PLATEAU'S PROBLEM AS A SINGULAR LIMIT OF CAPILLARITY PROBLEMS

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ABSTRACT. Soap films at equilibrium are modeled, rather than as surfaces, as regions of small total volume through the introduction of a capillarity problem with a homotopic spanning condition. This point of view introduces a length scale in the classical Plateau's problem, which is in turn recovered in the vanishing volume limit. This approximation of area minimizing hypersurfaces leads to an energy based selection principle for Plateau's problem, points at physical features of soap films that are unaccessible by simply looking at minimal surfaces, and opens several challenging questions.

CONTENTS

1. INTRODUCTION

1.1. **Overview.** The theory of minimal surfaces with prescribed boundary data provides the basic model for soap films hanging from a wire frame: given an $(n-1)$-dimensional surface $\Gamma \subset \mathbb{R}^{n+1}$ without boundary, one seeks n-dimensional surfaces M such that

$$H_M = 0\,, \qquad \partial M = \Gamma\,, \tag{1.1}$$

where H_M is the mean curvature of M (and $n = 2$ in the physical case). A limitation of (1.1) as a physical model is that, in general, (1.1) may be non-uniquely solvable, including unstable (and thus, not related to observable soap films) solutions. Area minimization can be used to construct stable (and thus, physical) solutions, providing a strong motivation for the study of *Plateau's problem*; see [CM11]. Here we are concerned with a more elementary physical limitation of (1.1), namely, the absence of a length scale: if M solves (1.1) for Γ, then $t\,M$ solves (1.1) for $t\,\Gamma$, no matter how large $t > 0$ is.

Following [MSS19], we introduce a length scale in the modeling of soap films by thinking of them as regions $E \subset \mathbb{R}^{n+1}$ with small volume $|E| = \varepsilon$. At equilibrium, the isotropic pressure at a point y interior to the liquid but immediately close to its boundary ∂E is

$$p(y) = p_0 + \sigma\,\vec{H}_{\partial E}(y) \cdot \nu_E(y)\,, \tag{1.2}$$

where p_0 is the atmospheric pressure, σ is the surface tension, ν_E the outer unit normal to E, and $\vec{H}_{\partial E}$ the mean curvature vector of ∂E; at the same time, for any two points y, z

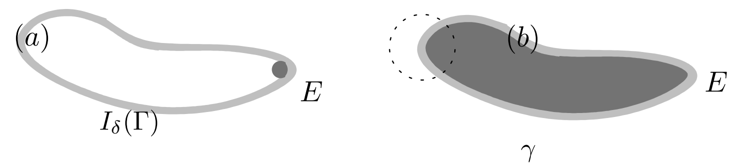

Figure 1.1. Minimizers of the capillarity problem in the unusual container Ω consisting of the complement of a δ-neighborhood $I_\delta(\Gamma)$ of a curve Γ (depicted in light gray). The shape of E is drastically different depending on whether or not a homotopic spanning condition is prescribed: (a) without a $\mathcal{C}$-spanning condition, we observe tiny droplets sitting near points of maximal mean curvature of $\partial\Omega$; (b) with a $\mathcal{C}$-spanning condition, small rounds droplets will not be admissible, and a different region of the energy landscape is explored; minimizers are now expected to stretch out and look like soap films.

inside the film we have

$$p(y) - p(z) = \rho\, g\, (z - y) \cdot e_{n+1}\,, \tag{1.3}$$

where ρ is the density of the fluid, g the gravity of Earth and e_{n+1} is the vertical direction. In the absence of gravity, (1.2) and (1.3) imply that $H_E = \vec{H}_{\partial E} \cdot \nu_E$ is *constant* along ∂E. A heuristic analysis shows that if ∂E is representable, locally, by the two graphs $\{x \pm (h(x)/2)\, \nu_M(x) : x \in M\}$ defined by a positive function h over an ideal mid-surface M, then H_M should be *small, but non-zero* (even in the absence of gravity); see [MSS19, Section 2]. As it is well-known, one cannot prescribe non-vanishing mean curvature with arbitrarily large boundary data, see, e.g. [Giu78, DF90]. Hence this point of view can potentially capture physical features of soap films that are not accessible by modeling them as minimal surfaces.

The goal of this paper is starting the analysis of the variational problem playing for (1.2) and (1.3) the role that Plateau's problem plays for (1.1). The new aspect is not in the energy minimized, but in the boundary conditions under which the minimization occurs. Indeed, the equivalence between the constancy of H_E and the balance equations (1.2) and (1.3), leads us to work in the classical framework of Gauss' capillarity model for liquid droplets in a container. Given an open set $\Omega \subset \mathbb{R}^{n+1}$ (the container), the surface tension energy[1] of a droplet occupying the open region $E \subset \Omega$ is given by

$$\sigma\, \mathcal{H}^n(\Omega \cap \partial E)\,,$$

where $\mathcal{H}^n$ denotes n-dimensional Hausdorff measure (surface area if $n = 2$, length if $n = 1$). In the case of soap films hanging from a wire frame Γ, we choose as container Ω the set

$$\Omega = \mathbb{R}^{n+1} \setminus I_\delta(\Gamma)\,,$$

corresponding to the complement of the "solid wire" $I_\delta(\Gamma)$, where I_δ denotes the closed δ-neighborhood of a set. The minimization of $\mathcal{H}^n(\Omega \cap \partial E)$ among open sets $E \subset \Omega$ with $|E| = \varepsilon$ leads indeed to finding minimizers whose boundaries have constant mean curvature. However, these boundaries will not resemble soap films at all, but will rather consist of small "droplets" sitting at points of maximal curvature for $I_\delta(\Gamma)$; see Figure 1.1, and [BR05, Fal10, MM16] for more information.

To observe soap films, rather than droplets, we must require that ∂E stretches out to span $I_\delta(\Gamma)$. To this end, we exploit a beautiful idea introduced by Harrison and Pugh in [HP16a], as slightly generalized in [DLGM17]. The idea is fixing a **spanning class**, i.e. a

[1]For simplicity, we are setting to zero the adhesion coefficient with the container; see, e.g. [Fin86].

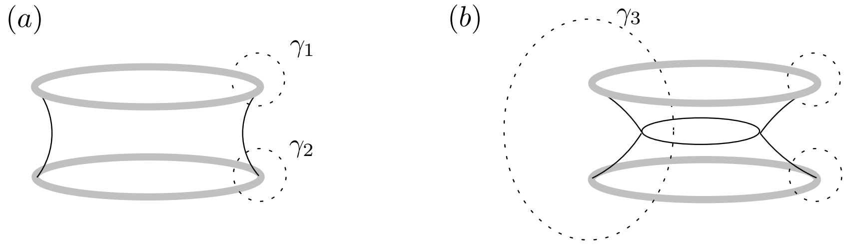

FIGURE 1.2. The variational problem (1.5) with Γ given by two parallel circles centered on the same axis at a mutual distance smaller than their common radius. Different choices of $\mathcal{C}$ lead to different minimizers S in ℓ: (a) if $\mathcal{C}$ is generated by the loops γ_1 and γ_2, then S is the area minimizing catenoid; (b) if we add to $\mathcal{C}$ the homotopy class of γ_3, then S is the *singular* area minimizing catenoid, consisting of two catenoidal necks, meeting at equal angles along a circle of Y-points bounding a "floating" disk. Such singular catenoid cannot be approximated in energy by smooth surfaces: hence the choice of casting ℓ in a class of non-smooth surfaces.

homotopically closed[2] family $\mathcal{C}$ of smooth embeddings of $\mathbb{S}^1$ into $\Omega = \mathbb{R}^{n+1} \setminus I_\delta(\Gamma)$, and to say[3] that a relatively closed set $S \subset \Omega$ is **$\mathcal{C}$-spanning $I_\delta(\Gamma)$** if

$$S \cap \gamma \neq \emptyset \qquad \forall \gamma \in \mathcal{C}\,. \tag{1.4}$$

Given a choice of $\mathcal{C}$, we have a corresponding version of Plateau's problem

$$\ell = \inf \Big\{ \mathcal{H}^n(S) : S \text{ is relatively closed in } \Omega, \text{ and } S \text{ is } \mathcal{C}\text{-spanning } I_\delta(\Gamma) \Big\}\,, \tag{1.5}$$

as illustrated in Figure 1.2. The variational problem $\psi(\varepsilon)$ studied here is thus a reformulation of ℓ as a capillarity problem with a homotopic spanning condition, namely:

$$\psi(\varepsilon) = \inf \Big\{ \mathcal{H}^n(\Omega \cap \partial E) : E \subset \Omega,\ |E| = \varepsilon,\ \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } I_\delta(\Gamma) \Big\}\,, \qquad \varepsilon > 0\,.$$

We now give *informal* statements of our main results (e.g., we make no mention to singular sets or comment on reduced vs topological boundaries); see section 1.2 for the formal ones.

Existence of generalized minimizers and Euler-Lagrange equations (Theorem 1.4 and Theorem 1.6): *There always exists a generalized minimizer (K, E) for $\psi(\varepsilon)$: that is, there exists a set $K \subset \Omega$, relatively closed in Ω and $\mathcal{C}$-spanning $I_\delta(\Gamma)$, and there exists an open set $E \subset \Omega$ with $\Omega \cap \partial E \subset K$ and $|E| = \varepsilon$, such that*

$$\psi(\varepsilon) = \mathcal{F}(K, E) = 2\,\mathcal{H}^n(K \setminus \partial E) + \mathcal{H}^n(\Omega \cap \partial E)\,.$$

Moreover, (K, E) minimizes $\mathcal{F}$ with respect to all its diffeomorphic images: in particular, $\Omega \cap \partial E$ has constant mean curvature $\lambda \in \mathbb{R}$ and $K \setminus \partial E$ has zero mean curvature.

Convergence to the Plateau's problem (Theorem 1.9): *We always have $\psi(\varepsilon) \to 2\,\ell$ when $\varepsilon \to 0^+$, and if (K_j, E_j) are generalized minimizers for $\psi(\varepsilon_j)$ with $\varepsilon_j \to 0^+$, then, up to extracting subsequences, we can find a minimizer S for ℓ with*

$$2 \int_{K_j \setminus \partial E_j} \varphi + \int_{\partial E_j} \varphi \to 2 \int_S \varphi \qquad \forall \varphi \in C^0_c(\Omega)\,,$$

as $j \to \infty$; in other words, generalized minimizers in $\psi(\varepsilon_j)$ with $\varepsilon_j \to 0^+$ converge as Radon measures to minimizers in the Harrison-Pugh formulation of Plateau's problem.

[2] By this we mean that if γ_0, γ_1 are smooth embeddings of $\mathbb{S}^1$ into Ω, $\gamma_0 \in \mathcal{C}$, and there exists a continuous map $f : [0,1] \times \mathbb{S}^1 \to \Omega$ with $f(t, \cdot) = \gamma_t$ for $t = 0, 1$, then $\gamma_1 \in \mathcal{C}$.

[3] Notice that, in stating condition (1.4), the symbol γ denotes the subset $\gamma(\mathbb{S}^1) \subset \Omega$. We are following here the same convention set in [DLGM17].

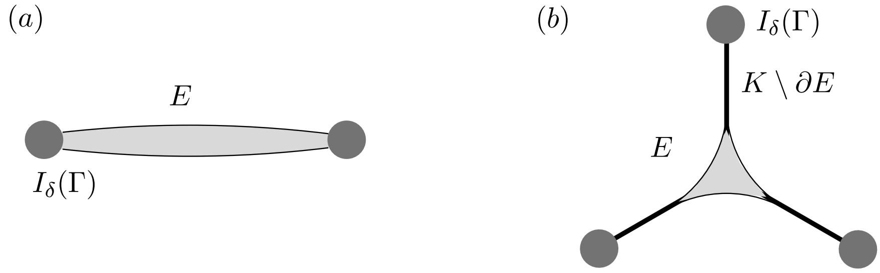

FIGURE 1.3. (a) If Γ consists of two points, then the minimizer is not collapsed, and is bounded by two very flat circular arcs; (b) when Γ consists of the vertices of an equilateral triangle, the generalized minimizer is indeed collapsed. The three segments defining $K \setminus \partial E$ are depicted in bold, and E is a negatively curved curvilinear triangle nested around the singular point of the unique minimizer of ℓ.

Example 1.1 (Volume and thickness in the non-collapsed case)**.** Let Γ consists of two points at distance r in the plane, or of an $(n-1)$-sphere of radius r in $\mathbb{R}^{n+1}$. For ε small enough, $\psi(\varepsilon)$ should admit a unique generalized minimizer (K,E), consisting of two almost flat spherical caps meeting orthogonally along the torus $I_\delta(\Gamma)$ (so that $K=\partial E$ and collapsing does not occur); see Figure 1.3-(a). In general, we expect that *when all the minimizers S in ℓ are smooth, then generalized minimizers in $\psi(\varepsilon)$ are not collapsed, and, for small ε, $K=\partial E$ is a two-sided approximation of S, with $H_E=\psi'(\varepsilon)\to 0$ and*

$$\psi(\varepsilon) = 2\,\ell + C\,\varepsilon^2 + \mathrm{o}(\varepsilon^2)\,, \qquad \text{as } \varepsilon \to 0^+\,, \tag{1.6}$$

for a positive C. This insight is consistent with the idea (see [MSS19]) that *almost minimal surfaces* arise in studying soap films with a thickness. In particular, *volume and thickness will be directly related* in terms of the geometry of Γ. Sending $\varepsilon \to 0^+$ with Γ fixed or, equivalently, considering $t\,\Gamma$ for large t at ε fixed, will make the thickness decrease until it reaches a threshold below which we do not expect soap films to be stable. A critical thickness can definitely be identified with the characteristic length scale of the molecules of surfactant, below which the model stops making sense. But depending on temperatures, actual soap films with even larger thicknesses should burst out due to the increased probability of fluctuations towards unstable configurations.

Example 1.2 (Volume and thickness in the collapsed case)**.** At small volumes, and in presence of singularities in the minimizers of ℓ, collapsing is energetically convenient, and allows $\psi(\varepsilon)$ to approximate $2\,\ell$ from below. If $\Gamma \subset \mathbb{R}^2$ consists of the three vertices of an equilateral triangle, for small δ the unique minimizer of ℓ consists of a Y-configuration. For small ε, we expect generalized minimizers (K,E) of $\psi(\varepsilon)$ to be collapsed, see Figure 1.3-(b): there, E is a curvilinear triangle made up of three circular arcs whose length is $\mathrm{O}(\sqrt{\varepsilon})$, and whose (negative) curvature is $\mathrm{O}(1/\sqrt{\varepsilon})$. The thickness of an actual soap film in this configuration should thus be considerably larger near the singularity than along the collapsed region, and the volume and the thickness of the film are somehow *independent* geometric quantities. This suggests, in presence of singularities, the need for introducing a second length scale in the model. A possibility is replacing the sharp interface energy $\mathcal{H}^n(\Omega \cap \partial E)$ with a diffused interface energy, like the Allen-Cahn energy

$$\mathcal{E}_\eta(u) = \eta \int_\Omega |\nabla u|^2 + \frac{1}{\eta}\int_\Omega W(u)\,, \qquad \eta > 0\,,$$

for a double-well potential with $\{W=0\}=\{-1,1\}$. We expect $\{u>0\}$ to (approximately) coincide with the union of a curvilinear triangle of area ε with three stripes having the collapsed segments as their mid-sections, and of width $\eta\,|\log\eta|$; cf. with [dPKW08].

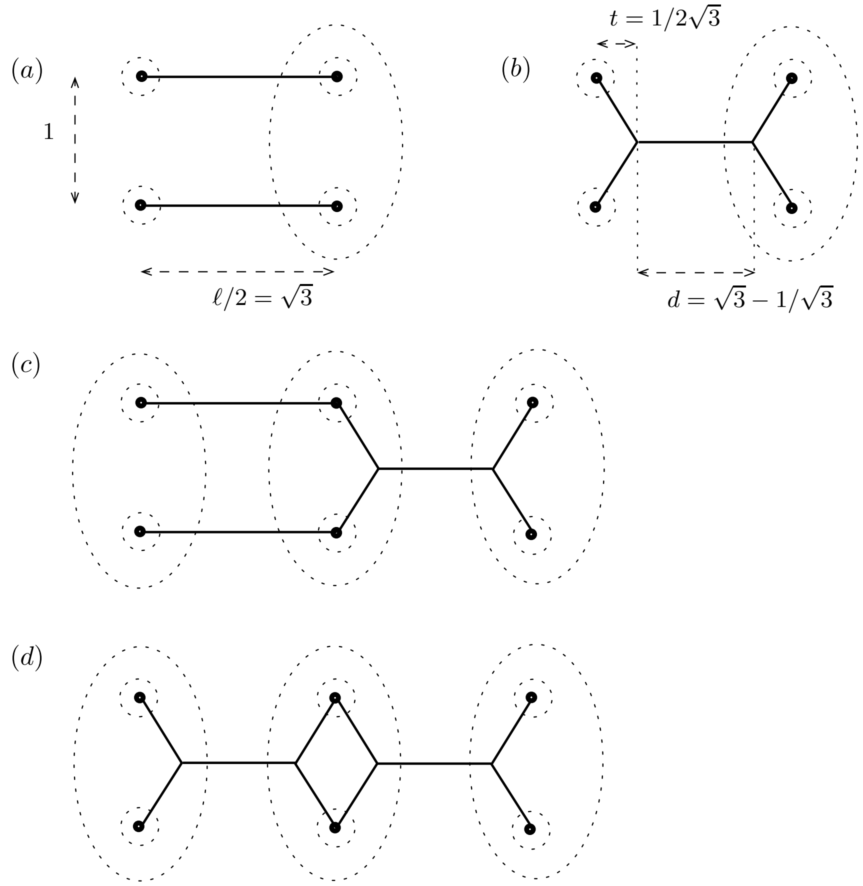

FIGURE 1.4. (a) and (b): a four points configuration Γ with a choice of $\mathcal{C}$ such that ℓ admits two minimizers, one with and one without singularities; (c) and (d): a six points configuration Γ with a choice of $\mathcal{C}$ such that ℓ admits many minimizers, possibly with a variable number of singularities; here we have depicted two of them, including the one with four singular points that is selected by the $\psi(\varepsilon)$ problems.

Example 1.3 (Capillarity as a selection principle for Plateau's problem)**.** The following statement holds (as a heuristic principle): *Generalized minimizers of $\psi(\varepsilon)$ converge to those minimizers of Plateau's problem* (1.5) *with larger singular set, and when no singular minimizers are present, they select those whose second fundamental form has maximal L^2-norm.* Since the second part of this selection principle is justified by standard second variation arguments, we illustrate the first part only. In Figure 1.4, Γ is either given by four or by six points, that are suitably spaced so that ℓ has different minimizers. As $\varepsilon \to 0^+$, $\psi(\varepsilon)$ selects those ℓ-minimizers with singularities over the ones without singularities; and when more minimizers with singularities are present, it selects the ones with the largest number of singularities. Indeed, the approximation of a smooth minimizer in ℓ will require an energy cost larger than $2\,\ell$. At the same time, each time a singularity is present, minimizers of $\psi(\varepsilon)$ can save length in the approximation, thus paying less than $2\,\ell$ in energy, and the more the singularities, the bigger the gain. To check this claim, pick N singularities, and denote by ε_i the volume placed near the i-th singularity and by r_i the radius of the three circular arcs enclosing ε_i. Each wetted singularity has area $c_1\,r_i^2$, while the total relaxed energy of the approximating configuration is $\mathcal{F} = 2\,\ell - c_2\,\sum_{i=1}^N\,r_i$.

Minimizing under the constraint $\varepsilon = c_1 \sum_{i=1}^N r_i^2$, we must take $r_i = \sqrt{\varepsilon/Nc_1}$, thus finding

$$\psi(\varepsilon) = 2\,\ell - c_2 \sqrt{\frac{\varepsilon\, N_{\max}}{c_1}}\,,$$

if $N_{\max}$ is the maximal number of singularities available among minimizers of ℓ. This example suggests that (in every dimension) *in the presence of singular minimizers of ℓ, one should have*

$$\psi'(\varepsilon) \to -\infty \text{ as } \varepsilon \to 0^+\,. \tag{1.7}$$

This is of course markedly different from what we expect to be the situation when ℓ has only smooth minimizers, see (1.6). We finally notice that a selection principle for the capillarity model (without homotopic spanning conditions) via its Allen-Cahn approximation has been recently obtained by Leoni and Murray, see [LM16, LM17].

1.2. **Statements of the results.** We now give a more technical introduction to our paper, with precise statements, more bibliographical references, and comments on the proofs.

Plateau's problem with homotopic spanning: We fix a compact set $W \subset \mathbb{R}^{n+1}$ (the "wire frame") and denote the region accessible by the soap film as

$$\Omega = \mathbb{R}^{n+1} \setminus W\,.$$

The typical case we have in mind is $W = I_\delta(\Gamma)$, as discussed in section 1.1, but this is not necessary. We fix a **spanning class** $\mathcal{C}$, that is a non-empty family of smooth embeddings of $\mathbb{S}^1$ into Ω which is closed by homotopy in Ω. We assume that W and $\mathcal{C}$ are such that the **Plateau's problem defined by** $\mathcal{C}$

$$\ell = \inf\big\{\mathcal{H}^n(S) : S \in \mathcal{S}\big\} \tag{1.8}$$

is such that[4] $\ell < \infty$. Here, for the sake of brevity, we have introduced

$$\mathcal{S} = \big\{S \subset \Omega : S \text{ is relatively closed in } \Omega \text{ and } S \text{ is } \mathcal{C}\text{-spanning } W\big\}\,.$$

As proved in [HP16a, DLGM17], if $\ell < \infty$, then there exists a compact, $\mathcal{H}^n$-rectifiable set S such that $\mathcal{H}^n(S) = \ell$; see also [Har14, Dav14, Fan16, HP16b, HP16c, DPDRG16, DLDRG19, GLF17, HP17, FK18, DR18] for related existence results. In addition, S minimizes $\mathcal{H}^n$ with respect to Lipschitz perturbations of the identity localized in Ω, so that: (i) S is a classical minimal surface outside of an $\mathcal{H}^n$-negligible, relatively closed set in Ω by [Alm76]; (ii) if $n = 1$, S consists of finitely many segments, possibly meeting in three equal angles at singular Y-points in Ω; (iii) if $n = 2$, S satisfies **Plateau's laws** by [Tay76]: namely, S is locally diffeomorphic either to a plane, or to a cone $Y = T^1 \times \mathbb{R}$, or to a cone T^2, where T^n is the cone over the origin defined by the $(n-1)$-dimensional faces of a regular tetrahedron in $\mathbb{R}^{n+1}$. The validity of Plateau's laws in this context makes (1.8) more suitable when one is motivated by physical considerations: indeed, minimizers of the codimension one Plateau's problem in the class of rectifiable currents are necessarily smooth if $n \le 6$. Although smoothness is desirable for geometric applications, it creates an *a priori* limitation when studying actual soap films; see also [Dav14, HP16a, DLGM17].

The capillarity problem and the relaxed energy: Next, we give a precise formulation of the capillarity problem $\psi(\varepsilon)$ at volume $\varepsilon > 0$, which is defined as

$$\psi(\varepsilon) = \inf\Big\{\mathcal{H}^n(\Omega \cap \partial E) : E \in \mathcal{E},\, |E| = \varepsilon,\, \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } W\Big\}\,. \tag{1.9}$$

Here we have introduced the family of sets

$$\mathcal{E} = \Big\{E \subset \Omega :\, E \text{ is an open set and } \partial E \text{ is } \mathcal{H}^n\text{-rectifiable}\Big\}\,. \tag{1.10}$$

[4]The condition $\ell < \infty$ clearly implies that no $\gamma \in \mathcal{C}$ is homotopic to a constant map.

If $E \in \mathcal{E}$, then ∂E is $\mathcal{H}^n$-finite and covered by countably many Lipschitz images of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$. Thus, E is of finite perimeter in Ω by a classical result of Federer, and its (distributional) perimeter $P(E;U)$ in an open set $U \subset \Omega$ is equal to $\mathcal{H}^n(U \cap \partial^* E)$, where $\partial^* E$ is the reduced boundary of E (notice that, in general, $P(E;U) \le \mathcal{H}^n(U \cap \partial E)$). The relaxed energy $\mathcal{F}$ is defined by

$$\begin{aligned} \mathcal{F}(K,E;U) &= \mathcal{H}^n(U \cap \partial^* E) + 2\,\mathcal{H}^n\big(U \cap (K \setminus \partial^* E)\big)\,, \qquad U \subset \Omega \text{ open}\,, \\ \mathcal{F}(K,E) &= \mathcal{F}(K,E;\Omega)\,, \end{aligned}$$

on every pair (K,E) in the family $\mathcal{K}$ given by

$$\mathcal{K} = \Big\{(K,E) : E \subset \Omega \text{ is open with } \Omega \cap \mathrm{cl}\,(\partial^* E) = \Omega \cap \partial E \subset K\,, \\ K \in \mathcal{S} \text{ and } K \text{ is } \mathcal{H}^n\text{-rectifiable in } \Omega\Big\}\,.$$

By the requirement $K \in \mathcal{S}$, K is $\mathcal{C}$-spanning W, while $\Omega \cap \partial E$, which is always a subset of K, may be not be $\mathcal{C}$-spanning W; we expect this when collapsing occurs, see Figure 1.3.

Assumptions on Ω: We make two main geometric assumptions on W and $\mathcal{C}$. Firstly, in constructing a system of volume-fixing variations for a given minimizing sequence of $\psi(\varepsilon)$ (see step two of the proof of Theorem 1.4) we shall assume that

$$\exists\, \tau_0 > 0 \text{ such that, for every } \tau < \tau_0,\ \mathbb{R}^{n+1} \setminus I_\tau(W) \text{ is connected}\,. \tag{1.11}$$

This is compatible with the idea that, in the physical case $n = 2$, W represents a "solid wire". Secondly, to verify the finiteness of $\psi(\varepsilon)$ (see step one in the proof of Theorem 1.4), we require that

$$\exists\, \eta_0 > 0 \text{ and a minimizer } S \text{ in } \ell \text{ s.t. } \gamma \setminus I_{\eta_0}(S) \neq \emptyset \text{ for every } \gamma \in \mathcal{C}\,. \tag{1.12}$$

This is clearly a generic situation, which (thanks to the convex hull property of stationary varifolds) is implied, for example, by the much more stringent condition that $\gamma \setminus Z \neq \emptyset$ for every $\gamma \in \mathcal{C}$ where Z is the closed convex hull of W. Finally, we shall also assume that "$\partial\Omega = \partial W$ is smooth": by this we mean that locally near each $x \in \partial\Omega$, Ω can be described as the epigraph of a smooth function of n-variables.

Existence of minimizers and Euler-Lagrange equations: Our first main result is the existence of generalized minimizers of $\psi(\varepsilon)$.

Theorem 1.4 (Existence of generalized minimizers)**.** *Let $\ell < \infty$, ∂W be smooth and let (1.11) and (1.12) hold. If $\{E_j\}_j$ is a minimizing sequence for $\psi(\varepsilon)$, then there exists a pair $(K,E) \in \mathcal{K}$ with $|E| = \varepsilon$ such that, up to possibly extracting subsequences, and up to possible modifications of each E_j outside a large ball containing W (with both operations resulting in defining a new minimizing sequence for $\psi(\varepsilon)$, still denoted by $\{E_j\}_j$), we have that*

$$\begin{aligned} &E_j \to E \text{ in } L^1(\Omega)\,, \\ &\mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \stackrel{*}{\rightharpoonup} \theta\, \mathcal{H}^n \llcorner K \qquad \text{as Radon measures in } \Omega\,, \end{aligned} \tag{1.13}$$

as $j \to \infty$, where $\theta : K \to \mathbb{R}$ is an upper semicontinuous function with

$$\theta = 2\ \mathcal{H}^n\text{-a.e. on } K \setminus \partial^* E, \qquad \theta = 1 \text{ on } \Omega \cap \partial^* E\,. \tag{1.14}$$

Moreover, $\psi(\varepsilon) = \mathcal{F}(K,E)$ and, for a suitable constant C, $\psi(\varepsilon) \le 2\,\ell + C\,\varepsilon^{n/(n+1)}$.

Remark 1.5. Whenever $(K,E) \in \mathcal{K}$ is such that $|E| = \varepsilon$, $\mathcal{F}(K,E) = \psi(\varepsilon)$ and there exists a minimizing sequence $\{E_j\}_j$ for $\psi(\varepsilon)$ which converges to (K,E) as in (1.13), we say that (K,E) is a **generalized minimizer of** $\psi(\varepsilon)$. We say that (K,E) is **collapsed** if $K \setminus \partial E \neq \emptyset$. If (K,E) is not collapsed, then E is a (standard) minimizer of $\psi(\varepsilon)$.

Next, we derive the Euler-Lagrange equations for a generalized minimizer and apply Allard's theorem.

Theorem 1.6 (Euler-Lagrange equation for generalized minimizers)**.** *Let $\ell < \infty$, ∂W be smooth and let* (1.11) *and* (1.12) *hold. If (K, E) is a generalized minimizer of $\psi(\varepsilon)$ and $f : \Omega \to \Omega$ is a diffeomorphism such that $|f(E)| = |E|$, then*

$$\mathcal{F}(K, E) \le \mathcal{F}(f(K), f(E)) . \tag{1.15}$$

In particular:

(i) *there exists $\lambda \in \mathbb{R}$ such that*

$$\lambda \int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\partial^* E} \operatorname{div}^K X \, d\mathcal{H}^n + 2 \int_{K \setminus \partial^* E} \operatorname{div}^K X \, d\mathcal{H}^n , \tag{1.16}$$

for every $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ with $X \cdot \nu_\Omega = 0$ on $\partial\Omega$, where div^K denotes the tangential divergence along K;

(ii) *there exists $\Sigma \subset K$, closed and with empty interior in K, such that $K \setminus \Sigma$ is a smooth hypersurface, $K \setminus (\Sigma \cup \partial E)$ is a smooth embedded minimal hypersurface, $\mathcal{H}^n(\Sigma \setminus \partial E) = 0$, $\Omega \cap (\partial E \setminus \partial^* E) \subset \Sigma$ has empty interior in K, and $\Omega \cap \partial^* E$ is a smooth embedded hypersurface with constant scalar (w.r.t. ν_E) mean curvature λ.*

Remark 1.7. Although we do not pursue this point here, we mention that we would expect (K, E) to be a proper minimizer of $\mathcal{F}$ among pairs $(K', E') \in \mathcal{K}$ with $|E'| = \varepsilon$ (and not just when $K' = f(K)$ for a diffeomorphism f, as proved in (1.15)). To show this we would need to approximate in energy a generic (K', E') by competitors $\{F_j\}_j$ for $\psi(\varepsilon)$. The natural *ansatz* for this approximation would be taking $F_j = U_{\eta_j}(K' \cup E') \setminus I_{\eta_j}(K' \cap E')$ for $\eta_j \to 0^+$, where U_η denotes the *open* η-neighborhood of a set. The convergence of this approximation is delicate, and can be made to work by elaborating on the ideas contained in [ACV08, Vil09] at least for (K', E') in certain subclasses of $\mathcal{K}$.

Remark 1.8. Theorem 1.6 points at two interesting free boundary problems. The first problem concerns the size and properties of $\partial E \setminus \partial^* E$, which is the transition region between constant and zero mean curvature; similar free boundary problems (on graphs rather than on unconstrained surfaces) have been considered, e.g., in [CJK02, CDSS16, CDSS17]. The second problem concerns the wetted region $\partial\Omega \cap \partial E$, which could either be $\mathcal{H}^n$-negligible or not, recall Figure 1.3: in the former case, $\partial\Omega \cap \partial E$ should be $(n-1)$-dimensional, while in the latter case $\partial\Omega \cap \partial E$ should be a set of finite perimeter inside $\partial\Omega$, and Young's law $\nu_\Omega \cdot \nu_E = 0$ should hold at generic boundary points of $\partial\Omega \cap \partial E$ relative to $\partial\Omega$; see for example [DPM15, DPM17].

Convergence towards Plateau's problem: The next theorem establishes the nature of Plateau's problem ℓ as the singular limit of the capillarity problems $\psi(\varepsilon)$ as $\varepsilon \to 0^+$.

Theorem 1.9 (Plateau's problem as a singular limit of capillarity problems)**.** *If $\ell < \infty$, ∂W be smooth, and* (1.11) *and* (1.12) *hold, then ψ is lower semicontinuous on* $(0, \infty)$ *and*

$$\lim_{\varepsilon \to 0^+} \psi(\varepsilon) = 2\,\ell . \tag{1.17}$$

In addition, if $\{(K_h, E_h)\}_h$ is a sequence of generalized minimizers of $\psi(\varepsilon_h)$ for $\varepsilon_h \to 0^+$ as $h \to \infty$, then there exists a minimizer S in ℓ such that, up to extracting subsequences and as $h \to \infty$,

$$\mathcal{H}^n \llcorner (\Omega \cap \partial^* E_h) + 2\,\mathcal{H}^n \llcorner (K_h \setminus \partial^* E_h) \stackrel{*}{\rightharpoonup} 2\,\mathcal{H}^n \llcorner S , \qquad \textit{as Radon measures in } \Omega . \tag{1.18}$$

Remark 1.10. The behavior of $\psi(\varepsilon) - 2\,\ell$ as $\varepsilon \to 0^+$ is expected to depend heavily on whether minimizers of ℓ have or do not have singularities, as noticed in (1.6) and (1.7). In particular, we expect $\psi'(\varepsilon) \to 0^+$ only in special situations: when this happens, we have a

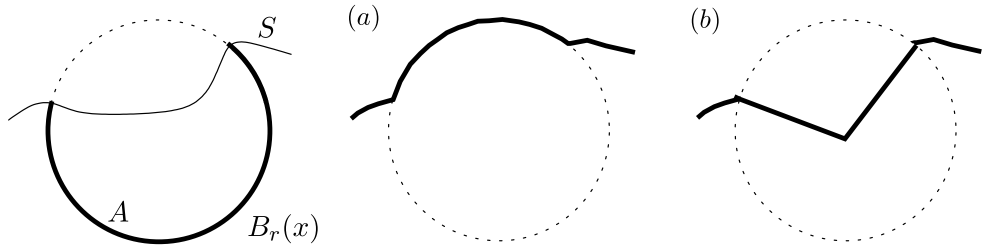

FIGURE 1.5. (a) the cup competitor of a set S in $B_r(x)$ relative to an $\mathcal{H}^n$-maximal connected component A of $\partial B_r(x) \setminus S$; (b) the cone competitor of S in $B_r(x)$.

vanishing mean curvature approximation of Plateau's problem which is related to Rellich's conjecture, see e.g. [BC84].

Remark 1.11. The Hausdorff convergence of K_h to S is not immediate (nor is the convergence in varifolds sense). Given (1.18), Hausdorff convergence would follow from an area lower bound on K_h. In turn, this could be deduced (thanks to area monotonicity) from a uniform L^p-bound, for some $p > n$, on the mean curvature vectors $\vec{H}_{V_h}$ of the integer varifolds V_h supported on K_h, with multiplicity 2 on $K_h \setminus \partial^* E_h$, and multiplicity 1 on $\partial^* E_h$. Notice however that, by (1.16), if λ_h is the Lagrange multiplier of (K_h, E_h), then $\vec{H}_{V_h} = \lambda_h \, \nu_{E_h} \, 1_{\partial^* E_h}$, so that, even when $n = 1$, the only uniform L^p-bound that can hold is the one with $p = 1$; see Example 1.2.

Proofs: We approach Theorem 1.4 with the method introduced in [DLGM17] to solve (1.8), which is now briefly summarized. The idea in [DLGM17] is considering a minimizing sequence $\{S_j\}_j$ for ℓ, which (up to extracting subsequences) immediately leads to a sequence of Radon measures $\mu_j = \mathcal{H}^n \llcorner S_j \stackrel{*}{\rightharpoonup} \mu$ as Radon measures in Ω, with $S = \operatorname{spt} \mu$ $\mathcal{C}$-spanning W. By comparing S_j with its cup competitors S_j', see Figure 1.5-(a), and then letting $j \to \infty$, it is shown that $\mu(B_r(x)) \ge \theta_0(n) \, r^n$ for every $x \in \operatorname{spt} \mu$; by comparing S_j with its cone competitors S_j', and then letting $j \to \infty$, it is proved that $r^{-n} \, \mu(B_r(x))$ is increasing in r. By Preiss' theorem [Pre87, DL08] it follows that $\mu = \theta \, \mathcal{H}^n \llcorner S$ and that S is $\mathcal{H}^n$-rectifiable. Finally, spherical isoperimetry and a geometric argument imply that $\theta \ge 1$ $\mathcal{H}^n$-a.e. on S, which in turn suffices to conclude that S is a minimizer in ℓ since, by lower semicontinuity, $\mathcal{H}^n(S) \le \mu(\Omega) \le \liminf_j \mu_j(\Omega) = \ell$, and because S is in the competition class of ℓ.

Adapting this approach to a minimizing sequence $\{E_j\}_j$ for $\psi(\varepsilon)$ requires the introduction of new ideas. First, cup and cone competitors for $\{E_j\}_j$ have to be defined as *boundaries*, a feature that requires taking into consideration two kind of cup competitors, and that also leads to other difficulties. Second, local variations need to be compensated by volume-fixing variations, which must be uniform along the elements of the minimizing sequence. At this stage, we can prove that $\mu_j = \mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \stackrel{*}{\rightharpoonup} \mu = \theta \, \mathcal{H}^n \llcorner K$ for an $\mathcal{H}^n$-rectifiable set K which is $\mathcal{C}$-spanning W. The same argument as in [DLGM17] shows that $\theta \ge 1$, and the lower bound $\theta \ge 2$ $\mathcal{H}^n$-a.e. on $K \setminus \partial^* E$ requires a further elaboration which takes into account that we are considering the convergence of boundaries. We cannot conclude that $\mathcal{F}(K, E) = \psi(\varepsilon)$ just by lower semicontinuity because clearly (K, E) is not in the competition class of $\psi(\varepsilon)$. We thus improve lower semicontinuity by some non-concentration estimates: at infinity, at the boundary and by folding against K. The latter are the most interesting ones, and they require a careful comparison argument based on the introduction of a third kind of competitors, called slab competitors. The construction of the various competitors is discussed in section 2, while the proof of Theorem 1.4 is contained in section 3. Slab competitors are also used in the delicate proof of

(1.15), whose starting point are some ideas originating in [DPH03], as further developed in [DLGM17] when addressing the formulation of Plateau's problem for David's sliding minimizers; see section 4. Finally, in section 5 we prove Theorem 1.9: the main difficulty, explained there in more detail, is that, at vanishing volume, we have no non-trivial local limit sets to be used for constructing uniform volume-fixing variations.

Structure of generalized minimizers: Theorem 1.4, Theorem 1.6 and Theorem 1.9 lay the foundations to study the properties of generalized minimizers of $\psi(\varepsilon)$. The most intriguing questions are concerned with the relations between the properties of minimizers in Plateau's problem ℓ, like the presence or the absence of singularities, and the properties of minimizers in $\psi(\varepsilon)$ at small ε: collapsing vs non-collapsing and the sign of λ, limiting behavior of λ as $\varepsilon \to 0^+$, dimensionality of the wetted part of the wire, etc. This is of course a very large set of problems, which will require further investigations. In the companion paper [KMS21], we start this kind of study by proving that collapsed minimizers have non-positive Lagrange multipliers, deduce from this property that they satisfy the convex hull property, and lay the ground for the forthcoming paper [KFS20], where we further investigate the regularity of the collapsed set $K \setminus \partial^* E$.

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2. Cone, cup and slab competitors, nucleation and collapsing

Section 2.1 contains the notation and terminology used in the paper. Section 2.2 collects some basic properties of $\mathcal{C}$-spanning sets. Sections 2.3, 2.4 and 2.5 deal with cup, slab and cone competitors. Section 2.6 contains the nucleation lemma for volume-fixing variations, and section 2.7 concerns density lower bounds for collapsing sequences of sets of finite perimeter.

2.1. **Notation and terminology.** We denote by $|A|$ and $\mathcal{H}^s(A)$ the Lebesgue and the s-dimensional Hausdorff measures of $A \subset \mathbb{R}^{n+1}$, by $I_\eta(A)$ and $U_\eta(A)$ the closed and open η-neighborhoods of A, by $B_r(x)$ the open ball of center at x and radius r. We work in the framework of [Sim83, AFP00, Mag12]. Given $k \in \mathbb{N}$, $1 \le k \le n$, a Borel set $M \subset \mathbb{R}^{n+1}$ is **countably $\mathcal{H}^k$-rectifiable** if it is covered by countably many Lipschitz images of $\mathbb{R}^k$; it is **(locally) $\mathcal{H}^k$-rectifiable** if, in addition, M is (locally) $\mathcal{H}^k$-finite. If M is locally $\mathcal{H}^k$-rectifiable, then for $\mathcal{H}^k$-a.e. $x \in M$ there exists a unique k-plane T_xM such that, as $r \to 0^+$, $\mathcal{H}^k \llcorner (M-x)/r \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner T_xM$ as Radon measures in $\mathbb{R}^{n+1}$; T_xM is called the **approximate tangent plane to M at x**. Given a Lipschitz map $f:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$, we denote by $J^M f$ its **tangential jacobian along** M, so that if f is smooth and $f(x) = x + t\,X(x) + \mathrm{o}(t)$ in C^1 as $t \to 0^+$, then $J^M f = 1 + t\,\mathrm{div}^M X + \mathrm{o}(t)$ where $\mathrm{div}^M X$ is the **tangential divergence of X along** M; moreover, M has **distributional mean curvature vector $\vec{H} \in L^1_{\mathrm{loc}}(U;\mathcal{H}^k \llcorner M)$ in** U open, if

$$\int_M \mathrm{div}^M X \, d\mathcal{H}^k = \int_M X \cdot \vec{H} \, d\mathcal{H}^k \,, \qquad \forall X \in C^\infty_c(U;\mathbb{R}^{n+1}) \,,$$

see [Sim83, Sections 8 and 9]. A Borel set $E \subset \mathbb{R}^{n+1}$ has **finite perimeter** if there exists an $\mathbb{R}^{n+1}$-valued Radon measure on $\mathbb{R}^{n+1}$, denoted by μ_E, such that $\langle \mu_E, X\rangle = \int_E \mathrm{div}\,X$ whenever $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ and $P(E;\mathbb{R}^{n+1}) = |\mu_E|(\mathbb{R}^{n+1}) < \infty$. The set of points

$x \in \mathbb{R}^{n+1}$ such that $|\mu_E|(B_r(x))^{-1}\,\mu_E(B_r(x)) \to \nu_E(x) \in \mathbb{S}^n$ as $r \to 0^+$ is denoted by $\partial^* E$, and called the **reduced boundary** $\partial^* E$ of E. Then $\mu_E = \nu_E\,\mathcal{H}^n \llcorner \partial^* E$, $\partial^* E$ is $\mathcal{H}^n$-rectifiable in $\mathbb{R}^{n+1}$, and $T_x\partial^* E = \nu_E(x)^\perp$ for every $x \in \partial^* E$. The **set** $E^{(t)}$ **of points of density** $t \in [0,1]$ **of** E is given by those $x \in \mathbb{R}^{n+1}$ with $|E \cap B_r(x)|/|B_r(x)| \to t$ as $r \to 0^+$, and (see, e.g., see [Mag12, Theorem 16.2]),

$$\{\partial^* E, E^{(0)}, E^{(1)}\} \text{ is a partition of } \mathbb{R}^{n+1} \text{ modulo } \mathcal{H}^n\,. \tag{2.1}$$

Federer's criterion [Fed69, 4.5.11] states that if the **essential boundary** $\partial^{\rm e} E = \mathbb{R}^{n+1} \setminus (E^{(0)} \cup E^{(1)})$ is $\mathcal{H}^n$-finite, then E is of finite perimeter in $\mathbb{R}^{n+1}$. If E is open, then $\partial^{\rm e} E \subset \partial E$: hence, if $E \in \mathcal{E}$ and $\mathcal{H}^n(\partial\Omega) < \infty$, then E is of finite perimeter.

2.2. **Some preliminary results.** In the following, W is a compact set, $\mathcal{C}$ a spanning class for W and $\Omega = \mathbb{R}^{n+1} \setminus W$.

Lemma 2.1. *If $\{K_j\}_j$ are relatively closed sets in Ω, such that each K_j is $\mathcal{C}$-spanning W and $\mathcal{H}^n \llcorner K_j \overset{*}{\rightharpoonup} \mu$ as Radon measures in Ω, then $K = \Omega \cap \mathrm{spt}\mu$ is $\mathcal{C}$-spanning W.*

Proof. See [DLGM17, Step 2, proof of Theorem 4]. □

Lemma 2.2. *Let K be relatively closed in Ω and let $B_r(x) \subset\subset \Omega$. Then K is $\mathcal{C}$-spanning W if and only if, whenever $\gamma \in \mathcal{C}$ is such that $\gamma \cap K \setminus B_r(x) = \emptyset$, then there exists a connected component of $\gamma \cap \mathrm{cl}\,(B_r(x))$ which is diffeomorphic to an interval, and whose end-points belong to distinct connected components of $\mathrm{cl}\,(B_r(x)) \setminus K$, as well as to distinct components of $\partial B_r(x) \setminus K$.*

Proof. This is [DLGM17, Lemma 10]. □

Lemma 2.3. *If K is $\mathcal{C}$-spanning W, $B_r(x) \subset\subset \Omega$, and $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a bi-Lipschitz map with $\{f \neq \mathrm{id}\} \subset\subset B_r(x)$ and $f(B_r(x)) \subset B_r(x)$, then $f(K)$ is $\mathcal{C}$-spanning W.*

Proof. By $f(K) \setminus B_r(x) = K \setminus B_r(x)$, if $f(K)$ is not $\mathcal{C}$-spanning W, then there exists $\gamma \in \mathcal{C}$ with $\gamma \cap K \setminus B_r(x) = \emptyset$ such that $\gamma \cap f(K) = \emptyset$. Hence, the curve $\tilde\gamma := f^{-1} \circ \gamma$ is a continuous embedding of $\mathbb{S}^1$ in Ω, homotopic to γ in Ω, and such that $\tilde\gamma \cap K = \emptyset$. Since $\tilde\gamma$ and W are compact and K is closed, $\tilde\gamma$ has positive distance from $K \cup W$, and by smoothing out $\tilde\gamma$ we define a smooth embedding $\hat\gamma$ of $\mathbb{S}^1$ into Ω, disjoint from K, and homotopic to $\tilde\gamma$ (and therefore to γ) in Ω, a contradiction. □

Lemma 2.4. *If $\partial\Omega$ is smooth, then there exists $r_0 > 0$ with the following property. If $x \in \partial\Omega$, $\Omega \subset \Omega'$, $f : \mathrm{cl}\,(\Omega) \to \mathrm{cl}\,(\Omega') = f(\mathrm{cl}\,(\Omega))$ is a homeomorphism with $f(\partial\Omega) = \partial\Omega'$, $\{f \neq \mathrm{id}\} \subset\subset B_{r_0}(x)$, and $f(B_{r_0}(x) \cap \mathrm{cl}\,(\Omega)) = B_{r_0}(x) \cap \mathrm{cl}\,(\Omega')$, and if K is $\mathcal{C}$-spanning W, then $K' = f(K \cap \Omega^*)$ is relatively closed in Ω and is $\mathcal{C}$-spanning W, where $\Omega^* = f^{-1}(\Omega)$.*

Proof. Step one: We show that, for K relatively closed in Ω and $B_{r_0}(x)$ as in the statement, K is $\mathcal{C}$-spanning W if and only if, whenever $\gamma \in \mathcal{C}$ is such that $\gamma \cap K \setminus B_{r_0}(x) = \emptyset$, then there exists a connected component of $\gamma \cap \mathrm{cl}\,(B_{r_0}(x))$, diffeomorphic to an interval, and whose end-points belong to distinct connected components of $\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K$. We only prove the "only if" part. First of all, we notice that γ cannot be contained in $\Omega \cap B_{r_0}(x)$, because r_0 can be chosen small enough to ensure that $\Omega \cap B_{r_0}(x)$ is simply connected, and because $\ell < \infty$ implies that no element of $\mathcal{C}$ is homotopic to a constant. Arguing as in [DLGM17, Step two, proof of Lemma 10] we can assume that γ and $\partial B_{r_0}(x)$ intersect transversally, so that there exist finitely many disjoint $I_i = [a_i, b_i] \subset \mathbb{S}^1$ such that $\gamma \cap \mathrm{cl}\,(B_{r_0}(x)) = \bigcup_i \gamma(I_i)$ with $\gamma \cap \partial B_{r_0}(x) = \bigcup_i \{\gamma(a_i), \gamma(b_i)\}$ and $\gamma \cap B_{r_0}(x) = \bigcup_i \gamma((a_i, b_i))$. Assume by contradiction that for each i there exists a connected component A_i of $\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K$ such that $\gamma(a_i), \gamma(b_i) \in A_i$. If r_0 is small enough, then $\mathrm{cl}\,(\Omega \cap B_{r_0}(x))$ is diffeomorphic to $\mathrm{cl}\,(B_1(0) \cap \{x_1 > 0\})$ through a diffeomorphism mapping $B_{r_0}(x) \cap \partial\Omega$ into $B_1(0) \cap \{x_1 = 0\}$. Using this fact and the connectedness of each A_i, we define smooth embeddings $\tau_i : I_i \to A_i$

with $\tau_i(a_i) = \gamma(a_i)$, $\tau_i(b_i) = \gamma(b_i)$ and τ_i homotopic in $\Omega \cap B_{r_0}(x)$ to the restriction of γ to I_i. Moreover, this can be done with $\tau_i(I_i) \cap \tau_j(I_j) = \emptyset$. The new embedding $\bar{\gamma}$ of $\mathbb{S}^1$ obtained by replacing γ with τ_i on I_i is thus homotopic to γ in Ω, and such that $\bar{\gamma} \cap K = \emptyset$, a contradiction.

Step two: Since $K \cap \Omega^*$ is relatively closed in Ω^*, $K' = f(K \cap \Omega^*)$ is relatively closed in $\Omega = f(\Omega^*)$. Should K' not be $\mathcal{C}$-spanning W, given that $K' \setminus B_{r_0}(x) = K \setminus B_{r_0}(x)$, we could find $\gamma \in \mathcal{C}$ with $\gamma \cap K \setminus B_{r_0}(x) = \emptyset$ and $\gamma \cap K' = \emptyset$. By step one, there would be a connected component σ of $\gamma \cap \mathrm{cl}\,(B_{r_0}(x))$, diffeomorphic to an interval, and such that: (i) the end-points p and q of σ (which lie on $\partial B_{r_0}(x)$) belong to distinct connected components of $\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K$; and (ii) p and q belong to the same connected component of $\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K'$. Since f is a homeomorphism, $f(p) = p$, and $f(q) = q$, by (i) we would find that p and q belong to *distinct* connected components of

$$f\big(\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K\big) = \Omega' \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K)$$

while, by (ii), there would be an arc connecting p and q in $\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K'$, where

$$\begin{aligned}\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K' &= \Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K \cap \Omega^*)\\ &= \Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K) \;\subset\; \Omega' \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K)\,,\end{aligned}$$

and hence p and q would belong to a *same* component of $\Omega' \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K)$. □

2.3. **Cup competitors.** Given $E \in \mathcal{E}$, $B_r(x) \subset\subset \Omega$ and a connected component A of $\partial B_r(x) \setminus \partial E$, cup competitors are used to compare $\mathcal{H}^n(B_r(x) \cap \partial E)$ with $\mathcal{H}^n(\partial B_r(x) \setminus A)$. The construction is more involved than in the case of Plateau's problem considered in [DLGM17] as we need to construct cup competitors as *boundaries*, and we have to argue differently depending on whether $A \cap E = \emptyset$ or $A \subset E$.

Lemma 2.5 (Cup competitors)**.** *Let $E \in \mathcal{E}$ be such that $\Omega \cap \partial E$ is $\mathcal{C}$-spanning W, let $x \in \Omega$, $0 < r < \mathrm{dist}(x, \partial\Omega)$, and let A be a connected component of $\partial B_r(x) \setminus \partial E$. Assume that $\partial E \cap \partial B_r(x)$ is $\mathcal{H}^{n-1}$-rectifiable. Then, for every $\eta \in (0, r/2)$ there exists a set $F = F_\eta \in \mathcal{E}$ so that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W, and*

$$\partial F \setminus \mathrm{cl}\,(B_r(x)) = \partial E \setminus \mathrm{cl}\,(B_r(x))\,, \tag{2.2}$$

$$\lim_{\eta \to 0^+} \mathcal{H}^n\big((\partial B_r(x) \cap \partial F)\,\Delta\,(\partial B_r(x) \setminus A)\big) = 0\,, \tag{2.3}$$

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F) \le \mathcal{H}^n\big(\Omega \cap \partial E \setminus B_r(x)\big) + 2\,\mathcal{H}^n(\partial B_r(x) \setminus A)\,. \tag{2.4}$$

Moreover,

(i) *If $A \cap E = \emptyset$, then*

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(B_r(x) \cap \partial F) \le \mathcal{H}^n\Big(\partial B_r(x) \setminus \big(A \cup (E \cap \partial B_r)\big)\Big)\,; \tag{2.5}$$

(ii) *If $A \subset E$, then*

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(B_r(x) \cap \partial F) \le \mathcal{H}^n\big(E \cap \partial B_r(x) \setminus A\big)\,. \tag{2.6}$$

Remark 2.6. Before proceeding with the proof of the lemma, let us first provide some additional details on the construction of the competitors $F = F_\eta$, which, as anticipated, is different depending on whether $A \cap E = \emptyset$ or $A \subset E$. In what follows, given $Y \subset \partial B_r(x)$, we set

$$N_\eta(Y) = \Big\{y - t\,\nu_{B_r(x)}(y) : y \in Y\,, t \in (0, \eta)\Big\}\,, \qquad 0 < \eta < r\,.$$

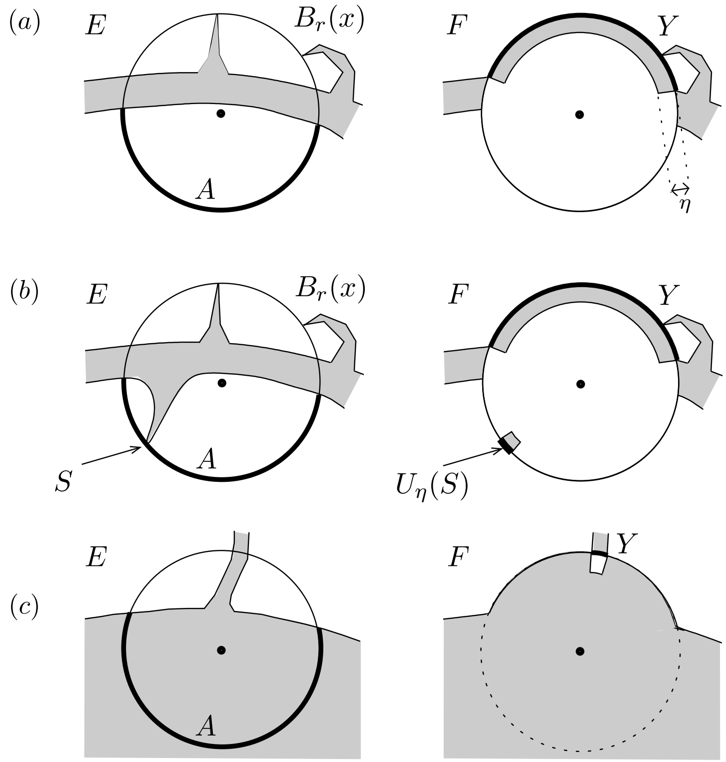

FIGURE 2.1. Cup competitors when: (a) $A \cap E = \emptyset$ and $S = \emptyset$; (b) $A \cap E = \emptyset$ and $S \neq \emptyset$; (c) $A \subset E$. Picture (b) really pertains to the case $n \geq 2$, in which the component A in the picture is not necessarily disconnected by the presence of S. In the situation of picture (b) the set F defined by (2.9) may fail to intersect a test curve γ which was intersecting with $\Omega \cap \partial E$ only at points in S.

The case when $A \cap E = \emptyset$: In this case, we define

$$Y = \partial B_r(x) \setminus \big(\mathrm{cl}\,(E \cap \partial B_r(x)) \cup \mathrm{cl}\,(A)\big)\,, \tag{2.7}$$

and then we further distinguish two scenarios, depending on whether the set

$$S = \partial E \cap \mathrm{cl}\,(A) \setminus [\mathrm{cl}\;(E \cap \partial B_r(x)) \cup \mathrm{cl}\,(Y)] \tag{2.8}$$

is empty or not. When $S = \emptyset$ the cup competitor defined by E and A is given by

$$F = \big(E \setminus \mathrm{cl}\,(B_r(x))\big) \cup N_\eta(Y)\,, \tag{2.9}$$

see Figure 2.1-(a), and step one of the proof. When $S \neq \emptyset$, see Figure 2.1-(b), if we define F as in (2.9), then $\Omega \cap \partial F$ may fail to be $\mathcal{C}$-spanning W; we thus need to modify (2.9), and to this end, denoting by d_S the distance function from S and by $U_\eta(S) = \partial B_r(x) \cap \{\mathrm{d}_S(y) < \eta\}$, we set

$$F = \big(E \setminus \mathrm{cl}\,(B_r(x))\big) \cup N_\eta(Z)\,, \qquad Z = Y \cup (U_\eta(S) \setminus \mathrm{cl}\,(E \cap \partial B_r(x)))\;, \tag{2.10}$$

see, again, Figure 2.1-(b). This situation, discussed in detail in step two of the proof, is made more delicate since we can prove that the sets defined in (2.10) are well-behaved in the limit as $\eta \to 0^+$ only along along a suitable sequence $\eta_k \downarrow 0^+$. For this reason, we will actually define F_η as in (2.10) only when $\eta = \eta_k$, and then extend the definition by setting $F_\eta = F_{\eta_k}$ for all $\eta \in (\eta_{k+1}, \eta_k)$ (so that, for the sake of homogenity, (2.4) can be stated as an $\eta \to 0^+$-limit in all three cases).

The case when $A \cap E = \emptyset$: Finally, when $A \subset E$ the cup competitor defined by E and A is given by

$$F = \big(E \cup B_r(x)\big) \setminus \mathrm{cl}\,\big(N_\eta(Y)\big)\,, \qquad Y = (E \cap \partial B_r(x)) \setminus \mathrm{cl}\,(A)\,, \tag{2.11}$$

see Figure 2.1-(c). We treat this case in step three of the proof.

Proof. Step one: We assume that $A \cap E = \emptyset$ and, after defining Y as in (2.7) and S as in (2.8), we suppose first that

$$S = \emptyset\,. \tag{2.12}$$

We then define F by (2.9). For the sake of brevity we set $B_r = B_r(x)$. We claim that (2.2) holds, and that we have

$$\begin{aligned}
B_r \cap \partial F &= B_r \cap \partial N_\eta(Y)\,, && (2.13)\\
Y &\subset \partial F \cap \partial B_r\,, && (2.14)\\
E \cap \partial B_r &\subset \partial F \cap \partial B_r\,, && (2.15)\\
\partial B_r \setminus \mathrm{cl}\,(A) &\subset \partial F \cap \partial B_r\,, && (2.16)\\
\partial E \cap \partial B_r &\subset \partial F \cap \partial B_r\,, && (2.17)\\
&A,\ E \cap \partial B_r,\ Y \text{ are open and disjoint in } \partial B_r\,, && (2.18)\\
\partial F \cap \partial B_r &\subset \partial B_r \setminus A\,, && (2.19)\\
\partial B_r \setminus \mathrm{cl}\,(E) &\subset A \cup Y\,, && (2.20)\\
\mathrm{cl}\,(Y) \setminus Y &\subset \partial B_r \cap \partial E\,, && (2.21)\\
\mathrm{cl}\,(A) \setminus A &\subset \partial B_r \cap \partial E\,, && (2.22)\\
\mathrm{cl}\,(E \cap \partial B_r) \setminus (E \cap \partial B_r) &= \partial B_r \cap \partial E\,. && (2.23)
\end{aligned}$$

Indeed, (2.2) and (2.13) follow from $F \cap B_r = N_\eta(Y) \cap B_r$ and $F \setminus \mathrm{cl}\,(B_r) = E \setminus \mathrm{cl}\,(B_r)$. To prove (2.14): $Y \subset \mathrm{cl}\,(N_\eta(Y))$ gives $Y \subset \mathrm{cl}\,(F)$, and $F \cap \partial B_r = \emptyset$ implies $Y \cap F = \emptyset$. To prove (2.15): $E \cap \partial B_r \subset \mathrm{cl}\,(E \setminus \mathrm{cl}\,(B_r))$, so that $E \cap \partial B_r \subset \mathrm{cl}\,(F)$, while $F \cap \partial B_r = \emptyset$ gives $(E \cap \partial B_r) \cap F = \emptyset$. (2.18) is obvious, and (2.16) follows from (2.14) and (2.15). (2.17) is then an immediate consequence of (2.14), (2.15), (2.16), and the condition in (2.12). To prove (2.19): A is open in $\partial B_r \setminus \partial E$ and $A \cap E = \emptyset$, thus $A \cap \mathrm{cl}\,(E) = \emptyset$; moreover, $A \cap \mathrm{cl}\,(Y) = \emptyset$ by (2.18), hence

$$\partial F \cap \partial B_r \;\subset\; \mathrm{cl}\,(F) \cap \partial B_r \;\subset\; \mathrm{cl}\,(E) \cup \Big(\mathrm{cl}\,(N_\eta(Y)) \cap \partial B_r\Big) \;= \mathrm{cl}\,(E) \cup \mathrm{cl}\,(Y)\,,$$

and we deduce (2.19). To prove (2.20): if $y \in \partial B_r \setminus \mathrm{cl}\,(E)$, then y belongs to one of the open connected components of $\partial B_r \setminus \partial E$, so it is either $y \in A$, or $y \in \partial B_r \setminus \mathrm{cl}\,(A) \subset Y$. To prove (2.21): by (2.18) we have $A \cap \mathrm{cl}\,(Y) = \emptyset$, so that by (2.20)

$$\mathrm{cl}\,(Y) \setminus Y \;\subset\; \partial B_r \setminus (A \cup Y) \;\subset\; \partial B_r \cap \mathrm{cl}\,(E)\,,$$

and we conclude by $(E \cap \partial B_r) \cap \mathrm{cl}\,(Y) = \emptyset$ (again, thanks to (2.18)). Finally, (2.22) and the inclusion "$\subset$" in (2.23) are obvious, while the other inclusion in (2.23) follows from (2.12). Having proved the claim, we complete the proof. By definition, $F \subset \Omega$ is open. We show that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W. Given $\gamma \in \mathcal{C}$, if $\gamma \cap \partial E \setminus \mathrm{cl}\,(B_r) \neq \emptyset$, then $\gamma \cap \partial F \neq \emptyset$ by (2.2); if instead $\gamma \cap \partial E \setminus \mathrm{cl}\,(B_r) = \emptyset$, then necessarily $\gamma \cap \partial E \cap \mathrm{cl}\,(B_r) \neq \emptyset$. Now, if $\gamma \cap \partial E \cap \partial B_r \neq \emptyset$ then $\gamma \cap \partial F \neq \emptyset$ by (2.17); otherwise we actually have $\gamma \cap \partial E \setminus B_r = \emptyset$, and thus, by Lemma 2.2, γ intersects two distinct connect components of $\partial B_r \setminus \partial E$, and at least one of them is contained in $\partial F \cap \partial B_r$: indeed, $\partial F \cap \partial B_r$ contains $\partial B_r \setminus \mathrm{cl}\,(A)$ by (2.16), where $\mathrm{cl}\,(A)$ is disjoint from all the connected components of $\partial B_r \setminus \partial E$ that are different from A.

Now, we prove (2.3), (2.4), and (2.5). First notice that (2.16), (2.19), (2.22), and $\mathcal{H}^n(\partial B_r \cap \partial E) = 0$ imply that

$$\partial F \cap \partial B_r \;=\; \partial B_r \setminus A \qquad \text{modulo } \mathcal{H}^n\,, \tag{2.24}$$

which in turn implies (2.3). Next, we claim that

$$\begin{aligned}\mathcal{H}^n(\Omega\cap\partial F) \;&\le\; \mathcal{H}^n\big(\Omega\cap\partial E\setminus B_r\big)+\mathcal{H}^n(E\cap\partial B_r) \\ &\quad+\big(2+C(n)\,\eta\big)\,\mathcal{H}^n\Big(\partial B_r\setminus\big(A\cup(E\cap\partial B_r)\big)\Big)+C(n)\,\eta\,\mathcal{H}^{n-1}\big(\partial E\cap\partial B_r\big)\,.\end{aligned}\tag{2.25}$$

To prove the claim, first by $\mathcal{H}^n(\partial E\cap\partial B_r)=0$, (2.2) and (2.19) we have

$$\begin{aligned}\mathcal{H}^n(\Omega\cap\partial F) \;&=\; \mathcal{H}^n(\Omega\cap\partial E\setminus B_r)+\mathcal{H}^n(\mathrm{cl}\,(B_r)\cap\partial F) \\ &\le\; \mathcal{H}^n(\Omega\cap\partial E\setminus B_r)+\mathcal{H}^n(\partial B_r\setminus A)+\mathcal{H}^n(B_r\cap\partial F)\,.\end{aligned}\tag{2.26}$$

If $g(y,t)=y-t\,\nu_{B_r}(y)$, then by (2.13)

$$B_r\cap\partial F=B_r\cap\partial N_\eta(Y)=g(Y,\eta)\,\cup\,g\Big(\big(\mathrm{cl}\,(Y)\setminus Y\big)\times[0,\eta]\Big)\,,$$

so that (2.21), the $\mathcal{H}^{n-1}$-rectifiability of $\partial E\cap\partial B_r$, and the area formula give us

$$\mathcal{H}^n(B_r\cap\partial F)\le(1+C(n)\,\eta)\,\mathcal{H}^n(Y)+C(n)\,\eta\,\mathcal{H}^{n-1}(\partial E\cap\partial B_r)\,.\tag{2.27}$$

By $\mathcal{H}^n(\partial E\cap\partial B_r)=0$, (2.22) and (2.23) we have

$$\mathcal{H}^n(Y)=\mathcal{H}^n\big(\partial B_r\setminus\big(A\cup(E\cap\partial B_r)\big)\big)\,,\tag{2.28}$$

so that (2.26), (2.27) and (2.28) imply (2.25). Letting $\eta\to 0^+$ in (2.25) we find (2.4), and doing the same in (2.27) and (2.28), we deduce (2.5).

Step two: In the case $A\cap E=\emptyset$, we now allow for the set S defined in (2.8) to be non-empty. In this case, if F is defined as in (2.9) then the inclusion (2.17) is not true in general, and $\Omega\cap\partial F$ may fail to be $\mathcal{C}$-spanning W. We then modify the construction as detailed in Remark 2.6, defining F as in (2.10). We notice that $F\subset\Omega$ is open, and that (2.2) holds true, since once again $F\setminus\mathrm{cl}\,(B_r)=E\setminus\mathrm{cl}\,(B_r)$. Moreover, we have

$$B_r\cap\partial F \;=\; B_r\cap\partial N_\eta(Z)\,,\tag{2.29}$$

$$Z \;\subset\; \partial F\cap\partial B_r\,,\tag{2.30}$$

$$E\cap\partial B_r \;\subset\; \partial F\cap\partial B_r\,,\tag{2.31}$$

$$\partial B_r\setminus\mathrm{cl}\,(A) \;\subset\; \partial F\cap\partial B_r\,,\tag{2.32}$$

$$\partial E\cap\partial B_r \;\subset\; \partial F\cap\partial B_r\,,\tag{2.33}$$

$$A,\ E\cap\partial B_r,\ Y \text{ are open and disjoint in } \partial B_r\,,\tag{2.34}$$

$$\partial F\cap\partial B_r \;\subset\; [\partial B_r\setminus A]\cup[\partial B_r\cap\{\mathrm{d}_S\le\eta\}]\,,\tag{2.35}$$

$$\partial B_r\setminus\mathrm{cl}\,(E) \;\subset\; A\cup Y\,,\tag{2.36}$$

$$\mathrm{cl}\,(Y)\setminus Y \;\subset\; \partial B_r\cap\partial E\,,\tag{2.37}$$

$$\mathrm{cl}\,(A)\setminus A \;\subset\; \partial B_r\cap\partial E\,,\tag{2.38}$$

$$\mathrm{cl}\,(E\cap\partial B_r)\setminus(E\cap\partial B_r) \;\subset\; \partial B_r\cap\partial E\,.\tag{2.39}$$

The proofs of (2.29), (2.30), (2.31), (2.32) are identical to the proofs of the corresponding statements in step one with Z replacing Y; (2.33) then follows from (2.30), (2.31), and (2.32), since $S\subset U_\eta(S)\setminus\mathrm{cl}\,(E\cap\partial B_r)\subset Z$; (2.34) is obvious. To prove (2.35): as in step one, $A\cap\mathrm{cl}\,(E)=\emptyset$ and $A\cap\mathrm{cl}\,(Y)=\emptyset$ by (2.34), and

$$\begin{aligned}\partial F\cap\partial B_r \;&\subset\; \mathrm{cl}\,(F)\cap\partial B_r\;\subset\;\mathrm{cl}\,(E)\cup(\mathrm{cl}\,(N_\eta(Z))\cap\partial B_r) \\ &\subset\; \mathrm{cl}\,(E)\cup\mathrm{cl}\,(Y)\cup\mathrm{cl}\,(U_\eta(S))\,,\end{aligned}$$

so that (2.35) follows from the fact that $\mathrm{cl}\,(U_\eta(S))\subset\partial B_r\cap\{\mathrm{d}_S\le\eta\}$. Next, we notice that (2.36), (2.37), (2.38), and (2.39) are shown analogously to step one (with the identity in (2.23) which becomes an inclusion in (2.39) due to S possibly being not empty). With the above at our disposal, we proceed now to verify the claims of the lemma. First, the proof that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W follows *verbatim* the argument from step one. Next, (2.32), (2.35), (2.38), and $\mathcal{H}^n(\partial E \cap \partial B_r) = 0$ imply that

$$\mathcal{H}^n\left((\partial F \cap \partial B_r) \,\Delta\, (\partial B_r \setminus A)\right) \le \mathcal{H}^n(\partial B_r \cap \{\mathrm{d}_S \le \eta\})\,. \tag{2.40}$$

In particular, since $\mathcal{H}^{n-1}(S) < \infty$, it holds

$$\lim_{\eta \to 0^+} \mathcal{H}^n\left((\partial F \cap \partial B_r) \,\Delta\, (\partial B_r \setminus A)\right) = 0\,, \tag{2.41}$$

that is (2.3). Next, we proceed with estimating $\mathcal{H}^n(\Omega \cap \partial F)$. We first notice that, by (2.2) and $\mathcal{H}^n(\partial E \cap \partial B_r) = 0$

$$\begin{aligned} \mathcal{H}^n(\Omega \cap \partial F) &= \mathcal{H}^n(\Omega \cap \partial E \setminus B_r) + \mathcal{H}^n(\mathrm{cl}\,(B_r) \cap \partial F) \\ &\le \mathcal{H}^n(\Omega \cap \partial E \setminus B_r) + \mathcal{H}^n(\partial F \cap \partial B_r) + \mathcal{H}^n(B_r \cap \partial F)\,. \end{aligned} \tag{2.42}$$

Setting, as in step one, $g(y,t) = y - t\,\nu_{B_r}(y)$, we then have from (2.29) that

$$B_r \cap \partial F = B_r \cap \partial N_\eta(Z) = g(Z,\eta) \cup g\left((\mathrm{cl}\,(Z) \setminus Z) \times [0,\eta]\right)\,. \tag{2.43}$$

By the area formula, we can easily estimate

$$\begin{aligned} \mathcal{H}^n(g(Z,\eta)) &\le (1 + C(n)\,\eta)\,\mathcal{H}^n(Z) \\ &\le (1 + C(n)\,\eta)\,\Big(\mathcal{H}^n(Y) + \mathcal{H}^n(\partial B_r \cap \{\mathrm{d}_S < \eta\})\Big) \\ &\le (1 + C(n)\,\eta)\,\Big(\mathcal{H}^n(\partial B_r \setminus (A \cup (E \cap \partial B_r))) + \mathcal{H}^n(\partial B_r \cap \{\mathrm{d}_S < \eta\})\Big)\,. \end{aligned} \tag{2.44}$$

On the other hand, it holds

$$\mathrm{cl}\,(Z) \setminus Z \subset [\mathrm{cl}\,(Y) \setminus (Y)] \cup [\mathrm{cl}\,(\hat{U}) \setminus \hat{U}]\,, \tag{2.45}$$

where $\hat{U} = U_\eta(S) \setminus \mathrm{cl}\,(E \cap \partial B_r)$. Since $\mathrm{cl}\,(\hat{U}) \subset \mathrm{cl}\,(U_\eta(S)) \setminus (E \cap \partial B_r)$, (2.39) implies that

$$\mathrm{cl}\,(\hat{U}) \setminus \hat{U} \subset (\partial B_r \cap \{\mathrm{d}_S = \eta\}) \cup (\partial B_r \cap \partial E)\,, \tag{2.46}$$

and thus (2.37) yields

$$\mathcal{H}^n\left(g((\mathrm{cl}\,(Z) \setminus Z) \times [0,\eta])\right) \le C(n)\,\eta\,\Big(\mathcal{H}^{n-1}(\partial B_r \cap \partial E) + \mathcal{H}^{n-1}(\partial B_r \cap \{\mathrm{d}_S = \eta\})\Big)\,. \tag{2.47}$$

By applying the coarea formula to d_S, it holds for every $0 < \sigma < r/2$

$$\int_0^\sigma \mathcal{H}^{n-1}(\partial B_r \cap \{\mathrm{d}_S = \eta\})\,d\eta = \mathcal{H}^n(\partial B_r \cap \{\mathrm{d}_S \le \sigma\}) < \infty\,, \tag{2.48}$$

and thus there exists a decreasing sequence $\{\eta_k\}_{k=1}^\infty$ with $\lim_{k\to\infty} \eta_k = 0$ such that $\partial B_r \cap \{\mathrm{d}_S = \eta_k\}$ is $\mathcal{H}^{n-1}$-rectifiable and

$$\lim_{k \to \infty} \eta_k\,\mathcal{H}^{n-1}(\partial B_r \cap \{\mathrm{d}_S = \eta_k\}) = 0\,. \tag{2.49}$$

If F_k is the sequence of cup competitors defined by (2.10) in correspondence with the choice $\eta = \eta_k$, we then have from (2.43), (2.45), (2.37), and (2.46) that $\Omega \cap \partial F_k$ is $\mathcal{H}^n$-rectifiable, and from (2.42), (2.41), (2.44), (2.47), and (2.49) that

$$\limsup_{k \to \infty} \mathcal{H}^n(B_r \cap \partial F_k) \le \mathcal{H}^n(\partial B_r \setminus (A \cup (E \cap \partial B_r)))\,, \tag{2.50}$$

$$\limsup_{k \to \infty} \mathcal{H}^n(\Omega \cap \partial F_k) \le \mathcal{H}^n(\Omega \cap \partial E \setminus B_r) + 2\,\mathcal{H}^n(\partial B_r \setminus A)\,. \tag{2.51}$$

Defining $F_\eta = F_{\eta_k}$ for all $\eta \in (\eta_{k+1}, \eta_k)$ then allows to conclude both (2.4) and (2.5).

Step three: We now assume that $A \subset E$, and define F by (2.11), that is

$$F = \big(E \cup B_r\big) \setminus \mathrm{cl}\,\big(N_\eta(Y)\big)\,, \qquad Y = (E \cap \partial B_r) \setminus \mathrm{cl}\,(A)\,. \tag{2.52}$$

We claim that (2.2) holds, as well as

$$\begin{aligned}
Y &\subset \partial F\cap\partial B_r\,, && (2.53)\\
\partial B_r\setminus E &\subset \partial F\cap\partial B_r\,, && (2.54)\\
\partial B_r\setminus \mathrm{cl}\,(A) &\subset \partial F\cap\partial B_r\,, && (2.55)\\
B_r\cap\partial F &\subset B_r\cap\partial N_\eta(Y)\,, && (2.56)\\
A,\partial B_r\setminus\mathrm{cl}\,(E),Y &\text{ are open and disjoint in }\partial B_r\,, && (2.57)\\
\partial F\cap\partial B_r &\subset \partial B_r\setminus A\,, && (2.58)\\
\mathrm{cl}\,(A)\setminus A &\subset \partial B_r\cap\partial E\,, && (2.59)\\
\mathrm{cl}\,(Y)\setminus Y &\subset \partial B_r\cap\partial E\,. && (2.60)
\end{aligned}$$

First, $F\setminus\mathrm{cl}\,(B_r)=E\setminus\mathrm{cl}\,(B_r)$ implies (2.2). To prove (2.53): since E is open we have $E\cap\partial B_r\subset\mathrm{cl}\,(E\setminus\mathrm{cl}\,(B_r))=\mathrm{cl}\,(F\setminus\mathrm{cl}\,(B_r))$ (by (2.52)), thus $Y\subset\mathrm{cl}\,(F)$; we conclude as $Y\cap F=\emptyset$. As $F\cap\partial B_r\subset E\cap\partial B_r$, to prove (2.54) we just need to show that $\partial B_r\setminus E\subset\mathrm{cl}\,(F)$: since $\mathrm{cl}\,(U)\setminus\mathrm{cl}\,(V)\subset\mathrm{cl}\,(U\setminus\mathrm{cl}\,(V))$ for every $U,V\subset\mathbb{R}^{n+1}$, by $\partial B_r\cap\mathrm{cl}\,(N_\eta(Y))\subset\mathrm{cl}\,(E)$,

$$\begin{aligned}
\partial B_r\setminus\mathrm{cl}\,(E) &\subset \mathrm{cl}\,(B_r)\setminus\mathrm{cl}\,(N_\eta(Y))\subset\mathrm{cl}\,\big(B_r\setminus\mathrm{cl}\,(N_\eta(Y)\big)\subset\mathrm{cl}\,(F)\,,\\
(\partial B_r\cap\partial E)\setminus\mathrm{cl}\,(N_\eta(Y)) &\subset \mathrm{cl}\,(E)\setminus\mathrm{cl}\,(N_\eta(Y))\subset\mathrm{cl}\,(E\setminus\mathrm{cl}\,(N_\eta(Y)))\subset\mathrm{cl}\,(F)\,,\\
\partial B_r\cap\partial E\cap\mathrm{cl}\,(N_\eta(Y)) &\subset \partial E\cap\mathrm{cl}\,(Y)\subset\partial F\,,
\end{aligned}$$

where the last inclusion follows by (2.53). Next, (2.55) follows by (2.53), (2.54) and

$$\partial B_r\setminus\mathrm{cl}\,(A)=[(E\cap\partial B_r)\setminus\mathrm{cl}\,(A)]\cup[\partial B_r\setminus(E\cup\mathrm{cl}\,(A))]\subset Y\cup(\partial B_r\setminus E)\,.$$

To prove (2.56): setting $V^c=\mathbb{R}^{n+1}\setminus V$, by $B_r\cap F=B_r\cap\mathrm{cl}\,(N_\eta(Y))^c$ we find $B_r\cap\partial F=B_r\cap\partial\,[\mathrm{cl}\,(N_\eta(Y))^c]$, where, as a general fact on open set $U\subset\mathbb{R}^{n+1}$, we have

$$\partial[\mathrm{cl}\,(U)^c]=\mathrm{cl}\,(\mathrm{cl}\,(U)^c)\setminus\mathrm{cl}\,(U)^c=\mathrm{cl}\,(U)\cap\mathrm{cl}\,(\mathrm{cl}\,(U)^c)\,,\qquad\mathrm{cl}\,(\mathrm{cl}\,(U)^c)\subset U^c\,,$$

and thus $\partial[\mathrm{cl}\,(U)^c]\subset\partial U$. Next, (2.57) is obvious, and implies $A\cap\mathrm{cl}\,(Y)=\emptyset$ where $\mathrm{cl}\,(Y)=\mathrm{cl}\,(N_\eta(Y))\cap\partial B_r$, so that $A\cap\partial B_r\subset E\cap\partial B_r\setminus\mathrm{cl}\,(N_\eta(Y))=F\cap\partial B_r$, and (2.58) follows. To prove (2.59), just notice that $A\subset E$ and A is a connected component of $\partial B_r\setminus\partial E$. To prove (2.60): trivially, $\mathrm{cl}\,(Y)\setminus Y\subset\mathrm{cl}\,(Y)\subset\partial B_r\cap\mathrm{cl}\,(E)$, while by definition of Y and by $\mathrm{cl}\,(Y)\cap A=\emptyset$

$$\begin{aligned}
E\cap(\mathrm{cl}\,(Y)\setminus Y) &= \big(\mathrm{cl}\,(Y)\cap(E\cap\partial B_r)\big)\setminus Y=\mathrm{cl}\,(Y)\cap(E\cap\partial B_r)\cap\mathrm{cl}\,(A)\\
&= (E\cap\partial B_r)\cap\mathrm{cl}\,(Y)\cap\partial A\subset E\cap(\mathrm{cl}\,(A)\setminus A)=\emptyset\,,
\end{aligned}$$

thanks to (2.59). We have completed the claim. Next, by (2.55), (2.58), (2.59), and by $\mathcal{H}^n(\partial B_r\cap\partial E)=0$, we deduce (2.24) and thus (2.3), while $\Omega\cap\partial F$ is $\mathcal{C}$-spanning W thanks to (2.2), Lemma 2.2, (2.55), and (2.54). Finally,

$$\begin{aligned}
\mathcal{H}^n(\Omega\cap\partial F) &\le \mathcal{H}^n\big(\partial E\setminus B_r\big)+\mathcal{H}^n(\partial B_r\setminus E) \qquad (2.61)\\
&\quad+\big(2+C(n)\,\eta\big)\,\mathcal{H}^n(E\cap\partial B_r\setminus A)+C(n)\,\eta\,\mathcal{H}^{n-1}\big(\partial E\cap\partial B_r\big)\,.
\end{aligned}$$

Indeed, by $\mathcal{H}^n(\partial E\cap\partial B_r)=0$, (2.2), and (2.58)

$$\begin{aligned}
\mathcal{H}^n(\Omega\cap\partial F) &\le \mathcal{H}^n(\partial E\setminus B_r)+\mathcal{H}^n(\partial F\cap\mathrm{cl}\,(B_r)) \qquad (2.62)\\
&\le \mathcal{H}^n(\partial E\setminus B_r)+\mathcal{H}^n(\partial B_r\setminus A)+\mathcal{H}^n(B_r\cap\partial F)\\
&\le \mathcal{H}^n(\partial E\setminus B_r)+\mathcal{H}^n(\partial B_r\setminus E)+\mathcal{H}^n((E\cap\partial B_r)\setminus A)+\mathcal{H}^n(B_r\cap\partial F)\,;
\end{aligned}$$

by (2.56), (2.60), the $\mathcal{H}^{n-1}$-rectifiability of $\partial E\cap\partial B_r$, and the area formula

$$\begin{aligned}
\mathcal{H}^n(B_r\cap\partial F) &\le \mathcal{H}^n(B_r\cap\partial N_\eta(Y)) \qquad (2.63)\\
&\le (1+C(n)\,\eta)\mathcal{H}^n(Y)+C(n)\,\eta\,\mathcal{H}^{n-1}(\partial E\cap\partial B_r)\,,
\end{aligned}$$

while (2.59) and $\mathcal{H}^n(\partial B_r\cap\partial E)=0$ give

$$\mathcal{H}^n(Y)=\mathcal{H}^n((E\cap\partial B_r)\setminus\mathrm{cl}\,(A))=\mathcal{H}^n((E\cap\partial B_r)\setminus A)\,.$$

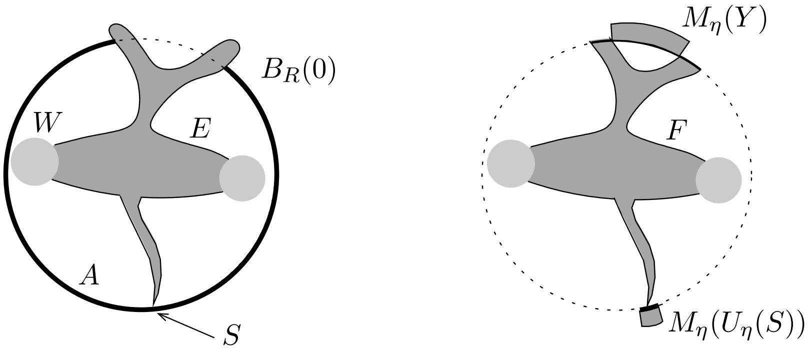

FIGURE 2.2. An exterior cup competitor. Notice that for S to be non-emtpy, and non-disconnecting A, it must be $n \geq 2$.

We thus deduce (2.61). As $\eta \to 0^+$ in (2.61) and in (2.63) we get (2.4) and (2.6). $\square$

In the following lemma we introduce the notion of exterior cup competitor. We set

$$M_\eta(Y) = \Big\{ y + t\,\nu_B(y) : y \in Y\,, t \in (0,\eta) \Big\}\,, \qquad \eta > 0\,,$$

whenever B is an open ball and $Y \subset \partial B$.

Lemma 2.7 (Exterior cup competitor)**.** *Let $E \in \mathcal{E}$ be such that $\Omega \cap \partial E$ is $\mathcal{C}$-spanning W, let $R > 0$ be such that $W \subset\subset B_R(0)$ and $\partial E \cap \partial B_R(0)$ is $\mathcal{H}^{n-1}$-rectifiable, and let A be a connected component of $\partial B_R(0) \setminus \partial E$ such that $A \cap E = \emptyset$. For every $\eta \in (0,1)$ there exists a set $F = F_\eta \in \mathcal{E}$ such that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W and*

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F) \leq \mathcal{H}^n\big(\Omega \cap B_R(0) \cap \partial E) + 2\,\mathcal{H}^n(\partial B_R(0) \setminus A)\,. \tag{2.64}$$

Proof. The proof consists of a minor modification of step one and step two in the proof of Lemma 2.5. Precisely, the exterior cup competitor defined by E and A is given by

$$F = \big(E \cap B_R(0)\big) \cup M_\eta(Z)\,, \tag{2.65}$$

where

$$\begin{aligned} Z &= Y \cup \Big(U_\eta(S) \setminus \mathrm{cl}\,(E \cap \partial B_R(0))\Big)\,, \\ Y &= \partial B_R(0) \setminus \Big(\mathrm{cl}\,(E \cap \partial B_R(0)) \cup \mathrm{cl}\,(A)\Big)\,, \\ U_\eta(S) &= \partial B_R(0) \cap \{\mathrm{d}_S < \eta\}\,, \\ S &= \partial E \cap \mathrm{cl}\,(A) \setminus \Big(\mathrm{cl}\,(E \cap \partial B_R(0)) \cup \mathrm{cl}\,(Y)\Big)\,; \end{aligned}$$

see Figure 2.2. If $\gamma \in \mathcal{C}$ is such that $\gamma \cap \partial E \cap \mathrm{cl}\,(B_R(0)) = \emptyset$, then an adaptation of step one in the proof of Lemma 2.4 shows that there exists a connected component of $\gamma \setminus B_R(0)$ which is diffeomorphic to an interval, and whose end-points belong to distinct connected components of $(\mathbb{R}^{n+1} \setminus B_R(0)) \setminus \partial E$. Using this fact, and since $\partial F \cap B_R(0) = \partial E \cap B_R(0)$, we just need to show that $\partial B_R(0) \cap \partial F$ contains $\partial B_R(0) \cap \partial E$ as well as $\partial B_R(0) \setminus \mathrm{cl}\,(A)$ in order to show that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W. This is done by repeating with minor variations the considerations contained in step two of the proof of Lemma 2.5. The proof of (2.64) is obtained in a similar way, and the details are omitted. $\square$

2.4. **Slab competitors.** Bi-Lipschitz deformations of cup competitors can be used to generate new competitors thanks to Lemma 2.3. We will crucially use this remark to replace balls with "slabs" (see Figures 3.2, 3.3 and 3.4) and obtain sharp area concentration estimates in step five of the proof of Theorem 1.4, as well as in the proof of Theorem 1.6, see e.g. (4.7). Given $\tau \in (0,1)$, $x \in \mathbb{R}^{n+1}$, $r > 0$, and $\nu \in \mathbb{S}^n$, we set

$$S^\nu_{\tau,r}(x) = \big\{ y \in B_r(x) : |(y-x)\cdot\nu| < \tau\, r \big\} ,$$

and we claim the existence of a bi-Lipschitz map $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with

$$\{\Phi \ne \mathrm{id}\,\} \subset\subset B_{2\,r}(x) , \qquad \Phi\big(B_{2\,r}(x)\big) = B_{2\,r}(x) , \qquad \Phi\big(\partial S^\nu_{\tau,t}(x)\big) = \partial B_t(x) \quad \forall t \in (0,r) ,$$

and such that $\mathrm{Lip}\,\Phi$ and $\mathrm{Lip}\,\Phi^{-1}$ depend only on n and τ. Indeed, assuming without loss of generality that $x = 0$, there is a convex, degree-one positively homogenous function $\varphi : \mathbb{R}^{n+1} \to [0,\infty)$ such that $S^\nu_{\tau,t}(0) = \{\varphi < t\}$ for every $t > 0$. Taking $\eta_r : [0,\infty) \to [0,\infty)$ smooth, decreasing and such that $\eta = 1$ on $[0, 4r/3]$ and $\eta = 0$ on $[5r/3, \infty)$, we set

$$\Phi(x) = \eta_r(|x|)\,\frac{\varphi(x)}{|x|}\,x + (1-\eta_r(|x|))\,x .$$

Noticing that Φ is a smooth interpolation between linear maps on each half-line $\{t\,x : t \ge 0\}$, and observing that the slopes of these linear maps change in a Lipschitz way with respect to the angular variable, one sees that Φ has the required properties.

Lemma 2.8 (Slab competitors)**.** *Let $E \in \mathcal{E}$ be such that $\Omega \cap \partial E$ is $\mathcal{C}$-spanning W, and let $B_{2r}(x) \subset\subset \Omega$, $\nu \in \mathbb{S}^n$, $\tau \in (0,1)$ with $\partial S^\nu_{\tau,r}(x) \cap \partial E$ $\mathcal{H}^{n-1}$-rectifiable. Let A be an open connected component of $\partial S^\nu_{\tau,r}(x) \setminus \partial E$. Then for every $\eta \in (0, r/2)$, there exists $F \in \mathcal{E}$ such that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W,*

$$F \setminus \mathrm{cl}\,(S^\nu_{\tau,r}(x)) = E \setminus \mathrm{cl}\,(S^\nu_{\tau,r}(x)) , \tag{2.66}$$

$$\lim_{\eta\to0^+} \mathcal{H}^n\,\big((\partial F \cap \partial S^\nu_{\tau,r}(x))\,\Delta\,(\partial S^\nu_{\tau,r}(x) \setminus A)\big) = 0 , \tag{2.67}$$

and such that if $A \cap E = \emptyset$, then

$$\limsup_{\eta\to0^+} \mathcal{H}^n(S^\nu_{\tau,r}(x) \cap \partial F) \le C(n,\tau)\,\mathcal{H}^n\big(\partial S^\nu_{\tau,r}(x) \setminus (A \cup E)\big) ; \tag{2.68}$$

while, if $A \subset E$, then

$$\limsup_{\eta\to0^+} \mathcal{H}^n(S^\nu_{\tau,r}(x) \cap \partial F) \le C(n,\tau)\,\mathcal{H}^n\big(E \cap \partial S^\nu_{\tau,r}(x) \setminus A\big) . \tag{2.69}$$

Proof. Let us set for brevity $S_r = S^\nu_{\tau,r}(x)$ and $B_r = B_r(x)$. By Lemma 2.3, $\Phi(E) \in \mathcal{E}$ and $\Omega \cap \partial\Phi(E)$ is $\mathcal{C}$-spanning W. Since Φ is an homeomorphism between ∂S_r and ∂B_r, $\Phi(A)$ is an open connected component of $\partial B_r \setminus \partial\Phi(E)$. Depending on whether $A \cap E = \emptyset$ or $A \subset E$, and thus, respectively, depending on whether $\Phi(A) \cap \Phi(E) = \emptyset$ or $\Phi(A) \cap \Phi(E) \ne \emptyset$, we consider the cup competitor G defined by $\Phi(E)$ and $\Phi(A)$, so that

$$G = \big(\Phi(E) \setminus \mathrm{cl}\,(B_r)\big) \cup N_\eta(Z) , \qquad Z = Y \cup \Big(U_\eta(S) \setminus \mathrm{cl}\,(\Phi(E) \cap \partial B_r)\Big) ,$$

where

$$Y = \partial B_r \setminus \big(\mathrm{cl}\,(\Phi(E) \cap \partial B_r) \cup \mathrm{cl}\,(\Phi(A))\big) , \qquad U_\eta(S) = \partial B_r \cap \{\mathrm{d}_S < \eta\} ,$$

with

$$S = \partial\Phi(E) \cap \mathrm{cl}\,(\Phi(A)) \setminus [\mathrm{cl}\,(\Phi(E) \cap \partial B_r) \cup \mathrm{cl}\,(Y)] ,$$

if $A \cap E = \emptyset$, see (2.10), and

$$G = \big(\Phi(E) \cup B_r\big) \setminus \mathrm{cl}\,\big(N_\eta(Y)\big) , \qquad Y = \big(\Phi(E) \cap \partial B_r\big) \setminus \mathrm{cl}\,(\Phi(A)) ,$$

if $A \subset E$, see (2.11). Finally, we set $F = \Phi^{-1}(G)$. Since $G \in \mathcal{E}$ and $\Omega \cap \partial G$ is $\mathcal{C}$-spanning W, by Lemma 2.3 we find that $F \in \mathcal{E}$ and that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W. By construction $G \setminus \mathrm{cl}\,(B_r) = \Phi(E) \setminus \mathrm{cl}\,(B_r)$, so that (2.66) follows by

$$F \setminus \mathrm{cl}\,(S_r) = \Phi^{-1}\big(G \setminus \mathrm{cl}\,(B_r)\big) = \Phi^{-1}\big(\Phi(E) \setminus \mathrm{cl}\,(B_r)\big) = E \setminus \mathrm{cl}\,(S_r) .$$

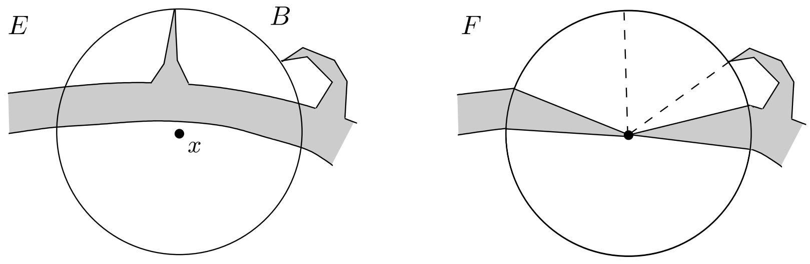

FIGURE 2.3. In this picture, the cone competitor F defined by $E \cap \partial B_r$ as in (2.70) may fail to be $\mathcal{C}$-spanning W. Notice that the dashed lines are part of the cone competitor K' defined by $K = \Omega \cap \partial E$ in $B_r(x)$, which is indeed strictly larger than $\Omega \cap \partial F$.

By (2.3), $\mathcal{H}^n\left((\partial B_r \cap \partial G)\,\Delta\,(\partial B_r \setminus \Phi(A))\right) \to 0$ as $\eta \to 0^+$, which gives (2.67) by the area formula. Finally, (2.68) and (2.69) are deduced by the area formula, (2.5) and (2.6). $\square$

2.5. **Cone competitors.** As customary in the analysis of area minimization problems, we want to compare $\mathcal{H}^n(B_r(x) \cap \partial E)$ with $\mathcal{H}^n(B_r(x) \cap \partial F)$, where F is the cone spanned by $E \cap \partial B_r(x)$ over x,

$$F = \big(E \setminus \mathrm{cl}\,(B_r(x))\big) \cup \big\{(1-t)\,x + t\,y : y \in E \cap \partial B_r(x)\,, t \in (0,1]\big\}\,. \tag{2.70}$$

Following the terminology of [DLGM17], given $K \in \mathcal{S}$, the cone competitor K' of K in $B_r(x)$ is similarly defined as

$$K' = (K \setminus B_r(x)) \cup \big\{(1-t)\,x + t\,y : y \in K \cap \partial B_r(x)\,, t \in [0,1]\big\}\,,$$

and is indeed $\mathcal{C}$-spanning W (since K was). However, for some values of r, $\partial F \cap B_r(x)$ may be strictly smaller than the cone competitor K' defined by the choice $K = \Omega \cap \partial E$ in $B_r(x)$, and thus it may fail to be $\mathcal{C}$-spanning; see Figure 2.3. By Sard's lemma, if E has smooth boundary in Ω this issue can be avoided as, for a.e. r, ∂E and ∂B_r intersect transversally, and thus $\partial E \cap \partial B_r(x)$ is the boundary of $E \cap \partial B_r(x)$ relative to $\partial B_r(x)$; but working with smooth boundary leads to other difficulties when constructing cup competitors. We thus approximate F (as defined in (2.70)) in energy by means of diffeomorphic images of E.

Lemma 2.9 (Cone competitors)**.** *Let $E \in \mathcal{E}$ be such that $\Omega \cap \partial E$ is $\mathcal{C}$-spanning W, and let $B = B_r(x) \subset\subset \Omega$ be such that $E \cap \partial B_r(x)$ is $\mathcal{H}^n$-rectifiable, $\partial E \cap \partial B_r(x)$ is $\mathcal{H}^{n-1}$-rectifiable and r is a Lebesgue point of the maps $t \mapsto \mathcal{H}^n(E \cap \partial B_t(x))$ and $t \mapsto \mathcal{H}^{n-1}(\partial E \cap \partial B_t(x))$. Then for each $\eta \in (0, r/2)$ there exists $F \in \mathcal{E}$ such that $F \Delta E \subset B_r(x)$, $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W, and*

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F) \le \mathcal{H}^n(\partial E \setminus B_r(x)) + \frac{r}{n}\,\mathcal{H}^{n-1}(\partial E \cap \partial B_r(x))\,, \tag{2.71}$$

$$\liminf_{\eta \to 0^+} |F| \ge |E \setminus B_r(x)| + \frac{r}{n+1}\,\mathcal{H}^n(E \cap \partial B_r(x))\,. \tag{2.72}$$

Proof. Let $x = 0$, $r = 1$, $B_r = B_r(0)$, and define a bi-Lipschitz map f_η by $f_\eta(0) = 0$ and $f_\eta(x) = u_\eta(|x|)\,\hat{x}$ if $x \ne 0$, where $\hat{x} = x/|x|$ and $u_\eta : \mathbb{R} \to [0,\infty)$ is given by

$$u_\eta(t) := \begin{cases} \max\{0, \eta\, t\}\,, & \text{for } t \le 1-\eta\,, \\ \eta(1-\eta) + \frac{t-(1-\eta)}{\eta}\,\left(1 - \eta(1-\eta)\right)\,, & \text{for } t \in [1-\eta, 1]\,, \\ t\,, & \text{for } t \ge 1\,, \end{cases} \tag{2.73}$$

so that $u_\eta(t) \le t$ for $t \ge 0$. Clearly, $\{f_\eta \ne \mathrm{id}\} \subset B_1$ and $f_\eta(B_1) \subset B_1$. The open set $F = f_\eta(E)$ is such that $\Omega \cap \partial F = f_\eta(\Omega \cap \partial E)$, so that $\Omega \cap \partial F$ is $\mathcal{H}^n$-rectifiable and, by

Lemma 2.3, $\mathcal{C}$-spanning W. Thanks to the area formula, (2.71) will follow by showing

$$\limsup_{\eta\to 0^+}\int_{B_1\cap\partial E} J^{\partial E} f_\eta\, d\mathcal{H}^n \le \frac{1}{n}\,\mathcal{H}^{n-1}(\partial E\cap\partial B_1)\,. \tag{2.74}$$

Trivially, the integral over $B_{1-\eta}\cap\partial E$ is bounded by $C(n)\,\eta^n\,\mathcal{H}^n(\Omega\cap\partial E)$. The integral over $B_1\setminus B_{1-\eta}$ is treated as in [DLGM17, Step two, Theorem 7]; by the coarea formula,

$$\begin{aligned}\int_{(B_1\setminus B_{1-\eta})\cap\partial E} J^{\partial E} f_\eta\, d\mathcal{H}^n = \int_{1-\eta}^1 dt \int_{\partial B_t\cap\partial E\cap\{|\nu_E\cdot\hat{x}|<1\}} \frac{J^{\partial E} f_\eta}{\sqrt{1-(\nu_E\cdot\hat{x})^2}}\, d\mathcal{H}^{n-1}\\ + \int_{(B_1\setminus B_{1-\eta})\cap\partial E\cap\{|\nu_E\cdot\hat{x}|=1\}} J^{\partial E} f_\eta\, d\mathcal{H}^n\,,\end{aligned} \tag{2.75}$$

where $\nu_E(x)\in T_x(\partial E)\cap\mathbb{S}^n$ at $\mathcal{H}^n$-a.e. $x\in\partial E$. By

$$\nabla f_\eta(x) = \frac{u_\eta(|x|)}{|x|}\,\mathrm{Id} + \left(u_\eta'(|x|) - \frac{u_\eta(|x|)}{|x|}\right)\hat{x}\otimes\hat{x}\,, \tag{2.76}$$

if $|\nu_E(x)\cdot\hat{x}|=1$, then $J^{\partial E} f_\eta = (u_\eta(|x|)/|x|)^n \le 1$. Since

$$\lim_{\eta\to 0^+}\mathcal{H}^n(\partial E\cap(B_1\setminus B_{1-\eta})) = 0\,, \tag{2.77}$$

the second term on the right-hand side of (2.75) converges to 0 as $\eta\to 0^+$. As for the first term, by (2.76), we have, as explained later on,

$$J^{\partial E} f_\eta(x) \le 1 + \sqrt{1-(\nu_E(x)\cdot\hat{x})^2}\,u_\eta'(|x|)\left(\frac{u_\eta(|x|)}{|x|}\right)^{n-1} \quad \text{for } \mathcal{H}^n\text{-a.e. } x\in\partial E\,. \tag{2.78}$$

The term corresponding to 1 in (2.78) converges to 0 as $\eta\to 0^+$ by (2.77). At the same time,

$$\limsup_{\eta\to 0^+}\left|\int_{1-\eta}^1 \Big(\mathcal{H}^{n-1}(\partial E\cap\partial B_t) - \mathcal{H}^{n-1}(\partial E\cap\partial B_1)\Big)\,u_\eta'\left(\frac{u_\eta}{t}\right)^{n-1} dt\right| = 0$$

since $t=1$ is a Lebesgue point of $t\mapsto\mathcal{H}^{n-1}(\partial B_t\cap\partial E)$, and since $u_\eta'(t)\le 1/\eta$ and $(u_\eta(t)/t)\le 1$ for $t\ge 0$. Finally,

$$\int_{1-\eta}^1 u_\eta'\left(\frac{u_\eta}{t}\right)^{n-1} dt \le \frac{1}{(1-\eta)^{n-1}}\,\frac{u_\eta(1)^n - u_\eta(1-\eta)^n}{n} = \frac{1}{(1-\eta)^{n-1}}\,\frac{1-\eta^n(1-\eta)^n}{n} \to \frac{1}{n}$$

as $\eta\to 0^+$, thus completing the proof of (2.71). The proof of (2.72) follows an analogous argument. The goal is to show that

$$\liminf_{\eta\to 0^+}\int_{E\cap B_1} Jf_\eta\, dx \ge \frac{1}{n+1}\,\mathcal{H}^n(E\cap\partial B_1)\,, \tag{2.79}$$

and by the coarea formula and (2.76) it is immediate to see that

$$\int_{E\cap B_1} Jf_\eta\, dx \ge \int_{1-\eta}^1 u_\eta'(t)\left(\frac{u_\eta(t)}{t}\right)^n \mathcal{H}^n(E\cap\partial B_t)\, dt\,.$$

The estimate in (2.79) then readily follows using that $t=1$ is a Lebesgue point for the map $t\mapsto\mathcal{H}^n(E\cap\partial B_t)$, together with

$$\int_{1-\eta}^1 u_\eta'(t)\left(\frac{u_\eta(t)}{t}\right)^n dt \ge \frac{1-\eta^{n+1}(1-\eta)^{n+1}}{n+1} \to \frac{1}{n+1} \qquad \text{as } \eta\to 0^+\,.$$

We finally explain how to deduce (2.78) from (2.76). For $x\in\partial^* E$, let $\{\tau_i\}_{i=1}^n$ be an orthonormal basis of $T_x\partial^* E$ such that $\{\tau_i\}_{i=1}^{n-1}\subset x^\perp$. In this way, we can take

$$\tau_n = \frac{\hat{x} - (\hat{x}\cdot\nu_E(x))\nu_E(x)}{\sqrt{1-(\hat{x}\cdot\nu_E(x))^2}}\,,$$

and therefore compute by (2.76) that

$$\begin{aligned}\nabla^{\partial E} f_\eta(x)[\tau_i] &= \frac{u_\eta(|x|)}{|x|}\,\tau_i\,, \qquad \forall i=1,...,n-1\,,\\ \nabla^{\partial E} f_\eta(x)[\tau_n] &= u'_\eta(|x|)\,\sqrt{1-(\hat{x}\cdot\nu_E)^2}\,\hat{x}-\frac{u_\eta(|x|)}{|x|}\,(\hat{x}\cdot\nu_E)\,\frac{\nu_E-(\hat{x}\cdot\nu_E)\hat{x}}{\sqrt{1-(\hat{x}\cdot\nu_E)^2}}\,,\end{aligned}$$

where we have set for brevity ν_E in place of $\nu_E(x)$. Therefore

$$\begin{aligned}J^{\partial E} f(x)^2 &= \Big|\bigwedge_{i=1}^n \nabla^{\partial E} f_\eta(x)[\tau_i]\Big|^2\\ &= \Big(\frac{u_\eta(|x|)}{|x|}\Big)^{2n}\,(\hat{x}\cdot\nu_E)^2\,\Big|\tau_1\wedge\cdots\wedge\tau_{n-1}\wedge\Big(\frac{\nu_E-(\hat{x}\cdot\nu_E)\hat{x}}{\sqrt{1-(\hat{x}\cdot\nu_E)^2}}\Big)\Big|^2\\ &\quad+\Big(\frac{u_\eta(|x|)}{|x|}\Big)^{2(n-1)}\,u'_\eta(|x|)^2\,\big(1-(\hat{x}\cdot\nu_E)^2\big)\,\Big|\tau_1\wedge\cdots\wedge\tau_{n-1}\wedge\hat{x}\Big|^2\\ &\le 1+\Big(\frac{u_\eta(|x|)}{|x|}\Big)^{2(n-1)}\,u'_\eta(|x|)^2\,\big(1-(\hat{x}\cdot\nu_E)^2\big)\,,\end{aligned}$$

from which (2.78) follows thanks to $\sqrt{1+a}\le 1+\sqrt{a}$ for $a\ge 0$. □

2.6. **Nucleation lemma.** The following nucleation lemma can be found, with slightly different statements, in [Alm76, VI(13)] or in [Mag12, Lemma 29.10].

Lemma 2.10. *Let $\xi(n)$ be the constant of Besicovitch's covering theorem in $\mathbb{R}^{n+1}$. If T is closed, $A=\mathbb{R}^{n+1}\setminus T$, $0<|E|<\infty$, $P(E;A)<\infty$, $\tau>0$, and*

$$\sigma=\min\Big\{\frac{|E\setminus I_\tau(T)|}{\tau\,P(E;A)},\frac{\xi(n)}{n+1}\Big\}>0$$

then there exists $x\in E^{(1)}\setminus I_\tau(T)$ such that

$$|E\cap B_\tau(x)|\ge\Big(\frac{\sigma}{2\xi(n)}\Big)^{n+1}\,\tau^{n+1}\,.$$

Proof. By contradiction one assumes that

$$|E\cap B_\tau(x)|<\Big(\frac{\sigma}{2\xi(n)}\Big)^{n+1}\,\tau^{n+1}\qquad\forall x\in E^{(1)}\setminus I_\tau(T)\,. \tag{2.80}$$

Setting $\alpha=\xi(n)/\sigma$, so that $\alpha\ge n+1$, we claim that (2.80) implies the existence, for each $x\in E^{(1)}\setminus I_\tau(T)$, of $\tau_x\in(0,\tau)$ such that

$$P(E;B_{\tau_x}(x))>\frac{\alpha}{\tau}\,|E\cap B_{\tau_x}(x)|\,. \tag{2.81}$$

In turn (2.81) is in contradiction with (2.80): indeed, by applying Besicovitch's theorem to $\{\mathrm{cl}\,(B_{\tau_x}(x)):x\in E^{(1)}\setminus I_\tau(T)\}$ we find an at most countable subset I of $E^{(1)}\setminus I_\tau(T)$ such that $\{\mathrm{cl}\,(B_{\tau_x}(x))\}_{x\in I}$ is disjoint and

$$\begin{aligned}|E\setminus I_\tau(T)| &\le \xi(n)\sum_{x\in I}|E\cap B_{\tau_x}(x)|<\frac{\xi(n)\,\tau}{\alpha}\sum_{x\in I}P(E;B_{\tau_x}(x))\\ &\le \frac{\xi(n)\,\tau\,P(E;A)}{\alpha}=\tau\,\sigma\,P(E;A)\le|E\setminus I_\tau(T)|\,,\end{aligned}$$

a contradiction. We show that (2.80) implies (2.81): indeed, if (2.80) holds but (2.81) fails, then there exists $x\in E^{(1)}\setminus I_\tau(T)$ such that, setting $m(r)=|E\cap B_r(x)|$ for $r>0$,

$$m>0 \text{ on } (0,\infty)\,,\qquad m(\tau)<\Big(\frac{\tau}{2\alpha}\Big)^{n+1} \tag{2.82}$$

and $(\alpha/\tau)\, m(r) \ge P(E; B_r(x))$ for every $r \in (0,\tau)$. Adding up $\mathcal{H}^n(\partial B_r(x) \cap E)$, which equals $m'(r)$ for a.e. $r > 0$ by the coarea formula, we obtain

$$m'(r) + \frac{\alpha}{\tau}\, m(r) \ge P(E \cap B_r(x)) \ge m(r)^{n/(n+1)}\,, \qquad \text{for a.e. } r \in (0,\tau)\,. \tag{2.83}$$

where in the last inequality we have used that $P(F) \ge |F|^{n/(n+1)}$ whenever $0 < |F| < \infty$; see e.g. [Mag12, Proposition 12.35]. Since $m > 0$ on $(0,\infty)$ we find

$$\begin{cases} \dfrac{\alpha}{\tau} m(r) \le (1/2) m(r)^{n/(n+1)} \\ \forall r \in (0,\tau) \end{cases} \quad \text{iff} \quad \begin{cases} m(r) \le (\tau/2\alpha)^{n+1} \\ \forall r \in (0,\tau) \end{cases} \quad \text{if} \quad m(\tau) \le \Big(\frac{\tau}{2\alpha}\Big)^{n+1},$$

where the last condition holds by (2.82). Thus (2.83) gives $m'(r) \ge (1/2) m(r)^{n/(n+1)}$ for a.e. $r \in (0,\tau)$, thus $m(\tau) \ge (\tau/2(n+1))^{n+1} \ge (\tau/2\alpha)^{n+1}$ as $\alpha \ge n+1$, a contradiction. □

2.7. **Isoperimetry, lower bounds and collapsing.** Given an L^1-converging sequence of sets of finite perimeter $\{E_j\}_j$, the boundary of the L^1-limit set E will be (in general) strictly included in $K = \operatorname{spt}\mu$, where μ is the weak-star limit of the Radon measures defined by the boundaries of the E_j's. In the next lemma we show that, under some mild bounds on μ and E_j, if μ is absolutely continuous with respect to $\mathcal{H}^n \llcorner K$ then the Radon-Nikodým density θ of μ is everywhere larger than 1, and is actually larger than 2 at a.e. point of $K \setminus \partial^* E$ (that is, a cancellation can happen only when boundaries are collapsing).

Lemma 2.11 (Collapsing lemma). *Let K be a relatively compact and $\mathcal{H}^n$-rectifiable set in Ω, let $E \subset \Omega$ be a set of finite perimeter with $\Omega \cap \partial^* E \subset K$, and let $\{E_j\}_j \subset \mathcal{E}$ such that $E_j \to E$ in $L^1_{\rm loc}(\Omega)$, and $\mu_j \overset{*}{\rightharpoonup} \mu$ as Radon measures in Ω, where $\mu_j = \mathcal{H}^n \llcorner (\Omega \cap \partial E_j)$ and $\mu = \theta\, \mathcal{H}^n \llcorner K$ for a Borel function θ. If $\Omega' \subset \Omega$ and $r_* > 0$ are such that for every $x \in K \cap \Omega'$ and a.e. $r < r_*$ with $B_r(x) \subset\subset \Omega'$ we have*

$$\mu(B_r(x)) \;\ge\; c(n)\, r^n\,, \tag{2.84}$$

$$\liminf_{j\to\infty} \mathcal{H}^n(B_r(x) \cap \partial E_j) \;\le\; C(n) \liminf_{j\to\infty} \mathcal{H}^n(\partial B_r(x) \setminus A^0_{r,j})\,, \tag{2.85}$$

where $A^0_{r,j}$ denotes an $\mathcal{H}^n$-maximal connected component of $\partial B_r(x) \setminus \partial E_j$, then $\theta(x) \ge 1$ for $\mathcal{H}^n$-a.e. $x \in K \cap \Omega'$, and $\theta(x) \ge 2$ for $\mathcal{H}^n$-a.e. $x \in (K \setminus \partial^* E) \cap \Omega'$.

The bound $\theta \ge 1$ follows by arguing exactly as in [DLGM17, Proof of Theorem 2, Step three], and has nothing to do with the fact that the measures μ_j are defined by boundaries; the latter information is in turn crucial in obtaining the bound $\theta \ge 2$, and requires a new argument. For the sake of clarity, we also give the details of the $\theta \ge 1$ bound, which in turn is based on spherical isoperimetry.

Lemma 2.12 (Spherical isoperimetry). *Let $\Sigma \subset \mathbb{R}^{n+1}$ denote a spherical cap[5] in the n-dimensional unit sphere $\mathbb{S}^n$, possibly with $\Sigma = \mathbb{S}^n$. If K is a compact set in $\mathbb{R}^{n+1}$ and $\{A^h\}_{h=0}^\infty$ is the family of the open connected components of $\Sigma \setminus K$, ordered so to have $\mathcal{H}^n(A^h) \ge \mathcal{H}^n(A^{h+1})$, then*

$$\mathcal{H}^n(\Sigma \setminus A^0) \le C(n)\, \mathcal{H}^{n-1}(\Sigma \cap K)^{n/(n-1)}\,. \tag{2.86}$$

Moreover, if $\Sigma = \mathbb{S}^n$, $\sigma_n = \mathcal{H}^n(\mathbb{S}^n)$ and $\mathcal{H}^{n-1}(\mathbb{S}^n \cap K) < \infty$, then each A^h is a set of finite perimeter in $\mathbb{S}^n$ and for every $\tau > 0$ there exists $\sigma > 0$ such that

$$\min\Big\{\mathcal{H}^n(A^0), \mathcal{H}^n(A^1)\Big\} = \mathcal{H}^n(A^1) \ge \frac{\sigma_n}{2} - \sigma \tag{2.87}$$

implies

$$\min\Big\{\mathcal{H}^{n-1}(\partial^* A^0), \mathcal{H}^{n-1}(\partial^* A^1)\Big\} \ge \sigma_{n-1} - \tau\,. \tag{2.88}$$

Here $\partial^* A^h$ denotes the reduced boundary of A^h in $\mathbb{S}^n$.

[5] That is, $\Sigma = \mathbb{S}^n \cap H$ where H is an open half-space of $\mathbb{R}^{n+1}$.

Proof. This is [DLGM17, Lemma 9]. However, (2.88) is stated in a weaker form in [DLGM17, Lemma 9], so we give the details. Arguing by contradiction, we can find $\tau > 0$ and $\{K_j\}_j$ such that, for $\alpha = 0, 1$, $\mathcal{H}^{n-1}(\partial^* A_j^\alpha) \le \sigma_{n-1} - \tau$ for every j, but $\mathcal{H}^n(A_j^\alpha) \to \sigma_n/2$ as $j \to \infty$. Since $\sigma_n = \mathcal{H}^n(\mathbb{S}^n)$ and $A_j^0 \cap A_j^1 = \emptyset$, we find that, for $\alpha = 0, 1$, $A_j^\alpha \to A^\alpha$ in $L^1(\mathbb{S}^n)$ where $A^0 \cap A^1 = \emptyset$ and $A^0 \cup A^1$ is $\mathcal{H}^n$-equivalent to $\mathbb{S}^n$. Therefore $\mathcal{H}^{n-1}(\partial^* A^0) = \mathcal{H}^{n-1}(\partial^* A^1) \le \sigma_{n-1} - \tau$, where we have used lower semicontinuity of perimeter. Since $\inf \mathcal{H}^{n-1}(\partial^* A)$ with $\mathcal{H}^n(A) = \sigma_n/2$ is equal to σ_{n-1} we have reached a contradiction. □

Proof of Lemma 2.11. Step one: We fix $x \in K \cap \Omega'$ such that $\mathcal{H}^n \llcorner (K - x)/r \overset{*}{\rightharpoonup} \mathcal{H}^n \llcorner T_x K$ as $r \to 0^+$. Setting $\nu(x)^\perp = T_x K$ for $\nu(x) \in \mathbb{S}^n$, by the lower density estimate (2.84) we easily find that for every $\sigma > 0$ there exists $r_0 = r_0(\sigma, x) \in (0, \min\{r_*, \mathrm{dist}(x, \partial\Omega')\})$ such that $|(y - x) \cdot \nu(x)| < \sigma\, r$ for every $y \in K \cap B_r(x)$ and every $r < r_0$. In particular,

$$\lim_{j\to\infty} \mathcal{H}^n(\partial E_j \cap \{y \in B_r(x) : \ |(y - x) \cdot \nu(x)| > \sigma\, r\}) = 0 \qquad \text{for every } r \le r_0\,,$$

and thus by the coarea formula (see [DLGM17, Equation (2.13)])

$$\lim_{j\to\infty} \mathcal{H}^{n-1}(\Sigma^\pm_{r,\sigma} \cap \partial E_j) = 0 \qquad \text{for a.e. } r \le r_0\,, \tag{2.89}$$

where we have set

$$\begin{aligned} \Sigma^+_{r,\sigma} &= \big\{y \in \partial B_r(x) : (y - x) \cdot \nu(x) > \sigma\, r\big\}\,, \\ \Sigma^-_{r,\sigma} &= \big\{y \in \partial B_r(x) : (y - x) \cdot \nu(x) < -\sigma\, r\big\}\,. \end{aligned}$$

Let $A^+_{r,j}$ be an $\mathcal{H}^n$-maximal connected component of $\Sigma^+_{r,\sigma} \setminus \partial E_j$, and define similarly $A^-_{r,j}$. Equations (2.89) and (2.86) imply that, for a.e. $r < r_0$,

$$\lim_{j\to\infty} \mathcal{H}^n(A^\pm_{r,j}) = \mathcal{H}^n(\Sigma^\pm_{r,\sigma})\,. \tag{2.90}$$

Now let $\{A^h_{r,j}\}_{h=0}^\infty$ denote the open connected components of $\partial B_r(x) \setminus \partial E_j$, ordered by decreasing $\mathcal{H}^n$-measure. We claim that

$$\text{if (2.90) holds, then either } A^+_{r,j} \text{ or } A^-_{r,j} \text{ is not contained in } A^0_{r,j}\,. \tag{2.91}$$

Indeed, if for some r we have $A^+_{r,j} \cup A^-_{r,j} \subset A^0_{r,j}$, then by (2.85) and (2.90) we find

$$\mu(B_r(x)) \le \liminf_{j\to\infty} \mu_j(B_r(x)) \le C(n) \liminf_{j\to\infty} \mathcal{H}^n(\partial B_r \setminus A^0_{r,j}) \le C(n)\, r^n\, \sigma\,, \tag{2.92}$$

a contradiction to (2.84) if $\sigma \le \sigma_0(n)$ for a suitable $\sigma_0(n)$. By (2.91) and (2.90),

$$\min\Big\{\mathcal{H}^n(A^0_{r,j}), \mathcal{H}^n(A^1_{r,j})\Big\} \ge \Big(\frac{\sigma_n}{2} - C(n)\, \sigma\Big)\, r^n \qquad \text{for a.e. } r < r_0\,. \tag{2.93}$$

By Lemma 2.12 and (2.93), given $\tau > 0$, if σ is small enough in terms of n and τ, then

$$\min\Big\{\mathcal{H}^{n-1}(\partial^* A^0_{r,j}), \mathcal{H}^{n-1}(\partial^* A^1_{r,j})\Big\} \ge \big(\sigma_{n-1} - \tau\big)\, r^{n-1} \qquad \text{for a.e. } r < r_0\,, \tag{2.94}$$

where $\partial^* A^\alpha_{r,j}$ is the reduced boundary of $A^\alpha_{r,j}$ as a subset of $\partial B_r(x)$. Since $A^0_{r,j}$ is a connected component of $\partial B_r(x) \setminus \partial E_j$ we have

$$\big(\sigma_{n-1} - \tau\big)\, r^{n-1} \le \mathcal{H}^{n-1}(\partial^* A^0_{r,j}) \le \mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_j)\,. \tag{2.95}$$

Now if $f_j(r) = \mu_j(B_r(x))$ and $f(r) = \mu(B_r(x))$ then by the coarea formula we easily find that $f_j \to f$ a.e. with $\liminf_{j\to\infty} f_j'(r) \le f'(r) \le Df$, where Df denotes the distributional derivative of f. Hence, letting $j \to \infty$ and $\tau \to 0^+$ in (2.95) we obtain $Df \ge \sigma_{n-1}\, r^{n-1}\, dr$ on $(0, r_0)$. As $\omega_n = n\, \sigma_{n-1}$, we conclude that $\theta(x) \ge 1$. We stress once more that so far we have just followed the argument of [DLGM17, Proof of Theorem 2, Step three].

Step two: We use the boundary structure to show that $\theta \ge 2$ $\mathcal{H}^n$-a.e. on $\Omega' \cap (K \setminus \partial^* E)$. Since $\{E^{(0)}, E^{(1)}, \partial^* E\}$ is an $\mathcal{H}^n$-a.e. partition of $\mathbb{R}^{n+1}$, we can assume that $x \in (E^{(0)} \cup E^{(1)}) \cap K \cap \Omega'$. We consider first the case $x \in E^{(0)}$. Given $\sigma > 0$, up to decreasing r_0,

$$\sigma\, r_0^{n+1} \ge \lim_{j\to\infty} |E_j \cap B_{r_0}(x)| = \lim_{j\to\infty} \int_0^{r_0} \mathcal{H}^n(E_j \cap \partial B_r(x))\, dr\,. \tag{2.96}$$

Let us consider the measurable set $I_j \subset (0, r_0)$

$$I_j = \big\{ r \in (0,r_0) : A^0_{r,j} \cup A^1_{r,j} \subset \partial B_r(x) \setminus \mathrm{cl}\,(E_j) \big\}\,.$$

We claim that

$$\mathcal{H}^{n-1}(\partial^* A^0_{r,j} \cap \partial^* A^1_{r,j}) = 0 \qquad \forall r \in I_j\,. \tag{2.97}$$

Indeed, if $r \in I_j$, then $A^0_{r,j}$, $A^1_{r,j}$, and $\partial B_r(x) \cap E_j$ are disjoint sets of finite perimeter in $\partial B_r(x)$, and in particular

$$\begin{aligned}
\nu_{A^0_{r,j}} &= -\nu_{A^1_{r,j}}\,, && \mathcal{H}^{n-1}\text{-a.e. on } \partial^* A^0_{r,j} \cap \partial^* A^1_{r,j}\,,\\
\nu_{A^0_{r,j}} &= -\nu_{\partial B_r(x)\cap E_j} && \mathcal{H}^{n-1}\text{-a.e. on } \partial^* A^0_{r,j} \cap \partial^*[\partial B_r(x) \cap E_j]\,,\\
\nu_{A^1_{r,j}} &= -\nu_{\partial B_r(x)\cap E_j} && \mathcal{H}^{n-1}\text{-a.e. on } \partial^* A^1_{r,j} \cap \partial^*[\partial B_r(x) \cap E_j]\,.
\end{aligned}$$

At the same time, since $\{A^h_{r,j}\}_{h=0}^\infty$ are connected components of $\partial B_r(x) \setminus \partial E_j$,

$$\partial^* A^h_{r,j} \subset \partial^*[\partial B_r(x) \cap E_j] \qquad \text{modulo } \mathcal{H}^n$$

and thus $\mathcal{H}^{n-1}$-a.e. on $\partial^* A^0_{r,j} \cap \partial^* A^1_{r,j}$ we have

$$\nu_{\partial B_r(x)\cap E_j} = -\nu_{A^0_{r,j}} = \nu_{A^1_{r,j}} = -\nu_{\partial B_r(x)\cap E_j}$$

a contradiction. By (2.94) and (2.97), given $\tau > 0$ and provided σ is small enough in terms of n and τ, for a.e. $r \in I_j$ we find

$$\begin{aligned}
f_j'(r) &\ge \mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_j) \ge \mathcal{H}^{n-1}(\partial^* A^0_{r,j} \cup \partial^* A^1_{r,j})\\
&= \mathcal{H}^{n-1}(\partial^* A^0_{r,j}) + \mathcal{H}^{n-1}(\partial^* A^1_{r,j}) \ge 2\,(\sigma_{n-1} - \tau)\, r^{n-1}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
f_j(r_0) &\ge 2\,(\sigma_{n-1}-\tau)\,\frac{r_0^n}{n} - C(n) \int_{(0,r_0)\setminus I_j} r^{n-1}\, dr\\
&\ge 2\,(\sigma_{n-1}-\tau)\,\frac{r_0^n}{n} - C(n)\, r_0^{1/n} \Big(\int_{(0,r_0)\setminus I_j} r^n\, dr \Big)^{(n-1)/n}\,.
\end{aligned} \tag{2.98}$$

We notice that for a.e. $r \in (0,r_0) \setminus I_j$, (2.93) gives

$$\mathcal{H}^n(E_j \cap \partial B_r(x)) \ge \min\Big\{ \mathcal{H}^n(A^0_{r,j}), \mathcal{H}^n(A^1_{r,j}) \Big\} \ge \Big(\frac{\sigma_n}{2} - C(n)\,\sigma \Big)\, r^n\,,$$

so that (2.96) implies

$$\sigma\, r_0^{n+1} \ge c(n)\, \limsup_{j\to\infty} \int_{(0,r_0)\setminus I_j} r^n\, dr\,. \tag{2.99}$$

If we combine (2.98) and (2.99) and let $j \to \infty$, then we find

$$f(r_0) = \lim_{j\to\infty} f_j(r_0) \ge 2\,(\sigma_{n-1}-\tau)\,\frac{r_0^n}{n} - C(n)\, r_0^{1/n} \Big(\sigma\, r_0^{n+1} \Big)^{(n-1)/n}$$

Dividing by r_0^n and letting $r_0 \to 0^+$, $\sigma \to 0^+$ and $\tau \to 0^+$ we find $\theta(x) \ge 2$ whenever $x \in E^{(0)} \cap K \cap \Omega'$. The case when $x \in E^{(1)}$ is analogous and the details are omitted. □

3. Existence of generalized minimizers: Proof of Theorem 1.4

Given the length of the proof, we provide a short overview. In step one, we check that $\psi(\varepsilon) < \infty$ by using the open neighborhoods of a minimizer S of ℓ as comparison sets for $\psi(\varepsilon)$. We remark that this is the only point of the proof where (1.12) is used. It is important here to allow for sufficiently non-smooth sets in the competition class $\mathcal{E}$: indeed, minimizers of ℓ are known to be smooth only outside of a close $\mathcal{H}^n$-negligible set in arbitrary dimension. Once $\psi(\varepsilon) < \infty$ is established, we consider a minimizing sequence $\{E_j\}_j$ for $\psi(\varepsilon)$, so that $E_j \in \mathcal{E}$, $|E_j| = \varepsilon$, $\Omega \cap \partial E_j$ is $\mathcal{C}$-spanning W and

$$\mathcal{H}^n(\Omega \cap \partial E_j) \le \mathcal{H}^n(\Omega \cap \partial F) + \frac{1}{j} \qquad \forall F \in \mathcal{E}\,,\ |F| = \varepsilon\,,\ \Omega \cap \partial F \text{ is } \mathcal{C}\text{-spanning } W\,. \tag{3.1}$$

We want to apply (3.1) to the comparison sets constructed in section 2, but, in general, those local variations do not preserve the volume constraint. A family of volume-fixing variations acting uniformly on $\{E_j\}_j$ is constructed through the nucleation lemma (Lemma 2.10) following some ideas introduced by Almgren in the existence theory of minimizing clusters [Alm76]; see steps two and three. In step four we exploit cup and cone competitors to show that, up to extracting subsequences, $\mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \stackrel{*}{\rightharpoonup} \mu = \theta\, \mathcal{H}^n \llcorner K$ as Radon measures in Ω, and $E_j \to E$ in $L^1_{\rm loc}(\Omega)$, for a pair $(K, E) \in \mathcal{K}$ and for an upper semicontinuous function $\theta \ge 1$ on K. An application of Lemma 2.11 shows that $\theta \ge 2$ $\mathcal{H}^n$-a.e. on $K \setminus \partial^* E$, thus proving $\psi(\varepsilon) \ge \mathcal{F}(K, E)$. In order to show that $\psi(\varepsilon) = \mathcal{F}(K, E)$, and thus that (K, E) is a generalized minimizer of $\psi(\varepsilon)$, we need to exclude that $\Omega \cap \partial E_j$ concentrates area by folding against K, at infinity, or against the wire frame. By using slab competitors we prove that $\Omega \cap \partial E_j$, in its convergence towards K, cannot fold at all near points in $\partial^* E$, and can fold at most twice near points in $K \cap (E^{(0)} \cup E^{(1)})$ (step five). In step six, concentration of area at the boundary is ruled out by a deformation argument based on Lemma 2.4. Finally, in step seven, we exclude area (and volume) concentration at infinity by using exterior cup competitors to construct a uniformly bounded minimizing sequence.

Proof of Theorem 1.4. Step one: We show that

$$\psi(\varepsilon) \le 2\,\ell + C(n)\, \varepsilon^{n/(n+1)} \qquad \forall \varepsilon > 0\,. \tag{3.2}$$

Let S be a minimizer of ℓ, and let $\eta_0 > 0$ be such that (1.12) holds. If $\eta \in (0, \eta_0)$, then the open η-neighborhood $U_\eta(S)$ of S is such that $\Omega \cap \partial U_\eta(S)$ is $\mathcal{C}$-spanning W: otherwise we could find $\eta \in (0, \eta_0)$ and $\gamma \in \mathcal{C}$ such that $\gamma \cap \partial U_\eta(S) = \emptyset$. Since γ is connected, we would either have $\gamma \subset \{x : \mathrm{dist}(x, S) > \eta\}$, against the fact that S is $\mathcal{C}$-spanning; or we would have $\gamma \subset U_\eta(S)$, against (1.12). Hence $\Omega \cap \partial U_\eta(S)$ is $\mathcal{C}$-spanning W.

As proved in [DLGM17], S is $\mathcal{H}^n$-rectifiable. Moreover, as shown in Theorem B.1 in the appendix, we have

$$\mathcal{H}^n(S \cap B_r(x)) \ge c(n)\, r^n \qquad \forall x \in \mathrm{cl}\,(S)\,, r < \rho_0 \tag{3.3}$$

where ρ_0 depends on W, so that $\mathcal{H}^n(S) < \infty$ implies that $\mathrm{cl}\,(S)$ is compact. This density estimate has two more consequences: first, combined with [Mag12, Corollary 6.5], it implies $\mathcal{H}^n(\mathrm{cl}\,(S) \setminus S) = 0$; second, it allows us to exploit [AFP00, Theorem 2.104] to find

$$|U_\eta(S)| = 2\,\eta\, \mathcal{H}^n(\mathrm{cl}\,(S)) + \mathrm{o}(\eta) = 2\,\eta\, \mathcal{H}^n(S) + \mathrm{o}(\eta) \qquad \text{as } \eta \to 0^+\,. \tag{3.4}$$

By the coarea formula for Lipschitz maps applied to the distance function from S, see [Mag12, Theorem 18.1, Remark 18.2], we have

$$|U_\eta(S) \cap A| = \int_0^\eta P(U_t(S); A)\, dt = \int_0^\eta \mathcal{H}^n(A \cap \partial U_t(S))\, dt\,, \qquad \forall A \subset \mathbb{R}^{n+1} \text{ open}\,,$$

so that $U_\eta(S)$ is a set of finite perimeter in $\mathbb{R}^{n+1}$ and $\mathcal{H}^n(\partial U_\eta(S)\setminus\partial^* U_\eta(S))=0$ for a.e. $\eta>0$. Summarizing, we have proved that, for a.e. $\eta\in(0,\eta_0)$,

$$F_\eta=\Omega\cap U_\eta(S)\in\mathcal{E}\,,\qquad \Omega\cap\mathrm{cl}\,(\partial^*F_\eta)=\Omega\cap\partial F_\eta \text{ is } \mathcal{C}\text{-spanning } W\,,$$

and, by (3.4),

$$f(\eta)=|F_\eta|=\int_0^\eta P(F_t;\Omega)\,dt=\int_0^\eta P(U_t(S);\Omega)\,dt\le 2\,\eta\,\mathcal{H}^n(S)+\mathrm{o}(\eta)\,.$$

Notice that $f(s)$ is absolutely continuous with $f(\eta)=\int_0^\eta f'(t)\,dt$ and $f'(t)=P(F_t;\Omega)$ for a.e. $t\in(0,\eta)$. Hence, for every $\eta>0$ there exist $t_1(\eta),t_2(\eta)\in(0,\eta)$ such that $f'(t_1(\eta))\le f(\eta)/\eta\le f'(t_2(\eta))$. Setting $F_j=F_{t_1(\eta_j)}$ for a suitable $\eta_j\to0^+$, we get

$$\limsup_{j\to\infty}P(F_j;\Omega)\le 2\,\ell\,,$$

where $|F_j|\to0^+$. Finally, given $\varepsilon>0$, we pick j such that $|F_j|<\varepsilon$, and construct a competitor for $\psi(\varepsilon)$ by adding to F_j a disjoint ball of volume $\varepsilon-|F_j|$. In this way, $\psi(\varepsilon)\le P(F_j;\Omega)+C(n)\big(\varepsilon-|F_j|\big)^{n/(n+1)}$, and (3.2) is found by letting $j\to\infty$.

Since $\psi(\varepsilon)<\infty$, we can now consider a minimizing sequence $\{E_j\}_{j=1}^\infty$ for $\psi(\varepsilon)$. Given that $P(E_j)\le\mathcal{H}^n(\partial\Omega)+\mathcal{H}^n(\Omega\cap\partial E_j)\le\mathcal{H}^n(\partial\Omega)+\psi(\varepsilon)+1$ for j large, and that $|E_j|=\varepsilon$ for every j, there exist a set of finite perimeter $E\subset\Omega$ and a Radon measure μ in Ω such that, up to extracting subsequences,

$$E_j\to E\quad\text{in } L^1_{\mathrm{loc}}(\Omega)\,,\qquad \mu_j=\mathcal{H}^n\llcorner(\Omega\cap\partial E_j)\stackrel{*}{\rightharpoonup}\mu\quad\text{as Radon measures on }\Omega\,,\tag{3.5}$$

as $j\to\infty$, see e.g. [Mag12, Section 12.4]. We consider the set, relatively closed in Ω, defined by

$$K=\Omega\cap\mathrm{spt}\mu=\big\{x\in\Omega:\mu(B_r(x))>0\quad\forall r>0\big\}\,,$$

and claim that

$$K \text{ is } \mathcal{C}\text{-spanning } W\,,\qquad \Omega\cap\partial^*E\subset K\,.\tag{3.6}$$

Indeed, the first claim in (3.6) is obtained by applying Lemma 2.1 to $K_j=\Omega\cap\partial E_j$; and if $x\in\Omega\cap\partial^*E$ and $B_r(x)\subset\Omega$, then

$$0<P(E;B_r(x))\le\liminf_{j\to\infty}P(E_j;B_r(x))\le\liminf_{j\to\infty}\mu_j(B_r(x))\le\mu(\mathrm{cl}\,(B_r(x)))$$

so that $x\in K$. Notice that, at this stage, we still do not know if $(K,E)\in\mathcal{K}$: we still need to show that K is $\mathcal{H}^n$-rectifiable and, possibly up to Lebesgue negligible modifications, that E is open with $\Omega\cap\mathrm{cl}\,(\partial^*E)=\Omega\cap\partial E$. Moreover, we just have $|E|\le\varepsilon$ (possible volume loss at infinity), and we know nothing about the structure of μ.

Step two: We show the existence of $\tau>0$ such that for every E_j there exist $x_j^1,x_j^2\in\mathbb{R}^{n+1}$ such that $\{\mathrm{cl}\,(B_{2\tau}(x_j^1)),\mathrm{cl}\,(B_{2\tau}(x_j^2)),W\}$ is disjoint and

$$|E_j\cap B_\tau(x_j^1)|=\kappa_1\,,\qquad |E_j\cap B_\tau(x_j^2)|=\kappa_2\,,\tag{3.7}$$

for some $\kappa_1,\kappa_2\in(0,|B_\tau|/2]$ depending on n, τ, ε and ℓ only. With τ_0 as in (1.11), for $M\in\mathbb{N}\setminus\{0\}$ to be chosen later on, and by compactness of W, we can pick $\tau>0$ so that

$$(M+1)\,\tau<\tau_0\,,\qquad |B_{M\,\tau}|<\frac{\varepsilon}{4}\,,\qquad |I_{(M+1)\tau}(W)\setminus W|<\frac{\varepsilon}{2}\,.\tag{3.8}$$

The value σ in Lemma 2.10 corresponding to E_j and $T=I_{M\,\tau}(W)$ is given by

$$\min\Big\{\frac{|E_j\setminus I_\tau(T)|}{\tau\,P(E_j;\mathbb{R}^{n+1}\setminus T)},\frac{\xi(n)}{n+1}\Big\}\ge\min\Big\{\frac{\varepsilon/2}{\tau\,(\psi(\varepsilon)+1)},\frac{\xi(n)}{n+1}\Big\}>0\,,$$

since $|E_j \setminus I_\tau(T)| \ge \varepsilon/2$ by (3.8), and since $P(E_j;\Omega) \le \psi(\varepsilon)+1$. Therefore, setting

$$\sigma_1 = \min\Big\{\frac{\varepsilon/2}{\tau\,(\psi(\varepsilon)+1)}, \frac{\xi(n)}{n+1}\Big\}\,,$$

an application of Lemma 2.10 yields $y_j \in \mathbb{R}^{n+1} \setminus I_{(M+1)\tau}(W)$ such that

$$|E_j \cap B_\tau(y_j)| \ge \min\Big\{\Big(\frac{\sigma_1}{2\xi(n)}\Big)^{n+1}\tau^{n+1}\,, \frac{|B_\tau|}{2}\Big\} = \kappa_1\,,$$

so that $\kappa_1 \in (0, |B_\tau|/2]$ depends on n, ℓ, ε, and τ only (observe that this is a consequence of (3.2)). The continuous map $x \mapsto |E_j \cap B_\tau(x)|$ takes a value larger than κ_1 at $y_j \in \mathbb{R}^{n+1} \setminus I_{(M+1)\,\tau}(W)$; at the same time, by (1.11), $\mathbb{R}^{n+1} \setminus I_{(M+1)\,\tau}(W)$ is open and connected, therefore it is pathwise connected [Dug66, Corollary 5.6], and $|E_j \cap B_\tau(x)| \to 0$ as $|x| \to \infty$ in $\mathbb{R}^{n+1} \setminus I_{(M+1)\,\tau}(W)$. Therefore we can find $x_j^1 \in \mathbb{R}^{n+1} \setminus I_{(M+1)\,\tau}(W)$ such that the first identity in (3.7) holds and $\{\mathrm{cl}\,(B_{(M+1)\,\tau}(x_j^1)), W\}$ is disjoint. Setting $B = \mathrm{cl}\,(B_{(M-2)\tau}(x_j^1))$, the value σ in Lemma 2.10 corresponding to E_j and $T = I_\tau(W) \cup B$ is given by

$$\min\Big\{\frac{|E_j \setminus I_\tau(T)|}{\tau\,P(E_j;\mathbb{R}^{n+1}\setminus T)}, \frac{\xi(n)}{n+1}\Big\} \ge \min\Big\{\frac{\varepsilon/4}{\tau\,(\psi(\varepsilon)+1)}, \frac{\xi(n)}{n+1}\Big\} > 0\,,$$

so that, after setting

$$\sigma_2 = \min\Big\{\frac{\varepsilon/4}{\tau\,(\psi(\varepsilon)+1)}, \frac{\xi(n)}{n+1}\Big\}\,,$$

we can find $z_j \in \mathbb{R}^{n+1} \setminus (I_{2\tau}(W) \cup \mathrm{cl}\,(B_{(M-1)\,\tau}(x_j^1)))$ such that

$$|E_j \cap B_\tau(z_j)| \ge \min\Big\{\Big(\frac{\sigma_2}{2\xi(n)}\Big)^{n+1}\tau^{n+1}\,, \frac{|B_\tau|}{2}\Big\} = \kappa_2\,,$$

with $\kappa_2 \in (0, |B_\tau|/2]$ depending on n, ℓ, ε, and τ only. Since $I_{2\tau}(W)$ and $\mathrm{cl}\,(B_{(M-1)\tau}(x_j^1))$ are disjoint and since $\mathbb{R}^{n+1} \setminus I_{2\tau}(W)$ is pathwise connected by (1.11), we easily check that $\mathbb{R}^{n+1} \setminus (I_{2\tau}(W) \cup \mathrm{cl}\,(B_{(M-1)\tau}(x_j^1)))$ is pathwise connected. By continuity,

$$\exists\, x_j^2 \in \mathbb{R}^{n+1} \setminus (I_{2\tau}(W) \cup \mathrm{cl}\,(B_{(M-1)\,\tau}(x_j^1))) \tag{3.9}$$

such that the second identity in (3.7) holds. Finally, (3.9) implies that the family of sets $\{\mathrm{cl}\,(B_{(M-3)\tau}(x_j^1)), \mathrm{cl}\,(B_{2\tau}(x_j^2)), W\}$ is disjoint. We pick $M = 5$ to conclude the proof.

Step three: In this step we show that (3.1) can be modified to allow for comparison with local variations F_j of E_j that do not necessarily preserve the volume constraint. More precisely, we prove the existence of positive constants r_* and C_* (depending on the whole sequence $\{E_j\}_j$, and thus uniform in j) such that if $x \in \Omega$, $r < r_*$ and $\{F_j\}_j$ is an **admissible local variation of** $\{E_j\}_j$ **in** $B_r(x)$, in the sense that

$$F_j \in \mathcal{E}\,, \qquad F_j \Delta E_j \subset\subset B_r(x)\,, \qquad \Omega \cap \partial F_j \text{ is } \mathcal{C}\text{-spanning } W\,, \tag{3.10}$$

(notice that we do not require $B_r(x) \subset \Omega$), then

$$\mathcal{H}^n(\Omega \cap \partial E_j) \le \mathcal{H}^n(\Omega \cap \partial F_j) + C_*\Big||E_j| - |F_j|\Big| + \frac{1}{j}\,. \tag{3.11}$$

We first claim that if $B_j \subset \Omega$ is a ball with $\mathrm{dist}(B_j, B_r(x)) > 0$, $\zeta : \Omega \to \Omega$ is a diffeomorphism with $\zeta(B_j) \subset B_j$ and $\{\zeta \ne \mathrm{id}\} \subset\subset B_j$, and if

$$G_j = \Big(F_j \cap B_r(x)\Big) \cup \Big(\zeta(E_j) \cap B_j\Big) \cup \Big(E_j \setminus (B_j \cup B_r(x))\Big) \tag{3.12}$$

then $G_j \in \mathcal{E}$ and $\Omega \cap \partial G_j$ is $\mathcal{C}$-spanning W. The fact that G_j is open is obvious since G_j is equal to E_j in a neighborhood of $\Omega \setminus (B_r(x) \cup B_j)$, to F_j in a neighborhood of $B_r(x)$, and to $\zeta(E_j)$ in a neighborhood of B_j, where E_j, F_j and $\zeta(E_j)$ are open, and where $\mathrm{dist}(B_j, B_r(x)) > 0$; this also shows that ∂G_j is equal to ∂E_j in a neighborhood of $\Omega \setminus (B_r(x) \cup B_j)$, to ∂F_j in a neighborhood of $B_r(x)$, and to $\partial \zeta(E_j) = \zeta(\partial E_j)$ in a neighborhood of B_j, so that $\Omega \cap \partial G_j$ is $\mathcal{H}^n$-rectifiable and, thanks to (3.10) and Lemma 2.3, that $\Omega \cap \partial G_j$ is $\mathcal{C}$-spanning W. Having proved the claim, we only have to construct sets G_j as in (3.12) and such that

$$|G_j| = \varepsilon\,, \qquad \mathcal{H}^n(\Omega \cap \partial G_j) \le \mathcal{H}^n(\Omega \cap \partial F_j) + C_* \big||E_j| - |F_j|\big|\,, \tag{3.13}$$

in order to deduce (3.11) from (3.1). To this aim, let $\{x_j^k\}_{k=1,2}$ be as in step two: the sets $\{(E_j - x_j^k) \cap B_\tau(0)\}_j$ are bounded in $B_\tau(0)$, and have uniformly bounded perimeters, so that, up to extracting a subsequence, for each $k = 1, 2$ there exists a set of finite perimeter $E_*^k \subset B_\tau(0)$ such that $(E_j - x_j^k) \cap B_\tau(0) \to E_*^k$ in $L^1(\mathbb{R}^{n+1})$. The crucial point is that, by (3.7) and since $\kappa_k \in (0, |B_\tau(0)|/2]$, we must have

$$B_\tau(0) \cap \partial^* E_*^k \ne \emptyset\,.$$

Hence, by arguing as in [Mag12, Section 29.6], we can find positive constants C_*' and ε_* such that for every set of finite perimeter $E' \subset B_\tau(0)$ with

$$|E' \Delta E_*^k| < \varepsilon_*\,,$$

there exists a C^1-map $\Phi_k : (-\varepsilon_*, \varepsilon_*) \times B_\tau(0) \to B_\tau(0)$ such that, for each $|v| < \varepsilon_*$: (i) $\Phi_k(v, \cdot)$ is a diffeomorphism with $\{\Phi_k(v, \cdot) \ne \mathrm{Id}\} \subset\subset B_\tau(0)$; (ii) $|\Phi_k(v, E')| = |E'| + v$; (iii) if Σ is an $\mathcal{H}^n$-rectifiable set in $B_\tau(0)$, then

$$\Big|\mathcal{H}^n(\Phi_k(v, \Sigma)) - \mathcal{H}^n(\Sigma)\Big| \le C_*'\, \mathcal{H}^n(\Sigma)\, |v|\,.$$

By taking $E' = (E_j - x_j^k) \cap B_\tau(0)$ (for j large enough), by composing the maps Φ_k with a translation by x_j^k, and then by extending the resulting maps as the identity map outside of $B_\tau(x_j^k)$, we prove the existence of C^1-maps $\Psi_k : (-\varepsilon_*, \varepsilon_*) \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that, for each $|v| < \varepsilon_*$: (i) $\Psi_k(v, \cdot)$ is a diffeomorphism with $\{\Psi_k(v, \cdot) \ne \mathrm{Id}\} \subset\subset B_\tau(x_j^k)$; (ii) $|\Psi_k(v, E_j)| = |E_j| + v$; (iii) if Σ is an $\mathcal{H}^n$-rectifiable set in $\mathbb{R}^{n+1}$, then

$$\Big|\mathcal{H}^n(\Psi_k(v, \Sigma)) - \mathcal{H}^n(\Sigma)\Big| \le C_*'\, \mathcal{H}^n(\Sigma)\, |v|\,.$$

Finally, we set

$$r_* = \min\Big\{\tau, \Big(\frac{\varepsilon_*}{2\,\omega_{n+1}}\Big)^{1/(n+1)}\Big\}\,, \qquad B_j = B_\tau(x_j^{k(j)})$$

where $k = k(j) \in \{1, 2\}$ is selected so that $\mathrm{dist}(B_r(x), B_j) > 0$ (this is possible because $r_x* \le \tau$ and $\{\mathrm{cl}\,(B_{2\tau}(x_j^1)), \mathrm{cl}\,(B_{2\tau}(x_j^2))\}$ are disjoint). We finally define G_j by (3.12) with

$$\zeta = \Psi_{k(j)}(v_j, \cdot)\,, \qquad v_j = |E_j \cap B_r(x)| - |F_j \cap B_r(x)|\,,$$

as we are allowed to do since $E_j \Delta F_j \subset\subset B_r(x)$ and thus $|v_j| \le \omega_{n+1}\, r_*^{n+1} \le \varepsilon_*/2$. To prove (3.13): first, we have $G_j \Delta F_j \subset\subset \Omega \setminus \mathrm{cl}\,(B_r(x))$, while property (ii) of $\Psi_{k(j)}$ gives

$$\begin{aligned}|G_j| - |E_j| &= |\Psi_{k(j)}(v_j, E_j) \cap B_j| + |F_j \cap B_r(x)| - |E_j \cap B_j| - |E_j \cap B_r(x)| \\ &= |\Psi_{k(j)}(v_j, E_j) \cap B_j| - v_j - |E_j \cap B_j| = 0\,;\end{aligned}$$

second, property (iii) applied to the $\mathcal{H}^n$-rectifiable set $\Sigma = B_j \cap \partial E_j$ gives

$$\begin{aligned}&\mathcal{H}^n(\Omega \cap \partial G_j) - \mathcal{H}^n(\Omega \cap \partial F_j) \\ =\;& \mathcal{H}^n\Big(\Psi_{k(j)}(v_j, B_j \cap \partial E_j)\Big) - \mathcal{H}^n(B_j \cap \partial E_j) \le C_*'\, |v_j|\, \mathcal{H}^n(B_j \cap \partial E_j)\end{aligned}$$

so that (3.13) follows by taking $C_* = C_*'\,(\psi(\varepsilon) + 1)$.

Step four: In this step we apply (3.11) to the cup and cone competitors constructed in section 2 and show that $K = \Omega \cap \mathrm{spt}\mu$ is relatively compact in Ω and $\mathcal{H}^n$-rectifiable, that

$\mu = \theta\,\mathcal{H}^n \llcorner K$ with $\theta \geq 1$ on K and $\theta \geq 2$ $\mathcal{H}^n$-a.e. on $K \setminus \partial^* E$, and, finally, that $(K, E) \in \mathcal{K}$. To this end, pick $x \in K$, set $d(x) = \mathrm{dist}(x, W) > 0$, and let

$$f_j(r) = \mu_j(B_r(x)) = \mathcal{H}^n(B_r(x) \cap \partial E_j)\,, \qquad f(r) = \mu(B_r(x))\,, \qquad \text{for every } r \in (0, d(x))\,.$$

Denoting by Df the distributional derivative of f, and by f' its classical derivative, the coarea formula (see [DLGM17, Step one, proof of Theorem 2] and [Fed69, Theorem 2.9.19]) gives

$$f_j \to f \text{ a.e. on } (0, d(x))\,, \quad Df_j \geq f_j'\,dr\,, \quad Df \geq f'\,dr\,, \quad f' \geq g = \liminf_{j\to\infty} f_j'\,, \tag{3.14}$$

$$f_j'(r) \geq \mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_j) \qquad \forall j \text{ and for a.e. } r \in (0, d(x))\,. \tag{3.15}$$

Now let $\eta \in (0, r/2)$, let A_j denote an $\mathcal{H}^n$-maximal open connected component of $\partial B_r(x) \setminus \partial E_j$, and let F_j be the cup competitor defined by E_j and A_j as in Lemma 2.5. More precisely, when $E_j \cap A_j = \emptyset$, we let $\{\eta_k^j\}_{k=1}^\infty$ be the decreasing sequence with $\lim_{k\to\infty} \eta_k^j = 0$ defined in step two of the proof of Lemma 2.5, and setting, for η_k^j such that $\eta \in \left(\eta_{k+1}^j, \eta_k^j\right]$,

$$\begin{aligned} Y_j &= \partial B_r(x) \setminus \big(\mathrm{cl}\,(E_j \cap \partial B_r(x)) \cup \mathrm{cl}\,(A_j)\big)\,, \\ S_j &= \partial E_j \cap \mathrm{cl}\,(A_j) \setminus (\mathrm{cl}\,(E_j \cap \partial B_r(x)) \cup \mathrm{cl}\,(Y_j))\,, \\ U_j &= \partial B_r(x) \cap \{\mathrm{d}_{S_j} < \eta_k^j\}\,, \end{aligned}$$

we define

$$F_j = \big(E_j \setminus \mathrm{cl}\,(B_r(x))\big) \cup N_{\eta_k^j}(Z_j)\,, \qquad Z_j = Y_j \cup (U_j \setminus \mathrm{cl}\,(E_j \cap \partial B_r(x)))\,. \tag{3.16}$$

When $A_j \subset E_j$, instead, we define

$$F_j = \big(E_j \cup B_r(x)\big) \setminus \mathrm{cl}\,\big(N_\eta(Y_j)\big)\,, \qquad Y_j = (E_j \cap \partial B_r(x)) \setminus \mathrm{cl}\,(A_j)\,; \tag{3.17}$$

see Figure 2.1. In both cases, $\{F_j\}_j$ is an admissible local variation of $\{E_j\}_j$ in $B_{r'}(x)$ for some $r' > r$, and by (2.4), for a.e. $r < d(x)$ we have

$$\limsup_{\eta\to 0^+} \mathcal{H}^n(\Omega \cap \partial F_j) \leq \mathcal{H}^n(\partial E_j \setminus B_r(x)) + 2\,\mathcal{H}^n(\partial B_r(x) \setminus A_j)$$

so that, by (3.11), for a.e. $r < \min\{d(x), r_*\}$, we have

$$f_j(r) \leq 2\,\mathcal{H}^n(\partial B_r(x) \setminus A_j) + C_* \limsup_{\eta\to 0^+} \big||E_j| - |F_j|\big| + \frac{1}{j}\,. \tag{3.18}$$

The estimate of $||E_j| - |F_j||$ is different depending on whether F_j is given by (3.16) or by (3.17). In both cases we make use of the Euclidean isoperimetric inequality

$$(n+1)\,|B_1|^{1/(n+1)}\,|U|^{n/(n+1)} \leq P(U)\,, \qquad \forall U \subset \mathbb{R}^{n+1}\,,$$

and we also need the perimeter identities

$$\begin{aligned} P(E_j \cap B_r(x)) &= P(E_j; B_r(x)) + \mathcal{H}^n(E_j \cap \partial B_r(x))\,, \\ P(B_r(x) \setminus E_j) &= P(E_j; B_r(x)) + \mathcal{H}^n(\partial B_r(x) \setminus E_j)\,, \end{aligned} \tag{3.19}$$

which hold for a.e. $r > 0$, with the exceptional set of r-values that can be made independent from j. We now take F_j as in (3.16): up to further decreasing the value of r_* so to entail

$C_* r_*/(n+1) \le 1/2$, and assuming that $r < r_*$, we have

$$
\begin{aligned}
C_* \Big| |E_j| - |F_j| \Big| &\le C_* |E_j \cap B_r(x)| + C_* c(n) r^n \eta_k^j \\
&\le C_* |B_1|^{1/(n+1)} r |E_j \cap B_r(x)|^{n/(n+1)} + C_* c(n) r^n \eta_k^j \\
&\le \frac{C_*}{n+1} r_* P(E_j \cap B_r(x)) + C_* c(n) r^n \eta_k^j \\
&\le \frac{1}{2}\Big\{P(E_j; B_r(x)) + \mathcal{H}^n(E_j \cap \partial B_r(x))\Big\} + C_* c(n) r^n \eta_k^j \\
&\le \frac{1}{2}\Big\{f_j(r) + \mathcal{H}^n(\partial B_r(x) \setminus A_j)\Big\} + C_* c(n) r^n \eta_k^j , \qquad (3.20)
\end{aligned}
$$

where in the last inequality we have used $\partial^* E_j \subset \partial E$ and $A_j \cap E_j = \emptyset$ (that is the assumption under which F_j is chosen as in (3.16)). If instead we take F_j as in (3.17), then

$$
\begin{aligned}
C_* \Big| |E_j| - |F_j| \Big| &= C_* \Big| |E_j \cap B_r(x)| - |F_j \cap B_r(x)| \Big| = C_* \Big| |B_r(x) \setminus E_j| - |B_r(x) \setminus F_j| \Big| \\
&\le C_* |B_1|^{1/(n+1)} r |B_r(x) \setminus E_j|^{n/(n+1)} + C_* |N_\eta(\partial B_r(x) \cap E_j \setminus \mathrm{cl}\,(A_j))| \\
&\le \frac{1}{2}\Big\{P(E_j; B_r(x)) + \mathcal{H}^n(\partial B_r(x) \setminus E_j)\Big\} + C_* c(n) r^n \eta \\
&\le \frac{1}{2}\Big\{f_j(r) + \mathcal{H}^n(\partial B_r(x) \setminus A_j)\Big\} + C_* c(n) r^n \eta , \qquad (3.21)
\end{aligned}
$$

where in the last inequality we have used $\partial^* E_j \subset \partial E_j$ and $A_j \subset E_j$ (the assumption corresponding to (3.17)). By combining (3.18) with (3.20) and (3.21), we conclude that

$$
\frac{f_j(r)}{2} \le 3\,\mathcal{H}^n(\partial B_r(x) \setminus A_j) + \frac{1}{j}, \qquad \text{for a.e. } r < \min\{r_*, d(x)\}. \qquad (3.22)
$$

By the spherical isoperimetric inequality, Lemma 2.12, and by (3.15), for a.e. $r < d(x)$,

$$
\mathcal{H}^n(\partial B_r(x) \setminus A_j) \le C(n)\,\mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_j)^{n/(n-1)} \le C(n)\, f_j'(r)^{n/(n-1)},
$$

which combined with (3.22) and (3.14), allows us to conclude (letting $j \to \infty$), that

$$
f(r) \le C(n)\, f'(r)^{n/(n-1)}, \qquad \text{for a.e. } r < \min\{r_*, d(x)\}. \qquad (3.23)
$$

Since $x \in \mathrm{spt}\mu$, f is positive, and thus (3.23) implies the existence of $\theta_0(n) > 0$ such that

$$
\mu(B_r(x)) \ge \theta_0\, \omega_n\, r^n \qquad \forall x \in K, r < r_*, B_r(x) \subset\subset \Omega. \qquad (3.24)
$$

Since $K = \Omega \cap \mathrm{spt}\mu$, by [Mat95, Theorem 6.9] and (3.24) we obtain

$$
\mu \ge \theta_0\, \mathcal{H}^n \llcorner K \qquad \text{on } \Omega. \qquad (3.25)
$$

As a consequence of $\mu(\Omega) < \infty$ and of (3.24) we deduce that K is bounded, thus relatively compact in Ω. In turn, $\partial^* E \subset K$ implies the boundedness of E. Notice that we have not excluded $|E| < \varepsilon$ yet.

To further progress in the analysis of μ, given $\eta \in (0, r/2)$ let use now denote by F_j the set corresponding to η constructed in Lemma 2.9, so that, by (2.71), for a.e. $r < d(x)$,

$$
\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F_j) \le \mathcal{H}^n(\partial E_j \setminus B_r(x)) + \frac{r}{n}\,\mathcal{H}^{n-1}(\partial E_j \cap \partial B_r(x)). \qquad (3.26)
$$

Using that $\{F_j\}_j$ is an admissible local variation of $\{E_j\}_j$ in $B_r(x)$, and combining (3.11) and (3.26) with $||E_j| - |F_j|| \le C(n)\, r^{n+1}$, we find that

$$
\mathcal{H}^n(B_r(x) \cap \partial E_j) \le \frac{r}{n} f_j'(r) + C_* r^{n+1} + \frac{1}{j},
$$

so that, as $j\to\infty$, $f(r)\le (r/n)\,f'(r)+C_*\,r^{n+1}$. By combining this last inequality with $Df\ge f'(r)\,dr$ and (3.24) we find that

$$
\begin{aligned}
D\,(e^{\Lambda\,r}f(r)/r^n) &= \frac{n\,e^{\Lambda\,r}}{r^{n+1}}\Big\{\frac{r}{n}\,Df+\Big(\frac{r\,\Lambda}{n}\,f(r)-\,f(r)\Big)\,dr\Big\}\\
&\ge \frac{n\,e^{\Lambda\,r}}{r^{n+1}}\Big\{f(r)-C_*\,r^{n+1}+\frac{r\,\Lambda}{n}\,f(r)-\,f(r)\Big\}\,dr\\
&= \frac{n\,e^{\Lambda\,r}}{r^{n}}\Big\{-C_*\,r^n+\frac{\Lambda\,f(r)}{n}\Big\}\,dr\ge n\;e^{\Lambda\,r}\Big\{-C_*+\frac{\Lambda\,\theta_0\,\omega_n}{n}\Big\}\,dr
\end{aligned}
$$

so that, setting $\Lambda\ge n\,C_*/(\theta_0\omega_n)$, we have proved

$$
e^{\Lambda\,r}\,\frac{\mu(B_r(x))}{r^n}\qquad\text{is non-decreasing on } r<\min\{r_*,d(x)\}\,. \tag{3.27}
$$

By (3.27) and (3.25) we find that

$$
\theta(x)=\lim_{r\to0^+}\frac{\mu(B_r(x))}{\omega_n\,r^n}\quad\text{exists in } (0,\infty) \text{ for every } x\in K\,.
$$

By Preiss' theorem, $\mu=\theta\,\mathcal{H}^n\llcorner K^*$ for a Borel function θ and a countably $\mathcal{H}^n$-rectifiable set $K^*\subset\Omega$. Since $K=\Omega\cap\mathrm{spt}\mu$, we have $\mathcal{H}^n(K^*\setminus K)=0$, while (3.25) gives $\mathcal{H}^n(K\setminus K^*)=0$. Thus K is countably $\mathcal{H}^n$-rectifiable and $\mu=\theta\,\mathcal{H}^n\llcorner K$. Moreover, θ is upper semicontinuous on K thanks to (3.27). Finally, consider the open set

$$
E^*=\big\{x\in\Omega:\exists r>0\text{ s.t. } |B_r(x)|=|E\cap B_r(x)|\big\}\,.
$$

The topological boundary of E^* is equal to

$$
\partial E^*=\big\{x\in\mathrm{cl}\,(\Omega):0<|E\cap B_r(x)|<|B_r(x)|\quad\forall r>0\big\}\,,
$$

so that $\Omega\cap\mathrm{cl}\,(\partial^*E)=\Omega\cap\partial E^*$ by [Mag12, Proposition 12.19]. Clearly $E^*\subset E^{(1)}$: moreover, if $x\in E^{(1)}\setminus E^*$, then $0<|E\cap B_r(x)|<|B_r(x)|$ for every $r>0$, and thus $x\in\partial E^*$. In particular,

$$
\Omega\cap(E^{(1)}\setminus E^*)\subset\Omega\cap\partial E^*=\Omega\cap\mathrm{cl}\,(\partial^*E)\subset K\,,
$$

where K is $\mathcal{H}^n$-rectifiable, and thus Lebesgue negligible. Since $\mathcal{H}^n(\partial\Omega)<\infty$, we have proved $\mathcal{H}^n(E^{(1)}\setminus E^*)<\infty$, and thus $|E^{(1)}\Delta E^*|=0$. By the Lebesgue's points theorem, E^* is equivalent to E, so that $\partial^*E=\partial^*E^*$. Replacing E with E^* we find $(K,E)\in\mathcal{K}$. Finally, the lower bounds $\theta\ge1$ $\mathcal{H}^n$-a.e. on K and $\theta\ge2$ $\mathcal{H}^n$-a.e. on $K\setminus\partial^*E$ follow by applying Lemma 2.11 with $\Omega'=\Omega$: notice indeed that assumptions (2.84) and (2.85) in Lemma 2.11 hold by (3.24) and by (3.22).

Step five: We show that $\theta(x)\le1$ at every $x\in\Omega\cap\partial^*E$ and that $\theta(x)\le2$ at every $x\in K\cap(E^{(0)}\cup E^{(1)})$ such that K admits an approximate tangent plane at x (thus, that $\theta\le2$ $\mathcal{H}^n$-a.e. on $K\setminus\partial^*E$). We choose $\nu(x)\in\mathbb{S}^n$ such that $T_xK=\nu(x)^\perp$ (notice that, necessarily, $\nu(x)=\nu_E(x)$ or $\nu(x)=-\nu_E(x)$ when, in addition, $x\in\partial^*E$), and let $B_{2\,r}(x)\subset\subset\Omega$. For $\tau\in(0,1)$ and $\sigma\in(0,\tau)$ we set

$$
\begin{aligned}
S_{\tau,r} &= \big\{y\in B_r(x):|(y-x)\cdot\nu(x)|<\tau\,r\big\}\,, \\
V_{\sigma,r} &= \big\{y\in B_r(x):|(y-x)\cdot\nu(x)|<\sigma\,|y-x|\big\}\;\subset\;S_{\sigma,r}\;\subset\;S_{\tau,r}\,,\\
W^{\pm}_{\tau,\sigma,r} &= \big(S_{\tau,r}\setminus\mathrm{cl}\,(V_{\sigma,r})\big)\cap\{y:(y-x)\cdot\nu_E(x)\gtrless0\}\,,\\
\Gamma^{\pm}_{\tau,\sigma,r} &= \partial S_{\tau,r}\cap\partial W^{\pm}_{\tau,\sigma,r}\,,
\end{aligned}
\tag{3.28}
$$

that are depicted in Figure 3.1. By (3.24) and since $\mathcal{H}^n\llcorner(K-x)/\rho\overset{*}{\rightharpoonup}\mathcal{H}^n\llcorner T_xK$ as $\rho\to0^+$, the approximate tangent plane T_xK is a classical tangent plane, and thus there exists $r_0=r_0(\sigma,x)>0$ such that $K\cap B_r(x)\subset S_{\sigma,r}$ for every $r<r_0$, or, equivalently,

$$
K\cap B_{r_0}(x)\subset V_{\sigma,r_0}\cup\{x\}\,. \tag{3.29}
$$

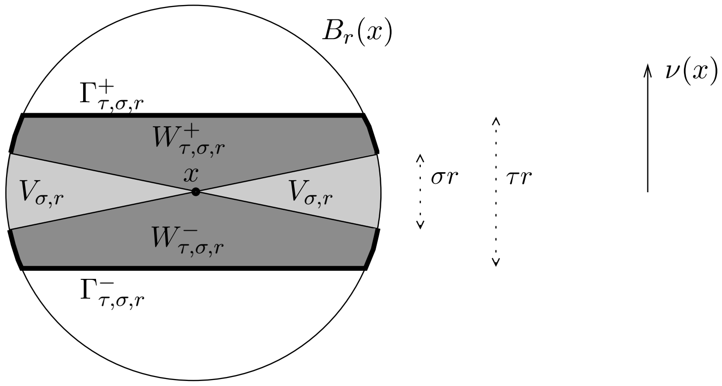

FIGURE 3.1. The sets defined in (3.28). Here $\sigma < \tau < 1$, and $S_{\tau,r}$ is decomposed into a central open cone $V_{\sigma,r}$ of small amplitude σ, the upper and lower open regions $W^{\pm}_{\tau,\sigma,r}$, and the closed cone $S_{\tau,r} \cap \partial V_{\sigma,r}$. For $r \le r_0(\sigma, x)$, $B_r(x) \cap K$ lies inside $V_{\sigma,r}$ by approximate differentiability of K at x and by the density estimate (3.24). When $x \in \partial^* E$, if we choose $\nu(x) = \nu_E(x)$, then the divergence theorem implies that E fills up the whole $W^-_{\tau,\sigma,r}$, and leaves empty $W^+_{\tau,\sigma,r}$.

In particular

$$\mu(S_{\tau,r}) = \mu(B_r(x)) \,, \qquad \forall r < r_0 \,. \tag{3.30}$$

We also notice that for a.e. value of r we have

$$\partial S_{\tau,r} \cap \partial E_j \text{ is } \mathcal{H}^{n-1}\text{-rectifiable} \qquad \forall j \,. \tag{3.31}$$

We now introduce the family of open sets

$$\begin{aligned}
\mathcal{A}^{\rm out}_{r,j} &= \Big\{ A \subset \partial S_{\tau,r} : A \text{ is an open connected component} \\
&\qquad\qquad \text{of } \partial S_{\tau,r} \setminus \partial E_j \text{ and } A \text{ is } \textbf{disjoint} \text{ from } E_j \Big\} \,, \\
\mathcal{A}^{\rm in}_{r,j} &= \Big\{ A \subset \partial S_{\tau,r} : A \text{ is an open connected component} \\
&\qquad\qquad \text{of } \partial S_{\tau,r} \setminus \partial E_j \text{ and } A \text{ is } \textbf{contained} \text{ in } E_j \Big\} \,,
\end{aligned}$$

and denote by $A^{\rm out}_{r,j}$ and $A^{\rm in}_{r,j}$ $\mathcal{H}^n$-maximal elements of $\mathcal{A}^{\rm out}_{r,j}$ and $\mathcal{A}^{\rm in}_{r,j}$ respectively. Finally, given $\eta \in (0, r/2)$, we let $F_j^\star$ be the slab competitor defined by E_j, $A^\star_{r,j}$ and τ in $B_{2r}(x)$ for $\star \in \{\text{out}, \text{in}\}$ as in Lemma 2.8: accordingly, $F_j^\star \in \mathcal{E}$, $\Omega \cap \partial F_j^\star$ is $\mathcal{C}$-spanning W,

$$F_j^\star \setminus {\rm cl}\,(S_{\tau,r}) = E_j \setminus {\rm cl}\,(S_{\tau,r}) \,, \tag{3.32}$$

$$\lim_{\eta \to 0^+} \mathcal{H}^n \left((\partial S_{\tau,r} \cap \partial F_j^\star) \,\Delta\, (\partial S_{\tau,r} \setminus A^\star_{r,j}) \right) = 0 \,, \tag{3.33}$$

and

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(S_{\tau,r} \cap \partial F_j^\star) \le C(n,\tau) \begin{cases} \mathcal{H}^n\big(\partial S_{\tau,r} \setminus (A^{\rm out}_{r,j} \cup E_j)\big) \,, & \text{if } \star = \text{out} \,, \\ \mathcal{H}^n\big((E_j \cap \partial S_{\tau,r}) \setminus A^{\rm in}_{r,j}\big) \,, & \text{if } \star = \text{in} \,; \end{cases} \tag{3.34}$$

see (2.66), (2.67), (2.68) and (2.69). By (3.11), $\mathcal{H}^n(\partial S_{\tau,r} \cap \partial E_j) = 0$ and (3.32),

$$\mathcal{H}^n(S_{\tau,r} \cap \partial E_j) \le \mathcal{H}^n({\rm cl}\,(S_{\tau,r}) \cap \partial F_j^\star) + C_*\, c(n)\, r^{n+1} + \frac{1}{j} \,, \qquad \forall \star \in \{\text{out}, \text{in}\} \,.$$

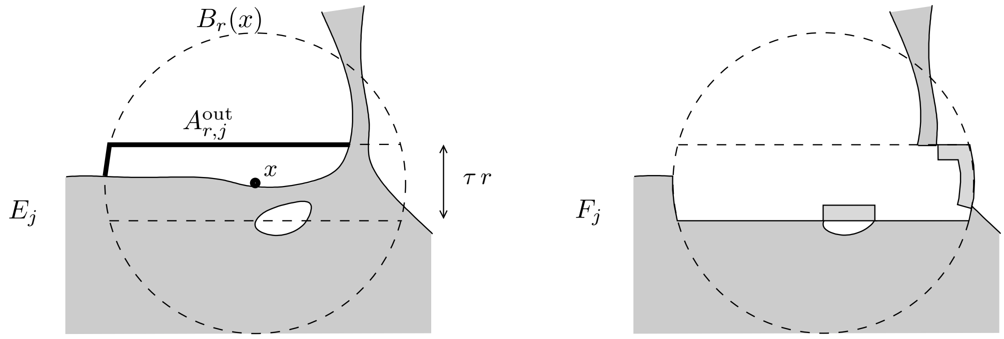

FIGURE 3.2. The slab competitor $F_j^{\rm out}$ is used in proving that $\theta(x) \le 1$. The fact that $x \in \partial^* E$ is used to show that $E_j \cap \partial S_{\tau,r}$ consists of a large connected component whose area is close to $\omega_n\, r^n$ up to a $\mathrm{o}(r^n)$ error as $r \to 0^+$.

By (3.33) and (3.34), taking the limit first as $\eta \to 0^+$ and then as $j \to \infty$, and by taking also into account that $\mu_j \overset{*}{\rightharpoonup} \mu$ and that (3.30) holds, we find, in the case $\star = \mathrm{out}$, that

$$\begin{aligned}\mu(B_r(x)) \;\le\; & \limsup_{j\to\infty} \mathcal{H}^n(E_j \cap \partial S_{\tau,r}) \\ & + C(n,\tau) \limsup_{j\to\infty} \mathcal{H}^n\big(\partial S_{\tau,r} \setminus (A^{\rm out}_{r,j} \cup E_j)\big) + C_*\, c(n)\, r^{n+1}\,,\end{aligned} \tag{3.35}$$

and, in the case $\star = \mathrm{in}$, that

$$\begin{aligned}\mu(B_r(x)) \;\le\; & \limsup_{j\to\infty} \mathcal{H}^n(\partial S_{\tau,r} \setminus E_j) \\ & + C(n,\tau) \limsup_{j\to\infty} \mathcal{H}^n\big((E_j \cap \partial S_{\tau,r}) \setminus A^{\rm in}_{r,j}\big) + C_*\, c(n)\, r^{n+1}\,.\end{aligned} \tag{3.36}$$

We now discuss the cases $x \in \partial^* E$, $x \in K \cap E^{(0)}$ and $x \in K \cap E^{(1)}$ separately.

The case $x \in \partial^* E$: We claim that, in this case, for every $\sigma \in (0,\tau)$ and for a.e. $r < r_0(\sigma,x)$,

$$\limsup_{j\to\infty} \mathcal{H}^n\Big(\partial S_{\tau,r} \setminus \big(A^{\rm out}_{r,j} \cup E_j\big)\Big) \;\le\; C(n)\, \sigma\, r^n\,, \tag{3.37}$$

$$\limsup_{j\to\infty} \Big|\mathcal{H}^n\big(E_j \cap \partial S_{\tau,r}\big) - \omega_n\, r^n\Big| \;\le\; C(n)\, \tau\, r^n\,; \tag{3.38}$$

see Figure 3.2. We notice that (3.37) and (3.38) combined with (3.35) imply

$$\frac{\mu(B_r(x))}{r^n} \le \omega_n + C(n)\,\tau + C(n,\tau)\,\sigma + C_*\, c(n)\, r\,, \qquad \text{for a.e. } r < r_0\,,$$

which gives $\theta(x) \le 1$ by letting, in the order, $r \to 0^+$, $\sigma \to 0^+$ and then $\tau \to 0^+$. We now prove (3.37) and (3.38). Since $x \in \partial^* E$, we can set $\nu(x) = \nu_E(x)$. As $\nu_E(x)$ is the outer normal to E, by $\partial^* E \subset K$, (3.29) and the divergence theorem, we obtain

$$|W^-_{\tau,\sigma,r_0} \setminus E| = |W^+_{\tau,\sigma,r_0} \cap E| = 0\,.$$

By $|W^-_{\tau,\sigma,r_0} \setminus E| = 0$, the coarea formula and Fatou's lemma, we deduce

$$\begin{aligned} 0 \;&=\; \lim_{j\to\infty} |W^-_{\tau,\sigma,r_0} \setminus E_j| = \lim_{j\to\infty} \int_0^{r_0} \mathcal{H}^n\Big(\partial S_{\tau,r} \cap \big(W^-_{\tau,\sigma,r_0} \setminus E_j\big)\Big)\, dr \\ &\ge\; \int_0^{r_0} \liminf_{j\to\infty} \mathcal{H}^n\Big(\Gamma^-_{\tau,\sigma,r} \setminus E_j\Big)\, dr\,,\end{aligned}$$

and by arguing similarly with $|W^+_{\tau,\sigma,r_0}\cap E|=0$ we conclude that, for a.e. $r<r_0$,

$$\lim_{j\to\infty}\mathcal{H}^n\big(\Gamma^+_{\tau,\sigma,r}\cap E_j\big)=0\,, \tag{3.39}$$

$$\lim_{j\to\infty}\mathcal{H}^n\big(\Gamma^-_{\tau,\sigma,r}\setminus E_j\big)=0\,. \tag{3.40}$$

By (3.39), (3.40), and since

$$\partial S_{\tau,r}=\Gamma^+_{\tau,\sigma,r}\cup\Gamma^-_{\tau,\sigma,r}\cup\big(\partial S_{\tau,r}\cap\partial S_{\sigma,r}\big) \tag{3.41}$$

we find that, as $j\to\infty$,

$$\begin{aligned}\big|\mathcal{H}^n(\partial S_{\tau,r}\cap E_j)-\omega_n\,r^n\big| &\le \mathcal{H}^n(\partial S_{\tau,r}\cap\partial S_{\sigma,r})+\big|\mathcal{H}^n(\Gamma^-_{\tau,\sigma,r}\cap E_j)-\omega_n\,r^n\big|+\mathrm{o}(1)\\ &\le C(n)\,\sigma\,r^n+\big|\mathcal{H}^n(\Gamma^-_{\tau,\sigma,r})-\omega_n\,r^n\big|+\mathrm{o}(1)\\ &\le C(n)\,\tau\,r^n+\mathrm{o}(1)\,,\end{aligned}$$

that is (3.38). At the same time, again by (3.29) and by the coarea formula, assuming without loss of generality that $r_0=r_0(\sigma,x)$ also satisfies $\mathcal{H}^n(K\cap\partial B_{r_0}(x))=0$ in addition to (3.29), we get

$$\begin{aligned}0 &= \mu(K\cap\mathrm{cl}\,(B_{r_0}(x))\setminus V_{\sigma,r_0})=\lim_{j\to\infty}\mathcal{H}^n\big(B_{r_0}(x)\cap\partial E_j\setminus V_{\sigma,r_0}\big)\\ &\ge \lim_{j\to\infty}\mathcal{H}^n\big(S_{\tau,r_0}\cap\partial E_j\setminus V_{\sigma,r_0}\big)\\ &\ge \lim_{j\to\infty}\int_0^{r_0}\mathcal{H}^{n-1}\big(\partial S_{\tau,r}\cap\partial E_j\setminus V_{\sigma,r_0}\big)\,dr\,,\end{aligned}$$

that is

$$\lim_{j\to\infty}\mathcal{H}^{n-1}\big(\partial S_{\tau,r}\cap\partial E_j\setminus V_{\sigma,r_0}\big)=0\qquad\text{for a.e. } r<r_0\,. \tag{3.42}$$

Notice that (3.42) implies in particular that

$$\lim_{j\to\infty}\mathcal{H}^{n-1}\big(\Gamma^+_{\tau,\sigma,r}\cap\partial E_j\big)=0\qquad\text{for a.e. } r<r_0\,. \tag{3.43}$$

Since $\Gamma^+_{\tau,\sigma,r}$ is a bi-Lipschitz image of a hemisphere, by Lemma 2.12,

$$\mathcal{H}^{n-1}(\Gamma^+_{\tau,\sigma,r}\cap J)^{n/(n-1)}\ge c(n,\tau,\sigma)\,\mathcal{H}^n(\Gamma^+_{\tau,\sigma,r}\setminus A)\,, \tag{3.44}$$

whenever J is relatively closed in $\Gamma^+_{\tau,\sigma,r}$, and A is an $\mathcal{H}^n$-maximal connected component of $\Gamma^+_{\tau,\sigma,r}\setminus J$. By (3.43) and (3.44) we find that, if

$$A^+_{r,j}\text{ is a maximal }\mathcal{H}^n\text{-component of }\Gamma^+_{\tau,\sigma,r}\setminus\partial E_j\,,$$

then

$$\lim_{j\to\infty}\mathcal{H}^n(\Gamma^+_{\tau,\sigma,r}\setminus A^+_{r,j})=0\,,\qquad\text{for a.e. } r<r_0\,. \tag{3.45}$$

By connectedness, $A^+_{r,j}$ is either contained in $A^{\rm out}_{r,j}$, or in E_j, or in

$$Y_{r,j}=\bigcup\big\{A:A\in\mathcal{A}^{\rm out}_{r,j}\,,A\ne A^{\rm out}_{r,j}\big\}\,.$$

By combining (3.39) with (3.45) we find that for a.e. $r<r_0$, if j is large enough, then

$$A^+_{r,j}\cap E_j=\emptyset\,.$$

Similarly, should there be a non-negligible set of values of r such that for infinitely many value of j the inclusion $A^+_{r,j}\subset Y_{r,j}$ holds, then by (3.40) and (3.45) there would be an element of $\mathcal{A}^{\rm out}_{r,j}$ different from $A^{\rm out}_{r,j}$ with $\mathcal{H}^n$-measure arbitrarily close to $\mathcal{H}^n(\Gamma^+_{\tau,\sigma,r})$; thanks to (3.40), we would then have $\mathcal{H}^n(A^{\rm out}_{r,j})\to 0$, against the $\mathcal{H}^n$-maximality of $A^{\rm out}_{r,j}$ itself. In conclusion, it must be

$$A^+_{r,j}\subset A^{\rm out}_{r,j}\ \text{for a.e. } r<r_0\text{ and for } j\text{ large enough}\,. \tag{3.46}$$

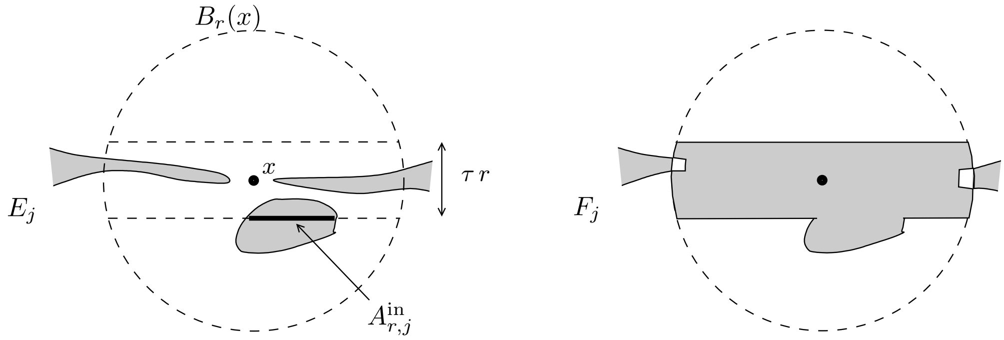

FIGURE 3.3. The slab competitor used in proving that $\theta(x)\le 2$ when $x\in E^{(0)}$ is the one defined by $A^{\rm in}_{r,j}$. Since $x\in E^{(0)}$ we can show that $E_j\cap\partial S_{\tau,r}$ is $\mathrm{o}(r^n)$ as $r\to 0^+$.

By combining (3.46) and (3.45) we conclude that

$$\lim_{j\to\infty}\mathcal{H}^n\big(\Gamma^+_{\tau,\sigma,r}\setminus A^{\rm out}_{r,j}\big)=0\,. \tag{3.47}$$

By (3.41), (3.40) and (3.47) we conclude that

$$\limsup_{j\to\infty}\mathcal{H}^n\Big(\partial S_{\tau,r}\setminus\big(A^{\rm out}_{r,j}\cup E_j\big)\Big)\le\mathcal{H}^n(\partial S_{\tau,r}\cap\partial S_{\sigma,r})\le C(n)\,\sigma\,r^n\,,$$

that is (3.37). This completes the proof of $\theta(x)\le 1$ for $x\in\partial^*E$.

The case $x\in E^{(0)}$: We claim that, in this case, for every $\sigma\in(0,\tau)$,

$$\limsup_{j\to\infty}\mathcal{H}^n(E_j\cap\partial S_{\tau,r})\le C(n)\,\sigma\,r^n\,, \tag{3.48}$$

$$\limsup_{j\to\infty}\big|\mathcal{H}^n\big(\partial S_{\tau,r}\setminus E_j\big)-2\,\omega_n\,r^n\big|\le C(n)\,\tau\,r^n\,, \tag{3.49}$$

for a.e. $r<r_0(\sigma,x)$, see Figure 3.3. The idea is using the competitor defined by $A^{\rm in}_{r,j}$: indeed, (3.48), (3.49), and (3.36) give

$$\begin{aligned}\frac{\mu(B_r(x))}{r^n}&\le\limsup_{j\to\infty}\frac{\mathcal{H}^n(\partial S_{\tau,r}\setminus E_j)}{r^n}\\&\quad+C(n,\tau)\,\limsup_{j\to\infty}\frac{\mathcal{H}^n\big((E_j\cap\partial S_{\tau,r})\setminus A^{\rm in}_{r,j}\big)}{r^n}+C_*\,c(n)\,r\\&\le 2\,\omega_n+C(n)\,\tau+C(n,\tau)\,\sigma+C_*\,c(n)\,r\,,\end{aligned}$$

and then $\theta(x)\le 2$ by letting, in the order, $r\to 0^+$, $\sigma\to 0^+$ and then $\tau\to 0^+$. The proof of (3.48) and (3.49) is simple: since $x\in E^{(0)}$ and $\partial^*E\subset K$, by (3.29) and by the divergence theorem we find that

$$|E\cap B_{r_0}(x)\setminus V_{\sigma,r_0}|=0\,.$$

In particular, by the coarea formula we find that for a.e. $r<r_0$,

$$0=\lim_{j\to\infty}\mathcal{H}^n\Big((E_j\setminus V_{\sigma,r_0})\cap\partial S_{\tau,r}\Big)=\lim_{j\to\infty}\mathcal{H}^n\Big(E_j\cap\big(\Gamma^+_{\tau,\sigma,r}\cup\Gamma^-_{\tau,\sigma,r}\big)\Big)\,,$$

so that, by (3.41),

$$\mathcal{H}^n(E_j\cap\partial S_{\tau,r})=\mathcal{H}^n(\partial S_{\tau,r}\cap\partial S_{\sigma,r})+\mathrm{o}(1)\le C(n)\,\sigma\,r^n+\mathrm{o}(1)\,,$$

as $j\to\infty$, that is (3.48), and

$$\begin{aligned}\big|\mathcal{H}^n(\partial S_{\tau,r}\setminus E_j)-2\,\omega_n\,r^n\big|&\le\mathcal{H}^n(\partial S_{\tau,r}\cap\partial S_{\sigma,r})+\big|\mathcal{H}^n(\Gamma^+_{\tau,\sigma,r}\cup\Gamma^-_{\tau,\sigma,r})-2\,\omega_n\,r^n\big|+\mathrm{o}(1)\\&\le C(n)\,\tau\,r^n+\mathrm{o}(1)\end{aligned}$$

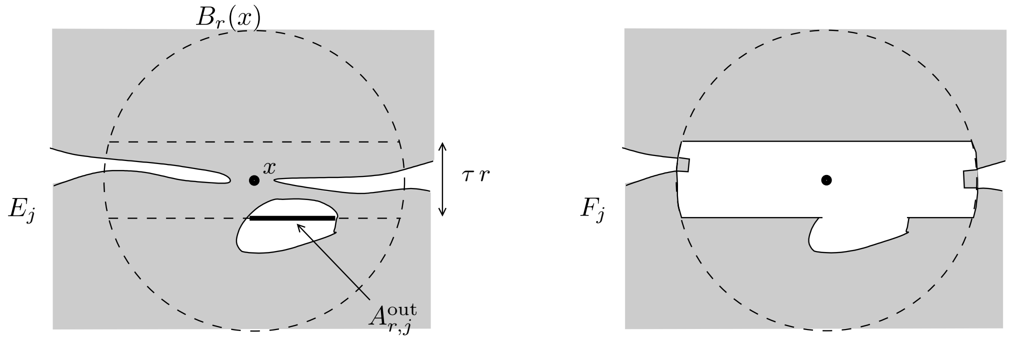

FIGURE 3.4. The slab competitor used in proving that $\theta(x) \le 2$ when $x \in E^{(1)}$ is the one defined by $A^{\rm out}_{r,j}$.

as $j \to \infty$, that is (3.49).

The case $x \in E^{(1)}$: We claim that for every $\sigma \in (0,\tau)$,

$$\limsup_{j\to\infty} \big|\mathcal{H}^n(E_j \cap \partial S_{\tau,r}) - 2\,\omega_n\, r^n\big| \le C(n)\,\tau\, r^n\,, \tag{3.50}$$

$$\limsup_{j\to\infty} \mathcal{H}^n\big(\partial S_{\tau,r} \setminus E_j\big) \le C(n)\,\sigma\, r^n\,, \tag{3.51}$$

for a.e. $r < r_0(\sigma,x)$, see Figure 3.4. Indeed, by using as in the case $x \in \partial^* E$ the competitor defined by $A^{\rm out}_{r,j}$, (3.50), (3.51) are combined with (3.35) to obtain

$$\begin{aligned}\frac{\mu(B_r(x))}{r^n} &\le \limsup_{j\to\infty} \frac{\mathcal{H}^n(E_j \cap \partial S_{\tau,r})}{r^n} \\ &\quad + C(n,\tau)\,\limsup_{j\to\infty} \frac{\mathcal{H}^n\big(\partial S_{\tau,r} \setminus (A^{\rm out}_{r,j} \cup E_j)\big)}{r^n} + C_*\,c(n)\,r \\ &\le 2\,\omega_n + C(n)\,\tau + C(n,\tau)\,\sigma + C_*\,c(n)\,r\,, \end{aligned} \tag{3.52}$$

which gives $\theta(x) \le 2$ by letting once again $r \to 0^+$, $\sigma \to 0^+$ and finally $\tau \to 0^+$. To prove (3.50) and (3.51), we notice that by $x \in E^{(1)}$, $\partial^* E \subset K$, (3.29) and the divergence theorem, we have

$$\big|B_{r_0}(x) \setminus \big(V_{\sigma,r_0} \cup E\big)\big| = 0\,.$$

By the coarea formula, for a.e. $r < r_0$ we find

$$0 = \lim_{j\to\infty} \mathcal{H}^n\big(\big(\Gamma^+_{\tau,\sigma,r} \cup \Gamma^-_{\tau,\sigma,r}\big) \setminus E_j\big)\,,$$

and conclude as in the previous case by exploiting (3.41).

Remark: We make an important remark on the constructions of step five, which will be needed in the proof of Theorem 1.6. We claim that, under the assumptions on x considered in step five, for a.e. $r < r_0(\sigma,x)$ we have

$$\begin{aligned}\limsup_{\eta\to 0^+} \Big|\mathcal{H}^n\Big(\Big\{y \in {\rm cl}\,(S_{\tau,r}) \cap \partial F_j^\star : T_y(\partial F_j^\star) = T_x K\Big\}\Big) - \theta(x)\,\omega_n\, r^n\Big| \\ \le C(n)\,\tau\, r^n + C(n,\tau)\,\sigma\, r^n + {\rm o}(1)\,, \qquad \text{as } j \to \infty\,.\end{aligned} \tag{3.53}$$

Here $\star =$ out if $x \in \partial^* E \cup (K \cap E^{(1)})$, $\star =$ in if $x \in K \cap E^{(0)}$, and $\theta(x) = 1$ if $x \in \partial^* E$ and $\theta(x) = 2$ if $x \in K \cap (E^{(0)} \cup E^{(1)})$. Consider, for example, the case when $x \in \partial^* E$. By (3.33), $\partial S_{\tau,r} \cap \partial F_j^{\rm out} \subset (\partial S_{\tau,r} \setminus A^{\rm out}_{r,j}) \cup N_j$ with $\lim_{\eta\to 0^+} \mathcal{H}^n(N_j) = 0$: thus, by taking into account that

$$T_y(\partial F_j^{\rm out}) = T_y(\partial S_{\tau,r}) \qquad \mathcal{H}^n\text{-a.e. on } \partial F_j^{\rm out} \cap \partial S_{\tau,r}$$

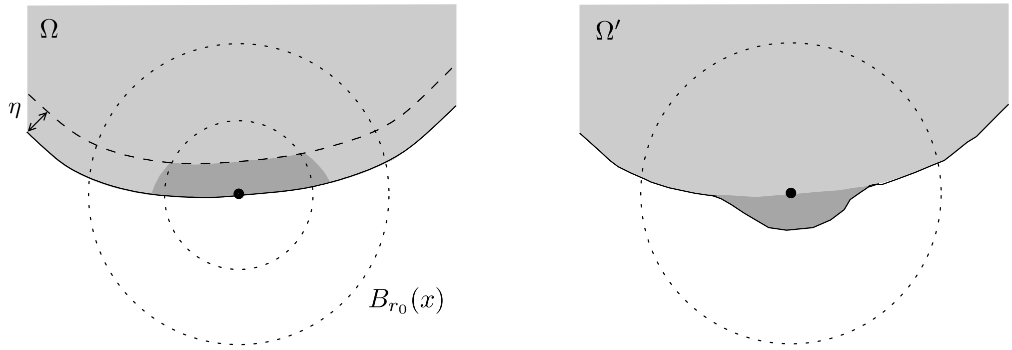

FIGURE 3.5. The boundary diffeomorphism f pushes out Ω into a larger open set Ω'. Regions depicted with the same color are mapped one into the other. Notice that the dark region on the left contains $\Omega\cap U_\eta(\partial\Omega)\cap B_{r_0/2}(x)$, and is mapped outside of Ω. The diffeomorphism f can be formally constructed by exploiting the local graphicality of Ω, and the simple details are omitted.

and that

$$\big\{y\in\partial S_{\tau,r}:T_y(\partial S_{\tau,r})=T_xK\big\}=\partial S_{\tau,r}\setminus\partial B_r(x)\,,$$

(recall that $T_xK=\nu(x)^\perp$), we have

$$\begin{aligned}
&\Big|\mathcal{H}^n\Big(\Big\{y\in\mathrm{cl}\,(S_{\tau,r})\cap\partial F_j^{\rm out}:T_y(\partial F_j^{\rm out})=T_xK\Big\}\Big)-\omega_n\,r^n\Big|\\
\le\;&\Big|\mathcal{H}^n\Big(\Big\{y\in\partial S_{\tau,r}\cap\partial F_j^{\rm out}:T_y(\partial F_j^{\rm out})=T_xK\Big\}\Big)-\omega_n\,r^n\Big|+\mathcal{H}^n(S_{\tau,r}\cap\partial F_j^{\rm out})\\
\le\;&\Big|\mathcal{H}^n\Big(\Big\{y\in\partial S_{\tau,r}\setminus A_{r,j}^{\rm out}:T_y(\partial S_{\tau,r})=T_xK\Big\}\Big)-\omega_n\,r^n\Big|+\mathcal{H}^n(N_j)+\mathcal{H}^n(S_{\tau,r}\cap\partial F_j^{\rm out})\\
=\;&\Big|\mathcal{H}^n\big(\partial S_{\tau,r}\setminus(\partial B_r(x)\cup A_{r,j}^{\rm out})\big)-\omega_n\,r^n\Big|+\mathcal{H}^n(N_j)+\mathcal{H}^n(S_{\tau,r}\cap\partial F_j^{\rm out})
\end{aligned}$$

so that, by (3.34), (3.37), and $\mathcal{H}^n(\partial S_{\tau,r}\cap\partial B_r(x))\le C(n)\,\tau\,r^n$,

$$\begin{aligned}
&\limsup_{\eta\to0^+}\Big|\mathcal{H}^n\Big(\Big\{y\in\mathrm{cl}\,(S_{\tau,r})\cap\partial F_j^{\rm out}:T_y(\partial F_j^{\rm out})=T_xK\Big\}\Big)-\omega_n\,r^n\Big|\\
\le\;&\Big|\mathcal{H}^n(\partial S_{\tau,r}\cap E_j)-\omega_n\,r^n\Big|+C(n,\tau)\mathcal{H}^n\big(\partial S_{\tau,r}\setminus(A_{r,j}^{\rm out}\cup E_j)\big)+C(n)\,\tau\,r^n\,.
\end{aligned}$$

By (3.37) and (3.38) we deduce (3.53) when $x\in\partial^*E$. The case when $x\in K\cap(E^{(0)}\cup E^{(1)})$ is treated analogously and the details are omitted.

Step six: We exclude area concentration near $\partial\Omega$, by showing that

$$\limsup_{\eta\to0^+}\limsup_{j\to\infty}\mu_j(\Omega\cap U_\eta(\partial\Omega))=0\,.\tag{3.54}$$

Exploiting the smoothness and boundedness of $\partial\Omega$, we can find $r_0>0$ such that Lemma 2.4 holds, and such that for every $x\in\partial\Omega$ there exists an open set Ω' with $\Omega\subset\Omega'$ and a homeomorphism $f:\mathrm{cl}\,(\Omega)\to\mathrm{cl}\,(\Omega')=f(\mathrm{cl}\,(\Omega))$ with $f(\partial\Omega)=\partial\Omega'$, $\{f\ne\mathrm{id}\,\}\subset\subset B_{r_0}(x)$, $f(B_{r_0}(x)\cap\mathrm{cl}\,(\Omega))=B_{r_0}(x)\cap\mathrm{cl}\,(\Omega')$, which is a diffeomorphism $f:\Omega\to\Omega'$, and such that

$$f\Big(\Omega\cap U_\eta(\partial\Omega)\cap B_{r_0/2}(x)\Big)\subset\Omega'\setminus\Omega\,,\qquad\|f-\mathrm{id}\,\|_{C^1(\Omega)}\le C\,\eta\,;\tag{3.55}$$

see Figure 3.5. Let $\Omega^*=f^{-1}(\Omega)$ and let $F_j=f(E_j\cap\Omega^*)=f(E_j)\cap\Omega$. Clearly $F_j\in\mathcal{E}$, and $f(\partial\Omega^*)=\partial\Omega$ and $\Omega^*\cap\partial(E_j\cap\Omega^*)=\Omega^*\cap\partial E_j$ give

$$\Omega\cap\partial F_j=f(\Omega^*)\cap\,f\big(\partial(E_j\cap\Omega^*)\big)=f\big(\Omega^*\cap\partial E_j\big)\,,$$

so that $\Omega\cap\partial F_j$ is $\mathcal{C}$-spanning W by Lemma 2.4. Assuming without loss of generality that $r_0<r_*$, by (3.11), $\{f\neq \mathrm{id}\}\subset\subset B_{r_0}(x)$ and $f(B_{r_0}(x)\cap\mathrm{cl}\,(\Omega))=B_{r_0}(x)\cap\mathrm{cl}\,(\Omega')$ we have

$$\begin{aligned}\mathcal{H}^n(\Omega\cap B_{r_0}(x)\cap\partial E_j) &\le \mathcal{H}^n\big(f(B_{r_0}(x)\cap\Omega^*\cap\partial E_j)\big)+C_*\,\big||F_j|-|E_j|\big|+\frac{1}{j}\\ &\le \big(1+C\,\eta\big)\,\mathcal{H}^n\big(B_{r_0}(x)\cap\Omega^*\cap\partial E_j\big)+C_*\,\big||F_j|-|E_j|\big|+\frac{1}{j}\,,\end{aligned}$$

where

$$\big||F_j|-|E_j|\big|\le\big||E_j\cap\Omega^*|-|E_j|\big|+\int_{E_j\cap\Omega^*}|Jf-1|\le\big|\Omega\setminus\Omega^*\big|+C\,\varepsilon\,\eta\le C\,\eta\,,$$

so that

$$\mathcal{H}^n(\Omega\cap B_{r_0}(x)\cap\partial E_j\setminus\Omega^*)\le C\,\eta\,\Big\{\mathcal{H}^n\big(\Omega\cap\partial E_j\big)+1\Big\}+\frac{1}{j}\le C\,\eta\,\Big\{\psi(\varepsilon)+2\Big\}+\frac{1}{j}\,.$$

Since $\Omega\cap U_\eta(\partial\Omega)\cap B_{r_0/2}(x)\subset\Omega\setminus\Omega^*$, by letting $j\to\infty$ we conclude that

$$\mu\big(B_{r_0/2}(x)\cap U_\eta(\partial\Omega)\big)\le C\,\eta\,,\qquad\forall x\in\partial\Omega\,.$$

By a covering argument we find $\mu(\Omega\cap U_\eta(\partial\Omega))\le C\,\eta$, and thus (3.54) follows.

Step seven: Let us now pick $R>0$ such that $W\cup K\cup E\subset\subset B_R(0)$. If $E_j\subset B_{R+1}(0)$ for infinitely many values of j, then $|E|=\varepsilon$ and $\mu_j(\Omega\setminus B_{R+1}(0))=0$, which combined with (3.54) implies $\mu_j(\Omega)\to\mu(\Omega)=\mathcal{F}(K,E)$ as $j\to\infty$, and thus $\psi(\varepsilon)=\mathcal{F}(K,E)$ with $(K,E)\in\mathcal{K}$ and $|E|=\varepsilon$: thus (K,E) is a generalized minimizer of $\psi(\varepsilon)$, as desired. We now assume without loss of generality that $|E_j\setminus B_{R+1}(0)|>0$ for every j. By (3.5),

$$\limsup_{j\to\infty}|E_j\cap(B_{R+1}(0)\setminus B_R(0))|=\limsup_{j\to\infty}\mathcal{H}^n((B_{R+1}(0)\setminus B_R(0))\cap\partial E_j)=0\,.$$

By the coarea formula, this implies that for a.e. $s\in(R,R+1)$,

$$\limsup_{j\to\infty}\mathcal{H}^n(E_j\cap\partial B_s(0))=\limsup_{j\to\infty}\mathcal{H}^{n-1}(\partial E_j\cap\partial B_s(0))=0\,.\tag{3.56}$$

We fix a value of s such that (3.56) holds, and we let A_j denote an $\mathcal{H}^n$-maximal connected component of $\partial B_s(0)\setminus\partial E_j$. It must be $A_j\cap E_j=\emptyset$: for, otherwise, by the spherical isoperimetric inequality, $A_j\subset E_j$ would imply

$$\begin{aligned}C(n)\,\mathcal{H}^{n-1}(\partial B_s(0)\cap\partial E_j)^{n/(n-1)} &\ge \mathcal{H}^n(\partial B_s(0)\setminus A_j)\ge\mathcal{H}^n(\partial B_s(0)\setminus E_j)\\ &\ge c(n)\,R^n-\mathcal{H}^n(E_j\cap\partial B_s(0))\,,\end{aligned}$$

a contradiction to (3.56). Since $A_j\cap E_j=\emptyset$, we can consider the exterior cup competitor defined by E_j and A_j. More precisely, for every j there exists a decreasing sequence $\{\eta_k^j\}_{k=1}^\infty$ with $\lim_{k\to\infty}\eta_k^j=0$ such that, setting

$$\begin{aligned}Y_j&=\partial B_s(0)\setminus\mathrm{cl}\,((E_j\cap\partial B_s(0))\cup A_j)\,, & S_j&=\partial E_j\cap\mathrm{cl}\,(A_j)\setminus(\mathrm{cl}\,((E_j\cap\partial B_s(0))\cup Y_j))\,,\\ U_{j,k}&=\partial B_s(0)\cap\{\mathrm{d}_{S_j}<\eta_k^j\}\,, & Z_{j,k}&=Y_j\cup(U_{j,k}\setminus\mathrm{cl}\,(E_j\cap\partial B_s(0)))\,,\end{aligned}$$

the sets

$$F_{j,k}=\big(E_j\cap B_s(0)\big)\cup M_{\eta_k^j}(Z_{j,k})$$

satisfy $F_{j,k}\in\mathcal{E}$, with $\Omega\cap\partial F_{j,k}$ $\mathcal{C}$-spanning W, $F_{j,k}\subset B_{R+1}$ and

$$\begin{aligned}\limsup_{k\to\infty}\mathcal{H}^n(\Omega\cap\partial F_{j,k}) &\le \mathcal{H}^n(\Omega\cap B_s(0)\cap\partial E_j)+2\,\mathcal{H}^n(\partial B_s(0)\setminus A_j)\\ &\le \mathcal{H}^n(\Omega\cap B_s(0)\cap\partial E_j)+C(n)\,\mathcal{H}^{n-1}(\partial B_s(0)\cap\partial E_j)^{n/(n-1)}\,.\end{aligned}\tag{3.57}$$

Since $|E_j\setminus B_{R+1}(0)|>0$ for every j, we can select $k(j)$ sufficiently large so that

$$\mathcal{H}^n(\Omega\cap\partial F_{j,k(j)})\le\mathcal{H}^n(\Omega\cap B_s(0)\cap\partial E_j)+C(n)\,\mathcal{H}^{n-1}(\partial B_s(0)\cap\partial E_j)^{n/(n-1)}+\frac{1}{j}\,,\tag{3.58}$$

as well as $|E_j \setminus B_s(0)| > |M_{\eta^j_{k(j)}}(Z_{j,k(j)})|$; then, after setting $F_j = F_{j,k(j)}$, define $\rho_j > 0$ by the equation

$$|B_{\rho_j}| = |E_j| - |F_j| = |E_j \setminus B_s(0)| - |M_{\eta^j_{k(j)}}(Z_{j,k(j)})|\,.$$

In particular, $|B_{\rho_j}| \le \varepsilon$, so that we can find $x \in \Omega$ such that $\mathrm{cl}\,(B_{\rho_j}(x)) \cap \mathrm{cl}\,(F_j) = \emptyset$ and

$$E_j^* = F_j \cup B_{\rho_j}(x) \subset B_{R+1+C(n)\,\varepsilon^{1/(n+1)}}(0) \qquad \forall j\,.$$

We notice that $E_j^* \in \mathcal{E}$ with $|E_j^*| = \varepsilon$ and $\Omega\cap\partial F_j \subset \Omega\cap\partial E_j^*$, so that $\Omega\cap\partial E_j^*$ is $\mathcal{C}$-spanning W: in particular, $\psi(\varepsilon) \le \mathcal{H}^n(\Omega \cap \partial E_j^*)$. By the Euclidean isoperimetric inequality, and since $|B_{\rho_j}| \le |E_j \setminus B_s(0)|$ by definition of ρ_j, we have

$$P(B_{\rho_j}) \le P(E_j \setminus B_s(0)) = \mathcal{H}^n(\partial E_j \setminus B_s(0)) + \mathcal{H}^n(E_j \cap \partial B_s(0))\,,$$

so that by (3.56) and (3.58) we get

$$\begin{aligned}\psi(\varepsilon) &\le \limsup_{j\to\infty} \mathcal{H}^n(\Omega \cap \partial E_j^*) \le \limsup_{j\to\infty} \mathcal{H}^n(\Omega\cap\partial F_j) + P(B_{\rho_j})\\ &\le \limsup_{j\to\infty} \mathcal{H}^n(\Omega\cap\partial E_j) + 2\,C(n)\,\limsup_{j\to\infty} \mathcal{H}^{n-1}(\partial B_s(0)\cap\partial E_j)^{n/(n-1)} = \psi(\varepsilon)\,.\end{aligned}$$

We have thus proved that $\{E_j^*\}_j$ is a minimizing sequence for $\psi(\varepsilon)$, with $E_j^* \subset B_{R^*}(0)$ for some R^* depending only on R, n and ε. By repeating the argument of the first six steps with E_j^* in place of E_j we see that $E_j^* \to E^*$ in $L^1(\Omega)$ and $\mu_j^* = \mathcal{H}^n \llcorner (\Omega\cap\partial E_j^*) \overset{*}{\rightharpoonup} \mu^*$ where $\mu^* = 2\,\mathcal{H}^n \llcorner (K^* \setminus \partial^* E^*) + \mathcal{H}^n \llcorner \partial^* E^*$, and where $(K^*, E^*) \in \mathcal{K}$ with $|E^*| = \varepsilon$ and with

$$\limsup_{\eta\to 0^+} \limsup_{j\to\infty} \mu_j^*(\Omega \cap\ U_\eta(\partial\Omega)) = 0\,.$$

Therefore $\mu_j^*(\Omega) \to \mu^*(\Omega) = \mathcal{F}(K^*, E^*)$ and in conclusion

$$\mathcal{F}(K^*, E^*) = \mu^*(\Omega) = \lim_{j\to\infty} \mu_j^*(\Omega) = \psi(\varepsilon)$$

so that, by $|E^*| = \varepsilon$, (K^*, E^*) is indeed a generalized minimizer of $\psi(\varepsilon)$. This concludes the proof of the theorem. □

4. The Euler-Lagrange equation: Proof of Theorem 1.6

Proof of Theorem 1.6. Let (K, E) be a generalized minimizer of $\psi(\varepsilon)$ and $f : \Omega \to \Omega$ be a diffeomorphism such that $|f(E)| = |E|$. We want to prove that

$$\mathcal{F}(K, E) \le \mathcal{F}(f(K), f(E))\,. \tag{4.1}$$

Let K' denote the set of points of approximate differentiability of K, so that $\mathcal{H}^n(K\setminus K') = 0$, and for $x \in K'$ denote by $T_x = T_x K = \nu_x^\perp$ the approximate tangent plane to K at x, where $\nu_x \in \mathbb{S}^n$ is chosen so that $\nu_x = \nu_E(x)$ if $x \in \partial^* E$. As in step five of the proof of Theorem 1.4, for every $\sigma > 0$ we introduce $r_0 = r_0(\sigma, x)$ such that

$$K \cap B_r(x) \subset S^x_{\sigma,r} = \Big\{y \in B_r(x) : |(y-x)\cdot\nu_x| < \sigma\, r\Big\} \qquad \forall r < r_0(\sigma, x)\,, \tag{4.2}$$

see (3.29). In fact, by Egoroff's theorem, we can find a compact set $K^* \subset K'$ with $\mathcal{H}^n(K \setminus K^*) < \sigma$ such that $r_*(\sigma) = \max\{r_0(\sigma, x) : x \in K^*\} \to 0^+$ as $\sigma \to 0^+$, that is, such that (4.2) holds uniformly on K^*,

$$K \cap B_r(x) \subset S^x_{\sigma,r} \qquad \forall x \in K^*\,, \forall r < r_*(\sigma)\,. \tag{4.3}$$

Similarly, if G_n denotes the family of the n-planes in $\mathbb{R}^{n+1}$, endowed with a distance d, by Lusin's theorem and up to further decreasing the size of K^* while keeping $\mathcal{H}^n(K\setminus K^*) < \sigma$, we can make sure that

$$\sup_{x,y\in K^*,|x-y|<r} d(T_x, T_y) + \sup_{x,y\in K^*,|y-x|<r} |\nabla f(x) - \nabla f(y)| \le \omega_*(r)\,, \tag{4.4}$$

for a function $\omega_*(r)\to 0^+$ as $r\to 0^+$. Finally, since

$$\begin{cases}\mathcal{H}^n(B_r(x)\cap\partial^*E)=\mathrm{o}(r^n)\,,\\ \mathcal{H}^n\big(B_r(x)\cap(K\setminus\partial^*E)\big)=\omega_n\,r^n+\mathrm{o}(r^n)\,,\end{cases}\qquad\text{for }\mathcal{H}^n\text{-a.e. }x\in K\setminus\partial^*E\,,$$

$$\begin{cases}\mathcal{H}^n(B_r(x)\cap\partial^*E)=\omega_n\,r^n+\mathrm{o}(r^n)\,,\\ \mathcal{H}^n\big(B_r(x)\cap(K\setminus\partial^*E)\big)=\mathrm{o}(r^n)\,,\end{cases}\qquad\text{for }\mathcal{H}^n\text{-a.e. }x\in\partial^*E\,,$$

as $r\to 0^+$, by Egoroff's theorem, up to decreasing K^* and increasing ω_*, we can also entail

$$\sup_{x\in K^*\setminus\partial^*E}\mathcal{H}^n(B_r(x)\cap\partial^*E)+\Big|\mathcal{H}^n\big(B_r(x)\cap(K\setminus\partial^*E)\big)-\omega_n\,r^n\Big|\le\omega_*(r)\,r^n\,,\tag{4.5}$$

$$\sup_{x\in K^*\cap\partial^*E}\Big|\mathcal{H}^n(B_r(x)\cap\partial^*E)-\omega_n\,r^n\Big|+\mathcal{H}^n\big(B_r(x)\cap(K\setminus\partial^*E)\big)\le\omega_*(r)\,r^n\,,\tag{4.6}$$

while still keeping $\mathcal{H}^n(K\setminus K^*)<\sigma$ and $\omega_*(r)\to 0^+$ as $r\to 0^+$.

Let $\{E_j\}_j$ be a minimizing sequence for $\psi(\varepsilon)$ converging to (K,E) as in (1.13), and consider a point $x\in K^*$. Given $\tau\in(0,1)$ and $\sigma\in(0,\tau)$, for a.e. $r<r_*(\sigma)$ such that $B_{2r}(x)\subset\subset\Omega$, we have that $\partial S^x_{\tau,r}\cap\partial E_j$ is $\mathcal{H}^{n-1}$-rectifiable for every j (with the exceptional set depending on x). For such values of r and for every $\eta\in(0,r/2)$, we can set

$$F_j^x=\begin{cases}F_j^{\rm out}\,, & \text{if }x\in\partial^*E\cup(K^*\cap E^{(1)})\,,\\ F_j^{\rm in}\,, & \text{if }x\in K^*\cap E^{(0)}\,,\end{cases}$$

with $F_j^{\rm out}$ and $F_j^{\rm in}$ defined as in step five of the proof of Theorem 1.4. In particular, $F_j^x\in\mathcal{E}$, $\Omega\cap\partial F_j^x$ is $\mathcal{C}$-spanning W, $F_j^x\setminus\mathrm{cl}\,(S^x_{\tau,r})=E_j\setminus\mathrm{cl}\,(S^x_{\tau,r})$ and, as proved in (3.53), for a.e. $r<r_*(\sigma)$ we have

$$\begin{aligned}\limsup_{\eta\to 0^+}\Big|\mathcal{H}^n\Big(\Big\{y\in\mathrm{cl}\,(S^x_{\tau,r})\cap\partial F_j^x:T_y(\partial F_j^x)=T_x\Big\}\Big)-\theta(x)\,\omega_n\,r^n\Big|\\ \le C(n)\,\tau\,r^n+C(n,\tau)\,\sigma\,r^n+\mathrm{o}(1)\qquad\text{as }j\to\infty\,,\end{aligned}\tag{4.7}$$

where $\theta(x)=1$ if $x\in\partial^*E$ and $\theta(x)=2$ if $x\in K\cap(E^{(0)}\cup E^{(1)})$, as well as

$$\limsup_{j\to\infty}\limsup_{\eta\to 0^+}\mathcal{H}^n(S^x_{\tau,r}\cap\partial F_j^x)\le C(n,\tau)\,\sigma\,r^n\,,\tag{4.8}$$

see (3.34), (3.37), (3.48), and (3.51). By Besicovitch-Vitali's covering theorem and by Federer's theorem (2.1), we can find a *finite* disjoint family of closed balls $\{B_i=\mathrm{cl}\,(B_{r_i}(x_i))\}_i$ such that $B_i\subset\subset\Omega$ and

$$\mathcal{H}^n\Big(K^*\setminus\bigcup B_{r_i}(x_i)\Big)<\sigma\,,\qquad x_i\in K^*\cap\big(E^{(0)}\cup E^{(1)}\cup\partial^*E\big)\,,\qquad r_i<r_*(\sigma)\,.\tag{4.9}$$

We let $\eta<\min_i\{r_i/2\}$, define $F_j^{x_i}$ accordingly, and set

$$S_i=S^{x_i}_{\tau,r_i}\subset\subset B_i\,,\qquad T_i=T_{x_i}\,,\qquad F^i_j=F^{x_i}_j\,.$$

Correspondingly, we define a sequence $\{F_j\}_j\subset\mathcal{E}$ with $\Omega\cap\partial F_j$ $\mathcal{C}$-spanning W by setting

$$F_j\setminus\bigcup_i B_i=E_j\setminus\bigcup_i B_i\,,\qquad F_j\cap B_i=F^i_j\cap B_i\,.\tag{4.10}$$

Since $F^i_j\setminus\mathrm{cl}\,(S_i)=E_j\setminus\mathrm{cl}\,(S_i)$ we find that

$$F_j\setminus\bigcup_i\mathrm{cl}\,(S_i)=E_j\setminus\bigcup_i\mathrm{cl}\,(S_i)\,,\tag{4.11}$$

and, setting,

$$\theta_i=1\quad\text{if }x_i\in\partial^*E\,,\qquad\theta_i=2\quad\text{if }x_i\in E^{(0)}\cup E^{(1)}\tag{4.12}$$

we deduce from (4.7) and (4.8) that, for each i,

$$\limsup_{\eta\to 0^+}\Big|\mathcal{H}^n\big(\{y\in \mathrm{cl}\,(S_i)\cap\partial F_j : T_y(\partial F_j)=T_i\}\big)-\theta_i\,\omega_n\,r_i^n\Big| \qquad (4.13)$$

$$\le C(n)\,\tau\,r_i^n+C(n,\tau)\,\sigma\,r_i^n+\mathrm{o}(1)$$

$$\limsup_{\eta\to 0^+}\mathcal{H}^n(S_i\cap\partial F_j)\le C(n,\tau)\,\sigma\,r_i^n+\mathrm{o}(1) \qquad (4.14)$$

as $j\to\infty$. Now let C_* and ε_* the volume-fixing variation constants defined by $f(E)$. By the monotonicity formula (3.27), which can be applied to $B_{r_i}(x_i)$ as $x_i\in K$, we have

$$e^{-\Lambda\,r_*(\sigma)}\,\theta_i\,\omega_n\,r_i^n\le e^{-\Lambda\,r_i}\,\theta_i\,\omega_n\,r_i^n\le\mu(B_{r_i}(x_i))=\mu(S_i)\,, \qquad (4.15)$$

where in the last identity we have used (4.3), and where Λ depends on E. By (4.15), $\theta_i\ge 1$, and $\mu=\theta\,\mathcal{H}^n\llcorner K$ with $\theta\le 2$,

$$\sum_i r_i^n\le C(n,E)\sum_i\mathcal{H}^n(K\cap B_i)\le C(n,E)\,\mathcal{H}^n(K)=C(n,E,K)\,, \qquad (4.16)$$

so that, by (4.11), $|S_i|\le C(n)\,\tau\,r_i^{n+1}$ and $r_i\le r_*(\sigma)\le 1$, we find

$$|F_j\Delta E_j|\le\sum_i|S_i|\le C(n,E,K)\,\tau\,.$$

Therefore,

$$|f(F_j)\Delta f(E)|\le C\big(n,E,\mathrm{Lip}(f),\mathcal{H}^n(K)\big)\,\Big\{\tau+|E_j\Delta E|\Big\}<\varepsilon_*\,,$$

provided j is large enough and τ is small enough depending on ε_*. By the volume-fixing variations construction, for each j large enough there exists a smooth map $\Phi_j:(-\varepsilon_*,\varepsilon_*)\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$, such that, for every $|v|<\varepsilon_*$, $\Phi_j(v,\cdot)$ is a diffeomorphism with $\Phi_j(v,\Omega)=\Omega$ and

$$|\Phi_j(v,f(F_j))|=v+|f(F_j)|\,,\qquad \mathcal{H}^n\big(\Phi_j(v,\Sigma)\big)\le\mathcal{H}^n(\Sigma)+C_*\,|v|\,\mathcal{H}^n(\Sigma)\,,$$

for every $\mathcal{H}^n$-rectifiable set $\Sigma\subset\Omega$. In particular, if we set

$$G_j=\Phi_j(v_j,f(F_j))\,,\qquad v_j=|f(E)|-|f(F_j)|=|E|-|f(F_j)|\,,$$

then we find that $G_j\in\mathcal{E}$, $|G_j|=|E|=\varepsilon$ and

$$\mathcal{H}^n(\Omega\cap\partial G_j)\le\Big(1+C\big(n,E,\mathrm{Lip}(f),\mathcal{H}^n(K)\big)\,\Big\{\tau+|E_j\Delta E|\Big\}\Big)\,\mathcal{H}^n(\Omega\cap\partial f(F_j))\,.$$

Since $\Omega\cap\partial F_j$ is $\mathcal{C}$-spanning W, so is $\Omega\cap\partial G_j$ thanks to Lemma 2.3, so that the minimizing sequence property of E_j implies

$$\mathcal{H}^n(\Omega\cap\partial E_j)\le\Big(1+C\,\Big\{\tau+|E_j\Delta E|\Big\}\Big)\,\mathcal{H}^n(\Omega\cap\partial f(F_j))+\frac{1}{j}\,, \qquad (4.17)$$

where, here and for the rest of the proof, C is a generic constant depending on K, E, f and n. We now claim that

$$\limsup_{\sigma\to 0^+}\limsup_{j\to\infty}\limsup_{\eta\to 0^+}\mathcal{H}^n(\Omega\cap\partial f(F_j))\le\mathcal{F}(f(K),f(E))+C\,\tau\,. \qquad (4.18)$$

Notice that by combining (4.17) and (4.18), and by finally letting $\tau\to 0^+$, we complete the proof of (4.1).

To prove (4.18), we notice that $f(\Omega)=\Omega$, $\Omega\cap\partial f(F_j)=f(\Omega\cap\partial F_j)$, and (4.11) yield

$$\mathcal{H}^n(\Omega\cap\partial f(F_j))\le\mathcal{H}^n\Big(f\Big(\Omega\cap\partial E_j\setminus\bigcup_i\mathrm{cl}\,(S_i)\Big)\Big)+\sum_i\int_{\mathrm{cl}\,(S_i)\cap\partial F_j}J^{\partial F_j}f\,d\mathcal{H}^n\,,$$

where

$$\limsup_{j\to\infty}\limsup_{\eta\to 0^+}\mathcal{H}^n\Big(f\Big(\Omega\cap\partial E_j\setminus\bigcup_i \mathrm{cl}\,(S_i)\Big)\Big)\le C\,\mathcal{H}^n\Big(K\setminus\bigcup_i S_i\Big)\le C\,\sigma$$

by (4.3), (4.9), and $\mathcal{H}^n(K\setminus K^*)<\sigma$. Hence, as

$$\mathcal{H}^n(\Omega\cap\partial f(F_j))\le\sum_i\int_{\mathrm{cl}\,(S_i)\cap\partial F_j}J^{\partial F_j}f\,d\mathcal{H}^n+C\,\sigma+\mathrm{o}(1)\,,\tag{4.19}$$

where $\mathrm{o}(1)\to 0^+$ if we let first $\eta\to 0^+$ and then $j\to\infty$.

If we set

$$Z_i=\big\{y\in\partial S_i\cap\partial F_j:T_y(\partial F_j)=T_i\big\}\,,$$

then by (4.13) and (4.14) we find

$$\begin{aligned}\mathcal{H}^n\big((\mathrm{cl}\,(S_i)\cap\partial F_j)\Delta Z_i\big)&\le C(n)\,\tau\,r_i^n+C(n,\tau)\,\sigma\,r_i^n+\mathrm{o}(1)\,,\\ \big|\mathcal{H}^n(Z_i)-\theta_i\,\omega_n\,r_i^n\big|&\le C(n)\,\tau\,r_i^n+C(n,\tau)\,\sigma\,r_i^n+\mathrm{o}(1)\,,\end{aligned}$$

where $\mathrm{o}(1)\to 0^+$ if we let first $\eta\to 0^+$ and then $j\to\infty$. Also, it follows from (4.15), the characterization of μ, and (4.6) that

$$e^{-\Lambda\,r_*(\sigma)}\,\theta_i\,\omega_n\,r_i^n\le\theta_i\,\mathcal{H}^n(S_i\cap K)+\omega_*(r_i)\,r_i^n\,.\tag{4.20}$$

By (4.4), (4.20), and $r_i<r_*(\sigma)$, we thus find

$$\begin{aligned}\int_{\mathrm{cl}\,(S_i)\cap\partial F_j}J^{\partial F_j}f&\le\int_{Z_i}J^{T_i}f+(\mathrm{Lip}\,f)^n\big\{C(n)\,\tau+C(n,\tau)\,\sigma\big\}\,r_i^n+\mathrm{o}(1)\\ &\le\theta_i\,\omega_n\,r_i^n\,\big\{J^{T_i}f(x_i)+C(n)\,\omega_*(r_i)\big\}+C\,\big\{\tau+C(n,\tau)\,\sigma\big\}\,r_i^n+\mathrm{o}(1)\\ &\le\Big\{J^{T_i}f(x_i)+C\,\Big(\omega_*(r_*(\sigma))+\tau+C(n,\tau)\,\sigma\Big)\Big\}\times\\ &\quad\times\big(\theta_i\,\mathcal{H}^n(S_i\cap K)+\omega_*(r_*(\sigma))\,r_i^n\big)\,e^{\Lambda\,r_*(\sigma)}+\mathrm{o}(1)\\ &=J^{T_i}f(x_i)\big(\theta_i\,\mathcal{H}^n(S_i\cap K^*)+\alpha_i+\omega_*(r_*(\sigma))\,r_i^n\big)\,e^{\Lambda\,r_*(\sigma)}\\ &\quad+C\,\Big\{\omega_*(r_*(\sigma))+\tau+C(n,\tau)\,\sigma\Big\}\,\big(\mathcal{H}^n(S_i\cap K)+\omega_*(r_*(\sigma))\,r_i^n\big)\,e^{\Lambda\,r_*(\sigma)}\\ &\quad+\mathrm{o}(1)\,,\end{aligned}\tag{4.21}$$

where we have set

$$\alpha_i=\theta_i\,\mathcal{H}^n(S_i\cap(K\setminus K^*))\quad\text{so that}\quad\sum_i\alpha_i<2\,\sigma\,.\tag{4.22}$$

Now, again by (4.4) we see that

$$\begin{aligned}\theta_i\,J^{T_i}f(x_i)\,\mathcal{H}^n(S_i\cap K^*)&\le\theta_i\int_{S_i\cap K^*}J^Kf\,d\mathcal{H}^n+C(n)\,\omega_*(r_i)\,\mathcal{H}^n(S_i\cap K^*)\\ &=\theta_i\,\mathcal{H}^n(f(S_i\cap K^*))+C(n)\,\omega_*(r_i)\,\mathcal{H}^n(S_i\cap K^*)\,.\end{aligned}$$

By combining this last relation with (4.16), (4.19), (4.21) and $r_i<r_*(\sigma)$, we find that

$$\begin{aligned}\mathcal{H}^n(\Omega\cap\partial f(F_j))&\le e^{\Lambda\,r_*(\sigma)}\sum_i\theta_i\,\mathcal{H}^n(f(S_i\cap K^*))\\ &\quad+C\,\Big\{\omega_*(r_*(\sigma))+\tau+C(n,\tau)\,\sigma\Big\}\,e^{\Lambda\,r_*(\sigma)}+\mathrm{o}(1)\,,\end{aligned}\tag{4.23}$$

with $\mathrm{o}(1)\to 0$ as first $\eta\to 0^+$ and then $j\to\infty$. If $x_i\in K^*\setminus\partial^*E$, then $\theta_i=2$ and by (4.5) we have

$$\begin{aligned}\theta_i\,\mathcal{H}^n(f(S_i\cap K^*))&\le 2\,\mathcal{H}^n\big(f\big(S_i\cap(K^*\setminus\partial^*E)\big)\big)+2\,\mathrm{Lip}(f)^n\,\omega_*(r_i)\,r_i^n\\ &\le 2\mathcal{H}^n\big(f\big(S_i\cap(K\setminus\partial^*E)\big)\big)+C\,\omega_*(r_*(\sigma))\,r_i^n\,;\end{aligned}$$

if, instead, $x_i \in \partial^* E$, then $\theta_i = 1$ and (4.6) give

$$\begin{aligned}\theta_i\,\mathcal{H}^n(f(S_i\cap K^*)) &\le \mathcal{H}^n\big(f\big(S_i\cap K^*\cap\partial^*E\big)\big)+\mathrm{Lip}(f)^n\,\omega_*(r_i)\,r_i^n\\ &\le \mathcal{H}^n\big(f(S_i\cap\partial^*E)\big)+C\,\omega_*(r_*(\sigma))\,r_i^n\,;\end{aligned}$$

combining these last two estimates with (4.16), we find

$$\begin{aligned}\sum_i \theta_i\,\mathcal{H}^n(f(S_i\cap K^*)) &\le \sum_i 2\,\mathcal{H}^n\big(f\big(S_i\cap(K\setminus\partial^*E)\big)\big)+\mathcal{H}^n\big(f(S_i\cap\partial^*E)\big)\\ &\quad +C\,\omega_*(r_*(\sigma))\sum_i r_i^n\\ &\le \mathcal{F}\Big(f(K),f(E);\bigcup_i f(S_i)\Big)+C\,\omega_*(r_*(\sigma))\,,\end{aligned}$$

where $f(\partial^*E)=\partial^*f(E)$ by Lemma A.1. Combining this last estimate with (4.23) we find

$$\mathcal{H}^n(\Omega\cap\partial f(F_j))\le e^{\Lambda\,r_*(\sigma)}\Big\{\mathcal{F}(f(K),f(E))+C\,\big\{\omega_*(r_*(\sigma))+\tau+C(n,\tau)\,\sigma\big\}\Big\}+\mathrm{o}(1)\,,$$

where $\mathrm{o}(1)\to 0$ as first $\eta\to 0^+$ and then $j\to\infty$; in particular, (4.18) holds.

We conclude the proof. As explained, (4.18) implies (4.1). By a classical first variation argument, see Appendix C, we deduce the existence of $\lambda\in\mathbb{R}$ such that

$$\lambda\int_{\partial^*E} X\cdot\nu_E\,d\mathcal{H}^n=\int_{\partial^*E}\mathrm{div}^K X\,d\mathcal{H}^n+2\int_{K\setminus\partial^*E}\mathrm{div}^K X\,d\mathcal{H}^n\,, \tag{4.24}$$

for every $X\in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ with $X\cdot\nu_\Omega=0$ on $\partial\Omega$. Let us now consider the integer rectifiable varifold V supported on K, with density 2 on $K\setminus\partial^*E$ and 1 on ∂^*E. By (4.24), we can compute the first variation of V as

$$\delta V(X)=\int \vec H\cdot X\,d\,\|V\|\qquad \forall X\in C^1_c(\Omega;\mathbb{R}^{n+1})$$

where $\vec H=0$ on $K\setminus\partial^*E$ and $\vec H=\lambda\,\nu_E$ on ∂^*E. In particular, $\vec H\in L^\infty(\|V\|)$, and by Allard's regularity theorem [Sim83, Chapter 5], we have $K=\Sigma\cup\mathrm{Reg}$, where $\Sigma\subset K$ is closed and has empty interior in K, and where for every $x\in\mathrm{Reg}$ there exists a $C^{1,\alpha}$-function u defined on $\mathbb{R}^n$ such that

$$B_{r_x/2}(x)\cap K=B_{r_x/2}(x)\cap\mathrm{Reg}=B_{r_x/2}(x)\cap\mathrm{graph}(u)\,. \tag{4.25}$$

By the divergence theorem, if $x\in\mathrm{Reg}\cap\partial E$, then, by (4.25) and by $\Omega\cap\partial E\subset K$,

$$E=\mathrm{epigraph}(u)\qquad \text{inside } B_{r_x/2}(x)\,, \tag{4.26}$$

$$K=\partial E=\mathrm{graph}(u)\qquad \text{inside } B_{r_x/2}(x)\,, \tag{4.27}$$

which imply $\mathrm{Reg}\cap\partial E\subset\Omega\cap\partial^*E$. Viceversa, if $x\in\Omega\cap\partial^*E$, then $\mathcal{H}^n(B_r(x)\cap(K\setminus\partial^*E))=\mathrm{o}(r^n)$ and $\mathcal{H}^n(B_r(x)\cap\partial^*E)=\omega_n\,r^n+\mathrm{o}(r^n)$ as $r\to 0^+$, so that Allard's regularity theorem implies $\Omega\cap\partial^*E\subset\mathrm{Reg}\cap\partial E$. Thus $\mathrm{Reg}\cap\partial E=\Omega\cap\partial^*E$, and, in particular, $\Omega\cap(\partial E\setminus\partial^*E)\subset\Sigma$, so that $\Omega\cap(\partial E\setminus\partial^*E)$ has empty interior in K. Moreover, by (4.26), (4.24) implies that the graph of u has constant mean curvature in $B_{r_x/2}(x)$, and thus that ∂^*E is a smooth hypersurface, see e.g. [GM05, Section 8.2]. Finally, (4.24) implies that $K\setminus\partial E$ is the support of a multiplicity one stationary varifold in the open set $\Omega\setminus\partial E$, so that $K\setminus(\Sigma\cup\partial E)$ is a smooth hypersurface with zero mean curvature, and $\mathcal{H}^n(\Sigma\setminus\partial E)=0$. The proof of Theorem 1.6 is complete. □

5. Convergence to Plateau's problem: Proof of Theorem 1.9

This section is devoted to showing that $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ and that a sequence $\{(K_h, E_h)\}_h$ of generalized minimizers for $\psi(\varepsilon_h)$ with $\varepsilon_h \to 0^+$ as $h \to \infty$ has to converge to a minimizer S for Plateau's problem ℓ counted with multiplicity 2 in the sense of Radon measures. If one could prove the latter assertion directly, then the former would follow at once by lower semicontinuity of weak-star converging Radon measures and by the upper bound $\psi(\varepsilon) \le 2\,\ell + C\,\varepsilon^{n/(n+1)}$ proved in (3.2). A possible direct approach to the convergence of (K_h, E_h) to a minimizer of Plateau's problem may be tried using White's compactness theorem [Whi09]. That would require proving an L^1-bound on the first variations of the varifolds V_h supported on K_h with density 1 on $\Omega \cap \partial^* E_h$ and with density 2 on $K_h \setminus \partial^* E_h$. The validity of such bound is supported by the analysis of simple examples like Example 1.1 and Example 1.2. However, Example 1.2 also indicates that when singularities are present in the limit Plateau minimizers S, then an L^1-bound for the mean curvatures of the varifolds V_h would result from a quantitative balance between the rate of divergence towards $-\infty$ of the constant mean curvatures of the reduced boundaries $\partial^* E_h$, and the rate of vanishing of the areas $\mathcal{H}^n(\Omega \cap \partial^* E_h)$. Validating a quantitative analysis of this kind in some generality would be of course very interesting per se as a way to describe the behavior of generalized minimizers; nonetheless, completing this analysis has so far eluded our attempts. Coming back to the proof of Theorem 1.9, we adopt a different approach. We prove directly that $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ by exploiting the same "compactness-by-comparison" strategy adopted in the proof of Theorem 1.4. An interesting point here is that because $|E_h| = \varepsilon_h \to 0^+$, we do not have a limit set that we can use to uniformly adjust volumes among local competitors of the elements of the minimizing sequence, and have to use a sort of "absolute minimality at vanishing volumes" of any sequence $\{(K_h, E_h)\}_h$ of generalized minimizers such that $\lim_{h\to\infty} \mathcal{F}(K_h, E_h)$ is equal to $\liminf_{\varepsilon\to 0^+} \psi(\varepsilon)$.

Proof of Theorem 1.9. Step one: We start proving that ψ is lower semicontinuous on $(0, \infty)$. Given $\varepsilon_0 > 0$, let $\varepsilon_j \to \varepsilon_0 > 0$ as $j \to \infty$ be such that

$$\lim_{j\to\infty} \psi(\varepsilon_j) = \liminf_{\varepsilon\to\varepsilon_0} \psi(\varepsilon)\,,$$

and let $E_j \in \mathcal{E}$ be such that $|E_j| = \varepsilon_j$ and $\mathcal{H}^n(\Omega \cap \partial E_j) \le \psi(\varepsilon_j) + 1/j$. By (3.2), $\psi(\varepsilon_j)$ is bounded in j, and thus by the compactness criteria for sets of finite perimeter and for Radon measures we have that, up to extracting subsequences, $\mu_j = \mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \stackrel{*}{\rightharpoonup} \mu$ as Radon measures in Ω and $E_j \to E$ in $L^1_{\rm loc}(\Omega)$, where μ is a Radon measure in Ω, and where $E \subset \Omega$ is a set of finite perimeter. We now repeat the proof of Theorem 1.4, with the only difference that while $|E_j|$ was constant in that proof, we know have that $|E_j| = \varepsilon_j \to \varepsilon_0$ for some $\varepsilon_0 > 0$. The modifications are minimal. In step two (nucleation of the sequence E_j), we repeat *verbatim* the argument, using the facts that $|E_j| \ge \varepsilon_0/2$ and that $\mathcal{H}^n(\Omega \cap \partial E_j) \le 2\,\ell + C\,\varepsilon_0^{n/(n-1)} + 1$ in place of $|E_j| = \varepsilon$ and $\mathcal{H}^n(\Omega \cap \partial E_j) \le \psi(\varepsilon) + 1$. Based on step two, in step three we construct volume-fixing variations with uniform constant ε_* and C_*, and then repeat the rest of the argument without modifications. As a consequence, we can show that $\mu = \theta\,\mathcal{H}^n \llcorner K$ and $(K, E) \in \mathcal{K}$ is a generalized minimizer of $\psi(\varepsilon_0)$, with

$$\psi(\varepsilon_0) = \mu(\Omega) = \lim_{j\to\infty} \mu_j(\Omega) \le \lim_{j\to\infty} \psi(\varepsilon_j) = \liminf_{\varepsilon\to\varepsilon_0} \psi(\varepsilon)\,,$$

as claimed. The key information here is of course that $|E_j| \ge \varepsilon_0/2$ where $\varepsilon_0 > 0$. If $\varepsilon_0 = 0$, then the nucleation lemma is inconsequential, and the argument cannot be used.

Step two: Thanks to (3.2), to prove $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ we just need to show that

$$\liminf_{\varepsilon\to 0^+} \psi(\varepsilon) \ge 2\,\ell\,. \tag{5.1}$$

To this end, we pick a sequence $\varepsilon_h \to 0^+$ such that

$$\liminf_{\varepsilon\to 0^+} \psi(\varepsilon) = \lim_{h\to\infty} \psi(\varepsilon_h)\,.$$

Notice that, in this way, given an arbitrary sequence $\sigma_h \to 0^+$, we have

$$\limsup_{h\to\infty} [\psi(\varepsilon_h) - \psi(\sigma_h)] \le 0\,. \tag{5.2}$$

Let $\{E_{h,j}\}_j$ be a minimizing sequence in $\psi(\varepsilon_h)$. By Theorem 1.4, there exists a generalized minimizer (K_h, E_h) in $\psi(\varepsilon_h)$ such that, up to extracting subsequences,

$$\begin{aligned}
&E_{h,j} \to E_h && \text{in } L^1(\Omega) \text{ as } j\to\infty\,,\\
&\mu_{h,j} = \mathcal{H}^n \llcorner (\Omega\cap\partial E_{h,j}) \overset{*}{\rightharpoonup} \mu_h && \text{as Radon measures in } \Omega \text{ as } j\to\infty\,,\\
&|E_{h,j}| = \varepsilon_h \text{ and } \mathcal{H}^n(\Omega\cap\partial E_{h,j}) \le \psi(\varepsilon_h) + \frac{1}{j}\,, && \forall j\in\mathbb{N}\,,
\end{aligned}$$

where, by (3.2) and up to extracting a further subsequence,

$$\mu_h = 2\,\mathcal{H}^n\llcorner(K_h\setminus\partial^* E_h) + \mathcal{H}^n\llcorner(\Omega\cap\partial^* E_h) \overset{*}{\rightharpoonup} \mu \qquad \text{as Radon measures in } \Omega \tag{5.3}$$

for some Radon measure μ in Ω. Given $x\in\Omega\cap\operatorname{spt}\mu$, we set $d(x) = \operatorname{dist}(x,\partial\Omega)$, and let

$$H_{x,r} = \big\{h\in\mathbb{N} : |E_h\setminus B_r(x)| > 0\big\}\,, \qquad I_x = \big\{r\in(0,d(x)) : H_{x,r} \text{ is infinite}\big\}\,. \tag{5.4}$$

We now look at local variations $F_{h,j}$ of $E_{h,j}$ such that $|F_{h,j}|$ has a positive limit volume σ_h as $j\to\infty$, which in turn satisfies $\sigma_h\to 0^+$ as $h\to\infty$. The idea is that, by (5.2), we will be able to use such variations to gather information on μ.

Claim: for every $r\in I_x$, if $\{F_{h,j}\}_{h\in H_{x,r},\,j\in\mathbb{N}} \subset \mathcal{E}$ is such that $\Omega\cap\partial F_{h,j}$ is $\mathcal{C}$-spanning W and $F_{h,j}\Delta E_{h,j}\subset \operatorname{cl}(B_r(x))$ for every $h\in H_{x,r}$ and every $j\in\mathbb{N}$, and if

$$\exists\ \sigma_h = \lim_{j\to\infty} |F_{h,j}| > 0\,, \qquad \text{and} \qquad \lim_{h\in H_{x,r},\,h\to\infty} \sigma_h = 0\,, \tag{5.5}$$

then

$$\mu(B_r(x)) \le \liminf_{h\in H_{x,r},\,h\to\infty}\ \liminf_{j\to\infty} \mathcal{H}^n\big(\operatorname{cl}(B_r(x))\cap\partial F_{h,j}\big)\,. \tag{5.6}$$

To prove this claim, we first notice that, for every $h\in H_{x,r}$,

$$\sigma_h = \lim_{j\to\infty} |F_{h,j}| \ge |E_h\setminus B_r(x)| > 0\,. \tag{5.7}$$

In particular, for j large enough, $|F_{h,j}|>0$, $\psi(|F_{h,j}|)$ is well-defined, and $F_{h,j}$ is a competitor for $\psi(|F_{h,j}|)$, so that

$$\begin{aligned}
\psi\big(|F_{h,j}|\big) &\le \mathcal{H}^n(\Omega\cap\partial F_{h,j}) = \mathcal{H}^n(\operatorname{cl}(B_r(x))\cap\partial F_{h,j}) + \mathcal{H}^n(\partial E_{h,j}\cap\Omega\setminus\operatorname{cl}(B_r(x)))\\
&\le \mathcal{H}^n(\operatorname{cl}(B_r(x))\cap\partial F_{h,j}) + \psi(\varepsilon_h) + \frac{1}{j} - \mathcal{H}^n(\partial E_{h,j}\cap B_r(x))
\end{aligned}$$

which can be recombined into

$$\mu_{h,j}(B_r(x)) \le \mathcal{H}^n(\operatorname{cl}(B_r(x))\cap\partial F_{h,j}) + \psi(\varepsilon_h) - \psi\big(|F_{h,j}|\big) + \frac{1}{j}\,.$$

Letting $j\to\infty$, by $\mu_{h,j}\overset{*}{\rightharpoonup}\mu_h$, $|F_{h,j}|\to\sigma_h>0$, and the lower semicontinuity of ψ on $(0,\infty)$, we find that

$$\mu_h(B_r(x)) \le \liminf_{j\to\infty} \mathcal{H}^n(\operatorname{cl}(B_r(x))\cap\partial F_{h,j}) + \psi(\varepsilon_h) - \psi(\sigma_h)\,.$$

Since $\sigma_h\to 0^+$ as $h\to\infty$ with $h\in H_{x,r}$, by $\mu_h\overset{*}{\rightharpoonup}\mu$ and (5.2) we deduce (5.6), and thus prove the claim.

Step three: We now fix $x \in \mathrm{spt}\mu$, set $f(r) = \mu(B_r(x))$, and prove that, for a.e. $r \in I_x$,

$$\begin{cases} \text{either } f'(r) \geq c(n)\, r^{n-1}\,, \\ \text{or } (f^{1/n})'(r) \geq c(n)\,, \end{cases} \tag{5.8}$$

$$f(r) \leq \frac{r}{n}\, f'(r)\,. \tag{5.9}$$

By using the coarea formula together with $|E_h| \to 0$ as $h \to \infty$ and $E_{h,j} \to E_h$ as $j \to \infty$, we find that for a.e. $r < d(x)$,

$$\partial E_{h,j} \cap \partial B_r(x) \text{ is } \mathcal{H}^{n-1}\text{-rectifiable}\,, \tag{5.10}$$

$$\lim_{j\to\infty} \mathcal{H}^n(E_{h,j} \cap \partial B_r(x)) = \mathcal{H}^n(E_h \cap \partial B_r(x))\,, \tag{5.11}$$

$$\lim_{h\to\infty}\lim_{j\to\infty} \mathcal{H}^n(E_{h,j} \cap \partial B_r(x)) = 0\,, \tag{5.12}$$

for every $h, j \in \mathbb{N}$. Moreover, if we set

$$f_{h,j}(r) = \mu_{h,j}(B_r(x))\,, \qquad f_h(r) = \mu_h(B_r(x))\,.$$

then, again by the coarea formula and by Fatou's lemma, for a.e. $r < d(x)$ we find

$$\begin{aligned} \mathcal{H}^{n-1}(\partial E_{h,j} \cap \partial B_r(x)) &\leq f'_{h,j}(r)\,, \\ g_h(r) = \liminf_{j\to\infty} f'_{h,j}(r) &\leq f'_h(r)\,, \\ g(r) = \liminf_{h\in H_{x,r}, h\to\infty} f'_h(r) &\leq f'(r)\,, \end{aligned} \tag{5.13}$$

for every $h, j \in \mathbb{N}$. We first prove (5.8). Let $r \in I_x$ be such that (5.10), (5.11), (5.12) and (5.13) hold, and let $A_{h,j}$ denote an $\mathcal{H}^n$-maximal connected component of $\partial B_r(x) \setminus \partial E_{h,j}$. If $A_{h,j} \subset E_{h,j}$, then, by spherical isoperimetry, by (5.13), and since the relative boundary to $A_{h,j}$ in $\partial B_r(x)$ is contained in $\partial B_r(x) \cap \partial E_{h,j}$, we find

$$f'_{h,j}(r) \geq c(n)\, \mathcal{H}^n(\partial B_r(x) \setminus A_{h,j})^{(n-1)/n}\,,$$

where the lower bound converges to $c(n)\, r^{n-1}$ if we let first $j \to \infty$ and then $h \to \infty$ thanks to (5.12); hence, if $A_{h,j} \subset E_{h,j}$, the first alternative in (5.8) holds. We now assume that $A_{h,j} \cap E_{h,j} = \emptyset$, and consider the corresponding cup competitor $F_{h,j}$ as defined in Lemma 2.5 starting from $E_{h,j}$, $A_{h,j}$. More precisely, if $\{\eta_k^{h,j}\}_{k=1}^\infty$ denotes the corresponding sequence as in (2.49), we choose $k(h,j)$ so that, setting

$$\begin{aligned} Y_{h,j} &= \partial B_r(x) \setminus \mathrm{cl}\,((E_{h,j} \cap \partial B_r(x)) \cup A_{h,j})\,, \\ S_{h,j} &= \partial E_{h,j} \cap \mathrm{cl}\,(A_{h,j}) \setminus (\mathrm{cl}\,((E_{h,j} \cap \partial B_r(x)) \cup Y_{h,j}))\,, \end{aligned}$$

we have that $\eta_j = \eta_{k(h,j)}^{h,j}$ satisfies $\eta_j \leq r/2j$, with

$$\mathcal{H}^n(\partial B_r(x) \cap \{\mathrm{d}_{S_{h,j}} \leq \eta_j\}) \leq \frac{1}{j}\,, \tag{5.14}$$

$$\eta_j\, \mathcal{H}^{n-1}(\partial B_r(x) \cap \{\mathrm{d}_{S_{h,j}} = \eta_j\}) \leq \frac{1}{j}\,. \tag{5.15}$$

Then, with the usual notation

$$U_{h,j} = \partial B_r(x) \cap \{\mathrm{d}_{S_{h,j}} < \eta_j\}\,, \qquad Z_{h,j} = Y_{h,j} \cup (U_{h,j} \setminus \mathrm{cl}\,(E_{h,j} \cap \partial B_r(x)))\,,$$

we define

$$F_{h,j} = \big(E_{h,j} \setminus \mathrm{cl}\,(B_r(x))\big) \cup N_{\eta_j}(Z_{h,j})\,.$$

By Lemma 2.5, $F_{h,j} \in \mathcal{E}$, $\Omega \cap \partial F_{h,j}$ is $\mathcal{C}$-spanning W and $E_{h,j} \Delta F_{h,j} \subset \mathrm{cl}\,(B_r(x))$. Since $\eta_j \to 0$ as $j \to \infty$, we find

$$\sigma_h = \lim_{j\to\infty} |F_{h,j}| = \lim_{j\to\infty} |E_{h,j} \setminus B_r(x)| = |E_h \setminus B_r(x)|\,,$$

so that $\sigma_h > 0$ if $h \in H_{x,r}$, and $\sigma_h \to 0^+$ if we let $h \to \infty$. Thus $F_{h,j}$ satisfies (5.5), and we can apply (5.6) to $F_{h,j}$. To estimate the upper bound in (5.6), we look back at (2.40), (2.43), (2.44), and (2.47), and find that

$$
\begin{aligned}
\mathcal{H}^n(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}) \le{}& (2 + C(n)\,\eta_j)\,\mathcal{H}^n(\partial B_r(x) \setminus A_{h,j}) \\
&+ (2 + C(n)\,\eta_j)\,\mathcal{H}^n(\partial B_r(x) \cap \{\mathrm{d}_{S_{h,j}} \le \eta_j\}) \\
&+ C(n)\,\eta_j\,\big(\mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_{h,j}) + \mathcal{H}^{n-1}(\partial B_r(x) \cap \{\mathrm{d}_{S_{h,j}} = \eta_j\})\big)\,.
\end{aligned}
\tag{5.16}
$$

By (5.6), (5.14), (5.15), and (5.16) we deduce that

$$
\begin{aligned}
f(r) = \mu(B_r(x)) &\le \liminf_{h \in H_{x,r},\, h \to \infty} \liminf_{j \to \infty} \mathcal{H}^n\big(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}\big) \\
&\le \liminf_{h \in H_{x,r},\, h \to \infty} \liminf_{j \to \infty} 2\,\mathcal{H}^n(\partial B_r(x) \setminus A_{h,j}) \qquad (5.17)\\
&\le C(n) \liminf_{h \in H_{x,r},\, h \to \infty} \liminf_{j \to \infty} f_{h,j}'(r)^{n/(n-1)} \le C(n)\, f'(r)^{n/(n-1)}\,,
\end{aligned}
$$

We have thus proved that the second alternative in (5.8) holds, as claimed. We now prove (5.9): let us now denote by $F_{h,j}$ the set defined by Lemma 2.9 as approximation of the cone competitor corresponding to $E_{h,j}$ in $B_r(x)$ with $\eta = \eta_j = r/2j$. We have that $F_{h,j} \in \mathcal{E}$ and that $\Omega \cap \partial F_{h,j}$ is $\mathcal{C}$-spanning W; furthermore, by (2.72) and (5.11) we find

$$
\sigma_h = \lim_{j \to \infty} |F_{h,j}| \ge |E_h \setminus B_r(x)| + \frac{r}{n+1}\,\mathcal{H}^n(E_h \cap \partial B_r(x))
$$

(in particular, $\sigma_h > 0$ if $h \in H_{x,r}$) and, by (5.12), $\sigma_h \to 0^+$ as $h \to \infty$. Thus (5.5) holds, and we can deduce from (5.6) and (2.71) that

$$
f(r) = \mu(B_r(x)) \le \liminf_{h \in H_{x,r},\, h \to \infty} \liminf_{j \to \infty} \frac{r}{n}\,\mathcal{H}^{n-1}(\partial E_{h,j} \cap \partial B_r(x)) \le \frac{r}{n}\, f'(r)\,,
$$

that is (5.9).

Step four: We now define a function $g : \Omega \to (0, \infty) \cup \{-\infty\}$ by letting

$$
\begin{aligned}
g(x) &= \sup\Big\{s > 0 : (0, s) \subset I_x\Big\} \\
&= \sup\Big\{t > 0 : \text{if } s < t, \text{ then } |E_h \setminus B_s(x)| > 0 \text{ for infinitely many } h\Big\}.
\end{aligned}
$$

We notice that

$$
g \text{ is lower semicontinuous on } \Omega\,, \tag{5.18}
$$

$$
\{g = -\infty\} \text{ contains at most one point}\,. \tag{5.19}
$$

(Notice that $\{g = -\infty\}$ may indeed contain one point: this is the case of the singular point of a triple junction, see Figure 1.3-(b)). To prove (5.18): if $g(x) \ne -\infty$, then $g(x) > 0$, and for every $s \in (0, g(x))$, $|E_h \setminus B_s(x)| > 0$ for infinitely many h. Thus, if $\eta \in (0, g(x))$ and m_η is such that $|x - x_m| < \eta$ for every $m \ge m_\eta$, then, for every $m \ge m_\eta$ and $s \in (0, g(x) - \eta)$,

$$
|E_h \setminus B_s(x_m)| \ge |E_h \setminus B_{s+\eta}(x)| > 0\,, \qquad \text{for intinitely many } h\,,
$$

that is $g(x) - \eta \le g(x_m)$ for every $m \ge m_\eta$; this proves (5.18). Next, if $g(x_1) = g(x_2) = -\infty$, then for every $s > 0$ there exists $h(s)$ such that

$$
|E_h \setminus B_s(x_1)| = |E_h \setminus B_s(x_2)| = 0 \qquad \forall h \ge h(s)\,.
$$

If $x_1 \ne x_2$ we can take $s = |x_1 - x_2|/2$ and deduce $|E_h| = 0$; thus (5.19) holds. Let us now consider the open set $\{g > s\} \subset \Omega$, $s > 0$, and set

$$
Z(s) = \mathrm{spt}\mu \cap \{g > s\}\,, \qquad Z = \mathrm{spt}\mu \cap \{g > 0\}\,.
$$

We claim that if $x \in Z(s)$, then

$$f(r) \geq c_0(n)\, r^n \qquad \forall r \in (0,s)\,, \qquad r^{-n}\, f(r) \text{ is increasing over } r \in (0,s)\,. \tag{5.20}$$

The second assertion is immediate from (5.9). To prove the first one, set

$$L_1 = \big\{ r \in (0,s) : f'(r) \geq c(n)\, r^{n-1} \big\}\,, \qquad L_2 = (0,s) \setminus L_1\,,$$

with $c(n)$ as in (5.8). If $x \in Z(s)$ is such that $\mathcal{H}^1(L_1) \geq s/2$, then for every $r \in (0,s)$

$$f(r) \geq \int_{L_1 \cap (0,r)} f' \geq c(n) \int_{L_1 \cap (0,r)} t^{n-1}\, dt \geq c(n) \int_0^{\min\{r,s/2\}} t^{n-1}\, dt \geq \frac{c(n)}{n\, 2^n}\, r^n\,;$$

if instead $\mathcal{H}^1(L_2) \geq s/2$, then for every $r \in (0,s)$,

$$f(r)^{1/n} \geq \int_{L_2 \cap (0,r)} (f^{1/n})' \geq c(n)\, \mathcal{H}^1(L_2 \cap (0,r)) \geq c(n)\, \min\big\{ r, \frac{s}{2} \big\} \geq \frac{c(n)}{2}\, r\,,$$

where we have used the fact that, by (5.8), we have $(f^{1/n})' \geq c(n)$ on L_2. Thanks to (5.20) we are in the position of using [Mat95, Theorem 6.9] and Preiss' theorem (as done in step four of the proof of Theorem 1.4) on each $Z(s)$, to find that Z is $\mathcal{H}^n$-rectifiable with

$$\mu \llcorner Z = \theta\, \mathcal{H}^n \llcorner Z\,, \tag{5.21}$$

where the density

$$\theta(x) = \lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_n\, r^n} \quad \text{exists in } [c_0(n), \infty) \text{ for every } x \in Z\,.$$

Moreover, by (5.19),

$$\mathcal{H}^0(\mathrm{spt}\mu \setminus Z) \leq 1\,. \tag{5.22}$$

By combining (5.21) and (5.22) we find that $K = \mathrm{spt}\mu$ is $\mathcal{H}^n$-rectifiable and such that $\mu = \theta\, \mathcal{H}^n \llcorner K$. Since $K_h = \mathrm{spt}\, \mu_h$ is $\mathcal{C}$-spanning W and $\mu_h \overset{*}{\rightharpoonup} \mu$, by Lemma 2.1 we find that K is $\mathcal{C}$-spanning W, and thus admissible in ℓ, so that

$$\liminf_{\varepsilon \to 0^+} \psi(\varepsilon) = \lim_{h \to \infty} \mu_h(\Omega) \geq \mu(\Omega) = \int_K \theta\, d\mathcal{H}^n \geq \min_K \theta\, \mathcal{H}^n(K) \geq \ell\, \min_K \theta\,. \tag{5.23}$$

Thus, to complete the proof of (5.1) we just need to show that

$$\theta \geq 2\ \mathcal{H}^n\text{-a.e. on } K\,. \tag{5.24}$$

Since $\mu_{h,j} = \mathcal{H}^n \llcorner (\Omega \cap \partial E_{h,j}) \overset{*}{\rightharpoonup} \mu_h$ as $j \to \infty$, with $\mu_h \overset{*}{\rightharpoonup} \theta\, \mathcal{H}^n \llcorner K$ as $h \to \infty$, we can extract a diagonal subsequence $j = j(h)$ so that, denoting $E_h^* = E_{h,j(h)}$, $\{E_h^*\}_h \subset \mathcal{E}$, $\Omega \cap \partial E_h^*$ $\mathcal{C}$-spanning W, and

$$\mu_h^* = \mathcal{H}^n \llcorner (\Omega \cap \partial E_h^*) \overset{*}{\rightharpoonup} \theta\, \mathcal{H}^n \llcorner K\,, \qquad \text{as } h \to \infty\,.$$

Moreover, $\mu(B_r(x)) \geq c(n)\, r^n$ for every $r \in (0,s)$ if $x \in K \cap \{g > s\}$ and, thanks to (5.16),

$$\liminf_{h \to \infty} \mathcal{H}^n(B_r(x) \cap \partial E_h^*) \leq C(n)\, \liminf_{h \to \infty}\, \mathcal{H}^n(\partial B_r(x) \setminus A^0_{r,h})\,,$$

where $A^0_{r,h}$ denotes an $\mathcal{H}^n$-maximal connected component of $\partial B_r(x) \setminus \partial E_h^*$, this time for every $x \in K$ and $B_r(x) \subset\subset \Omega$. We can thus apply Lemma 2.11 with the open set $\Omega' = \{g > s\}$ to deduce that

$$\theta \geq 2\ \mathcal{H}^n\text{-a.e. on } \{g > s\} \cap K \setminus \partial^* E^*$$

where $E^* = \emptyset$ is the L^1-limit of the sets E_h^*. Since $\partial^* E^* = \emptyset$, taking the union over $s > 0$ and recalling (5.22), we conclude that (5.24) holds.

Step five: Now that $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ has been proved, let (K_h, E_h) be a sequence of generalized minimizers of $\psi(\varepsilon_h)$ for an arbitrary sequence $\varepsilon_h \to 0^+$. Since the limit of $\psi(\varepsilon)$ as $\varepsilon \to 0^+$ exists, ε_h automatically satisfies (5.2), and the arguments of step two to four can be repeated *verbatim*. Correspondingly, up to extracting subsequences, (5.3) holds with $\mu = \theta\,\mathcal{H}^n \llcorner K$, $\theta \ge 2$ $\mathcal{H}^n$-a.e. on K, and K a relatively compact subset of Ω, $\mathcal{H}^n$-rectifiable, and $\mathcal{C}$-spanning W. By plugging $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ in (5.23), we find that $\theta = 2$ $\mathcal{H}^n$-a.e. on K, $2\,\mathcal{H}^n(K) = 2\,\ell$, so that K is a minimizer of ℓ, and thus, looking back at (5.3), we conclude that (1.18) holds. □

Appendix A. A technical fact on sets of finite perimeter

Lemma A.1. *If Ω is an open set in $\mathbb{R}^{n+1}$, E is a set of finite perimeter in Ω, and $f : \Omega \to \Omega$ is a diffeomorphism, then $f(E)$ is a set of finite perimeter in Ω with $\partial^* f(E) = f(\partial^* E)$ and*

$$\nu_{f(E)}(f(x)) = \frac{(\nabla g(f(x)))^T \nu_E(x)}{|(\nabla g(f(x)))^T \nu_E(x)|} \qquad \forall x \in \partial^* E\,, \tag{A.1}$$

where $g = f^{-1}$.

Proof. In [Mag12, Proposition 17.1, Remark 17.2] it is shown that $f(E)$ is a set of finite perimeter with

$$\mu_{f(E)} = f_\#\Big(Jf\,(\nabla g(f))^T\,\mu_E\Big)\,.$$

and that mapping by f preserves essential boundaries (thus just the $\mathcal{H}^n$-equivalence of $\partial^* f(E)$ and $f(\partial^* E)$ is deduced there). In order to prove $\partial^* f(E) = f(\partial^* E)$, we pick a ball $B_r(f(x)) \subset\subset \Omega$, and look at

$$\begin{aligned}\frac{\mu_{f(E)}(B_r(f(x)))}{|\mu_{f(E)}|(B_r(f(x)))} &= \frac{\int_{g(B_r(f(x)))\cap\partial^* E} Jf\,\nabla g(f)^T \nu_E\,d\mathcal{H}^n}{\int_{g(B_r(f(x)))\cap\partial^* E} Jf\,|\nabla g(f)^T \nu_E|\,d\mathcal{H}^n} \\ &= \frac{\int_{(\partial^* E - x)/r} u_r(z)\nu_E(x + r\,z)\,d\mathcal{H}^n_z}{\int_{(\partial^* E - x)/r} |u_r(z)\,\nu_E(x + r\,z)|\,d\mathcal{H}^n_z}\,.\end{aligned} \tag{A.2}$$

where we have set

$$F_r = \frac{g(B_r(f(x))) - x}{r}\,, \qquad u_r(z) = 1_{F_r}(z)\,Jf(x + r\,z)\,\nabla g(f(x + r\,z))^T\,.$$

If we set $F = L(B_1(0))$ for the linear map $L = \nabla g(f(x))$, then for every $\varepsilon > 0$ we have

$$L(B_{1-\varepsilon}(0)) \subset F_r \subset L(B_{1+\varepsilon}(0)) \qquad \text{provided } r \text{ is small enough}\,,$$

and thus, as $r \to 0^+$,

$$1_{F_r} \to 1_F \qquad \text{uniformly on } \mathbb{R}^{n+1} \setminus X_\varepsilon\,,$$

where we have set

$$X_\varepsilon = L(B_{1+\varepsilon}(0) \setminus B_{1-\varepsilon}(0))\,.$$

Since $F_r, F \subset B_{\mathrm{Lip} g}(0)$, and since for every $R > 0$

$$Jf(x + r\,z)\,\nabla g(f(x + r\,z))^T \to Jf(x)\,\nabla g(f(x))^T \qquad \text{uniformly on } |z| \le R\,,$$

as $r \to 0^+$, we conclude that

$$u_r(z) \to u(z) := 1_F(z)\,Jf(x)\,\nabla g(f(x))^T \qquad \text{uniformly on } \mathbb{R}^{n+1} \setminus X_\varepsilon\,. \tag{A.3}$$

We now decompose the integrals over $(\partial^* E - x)/r$ appearing in (A.2) through X_ε. By (A.3),

$$\begin{aligned}&\Big|\int_{[(\partial^* E - x)/r]\setminus X_\varepsilon} u_r(z)\nu_E(x + r\,z)\,d\mathcal{H}^n_z - \int_{[(\partial^* E - x)/r]\setminus X_\varepsilon} u(z)\nu_E(x + r\,z)\,d\mathcal{H}^n_z\Big| \\ \le\ &\omega(r)\,\mathcal{H}^n(B_{\mathrm{Lip} g}(0) \cap [(\partial^* E - x)/r] \setminus X_\varepsilon) \\ \le\ &\omega(r)\,P(E; B_{r\,\mathrm{Lip} g}(x)) \to 0\end{aligned}$$

as $r\to 0^+$, while $x\in\partial^*E$ gives

$$\lim_{r\to 0^+}\int_{[(\partial^*E-x)/r]\setminus X_\varepsilon}u(z)\nu_E(x+r\,z)\,d\mathcal{H}^n_z=\int_{T_x(\partial^*E)\setminus X_\varepsilon}u(z)\nu_E(x)\,d\mathcal{H}^n_z\,.$$

At the same time, since $|u_r|\le C$ for a constant C independent of r, we have

$$\Big|\int_{X_\varepsilon\cap[(\partial^*E-x)/r]}u_r(z)\nu_E(x+r\,z)\,d\mathcal{H}^n_z\Big|\le C\,\mathcal{H}^n\Big(X_\varepsilon\cap\frac{\partial^*E-x}{r}\Big)\to C\,\mathcal{H}^n(X_\varepsilon\cap T_x(\partial^*E))$$

as $r\to 0^+$. Combining the above estimates with $|u|\le C$ we finally find

$$\begin{aligned}&\limsup_{r\to 0^+}\Big|\int_{(\partial^*E-x)/r}u_r(z)\nu_E(x+r\,z)\,d\mathcal{H}^n_z-\int_{T_x(\partial^*E)}u(z)\nu_E(x)\,d\mathcal{H}^n_z\Big|\\ \le\;& C\,\mathcal{H}^n(X_\varepsilon\cap T_x(\partial^*E))\,,\qquad\forall\varepsilon>0\,.\end{aligned}$$

Letting $\varepsilon\to 0^+$, we find $\mathcal{H}^n(X_\varepsilon\cap T_x(\partial^*E))\to\mathcal{H}^n(X\cap T_x(\partial^*E))$ where

$$X=L(\partial B_1(0))\,.$$

Since L is invertible, $L(\partial B_1(0))$ intersects transversally any plane through the origin, and in particular $T_x(\partial^*E)$. Therefore $\mathcal{H}^n(X\cap T_x(\partial^*E))=0$ and we have proved

$$\begin{aligned}\lim_{r\to 0^+}\int_{(\partial^*E-x)/r}u_r(z)\nu_E(x+r\,z)\,d\mathcal{H}^n_z&=\int_{T_x(\partial^*E)}u(z)\,\nu_E(x)\,d\mathcal{H}^n_z\\&=Jf(x)\,L^T\,\nu_E(x)\,\mathcal{H}^n(F\cap T_x(\partial^*E))\,.\end{aligned}$$

An analogous argument shows

$$\lim_{r\to 0^+}\int_{(\partial^*E-x)/r}|u_r(z)\nu_E(x+r\,z)|\,d\mathcal{H}^n_z=Jf(x)\,|L^T\,\nu_E(x)|\,\mathcal{H}^n(F\cap T_x(\partial^*E))\,,$$

and finally we conclude that if $x\in\partial^*E$, then

$$\lim_{r\to 0^+}\frac{\mu_{f(E)}(B_r(f(x)))}{|\mu_{f(E)}|(B_r(f(x)))}=\frac{L^T\,\nu_E(x)}{|L^T\,\nu_E(x)|}\in\mathbb{S}^n\,.$$

In particular, $f(x)\in\partial^*f(E)$ and (A.1) holds. □

Appendix B. Boundary density estimates for the Harrison–Pugh minimizers

In this appendix we prove that when ∂W is smooth and $\ell<\infty$, then every minimizer S of ℓ satisfies uniform lower density estimates up to the boundary of Ω.

Theorem B.1. *If $\ell<\infty$, ∂W is smooth, and S is a minimizer of ℓ, then*

$$\mathcal{H}^n(B_r(x)\cap S)\ge c(n)\,r^n\,,\qquad\forall x\in\mathrm{cl}\,(S)\,,r\in(0,r_0)\,,\tag{B.1}$$

for a value of r_0 depending on W.

Proof. By Lemma 2.4, and since S minimizes $\mathcal{H}^n$ with respect to every relatively closed subset of Ω which is $\mathcal{C}$-spanning W, recall (1.8), we have

$$\mathcal{H}^n(S)\le\mathcal{H}^n(f(S))\tag{B.2}$$

whenever $f:\mathrm{cl}\,(\Omega)\to\mathrm{cl}\,(\Omega)$ is a homeomorphism with $f(\partial\Omega)=\partial\Omega$, $\{f\ne\mathrm{id}\,\}\subset\subset B_{r_0}(x)$ for $x\in\partial\Omega$, and $f(B_{r_0}(x)\cap\mathrm{cl}\,(\Omega))=B_{r_0}(x)\cap\mathrm{cl}\,(\Omega)$ for r_0 depending on W. We immediately deduce from (B.2), that

$$\int_S\mathrm{div}\,^SX\,d\mathcal{H}^n=0\tag{B.3}$$

for every $X\in C^1_c(B_{r_0}(x);\mathbb{R}^{n+1})$ with $X\cdot\nu_\Omega=0$ on $\partial\Omega$. Since S is an Almgren minimizer in Ω, (B.3) also holds for every $X\in C^1_c(\Omega;\mathbb{R}^{n+1})$. Finally, we deduce the validity of (B.3) for every $X\in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ with $X\cdot\nu_\Omega=0$ on $\partial\Omega$ by a standard covering argument.

The validity of (B.3) for every $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ with $X\cdot\nu_\Omega = 0$ on $\partial\Omega$ is a distributional formulation of Young's law, which has been extensively studied in the classical work of Grüter and Jost [GJ86], and has been recently extended to arbitrary contact angles by Kagaya and Tonegawa [KT17]. The main consequence of (B.3) we shall need here is an adapted monotonicity formula which takes care of the local geometry of ∂W. We now introduce this tool and then complete the proof.

Let r_0 be sufficiently small, so that $I_{r_0}(\partial W)$ admits a well-defined nearest point projection map $\Pi\colon I_{r_0}(\partial W)\to\partial W$ of class C^1. By [KT17, Theorem 3.2], there exists a constant $C = C(n, r_0)$ such that for any $x \in I_{r_0/6}(\partial W)\cap \mathrm{cl}(\Omega)$ the map

$$r\in(0,r_0/6)\mapsto \frac{\mathcal{H}^n(S\cap B_r(x))+\mathcal{H}^n(S\cap\tilde B_r(x))}{\omega_n\, r^n}\,e^{C\,r} \tag{B.4}$$

is increasing, where

$$\tilde B_r(x) := \left\{y\in\mathbb{R}^{n+1}:\ \tilde y\in B_r(x)\right\}\,,\qquad \tilde y := \Pi(y)+(\Pi(y)-y) \tag{B.5}$$

denotes a sort of nonlinear reflection of $B_r(x)$ across ∂W. In particular, the limit

$$\sigma(x)=\lim_{r\to0^+}\frac{\mathcal{H}^n(S\cap B_r(x))+\mathcal{H}^n(S\cap\tilde B_r(x))}{\omega_n\, r^n} \tag{B.6}$$

exists for every $x\in I_{r_0/6}(\partial W)\cap\mathrm{cl}(\Omega)$, and the map $x\mapsto\sigma(x)$ is upper semicontinuous in there; see [KT17, Corollary 5.1].

Next, we recall from [KT17, Lemma 4.2] a simple geometric fact: if $x\in I_{r_0}(\partial W)$, and $\rho>0$ is such that $\mathrm{dist}(x,\partial W)\le\rho$ and $B_\rho(x)\subset I_{r_0}(\partial W)$, then

$$\tilde B_\rho(x)\subset B_{5\rho}(x)\,. \tag{B.7}$$

We are now in the position to prove (B.1). First of all we recall that, since S defines a multiplicity one stationary varifold in Ω, we have

$$\mathcal{H}^n(S\cap B_r(x))\ge\omega_n\, r^n\,,\qquad \forall x\in S\,, B_r(x)\subset\subset\Omega\,. \tag{B.8}$$

In particular, (B.1) holds with $c=\omega_n$ for all $x\in S\setminus I_{r_0/6}(\partial W)$ as soon as $r<r_0/6$. Therefore we can assume that

$$x\in\mathrm{cl}\,(S)\cap I_{r_0/6}(\partial W)\,. \tag{B.9}$$

We first notice that we have $\sigma(x)\ge1$: by upper semicontinuity of σ on $\mathrm{cl}\,(S)\cap I_{r_0/6}(\partial W)$ we just need to show this when, in addition to (B.9), we have $x\in S$, and indeed in this case,

$$\sigma(x)\ge\lim_{\rho\to0^+}\frac{\mathcal{H}^n(S\cap B_\rho(x))}{\omega_n\,\rho^n}\ge1$$

thanks to (B.8); this proves $\sigma(x)\ge1$. Now we fix $r<5\,r_0/6$ and distinguish two cases depending on the validity of

$$\mathrm{dist}(x,\partial W)>\frac{r}{5}\,. \tag{B.10}$$

If (B.10) holds, then by (B.8)

$$\mathcal{H}^n(S\cap B_r(x))\ge\mathcal{H}^n(S\cap B_{r/5}(x))\ge\omega_n\left(\frac{r}{5}\right)^n,$$

thus proving (B.1). If $\mathrm{dist}(x,\partial W)\le r/5$, then, thanks to the obvious inclusion $B_r(x)\subset I_{r_0}(\partial W)$, we can apply (B.7) with $\rho=r/5$ to find $\tilde B_{r/5}(x)\subset B_r(x)$. In this way, by exploiting $\sigma(x)\geq 1$ and (B.4), we get

$$\begin{aligned} c(n)\,r^n &\leq \sigma(x)\,\omega_n\left(\frac{r}{5}\right)^n \\ &\leq \Big(\mathcal{H}^n(S\cap B_{r/5}(x))+\mathcal{H}^n(S\cap\tilde{B}_{r/5}(x))\Big)\,e^{C\,r/5} \\ &\leq 2\,\mathcal{H}^n(S\cap B_r(x))\,e^{C\,r_0}\leq 4\,\mathcal{H}^n(S\cap B_r(x)) \end{aligned}$$

up to further decreasing r_0. □

Appendix C. A classical variational argument

Let (K,E) be a generalized minimizer of $\psi(\varepsilon)$. In Theorem 1.6, we have proved that if $f:\Omega\to\Omega$ is a diffeomorphism such that $|f(E)|=|E|$, then

$$\mathcal{F}(K,E)\leq\mathcal{F}(f(K),f(E))\,. \tag{C.1}$$

Here we show how to deduce from (C.1) the existence of $\lambda\in\mathbb{R}$ such that

$$\lambda\int_{\partial^*E}X\cdot\nu_E\,d\mathcal{H}^n=\int_{\partial^*E}\operatorname{div}^K X\,d\mathcal{H}^n+2\int_{K\setminus\partial^*E}\operatorname{div}^K X\,d\mathcal{H}^n\,, \tag{C.2}$$

for every $X\in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ with $X\cdot\nu_\Omega=0$ on $\partial\Omega$. This is proved following a classical argument, see e.g. [Mag12, Theorem 17.20]. We first treat the case when we also have

$$\int_{\partial^*E}X\cdot\nu_E\,d\mathcal{H}^n=0\,. \tag{C.3}$$

In this case, let $Y\in C^1_c(\Omega;\mathbb{R}^{n+1})$ be such that

$$\int_{\partial^*E}Y\cdot\nu_E\,d\mathcal{H}^n=1\,,$$

and set

$$f_{t,s}(x)=x+t\,X(x)+s\,Y(x)\,,\qquad x\in\Omega\,.$$

Given that $X\cdot\nu_\Omega=0$ on $\partial\Omega$ and that $\partial\Omega$ is smooth, it is easily seen that for t and s sufficiently small, $f_{t,s}$ is a diffeomorphism from Ω to Ω. In particular, the map

$$\varphi(t,s)=|f_{t,s}(E)|$$

is such that $\varphi(0,0)=|E|$, $(\partial\varphi/\partial t)(0,0)=0$ by (C.3) and $(\partial\varphi/\partial s)(0,0)=1$ by the assumption on Y, so that, by the implicit function theorem we have $\varphi(t,s(t))=|E|$ for every t sufficiently small and for $s(t)=\mathrm{O}(t^2)$. Setting $g_t=f_{t,s(t)}$, by (C.1), we find that

$$m(t)=2\,\mathcal{H}^n(g_t(K)\setminus\partial^*g_t(E))+\mathcal{H}^n(\Omega\cap\partial^*g_t(E))$$

has a minimum at $t=0$. By Lemma A.1, we can write

$$m(t)=2\,\mathcal{H}^n(g_t(K\setminus\partial^*E))+\mathcal{H}^n(g_t(\Omega\cap\partial^*E))\,.$$

By the area formula, and since $s(t)=\mathrm{O}(t^2)$ gives $(\partial g_t/\partial t)|_{t=0}=X$, we deduce the validity of (C.2) when (C.3) holds. Let us now consider two fields $X_k\in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, $k=1,2$, with $X_k\cdot\nu_\Omega=0$ on $\partial\Omega$ and set

$$X=X_1-\frac{\int_{\partial^*E}X_1\cdot\nu_E\,d\mathcal{H}^n}{\int_{\partial^*E}X_2\cdot\nu_E\,d\mathcal{H}^n}\,X_2\,.$$

In this way X satisfies (C.3), and thus (C.2); as a consequence the quantity

$$\frac{\int_{\partial^*E}\operatorname{div}^K X_k\,d\mathcal{H}^n+2\int_{K\setminus\partial^*E}\operatorname{div}^K X_k\,d\mathcal{H}^n}{\int_{\partial^*E}X_k\cdot\nu_E\,d\mathcal{H}^n}$$

has the same value for $k=1,2$.

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