SINGULAR HERMITIAN METRICS AND THE DECOMPOSITION THEOREM OF CATANESE, FUJITA, AND KAWAMATA

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Abstract. We prove that a torsion-free sheaf $\mathcal{F}$ endowed with a singular hermitian metric with semi-positive curvature and satisfying the minimal extension property admits a direct-sum decomposition $\mathcal{F} \simeq \mathcal{U} \oplus \mathcal{A}$ where $\mathcal{U}$ is a hermitian flat bundle and $\mathcal{A}$ is a generically ample sheaf. The result applies to the case of direct images of relative pluricanonical bundles $f_*\omega_{X/Y}^{\otimes m}$ under a surjective morphism $f\colon X \to Y$ of smooth projective varieties with $m \geq 2$. This extends previous results of Fujita, Catanese–Kawamata, and Iwai.

1. Introduction

Let $f\colon X \to Y$ be a fibration of smooth projective varieties (over the complex numbers) and let $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$ be the relative canonical bundle. Motivated by earlier work of Fujita, Catanese and Kawamata [CK19, Theorem 1.2] proved a direct-sum decomposition

$$f_*\omega_{X/Y} \simeq \mathcal{U} \oplus \mathcal{A} \tag{1}$$

where $\mathcal{U}$ is a hermitian flat bundle and $\mathcal{A}$ is a generically ample sheaf, if not zero. We recall that a coherent torsion-free sheaf $\mathcal{A}$ on a smooth projective variety is *generically ample* if its restriction to a general complete intersection smooth curve is an ample vector bundle. In dimension one we say that $\mathcal{A}$ is generically ample if it is ample. When Y is a smooth projective curve, the decomposition (1) is Fujita's second decomposition in [Fuj78] (see also [CD17, Theorem 3.3]). Motivated by (1), we introduce the following definition.

Definition 1. A coherent torsion-free sheaf $\mathcal{F}$ admits a *Catanese–Fujita–Kawamata decomposition* if there exists an isomorphism $\mathcal{F} \simeq \mathcal{U} \oplus \mathcal{A}$ where $\mathcal{U}$ is a hermitian flat bundle, and $\mathcal{A}$ is either a generically ample sheaf or the zero sheaf.

In this paper we extend decomposition (1) to direct images of relative *pluricanonical* bundles.

Theorem 2. *If $f\colon X \to Y$ is a surjective morphism of smooth projective complex varieties, then $f_*\omega_{X/Y}^{\otimes m}$ admits a Catanese–Fujita–Kawamata decomposition for every $m \geq 2$.*

When Y is a smooth projective curve, Theorem 2 was proved by Iwai in [Iwa22, Theorem 1.4] by showing in greater generality that the reflexive hull $\left(f_*\omega_{X/Y}^{\otimes m}\right)^{**}$ admits a Catanese–Fujita–Kawamata decomposition for any smooth projective variety Y and $m \geq 1$.

While the proof of (1) is Hodge-theoretic, the proof of Theorem 2 is based on the fact that, for any $m \geq 2$, the sheaf $f_*\omega_{X/Y}^{\otimes m}$ carries a canonical singular hermitian metric with semi-positive curvature. Furthermore, this metric satisfies the so-called *minimal extension property* (*cf.* §2, [PT18, Theorem 1.1] and [HPS18, Theorem 27.1]). This is a property that stems from Ohsawa–Takegoshi's extension theorem with optimal bounds and allows one to extend local sections across subsets of measure zero with a control on the L^2-norm. Instances of bundles satisfying the minimal extension property are pseudo-effective line bundles and Nakano semi-positive vector bundles. Theorem 2 is an application of the following theorem.

Theorem 3. *Let $\mathcal{F}$ be a coherent torsion-free sheaf on a smooth projective variety Y endowed with a singular hermitian metric with semi-positive curvature and satisfying the minimal extension property. Then $\mathcal{F}$ admits a Catanese–Fujita–Kawamata decomposition.*

Proof of Theorem 2. By [PT18, Theorem 1.1] and [HPS18, Theorem 27.1] the push-forward $f_*\omega_{X/Y}^{\otimes m}$ admits a singular hermitian metric with semi-positive curvature and satisfying the minimal extension property for any $m \geq 2$. The result follows by Theorem 3. □

As an application of Theorem 2, we prove a structure theorem for the sheaves $f_*\omega_{X/Y}^{\otimes m}$ with $m \geq 2$.

Theorem 4. *Let $f\colon X \to Y$ be a fibration of smooth projective complex varieties and denote $J = \{\, m \in \mathbb{N}_{\geq 2} \,|\, f_*\omega_{X/Y}^{\otimes m} \neq 0 \,\}$. Suppose there exists an open subset $U \subset Y$ such that:* $\operatorname{codim}(Y \smallsetminus U) \geq 2$*, the morphism f is smooth over U, and $\omega_{V/U}$ is $f|_V$-semi-ample where $V = f^{-1}(U)$. Then $f_*\omega_{X/Y}^{\otimes m}$ is either generically ample for all $m \in J$, or hermitian flat for all $m \in J$.*

In §4 we collect further instances of Catanese–Fujita–Kawamata decompositions.

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2. The minimal extension property

Let Y be a complex manifold of positive dimension n and let $(\mathcal{F}, h)$ be a torsion-free sheaf endowed with a singular hermitian metric. We refer to [HPS18, §19] for the definition of singular hermitian metrics with semi-positive curvature on torsion-free sheaves. Denote by $B \subset \mathbb{C}^n$ the open unit ball centered at the origin with volume $\mu(B) = \pi^n/n!$. Moreover let F_y be the fiber of $\mathcal{F}$ at a point $y \in Y$ over which $\mathcal{F}$ is locally free. We recall the definition of the minimal extension property introduced in [HPS18, Definition 20.1].

Definition 5. The pair $(\mathcal{F}, h)$ satisfies the *minimal extension property* if there exists an analytic nowhere dense closed subset $Z \subset Y$ such that $\mathcal{F}$ is locally free on $Y \setminus Z$, and for every embedding $\iota \colon B \hookrightarrow Y$ of the open unit ball centered at $\iota(0) = y \in Y \setminus Z$, and vector $v \in F_y$ of length $|v|_{h,y} = 1$, there exists a holomorphic section $s \in H^0(B, \iota^* \mathcal{F})$ such that

$$s(0) = v \quad \text{and} \quad \frac{1}{\mu(B)} \int_B |s|_h^2 \, d\mu \leq 1.$$

We recall a few properties of the minimal extension property from [HPS18].

Proposition 6. *Let $(\mathcal{F}, h)$ be a torsion-free sheaf endowed with a singular hermitian metric satisfying the minimal extension property and let $b \colon \mathcal{F} \hookrightarrow \mathcal{G}$ be an inclusion of torsion-free sheaves. If b is generically an isomorphism, then h extends to a singular hermitian metric $h_{\mathcal{G}}$ on $\mathcal{G}$ satisfying the minimal extension property. Moreover, if h has semi-positive curvature, then $h_{\mathcal{G}}$ has semi-positive curvature as well.*

Proof. The proposition is essentially proved in [HPS18, Proposition 19.3] where it is stated in greater generality. It only remains to note that the minimal extension property holds for $(\mathcal{G}, h_{\mathcal{G}})$, but this is true because every section of $\mathcal{F}$ is also a section of $\mathcal{G}$. □

Proposition 7. *Let $(\mathcal{F}, h)$ be a torsion-free sheaf endowed with a singular hermitian metric satisfying the minimal extension property. If $\varphi \colon \mathcal{F} \twoheadrightarrow \mathcal{E}$ is a quotient onto a torsion-free sheaf and h' is the induced metric, then $(\mathcal{E}, h')$ satisfies the minimal extension property.*

Proof. Let $Z = Z(\mathcal{F})$ be as in Definition 5 and let $S(\mathcal{E})$ be the locus where $\mathcal{E}$ is not locally free. Set $Z' = Z \cup S(\mathcal{E})$ and let $y \in Y \setminus Z'$. For $w \in E_y$ the induced metric h' on $\mathcal{E}$ is defined by

$$|w|_{h',y} = \inf\{\, |v|_{h,y} \mid v \in F_y \text{ and } \varphi_y(v) = w \,\}. \tag{2}$$

(If $\varphi_y = 0$, then the metric is $+\infty$ for all $w \neq 0$.) Now let $w \in E_y$ be such that $|w|_{h',y} = 1$ and let $B \subset Y$ be the embedding of the unit ball centered at $y \in Y \setminus Z'$. Then there exists $v \in F_y$ such that $|v|_{h,y} = 1$ and, by the minimal extension property of $\mathcal{F}$, a holomorphic section $s \in H^0(B, \mathcal{F}|_B)$ such that $s(0) = v$ and $\frac{1}{\mu(B)} \int_B |s|_h^2 d\mu \leq 1$. As $|\varphi(s)|_{h',y} \leq |s|_{h,y}$ for almost every $y \in B$, this yields inequalities

$$\frac{1}{\mu(B)} \int_B |\varphi(s)|_{h'}^2 \, d\mu \leq \frac{1}{\mu(B)} \int_B |s|_h^2 \, d\mu \leq 1.$$

□

The following result generalizes [HPS18, Theorem 26.4].

Proposition 8. *Suppose Y is a compact complex manifold and let $(\mathcal{F}, h)$ be a torsion-free sheaf endowed with a singular hermitian metric of semi-positive curvature and satisfying the minimal extension property. If $f\colon \mathcal{F} \twoheadrightarrow \mathcal{U}$ is a quotient onto a vector bundle $\mathcal{U}$ endowed with a smooth hermitian flat metric, then there exists a morphism $s\colon \mathcal{U} \to \mathcal{F}$ such that $f \circ s = \mathrm{id}_{\mathcal{U}}$.*

Proof. Set $r = \mathrm{rk}\big(\mathcal{U}\big) > 0$. The bundle $\mathcal{U}$ is associated to a representation $\pi_1(Y) \to U(r)$ of the fundamental group of Y to the unitary group $U(r)$. Hence $\mathcal{U}$ decomposes as a direct sum of vector bundles arising from irreducible unitary representations. Without loss of generality we can suppose that $\mathcal{U}$ is irreducible.

Consider the quotients

$$\mathcal{F} \otimes \mathcal{U}^{\vee} \twoheadrightarrow \mathcal{U} \otimes \mathcal{U}^{\vee} \overset{tr}{\twoheadrightarrow} \mathcal{O}_Y$$

where the first is induced by f and the second is the trace map of $\mathcal{U}$. Since the metric on $\mathcal{U}$, and hence on $\mathcal{U}^{\vee}$, is flat, the induced singular hermitian metric on the sheaf $\mathcal{F}\otimes\mathcal{U}^{\vee}$ has semi-positive curvature and the minimal extension property. By [HPS18, Theorem 26.4] there exists a splitting $s'\colon \mathcal{O}_Y \to \mathcal{F} \otimes \mathcal{U}^{\vee}$, and hence a non-trivial morphism $s''\colon \mathcal{U} \to \mathcal{F}$ such that $f \circ s'' \neq 0$. By Schur's Lemma the composition $f \circ s''$ is an isomorphism and $s := s'' \circ (f \circ s'')^{-1}$ splits f. □

Remark 9. Following [HIM22, Theorem 1.4] the previous proposition remains valid when $\mathcal{F}$ is reflexive and h does not necessarily satisfy the minimal extension property.

Finally we recall the following theorem [HPS18, Theorem 26.1] which is based on an earlier result of Cao and Păun [CP17, Lemma 5.3]. We define the determinant of a torsion-free sheaf $\mathcal{F} \neq 0$ as $\det(\mathcal{F}) = \big(\bigwedge^{\mathrm{rk}(\mathcal{F})} \mathcal{F}\big)^{**}$.

Theorem 10. *Let Y be a compact complex manifold and let $(\mathcal{F}, h)$ be a nonzero torsion-free sheaf endowed with a singular hermitian metric. Suppose h has semi-positive curvature and satisfies the minimal extension property. If $c_1\big(\det(\mathcal{F})\big) = 0$ in $H^2(Y, \mathbb{R})$, then $\mathcal{F}$ is locally free, h is smooth and $(\mathcal{F}, h)$ is hermitian flat.*

3. Catanese–Fujita–Kawamata decompositions

3.1. **Proof of Theorem 3.** Set $n = \dim Y$. We may suppose $n > 0$ and $\mathcal{F} \neq 0$. By [HPS18, Proposition 25.1, §26 and Definition 19.1] the line bundle $\det(\mathcal{F})$ admits a singular hermitian metric with semi-positive curvature. Hence $\det(\mathcal{F})$ is pseudo-effective and for any very ample line bundle A on Y the *degree* of $\mathcal{F}$ satisfies

$$\deg_A(\mathcal{F}) \;:=\; \big(A^{n-1} \cdot \mathcal{F}\big) \;\geq\; 0$$

by Nakai–Moishezon's Theorem. We define the *A-slope* of $\mathcal{F}$ as

$$\mu^A(\mathcal{F}) \;:=\; \frac{\deg_A(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$$

and say that it is *semistable* (with respect to A) if for every nonzero coherent sub-module $\mathcal{E} \subset \mathcal{F}$ the inequality $\mu^A(\mathcal{E}) \leq \mu^A(\mathcal{F})$ holds. Let

$$0 = \mathcal{N}_0 \subsetneq \mathcal{N}_1 \subsetneq \ldots \subsetneq \mathcal{N}_d = \mathcal{F} \tag{3}$$

be the Harder–Narasimhan filtration of $\mathcal{F}$. Hence for any $i = 1, \ldots, d$ the quotients $\mathcal{N}_i/\mathcal{N}_{i-1}$ are torsion-free and semistable. Moreover the slopes

$$\mu_i^A := \mu^A(\mathcal{N}_i/\mathcal{N}_{i-1})$$

satisfy

$$\mu_1^A > \mu_2^A > \ldots > \mu_d^A$$

[Mar81, Proposition-Definition 1.13]. We denote by

$$\mathcal{Q} := \mathcal{F}/\mathcal{N}_{d-1} = \mathcal{N}_d/\mathcal{N}_{d-1}$$

the minimal destabilizing quotient of $\mathcal{F}$ and set

$$\mathcal{L} := \det(\mathcal{Q})$$

for the determinant of $\mathcal{Q}$. By [PT18, Lemma 2.4.3] and [HPS18, Proposition 25.1] both $\mathcal{Q}$ and $\mathcal{L}$ admit a singular hermitian metric with semi-positive curvature. Hence $\mathcal{L}$ is pseudo-effective and $\deg_A(\mathcal{L}) \geq 0$. We distinguish two cases: $\deg_A(\mathcal{L}) > 0$ and $\deg_A(\mathcal{L}) = 0$.

Let's begin with the case $\deg_A(\mathcal{L}) > 0$. We are going to show that $\mathcal{F}$ is already generically ample. Let $H = A^{\otimes a}$ be a very ample line bundle with $a \gg 0$ so that Flenner's Theorem [HL10, Theorem 7.1.1] applies to a general complete intersection smooth curve C cut out by divisors in $|H|$ and contained in the locus where $\mathcal{F}$ is locally free. It follows that the Harder–Narasimhan filtration (3) of $\mathcal{F}$ restricts to the Harder–Narasimhan filtration

$$0 = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \ldots \subsetneq \mathcal{M}_d = \mathcal{F}|_C$$

of $\mathcal{F}|_C$. Here the sheaves $\mathcal{M}_i := \mathcal{N}_i|_C$ are locally free and semistable, and

$$\frac{\deg\big(\mathcal{M}_i/\mathcal{M}_{i-1}\big)}{\operatorname{rk}\big(\mathcal{M}_i/\mathcal{M}_{i-1}\big)} > \frac{\deg\big(\mathcal{M}_{i+1}/\mathcal{M}_i\big)}{\operatorname{rk}\big(\mathcal{M}_{i+1}/\mathcal{M}_i\big)}$$

for all $i = 1, \ldots, d-1$. Since

$$\deg\big(\mathcal{M}_i/\mathcal{M}_{i-1}\big) = \Big(H^{n-1} \cdot \mathcal{N}_i/\mathcal{N}_{i-1}\Big) = a^{n-1}\Big(A^{n-1} \cdot \mathcal{N}_i/\mathcal{N}_{i-1}\Big),$$

the *minimal slope* of $\mathcal{F}|_C$ satisfies

$$\mu_{\min}\big(\mathcal{F}|_C\big) := \frac{\deg\big(\mathcal{M}_d/\mathcal{M}_{d-1}\big)}{\operatorname{rk}\big(\mathcal{M}_d/\mathcal{M}_{d-1}\big)} = a^{n-1}\mu_d^A = a^{n-1}\frac{\deg_A(\mathcal{L})}{\operatorname{rk}(\mathcal{Q})} > 0.$$

By [Bre04, Theorem 2.1] $\mathcal{F}|_C$ is an ample bundle. In this case we set $\mathcal{U} = 0$ and $\mathcal{A} = \mathcal{F}$.

Now let us suppose that $\deg_A(\mathcal{L}) = 0$. We will first show that $c_1(\mathcal{L}) = 0$ in $H^2(Y, \mathbb{R})$. Let $H = A^{\otimes a}$ be a very ample line bundle as before with a sufficiently large. Let $D_1, \ldots, D_{n-2} \in |H|$ be general members such that for all $i = 2, \ldots, n-2$ each partial intersection

$$V_i := D_1 \cap D_2 \cap \ldots \cap D_i$$

is a smooth and irreducible ample divisor in V_{i-1}. Hence $S := V_{n-2}$ is a smooth surface with $[S] = H^{n-2}$ in $H_4(Y;\mathbb{Z})$ such that the restriction map

$$H^2(Y;\mathbb{Z}) \to H^2(S;\mathbb{Z}) \quad \text{is injective} \tag{4}$$

(see Lefschetz's Hyperplane Theorem [Laz04, Theorem 3.1.17]). Moreover, we can choose the general divisors $D_i \in |H|$ such that the restrictions $\mathcal{L}|_{V_i}$ are pseudo-effective. Hence

$$0 \;=\; \big(H^{n-1} \cdot \mathcal{L}\big) \;=\; \big(H|_S \cdot \mathcal{L}|_S\big)$$

from which we deduce that $\mathcal{L}|_S$ is numerically trivial as $H|_S$ is ample and $\mathcal{L}|_S$ is a limit of effective classes (*cf.* [Laz04, Theorem 1.4.29]). By (4) the claim follows.

Since $c_1(\mathcal{L}) = 0$, by Proposition 7 and Theorem 10 the bundle $\mathcal{Q}$ is hermitian flat. Moreover, by Proposition 8 there exists a decomposition $\mathcal{F} \simeq \mathcal{Q} \oplus \mathcal{N}_{d-1}$ so that we only need to prove that $\mathcal{N}_{d-1}$ is generically ample (or zero). Let C be a general complete intersection smooth curve cut out by divisors in $|H|$ and consider the Harder–Narasimhan filtration

$$0 \;=\; \mathcal{M}_0 \;\subsetneq\; \mathcal{M}_1 \;\subsetneq\; \ldots \;\subsetneq\; \mathcal{M}_{d-2} \;\subsetneq\; \mathcal{M}_{d-1},$$

of $\mathcal{M}_{d-1} \;:=\; \big(\mathcal{N}_{d-1}\big)|_C$ where, as before, $\mathcal{M}_i \;:=\; \mathcal{N}_i|_C$. As the minimal slope of $\mathcal{M}_{d-1}$ satisfies

$$\mu_{\min}\big(\mathcal{M}_{d-1}\big) \;:=\; \frac{\deg\big(\mathcal{M}_{d-1}/\mathcal{M}_{d-2}\big)}{\mathrm{rk}\big(\mathcal{M}_{d-1}/\mathcal{M}_{d-2}\big)} \;=\; a^{n-1}\mu^A_{d-1} \;>\; a^{n-1}\mu^A_d \;=\; a^{n-1}\frac{\deg_A(\mathcal{L})}{\mathrm{rk}(\mathcal{Q})} \;=\; 0,$$

the bundle $\mathcal{M}_{d-1}$ is ample. In this case we set $\mathcal{U} = \mathcal{Q}$ and $\mathcal{A} = \mathcal{N}_{d-1}$.

4. Further applications

Corollary 11. *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties and let L be a semiample line bundle on X. Then $f_*(\omega_{X/Y} \otimes L)$ admits a Catanese–Fujita–Kawamata decomposition.*

Proof. As a consequence of the semiampleness, the line bundle L admits a smooth hermitian metric h with semi-positive curvature; in particular, the multiplier ideal $\mathcal{I}(h) = \mathcal{O}_X$ is trivial. By Păun–Takayama's theorem [PT18, Theorem 1.1] and [HPS18, Theorem 21.1], $f_*(\omega_{X/Y} \otimes L)$ admits a singular hermitian metric with semi-positive curvature and the minimal extension property. The corollary follows from Theorem 3. □

Corollary 12. *If $f\colon X \to Y$ is a smooth fibration of smooth projective varieties and E is a Nakano semi-positive vector bundle on X, then $R^j f_*(\omega_{X/Y} \otimes E)$ admits a Catanese–Fujita–Kawamata decomposition for every $j \geq 0$.*

Proof. The main result of [MT08, Theorem 1.1] proves that $R^j f_*(\omega_{X/Y} \otimes E)$ is a Nakano semi-positive vector bundle for all $j \geq 0$. On the other hand a Nakano semi-positive vector bundle satisfies the minimal extension property (*cf.* [SY23, Example 2.16]). The result follows by Theorem 3.

□

Remark 13. As a further example, Iwai and Matsumura prove that a nef cotangent bundle Ω^1_X of a smooth projective variety admits a Catanese–Fujita–Kawamata decomposition (*cf.* [IM22, Proposition 4.2]).

Remark 14. Let $f\colon X \to A$ be a surjective morphism from a smooth projective complex variety X to an abelian variety A. By employing [LPS20, Theorem C] it is possible to check that the hermitian flat part of the Catanese–Fujita–Kawamata decomposition of $f_*\omega_{X/A}^{\otimes m}$ $(m \geq 1)$ consists of a direct-sum of finitely many torsion line bundles in $\mathrm{Pic}^0(A)$.

5. Proof of Theorem 4

In this section we prove Theorem 4. In order to do so, we begin by recalling the following result of Esnault and Viehweg, which in fact holds under more general assumptions.

Given a fibration $f\colon X \to Y$ of varieties, a line bundle L on X is *f-semi-ample* if for some positive integer $N > 0$ the natural morphism $f^*f_*L^{\otimes N} \to L^{\otimes N}$ is surjective.

Theorem 15. *Let $g\colon X \to C$ be a smooth fibration from a smooth projective variety to a smooth projective curve such that $\omega_{X/C}$ is g-semi-ample. If $\det(g_*\omega_{X/C}^{\otimes m})$ is an ample line bundle for some $m \geq 1$, then for all $m \geq 2$ the bundle $g_*\omega_{X/C}^{\otimes m}$ is ample, if not the zero sheaf.*

Proof. The proof is a special case of [EV91, Theorem 0.1]. □

Corollary 16. *Let $g\colon X \to C$ be a smooth fibration from a smooth projective variety to a smooth projective curve such that $\omega_{X/C}$ is g-semi-ample. Denote $J = \{m \in \mathbb{N}_{\geq 2} \,|\, g_*\omega_{X/C}^{\otimes m} \neq 0\}$. Then either $g_*\omega_{X/C}^{\otimes m}$ is ample for every $m \in J$, or hermitian flat for every $m \in J$.*

Proof. By Theorem 3 $g_*\omega_{X/C}^{\otimes m}$ decomposes as $g_*\omega_{X/C}^{\otimes m} \simeq \mathcal{U}_m \oplus \mathcal{A}_m$ with $\mathcal{U}_m$ hermitian flat (or zero), and $\mathcal{A}_m$ ample (or zero). If $\mathcal{A}_m = 0$ for all $m \in J$, then $g_*\omega_{X/C}^{\otimes m}$ is hermitian flat for every $m \in J$ and the proof is complete. On the other hand, if $\mathcal{A}_m \neq 0$ for some $m \in J$, then $\det\big(g_*\omega_{X/C}^{\otimes m}\big)$ is ample, and by Theorem 15 the bundle $g_*\omega_{X/C}^{\otimes m}$ is itself ample for every $m \in J$. □

Lemma 17. *Let $f\colon X \to Y$ be a fibration of smooth projective varieties. Then for any sufficiently general hyperplane section $H \subset Y$ the variety $X_H := f^{-1}(H)$ is smooth and irreducible. Moreover, if m is a positive integer, then there is an isomorphism $f_*\omega_{X/Y}^{\otimes m}|_H \simeq g_*\omega_{X_H/H}^{\otimes m}$ where $g := f|_{X_H}$.*

Proof. The fact that X_H is smooth and irreducible follows by Bertini theorem, as stated in [Jou83, Theorem 6.3]. The following argument is inspired by [Kol86]. Consider the following cartesian diagram

$$\begin{array}{ccc} X_H & \overset{j}{\hookrightarrow} & X \\ \downarrow g & & \downarrow f \\ H & \overset{i}{\hookrightarrow} & Y \end{array}$$

where i and j are the natural inclusions. There is a short exact sequence for any $m \geq 1$

$$0 \to \omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_X(-X_H) \to \omega_{X/Y}^{\otimes m} \to j_*\omega_{X_H/H}^{\otimes m} \to 0$$

since $N_{X_H/X} \simeq g^* N_{H/Y}$ and $\omega_{X/Y}^{\otimes m}|_{X_H} \simeq \omega_{X_H/H}^{\otimes m}$. By taking higher direct images, there is a long exact sequence

$$0 \to f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \to f_*\omega_{X/Y}^{\otimes m} \to i_*g_*\omega_{X_H/H}^{\otimes m} \to$$
$$\to R^1 f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \to R^1 f_*\omega_{X/Y}^{\otimes m} \to i_*R^1 g_*\omega_{X_H/H}^{\otimes m} \to \dots$$
$$\dots \to R^j f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \to R^j f_*\omega_{X/Y}^{\otimes m} \to i_*R^j g_*\omega_{X_H/H}^{\otimes m} \to \dots$$

For every index $j \geq 0$ the morphism $\psi_j \colon R^j f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \to R^j f_*\omega_{X/Y}^{\otimes m}$ is obtained by tensoring the natural inclusion $\mathcal{O}_Y(-H) \to \mathcal{O}_Y$ with $R^j f_*\omega_{X/Y}^{\otimes m}$. Hence, for a general hyperplane section H, which does not contain any associated subvariety of $R^j f_*\omega_{X/Y}^{\otimes m}$, the morphism ψ_j is injective. For $j = 0$ it follows that the following sequence

$$0 \to f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \to f_*\omega_{X/Y}^{\otimes m} \to i_*g_*\omega_{X_H/H}^{\otimes m} \to 0 \tag{5}$$

is short exact. The desired isomorphism is obtained by restricting (5) to H. □

Repeated applications of the previous lemma yield the following corollary.

Corollary 18. *Let $f \colon X \to Y$ be a fibration of smooth projective varieties. Then for a general complete intersection smooth curve C in Y the variety $X_C = f^{-1}(C)$ is smooth and irreducible. Moreover, if m is a positive integer, then there is an isomorphism $f_*\omega_{X/Y}^{\otimes m}|_C \simeq g_*\omega_{X_C/C}^{\otimes m}$ where $g := f|_{X_C}$.*

Proof of Theorem 4. Set $n = \dim Y$. Thanks to Corollary 16 we can assume that $n \geq 2$. Moreover, without loss of generality we can suppose that $J \neq \emptyset$. Recall that for every $m \in J$ there exists a decomposition

$$\mathcal{F}_m := f_*\omega_{X/Y}^{\otimes m} \simeq \mathcal{U}_m \oplus \mathcal{A}_m \tag{6}$$

where $\mathcal{U}_m$ is hermitian flat (or zero), and $\mathcal{A}_m$ is generically ample (or zero). If $\mathcal{A}_m = 0$ for all $m \in J$, then $\mathcal{F}_m$ is hermitian flat for every $m \in J$ and the proof is complete. We may suppose that there exists an index $m_0 \in J$ such that $\mathcal{A}_{m_0} \neq 0$. We aim to prove that $\mathcal{U}_m = 0$ for all $m \in J$.

Let $U \subset Y$ be an open subset as in the statement of the theorem and fix a very ample line bundle H on Y. Moreover, let $C \subset U$ be a general complete intersection smooth curve cut out by divisors in $|H|$ such that $\deg(\mathcal{A}_{m_0}|_C) > 0$. If $g \colon X_C \to C$ denotes the restriction of f to $X_C := f^{-1}(C)$, then by Corollary 18 there is an isomorphism

$$0 \neq \mathcal{F}_{m_0}|_C \simeq g_*\omega_{X_C/C}^{\otimes m_0} \simeq \mathcal{A}_{m_0}|_C \oplus \mathcal{U}_{m_0}|_C. \tag{7}$$

Hence we have

$$\deg\big(\det(g_*\omega_{X_C/C}^{\otimes m_0})\big) \;=\; \big(\det(\mathcal{A}_{m_0})\cdot C\big) \;>\; 0$$

and by Corollary 16 we conclude that

$$g_*\omega_{X_C/C}^{\otimes k} \text{ is ample for all } k \in J. \tag{8}$$

If $m \in J$ is an arbitrary index, we can select a curve C (depending on m) as above, and general enough so that also the following isomorphism

$$0 \neq \mathcal{F}_m|_C \;\simeq\; g_*\omega_{X_C/C}^{\otimes m} \tag{9}$$

holds. By (6), (8) and (9) the bundle $g_*\omega_{X_C/C}^{\otimes m} \simeq \mathcal{U}_m|_C \oplus \mathcal{A}_m|_C$ is ample, and therefore $\mathcal{U}_m|_C$ itself is ample, if not trivial. Since $\big(\det(\mathcal{U}_m)\cdot C\big) = 0$, this forces $\det(\mathcal{U}_m)|_C = 0$, and thus $\mathcal{U}_m = 0$. □

References

[Bre04] Holger Brenner, *Slopes of vector bundles on projective curves and applications to tight closure problems*, Trans. Amer. Math. Soc. **356** (2004), no. 1, 371–392.

[CD17] Fabrizio Catanese and Michael Dettweiler, *Answer to a question by Fujita on variation of Hodge structures*, Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata's sixtieth birthday, Adv. Stud. Pure Math., vol. 74, Math. Soc. Japan, Tokyo, 2017, pp. 73–102.

[CK19] Fabrizio Catanese and Yujiro Kawamata, *Fujita decomposition over higher dimensional base*, Eur. J. Math. **5** (2019), no. 3, 720–728.

[CP17] Junyan Cao and Mihai Păun, *Kodaira dimension of algebraic fiber spaces over abelian varieties*, Invent. Math. **207** (2017), no. 1, 345–387.

[EV91] Hélène Esnault and Eckart Viehweg, *Ample sheaves on moduli schemes*, Algebraic geometry and analytic geometry (Tokyo, 1990), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, pp. 53–80.

[Fuj78] Takao Fujita, *The sheaf of relative canonical forms of a Kähler fiber space over a curve*, Proc. Japan Acad. Ser. A Math. Sci. **54** (1978), no. 7, 183–184.

[HIM22] Genki Hosono, Masataka Iwai, and Shin-ichi Matsumura, *On projective manifolds with pseudo-effective tangent bundle*, J. Inst. Math. Jussieu **21** (2022), no. 5, 1801–1830.

[HL10] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010.

[HPS18] Christopher Hacon, Mihnea Popa, and Christian Schnell, *Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun*, Local and global methods in algebraic geometry, Contemp. Math., vol. 712, Amer. Math. Soc., [Providence], RI, [2018] ©2018, pp. 143–195.

[IM22] Masataka Iwai and Shin-ichi Matsumura, *Abundance theorem for minimal compact Kähler manifolds with vanishing second Chern class*, arXiv:2205.10613 (2022).

[Iwa22] Masataka Iwai, *Almost nef regular foliations and Fujita's decomposition of reflexive sheaves*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **23** (2022), no. 2, 719–743.

[Jou83] Jean-Pierre Jouanolou, *Théorèmes de Bertini et applications*, Progress in Mathematics, vol. 42, Birkhäuser Boston, Inc., Boston, MA, 1983.

[Kol86] János Kollár, *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), no. 1, 11–42.

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[LPS20] Luigi Lombardi, Mihnea Popa, and Christian Schnell, *Pushforwards of pluricanonical bundles under morphisms to abelian varieties*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 8, 2511–2536.

[Mar81] Masaki Maruyama, *The theorem of Grauert-Mülich-Spindler*, Math. Ann. **255** (1981), no. 3, 317–333.

[MT08] Christophe Mourougane and Shigeharu Takayama, *Hodge metrics and the curvature of higher direct images*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 6, 905–924.

[PT18] Mihai Păun and Shigeharu Takayama, *Positivity of twisted relative pluricanonical bundles and their direct images*, J. Algebraic Geom. **27** (2018), no. 2, 211–272.

[SY23] Christian Schnell and Ruijie Yang, *Hodge modules and singular hermitian metrics*, Math. Z. **303** (2023), no. 2, Paper No. 28, 20.

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