed with this method is maximal.

1.4 | Structure of the paper

In Section 2, we review the necessary background on Gross' construction of torus fibrations, topological mirror symmetry and the Leray spectral sequence. In Section 3, we recall the setup in [8] where the standard and twisted real structures are defined and we discuss the results in [6] and [2] for the standard real structure. In Section 4, we prove Theorem 1 (i.e., Theorem 4.1). In Section 5, we prove Theorem 2 (i.e., Theorem 5.1). In Section 6, we prove some consequences, such as connectedness and bounds on the Betti numbers. In Section 7, we discuss the relationship between our short exact sequences and the spectral sequence constructed by Renaudineau and Shaw. In Section 8, we explain the construction of the connected $(M-2)$ real quintic.

2 | LAGRANGIAN FIBRATIONS AND MIRROR SYMMETRY

We explain the construction of Lagrangian torus fibrations starting from the data of an integral affine manifold with singularities. The topological construction was done by Gross [11] for the

threefold case and an extension to all dimensions was announced by Ruddat and Zharkov [21]. It was shown in [7] that a variant of Gross' topological fibrations are indeed Lagrangian with respect to a symplectic form on X, induced by the integral affine structure on B.

Definition 2.1. An integral affine manifold with singularities is a triple $(B, \Delta, \mathcal{A})$ where B is an n-dimensional topological manifold; $\Delta \subset B$ a closed, codimension 2 subset and $\mathcal{A}$ a maximal atlas on $B_0 = B - \Delta$ whose change of coordinate maps are in $\mathbb{R}^n \rtimes \mathrm{SL}(\mathbb{Z}, n)$. The set Δ is called the discriminant locus.

Given $B_0 = B - \Delta$, we denote by $j : B_0 \to B$ the inclusion. The cotangent bundle $T^* B_0$ carries the standard symplectic form, moreover we have the lattice

$$\Lambda^* = \mathrm{span}_{\mathbb{Z}} \langle dx_1, \dots, dx_n \rangle,$$

where $(x_1, \dots, x_n)$ are local integral affine coordinates. This defines the symplectic manifold

$$X_0 = T^* B_0 / \Lambda^*$$

together with the Lagrangian torus fibration $f_0 : X_0 \to B_0$ given by the standard projections. A (partial) compactification of X_0 is given by a $2n$-dimensional manifold X together with map $f : X \to B$ and a commutative diagram

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\
B_0 & \xrightarrow{\;j\;} & B
\end{array}
\tag{1}
$$

where the top arrow is a homeomorphism onto its image. In dimension $n = 3$, Gross shows that under certain hypothesis on the set Δ and the affine structure around it, such a compactification can be carried out topologically in a canonical way. In the same dimension and with the same hypothesis, Castaño–Bernard and Matessi [7] prove that, after a small thickening of Δ, X has a symplectic structure such that the inclusion $X_0 \to X$ is a symplectomorphism and f is a Lagrangian fibration.

The hypothesis on Δ and on the affine structure is that they are locally isomorphic to certain prescribed local models. When these hypothesis are satisfied we will say that B is *simple*. We describe below the local models in dimensions 2 and 3. In higher dimensions there is a longer list. The models are characterized by two key properties of the monodromy of Λ^* around Δ that are called simplicity and positivity, see [13] for further details.

2.1 | Dimension 2: Focus–focus points

In dimension 2, we ask for Δ to be a discrete set of points with an affine structure around them locally isomorphic to the one depicted in Figure 1. Such singular points are called focus–focus points. More precisely let B consist of two simplices glued along an edge and let the singular point be the barycenter of the common edge, that is, the (red) cross in Figure 1. Two charts are defined on the open subsets given by removing the "cuts" depicted in Figure 1 as the (blue) wiggled segments. The intersection between the two charts is given by removing the whole edge. The two sides of
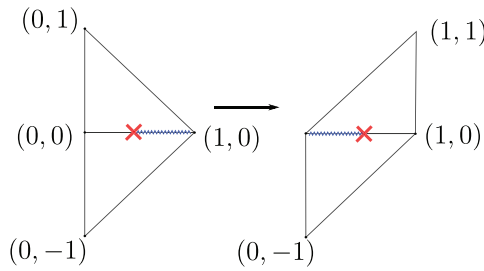
FIGURE 1 Charts around a focus–focus point.

Figure 1, viewed as embedded in $\mathbb{R}^2$, give the affine coordinates on these two charts. The change of coordinates map is given by the unique integral affine maps that match the two simplices on the left with the two on the right. More detailed descriptions are given in [13, Example 1.16] or [7, Example 3.7].

2.2 | The affine quartic

This is a global compact example. Take the simplex P in $\mathbb{R}^3$ whose vertices are

$$p_0 = (-1, -1, -1), \quad p_1 = (3, -1, -1),$$
$$p_2 = (-1, 3, -1), \quad p_3 = (-1, -1, 3).$$

We take $B = \partial P$. Each edge of P contains five integral points and is subdivided by these into four segments. Define Δ to be the set of midpoints of these segments. In total Δ consists of 24 points. We can define charts on B as follows. We have four charts consisting of the interior of each 2-face of P together with their natural integral affine structure. For every integral point q on some edge consider a neighborhood $U_q \subset B - \Delta$. Define a chart by the projection $U_q \to \mathbb{R}^3/\mathbb{R}q$, where $\mathbb{R}q$ is the line generated by q. By choosing these neighborhoods so that they cover B_0, we obtain an integral affine structure on B_0.

Every $p \in \Delta$ is of focus–focus type. One can prove this as follows. Let q_1 and q_2 two consecutive integral points on an edge of P such that p is the midpoint between them. There are two 2-faces of ∂P containing the segment between q_1 and q_2. Choose, on each of these faces, integral points q_3 and q_4, respectively, such that $\{q_1, q_2, q_3\}$ and $\{q_1, q_2, q_4\}$ both form bases of $\mathbb{Z}^3$. We must have that $q_4 = aq_1 + bq_2 - q_3$ for some $a, b \in \mathbb{Z}$. Notice that the union of the two triangles with vertices $\{q_1, q_2, q_3\}$ and $\{q_1, q_2, q_4\}$, respectively, forms a neighborhood of p in B. Consider the two quotients $\mathbb{R}^3/\mathbb{R}q_1$ and $\mathbb{R}^3/\mathbb{R}q_2$. On the first quotient choose the basis $e_1 = [q_2 - q_1]$ and $e_2 = [q_3 - q_1]$ and on the second quotient the basis $f_1 = [q_2 - q_1]$ and $f_2 = [q_3 - q_2]$. On both coordinate systems fix the origin to be $[q_1]$. Then the two triangles are represented in Figure 2: on the left with respect to the first quotient and on the right with respect to the second one. The change of coordinates map is given by the unique (piecewise) affine map that matches the two pairs of triangles. Notice that P is a Delzant polytope and this implies that $a + b = 1$. Therefore, the two bottom triangles are identified by the identity map. This shows that p is a focus–focus point.
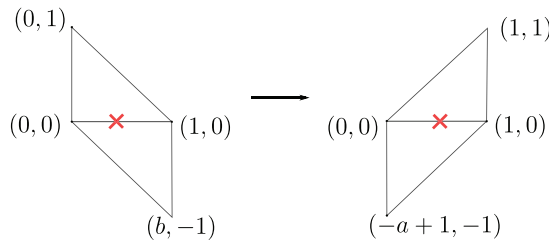
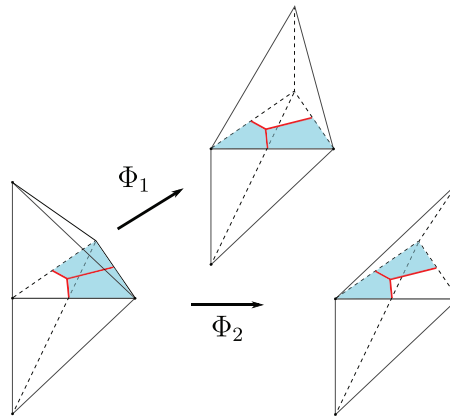**FIGURE 2**   Charts on the affine quartic.

FIGURE 3   Charts near a negative vertex.

This example is called the affine quartic because it is the affine structure associated to a toric degeneration of a quartic in $\mathbb{P}^3$ (see, for instance, the example in [13, Introduction]).

2.3 │ Dimension 3: Positive and negative vertices

In dimension 3, Δ must be a trivalent graph. We have three local models. One for a generic point along an edge of Δ and two local models for vertices, which can be either of positive or negative type.

The affine structure along an edge of Δ has the following description. Take the focus–focus 2-dimensional model, denote it by (B_{ff}, p), where p is the focus–focus singular point. Then along the interior of an edge of Δ we want the affine structure to be locally isomorphic to $B_{ff} \times \mathbb{R}$, where now $\Delta = \{p\} \times \mathbb{R}$.

A *negative vertex* is depicted in Figure 3. Here B is the union of two standard simplices and Δ is the trivalent graph (with just one vertex) depicted in red inside the common face. Figure 3 depicts the affine structure on B_0. It has three charts, one for each vertex of the common face. In the figure, the shaded regions are not part of the charts.

The map Φ_1 is the identity on the bottom simplex and on the top simplex it is the linear map given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The map Φ_2 is the identity on the bottom simplex and on the top simplex it is the linear map given by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the *positive vertex*, we let $B = \mathbb{R} \times \mathbb{R}^2$ and take Δ inside $\{0\} \times \mathbb{R}^2$ given by the set

$$\{y = 0, \ x \geqslant 0\} \cup \{x = 0, \ y \geqslant 0\} \cup \{x = y, \ x \leqslant 0\}.$$

Define the sets

$$R^+ = \mathbb{R}_{\geqslant 0} \times \Delta, \quad R^- = \mathbb{R}_{\leqslant 0} \times \Delta$$

The affine structure on B_0 has two charts. The open sets are

$$U_1 = (\mathbb{R} \times \mathbb{R}^2) - R^-, \quad U_2 = (\mathbb{R} \times \mathbb{R}^2) - R^+.$$

On $\mathbb{R}^2$ define the piecewise linear function

$$\nu(x, y) = \min\{0, x, y\}$$

Define the coordinate map ϕ_1 on U_1 to be the identity and the coordinate map ϕ_2 on U_2 to be

$$\phi_2(t, x, y) = (t + \nu(x, y), x, y).$$

2.4 | The affine quintic

This example is similar to the affine quartic, but one dimension higher. Take the simplex P in $\mathbb{R}^4$ with vertices

$$p_0 = (-1, -1, -1, -1), \quad p_1 = (4, -1, -1, -1), \quad p_2 = (-1, 4, -1, -1),$$

$$p_3 = (-1, -1, 4, -1), \quad p_4 = (-1, -1, -1, 4).$$

Let $B = \partial P$. Inside every 2-face of P, consider the honeycomb (red) graph depicted in Figure 4. Define Δ to be union of such graphs over all 2-faces of P. The interior of each 2-face contains 25 trivalent vertices. There are also five trivalent vertices in the interior of each edge. These are the points where the honeycomb graph intersects an edge, indeed each edge is contained in exactly three 2-faces. We can define charts on B as follows. We have obvious charts consisting of the interior of each 3-face of P. For every integral point q in a 2-face, consider a neighborhood $U_q \subset B - \Delta$. Define a chart by the projection $U_q \to \mathbb{R}^4/\mathbb{R}q$, where $\mathbb{R}q$ is the line generated by q. By choosing these neighborhoods so that they cover B_0, we obtain an integral affine structure. It can be shown that vertices in the interior of 2-faces are of negative type and vertices in the interior of edges are of positive type. This example is called the affine quintic because it is the affine structure associated to a toric degeneration of a quintic in $\mathbb{P}^4$. More examples of similar affine manifolds with singularities associated to toric degenerations of Calabi–Yau complete intersections in Fano toric varieties are constructed in [12].
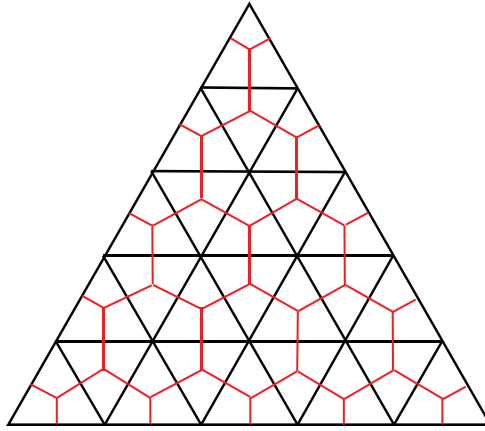
FIGURE 4   Discriminant of an affine quintic.

2.5 | Singular fibers

The compactification in diagram (1) is obtained by gluing suitable singular fibers over Δ. For instance, in dimension 2, the singular fiber over a focus–focus point is a once pinched torus. In dimension 3, the singular fiber over a point in the interior of an edge of Δ is $F \times S^1$ where F is a once pinched torus. The fiber over a positive vertex is obtained by considering a three torus $T^2 \times S^1$, where T^2 is a two torus, and collapsing a two torus $T^2 \times \{p\}$ to a point. The singular fiber over a negative vertex is more complicated, we refer to [11] or [7] for the Lagrangian models. In the case of the affine quartic the compactified manifold X is homeomorphic to a K3 surface, that is, to a quartic and the affine quintic is homeomorphic to a quintic Calabi–Yau, as proved by Gross in [11]. It is expected that when X is constructed from affine manifolds with singularities associated to toric degenerations of Calabi–Yau complete intersections in Fano toric varieties as in [12], then it is homeomorphic to the given Calabi–Yau.

2.6 | Topological mirror symmetry

In [11], Gross constructs the topological mirror $\check{X}$ of X. Given the lattice $\Lambda \subset TB_0$, dual to Λ^*, we can form

$$\check{X}_0 = TB_0/\Lambda$$

together with projection $\check{f}_0 : \check{X}_0 \to B_0$. Gross proved that when B is simple, also $\check{X}_0$ can be compactified to a manifold $\check{X}$ with a map $\check{f} : \check{X} \to B$ extending $\check{f}_0$. Indeed, in dimension 3, the positive fibers in X must be replaced by negative fibers in $\check{X}$ and viceversa.

As the tangent bundle does not have a natural symplectic structure, to construct a Lagrangian fibration on $\check{X}$ one needs the additional data of a potential ϕ. This is a multivalued strictly convex function that can be used to define a symplectic form on TB_0 or, equivalently, to define a mirror affine structure on B_0 via a Legendre transform. For the purpose of this paper, it will be enough to consider the mirror $\check{X}$ as the topological manifold obtained from TB_0.

2.7 | The Leray spectral sequence

The cohomology of X can be computed by the Leray spectral sequence associated to the map $f : X \to B$. Recall that given a group G we denote by $R^p f_* G$ the sheaf on B associated to the presheaf $U \mapsto H^p(f^{-1}(U), G)$. The fibration is called *G-simple* if

$$j_* R^p f_{0_*} G = R^p f_* G.$$

This essentially means that the cohomology of the singular fibers is determined by the local monodromy of $\Lambda^* \otimes G$. Gross proves that the fibrations constructed above (i.e., from a simple B) are G-simple for $G = \mathbb{Z}, \mathbb{Q}$ and $\mathbb{Z}_n$.

The E_2 page is given by the cohomology groups $H^q(B, R^p f_* G)$. As the fibers are connected, we have that

$$R^0 f_* G \cong G. \tag{2}$$

Let us now consider $G = \mathbb{Z}$. The fact that transition maps of the affine structure are in $\mathbb{R}^n \rtimes \mathrm{SL}(\mathbb{Z}, n)$ implies that the fibers are oriented, in particular

$$R^n f_* \mathbb{Z} \cong \mathbb{Z}. \tag{3}$$

This is equivalent to the fact that B_0 has a global integral volume form. If $b \in B_0$, we have that

$$(R^p f_* \mathbb{Z})_b = \bigwedge^p \Lambda_b. \tag{4}$$

On the other hand, if we consider the Leray spectral sequence for the mirror we have

$$(R^p \check{f}_* \mathbb{Z})_b = \bigwedge^p \Lambda_b^*. \tag{5}$$

By contraction with the global volume form (or equivalently by Poincaré duality on the fibers) we have the natural isomorphism

$$R^p f_{0_*} \mathbb{Z} \cong R^{n-p} \check{f}_{0_*} \mathbb{Z}.$$

This extends to an isomorphism

$$R^p f_* \mathbb{Z} \cong R^{n-p} \check{f}_* \mathbb{Z} \tag{6}$$

by $\mathbb{Z}$-simplicity.

We now consider $n = 3$ and $G = \mathbb{Q}$. With the additional assumptions that B is a $\mathbb{Q}$-homology sphere and that $b_1(X) = b_1(\check{X}) = 0$ it can be shown that the E_2 page for X looks as follows

$\mathbb{Q}$	0	0	$\mathbb{Q}$
0	$H^1(B, R^2 f_* \mathbb{Q})$	$H^2(B, R^2 f_* \mathbb{Q})$	0
0	$H^1(B, R^1 f_* \mathbb{Q})$	$H^2(B, R^1 f_* \mathbb{Q})$	0
$\mathbb{Q}$	0	0	$\mathbb{Q}$

The bottom and top rows follow from (2) and (3) and the assumption that B is a $\mathbb{Q}$-homology sphere. The vanishing of $H^0(B, R^1 f_* \mathbb{Q})$ and $H^3(B, R^2 f_* \mathbb{Q})$ follow from the assumption $b_1(X) =$

$b_5(X) = 0$. The vanishing of $H^0(B, R^2 f_* \mathbb{Q})$ and $H^3(B, R^1 f_* \mathbb{Q})$ follow from (6) and the assumption $b_1(\check{X}) = b_5(\check{X}) = 0$.

Gross [9–11] proves that, with the given hypothesis, the spectral sequence degenerates at the E_2 page. In particular, we have

$$H^2(X, \mathbb{Q}) \cong H^1(B, R^1 f_* \mathbb{Q}) \cong H^2(B, R^2 f_* \mathbb{Q}) \cong H^4(X, \mathbb{Q})$$

and similarly for $\check{X}$. Using (6), we also have

$$H^1(B, R^2 f_* \mathbb{Q}) \cong H^1(B, R^1 \check{f}_* \mathbb{Q}) \cong H^2(B, R^2 \check{f}_* \mathbb{Q}) \cong H^2(B, R^1 f_* \mathbb{Q}).$$

If X and $\check{X}$ are Calabi–Yau manifolds, we have that the Hodge numbers satisfy

$$h^{1,1}(X) = \dim H^1(B, R^1 f_* \mathbb{Q}) = \dim H^1(B, R^2 \check{f}_* \mathbb{Q}) = h^{1,2}(\check{X}).$$

In particular, we have the celebrated mirror symmetry of the Hodge diamonds of X and $\check{X}$.

In this paper, we will be concerned with cohomology with $\mathbb{Z}_2$ coefficients. Also in this case the spectral sequence degenerates at the E_2 page and if we assume that B is a $\mathbb{Z}_2$-cohomology sphere and $H^1(X, \mathbb{Z}_2) \cong H^1(\check{X}, \mathbb{Z}_2) \cong 0$, the E_2 page becomes

$$
\begin{array}{cccc}
\mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2 \\
0 & H^1(B, R^2 f_* \mathbb{Z}_2) & H^2(B, R^2 f_* \mathbb{Z}_2) & 0 \\
0 & H^1(B, R^1 f_* \mathbb{Z}_2) & H^2(B, R^1 f_* \mathbb{Z}_2) & 0 \\
\mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2
\end{array}
$$

Again we have

$$H^2(X, \mathbb{Z}_2) \cong H^1(B, R^1 f_* \mathbb{Z}_2),$$

$$H^1(B, R^2 f_* \mathbb{Z}_2) \cong H^2(B, R^1 f_* \mathbb{Z}_2).$$

In particular, the $\mathbb{Z}_2$-Betti numbers of X satisfy

$$b_2(X, \mathbb{Z}_2) = \dim H^1(B, R^1 f_* \mathbb{Z}_2),$$

$$b_3(X, \mathbb{Z}_2) = 2 + 2 \dim H^1(B, R^2 f_* \mathbb{Z}_2).$$

The relation between the hodge numbers of X and the groups $H^q(B, R^p f_* \mathbb{Z}_2)$ depends on whether $H^{p+q}(X, \mathbb{Z})$ has torsion.

3 | REAL STRUCTURES

3.1 | The standard real structure

Consider the involution $\iota_0 : X_0 \to X_0$ induced from the map $\alpha \mapsto -\alpha$ on the fibers of the cotangent bundle of B_0. It is clearly anti-symplectic in the sense that on the symplectic form ω it acts as $\iota_0^* \omega = -\omega$. It was proved in [8] that ι_0 extends to a smooth fiber preserving anti-symplectic involution $\iota : X \to X$. It is clear that ι fixes the zero section. We call ι the standard real structure. We denote by Σ the fixed point set of ι, which we can think as the real part of X. The zero section is

a connected component of Σ. We also denote by $\pi : \Sigma \to B$ the restriction of f, that is, $\pi = f|_\Sigma$. Notice that π is generically a 2^n to 1 covering.

3.2 | Twisted real structures

In [8], we constructed real structures that can be viewed as a twist of ι by a Lagrangian section. For technical reasons, in [8] we only considered sections that coincide with the zero section in a small neighborhood of the negative fibers. We called these sections of type $\mathfrak{C}$. For simplicity we will assume that all sections are of this type also in this paper, we do not believe this to be a restrictive condition.

Let $\tau : B \to X$ be a Lagrangian section. Consider on X_0 the translation by τ, that is, the map which on the fibers acts by $\alpha \mapsto \alpha + \tau$. It was shown in [8] that this map extends smoothly to a fiber preserving symplectomorphism of X, for simplicity we continue to denote it by τ. Now assume that τ is not a square, that is, that there does not exist another section τ' such that $\tau = 2\tau'$. Define

$$\iota_\tau = \iota \circ \tau. \tag{7}$$

Clearly, ι_τ is a fiber preserving antisymplectic map. It was shown in [8, Proposition 1.10] that this implies that ι_τ is an involution. Indeed we can see this as follows. Consider the map $\iota \circ \tau \circ \iota$. It is a fiber preserving symplectomorphism. As such, it must be given by the translation by a section. In fact, it is given by

$$\iota \circ \tau \circ \iota = -\tau.$$

One can easily see that this holds on smooth fibers, where we have

$$[\alpha] \xmapsto{\iota} [-\alpha] \xmapsto{\tau} [-\alpha + \tau] \xmapsto{\iota} [\alpha - \tau].$$

As smooth fibers are dense, the relation extends to X. In particular,

$$\iota_\tau^2 = (\iota \circ \tau) \circ (\iota \circ \tau) = (\iota \circ \tau \circ \iota) \circ \tau = (-\tau) \circ (\tau) = \mathrm{Id}_X.$$

Therefore, ι_τ is an involution.

The fact that τ is not a square implies that ι_τ does not fix any section (see [8, Remarks 1.11 and 1.12]). We call ι_τ a *twisted real structure*, where τ is the twist. We denote by Σ_τ the fixed point set of ι_τ. We also denote by $\pi_\tau : \Sigma_\tau \to B$ the map given by the restriction of f. Also in this case π_τ is generically an 2^n to 1 covering.

3.3 | A long exact sequence

Let us restrict to the 3-dimensional case $n = 3$. We describe the long exact sequence that was found in [6] relating the $\mathbb{Z}_2$-cohomology of Σ with the cohomology of X. Consider the projection $\pi : \Sigma \to B$. In particular, as the preimage by π of a point in B is finite, we have that the Leray spectral sequence of π is quite simple and

$$H^q(\Sigma, G) \cong H^q(B, \pi_* G).$$

We restrict to the case $G = \mathbb{Z}_2$. In [6], the following result is proved

Lemma 3.1. *There exists a short exact sequence of sheaves on B:*

$$0 \longrightarrow R^1 f_* \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \pi_* \mathbb{Z}_2 \longrightarrow R^2 f_* \mathbb{Z}_2 \longrightarrow 0. \tag{8}$$

Proof. Let us consider the same sheaves restricted to B_0. Notice that for every $b \in B_0$, we have

$$\pi^{-1}(b) = \tfrac{1}{2} \Lambda^* \quad \text{mod } \Lambda^*$$
$$\cong \Lambda^* \otimes \mathbb{Z}_2$$

In particular, it is a subgroup of the fiber $f^{-1}(b)$ isomorphic to $\mathbb{Z}_2^3$. On the other hand, the stalk of $\pi_* \mathbb{Z}_2$ at b is canonically identified with

$$(\pi_* \mathbb{Z}_2)_b = \mathrm{Maps}(\pi^{-1}(b), \mathbb{Z}_2).$$

Moreover, we have isomorphisms (4) for $\mathbb{Z}_2$ coefficients

$$(R^1 f_* \mathbb{Z}_2)_b = \Lambda \otimes \mathbb{Z}_2,$$
$$(R^2 f_* \mathbb{Z}_2)_b \cong \bigwedge^2 \Lambda \otimes \mathbb{Z}_2. \tag{9}$$

Notice that $(R^1 f_* \mathbb{Z}_2)_b$ is the space of linear maps from $\pi^{-1}(b)$ to $\mathbb{Z}_2$, so it injects in $(\pi_* \mathbb{Z}_2)_b$. Concerning the two $\mathbb{Z}_2$ summands in the left-hand side of the sequence we have

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle 1 \rangle \oplus \langle \delta_0 \rangle,$$

where 1 is the constant map equal to 1 and δ_0 is the delta function at 0 (i.e., the map that maps $0 \in \pi^{-1}(b)$ to 1 and everything else to 0).

From (5) and (6), we have the isomorphism

$$(R^2 f_* \mathbb{Z}_2)_b \cong \Lambda^* \otimes \mathbb{Z}_2 \cong \pi^{-1}(b). \tag{10}$$

One can identify $\pi^{-1}(b)$ with the quotient of the first two groups of the above sequence by identifying a nonzero point $y \in \pi^{-1}(b)$ with the class (in the quotient) of the map δ_y, the delta function at y. $\square$

Let us assume that B is a homology $\mathbb{Z}_2$ sphere. Then the short exact sequence induces the long exact sequence in cohomology:

$$0 \longrightarrow H^0(B, R^1 f_* \mathbb{Z}_2) \oplus (\mathbb{Z}_2)^2 \longrightarrow H^0(\Sigma, \mathbb{Z}^2) \longrightarrow H^0(B, R^2 f_* \mathbb{Z}_2) \longrightarrow$$

$$\longrightarrow H^1(B, R^1 f_* \mathbb{Z}_2) \longrightarrow H^1(\Sigma, \mathbb{Z}^2) \longrightarrow H^1(B, R^2 f_* \mathbb{Z}_2) \stackrel{\beta}{\longrightarrow} \tag{11}$$

$$\longrightarrow H^2(B, R^1 f_* \mathbb{Z}_2) \longrightarrow H^2(\Sigma, \mathbb{Z}_2) \longrightarrow H^2(B, R^2 f_* \mathbb{Z}_2) \longrightarrow \dots$$

The map $\beta : H^1(B, R^2 f_* \mathbb{Z}_2) \to H^2(B, R^1 f_* \mathbb{Z}_2)$ is the connecting homomorphism and it essentially determines the cohomology of Σ.

In particular, the following corollaries follow from the properties of the Leray spectral sequence described in Subsection 2.7.

Corollary 3.2. *If B is a cohomology $\mathbb{Z}_2$-sphere and $H^1(X, \mathbb{Z}_2)$ and $H^1(\check{X}, \mathbb{Z}_2)$ are both zero, then Σ has two connected components and the long exact sequence (11) splits as*

$$0 \to H^1(B, R^1 f_* \mathbb{Z}_2) \to H^1(\Sigma, \mathbb{Z}_2) \to H^1(B, R^2 f_* \mathbb{Z}_2) \overset{\beta}{\to} H^2(B, R^1 f_* \mathbb{Z}_2)$$
$$\to H^2(\Sigma, \mathbb{Z}_2) \to H^2(B, R^2 f_* \mathbb{Z}_2) \to 0$$

Corollary 3.3. *Under the same hypothesis of Corollary 3.2, with the additional assumption that X is a Calabi–Yau variety and the cohomologies of X and $\check{X}$ have no 2-torsion, we have that the $\mathbb{Z}_2$ Betti numbers of Σ satisfy*

$$b_q(\Sigma, \mathbb{Z}_2) \leqslant h^{q,3-q}(X) + h^{q,q}(X).$$

Indeed the hypothesis and the properties of the Leray spectral sequence imply

$$\dim H^p(B, R^q f_* \mathbb{Z}_2) = h^{p,q}(X).$$

These inequalities coincide with those proved by Renaudineau–Shaw [20] for any real hypersurface arising from primitive patchworking in a toric variety. Notice, however, that in our case X is not necessarily a hypersurface (see, for instance, the case of Schoen's Calabi–Yau, [6] and [1]).

3.4 | Mirror symmetry

Using the isomorphism (6), we can interpret the connecting homomorphism β in (11) as a map on the cohomology of the mirror $\check{X}$:

$$\beta : H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \to H^2(B, R^2 \check{f}_* \mathbb{Z}_2).$$

In [2], Arguz and Prince proved the following remarkable result

Theorem 3.4. *The connecting homomorphism β in the long exact sequence (11) coincides with the squaring map*

$$\mathrm{Sq} : H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \longrightarrow H^2(B, R^2 \check{f}_* \mathbb{Z}_2)$$
$$D \longmapsto D^2.$$

Notice that if B is a $\mathbb{Z}_2$ homology sphere and $H^1(X, \mathbb{Z}_2) = H^1(\check{X}, \mathbb{Z}_2) = 0$, then $H^p(B, R^p \check{f}_* \mathbb{Z}_2) \cong H^{2p}(\check{X}, \mathbb{Z}_2)$. The map β in this case is the squaring with respect to the usual cup product in cohomology.

4 │ SHORT EXACT SEQUENCES

Our first goal is to generalize the short exact sequence (8) to the case of twisted real structures. We will prove the following result

Theorem 4.1. *There exist sheaves $\mathcal{L}_\tau^1$ and $\mathcal{L}_\tau^2$ over B and a short exact sequence*

$$0 \longrightarrow \mathcal{L}_\tau^1 \longrightarrow \pi_{\tau_*}\mathbb{Z}_2 \longrightarrow \mathcal{L}_\tau^2 \longrightarrow 0, \tag{12}$$

such that $\mathcal{L}_\tau^1$ and $\mathcal{L}_\tau^2$ are related to the topology of X by the following short exact sequences

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{L}_\tau^1 \longrightarrow R^1 f_* \mathbb{Z}_2 \longrightarrow 0, \tag{13}$$

$$0 \longrightarrow R^2 f_* \mathbb{Z}_2 \longrightarrow \mathcal{L}_\tau^2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{14}$$

4.1 │ Classification of Lagrangian sections

The Lagrangian sections of $f : X \to B$ are classified, up to Hamiltonian equivalence, by the group $H^1(B, j_*\Lambda^*)$. Let us sketch how to associate a class in $H^1(B, j_*\Lambda^*)$ to every Lagrangian section τ (for more details, see [9, Theorem 3.6] or [10, Theorem 2.4]). We assume, as in [8], that τ coincides with the zero section in a neighborhood of a negative fiber. Let $B^\#$ be the complement, inside B, of the set of negative vertices and let $\mathrm{Crit}(f) \subseteq X$ be the set of critical points of f. Define

$$X^\# = f^{-1}(B^\#) - \mathrm{Crit}\, f.$$

It follows from the analysis in [10] or [7, section 5], that there is a short exact sequence

$$0 \to j_*\Lambda^* \to T^*B^\# \to X^\# \to 0.$$

This essentially means that the local monodromy invariant elements of Λ^* and angle coordinates on the fibers extend over the discriminant. Take some good covering $\{U_i\}_{i \in I}$ of B. We can assume that each U_i contains at most one vertex and double intersections $U_i \cap U_j$ contain no vertices. Let τ be a Lagrangian section. If U_i does not contain a negative vertex consider $\tau_{|U_i} : U_i \to T^*U_i/j_*\Lambda^*$. As U_i is homeomorphic to a 3-ball, we can find a Lagrangian lift $\tilde{\tau}_{U_i} : U_i \to T^*U_i$ of $\tau_{|U_i}$. If U_i contains a negative vertex, then we can assume that $\tau|_{U_i}$ is the zero section. We can therefore define $\tilde{\tau}|_{U_i} = 0$. Then, as τ is a global section, we must have that on overlaps $U_i \cap U_k$

$$\tilde{\tau}_{U_i} - \tilde{\tau}_{U_k} \in j_*\Lambda^*.$$

Therefore, to the section τ we can associate the Cech 1-cocycle $\{U_i \cap U_k, \tilde{\tau}_{U_i} - \tilde{\tau}_{U_k}\}$ giving a class in $H^1(B, j_*\Lambda^*)$, which we continue to denote by τ. It can be shown that any class can be represented by a Lagrangian section and that two Lagrangian sections represent the same class if and only if they are Hamiltonian isotopic.

4.2 | Local description of Σ_τ

Given the above local description of a Lagrangian section τ, it is easy to describe the involution ι_τ locally. We will do this away from singularities, that is, when $U_i \cap \Delta = \emptyset$. Indeed

$$\iota_\tau : T^* U_i / \Lambda^* \to T^* U_i / \Lambda^*$$
$$[\alpha] \mapsto [-(\alpha + \tilde{\tau}_{U_i})],$$

where α is a 1-form and $[\cdot]$ denotes the class in the quotient by Λ^*. Then, locally, we have

$$\Sigma_\tau|_{U_i} = -\frac{\tilde{\tau}_{U_i}}{2} + \frac{1}{2}\Lambda^* \mod \Lambda^*.$$

In particular, the fiber $\pi_\tau^{-1}(b)$ has the structure of an affine space modeled on $\Lambda^* \otimes \mathbb{Z}_2$.

4.3 | The sheaves $\mathcal{L}_\tau^1$ and $\mathcal{L}_\tau^2$

The sheaf $\mathcal{L}_\tau^1$ is easily described. Its stalks over $b \in B_0$ are the affine maps $\pi^{-1}(b) \to \mathbb{Z}_2$, which embed inside $\pi_{\tau*}\mathbb{Z}_2$. Moreover, on B_0, we have an obvious sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{L}_\tau^1 \longrightarrow \Lambda \otimes \mathbb{Z}_2 \longrightarrow 0,$$

where the left-hand side is the inclusion of the constant maps, while the right-hand side is given by taking the linear part of an affine map. If we push this sequence forward by $j : B_0 \to B$ and we use simplicity, we get the sequence (13). By definition, $\mathcal{L}_\tau^2$ is the quotient of the inclusion $\mathcal{L}_\tau^1 \to \pi_{\tau*}\mathbb{Z}_2$.

4.4 | Proof of Theorem 4.1

Let S be an affine space modeled on a $\mathbb{Z}_2$-vector space V of dimension 3. Let $L_S^1 = \text{Aff}(S)$ be the space of affine functions on S. Let L_S^2 be the quotient between $\text{Maps}(S, \mathbb{Z}_2)$ and L_S^1. Given a subset $A \subset S$, let δ_A denote the function which is 1 on A and 0 elsewhere and by $[\delta_A]$ the class of δ_A in L_S^2. An affine function on S is of the type δ_W or $1 + \delta_W$ for some affine subspace $W \subseteq S$ of codimension 1 or 0. It is not hard prove that L_S^2 is generated by the elements of type $[\delta_Z]$, where $Z \subset S$ is either a line or a point. Moreover, given two lines Z_1 and Z_2, then $[\delta_{Z_1}] = [\delta_{Z_2}]$ if and only if Z_1 and Z_2 are parallel. We then have an exact sequence

$$0 \to V \to L_S^2 \to \mathbb{Z}_2 \to 0, \tag{15}$$

where a vector $v \in V$ is mapped to $[\delta_{Z_v}]$, where Z_v is a line with direction v if $v \neq 0$ or the empty set if $v = 0$. Let $v, w \in V$ be linearly independent. Then we can assume that Z_v, Z_w and Z_{v+w} are three coplanar lines intersecting at one point. In particular,

$$\delta_{Z_{v+w}} + \delta_{Z_v} + \delta_{Z_w} = \delta_W,$$

where W is the plane containing the three lines. This implies that

$$[\delta_{Z_{v+w}}] = [\delta_{Z_v}] + [\delta_{Z_w}].$$

The same equality holds if v and w are linearly dependent. In particular, the first map in (15) is linear. The quotient of L_S^2 by V is $\mathbb{Z}_2$ and it is generated by the class of $[\delta_q]$ where q is a point in S. Given $b \in B_0$, let $S = \pi_\tau^{-1}(b)$. As we said, S is an affine space modeled on $\Lambda^* \otimes \mathbb{Z}_2$. Using the isomorphism $(R^2 f_* \mathbb{Z}_2)_b \cong \Lambda^* \otimes \mathbb{Z}_2$ (see (9) and (10)), the sequence (15) becomes the sequence (14).

4.5 | The 2-dimensional case

If B is 2-dimensional (hence the affine base of a K3 surface), then we only have the sheaf $\mathcal{L}_\tau^1$ of affine maps inside $\pi_* \mathbb{Z}_2$ that satisfies (13) and

$$0 \longrightarrow \mathcal{L}_\tau^1 \longrightarrow \pi_* \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{16}$$

This follows from the fact that if S is an affine space over $\mathbb{Z}_2$ of dimension 2 then we have

$$0 \to \mathrm{Aff}(S) \to \mathrm{Maps}(S, \mathbb{Z}_2) \to \mathbb{Z}_2 \to 0,$$

where $\mathbb{Z}_2$ is generated by $[\delta_q]$, with $q \in S$.

5 | THE CONNECTING HOMOMORPHISM

Also in this section, we assume we are in dimension $n = 3$. The sequence (12) gives the long exact sequence in cohomology

$$0 \longrightarrow H^0(B, \mathcal{L}_\tau^1) \longrightarrow H^0(\Sigma_\tau, \mathbb{Z}^2) \longrightarrow H^0(B, \mathcal{L}_\tau^2) \longrightarrow$$

$$H^1(B, \mathcal{L}_\tau^1) \longrightarrow H^1(\Sigma_\tau, \mathbb{Z}^2) \longrightarrow H^1(B, \mathcal{L}_\tau^2) \xrightarrow{\beta} \tag{17}$$

$$H^2(B, \mathcal{L}_\tau^1) \longrightarrow H^2(\Sigma_\tau, \mathbb{Z}_2) \longrightarrow H^2(B, \mathcal{L}_\tau^2) \longrightarrow \dots$$

Combining this with the maps induced by the sequences (13) and (14), we obtain the diagram

$$\begin{array}{ccc} H^1(B, R^2 f_* \mathbb{Z}_2) & \xrightarrow{\beta'} & H^2(B, R^1 f_* \mathbb{Z}_2) \\ \downarrow & & \uparrow \\ H^1(B, \mathcal{L}_\tau^2) & \xrightarrow{\beta} & H^2(B, \mathcal{L}_\tau^1) \end{array} \tag{18}$$

where β' is obtained by composition.

Using the isomorphisms (6), we interpret β' as a map on the cohomology of the mirror

$$\beta' : H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \to H^2(B, R^2 \check{f}_* \mathbb{Z}_2).$$

As explained in Subsection 4.1, the twist τ is a class in $H^1(B, j_* \Lambda^*)$. Notice that

$$H^1(B, j_* \Lambda^*) = H^1(B, R^1 \check{f}_* \mathbb{Z}).$$

This allows us to interpret τ as the class of a line bundle L_τ on $\check{X}$ (indeed, it is conjectured that Lagrangian sections are mirror to line bundles). The assumption that τ is not a square (see Subsection 3.2) implies that L_τ is not zero after reduction modulo two. Therefore, our assumption is that the twist τ is such that the mirror line bundle L_τ satisfies

$$0 \neq L_\tau \in H^1(B, R^1 \check{f}_* \mathbb{Z}_2).$$

We can now state our main result

Theorem 5.1. *The map $\beta' : H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \to H^2(B, R^2 \check{f}_* \mathbb{Z}_2)$, related to the connecting homomorphism in the long exact sequence (17) via diagram (18), coincides with the map*

$$S_\tau : H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \longrightarrow H^2(B, R^2 \check{f}_* \mathbb{Z}_2)$$
$$D \longmapsto D^2 + DL_\tau$$

Proof. The proof is similar to the proof of [2, Theorem 3.4]. Fix an open covering $\mathfrak{U} = \{\mathcal{U}_i\}_{i \in I}$ that is Leray for all the sheaves and such that all triple intersections do not intersect the discriminant Δ. We will denote multiple intersections of open sets by

$$U_{i_0, \dots, i_k} = U_{i_0} \cap \dots \cap U_{i_k}.$$

The cup product in Cech cohomology has the following description. Let $\alpha \in H^p(B, \mathcal{F})$ and $\beta \in H^q(B, \mathcal{G})$ then the cup product $\alpha \cup \beta \in H^{p+q}(B, \mathcal{F} \otimes G)$ is represented by the cochain

$$(\alpha \cup \beta)_{i_0 \dots i_{p+q}} = \sum_{r=0}^{p+q} \alpha_{i_r, \dots, i_{r+p}} \otimes \beta_{i_{r+p}, \dots, i_{r+p+q}},$$

where we have chosen cocycles representing α and β. The indices in this formula should be interpreted cyclically.

In the following, when we take a local section of $\Lambda^* \otimes \mathbb{Z}_2$ and denote it by λ, we will mean that $\lambda \in \Lambda^*$ reduced mod 2, that is, λ will be short for $\lambda \otimes 1$. On the other hand, we can also identify

$$\Lambda^* \otimes \mathbb{Z}_2 = \tfrac{1}{2}\Lambda^* \quad \mathrm{mod} \ \Lambda^*$$

and therefore we may identify λ with the point $\frac{1}{2}\lambda$ in the fiber $f^{-1}(b)$. Take a class $D \in H^1(B, R^1 \check{f}_* \mathbb{Z}_2)$ represented by a 1-cycle $\{U_{ij}, D_{ij}\}$. The twisting cycle $L_\tau \in H^1(B, R^1 \check{f}_* \mathbb{Z}_2)$ is represented by $\{U_{ij}, \tau_{ij}\}$ as described in Subsection 4.1. The product $D^2 + DL_\tau$ is then represented by the 2-cycle

$$(D^2 + DL_\tau)_{ijk} = D_{ij} \wedge (D_{jk} + \tau_{jk}) + D_{jk} \wedge (D_{ki} + \tau_{ki})$$
$$+ D_{ki} \wedge (D_{ij} + \tau_{ij}).$$

We now compare this cycle with $\beta'(D)$. First, we need to view D as a class in $H^1(B, \mathcal{L}_\tau^2)$. As described in Subsection 4.4, $\{U_{ij}, D_{ij}\}$ is sent to the cycle $\bar{D} = \{U_{ij}, [\delta_{Z_{D_{ij}}}]\}$, where $Z_{D_{ij}}$ is a line with direction D_{ij} if $D_{ij} \neq 0$ or the empty set if $D_{ij} = 0$. Now we need a cochain $\Gamma \in C^1(\mathfrak{U}, \pi_{\tau_*}\mathbb{Z}_2)$ representing $\bar{D}$. Using the local description of $\Sigma_\tau|_{U_i}$ given in Subsection 4.2, we can represent $Z_{D_{ij}}$

as the line through the point $-\frac{\tilde{\tau}_i}{2}$ and direction D_{ij}. Therefore, we lift $[\delta_{Z_{D_{ij}}}]$ to the map

$$\Gamma_{ij} = \delta_{Z_{D_{ij}}} = \delta_{-\frac{\tilde{\tau}_i}{2}} + \delta_{-\frac{\tilde{\tau}_i + D_{ij}}{2}}.$$

Then we consider $\partial\Gamma \in C^2(\mathfrak{U}, \pi_{\tau*}\mathbb{Z}_2)$. On U_{ijk} we have

$$
\begin{aligned}
(\partial\Gamma)_{ijk} &= \delta_{Z_{D_{ij}}} + \delta_{Z_{D_{jk}}} + \delta_{Z_{D_{ki}}} = \\
&= \delta_{-\frac{\tilde{\tau}_i}{2}} + \delta_{-\frac{\tilde{\tau}_i + D_{ij}}{2}} \\
&\quad + \delta_{-\frac{\tilde{\tau}_j}{2}} + \delta_{-\frac{\tilde{\tau}_j + D_{jk}}{2}} \\
&\quad + \delta_{-\frac{\tilde{\tau}_k}{2}} + \delta_{-\frac{\tilde{\tau}_k + D_{ki}}{2}}
\end{aligned}
\tag{19}
$$

The next steps consist first in describing $(\partial\Gamma)_{ijk}$ as an affine function α_{ijk} on $\pi_\tau^{-1}(b)$. Thus, giving a cycle $\alpha \in H^2(B, \mathcal{L}_\tau^1)$. Notice that $\beta(\bar{D}) = \alpha$. Then we need to take the linear part $\beta_{ijk} = \mathrm{Lin}(\alpha_{ijk})$ so that $\beta'(D) = \{U_{ijk}, \beta_{ijk}\}$. Then we compare β_{ijk} with $(D^2 + DL_\tau)_{ijk}$.

Let us identify $\pi_\tau^{-1}(b)$ with the vector space $V = \Lambda^* \otimes \mathbb{Z}_2$ by declaring the point $-\frac{\tilde{\tau}_i}{2}$ to be the origin. Moreover, let us denote by Z_0, Z_1, Z_2, respectively, the sets $Z_{D_{ij}}, Z_{D_{jk}}$ and $Z_{D_{ki}}$. Let

$$e_1 = D_{jk}, \quad e_2 = D_{ki}, \quad f_1 = \tau_{ij}, \quad f_2 = \tau_{ki}.$$

The cocycle condition implies $D_{ij} = e_1 + e_2$ and $\tau_{jk} = f_1 + f_2$. It is then easy to see that

$$
\begin{aligned}
\alpha_{ijk} = (\partial\Gamma)_{ijk} &= \delta_{Z_0} + \delta_{Z_1} + \delta_{Z_2} = \\
&= \delta_0 + \delta_{e_1 + e_2} \\
&\quad + \delta_{f_1} + \delta_{f_1 + e_1} \\
&\quad + \delta_{f_2} + \delta_{f_2 + e_2}
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(D^2 + DL_\tau)_{ijk} &= (e_1 + e_2) \wedge (e_1 + f_1 + f_2) + e_1 \wedge (e_2 + f_2) + \\
&\quad + e_2 \wedge (e_1 + e_2 + f_1) = \\
&= e_1 \wedge e_2 + e_1 \wedge f_1 + e_2 \wedge f_2
\end{aligned}
\tag{20}
$$

We study four different cases.

Case 1: e_1 and e_2 are linearly dependent. If $e_1 = e_2 = 0$, then both $(\partial\Gamma)_{ijk}$ and $(D^2 + DL_\tau)_{ijk}$ are zero, in particular they match. Otherwise we may assume w.l.o.g. that $e_1 = e_2 \neq 0$. Then $Z_0 = \emptyset$ and Z_1 and Z_2 either coincide or are parallel. In the first case e_1 and $f_1 + f_2$ are linearly dependent, therefore $(\partial\Gamma)_{ijk}$ and $(D^2 + DL_\tau)_{ijk}$ both vanish. Otherwise if Z_1 and Z_2 are parallel and distinct, then e_1 and $f_1 + f_2$ are linearly independent. In particular,

$$(D^2 + DL_\tau)_{ijk} = e_1 \wedge (f_1 + f_2)$$

is a nonzero two form. On the other hand, $\alpha_{ijk} = (\partial\Gamma)_{ijk} = \delta_W$, where W is the unique 2-plane containing the two lines. In particular, α_{ijk} is a nonconstant affine function. Let e_3 be a third vector so that $\{e_1, f_1 + f_2, e_3\}$ forms a basis of V. Let $\{e_1^*, e_2^*, e_3^*\}$ form the dual basis of $V^* = \Lambda \otimes \mathbb{Z}_2$. Taking the linear part of α_{ijk} we have

$$\beta_{ijk} = \text{Lin}(\alpha_{ijk}) = e_3^*.$$

Consider $\Omega = e_1 \wedge (f_1 + f_2) \wedge e_3$, which coincides with the global 3-form on B. Contracting Ω with e_3^* gives precisely $e_1 \wedge (f_1 + f_2)$. Therefore, after applying the isomorphism (6) we have

$$\beta_{ijk} = (D^2 + DL_\tau)_{ijk}.$$

We now assume e_1 and e_2 are linearly independent. Let e_3 be a third vector so that $\{e_1, e_2, e_3\}$ form a basis of V and let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis of $V^* = \Lambda \otimes \mathbb{Z}_2$. As above $\Omega = e_1 \wedge e_2 \wedge e_3$ coincides with the global three form on B. We discuss the following three cases.

Case 2: Z_0, Z_1, Z_2 are coplanar and pass through the same point. In this case, $\alpha_{ijk} = (\partial\Gamma)_{ijk} = \delta_W$, where W is the unique plane containing the three lines. We have

$$\beta_{ijk} = \text{Lin}(\alpha_{ijk}) = e_3^*.$$

Let q be the common point of the three lines. We must have $q = 0$ or $q = e_1 + e_2$. Notice that $f_j = q + \epsilon_j e_j$, where $\epsilon_j = 1, 0$. Then

$$(D^2 + DL_\tau)_{ijk} = e_1 \wedge e_2 + (e_1 + e_2) \wedge q = e_1 \wedge e_2.$$

As contracting Ω with e_3^* gives $e_1 \wedge e_2$, we have $\beta_{ijk} = (D^2 + DL_\tau)_{ijk}$.

Case 3: Z_0, Z_1, Z_2 are coplanar and intersect pairwise at three different points. In this case, $\alpha_{ijk} = 0$. Let $q = Z_0 \cap Z_1$. Then $f_1 = q + \epsilon_1 e_1$ and $f_2 = q + e_1 + \epsilon_2 e_2$. Therefore,

$$(D^2 + DL_\tau)_{ijk} = e_1 \wedge e_2 + e_1 \wedge q + e_2 \wedge (q + e_1) = (e_1 + e_2) \wedge q = 0,$$

so that $\beta_{ijk} = (D^2 + DL_\tau)_{ijk}$.

Case 4: Z_1, Z_2 are coplanar and Z_0 is disjoint from Z_1 and Z_2. Let $q = Z_1 \cap Z_2$. We must have that q is linearly independent from e_1 and e_2, therefore we may assume that $e_3 = q$. We have that $\alpha_{ijk} = \delta_W$, where W is the unique 2-plane containing Z_0 and the points $e_1 + e_3$ and $e_2 + e_3$. It can be easily seen that

$$\beta_{ijk} = \text{Lin}(\alpha_{ijk}) = e_1^* + e_2^* + e_3^*$$

On the other hand, we have $f_j = e_3 + \epsilon_j e_j$, which gives

$$(D^2 + DL_\tau)_{ijk} = e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3.$$

This is precisely the two form obtained by contracting Ω with $e_1^* + e_2^* + e_3^*$. Therefore, $\beta_{ijk} = (D^2 + DL_\tau)_{ijk}$ also in this case.

This concludes the proof. $\qquad\square$

6 │ CONNECTEDNESS AND BOUNDS ON BETTI NUMBERS

We will compute some consequences on the cohomology of Σ_τ with the assumption that the base B is a homology $\mathbb{Z}_2$-sphere and that $H^1(X, \mathbb{Z}_2) = H^1(\check{X}, \mathbb{Z}_2) = 0$. We will prove the following

Theorem 6.1. *With the above assumptions, if τ is a nontrivial twist, then Σ_τ is connected and its Betti numbers satisfy*

$$b_1(\Sigma_\tau, \mathbb{Z}_2) \leqslant \dim H^1(B, R^1 f_* \mathbb{Z}_2) + \dim H^1(B, R^2 f_* \mathbb{Z}_2) - 1,$$

where equality holds if and only if the connecting homomorphism β in (17) is zero. If the integral cohomologies of X and $\check{X}$ have no $\mathbb{Z}_2$ torsion, then

$$b_q(\Sigma_\tau, \mathbb{Z}_2) \leqslant h^{q,3-q}(X) + h^{q,q}(X) - 1.$$

Notice in particular that Σ_τ is never maximal, in fact we have

$$\sum b_j(\Sigma_\tau, \mathbb{Z}_2) \leqslant \sum b_j(X, \mathbb{Z}_2) - 4.$$

When this inequality is an equality, Σ_τ is called an $(M - 2)$ real variety (M stands for maximal, see [25, section 3]).

6.1 │ Cohomology of $\mathcal{L}_\tau^2$

We prove the following

Lemma 6.2. *In the hypothesis of Theorem 6.1, we have*

$$H^0(B, \mathcal{L}_\tau^2) = 0$$

$$H^1(B, \mathcal{L}_\tau^2)) \cong \frac{H^1(B, R^2 f_* \mathbb{Z}_2)}{\langle \tau \rangle}$$

$$H^2(B, \mathcal{L}_\tau^2) \cong H^2(B, R^2 f_* \mathbb{Z}_2)$$

$$H^3(B, \mathcal{L}_\tau^2) \cong \mathbb{Z}_2.$$

Proof. It follows from the discussion in Subsection 2.7 that

$$H^0(B, R^2 f_* \mathbb{Z}_2) \cong H^0(B, R^1 \check{f}_* \mathbb{Z}_2) = 0$$

$$H^3(B, R^2 f_* \mathbb{Z}_2) = 0.$$

Moreover,

$$H^0(B, \mathbb{Z}_2) = H^3(B, \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(B, \mathbb{Z}_2) = 0$$

as B is a homology $\mathbb{Z}_2$ sphere. Hence, the long exact sequence associated to (14) splits as follows

$$0 \longrightarrow H^0(B, \mathcal{L}_\tau^2) \longrightarrow \mathbb{Z}_2 \longrightarrow H^1(B, R^2 f_* \mathbb{Z}_2) \longrightarrow H^1(B, \mathcal{L}_\tau^2) \longrightarrow 0$$

$$0 \longrightarrow H^2(B, R^2 f_* \mathbb{Z}_2) \longrightarrow H^2(B, \mathcal{L}_\tau^2) \longrightarrow 0 \tag{21}$$

$$0 \longrightarrow H^3(B, \mathcal{L}_\tau^2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

The last two lines give the last two statements of the lemma. In the first line we have two possibilities, either $H^0(B, \mathcal{L}_\tau^2) = 0$ or $H^0(B, \mathcal{L}_\tau^2) \cong H^0(B, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Let us prove that the former holds. Take some covering $\mathfrak{U} = \{U_i\}$ over which the cycle τ and Σ_τ can be described as in Subsection 4.1. Then, over each U_i we have identifications

$$(\Lambda^* \otimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \xrightarrow{\phi_i} \mathcal{L}^2|_{U_i}.$$

In fact for every nonzero $v \in (\Lambda^* \otimes \mathbb{Z}_2)$ let Z_v be the line with direction v and passing through $-\frac{\bar{\tau}_i}{2}$. Then we define

$$\phi_i(v, \epsilon) = \left[\delta_{Z_v} + \epsilon \delta_{-\frac{\bar{\tau}_i}{2}} \right].$$

It is then easy to check that over U_{ij}

$$\phi_j^{-1} \circ \phi_i(v, \epsilon) = (v + \epsilon \tau_{ij}, \epsilon).$$

Suppose by contradiction that $\mathcal{L}_\tau^2$ has a nontrivial section α that is mapped to 1 under the homomorphism $H^0(B, \mathcal{L}_\tau^2) \to \mathbb{Z}_2$. Then, locally with the above identifications ϕ_i, we have

$$\alpha_i = \alpha|_{U_i} = (v_i, 1)$$

for some local section v_i of $\Lambda^* \otimes \mathbb{Z}_2$. But as α is a section, we must have that on U_{ij}

$$(v_i + \tau_{ij}, 1) = (v_j, 1).$$

This implies that $\tau_{ij} = v_j - v_i$, that is, that τ is the trivial class, contradicting our assumption.

The first line of (21) becomes

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow H^1(B, R^2 f_* \mathbb{Z}_2) \longrightarrow H^1(B, \mathcal{L}_\tau^2) \longrightarrow 0.$$

We now prove that image of $\mathbb{Z}_2$ inside $H^1(B, R^2 f_* \mathbb{Z}_2)$ is generated by τ. Given $1 \in \mathbb{Z}_2 = H^0(B, \mathbb{Z}_2)$, we first lift it to a cochain $\gamma \in C^0(\mathfrak{U}, \mathcal{L}_\tau^2)$. Then $\partial \gamma$ comes from a cocycle λ in $C^1(\mathfrak{U}, R^2 f_* \mathbb{Z}_2)$ whose class is the image of 1. We can define γ on each U_i by

$$\phi_i(\gamma_i) = (0, 1).$$

Now we have

$$\phi_j(\gamma_i - \gamma_j) = \phi_i(\gamma_i - \gamma_j) = (\tau_{ij}, 0).$$

Therefore, the cocycle λ coincides with τ. This proves the second isomorphism in this lemma. □

6.2 | Cohomology of $\mathcal{L}_\tau^1$

We can do a similar analysis of the cohomology of $\mathcal{L}_\tau^1$.

Lemma 6.3. *In the hypothesis of Theorem 6.1, we have that*

$$H^0(B, \mathcal{L}_\tau^1) = \mathbb{Z}_2$$

$$H^1(B, \mathcal{L}_\tau^1) \cong H^1(B, R^1 f_* \mathbb{Z}_2).$$

Proof. Both isomorphisms follow from the long exact sequence associated to (13). □

6.3 | Proof of Theorem 6.1

We apply the isomorphisms of Lemmas 6.2 and 6.3 to the long exact sequence (17). The first line becomes

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow H^0(\Sigma_\tau, \mathbb{Z}_2) \longrightarrow 0. \tag{22}$$

This proves that Σ_τ is connected. The rest of (17) becomes

$$0 \longrightarrow H^1(B, R^1 f_* \mathbb{Z}_2) \longrightarrow H^1(\Sigma_\tau, \mathbb{Z}_2) \longrightarrow H^1(B, \mathcal{L}_\tau^2) \xrightarrow{\beta}$$
$$H^2(B, \mathcal{L}_\tau^1) \longrightarrow H^2(\Sigma_\tau, \mathbb{Z}_2) \longrightarrow H^2(B, \mathcal{L}_\tau^2) \longrightarrow \dots, \tag{23}$$

which gives

$$b_1(\Sigma_\tau, \mathbb{Z}_2) \leqslant \dim H^1(B, R^1 f_* \mathbb{Z}_2) + \dim H^1(B, \mathcal{L}_\tau^2)$$
$$= \dim H^1(B, R^1 f_* \mathbb{Z}_2) + \dim H^1(B, R^2 f_* \mathbb{Z}_2) - 1.$$

Obviously the equality holds if and only if $\beta = 0$. If the integral cohomology of X has no $\mathbb{Z}_2$ torsion then the dimensions of the spaces on the right-hand side equal the corresponding Hodge numbers.

6.4 | Topology of twisted real $K3$ surfaces

Assume that B has dimension 2 and that it is the affine base of a $K3$ surface. Let Σ_τ be the real twisted $K3$ associated to some twist $\tau \in H^1(B, R^1 \check{f}_* \mathbb{Z}_2)$.

Theorem 6.4. *If the twist is nontrivial, then the real twisted K3 surface Σ_τ is connected and has genus 9.*

Proof. We use the sheaf $\mathcal{L}_\tau^1$ with the properties described in Subsection 4.5. The long exact sequence associated to (16) splits as

$$0 \longrightarrow H^0(B, \mathcal{L}_\tau^1) \longrightarrow H^0(\Sigma_\tau, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \xrightarrow{\alpha}$$
$$\longrightarrow H^1(B, \mathcal{L}_\tau^1) \longrightarrow H^1(\Sigma_\tau, \mathbb{Z}_2) \longrightarrow 0,$$

and

$$0 \longrightarrow H^2(B, \mathcal{L}_\tau^1) \longrightarrow H^2(\Sigma_\tau, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

The sequence (13) gives that $H^0(B, \mathcal{L}_\tau^1) \cong \mathbb{Z}_2$ and

$$0 \longrightarrow H^1(B, \mathcal{L}_\tau^1) \longrightarrow H^1(B, R^1 f_* \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \longrightarrow H^2(B, \mathcal{L}_\tau^1) \longrightarrow 0.$$

Let us prove that the homomorphism α in the first sequence is injective if and only if τ is nontrivial. The argument is similar to the proof of Lemma 6.2. Take some covering $\mathfrak{U} = \{U_i\}$ over which the cycle τ and Σ_τ can be described as in Subsection 4.1. Then, over each U_i we have identifications

$$\mathcal{L}_\tau^1|_{U_i} \xrightarrow{\phi_i} (\Lambda \otimes \mathbb{Z}_2) \oplus \mathbb{Z}_2,$$

which map an affine function β on $\Sigma_\tau|_{U_i}$ to $(v_\beta, \epsilon_\beta)$, where v_β is the linear part of β and $\epsilon_\beta = \beta(-\tilde{\tau}_{U_i}/2)$. It is easy to show that

$$\phi_j \circ \phi_i^{-1}(v, \epsilon) = (v, \epsilon + v(\tau_{ij})).$$

Using the mirror symmetry isomorphism (6), the twist τ has a mirror $\check{\tau} \in H^1(B, R^1 \check{f}_* \mathbb{Z}_2)$, which is represented by $\{\check{\tau}_{ij}\}$, where $\check{\tau}_{ij} \in (\Lambda \otimes \mathbb{Z}_2)_{U_{ij}}$ is obtained by contracting τ_{ij} with a global integral 2-form.

The generator 1 of $\mathbb{Z}_2$ (i.e., the last term in the sequence (16)) can be represented on each U_i as $[\delta_{-\frac{\tilde{\tau}_{U_i}}{2}}]$. Then one can check that $\alpha(1)$ is represented by the 1-cycle $\{(\check{\tau}_{ij}, 1)\}$. This cycle is nonzero if and only if τ is nonzero. Now the statement follows from the above sequences, where we use the fact that $\dim H^1(B, R^1 f_* \mathbb{Z}_2) = 20$. $\qquad\square$

7 | THE RENAUDINEAU–SHAW SPECTRAL SEQUENCE

We will compare our results with the work [20] by Renaudineau and Shaw, where they construct a spectral sequence computing the Betti numbers of real hypersurfaces in toric varieties arising from primitive patchworking. In their case, they consider the n-dimensional tropical hypersurface X, corresponding to the complex hypersurface $\mathbb{C}X$, and they compute the homology with $\mathbb{Z}_2$ coefficients of the real hypersurface $\mathbb{R}X$.

7.1 | The results of Renaudineau–Shaw

The patchworking data (i.e., the choice of signs) are encoded in a cosheaf $\mathcal{S}$ of $\mathbb{Z}_2$ modules over X, called the sign cosheaf. Its homology satisfies $H_q(X, \mathcal{S}) = H_q(\mathbb{R}X, \mathbb{Z}_2)$. Then they construct a filtration of $\mathcal{S}$

$$0 = \mathcal{K}_{n+1} \subset \mathcal{K}_n \subset \dots \subset \mathcal{K}_0 = \mathcal{S}$$

by cosheaves whose quotients

$$\mathcal{F}_p = \frac{\mathcal{K}_p}{\mathcal{K}_{p+1}}$$

are a $\mathbb{Z}_2$ version of the cosheaves defined by Itenberg, Katzarkov, Mikhalkin, and Zharkov (IKMZ) in [18]. The tropical homology groups of X are defined as $H_q(X, \mathcal{F}_p)$. The above filtration induces a spectral sequence, converging to the homology of $\mathbb{R}X$, whose first page is given by $E^1_{q,p} = H_q(X, \mathcal{F}_p)$. The boundary morphisms are maps $\partial : H_{q+1}(X, \mathcal{F}_p) \to H_q(X, \mathcal{F}_{p+1})$. In the original definition of [18], the cosheaves $\mathcal{F}_p$ are defined over $\mathbb{Q}$ and in this case IKMZ prove that

$$\dim(H_q(X, \mathcal{F}_p)) = h^{p,q}(\mathbb{C}X).$$

Under the assumption that the integral tropical homology of X has no torsion, the spectral sequence implies that

$$b_q(\mathbb{R}X, \mathbb{Z}_2) \leqslant \begin{cases} h^{q,q}(\mathbb{C}X) \text{ if } q = n/2, \\ h^{q,n-q}(\mathbb{C}X) + h^{q,q}(\mathbb{C}X) \text{ otherwise.} \end{cases} \tag{24}$$

It follows from [3] that indeed these inequalities hold when the ambient toric variety is smooth. This proves a generalization of the Itenberg conjecture [17]. Renaudineau and Shaw also prove that $\mathbb{R}X$ is maximal if and only if the spectral sequence degenerates at the first page, that is, if and only if all the boundary maps vanish. They conjecture that the spectral sequence degenerates at the second page.

7.2 | Comparison with our case

In our case, B plays the role of the tropical variety X and the sheaf $\pi_{\tau*}\mathbb{Z}_2$ is the replacement for the sign cosheaf $\mathcal{S}$. Notice that we use a sheaf because we compute cohomology, instead of homology. Let us define a filtration of $\pi_{\tau*}\mathbb{Z}_2$ analogous to the one described above. Let V be an n-dimensional $\mathbb{Z}_2$-vector space and let $S_V = \text{Maps}(V, \mathbb{Z}_2)$. We define the subspace $K^p \subset S_V$ as the set of maps that can be defined by a polynomial on V of degree less or equal to p. Then $K^0 \cong \mathbb{Z}^2$ is given by the constant maps, K^1 is the space of affine maps and it can be shown that $K^n = S_V$. Therefore, we have a filtration

$$0 \subset K^0 \subset K^1 \subset \dots \subset K^n = S_V$$

It can be shown (e.g., compare with [20, section 4]) that

$$K^p/K^{p-1} \cong \overset{p}{\bigwedge} V^*.$$

If V is an affine space instead of a vector-space, choosing a point V as the origin turns it into a vector space and thus we can still define the filtration on $\text{Maps}(V, \mathbb{Z}_2)$. It can be shown that this filtration is independent of the chosen point.

We have seen that at a smooth point $b \in B_0$, $\pi_\tau^{-1}(b)$ is an affine space modeled on $\Lambda_b^* \otimes \mathbb{Z}_2$ and therefore we can define a filtration by sheaves on $\pi_{\tau*}\mathbb{Z}_2$, as $(\pi_{\tau*}\mathbb{Z}_2)_b = \text{Maps}(\pi_\tau^{-1}(b), \mathbb{Z}_2)$:

$$0 \subset \mathcal{K}^1 \subset \mathcal{K}^2 \subset \dots \subset \mathcal{K}^n = \pi_{\tau*}\mathbb{Z}_2.$$

Let us denote

$$\mathcal{F}^p = \mathcal{K}^p / \mathcal{K}^{p-1}.$$

Then we have

$$\mathcal{F}^p \cong \bigwedge^p \Lambda = R^p f_* \mathbb{Z}_2,$$

where the last equality follows from (4). Therefore, the sheaves $R^p f_* \mathbb{Z}_2$ play the same role as the cosheaves $\mathcal{F}_p$ in the Renaudineau–Shaw sequence. The above filtration induces a spectral sequence computing the cohomology of Σ_τ. The first page is $E^1_{q,p} = H^q(B, R^p f_* \mathbb{Z}_2)$ and the boundary maps are

$$\partial_\tau : H^q(B, R^{p+1} f_* \mathbb{Z}_2) \to H^{q+1}(B, R^p f_* \mathbb{Z}_2)$$

Let us now compare this spectral sequence with our sequences. We restrict to the case $n = 3$, so that $\mathcal{K}^3 = \pi_{\tau*} \mathbb{Z}_2$. Notice that by definition

$$\mathcal{L}^1_\tau = \mathcal{K}^1.$$

Moreover, the sequence (12) defining $\mathcal{L}^2_\tau$ implies that

$$\mathcal{L}^2_\tau = \mathcal{K}_3 / \mathcal{K}_1.$$

The sequence (13) is the sequence

$$0 \to \mathcal{K}^0 \to \mathcal{K}^1 \to \mathcal{F}^1 \to 0$$

defining $\mathcal{F}^1$. The sequence (14) corresponds to

$$0 \to \mathcal{F}^2 \to \mathcal{K}^3 / \mathcal{K}^1 \to \mathcal{F}^3 \to 0. \tag{25}$$

It is not difficult to show that the homomorphism β' defined in (18) coincides with the corresponding boundary map from the spectral sequence. Moreover, the connecting homomorphism

$$H^0(B, \mathcal{F}^3) \to H^1(B, \mathcal{F}^2)$$

from sequence (25) is also a boundary map from the spectral sequence and it is equivalent to the connecting homomorphism

$$\mathbb{Z}_2 \to H^1(B, R^2 f_* \mathbb{Z}_2)$$

from (14). In particular, Lemma 6.2 shows that, with the given hypothesis, this map is always injective when the twist τ is nontrivial. Therefore, in the hypothesis of Section 6, the spectral sequence never degenerates at the first page when τ is nontrivial.

7.3 | Mirror symmetry

Let us now consider mirror symmetry. Applying the isomorphism (6), the boundary homomorphisms for the first page of the spectral sequence become

$$\partial_\tau : H^q(B, R^{n-p-1}\check{f}_*\mathbb{Z}_2) \to H^{q+1}(B, R^{n-p}\check{f}_*\mathbb{Z}_2).$$

When $n = 3$, Theorem 5.1 and Lemma 6.2 explicitly describe these homomorphisms in the cases $(q, p) = (1, 1)$ and $(q, p) = (0, 2)$.

Question. Can we explicitly describe these homomorphisms for all n, p and q?

For instance, in three dimensions, there is one more case to compute in order to determine the spectral sequence, that is, $(q, p) = (2, 0)$. In fact, if the boundary map in this case is injective, then the spectral sequence degenerates at the second page. We were not able to find an explicit description of this case.

8 | A CONNECTED $(M - 2)$-REAL QUINTIC

In the case of the quintic threefold, with the torus fibration described by Gross in [11], Arguz–Prince [2] computed the cohomology of the untwisted real quintic (i.e., $L_\tau = 0$). They found that the map β has rank 73, and hence that $b_1(\Sigma) = 29$. In particular, the untwisted real quintic is not maximal. As also the twisted quintics are not maximal, as proved in Section 6, none of the real quintics constructed in this way is maximal. The highest possible value of b_1 for our twisted real Calabi–Yau's is when $\beta_\tau = 0$. In this case, as proved in Section 6, the real Calabi–Yau is an $(M - 2)$-real variety. We prove the following.

Theorem 8.1. *There exists a connected $(M - 2)$ real twisted quintic Σ_τ. In particular,*

$$b_1(\Sigma_\tau, \mathbb{Z}_2) = 101.$$

We will prove this by finding a divisor L in the mirror quintic $\check{X}$ such that

$$D^2 + DL = 0 \quad \forall D \in H^2(\check{X}, \mathbb{Z}_2). \tag{26}$$

8.1 | The mirror quintic

Consider the same simplex P as in Subsection 2.4, that is, the one with vertices

$$V_0 = (-1, -1, -1, -1), \quad V_1 = (4, -1, -1, -1), \quad V_2 = (-1, 4, -1, -1),$$

$$V_3 = (-1, -1, 4, -1), \quad V_4 = (-1, -1, -1, 4).$$

It is a reflexive polytope with the origin as its only interior integral point. The fan whose cones are the cones over the faces of P gives a singular toric variety $\check{Y}_P$. We can resolve $\check{Y}_P$ by taking
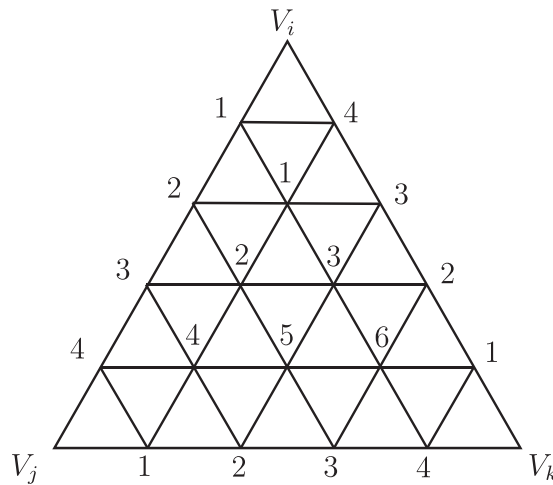
FIGURE 5   Triangulation of 2-dimensional faces.

a unimodular regular subdivision of the boundary ∂P and take the associated fan, which gives a smooth toric variety $\check{Y}$. The mirror quintic $\check{X}$ is a smooth anti-canonical divisor in $\check{Y}$. Let us take, as in Gross [11], a subdivision of ∂P which on 2-dimensional faces looks like Figure 5.

8.2 | Divisors in the mirror quintic

Each vertex in the subdivision of ∂P corresponds to a one dimensional cone of the fan, hence to a toric divisor in $\check{Y}$. The divisors corresponding to vertices inside 2-dimensional faces of ∂P are precisely the ones that have nontrivial intersection with the mirror quintic $\check{X}$, therefore they correspond to nonzero divisors in $\check{X}$. Gross shows that these divisors generate $H^2(\check{X}, \mathbb{Z})$, and hence also $H^2(\check{X}, \mathbb{Z}_2)$ (see [11, Lemma 4.3]). With some abuse of notation, we denote by $V_0, \ldots, V_4$ also the divisors corresponding to the vertices of ∂P. Moreover, we denote by E_{ij}^{ℓ} the divisor corresponding to the ℓth interior vertex along the edge from V_i to V_j, where interior vertices of edges are numbered as in Figure 5. The divisors in the interior of the 2-face with vertices V_i, V_j, and V_k, numbered as in Figure 5, are denoted by F_{ijk}^{ℓ}. In [11, Proposition 4.2], Gross also computes triple intersection numbers $D_1 D_2 D_3$ between these divisors. We report here the mod 2 version.

Proposition 8.2 [11, Proposition 4.2]. *The   mod 2 triple intersection numbers between the above divisors in $\check{X}$ are as follows.*

(1) *$(V_i)^3 = (E_{ij}^{\ell})^3 = 1$ and $(F_{ijk}^{\ell})^3 = 0$.*
(2) *Given two distinct divisors D_1 and D_2 lying on the same 2-face, then $(D_1)^2 D_2 = 1$ if and only if D_1 and D_2 are connected by one edge in the graph depicted in Figure 6.*
(3) *Given three distinct divisors D_1, D_2, D_3 lying on the same 2-face, then $D_1 D_2 D_3 = 1$ if and only if they are vertices of a two simplex in Figure 5.*
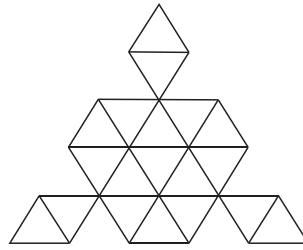
All other triple intersections are zero.

FIGURE 6   Triple intersection graph.

Let S be the set of all divisors of type V_i, E_{ij}^ℓ or F_{ijk}^ℓ. A general divisor in $H^2(\check{X}, \mathbb{Z}_2)$ can be written as

$$L = \sum_{D \in S} \epsilon_D D,$$

where $\epsilon_D \in \mathbb{Z}_2$. We can obviously view L as a subset of S, where $D \in L$ if and only if $\epsilon_D = 1$. We have that L satisfies (26) if and only if

$$D_1^2 D_2 = LD_1 D_2, \quad \forall D_1, D_2 \in S.$$

8.3 | Local configurations

It follows from Proposition 8.2 that if D_1 and D_2 do not belong to the same 2-face then $D_1^2 D_2 = LD_1 D_2 = 0$. We now consider the following cases (and subcases).

(1) D_1 and D_2 lie in the same 2-face but not in the same edge of ∂P.
    (1.1) $D_1 \neq D_2$ and they are connected by an edge of the subdivision.
    (1.2) $D_1 \neq D_2$ and they are not connected by an edge of the subdivision.
    (1.3) $D_1 = D_2$ and it is in the interior of a 2-face.
(2) D_1 and D_2 lie inside the same edge.
    (2.1) $D_1 = D_2$.
    (2.2) $D_1 \neq D_2$ and they are connected by an edge in the graph of Figure 6.
    (2.3) $D_1 \neq D_2$ and they are not connected by an edge in the graph of Figure 6.

We now see how in the above cases the condition $D_1^2 D_2 = LD_1 D_2$ imposes certain local configurations on L.

Case 1.1. Let D_3 and D_4 be the other vertices of the two simplices that contain the edge from D_1 to D_2 as in Figure 7. Then, using Proposition 6, we have

$$D_1^2 D_2 = 1 \quad \text{and} \quad LD_1 D_2 = \sum_{j=1}^{4} \epsilon_{D_j}.$$

Therefore, we have $D_1^2 D_2 = LD_1 D_2$ if and only if only an odd number of the vertices D_1, D_2, D_3, D_4 belong to L.
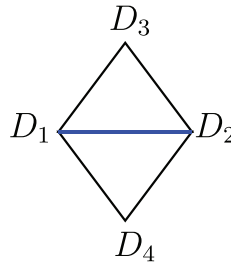
FIGURE 7   If D_1 and D_2 are on the same 2-face and are connected by an edge, then only an odd number of the D_1, D_2, D_3, D_4 can be in L.
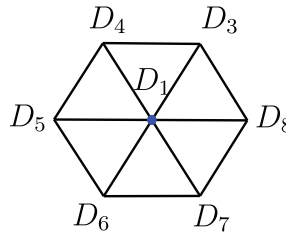

FIGURE 8   If $D_1 = D_2$ is in the interior of a 2-face, then an even number of the $D_3, \dots, D_8$ can be in L.

Case 1.2. It is easy to see that in this case, both $D_1^2 D_2$ and $LD_1 D_2$ are always zero.

Case 1.3. If $D_1 = D_2$ and it is in the interior of a 2-faces of ∂P, then $D_1^2 D_2 = D_1^3 = 0$. On the other hand,

$$LD_1^2 = \sum_{j=3}^{8} \epsilon_{D_j},$$

where $D_3, \dots, D_8$ are the six vertices adjacent to D_1.

Therefore, in this case, $D_1^2 D_2 = LD_1 D_2$ if and only if an even number of the $D_3, \dots, D_8$ belong to L (see Figure 8).

Case 2.1. If $D_1 = D_2$ and it lies on an edge of ∂P, then $D_1^2 D_2 = D_1^3 = 1$. Let S_{D_1} be the subset of S consisting of D_1 and of all the vertices that are connected to D_1 via an edge of the graph in Figure 6 for some 2-face containing D_1. For example, if D_1 is of type V_j, that is, it is a vertex of ∂P, then S_{D_1} contains five elements, otherwise if D_1 is in the interior of an edge, S_{D_1} will contain eight elements. Then we have

$$LD_1 D_2 = LD_1^2 = \sum_{D \in S_{D_1}} \epsilon_D.$$

Therefore, in this case $D_1^2 D_2 = LD_1 D_2$ if and only if $S_{D_1} \cap L$ contains an odd number of elements.

Case 2.2. Let $D_1 \neq D_2$ belong to some edge of ∂P and assume they are connected by an edge in the graph of Figure 6 (e.g., $D_1 = E_{ij}^2$ and $D_2 = E_{ij}^3$). Let $S_{D_1 D_2}$ be the subset of S consisting of D_1,
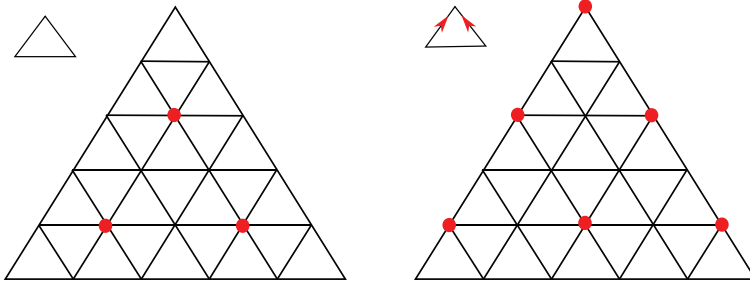
FIGURE 9   The "empty" face and the "arrow".

D_2 and the vertices D such that D_1, D_2 and D are vertices of a 2-simplex of the subdivision. In particular, $S_{D_1 D_2}$ contains five elements. Then we have $D_1^2 D_2 = 1$ and

$$L D_1 D_2 = \sum_{D \in S_{D_1 D_2}} \epsilon_D.$$

Therefore, in this case $D_1^2 D_2 = L D_1 D_2$ if and only if $S_{D_1 D_2} \cap L$ contains an odd number of elements.

Case 2.3. Let $D_1 \neq D_2$ belong to some edge of ∂P and assume they are not connected by an edge in the graph of Figure 6. We have two possibilities: either D_1 and D_2 are not adjacent (e.g., $D_1 = E_{ij}^2$ and $D_2 = E_{ij}^4$) or they are adjacent but the edge between them is not part of the graph in Figure 6 (e.g., $D_1 = E_{ij}^1$ and $D_2 = E_{ij}^2$). In both cases we have $D_1^2 D_2 = 0$. Let $S_{D_1 D_2}$ be the subset of S consisting of the vertices D such that D_1, D_2 and D are vertices of a 2-simplex of the subdivision. Obviously $S_{D_1 D_2}$ is empty in the first case and consists of three elements in the second case. Then

$$L D_1 D_2 = \sum_{D \in S_{D_1 D_2}} \epsilon_D.$$

Therefore, in this case $D_1^2 D_2 = L D_1 D_2$ if and only if $S_{D_1 D_2} \cap L$ contains an even number of elements.

8.4 | Proof of Theorem 8.1

In Figure 9, we give two examples of a 2-face and divisor L (the dotted "red" vertices) such that all interior edges satisfy the configurations of Case 1.1 and all interior vertices satisfy the configurations of Case 1.3. We call the picture on the left an "empty face" (because its edges are empty) and the one on the right the "arrow" and we also depict their symbols. The arrows, in the symbol for the arrow, correspond to nonempty edges and they point toward the vertex that lies in L.

   In Figure 10, we give two examples of a neighborhood of an edge of ∂P and a divisor L such that every vertex of the edge satisfies the configurations of Case 2.1 and every edge satisfies the configurations of either Case 2.2 or Case 2.3. As each edge of ∂P is contained in three 2-faces, we have glued together three copies (blue, black and red) of the graph in Figure 6 along an edge. Notice that the configuration on the left can be obtained by gluing three copies of an "arrow" along a nonempty edge. The configuration on right can be obtained by gluing two copies of an "empty" face and one copy of an "arrow", along their empty edges.
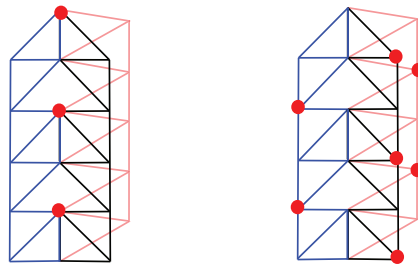
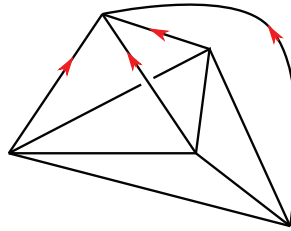**FIGURE 10**   Two configurations along an edge.

FIGURE 11   A global configuration with empty faces and arrows.

Figure 11 describes a global example of a divisor $L \in H^2(\check{X}, \mathbb{Z}_2)$ satisfying $D^2 + LD = 0$ for all $D \in H^2(\check{X}, \mathbb{Z}_2)$. We have depicted the graph formed by the edges of ∂P. Each triple of vertices corresponds to a 2-face of ∂P and this 2-faces is either an "empty" face or an "arrow" depending on whether the decoration of the edges matches the corresponding symbols. It is clear that all nonempty edges will satisfy the configuration on the left in Figure 10 and all empty edges will satisfy the configuration on the right.

8.5 │ The twisted real mirror quintic

Let $\check{X}$ be the mirror of the quintic. We study the topology of a twisted real mirror quintic in $\check{X}$, which we denote by $\check{\Sigma}_\tau$. In this case,

$$H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{101} \quad \text{and} \quad H^1(B, R^2 \check{f}_* \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

In the untwisted case, $\check{\Sigma}$ has two connected components and it follows from the result of Arguz and Prince that $b_1(\check{\Sigma}_0) = 101$ (see [2, Example 4.10]). As $H^1(B, R^1 f_* \mathbb{Z}_2) \cong \mathbb{Z}_2$, there is only one twisted real mirror quintic $\check{\Sigma}_\tau$. It follows from Lemmas 6.2 and 6.3 that $H^1(B, \mathcal{L}_\tau^2) = 0$ and $H^1(B, \mathcal{L}_\tau^1) = (\mathbb{Z}_2)^{101}$. Then, sequence (17) implies

$$b_1(\Sigma_\tau) = 101.$$

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Diego Matessi https://orcid.org/0000-0002-4603-2214

REFERENCES

1. H. Argüz and T. Prince, *A note on Schoen's Calabi–Yau threefolds*, Rend. Semin. Mat. Univ. Politec. Torino **79** (2021), no. 1, 15–20. arXiv:1807.10172.

2. H. Argüz and T. Prince, *Real Lagrangians in Calabi–Yau threefolds*, Proc. Lond. Math. Soc. (3) **121** (2020), no. 2, 287–311.

3. C. Arnal, A. Renaudineau, and K. Shaw, *Lefschetz section theorems for tropical hypersurfaces*, Ann. H. Lebesgue **4** (2021), 1347–1387. arXiv:1907.06420.

4. P. S. Aspinwall, T. Bridgeland, A. Craw, M. R. Douglas, M. Gross, A. Kapustin, G. W. Moore, G. Segal, B. Szendroi, and P. M. H. Wilson, *Dirichlet branes and mirror symmetry*, Clay Mathematics Monographs, vol. 4, American Mathematical Society, Providence, RI, 2009.

5. V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), no. 3, 1056–3911.

6. R. C. Bernard and D. Matessi, *The fixed point set of anti-symplectic involutions of Lagrangian fibrations*, Rend. Semin. Mat. Univ. Politec. Torino **68** (2010), no. 3, 235–250.

7. R. Castano-Bernard and D. Matessi, *Lagrangian 3-torus fibrations*, J. Differential Geom. **81** (2009), no. 3, 483–573. arXiv:math/0611139.

8. R. Castano-Bernard, D. Matessi, and J. Solomon, *Symmetries of Lagrangian Fibrations*, Adv. in Math. **225** (2010), no. 3, 1341–1386.

9. M. Gross, *Special Lagrangian fibrations. I. Topology*, AMS/IP Stud. Adv. Math., vol. 23, American Mathematical Society, Providence, RI, 2001, pp. 65–93.

10. M. Gross, *Special Lagrangian fibrations II. Geometry*, Surv. Differ. Geom. **5** (1999), 341–403. math.AG/9809072.

11. M. Gross, *Topological mirror symmetry*, Invent. Math. **144** (2001), 75–137.

12. M. Gross, *Toric degenerations and Batirev–Borisov duality*, Math. Ann. **333** (2005), 645–688. math.AG/0406171.

13. M. Gross and B. Siebert, *Mirror symmetry via logarithmic degeneration data I*, J. Differential Geom. **72** (2006), no. 2, 169–338. math.AG/0309070.

14. M. Gross and B. Siebert, *Mirror symmetry via logarithmic degeneration data, II*, J. Algebraic Geom. **19** (2010), no. 4, 679–780. arXiv:0709.2290.

15. I. Itenberg, *Contre-examples à la conjecture de Ragsdale*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 3, 277–282.

16. I. Itenberg, *Topology of real algebraic T-surfaces*, Rev. Mat. Univ. Complut. Madrid **10** (1997), 131–152.

17. I. Itenberg, *Tropical homology and Betti numbers of real algebraic varieties.* https://web.ma.utexas.edu/users/sampayne/pdf/Itenberg-Simons2017.pdf, 2017.

18. I. Itenberg, L. Katzarkov, G. Mikhalkin, and I. Zharkov, *Tropical homology*, Math. Ann. **374** (2019), no. 1–2, 963–1006. arXiv:1604.01838.

19. I. Itenberg, G. Mikhalkin, and E. Shustin, *Tropical algebraic geometry*, Oberwolfach Seminars, Birkhäuser Basel, 2008.

20. A. Renaudineau and K. Shaw, *Bounding the Betti numbers of real hypersurfaces near the tropical limit*, Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), no. 3, 945–980.

21. H. Ruddat and I. Zharkov, *Compactifying torus fibrations over integral affine manifolds with singularities*, MATRIX Book Ser., 4, Springer, Cham, 2021, pp. 609–622.

22. E. Strominger, S-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nucl. Phys. B **479** (1996), 243–259. hep-th/9606040.

23. O. Viro, *Patchworking real algebraic varieties*, arXiv:math/0611382, 2006.

24. O. Y. Viro, *Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7*, Topology (Leningrad, 1982), Lecture Notes in Math., vol. 1060, Springer, Berlin, 1984, pp. 187–200.

25. G. Wilson, *Hilbert's sixteenth problem*, Topology **17** (1978), no. 1, 53–73.