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Mixed graded modules in homotopy Lie theory

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El círculo del cielo mide mi gloria,
las bibliotecas del Oriente se disputan
mis versos,
los emires me buscan para llenarme de
oro la boca,
los ángeles ya saben de memoria mi
último zéjel.
Mis instrumentos de trabajo son la
humillación y la angustia;
ojalá yo hubiera nacido muerto.

Jorge Luis Borges

This thesis is dedicated to everybody who, for the whole duration of my PhD or just for some time of it, has been part of my life in the last three years and a half. Without my parents, my sister, my close relatives, all my friends, and many other people that *this margin is too narrow to contain*, I could have not come this far in the first place.

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Introduction

Historical framework and motivations

Why derived algebraic geometry?

Derived algebraic geometry has provided the theoretic framework and the technical formalism to treat many "singular" phenomena of algebraic geometry. While being a relatively modern area of mathematics (the first systematic treatment of the theoretic foundations is laid in [TV05; TV08]) its main ideas and motivations date back to the half of the last century. Its origins are usually traced back to Serre's celebrated intersection formula ([Ser65]): let Y and Z be two closed subschemes of an ambient space X, cut out by two sheaves of ideals $\mathcal{I}_Y$ and $\mathcal{I}_Z$, respectively. Then the intersection multiplicity at a point x in the set-theoretic intersection $Y \cap X$ is computed by the Euler characteristic

$$m := m(x, Y, Z) = \sum_{i \geqslant 0} (-1)^i \operatorname{length}_{\mathcal{O}_{X,x}} \left(\operatorname{Tor}_i^{\mathcal{O}_X} (\mathcal{O}_X / \mathcal{I}_Y,\ \mathcal{O}_X / \mathcal{I}_Z) \right).$$

The groundbreaking idea of this formula is that, unless Y and Z are both smooth and meet transversally, the merely scheme-theoretic data are not enough to compute the multiplicity, which takes into account some higher order corrections of homological nature.

However, one of the problems in algebraic geometry that most crucially inspired the development of *derived* ideas and techniques is *deformation theory,* and in particular the theory of formal moduli problems. First, Kodaira and Spencer ([KS58]) proved a relation between first order deformations of smooth projective complex manifolds and first homology classes in classes with coefficients in the holomorphic tangent sheaf T_X in $\mathrm{H}^1(X,\ \mathrm{T}_X)$; later, following the philosophy that the tangent sheaf should control deformation and obstruction theory, Quillen developed a homology theory for commutative rings (*André-Quillen homology*) in [Qui70], which was later globalized in [Ill71; Ill72], replacing the module of Kähler differentials with a more well-behaved object: the *cotangent complex*. The key idea is the following: given an affine covering $\{\operatorname{Spec}(R_i)\}_i$ of a scheme X, one approximates every commutative ring R_i by a simplicial diagram of smooth commutative rings with an augmentation morphism $R_{\bullet,i} \to R_i$ which provides a homotopy equivalence (as a simplicial set) between $R_{\bullet,i}$ and R_i, seen as a constant simplicial diagram of commutative rings. Such simplicial commutative rings $R_{\bullet,i}$

were then glued along intersections via simplicial resolutions involving infinite dimensional polynomial algebras, yielding a sheaf of simplicial commutative rings over X.

This idea was not only the first step towards considering *derived schemes*, but also provided a strong insight into deformation theory. In particular, for a $\Bbbk$-algebra A, André-Quillen homology yields an isomorphism of A-modules

$$\mathrm{Ext}^i_A\left(\mathbb{L}_{A/\Bbbk},\ M\right) \cong \mathrm{Hom}_{\mathrm{h}\left(\mathrm{CAlg}_{\Bbbk}^{\Delta^{\mathrm{op}}}\right)}(A,\ A \oplus M[i]).$$

Here, $\mathbb{L}_{A/\Bbbk}$ is the cotangent complex of A, M is an A-module, $A \oplus M[i]$ is the square-zero extension of A by the simplicial A-module $M[i]$ corresponding to the Eilenberg-Maclane space $K(M,i)$, and the Hom-set on the right hand side is the module of maps in the homotopy category of simplicial commutative $\Bbbk$-algebras.

The close relationship between derived algebraic geometry and deformation theory was then hinted in the now famous letter by Drinfeld to Schechtman ([Dri14]), from which stem the whole branch of *derived deformation theory* - including, of course, the work of this thesis as well. In this letter, Drinfeld suggested the following principle: let $\mathcal{X}$ be a pointed formal moduli problem, which we should think of as a generalized moduli space with a point $x\colon \mathrm{Spec}(\Bbbk) \to \mathcal{X}$ that parametrizes families of objects defined over formal thickenings of x. More precisely, $\mathcal{X}$ is a functor from the category of Artinian commutative $\Bbbk$-algebras $\mathrm{Art}_{\Bbbk}$ to the category of sets, satisfying some axioms. Then, one could be able to recover $\mathcal{X}$ up to isomorphism from the tangent differential graded Lie algebra $\mathbb{T}_{\mathcal{X},x}$ at the given point x. While this is not true in the classical sense, this becomes true (and it is indeed now a Theorem, due to [Pri10] and [Lur11c]) with the derived formalism, which Drinfeld uses in his letter.

Let us spell briefly the main idea. Consider any differential graded Lie algebra $\mathfrak{g}_\bullet$ defined over a field of characteristic 0. Then, the cohomological Chevalley-Eilenberg complex $\mathrm{CE}^\bullet(\mathfrak{g}_\bullet)$, considered as a differential graded commutative pro-Artinian $\Bbbk$-algebra, should pro-represent the derived formal moduli problem

$$\mathcal{X}\colon\ \mathrm{dgArt}^{\mathrm{sm}}_{\Bbbk} \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$$

from differential graded Artinian $\Bbbk$-algebras to simplicial sets sending a differential graded commutative Artinian algebra $A_\bullet$ with maximal augmentation ideal $\mathfrak{m}_{A_\bullet}$ to some simplicial set $\mathrm{MC}\left(\mathfrak{g}_\bullet \otimes_{\Bbbk} \mathfrak{m}_{A_\bullet}\right)$ counting *solutions of Maurer-Cartan equations of the differential graded Lie algebra* $\mathfrak{g}_\bullet \otimes_{\Bbbk} \mathfrak{m}_{A_\bullet}$, i.e., elements of degree -1 in $\mathfrak{g}_\bullet \otimes_{\Bbbk} \mathfrak{m}_{A_\bullet}$ satisfying the Maurer-Cartan equation

$$\mathrm{d}(g \otimes a) + \frac{1}{2}[g \otimes a,\ g \otimes a] = 0,$$

modulo gauge equivalence. Here, $\mathfrak{g}_\bullet \otimes_{\Bbbk} \mathfrak{m}_{A_\bullet}$ is the usual Künneth tensor product of chain complexes, which is endowed with the Lie bracket

$$\left[g \otimes a,\ g' \otimes a'\right] := \left[g, g'\right] \otimes (a \cdot a'),$$

up to Koszul sign rule.

This is an example strongly advocating the necessity of derived tools in algebraic geometry. Indeed, if one restricts a derived formal moduli problem $\mathcal{X}$ to ordinary Artinian $\Bbbk$-algebras, and considers just the set of connected components

$$X(A) := \pi_0 \mathcal{X}(A)$$

of the simplicial set $\mathcal{X}(A)$ associated to A, one obtains a classical formal moduli problem X. However, even if X is representable by a classical scheme M, its shifted tangent complex $\mathbb{T}_M[-1]$ is *not* equivalent to the differential graded Lie algebra $\mathfrak{g}_\bullet$ which is the tangent complex of $\mathcal{X}$. Moreover, $\mathfrak{g}_\bullet$ cannot be recovered as the tangent complex of some ordinary scheme because it is *too well-behaved* - for example, it is often of finite homological dimension, contrarily to tangent complexes of schemes of finite presentation. For a way more detailed and formal survey of the ideas that led deformation theory to embrace derived algebraic geometry methods, we suggest the beautiful survey [Toë14] and the introduction to [CG19].

Why mixed graded complexes?

It is well known to researchers with a background in either algebraic geometry or number theory that, in characteristic 0, an action of the circle $S^1 := \mathrm{B}\mathbb{Z}$ (seen as the classifying derived stack for the constant stack $\mathbb{Z}$) over a derived stack X is linked to differential forms and de Rham theory over X. Namely, in [BN12] the authors established that p-forms over a derived stack X can be interpreted via functions on the derived loop stack $\mathscr{L}X$, which is canonically endowed with an action of the circle S^1 by rotating loops. The condition of being closed is then encoded in the property that the function is S^1-equivariant.

However, the homotopy theory of S^1-complexes is equivalent to the homotopy theory of *mixed graded complexes*, i.e., chain complexes $M_\bullet$ endowed with a decomposition

$$M_\bullet := \bigoplus_{p \in \mathbb{Z}} M_p$$

endowed with a *mixed differential* $\varepsilon_p \colon M_p \to M_{p-1}[-1]$ satisfying the usual square-to-zero property. The theory of mixed graded complexes in characteristic 0, which was developed in [PTVV13] and [CPT+17], has been exploited extensively in the past years, and has also been linked to the usual derived filtered category of Beilinson in [MRT19], [TV20a; TV20b] and [CCN21]. Indeed, the de Rham algebra of a differential graded commutative ring $A_\bullet$, with its grading given by

$$\mathrm{dR}^p(A_\bullet) = \mathrm{Sym}^p_{\Bbbk}\big(\mathbb{L}_{A_\bullet/\Bbbk}[1]\big),$$

is a *mixed graded commutative algebra*, where the mixed differential is provided exactly by the de Rham differential. One can then define (shifted) p-forms and closed p-forms on a differential graded commutative algebra $A_\bullet$ in terms of elements of $\mathrm{dR}^p(A_\bullet)$ and homotopy fixed points for the de Rham differentials in $\mathrm{dR}^p(A_\bullet)$, respectively. Moreover, all these constructions satisfy descent, and make perfectly sense also for more general derived stacks.

This theoretical framework yields new perspectives over derived symplectic geometry and deformation quantization. Using mixed graded complexes and mixed graded cdga's one can define:

1. shifted symplectic forms over a derived stack as shifted closed 2-forms which are non-degenerate in some suitable sense;
2. Lagrangian structures on morphisms of derived stacks;
3. Poisson structures and their deformation quantization;
4. derived algebraic foliations.

In the last years, however, mixed graded complexes have been employed also in the homotopy theory of Lie algebras and Lie algebroids. Given a differential graded Lie algebra $\mathfrak{g}_\bullet$, it is known that the Chevalley-Eilenberg algebra has a richer structure of *mixed graded commutative algebra* (see also [CG19], [Nui19]). In particular, it is expected that mixed graded complexes can provide a natural setting where to work with formal geometry and deformation theory.

Outline of the thesis

In this thesis, following the *leitmotif* sketched above, we studied extensively the homotopy theory of mixed graded complexes and their relationship with the homotopy theory of Lie algebras.

In Chapter 1 we formalized, in a purely model-independent ∞-categorical setting, many important properties of the stable ∞-category $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ of mixed graded complexes such as the computation of limits and colimits, the closed monoidal structure, and the relationship of mixed graded complexes with purely graded complexes and non-graded complexes (Section 1.1). The main interest lied in the ∞-categories of algebras and coalgebras for some suitable operads and cooperads and the behaviour of some functors of interest with respect to the algebra and coalgebra structures (Section 1.3). The main result of this chapter is, however, the characterization of mixed graded modules as a full sub-∞-category of the ∞-category $\mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}$ of filtered modules. While this result was expected from experts, and has already been proved ([TV20b] and [CCN21]), our work offers a deeper insight on such embedding, namely:

Theorem (Theorems 1.4.1 and 1.5.25)**.** *There exists a left and right complete t-structure on the stable ∞-category of mixed graded modules whose heart is equivalent to the usual abelian 1-category of chain complexes. Moreover, the embedding of the ∞-category of mixed graded modules into the ∞-category of filtered modules admits a left adjoint $(-)^{\mathrm{gr}}_{\varepsilon}\colon \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk} \to \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ which identifies the ∞-category of mixed graded $\Bbbk$-modules with the left completion $\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}$ of the Beilinson t-structure of [Bei87] on the ∞-category of filtered $\Bbbk$-modules (which is the full sub-∞-category of modules with complete filtration).*

In Chapter 2, after recollecting the classical theory of Chevalley-Eilenberg complexes of Lie algebras and the fundamentals of derived Lie algebras defined over a base field of characteristic 0 (Sections 2.1 to 2.3), we constructed with purely ∞-categorical techniques both homological (Section 2.4) and cohomological (Section 2.5) mixed graded versions of the Chevalley-Eilenberg complexes, which provide a mixed graded cocommutative coalgebra and a mixed graded commutative algebra respectively. This has to be interpreted as a model independent construction of the mixed graded structure on the Chevalley-Eilenberg cohomological complex which has been provided, in terms of explicit mixed differential for a graded chain complex, in [CG19, Appendix B]. Moreover, in Section 2.6 we constructed a mixed graded version of the Chevalley-Eilenberg modules of a homotopy Lie algebra with coefficients in some representation. While this description of the Chevalley-Eilenberg complexes is known, to our knowledge there is no description of this functor in a model independent way in existing literature. Moreover, our construction highlights some important features of the Chevalley-Eilenberg functors, such as the preservation of homotopy limits and/or colimits. We can sum up the results of the second chapter as follows.

Theorem (Propositions 2.4.26, 2.5.9, 2.6.3)**.**

1. *There exists a covariant homological Chevalley-Eilenberg mixed graded coalgebra ∞-functor*

$$\mathrm{CE}_{\varepsilon}\colon \mathrm{Lie}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$$

from the ∞-category of Lie algebras to the ∞-category of coaugmented mixed graded cocommutative coalgebras, such that for any Lie algebra $\mathfrak{g}$ the underlying graded cocommutative coalgebra of $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ is equivalent to the graded symmetric coalgebra on $\mathfrak{g}[-1]$, with $\mathfrak{g}[-1]$ sitting in weight 1. Moreover, the totalization of $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ agrees with the classical Chevalley-Eilenberg homology of $\mathfrak{g}$.

2. *There exists a contravariant cohomological Chevalley-Eilenberg mixed graded algebra ∞-functor*

$$\mathrm{CE}^{\varepsilon}\colon \mathrm{Lie}^{\mathrm{op}}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$$

from the ∞-category of Lie algebras to the ∞-category of augmented mixed graded commutative algebras, such that for any Lie algebra $\mathfrak{g}$ the underlying graded commutative algebra of $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ is equivalent to the graded symmetric algebra on $\mathfrak{g}^{\vee}[1]$, with $\mathfrak{g}^{\vee}[1]$ sitting in weight -1. Moreover, the totalization of $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ agrees with the classical Chevalley-Eilenberg cohomology of $\mathfrak{g}$.

3. *Fixing a Lie algebra $\mathfrak{g}$ which is perfect as a $\mathbb{k}$-module, there exists a covariant Chevalley-Eilenberg cohomology ∞-functor*

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};-)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\mathbb{k}}^{\mathrm{gr}}\big)$$

from the ∞-category of representations of $\mathfrak{g}$ to the ∞-category of mixed graded $\mathbb{k}$-modules which are modules for the commutative algebra object $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$, such that for any representation M of $\mathfrak{g}$ the underlying graded module of $\mathrm{CE}^{\varepsilon}(\mathfrak{g};M)$ is equivalent to the (graded) tensor product of the symmetric algebra over $\mathfrak{g}^{\vee}[1]$ with M, with naturally induced grading.

Finally, in Chapter 3, we describe some conjectures regarding the fully faithfulness of the Chevalley-Eilenberg ∞-functors (Conjectures 3.1.7, 3.1.8, 3.2.3) and their essential image, hinting at possible strategies to tackle the proof.

Possible future developments of the theory

The work of this thesis started by investigating possible frameworks where to study formal derived algebraic geometry. In particular, we expect that our constructions can be carried out, with only minor modifications, in the more general setting of Chevalley-Eilenberg complexes of Lie algebroids.

Moreover, the proof of one of our conjectures (namely, Conjecture 3.2.3) should provide an insight to the problem of studying $\mathcal{D}$-modules for possibly non-smooth derived schemes. In particular, in [PT21] the authors show that given the de Rham algebra $\mathrm{dR}(X)$ of some smooth, affine $\mathbb{k}$-scheme $X := \mathrm{Spec}(A)$, with its natural mixed structure given by the de Rham differential, then there exists a fully faithful embedding

$$\mathrm{Mod}_{\mathcal{D}_X} \hookrightarrow \mathrm{Mod}_{\mathrm{dR}(X)}\big(\varepsilon\text{-}\mathrm{Mod}_{\mathbb{k}}^{\mathrm{gr}}\big)$$

whose essential image is spanned by objects $M_\bullet$ which are equivalent, at the level of the underlying graded object, to $M \otimes_A \mathrm{Sym}_{\mathbb{k}}\big(\mathbb{L}_{X/\mathbb{k}}[1](-1)\big)$ for some A-module M. This leads naturally to seek for an analogous result also for (possibly non-smooth) derived affine schemes, using the computation of the left adjoint to the inclusion of constant mixed graded modules over the Chevalley-Eilenberg algebra into all mixed graded modules described in Section 3.2.

Notations, conventions and main references

- Throughout all this thesis, we employ freely the language of derived algebraic geometry, ∞-categories, and homotopical algebra provided by [Lur09] and [Lur17], from which we borrow the formalism and most notations. Our language is *innerly* derived: every definition and construction has to be interpreted, without further indication suggesting the contrary, in the context of higher algebra. In particular, by *module* over a discrete commutative ring $\Bbbk$ we mean an object of the stable derived ∞-category of $\Bbbk$-modules, by *limits and colimits* we mean homotopy limits and colimits, by *tensor product* we mean derived tensor product, and so forth.

- Our main references for the homotopy theory of mixed graded complexes are provided by [PTVV13] and [CPT+17].

- Our main references for the homotopy theory of Lie algebras in characteristic 0, and its relationship with the derived deformation theory, are provided by [Lur11c] and [GR17b].

- When dealing with explicit models provided by chain complexes of $\Bbbk$-modules, we use a homological notation.

- Our standing assumption is that we work in characteristic 0, over a fixed commutative ring $\Bbbk$ (which throughout all Chapters 2 and 3 is even a field).

- Throughout this thesis, we shall often work with closed symmetric monoidal ∞-categories $\mathscr{C}^{\otimes}$ enriched over $\Bbbk$-modules. In particular, such ∞-categories are endowed with both internal mapping objects, obtained as a right adjoint to $\otimes$, and mapping $\Bbbk$-modules providing the enrichment over $\mathrm{Mod}_{\Bbbk}$. In order to avoid confusion, we shall denote the former with $\underline{\mathrm{Map}}$ and the latter with Map.

Chapter 1

Mixed graded modules

The fundamental objects of this thesis are *mixed graded* $\Bbbk$*-modules*, which generalize the concept of *mixed complexes* and provide - at least in characteristic 0 - a very useful analogue to complexes endowed with a complete and exhaustive filtration. Mixed graded modules have been studied extensively in the last years in the field of derived differential geometry and theory of Lie algebroids; yet, they are not as well known as filtered $\Bbbk$-modules, of which they provide a more well-behaved analogue in characteristic 0. In this chapter, we shall first gather some important definitions and properties of the ∞-category of mixed graded modules (and algebras and coalgebras over it), and fix our notations. For this scope, our main sources are [PTVV13] and [CPT+17].

The last sections of this chapter feature the most recent material. We first construct an accessible and both left and right complete t-structure on the ∞-category of mixed graded $\Bbbk$-modules, whose heart is equivalent to the usual abelian category of chain complexes. While we believe this to be a well known fact among experts, to our knowledge the construction has not been written down yet. Finally, we provide a fully-faithful embedding of the ∞-category of mixed graded $\Bbbk$-modules into the ∞-category of filtered $\Bbbk$-modules. This result is known, and has been proved and employed both in [TV20b] and [CCN21]; however, the relationship between their t-structures is not documented in the existing literature.

1.1 Basic definitions and notations

In order to capture the idea behind the notion of a mixed graded $\Bbbk$-module, we first recall the concept of *mixed* $\Bbbk$*-modules* (or *mixed complexes*, as they are classically called), which - in the words of [Kas87], are objects that are *both chain and cochain complexes in a compatible way*. Mixed complexes were first introduced in [BO86], as *algebraic* S^1*-chain complexes* (or *chain complexes with an algebraic circle action*), in order to study Hochschild and cyclic homology of unital associative algebras in characteristic 0, which naturally come equipped with a mixed structure at the level of chains.

Definition 1.1.1 (Mixed complexes, [Kas87]). A *mixed complex* over a base ring $\Bbbk$ of characteristic 0 is a chain complex $(C_\bullet, \partial_\bullet)$ together with morphisms $\beta_n \colon C_n \to C_{n+1}$ such that $\beta_{n+1} \circ \beta_n = \partial_{n+1} \circ \beta_n + \beta_{n-1} \circ \partial_n = 0$.

Equivalently, mixed complexes are modules over the free differential graded commutative algebra $\Bbbk[\eta] := \Bbbk[t]/(t^2)$, where $\eta := \bar{t}$ is a generator in homological degree 1 and $\partial(\eta) = 0$. Alternatively, they are comodules over the differential graded cocommutative coalgebra $\Bbbk[\varepsilon] = (\Bbbk[\eta])^\vee$ which is the (differential graded) $\Bbbk$-linear dual of $\Bbbk[\eta]$. This will be the stepping stone for generalizing the idea of mixed complexes to the derived setting.

1.1.2. Let $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk}$ the classifying stack for the affine smooth group scheme $\mathbb{A}^1 := \operatorname{Spec}(\Bbbk[t])$: it is an affine group stack which, in any characteristic, is equivalent to the spectrum of the derived commutative ring $\operatorname{Sym}_\Bbbk(\Bbbk[-1])$. However, when $\Bbbk$ is a base ring which contains $\mathbb{Q}$, it is well-known that

$$\operatorname{Sym}_\Bbbk(\Bbbk[-1]) \simeq \Bbbk \oplus \Bbbk[-1] =: \Bbbk[\varepsilon]$$

with its square-zero extension commutative algebra structure. In this case, we can describe the semi-direct product $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}$ of the affine group stacks $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk}$ and $\mathbb{G}_{\mathrm{m},\Bbbk}$ as the affine group stack whose algebra of functions is equivalent, as a commutative $\Bbbk$-algebra, to the formal $\Bbbk$-algebra

$$\Bbbk\left[t,\, t^{-1}\right] \otimes_\Bbbk (\Bbbk \oplus \Bbbk[-1]) \simeq \Bbbk\left[t,\, t^{-1}\right] \oplus \Bbbk\left[t,\, t^{-1}\right][-1].$$

Denoting again the generator in degree -1 with ε, the comultiplication for its cocommutative Hopf structure is given by the assignations $t \mapsto t \otimes t$ and $\varepsilon \mapsto t \otimes \varepsilon$.

Definition 1.1.3. The ∞*-category of mixed graded* $\Bbbk$-modules

$$\varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}} := \mathrm{Rep}_{\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}} \simeq \operatorname{QCoh}\big(\mathrm{B}\big(\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}\big)\big)$$

is the ∞-category of representations of the derived group stack $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}$. Equivalently, it is the ∞-category of comodule objects for the Hopf algebra $\mathcal{O}_{\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}}$ in $\mathrm{Mod}_\Bbbk$.

Remark 1.1.4. In the setting of commutative differential graded $\Bbbk$-algebras and chain complexes, a mixed graded $\Bbbk$-module can be thought as a chain complex of $\Bbbk$-modules $M_\bullet$, equipped with a decomposition of chain complexes of $\Bbbk$-modules $\left\{(M_\bullet)_p\right\}_{p\in\mathbb{Z}}$ and with a morphism of chain complexes

$$\varepsilon_p \colon (M_\bullet)_p \longrightarrow (M_\bullet)_{p-1}[-1]$$

such that $\varepsilon_{p-1}[-1] \circ \varepsilon_p = 0$ for all $p \in \mathbb{Z}$. The chain complex $(M_\bullet)_p$ is the p *weight component* of the mixed graded $\Bbbk$-module $M_\bullet$, while the morphism ε is the *mixed differential*. In the description given in Definition 1.1.3, the action of $\mathbb{G}_{\mathrm{m},\Bbbk}$ yields the weight grading, while the action of $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk}$ yields the mixed differential; the fact that we are considering the semi-direct product assures us that the mixed differential decreases the weight grading by -1, i.e. the two actions are intertwined. See also [PTVV13, Remark 1.1].

Remark 1.1.5. One could notice that in the explicit models provided by Remark 1.1.4 the internal differential and the mixed differential *commute*, while in the classical notion of mixed complexes of Definition 1.1.1 they were required to *anti-commute*. But since we are working with bi-graded objects, it is a standard computation to show that the two formalisms are completely equivalent, up to suitably changing the signs of the mixed differential.

1.1.6. The ∞-category of mixed graded $\Bbbk$-modules, being the ∞-category of quasi-coherent sheaves over a derived stack, is naturally stable and it is endowed with a symmetric closed monoidal structure. Viewing $\mathbb{O}_{\mathrm{B}\mathbb{G}_{a,\Bbbk}\rtimes\mathbb{G}_{m,\Bbbk}}$-comodules as graded $\Bbbk$-modules endowed with a mixed differential, we can describe the internal tensor product, the internal mapping space and the unit for such monoidal structure as follows.

1. Given two mixed graded $\Bbbk$-modules $M_\bullet$ and $N_\bullet$, the tensor product $M_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet$ is the mixed graded $\Bbbk$-module whose p-th weight component is given by the formula

$$\left(M_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet\right)_p := \bigoplus_{i+j=p} M_i \otimes_{\Bbbk} N_j$$

   with mixed differential defined on every summand by the formula

$$\varepsilon_M \otimes \mathrm{id}_N + \mathrm{id}_M \otimes \varepsilon_N : M_i \otimes N_j \longrightarrow \left(M_{i-1} \otimes N_j\right) \bigoplus \left(M_i \otimes N_{j-1}\right)[-1].$$

2. The unit for $\otimes_{\Bbbk}^{\varepsilon\text{-gr}}$ is the mixed graded $\Bbbk$-module $\Bbbk(0)$, consisting of $\Bbbk$ sitting in pure weight 0 with trivial mixed structure.

3. Given two mixed graded $\Bbbk$-modules $M_\bullet$ and $N_\bullet$, the internal mapping space $\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)$ is the mixed graded $\Bbbk$-module whose p-th weight component is given by the formula

$$\left(\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)\right)_p := \prod_{q\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_q, N_{q+p}\right)$$

   with mixed differential

$$\varepsilon_p : \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)_p \longrightarrow \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)_{p-1}[-1]$$

   given by the morphism whose r-th component is the sum of the morphism

$$\begin{tikzcd} \prod_{q\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_q, N_{q+p}\right) \arrow[r, "\pi_r"] & \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_r, N_{r+p}\right) \arrow[d, "\varepsilon^N_{r+p}\circ -"] \\ & \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_r, N_{r+p-1}[-1]\right) \simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_r, N_{r+p-1}\right)[-1] \end{tikzcd}$$

with the morphism

$$\begin{tikzcd} \prod_{q\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(M_q,\, N_{q+p}\big) \arrow[r, "\pi_r"] & \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(M_{r-1},\, N_{r-1+p}\big) \arrow[d, "-\circ\varepsilon_r^M[1]"] \\ & \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(M_r[1],\, N_{r-1+p}\big) \simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(M_r,\, N_{r+p-1}\big)[-1]. \end{tikzcd}$$

The enrichment of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ over $\mathrm{Mod}_{\Bbbk}$ is then given by

$$\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(M_\bullet,\, N_\bullet) := \mathrm{fib}\Big(\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet,\, N_\bullet)_0 \xrightarrow{\varepsilon_0} \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet,\, N_\bullet)_{-1}[-1]\Big).$$

See also [CPT+17, Section 1.1].

For future reference, we provide also a simple result about fully dualizable objects (in the sense of [Lur17, Section 4.6.1]).

Notation 1.1.7. In the following, we shall denote by $\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}$ the full sub-∞-category of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ spanned by those mixed graded $\Bbbk$-modules $M_\bullet$ which are perfect in each weight and such that $M_p \simeq 0$ for all $p \gg 0$. Dually, we shall denote by $\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},+}$ the full sub-∞-category of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ spanned by those mixed graded $\Bbbk$-modules $M_\bullet$ which are perfect in each weight and such that $M_p \simeq 0$ for all $p \ll 0$. In a similar fashion, we shall denote by $\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},\geqslant p}$ (respectively, $\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},\leqslant q}$) the full sub-$\infty$-category of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ spanned by those mixed graded $\Bbbk$-modules $M_\bullet$ which are perfect in each weight and such that $M_n \simeq 0$ for all $n < p$ (respectively, for all $n > q$). Let us remark that we have inclusions of ∞-categories

$$\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},\geqslant p} \subseteq \varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},+}$$

and

$$\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},\leqslant q} \subseteq \varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}$$

for all integers p and q.

Proposition 1.1.8. *The full sub-∞-category of fully dualizable objects of mixed graded $\Bbbk$-modules coincides with the ∞-category*

$$\big(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},+}\big)\bigcap\big(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}\big).$$

Proof. Let us denote by $M_\bullet^\vee$ the mixed graded $\Bbbk$-linear dual $\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet,\, \Bbbk(0))$ of $M_\bullet$. We have an obvious evaluation morphism $M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} M_\bullet \longrightarrow \Bbbk(0)$ given by the adjoint to the identity of $M_\bullet^\vee$. In virtue of [Lur17, Lemma 4.6.1.6], extending such evaluation morphism to a datum of full dualizability is equivalent to showing that tensoring with the evaluation produces an equivalence of $\Bbbk$-modules

$$\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\big(P_\bullet,\, M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet\big) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(P_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} M_\bullet,\, N_\bullet\big)$$

for any mixed graded $\Bbbk$-modules $N_\bullet$ and $P_\bullet$. This will be a consequence of the following Lemma.

Lemma 1.1.9. *For any $M_\bullet$ in $\left(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},+}\right) \cap \left(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}\right)$ and for an arbitrary mixed graded $\Bbbk$-module $N_\bullet$, the natural map of mixed graded $\Bbbk$-modules*

$$M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet \longrightarrow \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)$$

is an equivalence.

Proof. Let us recall how this map is defined. We have a chain of equivalences of mapping $\Bbbk$-modules

$$\begin{aligned}\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\left(M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet,\, \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)\right) &\simeq \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\left(M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} M_\bullet,\, N_\bullet\right)\\ &\simeq \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\left(M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} M_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet,\, N_\bullet\right)\end{aligned}$$

and so, tensoring the evaluation map $M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} M_\bullet \longrightarrow \Bbbk(0)$ with the identity of $N_\bullet$ one has the desired map $M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet \longrightarrow \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)$. To prove it is an equivalence, we check it weight-wise: the left hand side is described in weight p by the formula

$$\left(M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet\right)_p \simeq \bigoplus_{i+j=p} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(M_{-i}, \Bbbk) \otimes_{\Bbbk} N_j$$

whereas the right hand side is described in weight p by the formula

$$\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)_p \simeq \prod_{h\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_h, N_{h+p}\right).$$

With a change of indices $h := -i$ (hence $h+p = p-i = j$), since M_{-i} is perfect in $\mathrm{Mod}_{\Bbbk}$ for any integer i, we can rewrite the latter as

$$\prod_{i\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_{-i}, N_j\right) \simeq \prod_{i\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(M_{-i}, \Bbbk) \otimes_{\Bbbk} N_j.$$

Finally, since the grading of $M_\bullet$ is bounded above and below, we know that $\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(M_{-i}, \Bbbk)$ can be non-zero only for finitely many indices, hence the product is actually a direct sum. Therefore, the map is an equivalence. □

Lemma 1.1.9 shows that $M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet \simeq \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)$ for any mixed graded $\Bbbk$-module $N_\bullet$ and for any mixed graded $\Bbbk$-module $M_\bullet$ which is bounded and perfect in each weight, and this equivalence is provided exactly by tensoring the adjoint map to the evaluation $M_\bullet^\vee \otimes_{\Bbbk}^{\varepsilon\text{-gr}} M_\bullet \to \Bbbk(0)$ with the identity of $N_\bullet$. This proves the second assertion of [Lur17, Lemma 4.6.1.6], hence the dualizability of $M_\bullet$.

Proving that any fully dualizable mixed graded $\Bbbk$-module lies in $\left(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},+}\right) \cap \left(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}\right)$

relies on the description of dualizable graded $\Bbbk$-modules. Indeed, we have a forgetful ∞-functor

$$\mathrm{oblv}_\varepsilon\colon \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}} \longrightarrow \mathrm{Mod}_\Bbbk^{\mathrm{gr}},$$

described (model independently) in the following way. Let us interpret mixed graded $\Bbbk$-modules as quasi-coherent $\mathcal{O}$-sheaves on $\mathrm{B}\big(\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}\big)$. We have a right split extension of group stacks

$$\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \longrightarrow \mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk} \longrightarrow \mathbb{G}_{\mathrm{m},\Bbbk}. \tag{1.1.10}$$

The splitting morphism $\mathbb{G}_{\mathrm{m},\Bbbk} \to \mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}$ induces in this way a morphism $\mathrm{B}\mathbb{G}_{\mathrm{m},\Bbbk} \to \mathrm{B}\big(\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}\big)$, which induces a pullback ∞-functor

$$\mathrm{oblv}_\varepsilon\colon \mathrm{QCoh}\Big(\mathrm{B}\big(\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}\big)\Big) \longrightarrow \mathrm{QCoh}\big(\mathrm{B}\mathbb{G}_{\mathrm{m},\Bbbk}\big). \tag{1.1.11}$$

By the known equivalence between $\mathbb{G}_{\mathrm{m},\Bbbk}$-equivariant quasi-coherent sheaves on $\mathrm{Spec}(\Bbbk)$ and graded $\Bbbk$-modules, this reduces to the forgetful ∞-functor that sends a graded mixed $\Bbbk$-module $M_\bullet$ to the underlying graded $\Bbbk$-module $M_\bullet$ by forgetting the mixed structure. In particular, the forgetful ∞-functor $\mathrm{oblv}_\varepsilon\colon \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}} \to \mathrm{Mod}_\Bbbk^{\mathrm{gr}}$, being a quasi-coherent pullback ∞-functor, preserves both tensor products and internal mapping spaces. Hence, any dualizable object in $\varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}}$, after forgetting the mixed structure, must become dualizable also in $\mathrm{Mod}_\Bbbk^{\mathrm{gr}}$. So we just need to prove that any dualizable graded $\Bbbk$-module $M_\bullet$ is perfect in each weight and endowed with bounded grading. Indeed a map $\Bbbk(0) \to M_\bullet \otimes_\Bbbk^{\mathrm{gr}} M_\bullet^\vee$ corresponds to an element in the weight 0 part of the $\Bbbk$-module

$$\big(M_\bullet \otimes_\Bbbk^{\mathrm{gr}} M_\bullet^\vee\big)_0 \simeq \bigoplus_{n\in\mathbb{Z}} M_n \otimes_\Bbbk \mathrm{Map}_{\mathrm{Mod}_\Bbbk}(M_n,\ \Bbbk).$$

In particular, a coevaluation morphism for $M_\bullet$ in $\mathrm{Mod}_\Bbbk^{\mathrm{gr}}$ provides, after post-composing with the natural map

$$\bigoplus_{n\in\mathbb{Z}} M_n \otimes_\Bbbk \mathrm{Map}_{\mathrm{Mod}_\Bbbk}(M_n,\ \Bbbk) \longrightarrow \prod_{n\in\mathbb{Z}} M_n \otimes_\Bbbk \mathrm{Map}_{\mathrm{Mod}_\Bbbk}(M_n,\ \Bbbk)$$

and then projecting on the p-th direct summand, a coevaluation $\Bbbk \to M_p \otimes_\Bbbk \mathrm{Map}_{\mathrm{Mod}_\Bbbk}\big(M_p,\ \Bbbk\big)$ for any integer p. The fact that all the coherences of the definition of a coevaluation morphism are satisfied is a consequence of the fact that $\Bbbk(0) \to M_\bullet \otimes_\Bbbk^{\mathrm{gr}} M_\bullet^\vee$ is assumed to be a coevaluation itself: one can easily see it by checking the coherence diagram for the component in weight $-p$ of

$$\mathrm{id}_{M_\bullet^\vee}\colon M_\bullet^\vee \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} M_\bullet^\vee \otimes_\Bbbk^{\varepsilon\text{-}\mathrm{gr}} M_\bullet \otimes_\Bbbk^{\varepsilon\text{-}\mathrm{gr}} M_\bullet^\vee \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} M_\bullet^\vee.$$

By the characterization of dualizable objects in $\mathrm{Mod}_\Bbbk$, this shows that M_p must be perfect for any integer p.

For the statement about the upper and lower bound of the grading, let us remark that an analogous argument to the one in the proof of Lemma 1.1.9 provides always a map

$$\underline{\mathrm{Map}}^{\mathrm{gr}}_{\Bbbk}(M_\bullet,\ \Bbbk(0)) \otimes^{\mathrm{gr}}_{\Bbbk} N_\bullet \longrightarrow \underline{\mathrm{Map}}^{\mathrm{gr}}_{\Bbbk}(M_\bullet,\ N_\bullet)$$

where $\underline{\mathrm{Map}}^{\mathrm{gr}}_{\Bbbk}$ is the internal graded mapping $\Bbbk$-module ∞-functor for the closed symmetric monoidal ∞-category $\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$. If $M_\bullet$ is not bounded in both directions, in general it can fail to be an equivalence. For example, if $M_p \not\simeq 0$ for all non-negative integers p, considering $N_\bullet := M_\bullet$, then the weight 0 component of the above map is described by

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(M_n,\ \Bbbk) \otimes_{\Bbbk} M_n \longrightarrow \prod_{n\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(M_n,\ M_n)$$

which of course can never be an equivalence without the boundness assumption. □

Remark 1.1.12. The ∞-category $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ is equivalent, as a stable symmetric monoidal ∞-category, to the ∞-category of comodules on $\mathbb{O}_{\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk}\rtimes\mathbb{G}_{\mathrm{m},\Bbbk}}$. We can alternatively consider the ∞-category of comodules on $\mathbb{O}_{\Omega_0\mathbb{G}_{\mathrm{a},\Bbbk}\rtimes\mathbb{G}_{\mathrm{m},\Bbbk}}$, that is the ∞-category of comodules on the algebra of functions on the affine group stack

$$\Omega_0\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk} \simeq \mathrm{Spec}\big(\Bbbk\left[t,\ t^{-1}\right] \otimes_{\Bbbk} (\Bbbk \oplus \Bbbk[1])\big).$$

The two theories are equivalent: in the latter case, the mixed differential ε is a morphism of degree -1 instead of degree 1 (i.e., the mixed structure is the datum of a map $\varepsilon_p\colon M_p \to M_{p-1}[1]$). An explicit equivalence between the two comodule theories simply sends a comodule $M_\bullet$ over $\mathbb{O}_{\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk}\rtimes\mathbb{G}_{\mathrm{m},\Bbbk}}$ to the comodule over $\mathbb{O}_{\Omega_0\mathbb{G}_{\mathrm{a},\Bbbk}\rtimes\mathbb{G}_{\mathrm{m},\Bbbk}}$ given in weight p by the $\Bbbk$-module $M_p[-2p]$. See also [CPT+17, Remark 1.1.3].

1.1.13. Let $\Bbbk(p)$ denote the mixed graded $\Bbbk$-module consisting of $\Bbbk$ sitting in pure weight p and homological degree 0. We have an adjunction

$$-\otimes_{\Bbbk}\Bbbk(0)\colon \mathrm{Mod}_{\Bbbk} \rightleftarrows \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} : |-| \tag{1.1.14}$$

where the left adjoint simply sends a $\Bbbk$-module M to the mixed graded $\Bbbk$-module consisting of M concentrated in weight 0, and the right adjoint $|-|$ is the ∞-functor that sends a mixed graded $\Bbbk$-module $M_\bullet$ to the mapping $\Bbbk$-module

$$|M_\bullet| := \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}}(\Bbbk(0),\ M_\bullet).$$

This right adjoint is called the *realization ∞-functor*: a strict model of $|M_\bullet|$ is provided by the chain complex of $\Bbbk$-modules

$$\prod_{p\geqslant 0} M_{-p}[-2p]$$

endowed with the total differential, sum of the usual differential of chain complexes and the mixed differential ([CPT+17, Proposition 1.5.1]).

Notation 1.1.15. In the remainder of this work, the ∞-functor $-\otimes_{\Bbbk}\Bbbk(q)\colon \mathrm{Mod}_{\Bbbk}\to \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ which sends a $\Bbbk$-module M to the mixed graded $\Bbbk$-module M concentrated in weight q with trivial mixed differential shall be denoted simply as $(-)(q)$.

Construction 1.1.16. For our purposes, it will be convenient to introduce another realization ∞-functor that can keep track of the $\Bbbk$-modules in positive weights of a mixed graded module $M_\bullet$. Let us recall ([CPT+17, Section 1.5]) that for all $p\in\mathbb{Z}$ we have that

$$\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\Bbbk(i)[-2i],\ \Bbbk(i-1)[-2(i-1)])\simeq\Bbbk. \tag{1.1.17}$$

So we have a pro-object in $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$, defined by

$$\Bbbk(\infty):=\{\ldots\to\Bbbk(i)[-2i]\to\Bbbk(i-1)[-2(i-1)]\to\ldots\to\Bbbk(1)[-2]\to\Bbbk(0)\} \tag{1.1.18}$$

where the morphism $\Bbbk(i)[-2i]\to\Bbbk(i-1)[-2(i-1)]$ is the unique morphism corresponding to the unit 1 of $\Bbbk$ under the equivalence 1.1.17.

Definition 1.1.19 ([CPT+17, Definition 1.5.2])**.** The *Tate* or *stabilized realization* ∞-functor is defined as

$$|-|^{\mathrm{t}}:=\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\Bbbk(\infty),\ -)\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\longrightarrow \mathrm{Ind}\,(\mathrm{Mod}_{\Bbbk})\xrightarrow{\mathrm{colim}}\mathrm{Mod}_{\Bbbk}.$$

1.1.20. Again, working with explicit models given by graded chain complexes and mixed differentials, the ∞-functor of Definition 1.1.19 sends a mixed graded $\Bbbk$-module $M_\bullet=\{M_p\}_p$ to the $\Bbbk$-module

$$|M_\bullet|^{\mathrm{t}}:=\operatorname*{colim}_{i\leqslant 0}\prod_{p\geqslant i}M_{-p}[-2p]$$

again endowed with the total differential. There is a natural transformation of ∞-functors

$$|-|\Rightarrow|-|^{\mathrm{t}}:\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\longrightarrow\mathrm{Mod}_{\Bbbk}$$

which is induced by the map of pro-objects $\Bbbk(\infty)\to\Bbbk(0)$ (the latter seen as a constant pro-object). Working with explicit models, we easily see that the natural transformation above is described, on a given mixed graded $\Bbbk$-module $M_\bullet$, as the inclusion

$$\prod_{p\geqslant 0}M_{-p}[-2p]\hookrightarrow\operatorname*{colim}_{i\leqslant 0}\prod_{p\geqslant i}M_{-p}[-2p].$$

In particular, the map above is an equivalence whenever $M_\bullet$ is trivial in all positive weights.

Construction 1.1.21. For all $p\in\mathbb{Z}$ we have a *p weight part ∞-functor*

$$(-)_p\colon\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\longrightarrow\mathrm{Mod}_{\Bbbk}$$

described (model independently) in the following way. Let us recall the forgetful ∞-functor

$\mathrm{oblv}_\varepsilon\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \to \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ described in 1.1.11 as the quasi-coherent pullback ∞-functor induced by atlas $\mathrm{B}\mathbb{G}_{\mathrm{m},\Bbbk} \to \mathrm{B}\big(\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}\big)$. Since we have an equivalence of ∞-categories

$$\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \simeq \prod_p \mathrm{Mod}_{\Bbbk},$$

one can project onto the p-th coordinate: this composition yields the desired ∞-functor. The ∞-functor $(-)_0$ has a left adjoint

$$\mathrm{Free}_\varepsilon\colon \mathrm{Mod}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \tag{1.1.22}$$

(the free mixed graded $\Bbbk$-module construction, see [CPT+17, 1.4.1]).

1.1.23. By pre-composing $(-)_0$ with the weight shift by q on the left ∞-endofunctor

$$(-)(\!(q)\!) := - \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \Bbbk(q)\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

which informally sends a mixed graded $\Bbbk$-module $M_\bullet = \big\{M_p\big\}_{p\in\mathbb{Z}}$ to the mixed graded $\Bbbk$-module $M(\!(q)\!)_\bullet := \big\{M_{p-q}\big\}_{p\in\mathbb{Z}}$, and by post-composing $\mathrm{Free}_\varepsilon$ with the shift ∞-functor $(\!(q)\!)$ we obtain for all $q \in \mathbb{Z}$ another adjunction

$$\mathrm{Free}_\varepsilon(\!(q)\!)\colon\ \mathrm{Mod}_{\Bbbk} \rightleftarrows \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\ : (-)_q \tag{1.1.24}$$

which generalizes the adjunction 1.1.22 to all weights. Let us remark that the description of this left adjoint is very explicit: indeed, the proof of [CPT+17, Proposition 1.3.8] together with the observations made in [CPT+17, Section 1.4.1] shows that such left adjoint simply sends a $\Bbbk$-module M to the mixed graded $\Bbbk$-module $\mathrm{Free}_\varepsilon(M)(\!(q)\!)$ consisting of M in weight q, $M[1]$ in weight $q-1$, and with mixed structure given by the natural equivalence $M \simeq M[1][-1]$.

Let us remark that, for any integer q, one has an adjoint pair

$$(-)(q)\colon\ \mathrm{Mod}_{\Bbbk} \rightleftarrows \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\ : (-)_q$$

which is an ambidextrous adjunction. The ∞-functor $(-)_q\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \to \mathrm{Mod}_{\Bbbk}$ is canonically equivalent to the composition

$$\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \xrightarrow{\mathrm{oblv}_\varepsilon} \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \xrightarrow{(-)_q} \mathrm{Mod}_{\Bbbk},$$

so the existence of the adjoint $\mathrm{Free}_\varepsilon(\!(q)\!)$ yields the following result.

Lemma 1.1.25. *Limits and colimits in $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ are computed weight-wise.*

Proof. Let us recall that the forgetful functor $\mathrm{oblv}_\varepsilon\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ corresponds geometrically to the pullback functor 1.1.11. Every pullback ∞-functor commutes with colimits ([Lur11b, Proposition 2.7.17]) and thus colimits in $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ are computed as in $\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$. But

$$\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \simeq \prod_{q\in\mathbb{Z}} \mathrm{Mod}_{\Bbbk},$$

and colimits on the right hand side are computed weight-wise.
The assertion for limits is more subtle, since in general pullback ∞-functors do not preserve them. But in this case, we can use the discussion of 1.1.23: indeed, the composition

$$(-)_p \colon \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \xrightarrow{\mathrm{oblv}_\varepsilon} \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \longrightarrow \mathrm{Mod}_{\Bbbk}$$

preserves limits for all p's, since it admits an explicit left adjoint. The product

$$\prod_p (-)_p \colon \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \xrightarrow{\mathrm{oblv}_\varepsilon} \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \xrightarrow{\{(-)_p\}_p} \prod_{p\in\mathbb{Z}} \mathrm{Mod}_{\Bbbk} \simeq \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$$

is canonically equivalent to the forgetful ∞-functor $\mathrm{oblv}_\varepsilon$, and it commutes with limits since it is the product of ∞-functors which commute with limits. □

Porism 1.1.26. Since both $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ and $\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ are presentable ∞-categories, the forgetful ∞-functor $\mathrm{oblv}_\varepsilon$ is both a left and right adjoint in virtue of the Adjoint Functor Theorem ([Lur09, Corollary 5.5.2.9]). Such adjoints L_ε and R_ε can be described with explicit models in the following way.

- The left adjoint L_ε sends a graded $\Bbbk$-module $M_\bullet$ to the mixed graded $\Bbbk$-module $\mathrm{L}_\varepsilon M_\bullet$ defined in weight p by the formula

  $$(\mathrm{L}_\varepsilon M_\bullet)_p \simeq M_p \oplus M_{p+1}[1],$$

  and whose $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk}$-action is described by the morphism

  $$\varepsilon_p \colon M_p \oplus M_{p+1}[1] \longrightarrow \big(M_{p-1} \oplus M_{p-1+1}[1]\big)[-1] \simeq M_{p-1}[-1] \oplus M_p$$

  given by the canonical equivalence $M_p \simeq M_p[1][-1]$ and the zero map on $M_{p+1}[1]$.

- The right adjoint R_ε sends a graded $\Bbbk$-module $M_\bullet$ to the mixed graded $\Bbbk$-module $\mathrm{R}_\varepsilon M_\bullet$ defined in weight p by the formula $(\mathrm{R}_\varepsilon M_\bullet)_p \simeq M_p \oplus M_{p-1}[-1]$, and whose $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk}$-action is described by the morphism

  $$\varepsilon_p \colon M_p \oplus M_{p-1}[-1] \longrightarrow \big(M_{p-1} \oplus M_{p-2}[-1]\big)[-1] \simeq M_{p-1}[-1] \oplus M_{p-2}[-2]$$

  given by the canonical equivalence $M_{p-1}[-1] \simeq M_{p-1}[-1]$ and the zero map on M_p.

It is now clear that the ∞-functor $\mathrm{Free}_\varepsilon((q))$ of 1.1.24 is canonical equivalent to the composition

$$\mathrm{L}_\varepsilon \circ (-)(q) \colon \mathrm{Mod}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}.$$

Construction 1.1.27. We shall provide one last, but very useful, construction that highlights how mixed graded $\Bbbk$-modules are to be thought as graded $\Bbbk$-modules with extra structure. Recall the right split extension of group stacks 1.1.10. The morphism $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk} \to \mathbb{G}_{\mathrm{m},\Bbbk}$ induces a pullback ∞-functor at the level of quasi-coherent sheaves

$$\mathrm{triv}_\varepsilon \colon \mathrm{QCoh}\big(\mathrm{B}\mathbb{G}_{\mathrm{m},\Bbbk}\big) \longrightarrow \mathrm{QCoh}\big(\mathrm{B}\big(\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk} \rtimes \mathbb{G}_{\mathrm{m},\Bbbk}\big)\big).$$

By functoriality, one has a natural equivalence

$$\mathrm{id}_{\mathrm{Mod}_\Bbbk^{\mathrm{gr}}} \simeq \mathrm{oblv}_\varepsilon \circ \mathrm{triv}_\varepsilon, \tag{1.1.28}$$

i.e., this ∞-functor is a section of the forgetful ∞-functor $\mathrm{oblv}_\varepsilon \colon \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}} \to \mathrm{Mod}_\Bbbk^{\mathrm{gr}}$. In general, $\mathrm{triv}_\varepsilon$ has to be understood as the ∞-functor that endows a graded $\Bbbk$-module with the trivial mixed structure, i.e., the zero morphism between each weight.

Warning 1.1.29. The ∞-functor $\mathrm{triv}_\varepsilon \colon \mathrm{Mod}_\Bbbk^{\mathrm{gr}} \to \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}}$ commutes with all limits and colimits since the forgetful ∞-functor $\mathrm{oblv}_\varepsilon$ creates them, and $\mathrm{oblv}_\varepsilon \circ \mathrm{triv}_\varepsilon \simeq \mathrm{id}_{\mathrm{Mod}_\Bbbk^{\mathrm{gr}}}$. Moreover, it is a straight-forward consequence of the discussion in 1.1.6 that $\mathrm{triv}_\varepsilon$ is also strongly monoidal. It is, however, *not* fully faithful: such pullback ∞-functor is in fact the left adjoint to the push-forward ∞-functor $\varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}} \to \mathrm{Mod}_\Bbbk^{\mathrm{gr}}$ which agrees with the *weighted negative cyclic ∞-functor* $\mathrm{NC}^{\mathrm{w}} \colon \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}} \to \mathrm{Mod}_\Bbbk^{\mathrm{gr}}$ of [PTVV13, Remark 1.5]. Unraveling all definitions, this ∞-functor sends any mixed graded $\Bbbk$-module $M_\bullet$ to the graded $\Bbbk$-module $\mathrm{NC}^{\mathrm{w}}(M_\bullet)$ described in each weight by the formula

$$\mathrm{NC}^{\mathrm{w}}(M_\bullet)_p \coloneqq |M_\bullet((p))|\,.$$

In particular, the unit for the adjunction $\mathrm{triv}_\varepsilon \dashv \mathrm{NC}^{\mathrm{w}}$ cannot be an equivalence. For example, if we consider the graded $\Bbbk$-module consisting of $\Bbbk$ sitting in pure weight 0, then the formula provided in [PTVV13] for the weighted negative cyclic ∞-functor yields that

$$\mathrm{NC}^{\mathrm{w}}(\mathrm{triv}_\varepsilon \Bbbk(0))_p \simeq \Bbbk[-2p]$$

for all $p \geqslant 0$. However, the ∞-functor $\mathrm{triv}_\varepsilon$ is fully faithful on those graded $\Bbbk$-modules which are concentrated in a single weight: indeed, the right adjoint of the ∞-functor

$$(-)(q) \coloneqq \mathrm{triv}_\varepsilon \circ (-)(q) \colon \mathrm{Mod}_\Bbbk \longrightarrow \mathrm{Mod}_\Bbbk^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}} \tag{1.1.30}$$

is the composition of NC^{w} with the projection on the q-th weight component. In particular, the unit map

$$M \longrightarrow (\mathrm{NC}^{\mathrm{w}}(\mathrm{triv}_\varepsilon M(q)))_q$$

for the adjunction $\mathrm{triv}_\varepsilon \circ (-)(q) \dashv (-)_q \circ \mathrm{NC}^{\mathrm{w}}$ is an equivalence, and since $(-)(q)$ is fully faithful for any integer q we have a chain of equivalences of mapping spaces

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(M(q),\, N(q)) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(M,\, N)\\ &\simeq \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\mathrm{triv}_{\varepsilon}(M(q)),\, \mathrm{triv}_{\varepsilon}(N(q))).\end{aligned}$$

1.2 Naive and clever truncations

For any integer p, we can consider the full sub-∞-category $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant p}$ of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ spanned by all those mixed graded $\Bbbk$-modules whose q weight component is 0 whenever $q > p$.
The inclusion

$$\iota_{\leqslant p} \colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant p} \subseteq \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

commutes with all limits and colimits, since they are computed weight-wise (Lemma 1.1.25). It follows that for all integers p the inclusion $\iota_{\leqslant p}$, being a limit and colimit preserving ∞-functor between presentable ∞-categories admits both a left and a right adjoint, again in virtue of the Adjoint Functor Theorem.

Definition 1.2.1.

1. The ∞-functor

$$\sigma_{\leqslant p} \colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant p}$$

   right adjoint to the inclusion

$$\iota_{\leqslant p} \colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant p} \subseteq \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

   is the *naive truncation in weights* $\leqslant p$.

2. The ∞-functor

$$\theta_{\leqslant p} \colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant p}$$

   left adjoint to the inclusion

$$\iota_{\leqslant p} \colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant p} \subseteq \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

   is the *clever truncation in weights* $\leqslant p$.

The ∞-functor of Lemma 1.2.1.1 must be understood as the ∞-functor which sends a mixed graded $\Bbbk$-module $M_\bullet$ to the mixed graded $\Bbbk$-module $\sigma_{\leqslant p} M_\bullet$ whose q weight component is M_p if $q \leqslant p$, and 0 otherwise, with obviously induced mixed differential. This is different from the ∞-functor of Lemma 1.2.1.2 which in turn is an analogue of the clever truncation of chain complexes in the context of mixed graded $\Bbbk$-modules.

Remark 1.2.2. By the adjunction $\iota_{\leqslant p} \dashv \sigma_{\leqslant p}$, for any mixed graded $\Bbbk$-module $M_\bullet$ we have a counit morphism

$$\epsilon_{M_\bullet}\colon \iota_{\leqslant p}\left(\sigma_{\leqslant p} M_\bullet\right) \longrightarrow M_\bullet$$

which includes its part concentrated in weights $\leqslant p$ in itself.
For any $q \leqslant p$, we have that

$$\sigma_{\leqslant q}\left(\iota_{\leqslant p}\left(\sigma_{\leqslant p} M_\bullet\right)\right) \simeq \sigma_{\leqslant q} M_\bullet$$

so the counit morphisms yield inclusions

$$\epsilon_{\iota_{\leqslant p}\left(\theta_{\leqslant p} M_\bullet\right)}\colon \iota_{\leqslant q}\left(\sigma_{\leqslant q}\left(\iota_{\leqslant p}\left(\sigma_{\leqslant p} M_\bullet\right)\right)\right) \simeq \iota_{\leqslant q}\left(\sigma_{\leqslant q} M_\bullet\right) \longrightarrow \iota_{\leqslant p}\left(\sigma_{\leqslant p} M_\bullet\right).$$

We shall commit a slight abuse of notation, and identify $\sigma_{\leqslant q} M_\bullet$ with its inclusion $\iota_{\leqslant q}\left(\sigma_{\leqslant q} M_\bullet\right)$ in $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$; in particular, we shall write the inclusion morphisms above simply as

$$\sigma_{\leqslant q} M_\bullet \hookrightarrow \sigma_{\leqslant p} M_\bullet \hookrightarrow M_\bullet. \tag{1.2.3}$$

Using the dual properties of the adjunction $\theta_{\leqslant p} \dashv \iota_{\leqslant p}$, we easily obtain dual properties for the clever truncation ∞-functor.

1.2.4. A completely dual discussion can be carried out by replacing $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant p}$ with $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant p}$, which is now the full sub-∞-category consisting of those mixed graded $\Bbbk$-modules which are trivial in weights $q < p$. Now, the inclusion

$$\iota_{\geqslant p}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant p} \subseteq \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

preserves all limits and colimits and hence admits both left and right adjoints.

Definition 1.2.5.

1. The ∞-functor

   $$\sigma_{\geqslant p}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant p}$$

   left adjoint to the inclusion

   $$\iota_{\geqslant p}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant p} \subseteq \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

   is the *naive truncation in weights* $\geqslant p$.

2. The ∞-functor

   $$\theta_{\geqslant p}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant p}$$

right adjoint to the inclusion

$$\iota_{\geqslant p}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant p} \subseteq \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

is the *naive truncation in weights* $\geqslant p$.

Again, the notation and the nomenclature suggest the behavior of the ∞-functors on a mixed graded $\Bbbk$-module: the one in Definition 1.2.5.1 simply kills M_q for any $q < p$, while the one in Definition 1.2.5.2 changes the underlying graded $\Bbbk$-module even in weight $q \geqslant p$, since it involves some sort of totalization.

1.3 Algebras and coalgebras in the mixed graded setting

In virtue of 1.1.6, the ∞-category $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ is a symmetric monoidal ∞-category (which is also presentable and stable, thanks to the presentation via a Dwyer-Kan localization along weak equivalences of a stable model category provided in [CPT+17, Definition 1.2.1]), and analogous results hold also for the ∞-category of graded $\Bbbk$-modules $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$. This allows us to consider algebras and coalgebras for various operads and cooperads in the graded and mixed graded setting: the aim of this section is just to fix notations and establish the lax, oplax, or even strong monoidal structures of the ∞-functors we introduced earlier in this chapter, in order to check which ∞-functors can be defined also at the level of mixed graded algebras and/or coalgebras.

This section serves mainly as a reference for Chapter 2, where we shall consider Lie algebras, (augmented) commutative algebras, (coaugmented) cocommutative coalgebras, and associative algebras in $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ and $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$; all the results collected here are quite straight-forward and easy to prove. The only exception is probably Lemma 1.3.24, which concerns the fully faithfulness of the *graded* symmetric coalgebra ∞-functor, and requires some auxiliary technical results.

1.3.1. We shall begin by introducing the ∞-categories of ∞-*operads* and ∞-*cooperads*. Even if [Lur17, Chapters 2 and 3] studies carefully the properties of the former, we shall also be interested in the formalism developed in [GR17b, Chapter 6, Section 1] and [FG12], especially when dealing with coalgebras. Let

$$\mathrm{Mod}_{\Bbbk}^{\Sigma} := \prod_{n\geqslant 1} \mathrm{Rep}_{\Sigma_n}$$

be the ∞-category of symmetric sequences in the ∞-category of $\Bbbk$-modules. This ∞-category has a natural (non-symmetric) monoidal structure, called the *composition monoidal structure*, which makes the ∞-functor

$$\begin{aligned} \mathrm{Mod}_{\Bbbk}^{\Sigma} &\longrightarrow \mathrm{Fun}(\mathrm{Mod}_{\Bbbk}, \mathrm{Mod}_{\Bbbk}) \\ \{M_n\}_{n\geqslant 1} &\mapsto \left\{ N \mapsto \bigoplus_{n\geqslant 1} \left(M_n \otimes_{\Bbbk} N^{\otimes n}\right)_{\Sigma_n} \right\} \end{aligned}$$

strongly monoidal. Let us remark that this is just a "convenient" way to think about this monoidal structure: see [Hau22, Section 4] for a more technical (and purely ∞-categorical) construction of the composition product on symmetric sequences.

Definition 1.3.2.

1. An *augmented ∞-operad in* $\mathrm{Mod}_{\Bbbk}$ is an augmented associative algebra object in $\mathrm{Mod}_{\Bbbk}^{\Sigma}$ for the above monoidal structure.

2. A *coaugmented ∞-cooperad in* $\mathrm{Mod}_{\Bbbk}$ is coaugmented coassociative coalgebra object in $\mathrm{Mod}_{\Bbbk}^{\Sigma}$ for the above monoidal structure.

1.3.3. Given an augmented ∞-operad $\mathbb{O}$ in $\mathrm{Mod}_{\Bbbk}$ as in Definition 1.3.2, we can consider the *∞-category of $\mathbb{O}$-algebra objects* in any stable symmetric monoidal ∞-category $\mathscr{C}$ suitably tensored over $\mathrm{Mod}_{\Bbbk}$ (such as $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ or $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$) via the usual formalism provided in [Lur17, Chapter 3]. This agrees with taking modules for the monad over $\mathscr{C}$ defined by

$$\begin{aligned} \mathrm{Mod}_{\Bbbk}^{\Sigma} &\longrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{C}) \\ \{M_n\}_{n\geqslant 1} &\mapsto \left\{ C \mapsto \bigoplus_{n\geqslant 1} \left(M_n \otimes_{\Bbbk} C^{\otimes n}\right)_{\Sigma_n} \right\}. \end{aligned} \tag{1.3.4}$$

Under these assumptions, we shall denote with $\mathrm{Alg}_{\mathbb{O}}(\mathscr{C})$ the ∞-category of $\mathbb{O}$-algebras in $\mathscr{C}$.

Warning 1.3.5 ([FG12, Section 3.5])**.** Given an augmented ∞-cooperad $\mathbb{Q}$ in $\mathrm{Mod}_{\Bbbk}$, we would be tempted to define the ∞-category of $\mathbb{Q}$-algebra objects in $\mathscr{C}$ as the comodules for the comonad on $\mathscr{C}$ induced by 1.3.4. However, this turns out to be the *wrong* ∞-category: indeed, there are *four* different and *inequivalent* types of coalgebras in a stable symmetric monoidal ∞-category $\mathscr{C}$ for a given coaugmented cooperad $\mathbb{Q}$ in $\mathrm{Mod}_{\Bbbk}$.

1. Comodules for the comonad $\mathscr{C} \to \mathscr{C}$ defined by the action 1.3.4 of $\mathrm{Mod}_{\Bbbk}^{\Sigma}$ over $\mathscr{C}$ yield *ind-nilpotent $\mathbb{Q}$-coalgebras with divided powers*, which we shall denote by $\mathrm{cAlg}_{\mathbb{Q}}^{\text{ind-nil}}(\mathscr{C})_{\mathrm{dp}}$.

2. Comodules for the right lax action

$$\begin{aligned} \mathrm{Mod}_{\Bbbk}^{\Sigma} &\longrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{C}) \\ \{M_n\}_{n\geqslant 1} &\mapsto \left\{ C \mapsto \prod_{n\geqslant 1} \left(M_n \otimes_{\Bbbk} C^{\otimes n}\right)_{\Sigma_n} \right\} \end{aligned} \tag{1.3.6}$$

   yield *coalgebras with divided powers*, which we shall denote with $\mathrm{cAlg}_{\mathbb{Q}}(\mathscr{C})_{\mathrm{dp}}$. This is another different ∞-category of coalgebras and the endofunctor $\mathscr{C} \to \mathscr{C}$ defined by this right lax action is not even comonadic. Because of this, the forgetful ∞-functor $\mathrm{oblv}_{\mathbb{Q}}^{\mathrm{dp}}\colon \mathrm{cAlg}_{\mathbb{Q}}(\mathscr{C})_{\mathrm{dp}} \to \mathscr{C}$ does not even admit a right adjoint in general.

3. Comodules for the action

$$\begin{aligned} \mathrm{Mod}_{\Bbbk}^{\Sigma} &\longrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{C}) \\ \{M_n\}_{n\geqslant 1} &\mapsto \left\{ C \mapsto \bigoplus_{n\geqslant 1} \left(M_n \otimes_{\Bbbk} C^{\otimes n} \right)^{\Sigma_n} \right\} \end{aligned} \tag{1.3.7}$$

yield *ind-nilpotent coalgebras*, which we shall denote by $\mathrm{cAlg}_{\mathbb{Q}}^{\text{ind-nil}}(\mathscr{C})$.

4. Finally, comodules for the right lax action

$$\begin{aligned} \mathrm{Mod}_{\Bbbk}^{\Sigma} &\longrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{C}) \\ \{M_n\}_{n\geqslant 1} &\mapsto \left\{ C \mapsto \prod_{n\geqslant 1} \left(M_n \otimes_{\Bbbk} C^{\otimes n} \right)^{\Sigma_n} \right\} \end{aligned} \tag{1.3.8}$$

yield the ∞-category of *all* coalgebras, which we shall denote simply by $\mathrm{cAlg}_{\mathbb{Q}}(\mathscr{C})$. The ∞-functor $\mathrm{oblv}_{\mathbb{Q}} \colon \mathrm{cAlg}_{\mathbb{Q}}(\mathscr{C}) \to \mathscr{C}$ is conservative, preserves all colimits, and preserves totalizations of $\mathrm{oblv}_{\mathbb{Q}}$-split co-simplicial objects, hence is comonadic and admits a right adjoint. However, such right adjoint it is not easy to describe because the endofunctor on $\mathscr{C}$ defined by this comonad *does not* agree with the one induced by the right lax action of $\mathbb{Q}$. However, this is the correct ∞-category to consider, at least in our two main cases of interest (namely, $\mathrm{CoComm}^{\mathrm{aug}}$ and $\mathrm{CoAssoc}^{\mathrm{aug}}$).

The trace map $(-)_{\Sigma_n} \to (-)^{\Sigma_n}$ yields natural maps of right lax actions 1.3.4 → 1.3.7 and 1.3.6 → 1.3.8. In characteristic 0, both maps are homotopy equivalences by the usual symmetrization argument: i.e., every coalgebra has naturally a divided power structure. Since this is our standing assumption, we shall omit all references to divided powers in our notations from now on. Still, the two actions 1.3.4 ≃ 1.3.7 and 1.3.6 ≃ 1.3.8 are *not* equivalent: there is a natural map

$$\mathrm{res} \colon \mathrm{cAlg}_{\mathbb{Q}}^{\text{ind-nil}}(\mathscr{C}) \longrightarrow \mathrm{cAlg}_{\mathbb{Q}}(\mathscr{C}) \tag{1.3.9}$$

and in [GR17b, Chapter 6, Conjecture 2.8.4] it is conjectured that this map is fully faithful. It is known, however, that there is always an equivalence of ∞-functors from $\mathrm{cAlg}_{\mathbb{Q}}^{\text{ind-nil}}(\mathscr{C})$ to $\mathscr{C}$

$$\mathrm{oblv}_{\mathbb{Q}} \circ \mathrm{res} \simeq \mathrm{oblv}_{\mathbb{Q}}^{\text{ind-nil}} \tag{1.3.10}$$

([GR17b, Section 2.8.5]). Since $\mathrm{oblv}_{\mathbb{Q}}$ and $\mathrm{oblv}_{\mathbb{Q}}^{\text{ind-nil}}$ both commute with colimits and are conservative, it follows that res preserves all colimits as well, therefore it admits a right adjoint.

Notation 1.3.11. If $\mathrm{Alg}_{\mathbb{O}}$ is the ∞-category of $\mathbb{O}$-algebra objects in $\mathrm{Mod}_{\Bbbk}$, we shall denote by $\mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}}$ and $\varepsilon\text{-}\mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}}$ the ∞-categories of $\mathbb{O}$-algebras in $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ and $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$, respectively.
Dually, if $\mathrm{cAlg}_{\mathbb{Q}}$ is the ∞-category of $\mathbb{Q}$-coalgebra objects in $\mathrm{Mod}_{\Bbbk}$, we shall denote by $\mathrm{cAlg}_{\mathbb{Q}}^{\mathrm{gr}}$ and $\varepsilon\text{-}\mathrm{cAlg}_{\mathbb{Q}}^{\mathrm{gr}}$ the ∞-categories of $\mathbb{Q}$-coalgebras in $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ and $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$. For example, the ∞-category

of commutative mixed graded $\Bbbk$-algebras will be denoted by $\varepsilon\text{-}\mathrm{CAlg}_{\Bbbk}^{\mathrm{gr}} := \mathrm{CAlg}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$. Analogous notations will be used, without further explanation or definitions, for other algebra structures in (mixed) graded $\Bbbk$-modules in the remainder of the work.

1.3.12. Let $\mathbb{O}$ be an ∞-operad. As already observed in Section 1.1, the ∞-functors

$$\mathrm{oblv}_{\varepsilon}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

and

$$\mathrm{triv}_{\varepsilon}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

are strongly symmetric monoidal and accessible ∞-functors which preserve all limits and colimits, hence their right adjoints (which exist because of the Adjoint Functor Theorem) are lax symmetric monoidal. In particular, *all* these ∞-functors induce ∞-functors at the level of algebras and coalgebras: this means that forgetting a mixed structure, or setting a trivial one, does not alter the type of algebras considered. Moreover, forgetting the mixed structures preserves (and actually, creates) all limits and colimits.

1.3.13. Considering graded $\mathbb{O}$-algebras and mixed graded $\mathbb{O}$-algebras, we have two obvious forgetful ∞-functors

$$\mathrm{oblv}_{\mathbb{O}}^{\varepsilon\text{-}\mathrm{gr}}\colon \varepsilon\text{-}\mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

and

$$\mathrm{oblv}_{\mathbb{O}}^{\mathrm{gr}}\colon \mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}} \longrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}.$$

Both ∞-functors satisfy the hypothesis of [Lur17, Section 3.1], hence admit left adjoints

$$\mathrm{Free}_{\mathbb{O}}^{\varepsilon\text{-}\mathrm{gr}}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}}$$

and

$$\mathrm{Free}_{\mathbb{O}}^{\mathrm{gr}}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}}.$$

Moreover, the square of ∞-functors

$$\begin{tikzcd}
\varepsilon\text{-}\mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}} \arrow[r, shift left, "\mathrm{oblv}_{\varepsilon}"] \arrow[d, shift left, "\mathrm{oblv}_{\mathbb{O}}^{\varepsilon\text{-}\mathrm{gr}}"] & \mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}} \arrow[l, shift left, "\mathrm{triv}_{\varepsilon}"] \arrow[d, shift left, "\mathrm{oblv}_{\mathbb{O}}^{\mathrm{gr}}"] \\
\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \arrow[r, shift left, "\mathrm{oblv}_{\varepsilon}"] \arrow[u, shift left, "\mathrm{Free}_{\mathbb{O}}^{\varepsilon\text{-}\mathrm{gr}}"] & \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \arrow[l, shift left, "\mathrm{triv}_{\varepsilon}"] \arrow[u, shift left, "\mathrm{Free}_{\mathbb{O}}^{\mathrm{gr}}"]
\end{tikzcd}$$

commutes straight-forwardly in every direction.

1.3.14. As foretold in Warning 1.3.5, one needs to be careful when dealing with coalgebras. For sure, replacing $\mathbb{O}$ with an ∞-cooperad $\mathbb{Q}$ and considering the ∞-categories $\varepsilon\text{-}\mathrm{cAlg}_{\mathbb{Q}}^{\mathrm{gr,ind\text{-}nil}}$

and $\mathrm{cAlg}_{\mathbb{Q}}^{\mathrm{gr,ind\text{-}nil}}$ of (respectively) mixed graded and graded ind-nilpotent $\mathbb{Q}$-coalgebras, we have another commutative square of ∞-functors

$$\begin{tikzcd} \varepsilon\text{-}\mathrm{cAlg}_{\mathbb{Q}}^{\mathrm{gr,ind\text{-}nil}} \arrow[r, shift left, "\mathrm{oblv}_{\varepsilon}"] \arrow[d, shift right, "\mathrm{oblv}_{\mathbb{Q}}^{\varepsilon\text{-gr,ind-nil}}"'] & \mathrm{cAlg}_{\mathbb{Q}}^{\mathrm{gr,ind\text{-}nil}} \arrow[l, shift left, "\mathrm{triv}_{\varepsilon}"] \arrow[d, shift right, "\mathrm{oblv}_{\mathbb{Q}}^{\mathrm{gr,ind\text{-}nil}}"'] \\ \varepsilon\text{-}\mathrm{Mod}_{\mathbb{k}}^{\mathrm{gr}} \arrow[r, shift left, "\mathrm{oblv}_{\varepsilon}"] \arrow[u, shift right, "\mathrm{coFree}_{\mathbb{Q}}^{\varepsilon\text{-gr,ind-nil}}"'] & \mathrm{Mod}_{\mathbb{k}}^{\mathrm{gr}} \arrow[l, shift left, "\mathrm{triv}_{\varepsilon}"] \arrow[u, shift right, "\mathrm{coFree}_{\mathbb{Q}}^{\mathrm{gr,ind\text{-}nil}}"'] \end{tikzcd} \tag{1.3.15}$$

where now $\mathrm{coFree}_{\mathbb{Q}}^{\mathrm{gr,ind\text{-}nil}}$ and $\mathrm{coFree}_{\mathbb{Q}}^{\varepsilon\text{-gr,ind-nil}}$ are *right* adjoints to the forgetful ∞-functors $\mathrm{oblv}_{\mathbb{Q}}^{\mathrm{gr,ind\text{-}nil}}$ and $\mathrm{oblv}_{\mathbb{Q}}^{\varepsilon\text{-gr,ind-nil}}$ respectively. Again, since equivalences in all these ∞-categories are checked at the level of the underlying graded $\mathbb{k}$-module, this square commutes in every direction as well.

Remark 1.3.16. Let $\mathscr{C}$ be a symmetric monoidal ∞-category $\mathbb{k}$-linear over a base ring $\mathbb{k}$ of characteristic 0. Even if we do not know whether 1.3.9 is a fully faithful ∞-functor, thanks to [GR17b, Chapter 6, Theorem 2.9.4] we know that

$$\begin{aligned} \mathrm{Map}_{\mathscr{C}}(X,\ Y) \simeq \mathrm{Map}_{\mathscr{C}}\big(\mathrm{oblv}_{\mathbb{Q}}\big(\mathrm{triv}_{\mathbb{Q}}^{\mathrm{ind\text{-}nil}}(X)\big),\ Y\big) &\simeq \mathrm{Map}_{\mathrm{cAlg}_{\mathbb{Q}}^{\mathrm{ind\text{-}nil}}(\mathscr{C})}\big(\mathrm{triv}_{\mathbb{Q}}^{\mathrm{ind\text{-}nil}}(X),\ \mathrm{coFree}_{\mathbb{Q}}^{\mathrm{ind\text{-}nil}}(Y)\big) \\ &\simeq \mathrm{Map}_{\mathrm{cAlg}_{\mathbb{Q}}(\mathscr{C})}\big(\mathrm{triv}_{\mathbb{Q}}(X),\ \mathrm{res}\big(\mathrm{coFree}_{\mathbb{Q}}^{\mathrm{ind\text{-}nil}}(Y)\big)\big) \end{aligned}$$

at least when $\mathbb{Q}$ is either $\mathrm{coAssoc}^{\mathrm{aug}}$ or $\mathrm{coComm}^{\mathrm{aug}}$, which are the two main cooperads we shall be interested in. In particular, denoting by $\mathrm{coFree}_{\mathbb{Q}}$ the composition of ∞-functors $\mathrm{res}\circ\mathrm{coFree}_{\mathbb{Q}}^{\mathrm{ind\text{-}nil}}$, in these cases one can obtain an analogous square to 1.3.15 where now the ∞-categories of ind-nilpotent coalgebras are replaced by the ∞-categories of all $\mathbb{Q}$-coalgebras, thanks to the equivalence 1.3.10. The vertical arrows - while not being adjoints anymore - interact well one with the other just like in the ind-nilpotent case.

1.3.17. Many of the important ∞-functors relating usual $\mathbb{k}$-modules and (mixed) graded $\mathbb{k}$-modules can be defined at the level of $\mathbb{O}$-algebras and $\mathbb{Q}$-coalgebras.

- From the description of the tensor product of mixed graded $\mathbb{k}$-modules provided in 1.1.6, it follows that the ∞-functor $(-)(0)\colon \mathrm{Mod}_{\mathbb{k}} \to \varepsilon\text{-}\mathrm{Mod}_{\mathbb{k}}^{\mathrm{gr}}$ is obviously strongly monoidal, hence it defines an ∞-functor at the level of algebras and coalgebras objects

$$(-)(0)\colon \mathrm{Alg}_{\mathbb{O}} \longrightarrow \varepsilon\text{-}\mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}}$$

and

$$(-)(0)\colon\ \mathrm{cAlg}_{\mathbb{Q}} \longrightarrow \varepsilon\text{-}\mathrm{cAlg}_{\mathbb{Q}}^{\mathrm{gr}}\,.$$

As a consequence, its right adjoint $|-|$ is lax symmetric monoidal, and thus yields an ∞-functor

$$|-|\colon \mathrm{Alg}_{\mathbb{O}} \longrightarrow \varepsilon\text{-}\mathrm{Alg}_{\mathbb{O}}^{\mathrm{gr}}.$$

- Moreover, since $\Bbbk(\infty)$ is a cocommutative and counital coalgebra object ([CPT+17, Section 1.5]), the Tate realization acquires a lax symmetric monoidal structure as well. Again, it induces an ∞-functor
$$|-|^{\mathrm{t}}\colon \varepsilon\text{-}\mathrm{Alg}^{\mathrm{gr}}_{\mathbb{O}} \longrightarrow \mathrm{Alg}_{\mathbb{O}}.$$
Actually, if we restrict ourselves to the ∞-category $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr},\geqslant 0}_{\Bbbk}$ spanned by mixed graded $\Bbbk$-modules trivial in negative weights, such monoidal structure is strict, hence it also provides an ∞-functor
$$|-|^{\mathrm{t}}\colon \mathrm{cCcAlg}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr},\geqslant 0}_{\Bbbk}\right) \longrightarrow \mathrm{cCcAlg}_{\Bbbk}.$$
While one can show it directly via tedious computations with explicit models provided by chain complexes, we shall derive it from Theorem 1.5.25.

- Finally, consider the ∞-functor $(-)_0\colon \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$. In general, this is both lax and oplax monoidal, but not strongly so: in fact, given two mixed graded $\Bbbk$-modules $M_\bullet$ and $N_\bullet$, their tensor product $M_\bullet \otimes^{\varepsilon\text{-gr}}_{\Bbbk} N_\bullet$ exhibits $M_0 \otimes_{\Bbbk} N_0$ as a direct summand of $\left(M_\bullet \otimes^{\varepsilon\text{-gr}}_{\Bbbk} N_\bullet\right)_0$, and the natural inclusions and projections yield the desired lax and oplax monoidal structures. In particular, we have ∞-functors defined at the level of algebras and coalgebras objects
$$(-)_0\colon \varepsilon\text{-}\mathrm{Alg}^{\mathrm{gr}}_{\mathbb{O}} \longrightarrow \mathrm{Alg}_{\mathbb{O}}$$
and
$$(-)_0\colon \varepsilon\text{-}\mathrm{cAlg}^{\mathrm{gr}}_{\mathbb{Q}} \longrightarrow \mathrm{cAlg}_{\mathbb{Q}}.$$
Let us remark that this lax and oplax monoidal structure is actually strong when restricted to the full sub-∞-categories $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr},\geqslant 0}_{\Bbbk}$ and $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr},\leqslant 0}_{\Bbbk}$: this follows trivially because, in such cases, the only summand in $\left(M_\bullet \otimes^{\varepsilon\text{-gr}}_{\Bbbk} N_\bullet\right)_0$ is $M_0 \otimes_{\Bbbk} N_0$ itself.

Remark 1.3.18. As stated in [GR14, Chapter 6, Section 4.2], if one takes as $\mathbb{Q}$ the ∞-cooperad $\mathrm{CoComm}^{\mathrm{aug}}$ of coaugmented cocommutative coalgebras, then the composition
$$\mathrm{res}\circ\mathrm{coFree}^{\text{ind-nil}}\colon \mathrm{Mod}_{\Bbbk} \longrightarrow \mathrm{cCcAlg}^{\text{ind-nil}}_{\Bbbk//\Bbbk} \xrightarrow{1.3.9} \mathrm{cCcAlg}_{\Bbbk//\Bbbk} \tag{1.3.19}$$
coincides with the usual symmetric algebra ∞-functor with its cocommutative coalgebra structure given by its natural Hopf algebra structure.

It is known ([GR14, Chapter 6, Conjecture 2.9.3 and Theorem 2.9.4]) that it is possible to extract primitive elements from the symmetric (co)algebra, i.e., there exists an ∞-functor
$$\mathrm{Prim}\colon \mathrm{cCcAlg}_{\Bbbk//\Bbbk} \longrightarrow \mathrm{Mod}_{\Bbbk}$$
and a natural equivalence $M \simeq \mathrm{Prim}\left(\mathrm{Sym}_{\Bbbk}(M)\right)$ for any $\Bbbk$-module M. We want to show that, in the graded setting, the extraction of primitive elements is particularly simple, at least in the ind-nilpotent case.

1.3.20. Let $\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$ be the ∞-category $\mathrm{cAlg}_{\mathrm{CoComm}^{\mathrm{aug}}}\left(\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)$ of coaugmented cocommutative coalgebras in mixed graded $\Bbbk$-modules. Thanks to the discussion provided in 1.3.17, we know that the ∞-functor $(-)(0)\colon \mathrm{Mod}_{\Bbbk} \to \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$, is strongly monoidal, and its left/right adjoint $(-)_0\colon \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$ hence is both lax and oplax monoidal. So, consider the ∞-functor

$$\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}\colon \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \longrightarrow \mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk} \tag{1.3.21}$$

be the (graded version of the) ∞-functor 1.3.19, given by the composition of the cofree ind-nilpotent coalgebra ∞-functor for the operad $\mathrm{CoComm}^{\mathrm{aug}}$, right adjoint to the forgetful ∞-functor

$$\mathrm{oblv}^{\text{ind-nil}}_{\mathrm{cCcAlg}}\colon \mathrm{cCcAlg}^{\text{gr,ind-nil}}_{\Bbbk//\Bbbk} \longrightarrow \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$$

and the ∞-functor res of 1.3.9. We want to prove the following result.

Proposition 1.3.22. *The ∞-functor*

$$\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}((-)[-1](1))\colon \mathrm{Mod}_{\Bbbk} \overset{[-1]}{\simeq} \mathrm{Mod}_{\Bbbk} \xhookrightarrow{(-)(1)} \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \xrightarrow{\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}} \mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk} \tag{1.3.23}$$

is fully faithful.

We shall deduce Proposition 1.3.22 from the following Lemma.

Lemma 1.3.24. *The ∞-functor*

$$\mathrm{Mod}_{\Bbbk} \overset{[-1]}{\simeq} \mathrm{Mod}_{\Bbbk} \xhookrightarrow{(-)(1)} \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \xrightarrow{\mathrm{coFree}^{\text{ind-nil}}_{\mathrm{cCcAlg}}} \mathrm{cCcAlg}^{\text{gr,ind-nil}}_{\Bbbk//\Bbbk} \tag{1.3.25}$$

is fully faithful.

Proof. By abstract nonsense, the composition 1.3.25 admits a left adjoint

$$\mathrm{cCcAlg}^{\text{gr,ind-nil}}_{\Bbbk//\Bbbk} \xrightarrow{\mathrm{oblv}^{\text{ind-nil}}_{\mathrm{cCcAlg}}} \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \xrightarrow{(-)_1} \mathrm{Mod}_{\Bbbk} \overset{[1]}{\simeq} \mathrm{Mod}_{\Bbbk}$$

given by composing all left adjoints of each ∞-functor which makes up 1.3.25. So it will suffice to show that the counit

$$M \longrightarrow \mathrm{oblv}^{\text{ind-nil}}_{\mathrm{cCcAlg}}\left(\mathrm{coFree}^{\text{ind-nil}}_{\mathrm{cCcAlg}}(M[-1](1))\right)_1[1]$$

of such adjunction is an equivalence. Thanks to the equivalence 1.3.10, we know that

$$\mathrm{oblv}^{\text{ind-nil}}_{\mathrm{cCcAlg}} \simeq \mathrm{oblv}_{\mathrm{cCcAlg}} \circ \mathrm{res}$$

and Remark 1.3.18 assures us that the composition $\mathrm{coFree}_{\mathrm{cCcAlg}} := \mathrm{res} \circ \mathrm{coFree}^{\text{ind-nil}}_{\mathrm{cCcAlg}}$ is just the symmetric coalgebra ∞-functor $\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}$, with its natural grading. So, we can rewrite the counit above as

$$\begin{aligned} M \longrightarrow & \operatorname{oblv}^{\text{ind-nil}}_{\text{cCcAlg}}\left(\operatorname{coFree}^{\text{ind-nil}}_{\text{cCcAlg}}(M[-1](1))\right)_1[1] \\ \simeq & \operatorname{oblv}_{\text{cCcAlg}}\left(\operatorname{res}\left(\operatorname{coFree}^{\text{ind-nil}}_{\text{cCcAlg}}(M[-1](1))\right)\right)_1[1] \\ \simeq & \operatorname{Sym}^{\text{gr}}_{\Bbbk}(M[-1](1))_1[1] \simeq M \end{aligned}$$

which proves our assertion. □

Lemma 1.3.24 and [GR17b, Chapter 6, Corollary 2.10.7] together imply Proposition 1.3.22.

Remark 1.3.26. The analogous claim of Proposition 1.3.22 holds if we consider the graded symmetric algebra ∞-functor as the *free augmented commutative algebra* ∞-functor from $\mathrm{Mod}_{\Bbbk}$ to $\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$, where now the latter denotes the ∞-category of augmented commutative algebras in graded $\Bbbk$-modules. The proof is completely analogous to the one of Lemma 1.3.24, without needing the further technical auxiliary result of [GR17b].

1.4 t-structure on mixed graded modules

The main result of this section is the following Theorem, which provides a t-structure on mixed graded $\Bbbk$-modules describing explicitly both connective and coconnective objects.

Theorem 1.4.1. *Let $\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0}$ be the full ∞-subcategory of $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ spanned by those mixed graded k-modules $M_\bullet$ such that, for any integer q, the $\Bbbk$-module M_q is $(-q)$-connective. Dually, let $\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\leqslant 0}$ be the full ∞-subcategory spanned by those mixed graded $\Bbbk$-modules $M_\bullet$ such that, for all $q \in \mathbb{Z}$, the $\Bbbk$-module M_q is $(-q)$-coconnective. These sub-∞-categories determine an accessible t-structure on the stable ∞-category of mixed graded $\Bbbk$-modules, which we call the mixed graded Postnikov t-structure. When $\Bbbk$ is discrete, the heart of such t-structure is equivalent to the classical abelian 1-category of chain complexes of $\Bbbk$-modules* $\mathrm{dgMod}_{\Bbbk}$.

Remark 1.4.2. The name for such t-structure is borrowed from [AGH19], where the standard t-structure on modules over a connective $\mathbb{E}_1$-ring of [Lur17, Proposition 7.1.1.13] is referred to as the *Postnikov t-structure*.

Proof of Theorem 1.4.1. Consider the sub-∞-category $\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0}$ described in the statement of Theorem 1.4.1. It is a sub-∞-category of $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ which is closed under all colimits and extensions, since they are computed weight-wise (Lemma 1.1.25) and so it suffices to prove the claim in $\mathrm{Mod}_{\Bbbk}$, where q-connective objects are stable under colimits and extensions for any q (this is [Lur17, Proposition 7.1.1.13]). Moreover it is presentable: indeed, it fits in a

$(\infty, 2)$-pullback diagram of presentable ∞-categories

$$\begin{tikzcd} \left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \arrow[d, "\mathrm{oblv}_{\varepsilon}"] \\ \left(\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0} \arrow[r, hook] & \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \end{tikzcd}$$

where

$$\left(\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0} := \prod_{q \in \mathbb{Z}} (\mathrm{Mod}_{\Bbbk})_{\geqslant -q}\,.$$

Thus, the sub-∞-category $\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0}$ defines a t-structure such that $\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0}$ is precisely the ∞-category of connective objects, thanks to [Lur17, Proposition 1.4.4.11]. In detail: under these hypotheses, the ∞-category $\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0}$ is a colocalization of $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$, hence the Adjoint Functor Theorem provides an ∞-functor

$$\tau^{\varepsilon\text{-}\mathrm{gr}}_{\geqslant 0} : \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \longrightarrow \left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0} \tag{1.4.3}$$

right adjoint to the obvious inclusion. Thus, the (-1)-coconnective objects are defined to be those mixed graded $\Bbbk$-modules $N_\bullet$ such that $\tau^{\varepsilon\text{-}\mathrm{gr}}_{\geqslant 0} N_\bullet \simeq 0$ in $\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\geqslant 0}$, and so each mixed graded $\Bbbk$-module is sandwiched in a (∞-functorial) cofiber sequence

$$\tau^{\varepsilon\text{-}\mathrm{gr}}_{\geqslant 0} M_\bullet \longrightarrow M_\bullet \longrightarrow \tau^{\varepsilon\text{-}\mathrm{gr}}_{\leqslant -1} M_\bullet \tag{1.4.4}$$

where $\tau^{\varepsilon\text{-}\mathrm{gr}}_{\geqslant 0} M_\bullet \to M_\bullet$ is the adjoint to the identity of $\tau^{\varepsilon\text{-}\mathrm{gr}}_{\geqslant 0} M_\bullet$. The interesting part of the statement of the Theorem is the one that characterizes the coconnective objects: we have to prove that a mixed graded $\Bbbk$-module is coconnective $N_\bullet$ *precisely* if N_q is $(-q)$-coconnective for any integer q. [Lur17, Remark 1.2.1.3] assures us that (-1)-coconnective objects are uniquely determined by the property of being right orthogonal to any connective object. Since the enrichment in spaces of a $\Bbbk$-linear ∞-category $\mathscr{C}$ is given by the truncation in degrees $\geqslant 0$ of the mapping $\Bbbk$-module, this means that the $\Bbbk$-module $\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}}(M_\bullet, N_\bullet)$ has to be (-1)-coconnective for any $M_\bullet$ connective. Since in $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ colimits are computed weight-wise, and by definition

$$\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\leqslant -1} := \left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)_{\leqslant 0}[-1],$$

our claim is equivalent to proving that

$$\tau_{\geqslant 0}\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}}(M_\bullet, N_\bullet) \simeq \{*\}$$

for $N_\bullet$ such that N_q is $(-q-1)$-coconnective for any integer q. We shall need the following two Lemmas.

Lemma 1.4.5. *Let $N_\bullet$ such that N_q is $(-q-1)$-coconnective for any integer q. Then, $N_\bullet$ is a (-1)-coconnective object for the mixed graded Postnikov t-structure.*

Proof. We recall that the enrichment in $\Bbbk$-modules of mixed graded $\Bbbk$-modules is given by the fiber for the mixed structure morphism

$$\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)_0 \xrightarrow{\varepsilon_0} \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)_{-1}[-1].$$

The enrichment in spaces is now given by applying the truncation ∞-functor $\tau_{\geqslant 0}$ to such $\Bbbk$-module ([Lur17, Construction 1.3.1.13]). So, let $M_\bullet$ be such that M_q is $(-q)$-connective for all q's: we have a fiber sequence

$$\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}}(M_\bullet, N_\bullet) \longrightarrow \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)_0 \xrightarrow{\varepsilon_0} \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_\bullet, N_\bullet)_{-1}[-1].$$

Writing explicitly the second and third terms of such fiber sequence, using 1.1.6, we get

$$\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}}(M_\bullet, N_\bullet) \longrightarrow \prod_{q\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_\Bbbk}(M_q, N_q) \xrightarrow{\varepsilon} \prod_{p\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Mod}_\Bbbk}(M_p, N_{p-1})[-1].$$

The second term of such fiber sequence is (at least) (-1)-coconnective, because it is the product of (-1)-coconnective $\Bbbk$-modules (being M_q $(-q)$-connective and N_q $(-q-1)$-coconnective for all q's). Analogously, also the third term is (at least) (-2)-coconnective, hence in particular (-1)-coconnective. Being (-1)-coconnective objects stable under all limits, we get that $\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}}}(M_\bullet, N_\bullet)$ is now (-1)-coconnective, hence its truncation yields a contractible space. □

Lemma 1.4.6. *Suppose that $N_\bullet$ is (-1)-coconnective for the mixed graded Postnikov t-structure. Then N_q is $(-q-1)$-coconnective as a $\Bbbk$-module.*

Proof. Without loss of generality, by shifting weights, we can reduce ourselves to the case that $N_\bullet$ is such that, for some integer $n \geqslant 0$, there exists a non-trivial homotopy group $\pi_n N_0 \not\simeq 0$. Let $\Bbbk(0)$ be the mixed graded $\Bbbk$-module with trivial mixed structure in pure weight 0. Then by the adjunction 1.1.14, we have equivalences of $\Bbbk$-modules

$$\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}}}(\Bbbk(0), N_\bullet) \simeq \mathrm{Map}_{\mathrm{Mod}_\Bbbk}(\Bbbk, |N_\bullet|) \simeq |N_\bullet|\,.$$

If the realization of $N_\bullet$ is not (-1)-truncated, then $N_\bullet$ cannot be (-1)-coconnective for the mixed graded Postnikov t-structure, since $\Bbbk(0)$ is obviously connective. So, let us assume that $|N_\bullet|$ is (-1)-coconnective as a $\Bbbk$-module. Using the naive truncation ∞-functor in degree $\leqslant -1$ of 1.2.1.1, by adjunction we have a cofiber sequence

$$\sigma_{\leqslant -1} N_\bullet \longrightarrow N_\bullet \longrightarrow \mathrm{cofib}.$$

Applying the realization ∞-functor, that being a right adjoint between stable ∞-categories preserves cofiber sequences, we have another cofiber sequence of the form

$$\left|\sigma_{\leqslant -1} N_\bullet\right| \longrightarrow |N_\bullet| \longrightarrow |\mathrm{cofib}|\,.$$

Since limits and colimits are computed weight-wise in $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$, by the description of the realization ∞-functor provided in 1.1.13, we easily have that $|\mathrm{cofib}| \simeq N_0$, and that moreover $\left|\sigma_{\leqslant -1} N_\bullet\right| \simeq N_0[-1]$. In particular, $\left|\sigma_{\leqslant -1} N_\bullet\right|$ cannot be trivial, because this would contradict that $\pi_n N_0 \not\simeq 0$ for some $n \geqslant 0$. Therefore, considering the $\Bbbk$-module $\Bbbk[1](-1)$, we have

$$\begin{aligned} \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\Bbbk[1](-1),\ N_\bullet) &\simeq \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\big(\Bbbk(0),\ \sigma_{\leqslant -1} N_\bullet[-1](\!(1)\!)\big) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(\Bbbk,\ \left|\sigma_{\leqslant -1} N_\bullet[-1](\!(1)\!)\right|\big) \\ &\simeq \left|\sigma_{\leqslant -1} N_\bullet(\!(1)\!)[-1]\right| \simeq N_0. \end{aligned}$$

Now, $\Bbbk[1](-1)$ is again connective for the mixed graded Postnikov t-structure, and now we have a non-trivial map towards $N_\bullet$, which is given by any non-trivial homology class in $\pi_n N_0$. $\square$

Lemma 1.4.5 and Lemma 1.4.6 together imply that the class of (-1)-coconnective objects for the mixed graded Postnikov t-structure and the class of mixed graded $\Bbbk$-modules which are $(-q-1)$-coconnective in each weight q coincide, and this concludes the proof of the main statement. The claim about the heart of such t-structure can be proved directly. The idea is that a mixed graded $\Bbbk$-module lying in the heart of the t-structure is forced to be a mixed graded $\Bbbk$-module whose weight p component is a discrete $\Bbbk$-module concentrated in homological degree $-p$, and the differential for this chain complex is given by the mixed differential. Yet, we shall see this claim as a corollary of Theorem 1.5.25. $\square$

Porism 1.4.7. The proof of Theorem 1.4.1 *does not rely* on the assumption that $\Bbbk$ is a field. We can always define the ∞-category of mixed graded R-modules, for R a $\mathbb{Q}$-algebra in $\mathbb{E}_\infty$-rings, as the ∞-category of quasi-coherent sheaves on the affine group stack $\mathrm{B}\mathbb{G}_{\mathrm{a},R} \rtimes \mathbb{G}_{\mathrm{m},R}$. Let $\mathcal{S}\mathrm{p}^{\mathrm{gr}} := \prod_{p\in\mathbb{Z}} \mathcal{S}\mathrm{p}$ be the ∞-category of graded spectra, and let $\mathcal{S}\mathrm{p}^{\mathrm{gr}}_{\geqslant 0}$ be its full sub-∞-category spanned by graded spectra which are $(-q)$-connective for all integers q. In virtue of [Mou21, Theorem 4.1], we have an equivalence of ∞-categories

$$\mathcal{S}\mathrm{p}^{\mathrm{gr}} \simeq \mathrm{QCoh}\big(\mathrm{B}\mathbb{G}_{\mathrm{m},\mathbb{S}}\big),$$

where $\mathbb{G}_{\mathrm{m},\mathbb{S}} := \mathrm{Spec}(\mathbb{S}[\mathbb{Z}])$ is the flat multiplicative group (spectral) scheme. Hence, by the abstract nonsense recollected at the beginning of the proof of Theorem 1.4.1, the ∞-category

$$\big(\varepsilon\text{-}\mathrm{Mod}_R^{\mathrm{gr}}\big)_{\geqslant 0} := \varepsilon\text{-}\mathrm{Mod}_R^{\mathrm{gr}} \times_{\mathcal{S}\mathrm{p}^{\mathrm{gr}}} \mathcal{S}\mathrm{p}^{\mathrm{gr}}_{\geqslant 0}$$

is again presentable, closed under all colimits and extensions, and so it determines the connective part for an accessible t-structure over $\varepsilon\text{-}\mathrm{Mod}_R^{\mathrm{gr}}$. Even for an arbitrary derived scheme X defined over a derived $\mathbb{Q}$-algebra $\Bbbk$, one can define ([TV20b, Section 1.2.1]) the *∞-category of mixed graded modules over X* as the ∞-category of sheaves with values in $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ for the Zariski topology on X:

$$\varepsilon\text{-}\mathrm{Mod}_X^{\mathrm{gr}} := \mathrm{Shv}_{\mathrm{Zar}}\big(X,\ \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big).$$

Setting $\big(\varepsilon\text{-}\mathrm{Mod}_X^{\mathrm{gr}}\big)_{\geqslant 0}$ to be the ∞-category of those sheaves of mixed graded $\Bbbk$-modules over X which are connective as mixed graded S-modules for any S-point $\mathrm{Spec}(S) \to X$ (the fact that this yields, indeed, a t-structure is essentially [GR17a, Chapter 3, Section 1.5.1]). The only property of a field of characteristic 0 that we use in the proof of Theorem 1.4.1 is the fact that it is connective: indeed, *whenever R is connective*, then the coconnective part for such t-structure can be characterized as

$$\big(\varepsilon\text{-}\mathrm{Mod}_R^{\mathrm{gr}}\big)_{\leqslant 0} := \varepsilon\text{-}\mathrm{Mod}_R^{\mathrm{gr}} \times_{\mathcal{S}\mathrm{p}^{\mathrm{gr}}} \mathcal{S}\mathrm{p}^{\mathrm{gr}}_{\leqslant 0}.$$

This fact is not true in general if R is not connective (see also [Lur11c, Warning 3.5.9]). The analogous statement on the coconnective part for the t-structure on mixed graded modules over a non-affine derived scheme X case can be recovered under the hypothesis that the sheaf of $\mathbb{E}_\infty$-rings is connective (in the sense of [Lur11a, Definition 1.20]), using [Lur11b, Proposition 2.1.3], or in the non-connective case when X is at least an Artin stack (here, we use [GR17a, Chapter 3, Proposition 1.5.4]).

The characterization of the connective and coconnective objects of the t-structure of Theorem 1.4.1 allows us to describe pretty neatly how the connective cover and the coconnective cover ∞-functors behave on a mixed graded $\Bbbk$-module.

Corollary 1.4.8. *Let $\tau_{\geqslant 0}^{\varepsilon\text{-}\mathrm{gr}}$ be the connective cover ∞-functor described in 1.4.3, and let $\tau_{\leqslant 0}^{\varepsilon\text{-}\mathrm{gr}}$ be the cofiber ∞-functor described in 1.4.4. For any mixed graded $\Bbbk$-module $M_\bullet$, there exist natural equivalences of $\Bbbk$-modules*

$$\big(\tau_{\geqslant 0}^{\varepsilon\text{-}\mathrm{gr}} M_\bullet\big)_q \simeq \tau_{\geqslant -q} M_q$$

and

$$\big(\tau_{\leqslant 0}^{\varepsilon\text{-}\mathrm{gr}} M_\bullet\big)_q \simeq \tau_{\leqslant -q} M_q.$$

Proof. Since colimits are computed weight-wise, taking the $\Bbbk$-module in weight q is an exact ∞-functor from mixed graded $\Bbbk$-modules to $\Bbbk$-modules for any integer q. So, considering the canonical fiber sequence

$$\tau_{\geqslant 0}^{\varepsilon\text{-}\mathrm{gr}} M_\bullet \longrightarrow M_\bullet \longrightarrow \tau_{\leqslant -1}^{\varepsilon\text{-}\mathrm{gr}} M_\bullet$$

and taking the component in weight q we get another fiber sequence

$$\big(\tau_{\geqslant 0}^{\varepsilon\text{-}\mathrm{gr}} M_\bullet\big)_q \longrightarrow M_q \longrightarrow \big(\tau_{\leqslant -1}^{\varepsilon\text{-}\mathrm{gr}} M_\bullet\big)_q.$$

By suitably shifting, it follows that this fiber sequence is the essentially unique fiber sequence of $\Bbbk$-modules that extends M_q with a $(-q)$-connective part on the left and a $(-q-1)$-coconnective part on the right, hence $\big(\tau_{\geqslant 0}^{\varepsilon\text{-}\mathrm{gr}} M_\bullet\big)_q \simeq \tau_{\geqslant -q} M_q$ and $\big(\tau_{\leqslant -1}^{\varepsilon\text{-}\mathrm{gr}} M_\bullet\big)_q \simeq \tau_{\leqslant -q-1} M_q$. $\square$

1.5 Relationship with filtered modules

In this section, we state the main link between the ∞-categories of filtered $\mathbb{k}$-modules and mixed graded $\mathbb{k}$-modules, and their respective t-structures described in Theorems 1.5.14 and 1.4.1. The main result of this section is Theorem 1.5.25, which states that complete (in the homotopical sense) filtrations on $\mathbb{k}$-modules arise uniquely from a mixed structure on the associated graded of the filtration, and moreover that this assignment is t-exact. This can be seen as a slight improvement on [TV20b, Proposition 1.3.1] and [CCN21, Proposition 2.27], which exhibited a fully faithful embedding $\varepsilon\text{-}\mathrm{Mod}_{\mathbb{k}}^{\mathrm{gr}} \hookrightarrow \mathrm{Mod}_{\mathbb{k}}^{\mathrm{fil}}$ and characterized its essential image, without considering any t-structure on those ∞-categories.
We start by reviewing some important concepts concerning filtered $\mathbb{k}$-module, following [BMS19] and [GP18].

Definition 1.5.1 ([BMS19, Definition 5.1]). Let $\mathbb{Z}_{\geqslant} := \mathbb{Z}_{\leqslant}^{\mathrm{op}}$ be the 1-category on the poset $\mathbb{Z}$ with reverse order (i.e. there exists an arrow $n \to m$ whenever $n \geqslant m$). The *∞-category of filtered $\mathbb{k}$-modules* is the ∞-functor ∞-category

$$\mathrm{Mod}_{\mathbb{k}}^{\mathrm{fil}} := \mathrm{Fun}\big(\mathbb{Z}_{\geqslant},\, \mathrm{Mod}_{\mathbb{k}}\big).$$

Notation 1.5.2. An object of $\mathrm{Mod}_{\mathbb{k}}^{\mathrm{fil}}$ consists of a tower of morphisms

$$\ldots \longrightarrow M_{p+1} \longrightarrow M_p \longrightarrow M_{p-1} \longrightarrow \ldots$$

where M_p is a $\mathbb{k}$-module for any integer p. We shall denote the object associated to such tower with $M_\bullet$, while the $\mathbb{k}$-module M_p will be the p weight component of $M_\bullet$.
Consistently, a natural transformation between two filtered $\mathbb{k}$-modules $M_\bullet \to N_\bullet$ (that is, a sequence of morphisms $f_p\colon M_p \to N_p$ for any integer p with obvious compatibility with the transition morphisms $M_p \to M_{p-1}$ and $N_p \to N_{p-1}$) shall be denoted by $f_\bullet$.

1.5.3. The selection of the p weight component of a filtered $\mathbb{k}$-module $M_\bullet$ is functorial, i.e. there exists an ∞-functor

$$(-)_p\colon \mathrm{Mod}_{\mathbb{k}}^{\mathrm{fil}} \longrightarrow \mathrm{Mod}_{\mathbb{k}}, \tag{1.5.4}$$

informally described on objects by the assignation $M_\bullet \mapsto M_p$, given by the precomposition of an ∞-functor $M_\bullet\colon \mathbb{Z}_{\geqslant} \to \mathrm{Mod}_{\mathbb{k}}$ with the inclusion of the terminal 1-category $\{p\} \subseteq \mathbb{Z}_{\geqslant}$.

Construction 1.5.5. Given a filtered $\mathbb{k}$-module $M_\bullet$, we can do more than selecting its p weight component; we can also consider the $(-)_\infty$ and $(-)_{-\infty}$ ∞-functors, given respectively by

$$(-)_\infty\colon \mathrm{Mod}_{\mathbb{k}}^{\mathrm{fil}} := \mathrm{Fun}\big(\mathbb{Z}_{\geqslant},\, \mathrm{Mod}_{\mathbb{k}}\big) \xrightarrow{\lim} \mathrm{Mod}_{\mathbb{k}} \tag{1.5.6}$$

and

$$(-)_{-\infty}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} := \mathrm{Fun}\big(\mathbb{Z}_{\geqslant}, \mathrm{Mod}_{\Bbbk}\big) \xrightarrow{\mathrm{colim}} \mathrm{Mod}_{\Bbbk}. \tag{1.5.7}$$

Definition 1.5.8. A filtered $\Bbbk$-module $M_\bullet$ is *complete* if $M_\infty \simeq 0$.

The complete filtered $\Bbbk$-modules are gathered in a full sub-∞-category of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$, which we shall denote by $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$.

1.5.9. With the above notations, the p weight part M_p of a filtered $\Bbbk$-module $M_\bullet$ must be thought as the p-th part of an exhaustive filtration on the $\Bbbk$-module $M_{-\infty}$, while the $\Bbbk$-module M_∞ determines whether such filtration is separated Hausdorff *in the homotopical setting* (which is not always equivalent to classical one - see [GP18, Remark 2.13] for a classically separated filtration which is not complete in our sense).

Definition 1.5.10. Given a filtered $\Bbbk$-module $M_\bullet$, consider the inclusion of $\Delta^1 \simeq \{p+1 \to p\}$ in $\mathbb{Z}_{\geqslant}$. This provides an ∞-functor $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} := \mathrm{Fun}\big(\mathbb{Z}_{\geqslant}, \mathrm{Mod}_{\Bbbk}\big) \to \mathrm{Fun}\big(\Delta^1, \mathrm{Mod}_{\Bbbk}\big)$ which selects the morphism $M_{p+1} \to M_p$ in a filtered $\Bbbk$-module $M_\bullet$.
The *p-th graded piece ∞-functor* is the composition of the previous ∞-functor with the cofiber ∞-functor

$$\mathrm{Gr}_p\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \longrightarrow \mathrm{Fun}\big(\Delta^1, \mathrm{Mod}_{\Bbbk}\big) \xrightarrow{\mathrm{cofib}} \mathrm{Mod}_{\Bbbk}.$$

1.5.11. The ∞-category $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ is naturally a closed symmetric monoidal ∞-category ([GP18, § 2.23]); such monoidal structure is inherited in some sense by the closed symmetric monoidal structure on $\mathrm{Mod}_{\Bbbk}$ via the Day convolution. We briefly recall some important properties and useful explicit constructions.

1. Given two filtered $\Bbbk$-modules $M_\bullet$ and $N_\bullet$, the tensor product $M_\bullet \otimes_{\Bbbk}^{\mathrm{fil}} N_\bullet$ is the filtered $\Bbbk$-module whose p-th weight component is given by the formula

$$\big(M_\bullet \otimes_{\Bbbk}^{\mathrm{fil}} N_\bullet\big)_p := \operatorname*{colim}_{\substack{i+j \geqslant p \\ i,j \in \mathbb{Z}_{\geqslant}}} M_i \otimes_{\Bbbk} N_j.$$

2. The unit for $\otimes_{\Bbbk}^{\mathrm{fil}}$ is the sequence $\Bbbk^{\leqslant 0}$ given by $\Bbbk$ in non positive weights and by 0 otherwise, with only identities and trivial morphisms, i.e. by the sequence

$$\dots = 0 = 0 \to \Bbbk = \Bbbk = \Bbbk = \dots$$

where the first copy of $\Bbbk$ sits in weight 0.

3. Given two filtered $\Bbbk$-modules $M_\bullet$ and $N_\bullet$, the internal mapping space $\mathrm{Map}_{\Bbbk}^{\mathrm{fil}}(M_\bullet, N_\bullet)$ is the filtered $\Bbbk$-module whose p-th weight component is given by the end formula

$$\big(\mathrm{Map}_{\Bbbk}^{\mathrm{fil}}(M_\bullet, N_\bullet)\big)_p := \int_{q \in \mathbb{Z}_{\geqslant}} \mathrm{Map}_{\Bbbk}\big(M_q, N_{p+q}\big).$$

Lemma 1.5.12 ([BMS19, Lemma 5.2]).

1. *The collection of ∞-functors $\{\mathrm{Gr}_p\}_{p\in\mathbb{Z}}$ and $(-)_\infty$ is jointly conservative on $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$. On the sub-$\infty$-category $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$, the ∞-functors $\{\mathrm{Gr}_p\}_{p\in\mathbb{Z}}$ are already jointly conservative.*

2. *The inclusion $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}} \subseteq \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ admits a left adjoint*
$$\widehat{(-)}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \longrightarrow \widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}},$$
which sends $M_\bullet$ to its completion $\widehat{M}_\bullet$ described in its p weight component by
$$\widehat{M}_p := \mathrm{cofib}\big(M_\infty \to M_p\big).$$
Such completion ∞-functor does not alter the graded pieces of $M_\bullet$.

3. *Both $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ and $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ have all limits and colimits. On the former, both $\{(-)_p\}_{p\in\mathbb{Z}}$ and $\{\mathrm{Gr}_p\}_{p\in\mathbb{Z}}$ commute with all limits and colimits; on the latter, $\{\mathrm{Gr}_p\}_{p\in\mathbb{Z}}$ commute with all limits and colimits.*

4. *There exists a (unique) closed symmetric monoidal structure on $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ compatible with the one on $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ via the completion ∞-functor.*

5. *For any $M_\bullet$ and $N_\bullet$ in $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ or $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$, we have an equivalence*
$$\mathrm{Gr}_p\left(M_\bullet \otimes_{\Bbbk}^{\mathrm{fil}} N_\bullet\right) \simeq \bigoplus_{i+j=p} \mathrm{Gr}_i M_\bullet \otimes_{\Bbbk} \mathrm{Gr}_j N_\bullet.$$

Lemma 1.5.12 allows us to say something more about the sub-∞-category of complete filtered $\Bbbk$-modules: namely, it is stable.

Proposition 1.5.13. *The full sub-∞-category $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ is stable.*

Proof. The ∞-category $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ has a zero object, since the trivial filtration is clearly complete. Moreover, since their inclusion in all filtered $\Bbbk$-modules admits a left adjoint, it follows that all limits in $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ are computed as in $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$. So, we only need to prove that given a diagram of complete filtered $\Bbbk$-modules of the form
$$M_\bullet \longleftarrow N_\bullet \longrightarrow P_\bullet$$
its colimit $R_\bullet$ is again a complete filtered $\Bbbk$-module, i.e.,
$$\lim_{q\in\mathbb{Z}_{\geqslant}} R_q \simeq 0.$$

Since limits and colimits in $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ are computed weight-wise, we have that

$$\lim_{q \in \mathbb{Z}_{\geqslant}} R_q \simeq \lim_{q \in \mathbb{Z}_{\geqslant}} \left(M_q \coprod_{N_q} P_q \right).$$

Since in a stable ∞-category finite colimits commute with all limits ([Lur17, Proposition 1.1.4.1]), we can write

$$\lim_{q \in \mathbb{Z}_{\geqslant}} R_q \simeq \lim_{q \in \mathbb{Z}_{\geqslant}} M_q \coprod_{\lim_{q \in \mathbb{Z}_{\geqslant}} N_q} \lim_{q \in \mathbb{Z}_{\geqslant}} P_q$$

and now since $M_\bullet$, $N_\bullet$ and $P_\bullet$ are all complete, it follows that this limit is 0, hence their pushout is complete as well. In particular, any diagram in $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ is a pullback (resp. pushout) if and only if it is a pullback (resp. pushout) in $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$, and being the latter stable it follows that $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ has to be stable as well. □

We now state (without proof) this important and well known result on the t-structure on the (homotopy category of the) ∞-category of filtered $\Bbbk$-modules. Note that this theorem *does not need* the assumption that $\Bbbk$ is a classical ring in characteristic 0.

Theorem 1.5.14 (Beilinson t-structure, [Bei87, Appendix A] and [BMS19, Theorem 5.4]). *Let $\left(\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}\right)_{\geqslant 0}$ be the full ∞-subcategory of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ spanned by those filtered modules such that $\mathrm{Gr}_p M_\bullet$ is $(-p)$-connective for all integers p. Dually, let $\left(\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}\right)_{\leqslant 0}$ be the full ∞-subcategory spanned by those filtered modules such that M_p is $(-p)$-coconnective for all integers p. Then these two ∞-subcategories define a t-structure on $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ whose heart $\left(\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}\right)^{\heartsuit}$ is equivalent, as an abelian 1-category, to the usual 1-category of chain complexes of $\Bbbk$-modules $\mathrm{dgMod}_{\Bbbk}$ when $\Bbbk$ is a discrete ring.*

Remark 1.5.15. The t-structure of Theorem 1.5.14 is induced by a t-structure on graded $\Bbbk$-modules in the following sense. The ∞-category $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ is endowed with a t-structure described as follows: the connective objects are those graded $\Bbbk$-modules $M_\bullet$ such that M_p is $(-p)$-connective (for the Postnikov t-structure on $\mathrm{Mod}_{\Bbbk}$) for all p's, and dually the coconnective objects are those graded $\Bbbk$-modules $M_\bullet$ such that M_p is $(-p)$-coconnective for all p's. Such t-structure can be lifted via the ∞-functor

$$\mathrm{Gr}_\bullet \colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \longrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

to a t-structure on $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ that makes $\mathrm{Gr}_\bullet$ a t-exact ∞-functor between stable ∞-categories endowed with a t-structure. This is precisely the Beilinson t-structure.

The aim of this section is to prove that mixed graded $\Bbbk$-modules are not only fully faithfully embedded in filtered $\Bbbk$-modules, but that such inclusion is universal in some precise sense. Namely, the ∞-category of mixed graded $\Bbbk$-modules with the t-structure of Theorem 1.4.1 is the *left completion* of the ∞-category of filtered $\Bbbk$-modules with respect to the Beilinson

t-structure of Theorem 1.5.14. In order to precisely formalize and prove this assertion, we start by reviewing some important concepts concerning t-structures and left completions.

1.5.16. Let $\mathscr{C}$ be a stable ∞-category endowed with a t-structure $\left(\mathscr{C}_{\geqslant 0}, \mathscr{C}_{\leqslant 0}\right)$. For all integers n, we have the n-connective and n-coconnective cover ∞-functors $\tau_{\geqslant n}\colon \mathscr{C} \to \mathscr{C}_{\geqslant n}$ and $\tau_{\leqslant n}\colon \mathscr{C} \to \mathscr{C}_{\leqslant n}$. In particular, we have a tower

$$\dots \xrightarrow{\tau_{\leqslant n+1}} \mathscr{C}_{\leqslant n+1} \xrightarrow{\tau_{\leqslant n}} \mathscr{C}_{\leqslant n} \xrightarrow{\tau_{\leqslant n-1}} \mathscr{C}_{\leqslant n-1} \xrightarrow{\tau_{\leqslant n-2}} \dots \tag{1.5.17}$$

Definition 1.5.18 ([Lur17, Section 1.2.1])**.** The *left completion* $\widehat{\mathscr{C}}$ of $\mathscr{C}$ is the limit of the diagram 1.5.17.
We will say that $\mathscr{C}$ is *left complete* if the canonical morphism $\mathscr{C} \to \widehat{\mathscr{C}}$ is an equivalence of stable ∞-categories.

Let us recall this important result concerning left completions.

Proposition 1.5.19 ([Lur17, Proposition 1.2.1.17])**.** *Let $\mathscr{C}$ be a stable ∞-category endowed with a t-structure. Then:*

1. *The left completion $\widehat{\mathscr{C}}$ is also stable.*

2. *The left completion $\widehat{\mathscr{C}}$ is naturally endowed with a t-structure which can be described as follows. Given an identification of $\widehat{\mathscr{C}}$ with the full ∞-subcategory of $\mathrm{Fun}\left(\mathbb{Z}_{\leqslant}^{\mathrm{op}}, \mathscr{C}\right)$ spanned by those functors in which F_n factors through $\mathscr{C}_{\leqslant n}$ and such that $F_m \to F_n$ induces an equivalence $\tau_{\leqslant n}F_m \xrightarrow{\simeq} F_n$ for all $n \leqslant m$ in $\mathbb{Z}$, then the connective (resp. coconnective) objects of $\widehat{\mathscr{C}}$ are those ∞-functors which factor through $\mathscr{C}_{\geqslant 0}$ (resp. $\mathscr{C}_{\leqslant 0}$).*

3. *The canonical ∞-functor $\mathscr{C} \to \widehat{\mathscr{C}}$ is exact and induces an equivalence $\mathscr{C}_{\leqslant 0} \xrightarrow{\simeq} \widehat{\mathscr{C}}_{\leqslant 0}$.*

Proposition 1.5.20. *The stable ∞-category $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ is left complete with respect to the mixed graded Postnikov t-structure of Theorem 1.4.1.*

Proof. In order to prove our claim, we use the following criterion.

Proposition 1.5.21 ([Lur17, Proposition 1.2.1.19])**.** *Let $\mathscr{C}$ a stable ∞-category equipped with a t-structure. Suppose that $\mathscr{C}$ has all countable products, and $\mathscr{C}_{\geqslant 0}$ is closed under countable products. Then $\mathscr{C}$ is left exact with respect to its t-structure if and only if*

$$\mathscr{C}_{\geqslant \infty} := \bigcap_{n \in \mathbb{Z}} \mathscr{C}_{\geqslant n}$$

consists only of zero objects of $\mathscr{C}$.

Obviously $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$, being both complete and cocomplete, admits all countable products. Since all limits and colimits are computed weight-wise, a product of countably many mixed graded $\Bbbk$-modules $\{M_\bullet^\alpha\}_{\alpha\in\mathbb{N}}$, all of which are connective for the mixed graded Postnikov t-structure, is again connective. Indeed, in each weight we have an equivalence of $\Bbbk$-modules

$$\Big(\prod_{\alpha\in\mathbb{N}} M_\bullet^\alpha\Big)_q \simeq \prod_{\alpha\in\mathbb{N}} M_q^\alpha$$

where all the M_q^α's are now $(-q)$-connective, hence the statement follows from the fact that $\mathrm{Mod}_\Bbbk$ is left complete by [Lur17, Proposition 7.1.1.13]. □

1.5.22. On the converse, recall that the ∞-category $\mathrm{Mod}_\Bbbk^{\mathrm{fil}}$ endowed with the Beilinson t-structure of Theorem 1.5.14 is *not left complete* ([BMS19, Section 5]). In fact, the objects lying in $\big(\mathrm{Mod}_\Bbbk^{\mathrm{fil}}\big)_{\geqslant\infty}$ are those filtered $\Bbbk$-modules $M_\bullet$ such that $\mathrm{Gr}_n M_\bullet$ vanishes for all integers n, i.e. filtered $\Bbbk$-modules corresponding to essentially constant diagrams $\mathbb{Z}_\geqslant \longrightarrow \mathrm{Mod}_\Bbbk$. In particular, for any non trivial $\Bbbk$-module M the constant diagram on it is ∞-connective without being 0.

Yet, we can characterize what the left completion of the Beilinson t-structure on filtered $\Bbbk$-modules looks like: this is *precisely* $\widehat{\mathrm{Mod}}_\Bbbk^{\mathrm{fil}}$ (which justifies, *a posteriori*, the choice of notation for such sub-∞-category).

Proposition 1.5.23. *The full sub-∞-category $\widehat{\mathrm{Mod}}_\Bbbk^{\mathrm{fil}}$ of $\mathrm{Mod}_\Bbbk^{\mathrm{fil}}$ is endowed with a t-structure which is provided by the restriction of the Beilinson t-structure to complete filtered $\Bbbk$-modules. Moreover, the completion ∞-functor*

$$\widehat{(-)}\colon \mathrm{Mod}_\Bbbk^{\mathrm{fil}} \longrightarrow \widehat{\mathrm{Mod}}_\Bbbk^{\mathrm{fil}}$$

is a t-exact ∞-functor that naturally identifies $\widehat{\mathrm{Mod}}_\Bbbk^{\mathrm{fil}}$ with the left completion of the Beilinson t-structure on all filtered $\Bbbk$-modules.

In order to prove that the Beilinson t-structure restricts to $\widehat{\mathrm{Mod}}_\Bbbk^{\mathrm{fil}}$, we need the following important result.

Lemma 1.5.24. *If $M_\bullet$ is a filtered $\Bbbk$-module which is eventually coconnective (that is, n-coconnective for some n) for the Beilinson t-structure of Theorem 1.5.14, then $M_\bullet$ is complete.*

Proof. Since for any integer p, the $\Bbbk$-module M_p is $(n-p)$-coconnective for the usual t-structure on $\Bbbk$-modules by assumption, and coconnective objects are stable under all limits, we just need to observe that

$$M_\infty := \lim_{q\in\mathbb{Z}_\geqslant} M_q$$

must be $n-q$-coconnective for each integer q, i.e., it must belong to the ∞-category

$$(\mathrm{Mod}_\Bbbk)_{\leqslant-\infty} := \bigcap_{q\in\mathbb{Z}} (\mathrm{Mod}_\Bbbk)_{\leqslant -q}\,.$$

But the Postnikov t-structure on $\Bbbk$-modules is left complete, hence M_∞ must be trivial because of Proposition 1.5.21. □

Proof of Proposition 1.5.23. Since $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ is a stable full sub-∞-category of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ and its inclusion admits a right adjoint, it is closed under all limits and finite colimits, and in particular under all loops and suspensions. So, we only need to check is that if $M_\bullet$ is endowed with a complete filtration, then both $\tau_{\leqslant n}^{\mathrm{fil}}M_\bullet$ and $\tau_{\geqslant n}^{\mathrm{fil}}M_\bullet$ lie in $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$, where $\tau_{\geqslant n}^{\mathrm{fil}}$ and $\tau_{\leqslant n}^{\mathrm{fil}}$ denote the n-connective and n-coconnective cover ∞-functor for the Beilinson t-structure on $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$, respectively. This is clear for $\tau_{\leqslant n}^{\mathrm{fil}}$, since n-coconnective objects are always complete in virtue of Lemma 1.5.24, and this implies that the same holds also for $\tau_{\geqslant n}$: indeed, consider the left adjoint to the inclusion $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}} \subseteq \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ of Lemma 1.5.12.2. For any filtered module $M_\bullet$, the unit map of the adjunction yields a map of fiber sequences (given by the unit map of the adjunction)

$$\begin{tikzcd}
\tau_{\geqslant n+1}^{\mathrm{fil}}M_\bullet \arrow[r] \arrow[d] & M_\bullet \arrow[r] \arrow[d] & \tau_{\leqslant n}^{\mathrm{fil}}M_\bullet \arrow[d] \\
\widehat{\tau_{\geqslant n+1}^{\mathrm{fil}}M_\bullet} \arrow[r] & \widehat{M_\bullet} \arrow[r] & \widehat{\tau_{\leqslant n}^{\mathrm{fil}}M_\bullet}.
\end{tikzcd}$$

If $M_\bullet$ is complete, then both the second and third vertical arrows are equivalences, and so the first must be an equivalence as well. It follows that both the connective and coconnective cover ∞-functors restrict naturally to the stable full sub-∞-category $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$.

We shall now prove that $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ is left complete. Since it is closed under all limits existing in $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$, the hypotheses of the left completeness criterion of Proposition 1.5.21 are satisfied, so we can simply check what the ∞-connective objects of $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ are: they are ∞-connective objects of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ (hence, constant filtrations) which are also complete as filtered $\Bbbk$-modules. But it is immediate to see that the only constant filtration which is complete is the constant filtration on the trivial $\Bbbk$-module 0. Thus, $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ is left complete, that is (in virtue of [Lur17, Proposition 1.2.1.17]) the natural ∞-functor towards its left completion is an equivalence. As already described in Proposition 1.5.19, we can identify the left completion of the Beilinson t-structure on $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ with the full sub-∞-category $\widehat{\mathrm{Fun}}\left(\mathbb{Z}, \widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}\right) \subseteq \mathrm{Fun}\left(\mathbb{Z}, \widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}\right)$ spanned by those ∞-functors $F\colon \mathbb{Z} \to \widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ such that

- $F_n \in \left(\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}\right)_{\leqslant -n}$ for any integer n.
- For any $m \leqslant n$, the map $F_m \to F_n$ induces an equivalence $\tau_{\leqslant -n}^{\mathrm{fil}}F_m \xrightarrow{\simeq} F_n$.

Employing this canonical model, the canonical equivalence $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}} \to \widehat{\mathrm{Fun}}\left(\mathbb{Z}, \widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}\right)$ simply sends an object $M_\bullet$ to the tower of its truncations $\left\{\tau_{\leqslant -n}^{\mathrm{fil}}M_\bullet\right\}_{n\in\mathbb{Z}}$, while its inverse takes the limit of a functor over $\mathbb{Z}^{\mathrm{op}}_{\leqslant}$. So, we have a diagram

$$\begin{tikzcd} \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk} \arrow[r, "\simeq"] \arrow[d, hook] & \widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right) \arrow[d] \\ \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk} \arrow[r] & \widehat{\mathrm{Fun}}(\mathbb{Z},\, \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}) \end{tikzcd}$$

where the vertical arrow on the right is simply the post-composition with the natural inclusion. Since the coconnective truncation ∞-functors for filtered modules and complete filtered modules are the same, this diagram commutes; moreover, the functor

$$\widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right) \longrightarrow \widehat{\mathrm{Fun}}(\mathbb{Z},\, \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk})$$

is an equivalence as well, because a functor $F\colon \mathbb{Z} \to \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}$ lies in $\widehat{\mathrm{Fun}}(\mathbb{Z},\, \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk})$ if and only if F_n is $(-n)$-coconnective for all n. In particular, using again Lemma 1.5.24, it follows that at least as mere stable ∞-categories, $\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk} \simeq \widehat{\mathrm{Fun}}(\mathbb{Z},\, \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk})$.

The last thing we need to prove is that this equivalence is t-exact. Recall ([Lur17, Proposition 1.2.1.17.(2)]) that the natural t-structure on $\widehat{\mathrm{Fun}}(\mathbb{Z},\, \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk})$ is defined as follows: the connective objects are those ∞-functors $F\colon \mathbb{Z} \to \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}$ that factor through $\left(\mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}\right)_{\geqslant 0}$, while coconnective objects are those ∞-functors that factor through $\left(\mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}\right)_{\leqslant 0}$; an analogous statement holds for $\widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right)$. But, being complete with respect to its t-structure, the canonical ∞-functor

$$\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk} \longrightarrow \widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right)$$

is a t-exact equivalence, and

$$\widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right) \longrightarrow \widehat{\mathrm{Fun}}(\mathbb{Z},\, \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk})$$

is again t-exact, because by definition

$$\left(\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right)_{\geqslant 0} := \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk} \times_{\mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}} \left(\mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}\right)_{\geqslant 0}$$

and the dual claim holds also for $\left(\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right)_{\leqslant 0}$. In particular, $\widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right) \to \widehat{\mathrm{Fun}}(\mathbb{Z},\, \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk})$ sends connective objects to connective objects, and coconnective objects to coconnective objects. Now, since every complete filtered $\Bbbk$-module is canonically equivalent to the limit over the tower of its coconnective truncations, it is clear that the composition

$$\mathrm{Mod}^{\mathrm{fil}}_{\Bbbk} \longrightarrow \widehat{\mathrm{Fun}}(\mathbb{Z},\, \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}) \simeq \widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}\right) \simeq \widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}$$

agrees with the left adjoint to

$$\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}} \simeq \widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}\right) \simeq \widehat{\mathrm{Fun}}\left(\mathbb{Z},\, \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}\right) \longrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$$

which is canonically equivalent to the inclusion $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}} \subseteq \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$. Hence $\widehat{(-)}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \to \widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ does exhibit the ∞-category of complete filtered $\Bbbk$-modules as the left completion of the Beilinson t-structure on all filtered $\Bbbk$-modules. □

We can now state the main result of this section.

Theorem 1.5.25. *Let $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$ denote the full sub-∞-category of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ spanned by $\Bbbk$-modules with complete filtration. There exists a fully faithful embedding*

$$(-)^{\mathrm{fil}}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \hookrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$$

which is moreover t-exact with respect to both t-structures, such that the following diagram of ∞-functors commutes.

$$\begin{tikzcd} \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \arrow[rr, "\simeq"] \arrow[dr, bend right, "(-)^{\mathrm{fil}}"'] & & \widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}} \arrow[dl, hook, bend left] \\ & \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} & \end{tikzcd}$$

Moreover, such embedding admits a t-exact left adjoint

$$(-)^{\varepsilon\text{-}\mathrm{gr}}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

which exhibits the ∞-category $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$, endowed with the mixed graded Postnikov t-structure of Theorem 1.4.1, as the left completion of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ with respect to the Beilinson t-structure of Theorem 1.5.14.

Even if the ∞-functors that exhibit $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ as the left completion of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ are precisely the same as the ones considered in [TV20b], for the sake of clarity we recall all the details of their construction in this subsection.

1.5.26. Consider the naive truncation ∞-functor of mixed graded $\Bbbk$-modules of Definition 1.2.1.1. Thanks to Remark 1.2.2, for any mixed graded $\Bbbk$-module $M_\bullet$ and for any triple of integers $r \leqslant q \leqslant p$ we have natural morphisms in $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$

$$\sigma_{\leqslant r} M_\bullet \hookrightarrow \sigma_{\leqslant q} M_\bullet \hookrightarrow \sigma_{\leqslant p} M_\bullet.$$

This means that gathering all the naive truncations ∞-functors $\{\sigma_{\leqslant p}\}_{p \in \mathbb{Z}}$ we get an ∞-functor

$$\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \xrightarrow{\{\sigma_{\leqslant p}\}_p} \mathrm{Fun}\left(\mathbb{Z}_{\leqslant},\, \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\right).$$

Using the isomorphism $\mathbb{Z}_{\leqslant} \cong \mathbb{Z}_{\geqslant}$ given by changing signs, we can write the ∞-functor above as

$$\sigma := \left\{\sigma_{\leqslant -p}\right\}_{p\in\mathbb{Z}} : \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \mathrm{Fun}\big(\mathbb{Z}_{\geqslant},\ \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big).$$

Post-composing σ with the Tate realization ∞-functor of Definition 1.1.19, we land in the ∞-category $\mathrm{Fun}\big(\mathbb{Z}_{\geqslant},\ \mathrm{Mod}_{\Bbbk}\big) =: \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$.

Definition 1.5.27. The ∞-functor

$$(-)^{\mathrm{fil}} : \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \xrightarrow{\ \sigma\ } \mathrm{Fun}\big(\mathbb{Z}_{\geqslant},\ \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big) \xrightarrow{|-|^{\mathrm{t}}\circ -} \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$$

is the *associated filtered ∞-functor*.

Remark 1.5.28. Given a mixed graded $\Bbbk$-module $M_\bullet$, its associated filtered $\Bbbk$-module is given in weight p by the Tate realization of its truncation in weights $\leqslant -p$, i.e. $\left|\sigma_{\leqslant -p}M_\bullet\right|^{\mathrm{t}}$ and all the morphisms are the inclusions exhibited in 1.5.26. Let us study in greater detail the $\Bbbk$-module $\left|\sigma_{\leqslant -p}M_\bullet\right|^{\mathrm{t}}$: for any p, this mixed graded $\Bbbk$-module $\sigma_{\leqslant -p}M_\bullet$ is bounded above (i.e., its weight components are 0 for all $q > -p$), so its Tate realization is equivalent to

$$\operatorname*{colim}_{i\leqslant 0} \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\big(\Bbbk(-i)[2i],\ \sigma_{\leqslant -p}M_\bullet\big) \simeq \begin{cases} \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\big(\Bbbk(-p)[2p],\ \sigma_{\leqslant -p}M_\bullet\big) & \text{if } p<0 \\ \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\big(\Bbbk(0),\ \sigma_{\leqslant -p}M_\bullet\big) & \text{if } p\geqslant 0. \end{cases}$$

Either way, in terms of explicit models of chain complexes, this means that

$$\big(M_\bullet^{\mathrm{fil}}\big)_p \simeq \prod_{q\geqslant p} M_{-q}[-2q]$$

where the right hand side is endowed with the total differential. The transition maps, in terms of the above equivalence, are simply given by the obvious inclusions

$$\ldots \hookrightarrow \prod_{q\geqslant p+1} M_{-q}[-2q] \hookrightarrow \prod_{q\geqslant p} M_{-q}[-2q] \hookrightarrow \prod_{q\geqslant p-1} M_{-q}[-2q] \hookrightarrow \ldots$$

Notation 1.5.29. In the following, we shall often denote the object $(M_\bullet)^{\mathrm{fil}}$ with the suggestive notation $\prod_{q\geqslant p} M_{-q}[-2q]$, leaving implicit the fact that this is not actually a *product* of $\Bbbk$-modules.

We prove some important properties of the associated filtered ∞-functor.

Proposition 1.5.30. *The associated filtered ∞-functor of Definition 1.5.27 is a t-exact ∞-functor between stable ∞-categories with t-structure.*

Proof. Remark 1.5.28 allows us to describe neatly the graded pieces of the filtered $\Bbbk$-module $M_\bullet^{\mathrm{fil}}$ for any mixed graded $\Bbbk$-module $M_\bullet$. In fact, the cofiber sequence

$$\big(M_\bullet^{\mathrm{fil}}\big)_{p+1} \longrightarrow \big(M_\bullet^{\mathrm{fil}}\big)_p \longrightarrow \mathrm{Gr}_p M_\bullet^{\mathrm{fil}}$$

can be now represented by explicit models as

$$\prod_{q\geqslant p+1} M_{-q}[-2q] \hookrightarrow \prod_{q\geqslant p} M_{-q}[-2q] \longrightarrow \mathrm{Gr}_p M_\bullet^{\mathrm{fil}},$$

where again the product has to be thought as endowed with the total differential, and by direct inspection of the long exact sequence of homotopy groups we get that $\mathrm{Gr}_p M_\bullet^{\mathrm{fil}} \simeq M_{-p}[-2p]$.
So, let us assume that $M_\bullet$ was an n-connective mixed graded $\Bbbk$-module for the t-structure of Theorem 1.4.1: then, for any integer p, M_p was an $(n-p)$-connective $\Bbbk$-module. This implies that $\mathrm{Gr}_p M_\bullet^{\mathrm{fil}} \simeq M_{-p}[-2p]$ is an $(n-p)$-connective $\Bbbk$-module as well, so $(-)^{\mathrm{fil}}$ preserves connective objects.
On the converse, let $M_\bullet$ be an n-coconnective mixed graded $\Bbbk$-module. Then $\left(M_\bullet^{\mathrm{fil}}\right)_p$ is equivalent to a product of $\Bbbk$-modules sitting in $(\mathrm{Mod}_\Bbbk)_{\leqslant n-p}$, and since n-coconnective objects are stable under limits this yields that $\left(M_\bullet^{\mathrm{fil}}\right)_p$ is $(n-p)$-coconnective for any integer n. So $(-)^{\mathrm{fil}}$ preserves both connective and coconnective objects, hence it is t-exact. □

To prove Theorem 1.5.25, we have to show that $\varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}}$ is a localization of $\mathrm{Mod}_\Bbbk^{\mathrm{fil}}$, i.e. the ∞-functor $(-)^{\mathrm{fil}}$ is a fully faithful right adjoint whose essential image consists of complete objects.

Proposition 1.5.31. *The ∞-functor $(-)^{\mathrm{fil}}\colon \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}} \to \mathrm{Mod}_\Bbbk^{\mathrm{fil}}$ admits a left adjoint.*

Proof. Being $\varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}}$ and $\mathrm{Mod}_\Bbbk^{\mathrm{fil}}$ two ∞-categories which are both presentable and accessible, it is sufficient to show that $(-)^{\mathrm{fil}}$ commutes with all limits and with κ-filtered colimits for κ some regular cardinal. Since $\mathrm{Mod}_\Bbbk^{\mathrm{fil}}$ is an ∞- category of functors, hence limits and colimits are computed weight-wise, it suffices to show that for any integer p the ∞-functor $\left|\sigma_{\leqslant -p}(-)\right|^{\mathrm{t}}$ commutes with limits and is accessible, i.e., it commutes with κ-filtered colimits for some regular cardinal κ. Since such ∞-functor is the composition of two ∞-functors, one of which (namely, $\sigma_{\leqslant -p}$) commutes with limits and is accessible, being a left adjoint, we are left to prove that $|-|^{\mathrm{t}}$ commutes with limits and is accessible as well.

- The ∞-functor $|-|^{\mathrm{t}}$ commutes with all limits: this can be seen via direct computations from the description of $|-|^{\mathrm{t}}$ and by recalling that limits distribute over limits.
- The Tate realization ∞-functor is given explicitly by a countable product, so it commutes with κ_2-filtered colimits for some regular cardinal κ_2 bigger than $\aleph_0$.

The claim now follows, once again, from the Adjoint Functor Theorem. □

Proposition 1.5.31 allows us to introduce the following Definition.

Definition 1.5.32. The ∞-functor

$$(-)^{\varepsilon\text{-}\mathrm{gr}}\colon \mathrm{Mod}_\Bbbk^{\mathrm{fil}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}},$$

left adjoint to the ∞-functor

$$(-)^{\mathrm{fil}}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \to \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$$

is the *associated mixed graded ∞-functor*.

We want to study some properties of the ∞-functor of Definition 1.5.32 as well. One of its key features is described in the following Proposition.

Proposition 1.5.33. *Given a filtered $\Bbbk$-module $M_\bullet$, the p weight component of its associated mixed graded $\Bbbk$-module $(M_\bullet)^{\varepsilon\text{-gr}}$ is naturally equivalent to $\mathrm{Gr}_{-p}M_\bullet[-2p]$.*

Proof. Let

$$(-)^{\mathrm{gr}}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \longrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

denote the ∞-functor obtained by the composition of $(-)^{\varepsilon\text{-gr}}$ with the forgetful ∞-functor $\mathrm{oblv}_{\varepsilon}\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \to \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$. Let

$$\mathrm{Gr}_\bullet\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \longrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \tag{1.5.34}$$

denote the ∞-functor obtained by patching the ∞-functors $\mathrm{Gr}_{-p}(-)[-2p]\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \longrightarrow \mathrm{Mod}_{\Bbbk}$. In particular, the p-th component of this ∞-functor is $\mathrm{Gr}_{-p}(-)[-2p]$. We will prove that for any filtered $\Bbbk$-module $M_\bullet$, there is a canonical equivalence of $\Bbbk$-modules $\mathrm{Gr}_\bullet M_\bullet \simeq (M_\bullet)^{\mathrm{gr}}$. We first observe that $\mathrm{Gr}_\bullet\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \to \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ is the left adjoint to the ∞-functor

$$\mathrm{R}^{\mathrm{fil}}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$$

which sends a graded $\Bbbk$-module $N_\bullet$ to the filtered $\Bbbk$-module whose p-th component is $N_{-p}[-2p]$ and whose transition maps are all zero maps (this is essentially [GP18, Lemma 3.30], after an homological shift by $-2p$ and the equivalence $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \simeq \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ which swaps the graded parts in positive and negative degrees). In the same way, in virtue of Porism 1.1.26 the ∞-functor which forgets the mixed structure is the left adjoint to the ∞-functor

$$\mathrm{R}_{\varepsilon}\colon \mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}.$$

Now let $M_\bullet$ denote a filtered $\Bbbk$-module, and $N_\bullet$ denote a graded $\Bbbk$-module. By the adjunction $\mathrm{Gr}_\bullet \dashv \mathrm{R}^{\mathrm{fil}}$, we have the following equivalences of mapping spaces

$$\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\mathrm{Gr}_\bullet M_\bullet,\ N_\bullet) \simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}}\big(M_\bullet,\ \mathrm{R}^{\mathrm{fil}} N_\bullet\big)$$

On the other hand, by the adjunction $\mathrm{oblv}_{\varepsilon} \dashv \mathrm{R}_{\varepsilon}$ one has

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}((M_\bullet)^{\mathrm{gr}},\ N_\bullet) &\simeq \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}((M_\bullet)^{\varepsilon\text{-gr}},\ \mathrm{R}_{\varepsilon} N_\bullet)\\ &\overset{1.5.32}{\simeq} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}}\big(M_\bullet,\ (\mathrm{R}_{\varepsilon} N_\bullet)^{\mathrm{fil}}\big).\end{aligned}$$

By the fully faithfulness of the Yoneda embedding, it is enough to check that

$$\mathrm{R}^{\mathrm{fil}} N_\bullet \simeq (\mathrm{R}_\varepsilon N_\bullet)^{\mathrm{fil}}$$

for any graded $\Bbbk$-module $N_\bullet$. This is seen by direct inspection: in fact, $(\mathrm{R}_\varepsilon N_\bullet)^{\mathrm{fil}}$ is the filtered $\Bbbk$-module whose p-th component is obtained by the Tate realization of the naive truncation $\sigma_{\leqslant -p}(\mathrm{R}_\varepsilon N_\bullet)$. By the description of R_ε provided in Porism 1.1.26, $\left|\sigma_{\leqslant -p}(\mathrm{R}_\varepsilon N_\bullet)\right|^{\mathrm{t}}$ is equivalent to the $\Bbbk$-module $N_{-p}[-2p]$. The inclusion of $\sigma_{\leqslant -p}(\mathrm{R}_\varepsilon N_\bullet)$ into $\sigma_{\leqslant -p+1}(\mathrm{R}_\varepsilon N_\bullet)$ induces the zero map on the Tate realizations, and so we get exactly the filtered $\Bbbk$-module $\mathrm{R}^{\mathrm{fil}} N_\bullet$ we described above, which completes the proof. □

Remark 1.5.35. Working with explicit models, the mixed differential in the mixed graded $\Bbbk$-module $(M_\bullet)^{\varepsilon\text{-gr}}$ is given by the boundary morphism in the cofiber sequence

$$\mathrm{Gr}_{p+1} M_\bullet \longrightarrow \mathrm{cofib}\big(M_{p+1} \to M_p\big) \longrightarrow \mathrm{Gr}_p M_\bullet$$

after a *suitable* (i.e., correctly functorial) choice of the cofibers in each weight.

The above characterization of the weight components of the associated mixed graded ∞-functor leads immediately to two important consequences.

Corollary 1.5.36. *The ∞-functor $(-)^{\varepsilon\text{-gr}} \colon \mathrm{Mod}_\Bbbk^{\mathrm{fil}} \to \varepsilon\text{-}\mathrm{Mod}_\Bbbk^{\mathrm{gr}}$ is strongly monoidal.*

Proof. Let us recall that the ∞-functor $(-)^{\mathrm{fil}}$ is lax monoidal: it can be checked by straightforward computation, by using the fact that in each piece it is given by the mapping space out of a cocommutative coalgebra object (namely, $\Bbbk(\infty)$). Being the left adjoint of a lax monoidal ∞-functor, $(-)^{\varepsilon\text{-gr}}$ naturally possesses an oplax monoidal structure, and so given two filtered $\Bbbk$-modules $M_\bullet$ and $N_\bullet$ we have a natural map

$$\left(M_\bullet \otimes_\Bbbk^{\mathrm{fil}} N_\bullet\right)^{\varepsilon\text{-gr}} \longrightarrow (M_\bullet)^{\varepsilon\text{-gr}} \otimes_\Bbbk^{\varepsilon\text{-gr}} (N_\bullet)^{\varepsilon\text{-gr}}.$$

Since equivalences of mixed graded $\Bbbk$-modules are detected by the underlying graded $\Bbbk$-modules and forgetting the mixed structure is a strongly monoidal ∞-functor, it suffices to show that the underlying graded map

$$\mathrm{oblv}_\varepsilon\left(M_\bullet \otimes_\Bbbk^{\mathrm{fil}} N_\bullet\right)^{\varepsilon\text{-gr}} \simeq \left(M_\bullet \otimes_\Bbbk^{\mathrm{fil}} N_\bullet\right)^{\mathrm{gr}} \longrightarrow \mathrm{oblv}_\varepsilon\left((M_\bullet)^{\varepsilon\text{-gr}} \otimes_\Bbbk^{\varepsilon\text{-gr}} (N_\bullet)^{\varepsilon\text{-gr}}\right) \simeq (M_\bullet)^{\mathrm{gr}} \otimes_\Bbbk^{\mathrm{gr}} (N_\bullet)^{\mathrm{gr}}$$

is an equivalence. Now, Proposition 1.5.33 tells us that the graded $\Bbbk$-module on the left is described in its p weight component by the formula

$$\left(M_\bullet \otimes_\Bbbk^{\mathrm{fil}} N_\bullet\right)_p^{\varepsilon\text{-gr}} \simeq \mathrm{Gr}_{-p}\left(M_\bullet \otimes_\Bbbk^{\mathrm{fil}} N_\bullet\right).$$

But now Lemma 1.5.12.5 tells that the above object is equivalent to

$$\bigoplus_{i+j=p} \mathrm{Gr}_{-i} M_\bullet \otimes_\Bbbk \mathrm{Gr}_{-j} N_\bullet$$

which is exactly the p weight component of the tensor product of $(M_\bullet)^{\mathrm{gr}} \otimes_{\Bbbk}^{\mathrm{gr}} (N_\bullet)^{\mathrm{gr}}$. So, the natural map above is indeed an equivalence. $\square$

Corollary 1.5.37. *The ∞-functor $(-)^{\varepsilon\text{-gr}} \colon \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \to \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ is t-exact.*

Proof. Let $M_\bullet$ be an n-connective filtered $\Bbbk$-module. Then $\mathrm{Gr}_p M_\bullet$ is an $(n-p)$-connective $\Bbbk$-module for any integer p, therefore

$$(M_\bullet)_p^{\varepsilon\text{-gr}} \overset{\text{Prop. 1.5.31}}{\simeq} \mathrm{Gr}_{-p} M_\bullet[-2p]$$

is $(n-p)$-connective.
On the converse, if $M_\bullet$ is an n-coconnective filtered $\Bbbk$-module, then M_p is an $(n-p)$-coconnective $\Bbbk$-module for any p. This implies that $\mathrm{Gr}_p M_\bullet$ is an $(n-p)$-coconnective $\Bbbk$-module for any p as well: in fact, $\mathrm{Gr}_p M_\bullet$ is the cofiber of $M_{p+1} \longrightarrow M_p$. Since $M_\bullet$ is n-coconnective, M_{p+1} and M_p are respectively $(n-p-1)$-coconnective and $(n-p)$-coconnective $\Bbbk$-modules. By inspecting the long exact sequence induced on the homotopy groups by the cofiber sequence

$$M_{p+1} \longrightarrow M_p \longrightarrow \mathrm{Gr}_p M_\bullet$$

one has that $\pi_q \mathrm{Gr}_p M_\bullet \cong 0$ for all $q > n-p$, hence $\mathrm{Gr}_p M_\bullet$ is an $(n-p)$-coconnective $\Bbbk$-module and therefore

$$(M_\bullet)_p^{\varepsilon\text{-gr}} \overset{\text{Prop. 1.5.31}}{\simeq} \mathrm{Gr}_{-p} M_\bullet[-2p]$$

is $(n-p)$-coconnective for all p's. $\square$

The following two claims are the last ingredients we need to prove Theorem 1.5.25.

Proposition 1.5.38. *The counit morphism $(-)^{\varepsilon\text{-gr}} \circ (-)^{\mathrm{fil}} \to \mathrm{id}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}$ of the adjunction $(-)_{\varepsilon}^{\mathrm{gr}} \dashv (-)^{\mathrm{fil}}$ is an equivalence of ∞-functors.*

Proof. Let $M_\bullet$ be a mixed graded $\Bbbk$-module. By the description provided in Remark 1.5.28, its associated filtered $\Bbbk$-module is

$$\ldots \hookrightarrow \prod_{q \geqslant p+1} M_{-q}[-2q] \hookrightarrow \prod_{q \geqslant p} M_{-q}[-2q] \hookrightarrow \prod_{q \geqslant p-1} M_{-q}[-2q] \hookrightarrow \ldots$$

and for any p, $\mathrm{Gr}_p\,(M_\bullet)^{\mathrm{fil}}[-2p] \simeq M_{-p}[-2p]$. Applying $(-)^{\varepsilon\text{-gr}}$ to $(M_\bullet)^{\mathrm{fil}}$, we get a mixed graded $\Bbbk$-module whose p-th weight component is equivalent to

$$\mathrm{Gr}_{-p}\,(M_\bullet)^{\mathrm{fil}}[-2p] \overset{\text{Prop. 1.5.31}}{\simeq} M_{-(-p)}[-2(-p)][-2p] \simeq M_p.$$

Since equivalences of mixed graded $\Bbbk$-modules are detected by forgetting the mixed structure, we have that the counit is an equivalence because it induces on each weights precisely the equivalence $\mathrm{Gr}_{-p}(M_\bullet)^{\mathrm{fil}}[-2p] \simeq M_p$. $\square$

Proposition 1.5.39. *When restricted to n-coconnective objects of $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$, the unit morphism $\eta\colon \mathrm{id}_{(\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}})_{\leqslant n}} \to (-)^{\mathrm{fil}} \circ (-)^{\mathrm{gr}}_{\varepsilon}$ of the adjunction $(-)^{\varepsilon\text{-}\mathrm{gr}} \dashv (-)^{\mathrm{fil}}$ is an equivalence of ∞-functors for all integers n.*

Proof. Let $M_\bullet$ be a filtered $\Bbbk$-module which is n-coconnective for the Beilinson t-structure for some integer n. We briefly study what $((N_\bullet)^{\varepsilon\text{-}\mathrm{gr}})^{\mathrm{fil}}$ is: its p-th part is the $\Bbbk$-module given by the Tate realization

$$\left|\sigma_{\leqslant -p}(N_\bullet)^{\varepsilon\text{-}\mathrm{gr}}\right|^{\mathrm{t}} \simeq \prod_{q\geqslant p} (N_\bullet)^{\varepsilon\text{-}\mathrm{gr}}_{-q}[-2q].$$

But we have that

$$(N_\bullet)^{\varepsilon\text{-}\mathrm{gr}}_{-q} \overset{\text{Prop. 1.5.31}}{\simeq} \mathrm{Gr}_{-(-q)}N_\bullet[-2(-q)] \simeq \mathrm{Gr}_q N_\bullet[2q]$$

and since the shift ∞-functors commute with arbitrary products we have that

$$((N_\bullet)^{\varepsilon\text{-}\mathrm{gr}})^{\mathrm{fil}}_p \simeq \prod_{q\geqslant p} \mathrm{Gr}_q N_\bullet$$

The transition maps are just given by inclusions.
Let

$$F_p := \mathrm{fib}\left(N_p \xrightarrow{\eta_p} \prod_{q\geqslant p} \mathrm{Gr}_q N_\bullet\right)$$

denote the fiber of the p-th component of η. Then we have the following diagram, where every row and every column is a fiber sequence.

$$\begin{tikzcd}
F_{p+1} \arrow[r] \arrow[d] & F_p \arrow[r] \arrow[d] & \mathrm{cofib}\left(F_{p+1} \longrightarrow F_p\right) \arrow[d] \\
N_{p+1} \arrow[r] \arrow[d, "\eta_{p+1}"'] & N_p \arrow[r] \arrow[d, "\eta_p"] & \mathrm{Gr}_p N_\bullet \arrow[d, "\wr"] \\
\prod_{q\geqslant p+1} \mathrm{Gr}_q N_\bullet \arrow[r, hook] & \prod_{q\geqslant p} \mathrm{Gr}_q N_\bullet \arrow[r] & \mathrm{Gr}_p N_\bullet
\end{tikzcd}$$

The map on the bottom right is an equivalence, and this forces $\mathrm{cofib}\left(F_{p+1} \longrightarrow F_p\right)$ to be zero. This implies that $F_{p+1} \longrightarrow F_p$ is an equivalence; by induction, we can conclude that F_p is the fiber of

$$N_m \longrightarrow \prod_{q\geqslant m} \mathrm{Gr}_q N_\bullet$$

for any integer m. So, the fact that η_m is an equivalence for any m is equivalent to proving that F_p is 0, but this is a consequence of Lemma 1.5.24. Indeed, if F_p was not 0, we would have a non-trivial $\Bbbk$-module with a non-trivial map $F_p \to N_m$ for all m's. Such map would be forced to factor through N_∞; in particular, it would yield that N_∞ is non-trivial, hence that $N_\bullet$ is not complete. But this contradicts the (eventually) coconnectivity assumption on $N_\bullet$. □

Remark 1.5.40. In the proof of Proposition 1.5.39, we did not need to restrict ourselves to eventually coconnective objects. In fact, the key property of eventually coconnective filtered $\Bbbk$-modules that we used was their completeness (Lemma 1.5.24): the unit morphism of the adjunction $(-)^{\mathrm{gr}}_{\varepsilon} \dashv (-)^{\mathrm{fil}}$ is an equivalence on *all* complete filtered $\Bbbk$-modules, which are not necessarily eventually coconnective (take for example the filtered $\Bbbk$-module given by the 0 sequence everywhere, except for $\Bbbk[1]$ in weight 0). The fiber

$$F_p := \mathrm{fib}\left(N_p \to ((N_\bullet)^{\varepsilon\text{-}\mathrm{gr}})^{\mathrm{fil}}_p\right)$$

of the proof of Proposition 1.5.39 is, in fact, N_∞ itself: the unit η fails to be an equivalence *precisely* on non-complete filtered $\Bbbk$-modules.

1.5.41. All previous propositions, lemmas and remarks imply together Theorem 1.5.25. In fact, Proposition 1.5.23 allows us to identify the left completion $\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}$ with the ∞-category of complete filtered $\Bbbk$-modules in the sense of Definition 1.5.8, endowed with the restriction of the Beilinson t-structure. The equivalence of Theorem 1.5.25 is then provided by the composition of the natural inclusion $\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk} \subseteq \mathrm{Mod}^{\mathrm{fil}}_{\Bbbk}$ with the associated mixed graded ∞-functor of Definition 1.5.32. Its inverse is described by the composition of the associated filtered ∞-functor of Definition 1.5.27 with the completion ∞-functor of Lemma 1.5.12.2. The fact that these ∞-functors are indeed equivalences follows from Propositions 1.5.38 and 1.5.39. The t-exactness of the equivalence $\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk} \simeq \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ follows from the t-exactness of the ∞-functors $(-)^{\mathrm{fil}}$ and $(-)^{\mathrm{gr}}_{\varepsilon}$.

1.5.42. Finally, since $(-)^{\mathrm{fil}}$ and $(-)^{\mathrm{gr}}_{\varepsilon}$ are both t-exact ∞-functors which induce equivalences on coconnective objects, it follows that the hearts of the two t-structures are naturally equivalent and both yield, when $\Bbbk$ is a discrete ring, the usual 1-category of chain complexes of $\Bbbk$-modules: this is the last statement that we left unproven in Theorem 1.4.1.

Porism 1.5.43. The strongly monoidal structure of the ∞-functor $(-)^{\varepsilon\text{-}\mathrm{gr}}$ proved in Corollary 1.5.36, together with the description of the monoidal structure on complete filtered $\Bbbk$-modules provided by Lemma 1.5.12.4, implies that the equivalence of Theorem 1.5.25 is *strongly monoidal*. Explicitly, given two complete filtered $\Bbbk$-modules $M_\bullet$ and $N_\bullet$, their tensor product in $\widehat{\mathrm{Mod}}^{\mathrm{fil}}_{\Bbbk}$, given by the completion of their tensor product as mere filtered $\Bbbk$-modules ([GP18, Theorem 2.2.5]), is equivalently described as

$$M_\bullet \widehat{\otimes}^{\mathrm{fil}}_{\Bbbk} N_\bullet \simeq \left((M_\bullet)^{\varepsilon\text{-}\mathrm{gr}} \otimes^{\varepsilon\text{-}\mathrm{gr}}_{\Bbbk} (N_\bullet)^{\varepsilon\text{-}\mathrm{gr}}\right)^{\mathrm{fil}}.$$

Porism 1.5.44. In Section 1.1, we mentioned how the Tate realization ∞-functor was *strongly* monoidal when restricted to non-negatively mixed graded $\Bbbk$-modules. This claim follows immediately from Theorem 1.5.25: indeed, we have an equivalence in $\mathrm{Fun}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}, \mathrm{Mod}_{\Bbbk}\right)$

$$|-|^{\mathrm{t}} \simeq (-)_{-\infty} \circ (-)^{\mathrm{fil}}$$

and since $(-)^{\varepsilon\text{-gr}}$ is a localization ∞-functor, we have another equivalence

$$|-|^{\mathrm{t}} \circ (-)^{\varepsilon\text{-gr}} \simeq (-)_{-\infty}$$

if one restricts $(-)^{\varepsilon\text{-gr}}$ to $\widehat{\mathrm{Mod}}_{\Bbbk}^{\mathrm{fil}}$. Since the ∞-functor $(-)_{-\infty}$ is strongly monoidal ([GP18, Section 2.23]), one simply has to show that the tensor product of two complete filtered $\Bbbk$-modules $M_\bullet$ and $N_\bullet$ such that M_p and N_p are zero for all positive integers p is again complete. The explicit formula for the filtered tensor product provided in 1.5.11 yields that

$$\left(M_\bullet \otimes_{\Bbbk}^{\mathrm{fil}} N_\bullet\right)_n \simeq \operatorname*{colim}_{p+q\geqslant n} M_p \otimes_{\Bbbk} N_q.$$

If both $M_\bullet$ and $N_\bullet$ are endowed with a filtration bounded in non-positive weights, the above formula is 0 for any positive integer n. Hence the limit on the tower of $\Bbbk$-modules corresponding to $M_\bullet \otimes_{\Bbbk}^{\mathrm{fil}} N_\bullet$ is zero, hence the filtered tensor product in this case preserves completeness.

Porism 1.5.45. Arguing analogously to Porism 1.5.44, we can prove that the Tate realization ∞-functor preserves colimits when it is restricted to mixed non-negatively graded $\Bbbk$-modules. Indeed, we have a commutative square

$$\begin{tikzcd}
\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant 0} \arrow[r, "(-)^{\mathrm{fil}}"] \arrow[d, hook] & \mathrm{Mod}_{\Bbbk}^{\mathrm{fil},\leqslant 0} \arrow[d, hook] \\
\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \arrow[r, "(-)^{\mathrm{fil}}"] \arrow[dr, bend right, "|-|^{\mathrm{t}}"'] & \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \arrow[d, bend left, "(-)_{-\infty}"] \\
& \mathrm{Mod}_{\Bbbk}
\end{tikzcd}$$

where $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil},\leqslant 0}$ is the ∞-functor ∞-category $\mathrm{Fun}\left(\mathbb{N}_{\leqslant}, \mathrm{Mod}_{\Bbbk}\right)$ of $\Bbbk$-modules with filtration bounded in non-positive weights, and the inclusion is induced by the map of posets $\mathbb{N}_{\leqslant} \to \mathbb{Z}_{\geqslant}$ mapping n to $-n$. The ∞-functor $(-)_{-\infty}$ commutes with colimits because is itself a colimit ∞-functor, so it suffices to show that $(-)^{\mathrm{fil}} \colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant 0} \to \mathrm{Mod}_{\Bbbk}^{\mathrm{fil},\leqslant 0}$ commutes with colimits. This is true because colimits of non-positively filtered $\Bbbk$-modules are again non-positively filtered, since colimits are computed weight-wise. Hence they are again complete, and so the colimit in mixed graded $\Bbbk$-modules agrees with the colimit in filtered $\Bbbk$-modules. In particular, the ∞-functor

$$|-|^{\mathrm{t}} \colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant 0} \longrightarrow \mathrm{Mod}_{\Bbbk} \tag{1.5.46}$$

admits a right adjoint, which we can compute explicitly using the fact that the ∞-functor 1.5.46 is the composition of

$$\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\leqslant 0} \xrightarrow{\simeq} \mathrm{Mod}_{\Bbbk}^{\mathrm{fil},\leqslant 0} \hookrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \xrightarrow{(-)_{-\infty}} \mathrm{Mod}_{\Bbbk}.$$

In fact, each ∞-functor of this composition is a left adjoint, so the right adjoint to 1.5.46 is the composition of right adjoints

$$\mathrm{Mod}_{\Bbbk} \xrightarrow{(-)^{\mathrm{const}}} \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}} \xrightarrow{\sigma_{\leqslant 0}} \mathrm{Mod}_{\Bbbk}^{\mathrm{fil},\leqslant 0} \xrightarrow{\simeq} \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant 0}$$

where $(-)^{\mathrm{const}}$ is the diagonal ∞-functor sending a $\Bbbk$-module to the constant sequence, $\sigma_{\leqslant 0}$ is the naive truncation ∞-functor for filtered $\Bbbk$-modules (built similarly as the ∞-functor of Definition 1.2.1.1), and the last ∞-functor is the mixed graded $\Bbbk$-module construction (which is an equivalence because every filtered $\Bbbk$-module is complete if it is 0 for all positive integers). In particular, this right adjoint agrees with

$$(-)(0)\colon \mathrm{Mod}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr},\geqslant 0} \subseteq \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}.$$

Chapter 2

Mixed graded Chevalley-Eilenberg functors

In this chapter, we shall describe how Chevalley-Eilenberg chain complexes of differential graded Lie algebras (in the sense of [Lur11c]) are actually the shadow of objects endowed with a much richer structure. Namely, they arise as the Tate realization of mixed graded $\Bbbk$-modules, which are moreover either cocommutative coalgebras (in the case of the homological complex) or commutative algebras (in the case of the cohomological complex) in mixed graded $\Bbbk$-modules. This fact is a folklore result expected, if not known, by many experts (the most explicit formulation appears in [CG19, Appendix B]); however, the mixed graded structure on the Chevalley-Eilenberg algebra and coalgebra of a derived Lie algebra $\mathfrak{g}$ has until now appeared only in terms of explicit graded chain complexes equipped with mixed differentials, heavily relying on strict model choices. The main result of this chapter is the (completely ∞-categorical) construction of ∞-functors

$$\mathrm{CE}_\varepsilon\colon \mathrm{Lie}_\Bbbk \longrightarrow \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk},$$

$$\mathrm{CE}^\varepsilon\colon \mathrm{Lie}_\Bbbk^{\mathrm{op}} \longrightarrow \varepsilon\text{-}\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$$

and

$$\mathrm{CE}^\varepsilon(\mathfrak{g};\,-)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathrm{CE}^\varepsilon(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_\Bbbk\big),$$

computing respectively the homological Chevalley-Eilenberg mixed graded cocommutative coalgebra (Proposition 2.4.26), the cohomological Chevalley-Eilenberg mixed graded commutative algebra (Proposition 2.5.9), and the Chevalley-Eilenberg cohomology mixed graded $\Bbbk$-module of a fixed Lie algebra $\mathfrak{g}$ with values in any representation (Proposition 2.6.3).

This chapter is structured as follows. First, we shall recollect some important information about the homotopy theory of Lie algebras in characteristic 0, fixing notations and presenting some well-known constructions and definitions such as universal enveloping algebras, representations of Lie algebras, and classical Chevalley-Eilenberg complexes. The main reference is [Lur11c]. In the second half of the chapter (Sections 2.4 to 2.6) we shall provide a model independent construction of the mixed graded Chevalley-Eilenberg ∞-functors presented above. The construction is quite technical, but we shall emphasize the main conceptual tools and motivations of our construction. Finally, employing the theory of mixed graded $\Bbbk$-modules developed in Chapter 1, we shall study some formal properties of the Chevalley-Eilenberg mixed graded ∞-functors.

Throughout all this chapter, by $\Bbbk$ we mean a fixed base field of characteristic 0.

2.1 Lie algebras in characteristic 0

It is well known (see, for instance, [Lur17, Propositions 7.1.4.6 and 7.1.4.11]) that the homotopy theory of $\mathbb{E}_1$-ring spectra (respectively $\mathbb{E}_\infty$-ring spectra) and differential graded algebras (respectively commutative differential graded algebras) are equivalent, when working over a base provided by a discrete commutative ring $\Bbbk$ of characteristic 0. Following this philosophy, we can define Lie algebras in the derived setting in two different ways.

Definition 2.1.1 (Differential graded Lie algebras). A *differential graded Lie algebra* defined over a commutative ring $\Bbbk$ is a differential graded $\Bbbk$-module $\mathfrak{g}_\bullet$, endowed with a bracket

$$[-,-]\colon \mathfrak{g}_p \otimes_\Bbbk \mathfrak{g}_q \longrightarrow \mathfrak{g}_{p+q}$$

satisfying the following axioms.

- For $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$, we have that

$$[x, y] + (-1)^{pq}[y, x] = 0.$$

- For $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$ and $z \in \mathfrak{g}_r$, we have that

$$(-1)^{pr}[x, [y, z]] + (-1)^{pq}[y, [z, x]] + (-1)^{qr}[z, [x, y]] = 0.$$

- The differential $\mathrm{d}\colon \mathfrak{g} \to \mathfrak{g}[1]$ is a derivation with respect to the Lie bracket; i.e. for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$ we have that

$$\mathrm{d}[x, y] = [\mathrm{d}(x), y] + (-1)^{pq}[x, \mathrm{d}(y)].$$

A morphism of differential graded Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$ is a morphism of the underlying chain complexes preserving the Lie bracket, i.e. such that $f([x, y]) = [f(x), f(y)]$ for all x, y in $\mathfrak{g}$.

2.1.2. With the descriptions provided in Definition 2.1.1, it follows that differential graded Lie algebras are naturally gathered in a category, $\mathrm{dgLie}_\Bbbk$. This category is moreover endowed with a model structure where weak equivalences and fibrations are detected by the forgetful functor

$$\mathrm{oblv}_{\mathrm{Lie}} : \mathrm{dgLie}_{\Bbbk} \longrightarrow \mathrm{dgMod}_{\Bbbk},$$

see for example [Lur11c, Proposition 2.1.10]. In particular, using the formalism of ([DK80]), we can consider the Dwyer-Kan localization of $\mathrm{dgLie}_{\Bbbk}$ at the class $\mathscr{W}$ of weak equivalences (i.e., quasi-isomorphisms) and consider its simplicial nerve to get an ∞-category $\mathrm{N}_{\Delta}\left(\mathrm{L}^{\mathrm{H}}\left(\mathrm{dgLie}_{\Bbbk}, \mathscr{W}\right)\right)$, that we simply denote by $\mathrm{Lie}_{\Bbbk}$.

Definition 2.1.3. The ∞-category $\mathrm{Lie}_{\Bbbk}$ is the *∞-category of Lie algebras over* $\Bbbk$.

Alternatively, we can define Lie algebras via the operadic approach in the following way. Let us denote by Lie the ordinary Lie (1-)operad.

Definition 2.1.4. A *Lie algebra over a commutative ring* $\Bbbk$ is an algebra for the Lie operad in the ∞-category $\mathrm{Mod}_{\Bbbk}$ of $\Bbbk$-modules. The ∞-category $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\Bbbk})$ is the *∞-category of Lie algebras over* $\Bbbk$, and shall be denoted by $\mathrm{Lie}_{\Bbbk}$.

Remark 2.1.5. Definitions 2.1.1 and 2.1.4 agree in the characteristic 0 setting, hence the same notation. Let us briefly recall the main steps that provide the equivalence of these two homotopy theories: recall that homotopy algebras for the Lie operad are the same as $\mathbb{L}_{\infty}$-algebras in chain complexes, where $\mathbb{L}_{\infty}$ is any cofibrant replacement of the operad Lie in the model category of operads. There exists an obvious inclusion

$$\mathrm{dgLie}_{\Bbbk} \hookrightarrow \mathrm{Alg}_{\mathbb{L}_{\infty}}\left(\mathrm{dgMod}_{\Bbbk}\right).$$

There exist two different categories related to $\mathrm{dgLie}_{\Bbbk}$ and $\mathrm{Alg}_{\mathbb{L}_{\infty}}\left(\mathrm{dgMod}_{\Bbbk}\right)$.

1. The category $\mathrm{Pro}\left(\mathrm{dgArt}_{\Bbbk}\right)^{\mathrm{op}}$, which is the opposite category of the category of pro-objects in local differential graded Artinian commutative $\Bbbk$-algebras. This category is endowed with a model structure, whose fibrant objects are precisely $\mathbb{L}_{\infty}$-algebras (this is showed in the proof of [Pri10, Proposition 4.42]).

2. The category $\mathrm{dgcCcAlg}_{\Bbbk}^{\mathrm{un}}$, which is the category of unital differential graded cocommutative coalgebras (in the sense of [Hin01, Definition 2.1.1]). This category is endowed with a model structure as well, and in virtue of [Hin01, Theorem 3.2] there exists a Quillen equivalence
$$\mathrm{dgcCcAlg}_{\Bbbk}^{\mathrm{un}} \xrightarrow{\simeq} \mathrm{dgLie}_{\Bbbk}.$$

Moreover, there exists an equivalence of categories

$$\mathrm{Pro}\left(\mathrm{dgArt}_{\Bbbk}\right)^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{dgcCcAlg}_{\Bbbk}^{\mathrm{un}}$$

which identifies the two model structures (this is the content of [Pri10, Corollary 4.56]). In particular, the inclusion of fibrant differential graded Lie algebras into the category of fibrant

$\mathbb{L}_\infty$-algebras extends to a right Quillen functor, whose left adjoint agrees with the Quillen equivalence $\mathrm{dgcCcAlg}_{\Bbbk}^{\mathrm{un}} \simeq \mathrm{dgLie}_{\Bbbk}$. This induces the desired equivalence at the level of associated ∞-categories.

2.1.6. Let $\mathrm{Alg}_{\Bbbk}$ denote the ∞-category of associative $\Bbbk$-algebras. We have an ∞-functor

$$\mathrm{L}\colon \mathrm{Alg}_{\Bbbk} \longrightarrow \mathrm{Lie}_{\Bbbk}$$

given by the *underlying commutator Lie algebra*, which can be understood as the ∞-functor induced by the functor

$$\mathrm{dgAlg}_{\Bbbk} := \mathrm{Alg}^{\otimes}\left(\mathrm{dgMod}_{\Bbbk}\right) \longrightarrow \mathrm{dgLie}_{\Bbbk}$$

sending a differential graded $\Bbbk$-algebra to the differential graded Lie algebra given by the same underlying chain complex and endowed with the commutator Lie bracket. This ∞-functor is actually the right adjoint in the adjoint couple

$$\mathrm{U}\colon \mathrm{Lie}_{\Bbbk} \rightleftarrows \mathrm{Alg}_{\Bbbk} : \mathrm{L}\,. \tag{2.1.7}$$

Definition 2.1.8. The left adjoint in the adjunction 2.1.7 is the *universal enveloping algebra ∞-functor*.

Remark 2.1.9. The universal enveloping algebra ∞-functor has been observed from many perspectives, and a construction in the setting of *$\mathbb{L}_\infty$-algebras in chain complexes* (i.e., algebras over the operad $\mathbb{L}_\infty$, which provides a cofibrant replacement in the model category of operads for the operad Lie) is provided for instance in [Bar08]. Another interesting (and more general) approach to $\mathbb{E}_n$-enveloping algebras for arbitrary n can be found in [GH18, Section 2.7]. In [Lur11c, Remark 2.1.7], it is provided an explicit model in the context of differential graded Lie algebras.

In the remainder of this paper, however, we shall need a universal enveloping algebra ∞-functor also in the context of Lie algebra objects in less standard symmetric monoidal stable ∞-categories than the usual ∞-category $\mathrm{Mod}_{\Bbbk}$; in particular, we shall be interested in the case of $\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C})$ where $\mathscr{C}$ is the ∞-category of mixed graded $\Bbbk$-modules $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$. It appears that a universal enveloping algebra construction can be carried out also in this more general setting since we have a natural map of operads

$$\mathrm{Lie} \longrightarrow \mathrm{Assoc}^{\mathrm{aug}}$$

which induces a restriction ∞-functor between associative augmented algebra objects in a symmetric monoidal ∞-category $\mathscr{C}$ to Lie algebra objects. This ∞-functor admits a left adjoint which is the universal enveloping algebra ∞-functor we need: see [GR17b, Chapter 6, Section 5.1].

Construction 2.1.10. Let $\mathfrak{g}$ be a classical (i.e., discrete) Lie algebra defined over a field $\Bbbk$. Its enveloping algebra $\mathrm{U}(\mathfrak{g})$ is naturally endowed with a cocommutative Hopf structure over $\Bbbk$.

The morphims of $\Bbbk$-algebras $\epsilon\colon \mathrm{U}(\mathfrak{g}) \to \Bbbk$, $\mu\colon \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{U}(\mathfrak{g})$ and $\iota\colon \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ which provide respectively the counit, the comultiplication and the coinversion for such Hopf structure arise in the following way.

1. The counit $\epsilon\colon \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(0) \simeq \Bbbk$ is induced by applying the functor U to the trivial map of Lie algebras $\mathfrak{g} \to 0$.

2. The comultiplication $\mu\colon \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g} \times \mathfrak{g}) \simeq \mathrm{U}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{U}(\mathfrak{g})$ is induced by applying the functor U to the diagonal map $\Delta\colon \mathfrak{g} \hookrightarrow \mathfrak{g} \times \mathfrak{g}$.

3. The coinversion $\iota\colon \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ is induced by applying the functor U to the inversion map $-\mathrm{id}_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{g}$.

We can adapt this idea to the context of ∞-categories, and make the universal enveloping algebra of a Lie algebra in $\mathscr{C}$ into a Hopf algebra in $\mathscr{C}$ (see [Qui69, Appendix B] for the case of differential graded Lie algebras or, for a more modern and general perspective, [GR17b, Chapter 6, Section 5.1.4]). Given a symmetric monoidal stable ∞-category $\mathscr{C}$, we can consider algebras for the associative operad $\mathbb{E}_1$ (i.e., associative algebras) and algebras for the Lie operad. Both these ∞-categories of algebras are symmetric monoidal, with the monoidal structure given by the Cartesian monoidal structure on $\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C})$ and by the usual tensor product of algebras in $\mathrm{Alg}_{\mathbb{E}_1}(\mathscr{C})$. The universal enveloping algebra ∞-functor $\mathrm{U}\colon \mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C}) \to \mathrm{Alg}_{\mathbb{E}_1}(\mathscr{C})$ is then a *strongly monoidal ∞-functor* for these symmetric monoidal structures ([GR17b, Chapter 6, Lemma 5.2.8]), hence it induces an ∞-functor at the level of cocommutative coalgebras

$$\mathrm{U}\colon\ \mathrm{cCcAlg}\big(\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C})\big) \longrightarrow \mathrm{cCcAlg}\big(\mathrm{Alg}_{\mathbb{E}_1}(\mathscr{C})\big) =: \mathrm{cCBAlg}(\mathscr{C})$$

where the target is the ∞-category of cocommutative coalgebras in associative algebras, i.e., *cocommutative bialgebras*. Since the monoidal structure on $\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C})$ is Cartesian, there is an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C}) \simeq \mathrm{cCcAlg}\big(\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C})\big) \tag{2.1.11}$$

in virtue of [Lur17, Proposition 2.4.3.9]: essentially, the diagonal morphism $\Delta\colon \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ turns every Lie algebra into a cocommutative coalgebra object in Lie algebras. By precomposing with the equivalence 2.1.11, we get our universal enveloping algebra ∞-functor with its bialgebra structure

$$\mathrm{U}\colon \mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C}) \simeq \mathrm{cCcAlg}\big(\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C})\big) \longrightarrow \mathrm{cCBAlg}(\mathscr{C}). \tag{2.1.12}$$

The fact that the ∞-functor U factors through the ∞-category of Hopf algebras follows from the fact that

$$\mathrm{cCBAlg}(\mathscr{C}) := \mathrm{cCcAlg}\big(\mathrm{Alg}_{\mathbb{E}_1}(\mathscr{C})\big) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{cCcAlg}(\mathscr{C})) =: \mathrm{Mon}(\mathrm{cCcAlg}(\mathscr{C})),$$

since the monoidal structure on $\mathrm{cCcAlg}(\mathscr{C})$ is given by the tensor product in $\mathscr{C}$, which coincides with Cartesian monoidal structure for coalgebras, see[GR17b, Chapter 6, Appendix C.1.1.]. So the universal enveloping algebra ∞-functor preserves products of Lie algebras, hence gives rise to a ∞-functor at the level of group objects

$$\mathrm{U}\colon\ \mathrm{Grp}\left(\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C})\right) \longrightarrow \mathrm{Grp}\left(\mathrm{cCcAlg}(\mathscr{C})\right).$$

Since looping and delooping provides an equivalence $\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C}) \simeq \mathrm{Grp}\left(\mathrm{Alg}_{\mathrm{Lie}}(\mathscr{C})\right)$ for any symmetric monoidal stable ∞-category $\mathscr{C}$ ([GR17b, Proposition 1.6.4]), and that a Hopf algebra is precisely a group object in cocommutative coalgebras, we obtain the Hopf structured we asked for.

2.2 Representations of Lie algebras

In this section, we briefly review the definition and properties of representations of a Lie algebra. The most convenient and straight-forward definition for a representation of a Lie algebra in the derived setting is the following.

Definition 2.2.1. A *representation of a Lie algebra* $\mathfrak{g}$ *over* $\Bbbk$ is a left $\mathrm{U}(\mathfrak{g})$-module. The ∞-*category of representations of a Lie algebra* $\mathfrak{g}$ is then the ∞-category $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}$ of left $\mathrm{U}(\mathfrak{g})$-modules, and shall be denoted by $\mathrm{Rep}_{\mathfrak{g}}$.

Remark 2.2.2. The fact that this definition, motivated by the equivalence of left modules on the enveloping algebra of a classical (discrete) Lie algebra $\mathfrak{g}$ and classical representations of $\mathfrak{g}$, is still the right notion for derived representations of derived Lie algebras is explained by the following argument. Consider a differential graded Lie algebra $\mathfrak{g}_\bullet$ over $\Bbbk$; then one can consider, in analogy to the classical setting, a *differential graded representation of* $\mathfrak{g}_\bullet$ to be a chain complex of $\Bbbk$-modules $V_\bullet$ endowed with a left action

$$\mathfrak{g}_\bullet \otimes_{\Bbbk} V_\bullet \longrightarrow V_\bullet$$

such that

$$[x,y]\cdot v = x\cdot(y\cdot v) + (-1)^{pq} y\cdot(x\cdot v)$$

for any $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$. Differential graded representations of a differential graded Lie algebra $\mathfrak{g}_\bullet$ are gathered in a category, which we shall denote with $\mathrm{dgRep}_{\mathfrak{g}_\bullet}$. Similarly to the case of differential graded Lie algebras, also $\mathrm{dgRep}_{\mathfrak{g}_\bullet}$ is endowed with a model structure whose weak equivalences and fibrations are detected by the forgetful functor $\mathrm{dgRep}_{\mathfrak{g}_\bullet} \longrightarrow \mathrm{dgMod}_{\Bbbk}$ ([Lur11c, Proposition 2.4.5]). By Dwyer-Kan localization with respect to the class $\mathscr{W}$ of weak equivalences yields an ∞-category $\mathrm{N}_\Delta\left(\mathrm{L}^{\mathrm{H}}\left(\mathrm{dgRep}_{\mathfrak{g}_\bullet}, \mathscr{W}\right)\right)$ which we simply denote by $\mathrm{Rep}_{\mathfrak{g}}$.
It is known ([LM95, Theorem 5.4]) that giving a differential graded representation $V_\bullet$ of a differential graded Lie algebra $\mathfrak{g}_\bullet$ over $\Bbbk$ is equivalent to giving a morphism of differential

graded Lie algebras
$$\mathfrak{g}_\bullet \longrightarrow \mathrm{L}(\mathrm{End}_{\Bbbk}(V_\bullet))$$
where $\mathrm{L}(\mathrm{End}_{\Bbbk}(V_\bullet))$ is the associative differential graded $\Bbbk$-algebra of endomorphisms of $V_\bullet$ thought as a differential graded Lie algebra via the right adjoint 2.1.7. In particular, because of that adjunction, it follows that giving a morphism of associative differential graded $\Bbbk$-algebras $\mathfrak{g}_\bullet \to \mathrm{L}(\mathrm{End}_{\Bbbk}(V_\bullet))$ is equivalent to giving a differential graded $\Bbbk$-algebra morphism $\mathrm{U}(\mathfrak{g}_\bullet) \to \mathrm{End}_{\Bbbk}(V_\bullet)$, which is equivalent (see for instance [Lur17, Section 4.7.1]) to giving a left $\mathrm{U}(\mathfrak{g}_\bullet)$-module structure to the chain complex of $\Bbbk$-modules $V_\bullet$ in the category $\mathrm{dgMod}_{\Bbbk}$. In particular, it follows that the ∞-category $\mathrm{Rep}_{\mathfrak{g}}$ and the ∞-category $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}$ of left $\mathrm{U}(\mathfrak{g})$-modules are equivalent one to the other as stable $\Bbbk$-linear ∞-categories, for any Lie algebra $\mathfrak{g}$ over $\Bbbk$, hence the motivation for Definition 2.2.1.

Remark 2.2.3. Given a Lie algebra $\mathfrak{g}$, then $\mathrm{U}(\mathfrak{g})$ is equivalent as an associative algebra to its opposite $\mathrm{U}(\mathfrak{g})^{\mathrm{rev}}$ (in the sense of [Lur17, Remark 4.1.1.7]). Indeed, the antipode involution $-\mathrm{id}\colon \mathfrak{g} \to \mathfrak{g}$ induces an equivalence $\mathrm{U}(\mathfrak{g}) \xrightarrow{\simeq} \mathrm{U}(\mathfrak{g})^{\mathrm{rev}}$. In particular, pulling back along such equivalence induces a natural equivalence of ∞-categories
$$\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \simeq \mathrm{RMod}_{\mathrm{U}(\mathfrak{g})^{\mathrm{rev}}} \xrightarrow{\simeq} \mathrm{RMod}_{\mathrm{U}(\mathfrak{g})}.$$
This implies that any representation of a Lie algebra $\mathfrak{g}$ can be equivalently interpreted as a left or right $\mathrm{U}(\mathfrak{g})$-module, depending on our need.

2.2.4. Classically, given a Hopf algebra H over a base commutative ring $\Bbbk$ (such as the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over a base field of characteristic 0) one can define a monoidal structure on the abelian 1-category of left H-modules such that the forgetful functor
$$\mathrm{oblv}_H\colon \mathrm{LMod}_H \longrightarrow \mathrm{Mod}_{\Bbbk}$$
is strongly monoidal. The left H-action on the tensor product $M \otimes_{\Bbbk} N$ is induced by the comultiplication $\mu\colon H \to H \otimes_{\Bbbk} H$. Moreover, such monoidal structure on LMod_H is *symmetric* monoidal if H is a *cocommutative* Hopf algebra. The above discussion is actually part of a larger framework, namely *Tannaka duality*, which affirms that any symmetric monoidal category with a strongly monoidal fiber functor to the abelian category of $\Bbbk$-vector spaces is equivalent to the category of left modules on some cocommutative bialgebra (a nice account of this theory is provided in [Sch92]). While we do not need the whole Tannaka duality picture for Hopf algebras, we shall need the symmetric monoidal structure on the ∞-category of LMod_A, for A a cocommutative $\Bbbk$-bialgebra. This has been spelled out in great generality in [Bea18, Section 3.1], even if we need only the following special case.

Construction 2.2.5 ([Bea18, Corollary 3.19]). Let $\mathfrak{g}$ be a Lie algebra object in a symmetric monoidal stable ∞-category $\mathscr{C}$, which is $\Bbbk$-linear over a field $\Bbbk$ of characteristic 0. Then the ∞-category $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}(\mathscr{C})$ of left $\mathrm{U}(\mathfrak{g})$-modules in $\mathscr{C}$ is the ∞-category of left modules over a

Hopf algebra, hence of a cocommutative bialgebra: in particular, it is endowed with a symmetric monoidal structure and with a strongly monoidal forgetful ∞-functor

$$\mathrm{oblv}_{\mathrm{U}(\mathfrak{g})} \colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}(\mathscr{C}) \longrightarrow \mathscr{C}.$$

In particular, the action of $\mathrm{U}(\mathfrak{g})$ on the underlying object $M \otimes_{\mathscr{C}} N$ of $M \otimes_{\mathfrak{g}} N$ is given by

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathscr{C}} M \otimes_{\mathscr{C}} N \xrightarrow{\mu^*} \mathrm{U}(\mathfrak{g}) \otimes_{\mathscr{C}} \mathrm{U}(\mathfrak{g}) \otimes_{\mathscr{C}} M \otimes_{\mathscr{C}} N \simeq \mathrm{U}(\mathfrak{g}) \otimes_{\mathscr{C}} M \otimes_{\mathscr{C}} \mathrm{U}(\mathfrak{g}) \otimes_{\mathscr{C}} N \xrightarrow{\alpha_M \otimes \alpha_N} M \otimes_{\mathscr{C}} N$$

where $\mu \colon \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes_{\mathscr{C}} \mathrm{U}(\mathfrak{g})$ is the Hopf algebra comultiplication and α_M, α_N are the two natural left $\mathrm{U}(\mathfrak{g})$-actions ([Bea18, Remark 3.20]). In the following, we shall refer to such monoidal structure as the *tensor product of left* $\mathrm{U}(\mathfrak{g})$*-modules*. Abusing notations, fixed a left $\mathrm{U}(\mathfrak{g})$-module M, we shall refer to the ∞-functor

$$- \otimes_{\mathfrak{g}} M \colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}(\mathscr{C}) \longrightarrow \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}(\mathscr{C})$$

as the *tensor product with* M.

Remark 2.2.6. In the situation of Construction 2.2.5, given a Lie algebra object $\mathfrak{g}$ in a symmetric monoidal ∞-category $\mathscr{C}$ which is $\Bbbk$-linear over a field $\Bbbk$ of characteristic 0, the comultiplication map $\mu \colon \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes_{\mathscr{C}} \mathrm{U}(\mathfrak{g})$ is *always* a morphism of left $\mathrm{U}(\mathfrak{g})$-modules, i.e., it is actually a map

$$\mu \colon \mathrm{U}(\mathfrak{g}) \longrightarrow \mathrm{U}(\mathfrak{g}) \otimes_{\mathfrak{g}} \mathrm{U}(\mathfrak{g}).$$

Indeed, the fact that μ is a map of left $\mathrm{U}(\mathfrak{g})$-modules is equivalent to saying that it is compatible with the left action of $\mathrm{U}(\mathfrak{g})$ on itself, i.e., with the multiplication of $\mathrm{U}(\mathfrak{g})$. But now [GR17b, Proposition C.1.3] assures us that the ∞-category $\mathrm{cBAlg}(\mathscr{C})$ of cocommutative bialgebras of $\mathscr{C}$ can be described equivalently either as $\mathrm{Alg}(\mathrm{cCcAlg}(\mathscr{C}))$ or as $\mathrm{cCcAlg}(\mathrm{Alg}(\mathscr{C}))$, hence we have the desired compatibility.

2.3 Classical Chevalley-Eilenberg complexes

Classically, homology and cohomology of a Lie algebra $\mathfrak{g}$ can be interpreted respectively in terms of Tor and Ext groups, but in concrete computations is often more convenient to deal with standard, explicit presentations at the level of chains - the *homological* and *cohomological Chevalley-Eilenberg complexes*, respectively. In this section, we shall briefly review the classical construction of the Chevalley-Eilenberg complexes, providing motivations and ideas to generalize this construction at the ∞-categorical level.

2.3.1. Let $\mathfrak{g}$ be a classical finite dimensional Lie algebra defined over $\Bbbk$. We can consider the graded $\Bbbk$-module

$$\mathrm{V}_\bullet(\mathfrak{g}) \coloneqq \mathrm{U}(\mathfrak{g}) \otimes_{\Bbbk} \bigwedge\nolimits^{\bullet} \mathfrak{g}$$

endowed with differential

$$u\otimes(g_1\wedge\ldots\wedge g_n)\mapsto\sum_{1\leqslant i\leqslant n}(-1)^{i+1}u\cdot g_i\otimes\left(g_1\wedge\ldots\wedge\widehat{g}_i\wedge\ldots\wedge g_n\right)$$
$$+\sum_{1\leqslant i<j\leqslant n}(-1)^{i+j}u\otimes\left(g_1\wedge\ldots\wedge\widehat{g_i}\wedge\ldots\wedge g_{j-1}\wedge[g_i,g_j]\wedge\ldots\wedge g_n\right).$$

Definition 2.3.2. Let $\rho\colon\mathfrak{g}\to\mathrm{M}_n(M)$ be a representation of $\mathfrak{g}$ in some free classical $\Bbbk$-module M that, in virtue of Remark 2.2.3, we can view both as a left or right $\mathrm{U}(\mathfrak{g})$-module.

1. The *homological Chevalley-Eilenberg complex of* $\mathfrak{g}$ *with coefficients in* M is the chain complex

$$\mathrm{CE}_\bullet(\mathfrak{g};\,M):=M\otimes_{\mathrm{U}(\mathfrak{g})}\mathrm{V}_\bullet(\mathfrak{g}).$$

   The *homology of* $\mathfrak{g}$ *with coefficients in* M is the graded vector space $\mathrm{H}_\bullet\left(\mathrm{CE}_\bullet\left(\mathfrak{g};\,M\right)\right)$, and we will simply denote it as $\mathrm{H}_\bullet\left(\mathfrak{g};\,M\right)$.

2. The *cohomological Chevalley-Eilenberg complex of* $\mathfrak{g}$ *with coefficients in* M is the chain complex

$$\mathrm{CE}^\bullet(\mathfrak{g};\,M):=\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(\mathrm{V}_\bullet(\mathfrak{g}),\,M).$$

   The *cohomology of* $\mathfrak{g}$ *with coefficients in* M is the graded vector space $\mathrm{H}^\bullet\left(\mathrm{CE}^\bullet\left(\mathfrak{g};\,M\right)\right)$, and we sill simply denote it as $\mathrm{H}^\bullet\left(\mathfrak{g};\,M\right)$

Notation 2.3.3. If $M=\Bbbk$, we shall refer to $\Bbbk\otimes_{\mathrm{U}(\mathfrak{g})}\mathrm{V}_\bullet(\mathfrak{g})$ (respectively, $\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(\mathrm{V}_\bullet(\mathfrak{g}),\,\Bbbk)$) simply as the *homological* (respectively, the *cohomological*) *Chevalley-Eilenberg complex of* $\mathfrak{g}$, and denote it simply as $\mathrm{CE}_\bullet(\mathfrak{g})$ (respectively, $\mathrm{CE}^\bullet(\mathfrak{g})$).

Remark 2.3.4. It is well-known (see [Wei94, Theorem 7.7.2]) that the natural augmentation $\mathrm{V}_\bullet(\mathfrak{g})\to\Bbbk$ induced by the zero morphism $\mathfrak{g}\to\Bbbk$ exhibits $\mathrm{V}_\bullet(\mathfrak{g})$ as a projective resolution of the trivial representation $\Bbbk$. This means that $\mathrm{CE}_\bullet(\mathfrak{g};\,M)$ and $\mathrm{CE}^\bullet(\mathfrak{g};\,M)$ provide explicit models for the derived tensor product $\Bbbk\otimes^{\mathbb{L}}_{\mathrm{U}(\mathfrak{g})}\Bbbk$ and the derived hom-space $\mathbb{R}\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(\Bbbk,\,\Bbbk)$, respectively. In particular, the homology and cohomology of Lie algebras can be expressed using the language of derived functors as

$$\mathrm{H}_\bullet(\mathfrak{g};\,M):=\mathrm{Tor}_\bullet^{\mathrm{U}(\mathfrak{g})}(\Bbbk,\,M)$$

and

$$\mathrm{H}^\bullet(\mathfrak{g};\,M):=\mathrm{Ext}^\bullet_{\mathrm{U}(\mathfrak{g})}(\mathfrak{g},\,M).$$

Moreover, in the case $M=\Bbbk$, unraveling all the definitions one can recover the usual isomorphisms of graded $\Bbbk$-modules

$$\mathrm{CE}_\bullet(\mathfrak{g})=\Bbbk\otimes_{\mathrm{U}(\mathfrak{g})}\mathrm{U}(\mathfrak{g})\otimes_\Bbbk\bigwedge^\bullet\mathfrak{g}\cong\bigwedge^\bullet\mathfrak{g}$$

and

$$\mathrm{CE}^\bullet(\mathfrak{g})=\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}\Big(\mathrm{U}(\mathfrak{g})\otimes_\Bbbk\bigwedge^\bullet\mathfrak{g},\,\Bbbk\Big)\cong\mathrm{Hom}_\Bbbk\Big(\bigwedge^\bullet\mathfrak{g},\,\Bbbk\Big)\cong\bigwedge^{-\bullet}\mathfrak{g}^\vee.$$

In general these isomorphisms cannot be promoted to isomorphisms of chain complexes, unless the Lie algebra is abelian.

2.3.5. It is however true that the usual comultiplication and multiplication of $\bigwedge^\bullet \mathfrak{g}$ and $\bigwedge^{-\bullet} \mathfrak{g}^\vee$ are compatible with the differential, turning $\mathrm{CE}_\bullet(\mathfrak{g})$ and $\mathrm{CE}^\bullet(\mathfrak{g})$ into a differential graded cocommutative coalgebra and into a differential graded commutative algebra, respectively. It is moreover true that the set of Lie algebra structures on $\mathfrak{g}$ is in bijective correspondence with the set of differential graded cocommutative coalgebra structures of $\bigwedge^\bullet \mathfrak{g}$: in fact, a morphism

$$\mathrm{d}_\bullet \colon \bigwedge^\bullet \mathfrak{g} \longrightarrow \bigwedge^{\bullet-1} \mathfrak{g}$$

squares to zero if and only if $[-,-] := \mathrm{d}_2 \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is anti-symmetric, and it respects the coalgebra structure precisely if it satisfies the Leibniz rule. This statement can be rephrased as follows.

Proposition 2.3.6. *Let $\mathrm{Lie}_{\Bbbk}^{\mathrm{disc}}$ be the 1-category of discrete Lie algebras. Then the Chevalley-Eilenberg functor can be a promoted to a functor with values in differential graded cocommutative coalgebras*

$$\mathrm{CE}_\bullet \colon \mathrm{Lie}_{\Bbbk}^{\mathrm{disc}} \longrightarrow \mathrm{dgcCcAlg}_{\Bbbk}\,.$$

Such functor is fully faithful, and yields an equivalence of categories between the category of discrete Lie algebras and the full subcategory of differential graded cocommutative coalgebras which are semi-free (i.e., isomorphic as graded cocommutative coalgebras to the Grassmann algebra over some $\Bbbk$-vector space).

An analogous statement can be made taking into account the commutative algebra structures of $\bigwedge^{-\bullet} \mathfrak{g}$, once we add the assumption for $\mathfrak{g}$ to be finitely generated as a $\Bbbk$-module. In this case, the functor

$$\mathrm{CE}^\bullet \colon \mathrm{Lie}_{\Bbbk}^{\mathrm{disc}} \longrightarrow \mathrm{dgCAlg}_{\Bbbk}$$

is valued in differential graded commutative algebras.

2.3.7. The previous discussion can be made, almost *verbatim*, also in the context of differential graded Lie algebras, using [Lur11c, Section 2.2]. Using the fact that for any discrete vector space M there is an isomorphism of non-negatively graded vector spaces

$$\bigwedge^\bullet M \cong \mathrm{Sym}_{\Bbbk}^\bullet(M[1])$$

it is possible to define the *homological Chevalley-Eilenberg complex of a differential graded Lie algebra $\mathfrak{g}_\bullet$ with values in a differential graded representation $M_\bullet$* as the chain complex

$$\mathrm{CE}_\bullet(\mathfrak{g}_\bullet;\ M_\bullet) := M_\bullet \otimes_{\Bbbk} \mathrm{Sym}_{\Bbbk}^\bullet(\mathfrak{g}_\bullet[1])$$

with differential given by the formula

$$
\begin{aligned}
m \otimes (g_1 \dots g_n) \mapsto & \sum_{1\leqslant i\leqslant n} (-1)^{|x_1|+\dots+|x_{i-1}|} m \otimes \big(g_1 \dots \mathrm{d}_{\mathfrak{g}}(g_i) \dots g_n\big) \\
& + (-1)^{|x_i|(|x_{i+1}|+\dots+|x_n|)} \mathrm{d}_M(m) \otimes (g_1 \dots g_n) \\
& + \sum_{1\leqslant i\leqslant n} (-1)^{|x_i|(|x_{i+1}|+\dots+|x_n|)} m \cdot g_i \otimes \big(g_1 \dots \widehat{g}_i \dots \wedge g_n\big) \\
& + \sum_{1\leqslant i<j\leqslant n} (-1)^{|x_i|\left(|x_{i+1}|+\dots+|x_{j-1}|\right)} m \otimes \big(g_1 \dots \widehat{g_i} \dots g_{j-1} \cdot [g_i, g_j] \wedge \dots g_n\big).
\end{aligned}
\tag{2.3.8}
$$

In the same way, the *cohomological Chevalley-Eilenberg complex of a differential graded Lie algebra* $\mathfrak{g}_\bullet$ *with coefficients in a differential graded representation* $M_\bullet$ is the chain complex

$$\mathrm{CE}^\bullet(\mathfrak{g}_\bullet;\ M_\bullet) := \mathrm{Hom}_\Bbbk\big(\mathrm{Sym}_\Bbbk(\mathfrak{g}_\bullet[1]),\ M_\bullet\big).$$

Again, homological and cohomological Chevalley-Eilenberg complexes with coefficients in the trivial $\mathrm{U}(\mathfrak{g}_\bullet)$-module $\Bbbk$ inherit a cocommutative coalgebra and a commutative algebra structure, respectively, from the commutative and cocommutative bialgebra structure of the symmetric algebra. Moreover, they both preserve quasi-isomorphisms of differential graded Lie algebras, hence they provide ∞-functors

$$\mathrm{CE}_\bullet\colon \mathrm{Lie}_\Bbbk \longrightarrow \mathrm{cCcAlg}_\Bbbk$$

and

$$\mathrm{CE}^\bullet\colon \mathrm{Lie}_\Bbbk \longrightarrow \mathrm{CAlg}_\Bbbk.$$

Warning 2.3.9. There is *no hope* that these ∞-functors are again fully faithful: passing to the ∞-categorical setting, we identify strict commutative differential graded algebras with isomorphic homology $\Bbbk$-vector spaces. Thus, for example, the Lie algebras $\Bbbk$ with abelian Lie structure and $\Bbbk \oplus \Bbbk$ with its essentially unique non-abelian Lie bracket are both mapped, via the cohomological Chevalley-Eilenberg ∞-functor $\mathrm{CE}^\bullet$, onto the square-zero extension $\Bbbk \oplus \Bbbk[-1]$, even if they are obviously not equivalent as homotopy Lie algebras. In order to solve this issue, we need to keep track of the datum of the *filtration* of the Chevalley-Eilenberg complexes. This is the content of Section 2.4 and Section 2.5.

2.4 The homological Chevalley-Eilenberg ∞-functor

We shall now provide a mixed graded (hence, filtered) structure on the Chevalley-Eilenberg cocommutative coalgebra of a Lie algebra $\mathfrak{g}$ (Definition 2.3.2). Our main objective in this section is to show that the left $\mathrm{U}(\mathfrak{g})$-module $\mathrm{U}(\mathfrak{g})\otimes_\Bbbk \mathrm{Sym}_\Bbbk(\mathfrak{g}[-1])$, which is classically a projective resolution as a chain complex of left $\mathrm{U}(\mathfrak{g})$-modules of the discrete left $\mathrm{U}(\mathfrak{g})$-module $\Bbbk$, is naturally endowed with a mixed graded (hence, filtered) structure. What is more important, such mixed graded structure is naturally induced by the mixed graded structure on the *cone* of $\mathfrak{g}$ ([Lur11c, Construction 2.2.1]), which here naturally appears as some right adjoint to the ∞-functor $(-)_0\colon \varepsilon\text{-}\mathrm{Lie}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Lie}_{\Bbbk}$. All the Chevalley-Eilenberg $\Bbbk$-modules $\mathrm{CE}_{\bullet}(\mathfrak{g};\, N)$ and $\mathrm{CE}^{\bullet}(\mathfrak{g};\, M)$ for a right $\mathrm{U}(\mathfrak{g})$-module N or a left $\mathrm{U}(\mathfrak{g})$-module M (in particular, for $M = N = \Bbbk$) hence inherit a mixed graded / filtered structure as well, naturally induced by the one on $\mathrm{U}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{Sym}_{\Bbbk}(\mathfrak{g}[-1])$.

2.4.1. Let $\mathrm{Fun}\big(\Delta^1,\, \mathrm{Lie}_{\Bbbk}\big)$ be the ∞-category of morphisms in $\mathrm{Lie}_{\Bbbk}$, and consider the constant ∞-functor

$$\mathrm{const}\colon \mathrm{Lie}_{\Bbbk} \longrightarrow \mathrm{Fun}\big(\Delta^1,\, \mathrm{Lie}_{\Bbbk}\big), \tag{2.4.2}$$

right adjoint to the evaluation on the codomain and left adjoint to the evaluation on the domain, informally described by the assignation

$$\mathfrak{g} \mapsto \left\{\mathfrak{g} \xrightarrow{\mathrm{id}_{\mathfrak{g}}} \mathfrak{g}\right\}.$$

Thanks to the discussion of 1.3.17, we know that the fully faithful embedding 1.1.30, for $q = 0$, induces an ∞-functor at the level of the ∞-categories of Lie algebra objects

$$(-)(0)\colon \mathrm{Lie}_{\Bbbk} \longrightarrow \mathrm{Alg}_{\mathrm{Lie}}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big) =: \varepsilon\text{-}\mathrm{Lie}^{\mathrm{gr}}_{\Bbbk}. \tag{2.4.3}$$

The ∞-functor 2.4.3 simply sends a Lie algebra $\mathfrak{g}$ to the graded Lie algebra $\mathfrak{g}(0)$ with same Lie bracket, and trivial mixed structure. In particular, since the ∞-functor $(-)_0$ is an inverse for the ∞-functor $(-)(0)$, and it is moreover both lax and oplax monoidal (as already stated in 1.3.17), we can think the ∞-functor 2.4.2 to land in the ∞-category

$$\mathrm{Fun}\big(\Delta^1,\, \mathrm{Lie}_{\Bbbk}\big)_{(-)_0/} \simeq \mathrm{Fun}\big(\Delta^1,\, \mathrm{Lie}_{\Bbbk}\big)$$

of morphisms in $\mathrm{Lie}_{\Bbbk}$ such that the domain lies in the essential image of the ∞-functor $(-)_0\colon \varepsilon\text{-}\mathrm{Lie}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Lie}_{\Bbbk}$.

2.4.4. Recall now that the ∞-functor $(-)_0\colon \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$ is both a left and right adjoint in virtue of Lemma 1.1.25. Its right adjoint is given by

$$\mathrm{R}_{\varepsilon} \circ (-)(0)\colon \mathrm{Mod}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr},\geqslant 0}_{\Bbbk} \subseteq \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk},$$

where $\mathrm{R}_{\varepsilon}\colon \mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \to \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ is the ∞-functor described in Porism 1.1.26. Since it is right adjoint to an oplax monoidal ∞-functor, $\mathrm{R}_{\varepsilon} \circ (-)(0)$ is lax monoidal itself. The lax monoidal transformation

$$\mathrm{R}_{\varepsilon}(M(0)) \otimes^{\varepsilon\text{-}\mathrm{gr}}_{\Bbbk} \mathrm{R}_{\varepsilon}(N(0)) \longrightarrow \mathrm{R}_{\varepsilon}\left(M(0) \otimes^{\mathrm{gr}}_{\Bbbk} N(0)\right), \tag{2.4.5}$$

is, by definition, the map adjoint to the map given by tensoring the unit for M and N

$$\left(\mathrm{R}_{\varepsilon}(M(0)) \otimes^{\varepsilon\text{-}\mathrm{gr}}_{\Bbbk} \mathrm{R}_{\varepsilon}(N(0))\right)_0 \simeq \mathrm{R}_{\varepsilon}(M(0))_0 \otimes_{\Bbbk} \mathrm{R}_{\varepsilon}(N(0))_0 \simeq M \otimes_{\Bbbk} N \xrightarrow{\simeq} M \otimes_{\Bbbk} N.$$

In particular, using the explicit model for R_ε provided in Porism 1.1.26, one can see that the map 2.4.5 is given by the identity of $M \otimes_{\Bbbk} N$ in weight 0, and by the codiagonal $M \otimes_{\Bbbk} N \oplus M \otimes_{\Bbbk} N \to M \otimes_{\Bbbk} N$ in weight 1 (with an appropriate homological shift). In particular, this right adjoint $\mathrm{R}_\varepsilon \circ (-)(0)$ induces an ∞-functor at the level of Lie algebras

$$\mathrm{Cn}_\varepsilon^{\mathrm{gr}} \colon \mathrm{Lie}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{Lie}_{\Bbbk}^{\mathrm{gr}}. \tag{2.4.6}$$

For formal reasons (described in [GR17b, Chapter 6, Section 1.2]), the ∞-functor $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}$ is again the right adjoint to the ∞-functor $(-)_0 \colon \varepsilon\text{-}\mathrm{Lie}_{\Bbbk}^{\mathrm{gr}} \to \mathrm{Lie}_{\Bbbk}$, at least after restricting to non-negatively graded mixed Lie algebras. Using explicit models, such ∞-functor is given by sending a Lie algebra $\mathfrak{g}$ to the mixed graded Lie algebra described in the following way.

- As a mixed graded $\Bbbk$-module, $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$ is equivalent to the mixed graded $\Bbbk$-module $\mathrm{R}_\varepsilon(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}(0))$; in virtue of Porism 1.1.26, this mixed graded $\Bbbk$-module is given by a copy of $\mathfrak{g}$ in weight 0 and a copy of $\mathfrak{g}[-1]$ in weight 1, together with mixed differential given by the identity of $\mathfrak{g}[-1]$. This is true because $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}$ is the ∞-functor induced at the level of algebras for some operad by the lax monoidal ∞-functor R_ε.

- The Lie bracket of $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$ is given by the composition of the natural transformation 2.4.5 with the image under R_ε of the bracket $[-,-] \colon \mathfrak{g} \otimes_{\Bbbk} \mathfrak{g} \to \mathfrak{g}$. In particular, consider an explicit differential graded Lie algebra $\mathfrak{g}$, assumed to be fibrant and cofibrant for the model structure on $\mathrm{dgLie}_{\Bbbk}$, with Lie bracket $[-,-]$: the Lie bracket of $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$ is then given by the composition

$$\mathrm{R}_\varepsilon(\mathfrak{g}(0)) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{R}_\varepsilon(\mathfrak{g}(0)) \xrightarrow{2.4.5} \mathrm{R}_\varepsilon\left((\mathfrak{g} \otimes_{\Bbbk} \mathfrak{g})(0)\right) \xrightarrow{\mathrm{R}_\varepsilon([-,-])} \mathrm{R}_\varepsilon(\mathfrak{g}(0)).$$

  In particular, following this composition, we have that the Lie bracket on $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$ is given by taking two elements $x + \epsilon y$ and $x' + \epsilon y'$ in $\mathfrak{g} \oplus \mathfrak{g}[-1]$ and sending them to $[x, x'] + \epsilon\,([x, y'] + [y, x'])$.

This has to be understood as a model independent rewriting of the mixed structure on the free loop space $\mathcal{L}(\mathfrak{g})$ of a Lie algebroid $\mathfrak{g}$, in the case when $\mathfrak{g}$ is an ordinary Lie algebra, described in [Nui19, Section 6.3]. Indeed, in the case of an ordinary Lie algebras, the fiber $\mathfrak{n}$ of the anchor map $\mathfrak{g} \to 0$ is all $\mathfrak{g}$, hence we have an equivalence of mixed graded Lie algebras $\mathcal{L}(\mathfrak{g}) \simeq \mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$. The main ingredient in order to define a mixed structure on any Chevalley-Eilenberg ∞-functor is already contained in this right adjoint: all mixed structures on Chevalley-Eilenberg algebras, coalgebras and modules are straightly derived from the mixed structure of $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$.

Remark 2.4.7. Given a Lie algebra $\mathfrak{g}$ it is clear that the Tate realization $\left|\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right|^{\mathrm{t}}$ of the mixed graded Lie algebra $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$ yields again a Lie algebra (in virtue of 1.3.17) which as a explicit differential graded Lie algebra is presented by the contractible differential graded Lie algebra $\mathrm{Cn}(\mathfrak{g})$ of [Lur11c, Construction 2.2.1]. This explains our choice of notation for the ∞-functor 2.4.6.

2.4.8. Using Remark 2.1.9, we can consider the universal enveloping mixed graded algebra ∞-functor

$$\mathrm{U}_\varepsilon^{\mathrm{gr}}\colon \varepsilon\text{-}\mathrm{Lie}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \mathrm{Alg}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big) =: \varepsilon\text{-}\mathrm{Alg}_{\Bbbk}^{\mathrm{gr}}. \tag{2.4.9}$$

Such ∞-functor sits in a commutative diagram

$$\begin{tikzcd} \varepsilon\text{-}\mathrm{Lie}_{\Bbbk}^{\mathrm{gr}} \arrow[r, "\mathrm{U}^{\varepsilon\text{-}\mathrm{gr}}"] \arrow[d, "\mathrm{oblv}_\varepsilon"'] & \varepsilon\text{-}\mathrm{Alg}_{\Bbbk}^{\mathrm{gr}} \arrow[d, "\mathrm{oblv}_\varepsilon"] \\ \mathrm{Lie}_{\Bbbk}^{\mathrm{gr}} \arrow[r, "\mathrm{U}^{\mathrm{gr}}"'] & \mathrm{Alg}_{\Bbbk}^{\mathrm{gr}} \end{tikzcd}$$

where the bottom arrow is analogously given by applying Remark 2.1.9 at the ∞-category $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$. The fact that this square is commutative follows from the fact that one can check that the Beck-Chevalley morphism is an equivalence at the level of the underlying graded modules by forgetting both the algebra structure and the mixed structure, since they are both conservative ∞-functors. Then the commutativity becomes a corollary of the commutativity of the square of ∞-functors in 1.3.13.

As explained in [Lur11c, Remark 2.2.4], there is an obvious morphism $\mathfrak{g}(0) \hookrightarrow \mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$ which, after applying the universal enveloping algebra ∞-functor, equips $\mathrm{U}_\varepsilon^{\mathrm{gr}}\big(\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\big)$ with a left $\mathrm{U}(\mathfrak{g})(0)$-module structure. Applying the Tate realization ∞-functor and dealing with explicit presentations via chain complexes, one recovers the usual left $\mathrm{U}(\mathfrak{g})$-module structure on $\big|\mathrm{U}_\varepsilon^{\mathrm{gr}}\big(\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\big)\big|^{\mathrm{t}}$, which is a well known cofibrant replacement for the trivial left $\mathrm{U}(\mathfrak{g})$-module $\Bbbk$ in $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}$ (see [Wei94, Theorem 7.7.2] and [Lur11c, Remark 2.2.11]). Since both the $\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}_\varepsilon^{\mathrm{gr}}\big(\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\big)$ are assignations functorial in $\mathfrak{g}$, we need to find a way to keep track of the change of base associative ring and of the change of $\mathrm{U}_\varepsilon^{\mathrm{gr}}\big(\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\big)$ *simultaneously*. The tool we need in order to do so is the ∞-functorial assignation of the adjoint morphism

$$\Big\{\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}\Big\} \mapsto \Big\{\mathfrak{g}(0) \xrightarrow{\mathrm{id}^*} \mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\Big\}$$

via the adjunction $(-)_0 \dashv \mathrm{R}_\varepsilon(-)(0)$. This will provide an ∞-functor

$$\mathrm{Fun}\big(\Delta^1, \mathrm{Lie}_{\Bbbk}\big) \simeq \mathrm{Fun}\big(\Delta^1, \mathrm{Lie}_{\Bbbk}\big)_{(-)_0/} \longrightarrow \mathrm{Fun}\big(\Delta^1, \varepsilon\text{-}\mathrm{Lie}_{\Bbbk}^{\mathrm{gr}}\big)_{/\mathrm{Cn}_\varepsilon^{\mathrm{gr}}} \subseteq \mathrm{Fun}\big(\Delta^1, \varepsilon\text{-}\mathrm{Lie}_{\Bbbk}^{\mathrm{gr}}\big),$$

where again $\mathrm{Fun}\big(\Delta^1, \varepsilon\text{-}\mathrm{Lie}_{\Bbbk}^{\mathrm{gr}}\big)_{/\mathrm{Cn}_\varepsilon^{\mathrm{gr}}}$ is the ∞-category of morphisms of mixed graded Lie algebras with codomain of the form $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$ for some ordinary Lie algebra $\mathfrak{g}$. We show how this can be dealt with using the following general machinery.

Construction 2.4.10. Let $\mathrm{L}\colon \mathscr{C} \to \mathscr{D}$ be an ∞-functor with right adjoint $\mathrm{R}\colon \mathscr{D} \to \mathscr{C}$. By [Lur09, Definition 5.2.2.7 and Corollary 5.2.2.8], the datum of the adjunction $\mathrm{L} \dashv \mathrm{R}$ is encoded by a unit morphism $\eta\colon \mathrm{id}_{\mathscr{C}} \Rightarrow \mathrm{R}\circ\mathrm{L}$ in the ∞-category $\mathrm{Fun}(\mathscr{C},\mathscr{C})$, i.e., by an object

$$\begin{aligned}\eta &\in \mathrm{Fun}\big(\Delta^1, \mathrm{Fun}(\mathscr{C},\mathscr{C})\big)\\ &\simeq \mathrm{Fun}\big(\Delta^1\times\mathscr{C}, \mathscr{C}\big)\\ &\simeq \mathrm{Fun}\big(\mathscr{C}, \mathrm{Fun}(\Delta^1,\mathscr{C})\big).\end{aligned}$$

So, let $\mathrm{Fun}\big(\Delta^1, \mathscr{D}\big)_{\mathrm{L}/}$ be the full sub-∞-category of $\mathrm{Fun}\big(\Delta^1, \mathscr{D}\big)$ with domain lying in the image of L. Applying the ∞-functor R, we obtain an ∞-functor

$$\mathrm{Fun}\big(\Delta^1, \mathscr{D}\big)_{\mathrm{L}/} \longrightarrow \mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{\mathrm{RL}//\mathrm{R}}.$$

The latter ∞-category sits in a $(\infty,2)$-limit diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{\mathrm{RL}//\mathrm{R}} \arrow[r, "\Phi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{C} \arrow[d, "\mathrm{R}\circ\mathrm{L}"] \\ \mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{/\mathrm{R}} \arrow[r, "\mathrm{ev}_0"'] & \mathscr{C} \end{tikzcd}$$

Consider the ∞-functor $\Phi\colon \mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{\mathrm{RL}//\mathrm{R}} \to \mathscr{C}$: it is the ∞-functor which given a morphism $\mathrm{RL}(X)\to\mathrm{R}(Y)$ selects the object X itself. So we get an ∞-functor

$$\mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{\mathrm{RL}//\mathrm{R}} \xrightarrow{(\Phi,\,\mathrm{id})} \mathscr{C}\times\mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{\mathrm{RL}//\mathrm{R}} \xrightarrow{(\eta,\,\mathrm{id})} \mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{/\mathrm{R}}\times\mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{\mathrm{RL}//\mathrm{R}}.$$

Informally, this composition is given by the assignation

$$\{f\colon \mathrm{RL}(X)\to\mathrm{R}(Y)\}\mapsto(X,\,f)\mapsto\big(\{\eta_X\colon X\to\mathrm{RL}(X)\},\ \{f\colon \mathrm{RL}(X)\to\mathrm{R}(Y)\}\big).$$

It is clear that the image of such composition lies in the ∞-category $\mathrm{Fun}\big(\Lambda^2_1, \mathscr{C}\big)$, and so by the inner horn filling property of any ∞-category we have a well defined composition

$$\mathrm{Fun}\big(\Lambda^2_1, \mathscr{C}\big)\longrightarrow\mathrm{Fun}\big(\Delta^2, \mathscr{C}\big)\xrightarrow{|_{\{0,2\}}}\mathrm{Fun}\big(\Delta^1, \mathscr{C}\big).$$

This composition determines the adjoint morphism ∞-functor

$$\mathrm{Fun}\big(\Delta^1, \mathscr{D}\big)_{\mathrm{L}/}\longrightarrow\mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)_{/\mathrm{R}}.$$

By applying Construction 2.4.10 to the case of the adjunction $(-)_0\dashv\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}$, we get an ∞-functor

$$\mathrm{Lie}_{\Bbbk}\xrightarrow{2.4.2}\mathrm{Fun}\big(\Delta^1, \mathrm{Lie}_{\Bbbk}\big)\simeq\mathrm{Fun}\big(\Delta^1, \mathrm{Lie}_{\Bbbk}\big)_{(-)_0/}\longrightarrow\mathrm{Fun}\big(\Delta^1, \varepsilon\text{-}\mathrm{Lie}^{\mathrm{gr}}_{\Bbbk}\big)_{/\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}}. \tag{2.4.11}$$

Remark 2.4.12. This is a slight rewriting of [Lur11c, Remark 2.2.4] in the mixed graded setting.

Recall that $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$, as a graded $\Bbbk$-module, comes equipped with a decomposition

$$\mathrm{oblv}_{\mathrm{Lie}}\,\mathrm{oblv}_\varepsilon\left(\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right) \simeq \mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}(0) \oplus \mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1](1).$$

The natural inclusion of graded $\Bbbk$-modules $\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1](1) \hookrightarrow \mathrm{oblv}_{\mathrm{Lie}}\,\mathrm{oblv}_\varepsilon\left(\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right)$ agrees straight-forwardly with the Lie bracket of $\mathfrak{g}[-1](1)$ and $\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})$, hence can be promoted to a morphism of graded Lie algebras

$$\mathfrak{g}[-1](1) \longhookrightarrow \mathrm{oblv}_\varepsilon\left(\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right).$$

Applying the graded universal enveloping algebra ∞-functor, we have a morphism of graded associative algebras (and, in particular, of graded $\Bbbk$-modules)

$$\mathrm{U}^{\mathrm{gr}}(\mathfrak{g}[-1](1)) \simeq \mathrm{Sym}_\Bbbk^{\mathrm{gr}}\left(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1](1)\right) \longrightarrow \mathrm{U}^{\mathrm{gr}}\left(\mathrm{oblv}_\varepsilon\,\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right) \tag{2.4.13}$$

which in turn induces by adjunction a map of graded left $\mathrm{U}(\mathfrak{g})$-modules

$$\mathrm{U}(\mathfrak{g})(0) \otimes_\Bbbk^{\mathrm{gr}} \mathrm{Sym}_\Bbbk^{\mathrm{gr}}\left(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1](1)\right) \longrightarrow \mathrm{U}^{\mathrm{gr}}\left(\mathrm{oblv}_\varepsilon\,\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right).$$

This map is an equivalence: in order to prove this, we can check at the level of each weight

$$\mathrm{U}(\mathfrak{g}) \otimes_\Bbbk \mathrm{Sym}_\Bbbk^p(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]) \longrightarrow \mathrm{U}^p\left(\mathrm{oblv}_\varepsilon\,\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right).$$

The ∞-functor which selects the weight p part is obtained by first forgetting the associative algebra structure; in particular, by [GR17b, Corollary 6.1.7], we have that

$$\begin{aligned}
\mathrm{U}^p\left(\mathrm{oblv}_\varepsilon\,\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right) &\simeq \left(\mathrm{oblv}_{\mathrm{Alg}}\left(\mathrm{U}^{\mathrm{gr}}\left(\mathrm{oblv}_\varepsilon\,\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right)\right)\right)_p\\
&\simeq \left(\mathrm{Sym}_\Bbbk^{\mathrm{gr}}\left(\mathrm{oblv}_{\mathrm{Lie}}\left(\mathrm{oblv}_\varepsilon\,\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right)\right)\right)_p\\
&\simeq \left(\mathrm{Sym}_\Bbbk^{\mathrm{gr}}\left(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}(0) \oplus \mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1](1)\right)\right)_p\\
&\simeq \mathrm{Sym}_\Bbbk(\mathfrak{g})(0) \otimes_\Bbbk \mathrm{Sym}_\Bbbk^p(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]).
\end{aligned}$$

In particular, the map 2.4.13 is given by tensoring the equivalence of cocommutative coalgebras $\mathrm{U}(\mathfrak{g}) \simeq \mathrm{Sym}_\Bbbk(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g})$ with the identity on the p-th symmetric power of $\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]$. Unraveling all definitions, one can see that working with explicit models given by associative algebras in mixed graded complexes, the mixed differential of $\mathrm{U}^{\varepsilon\text{-gr}}\left(\mathrm{Cn}_\varepsilon^{\mathrm{gr}}(\mathfrak{g})\right)$ is given by

$$\begin{aligned}
u \otimes \left(x_1, \ldots, x_p\right) \mapsto &\sum_{1 \leqslant i \leqslant p} (-1)^{|x_i|\left(|x_{i+1}|+\ldots+|x_p|\right)}(x_i \cdot u) \otimes \left(x_1, \ldots, \hat{x}_i, \ldots, x_p\right) +\\
&\sum_{1 \leqslant i < j \leqslant p} (-1)^{|x_i|\left(|x_{i+1}|+\ldots+|x_{j-1}|\right)} u \otimes \left(x_1, \ldots, \hat{x}_i, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_p\right).
\end{aligned}$$

The above description of the mixed differential employs the following computations and remarks.

1. The mixed graded universal enveloping algebra of a mixed graded Lie algebra $\mathfrak{g}_\bullet$ is explicitly computed by the (mixed graded) tensor algebra over $\mathfrak{g}_\bullet$ modded out by the usual relations between the multiplication and the Lie bracket.

2. While the braiding on the category of chain complexes involves a change of signs, the braiding on the category of mixed graded complexes does not. In particular, the signs involved in the description of the mixed graded structure depend only on the homological (internal) degree of the elements $x_1, \dots, x_p$ in $\mathfrak{g}$.

3. Finally, the mixed differential on $\mathrm{U}^{\mathrm{gr}}_\varepsilon(\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g}))$ is induced by the mixed differential on $\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})$, which is the identity. By the usual computations that classically show how the universal enveloping algebra on the cone of a differential graded Lie algebra $\mathfrak{g}$ (with its natural Lie bracket) is isomorphic as a differential graded $\mathrm{U}(\mathfrak{g})$-module to the Chevalley-Eilenberg complex $\mathrm{U}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{Sym}_{\Bbbk}(\mathfrak{g}[1])$, it follows that the mixed differential on $\mathrm{U}^{\mathrm{gr}}_\varepsilon\big(\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})\big)$ agrees with the component of the differential on $\mathrm{U}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{Sym}_{\Bbbk}(\mathfrak{g}[1])$ which lowers the Sym-degree.

Example 2.4.14. Let $\mathfrak{g}$ be the essentially unique non abelian Lie algebra on $\Bbbk \oplus \Bbbk$ such that $[e_1, e_2] = e_1$. An explicit model for $\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})$ is the mixed graded Lie algebra consisting of $\mathfrak{g}$ in weight 0, $\mathfrak{g}[-1]$ in weight 1, with the canonical identity $\mathfrak{g}[-1] \cong \mathfrak{g}[-1]$ as mixed differential. Denoting by $\overline{e}_i$ the generator e_i of $\mathfrak{g}[-1]$ sitting in weight 1, the Lie bracket of $\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})$ is described by the formulas

$$\begin{aligned}[][e_i, e_j] &= [e_i, e_j]\\ [e_i, \overline{e}_j] &= \overline{[e_i, e_j]}\\ [\overline{e}_i, \overline{e}_j] &= 0.\end{aligned}$$

The mixed graded universal enveloping algebra on $\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})$ is then the mixed graded tensor algebra on $\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})$, modded out by the relations $x \otimes y - y \otimes x = [x, y]$. This implies that

$$\begin{aligned} \overline{e}_i \otimes \overline{e}_i &= 0 && \text{in weight 2,}\\ \overline{e}_1 \otimes \overline{e}_2 &= -\overline{e}_2 \otimes \overline{e}_1 && \text{in weight 2,}\\ e_1 \otimes \overline{e}_2 - \overline{e}_2 \otimes e_1 &= \overline{e}_1 && \text{in weight 1,}\\ e_2 \otimes \overline{e}_1 - \overline{e}_1 \otimes e_2 &= -\overline{e}_1 && \text{in weight 1}\end{aligned}$$

together with the usual relations for the universal enveloping algebra on $\mathfrak{g}$ in weight 0. Now, consider the summand

$$\Bbbk(0) \oplus \mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g}) \oplus \big(\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g}) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} \mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})\big)$$

of the mixed graded tensor algebra on $\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})$. This is a mixed graded complex concentrated in weights $[0,2]$, and its weight 2 component is isomorphic to

$$(\mathfrak{g}\otimes_{\Bbbk}\mathfrak{g})[-2]\cong\bigoplus_{1\leqslant i,j\leqslant 2}\langle\overline{e}_i\otimes\overline{e}_j\rangle\Bbbk[-2].$$

The mixed differential between weights 2 and 1 is then described as

$$\langle\mathrm{id},-\mathrm{id}\rangle:\mathfrak{g}\otimes\mathfrak{g}[-2]\longrightarrow(\mathfrak{g}\otimes\mathfrak{g}[-1]\oplus\mathfrak{g}[-1]\otimes\mathfrak{g})[-1].$$

However, by the relations written above, it follows that the weight 2 of this mixed graded complex is equivalent to the $\Bbbk$-module $\langle\overline{e}_1\otimes\overline{e}_2\rangle\Bbbk[-2]$, and under this identification the mixed differential above is defined by the assignation

$$\overline{e}_1\otimes\overline{e}_2\mapsto(\overline{e}_1\otimes e_2,-e_2\otimes\overline{e}_1).$$

Again, the previous relations imply that $-e_2\otimes\overline{e}_1=-\overline{e}_1\otimes e_2+\overline{e}_1$, and thus the mixed differential is uniquely defined by the assignation

$$\overline{e}_1\otimes\overline{e}_2\mapsto\overline{e}_1\otimes e_2-\overline{e}_1\otimes e_2+\overline{e}_1=\overline{e}_1.$$

But now $\overline{e}_1$ is just $\overline{[e_1,e_2]}$, i.e., the Lie bracket $[e_1,e_2]$ sitting in weight 1. On the other hand, the mixed differential between weights 1 and 0 is just the codiagonal

$$\nabla:\ (\mathfrak{g}[-1]\otimes_{\Bbbk}\mathfrak{g})\oplus(\mathfrak{g}\otimes_{\Bbbk}\mathfrak{g}[-1])\longrightarrow\mathfrak{g}[-1].$$

By the definition of the product on the universal enveloping algebra, this amounts to sending an element $u\otimes g$ to $u\cdot g$.
Now, considering *all* the summands of the universal enveloping algebra $\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big)$, we obtain $\mathrm{U}(\mathfrak{g})$ sitting in weight 0, $\mathrm{U}(\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}[-1]$ sitting in weight 1, and $\langle\overline{e}_1,\overline{e}_2\rangle\mathrm{U}(\mathfrak{g})[-2]$ sitting in weight 2. The mixed differential of such mixed graded associative algebra is induced by tensoring the mixed differential of $\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})$ with itself a suitable number of times: in particular, taking carefully into account the signs and considering all the identifications implied by the relations for the universal enveloping algebra, we obtain that an element $u\otimes\overline{e}_1\wedge\overline{e}_2$ in weight 2 is sent to $u\cdot\overline{e}_1\otimes\overline{e}_2-u\cdot\overline{e}_2\otimes\overline{e}_1-u\cdot\overline{[e_1,e_2]}$.

Applying the mixed graded version of the universal enveloping $\Bbbk$-algebra ∞-functor 2.4.9, we land in $\mathrm{Fun}\big(\Delta^1,\ \varepsilon\text{-}\mathrm{Alg}^{\mathrm{gr}}_{\Bbbk}\big)$ with an ∞-functor

$$\mathrm{Lie}_{\Bbbk}\longrightarrow\mathrm{Fun}\big(\Delta^1,\ \varepsilon\text{-}\mathrm{Alg}^{\mathrm{gr}}_{\Bbbk}\big).\tag{2.4.15}$$

2.4.16. Let us focus now on the ∞-category $\mathrm{Fun}\big(\Delta^1,\ \mathrm{Alg}_{\mathbb{E}_1}(\mathscr{C})\big)=:\mathrm{Fun}\big(\Delta^1,\ \mathrm{Alg}(\mathscr{C})\big)$, where $\mathscr{C}$ is a stable symmetric monoidal ∞-category which is $\Bbbk$-linear over a field of characteristic 0 (in our case, $\mathscr{C}$ will be $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$). By [Lur17, Corollary 4.2.3.2], we have a Cartesian fibration

$$\theta\colon \mathrm{LMod}(\mathscr{C}) \longrightarrow \mathrm{Alg}(\mathscr{C})$$

which classifies the ∞-functor

$$\mathrm{LMod}_{(-)}(\mathscr{C})\colon\ (\mathrm{Alg}(\mathscr{C}))^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$$

informally described by the assignation $\{f\colon A \to B\}$ to $\{f_*\colon \mathrm{LMod}_B(\mathscr{C}) \to \mathrm{LMod}_A(\mathscr{C})\}$, where f_* forgets the B-module structure to an A-module structure along f. In particular, in view of [Lur09, Remark 2.4.2.9], we can promote such assignation to an ∞-functor

$$\mathrm{Fun}\big(\Delta^1,\ \mathrm{Alg}(\mathscr{C})\big) \longrightarrow \mathrm{Fun}\big(\Delta^1,\ \mathrm{Cat}_\infty^{\mathrm{op}}\big).$$

Now, let

$$s\colon \mathrm{Alg}(\mathscr{C}) \longrightarrow \mathrm{LMod}(\mathscr{C})$$

be the section of the Cartesian fibration sending an algebra object A of $\mathscr{C}$ to the object (A, A) of $\mathrm{LMod}(\mathscr{C})$ where A is seen as an A-module object ([Lur17, Example 4.2.1.17]). Given an arrow $f\colon A \to B$, seen as an object of the ∞-category $\mathrm{Fun}\big(\Delta^1,\ \mathrm{Alg}(\mathscr{C})\big)$, we have a locally θ-Cartesian morphism with codomain $s(B)$ which lifts f ([Lur09, Proposition 2.4.2.8]). Let $\overline{f}$ be such θ-Cartesian lift. Then, the domain of $\overline{f}$ is described by the object $(A,\ B)$, where B is seen as an A-module along f. Considering the codomain of an arrow $f\colon A \to B$ and taking its image under the section s, we obtain an ∞-functor

$$\mathrm{Fun}\big(\Delta^1,\ \mathrm{Alg}(\mathscr{C})\big) \longrightarrow \mathrm{LMod}(\mathscr{C}).$$

Again, this can be promoted to an ∞-functor

$$\mathrm{Fun}\big(\Delta^1,\ \mathrm{Alg}(\mathscr{C})\big) \longrightarrow \mathrm{Fun}\big(\Delta^1,\ \mathrm{LMod}(\mathscr{C})\big). \tag{2.4.17}$$

Indeed, for any square of arrows

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

the Cartesian lift $\overline{f}\colon (A,\ B) \to (B,\ B)$ produces an arrow in $\mathrm{LMod}(\mathscr{C})$ given by $(A,\ B) \to (D,\ D)$. Considering now a θ-cartesian lift $\overline{g}$ of g, by the equivalence

$$\mathrm{LMod}(\mathscr{C})_{/\overline{g}} \simeq \mathrm{LMod}(\mathscr{C})_{/s(D)} \times_{\mathrm{Alg}(\mathscr{C})_{/D}} \mathrm{Alg}(\mathscr{C})_{/f}$$

(which is the definition of θ-Cartesian morphism, [Lur09, Definition 2.4.1.1]) we obtain an essentially unique arrow $(A, B) \to (C, D)$, where (C, D) is the domain of the Cartesian lift $\overline{g}$. This arrow has to be understood as the map of associative algebras $\phi : A \to C$, and the map $\psi : B \to D$ that, being compatible with the associative algebra structure, is also A-linear with respect to the A-module structure of B and D inherited by the maps $f : A \to B$ and $\psi \circ f : A \to D$, respectively. The functoriality of this construction is then a consequence of [Lur09, Remark 2.4.1.4].

2.4.18. In the case $\mathscr{C} = \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$, composing 2.4.15 with the ∞-functor 2.4.17, we obtain the ∞-functor

$$\mathrm{Lie}_{\Bbbk} \longrightarrow \mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right) \tag{2.4.19}$$

which can be interpreted as the ∞-functor sending a Lie algebra $\mathfrak{g}$ to the object $\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\left(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\right)$, seen as a left $\mathrm{U}(\mathfrak{g})(0)$-module.

2.4.20. Let us consider the ∞-category $\mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)$. Every mixed graded $\Bbbk$-module is a $\Bbbk(0)$-bimodule, since $\Bbbk(0)$ is the unit for the symmetric monoidal structure on $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$, and in particular it is both left and right $\Bbbk(0)$-module. Moreover, these $\Bbbk(0)$-module structures are clearly compatible with any left or right $A_\bullet$-module structure over an associative algebra $A_\bullet$ in $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$. So, by [Lur17, Theorem 4.3.2.7], the ∞-categories

$$\mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right) \subseteq \mathrm{LMod}\left(\mathrm{RMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)\right)$$

and

$$\mathrm{RMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right) \subseteq \mathrm{RMod}\left(\mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)\right)$$

can be seen as a sub-∞-categories of the ∞-category $\mathrm{BMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)$. The relative tensor product ∞-functor

$$T : \mathrm{BMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right) \times_{\mathrm{Alg}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)} \mathrm{BMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right) \longrightarrow \mathrm{BMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)$$

(see [Lur17, Definition 4.4.2.10]) sends an $(A_\bullet, B_\bullet)$-bimodule $M_\bullet$ and a $(B_\bullet, C_\bullet)$-bimodule $N_\bullet$ to the tensor product $M_\bullet \otimes^{\varepsilon\text{-gr}}_{B_\bullet} N_\bullet$, which is now an $(A_\bullet, C_\bullet)$-bimodule.

With all these notations, we can consider the relative tensor product ∞-functor T restricted to the sub-∞-category

$$\mathrm{RMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right) \times_{\mathrm{Alg}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)} \mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)$$

of

$$\mathrm{BMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right) \times_{\mathrm{Alg}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)} \mathrm{BMod}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)$$

spanned by those couples $(M_\bullet, N_\bullet)$ comprising of right and left module objects $M_\bullet$ and $N_\bullet$ in $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ such that $M_\bullet$ is a right module on the very same algebra for which $N_\bullet$ is a left module.

In this case, the relative tensor product ∞-functor lands naturally in the ∞-category of mixed graded $\Bbbk$-modules $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$.

2.4.21. By the discussion provided in 2.4.20, we have a family of ∞-functors parametrized by Lie algebras in the following way:

$$\{\mathfrak{g}\} \hookrightarrow \mathrm{Lie}_{\Bbbk} \xrightarrow{2.4.19} \varepsilon\text{-}\mathrm{LMod}^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)} \xrightarrow{T} \mathrm{Fun}\Big(\varepsilon\text{-}\mathrm{RMod}^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)},\ \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\Big). \tag{2.4.22}$$

Here, by $\varepsilon\text{-}\mathrm{LMod}^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)}$ and $\varepsilon\text{-}\mathrm{RMod}^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)}$ we mean, respectively, the ∞-categories

$$\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big)$$

and

$$\mathrm{RMod}_{\mathrm{U}(\mathfrak{g})(0)}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big).$$

Let us remark that, working with explicit models given by mixed graded complexes of $\Bbbk$-modules, these ∞-categories can be described as the ∞-categories whose objects are countable collections of chain complexes of left (resp. right) $\mathrm{U}(\mathfrak{g})$-modules such that the mixed differential is a morphism of left (resp. right) $\mathrm{U}(\mathfrak{g})$-modules, and whose 1-morphisms are countably many morphisms of left (resp. right) $\mathrm{U}(\mathfrak{g})$-modules commuting with the mixed differentials. Informally, the association 2.4.22 is defined by sending $\mathfrak{g}$ to the ∞-functor

$$\mathrm{CE}_{\varepsilon}(\mathfrak{g};\, -) := - \otimes^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)} \mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big).$$

We cannot in general promote the association 2.4.22 to an ∞-functor

$$\mathrm{Lie}_{\Bbbk} \longrightarrow \mathrm{Fun}\Big(\mathrm{RMod}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big),\ \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\Big)$$

because, for any $\mathfrak{g}$ in $\mathrm{Lie}_{\Bbbk}$, we cannot define the action of an ∞-functor over those right module objects in $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ which are right modules over algebras different from $\mathrm{U}(\mathfrak{g})(0)$. However, we can observe that $\Bbbk(0)$ is naturally a (both left and right) $\mathrm{U}(\mathfrak{g})(0)$-module for any $\mathfrak{g}$. This follows from the fact that $\mathrm{U}(\mathfrak{g})$ is an augmented $\Bbbk$-algebra, and so applying the ∞-functor $(-)(0)\colon \mathrm{Alg}_{\Bbbk} \hookrightarrow \varepsilon\text{-}\mathrm{Alg}^{\mathrm{gr}}_{\Bbbk}$, $\mathrm{U}(\mathfrak{g})(0)$ becomes naturally an augmented $\Bbbk(0)$-algebra, and this provides $\Bbbk(0)$ with a (trivial) left and right $\mathrm{U}(\mathfrak{g})(0)$-module structure for any Lie algebra $\mathfrak{g}$. In particular, we have an ∞-functor

$$\mathrm{Lie}_{\Bbbk} \xrightarrow{2.4.19} \mathrm{LMod}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big) \subseteq \{\Bbbk(0)\} \times \mathrm{LMod}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big) \xrightarrow{T} \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}. \tag{2.4.23}$$

Definition 2.4.24. The ∞-functor 2.4.23 is the *(mixed graded) homological Chevalley-Eilenberg ∞-functor*, and shall be denoted by $\mathrm{CE}_{\varepsilon}$.

Remark 2.4.25. The mixed graded homological Chevalley-Eilenberg ∞-functor $\mathrm{CE}_{\varepsilon}$ sends a

Lie algebra $\mathfrak{g}$ to a mixed graded $\Bbbk$-module given by the mixed graded tensor product

$$\mathrm{CE}_\varepsilon(\mathfrak{g}) := \Bbbk(0) \otimes^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)} \mathrm{U}^{\mathrm{gr}}_\varepsilon\big(\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})\big).$$

Since forgetting the mixed structure is a strongly monoidal ∞-functor, the underlying graded $\Bbbk$-module is equivalent, in virtue of the discussion of Remark 2.4.12, to

$$\begin{aligned}\mathrm{oblv}_\varepsilon(\mathrm{CE}_\varepsilon(\mathfrak{g})) &\simeq \mathrm{oblv}_\varepsilon\Big(\Bbbk(0) \otimes^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)} \mathrm{U}^{\mathrm{gr}}_\varepsilon\big(\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})\big)\Big)\\ &\simeq \mathrm{oblv}_\varepsilon(\Bbbk(0)) \otimes^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)} \mathrm{oblv}_\varepsilon\big(\mathrm{U}^{\mathrm{gr}}_\varepsilon\big(\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})\big)\big)\\ &\simeq \Bbbk(0) \otimes^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)} \mathrm{U}(\mathfrak{g})(0) \otimes^{\mathrm{gr}}_{\Bbbk(0)} \mathrm{Sym}^{\mathrm{gr}}_\Bbbk(\mathfrak{g}[-1](1)) \simeq \mathrm{Sym}^{\mathrm{gr}}_\Bbbk(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1](1)).\end{aligned}$$

In other words, for all integers p we have a natural equivalence of $\Bbbk$-modules

$$\mathrm{CE}_\varepsilon(\mathfrak{g})_p \simeq \mathrm{Sym}^p_\Bbbk(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1]).$$

Moreover, working with explicit models given by mixed graded complexes, the mixed differential of $\mathrm{CE}_\varepsilon(\mathfrak{g})$ is given by tensoring the mixed differential of $\mathrm{U}^{\mathrm{gr}}_\varepsilon\big(\mathrm{Cn}^{\mathrm{gr}}_\varepsilon(\mathfrak{g})\big)$ exhibited in Remark 2.4.12 with the trivial differential of $\Bbbk(0)$ which kills the part of the mixed differential which depends on the left $\mathrm{U}(\mathfrak{g})$-module structure. Hence, it follows that the Tate realization of $\mathrm{CE}_\varepsilon(\mathfrak{g})$ agrees with the usual Chevalley-Eilenberg complex $\mathrm{CE}_\bullet(\mathfrak{g})$, again by directly inspecting the usual explicit model for $|\mathrm{CE}_\varepsilon(\mathfrak{g})|^{\mathrm{t}}$.

Proposition 2.4.26. *The mixed graded homological Chevalley-Eilenberg ∞-functor can be promoted to an ∞-functor*

$$\mathrm{CE}_\varepsilon \colon \mathrm{Lie}_\Bbbk \longrightarrow \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$$

where $\varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$ is the ∞-category of coaugmented cocommutative coalgebra objects in $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_\Bbbk$.

Proof. First of all, let us remark that we have an equivalence of ∞-categories

$$\mathrm{Lie}_\Bbbk \simeq \mathrm{cCcAlg}(\mathrm{Lie}_\Bbbk)$$

in virtue of [Lur17, Proposition 2.4.3.9]: essentially, the diagonal morphism $\Delta\colon \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ turns every Lie algebra into a cocommutative coalgebra object in Lie algebras. So, our claim will follow by proving that CE_ε is a (actually, *strongly*) monoidal ∞-functor with respect to the Cartesian monoidal structure on $\mathrm{Lie}_\Bbbk$ and the monoidal structure on $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_\Bbbk$ described in 1.1.6. We have a natural map

$$\mathrm{CE}_\varepsilon(\mathfrak{g} \times \mathfrak{h}) \longrightarrow \mathrm{CE}_\varepsilon(\mathfrak{g}) \times \mathrm{CE}_\varepsilon(\mathfrak{h}) \longrightarrow \mathrm{CE}_\varepsilon \otimes^{\varepsilon\text{-gr}}_\Bbbk \mathrm{CE}_\varepsilon(\mathfrak{h})$$

and, since forgetting the mixed structure is a strongly monoidal operation, we can safely check that this is an equivalence by inspecting the map on the underlying graded $\Bbbk$-modules

$$\mathrm{oblv}_\varepsilon(\mathrm{CE}_\varepsilon(\mathfrak{g} \times \mathfrak{h})) \longrightarrow \mathrm{oblv}_\varepsilon(\mathrm{CE}_\varepsilon(\mathfrak{g})) \otimes^{\varepsilon\text{-gr}}_\Bbbk \mathrm{oblv}_\varepsilon(\mathrm{CE}_\varepsilon(\mathfrak{h})).$$

In virtue of Remark 2.4.25, we have an equivalence

$$\operatorname{oblv}_{\varepsilon}\left(\mathrm{CE}_{\varepsilon}(\mathfrak{g}\times\mathfrak{h})\right)\simeq \operatorname{Sym}^{\mathrm{gr}}_{\Bbbk}(\operatorname{oblv}_{\mathrm{Lie}}(\mathfrak{g}\times\mathfrak{h})[-1](1)),$$

where $\operatorname{Sym}^{\mathrm{gr}}_{\Bbbk}(\operatorname{oblv}_{\mathrm{Lie}}(\mathfrak{g}\times\mathfrak{h})[-1](1))$ is the free ind-nilpotent graded cocommutative coalgebra over $\mathfrak{g}\times\mathfrak{h}$ (which agrees on the usual symmetric coalgebra with its natural grading). On the other hand, the free ind-nilpotent cocommutative coalgebra ∞-functor is always a right adjoint, and the same holds for the inclusion of ind-nilpotent coalgebras into all coalgebras (Warning 1.3.5). Moreover, products in cocommutative coalgebras are described by tensor products ([GR17b, Chapter 6, Section 4.1.1] and [Lur18, Section 3.3]), therefore we have another equivalence

$$\operatorname{oblv}_{\varepsilon}\left(\mathrm{CE}_{\varepsilon}(\mathfrak{g})\right)\otimes^{\varepsilon\text{-gr}}_{\Bbbk}\operatorname{oblv}_{\varepsilon}\left(\mathrm{CE}_{\varepsilon}(\mathfrak{h})\right)\simeq \operatorname{Sym}^{\mathrm{gr}}_{\Bbbk}(\operatorname{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1](1))\otimes^{\varepsilon\text{-gr}}_{\Bbbk}\operatorname{Sym}^{\mathrm{gr}}_{\Bbbk}(\operatorname{oblv}_{\mathrm{Lie}}(\mathfrak{h})[-1](1)).$$

It follows that $\mathrm{CE}_{\varepsilon}$ is a strongly monoidal ∞-functor between $\mathrm{Lie}^{\times}_{\Bbbk}$ and $\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)^{\otimes^{\varepsilon\text{-gr}}}$, and so we can promote $\mathrm{CE}_{\varepsilon}$ to an ∞-functor

$$\mathrm{cCcAlg}(\mathrm{Lie}_{\Bbbk})\simeq\mathrm{Lie}_{\Bbbk}\longrightarrow\mathrm{cCcAlg}\left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)=:\varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk}. \tag{2.4.27}$$

The fact that every $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ is actually a *coaugmented* mixed graded cocommutative coalgebra follows from the obvious fact that $\mathrm{U}(0)\simeq\operatorname{Sym}_{\Bbbk}(0)\simeq\Bbbk$, and since 0 is a zero object in $\mathrm{Lie}_{\Bbbk}$ one has a natural coaugmentation, as desired. □

Notation 2.4.28. In the following, we shall denote both the ∞-functor 2.4.23 and the ∞-functor 2.4.27 as $\mathrm{CE}_{\varepsilon}$.

Porism 2.4.29. Unraveling all the constructions, it turns out that the mixed graded cocommutative coalgebra structure of $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ is induced simply by the diagonal morphism $\mathfrak{g}\to\mathfrak{g}\times\mathfrak{g}$, which depends only on the $\Bbbk$-module structure of $\mathfrak{g}$ and not on its Lie bracket. In fact, the cocommutative coalgebra structure of $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ is induced by the fact that $\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\left(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\right)$ is itself a mixed graded cocommutative coalgebra, since it is the universal enveloping algebra of $\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})$ (and $\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}$ preserves products of Lie algebras, trivially).
In other words, the proof of Proposition 2.4.26 shows that the underlying graded cocommutative coalgebra of $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ is *precisely* $\operatorname{Sym}^{\mathrm{gr}}_{\Bbbk}(\operatorname{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1](1))$. The Lie bracket plays a role only in setting a mixed structure of $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ which agrees with such cocommutative coalgebra structure.

We conclude this section by noticing that, working with explicit models, this construction actually yields the usual homological Chevalley-Eilenberg chain complex.

Proposition 2.4.30. *Given a Lie algebra $\mathfrak{g}$, the Tate realization $|\mathrm{CE}_{\varepsilon}(\mathfrak{g})|^{\mathrm{t}}$ agrees with the homological Chevalley-Eilenberg complex of [Lur11c, Construction 2.2.3].*

Proof. Our mixed graded Chevalley-Eilenberg ∞-functor is given by the mixed graded relative tensor product

$$\mathrm{CE}_{\varepsilon}(\mathfrak{g}):=\Bbbk(0)\otimes^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)}\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\left(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\right),$$

and $\Bbbk(0)$, $\mathrm{U}(\mathfrak{g})(0)$ and $\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big)$ are all mixed graded objects lying in the ∞-category $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr},\geqslant 0}_{\Bbbk}$ of non-negatively graded mixed $\Bbbk$-modules. The restriction of the Tate realization ∞-functor to non-negatively graded objects preserves colimits, hence geometric realizations, in virtue of Porism 1.5.45, and it is strongly monoidal in virtue of Porism 1.5.44. Since the relative tensor product is calculated by a two-sided Bar construction ([Lur17, Section 4.4.2]), it follows that

$$\begin{aligned}
|\mathrm{CE}_{\varepsilon}(\mathfrak{g})|^{\mathrm{t}} &:= \left|\Bbbk(0) \otimes^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)} \mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big)\right|^{\mathrm{t}} \\
&\simeq |\Bbbk(0)|^{\mathrm{t}} \otimes_{|\mathrm{U}(\mathfrak{g})(0)|^{\mathrm{t}}} \left|\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big)\right|^{\mathrm{t}} \\
&\simeq \Bbbk \otimes_{\mathrm{U}(\mathfrak{g})} \left|\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big)\right|^{\mathrm{t}}.
\end{aligned}$$

We are left to prove that the Tate realization of $\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big)$ is equivalent to $\Bbbk$, since the classical Chevalley-Eilenberg homology of a Lie algebra $\mathfrak{g}$ is given by $\Bbbk \otimes_{\mathrm{U}(\mathfrak{g})} \Bbbk$. This is a straightforward computation: the description of the mixed differential in 2.4.12 shows that the Tate realization is the usual universal enveloping algebra of the cone of $\mathfrak{g}$ (in the sense of [Lur11c, Construction 2.2.1]) which is a well known cofibrant replacement for the trivial $\mathrm{U}(\mathfrak{g})$-module $\Bbbk$. □

Corollary 2.4.31. *We have a natural equivalence*

$$\mathrm{CE}_{\varepsilon} \circ \mathrm{triv}_{\mathrm{Lie}} \xrightarrow{\simeq} \mathrm{triv}_{\varepsilon} \circ \mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}\big((-)[-1](1)\big)$$

of ∞-functors from $\Bbbk$-modules to mixed graded $\Bbbk$-modules.

Proof. If $\mathfrak{g} \simeq \mathrm{triv}_{\mathrm{Lie}}\, M$, since the Tate realization of $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ agrees with the usual Chevalley-Eilenberg homological complex, it turns out that

$$|\mathrm{CE}_{\varepsilon}(\mathfrak{g})|^{\mathrm{t}} \simeq \mathrm{CE}_{\bullet}(\mathfrak{g}) \simeq \mathrm{Sym}_{\Bbbk}(M[1]) := \bigoplus_{q\geqslant 0} \mathrm{Sym}^{q}_{\Bbbk}(M[1]),$$

where the latter equivalence is [GR17b, Chapter 6, Section 4.2.3]. We prove the following useful remark.

Lemma 2.4.32. *Let $M_{\bullet}$ be a non-negatively graded mixed $\Bbbk$-module. Then*

$$|M_{\bullet}|^{\mathrm{t}} \simeq \bigoplus_{q\geqslant 0} M_q[2q]$$

if and only if $M_{\bullet} \simeq \mathrm{triv}_{\varepsilon}\, \mathrm{oblv}_{\varepsilon}\, M_{\bullet}$.

Indeed, by the definition of Tate realization, it is clear that if $M_{\bullet}$ is trivial one has the desired description of its Tate realization. On the converse, suppose that $|M_{\bullet}|^{\mathrm{t}} \simeq \bigoplus_{q\geqslant 0} M_q[2q]$. In virtue of the adjunction $|-|^{\mathrm{t}} \dashv (-)(0)$ of Porism 1.5.45, for any $\Bbbk$-module N we have a chain of equivalences of mapping spaces

$$\begin{aligned}\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(M_\bullet,\, N(0)) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(|M_\bullet|^{\mathrm{t}},\, N\big)\\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\Big(\bigoplus_{q\geqslant 0} M_q[2q],\, N\Big)\\ &\simeq \prod_{q\geqslant 0}\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(M_q,\, N[-2q]\big).\end{aligned}$$

On the other hand, by the adjunction $\mathrm{triv}_\varepsilon \dashv \mathrm{NC}^{\mathrm{w}}$ of [PTVV13, Remark 1.5], and recalling that

$$\mathrm{NC}^{\mathrm{w}}(N_\bullet)_p \simeq \prod_{q\leqslant p} N_q[-2(p-q)],$$

we have a chain of equivalences of mapping spaces

$$\begin{aligned}\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\mathrm{triv}_\varepsilon\, \mathrm{oblv}_\varepsilon\, M_\bullet,\, N(0)) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\mathrm{oblv}_\varepsilon\, M_\bullet,\, \mathrm{NC}^{\mathrm{w}}(N))\\ &\simeq \prod_{q\in\mathbb{Z}}\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(M_q,\, \mathrm{NC}^{\mathrm{w}}(N)_q\big)\\ &\simeq \prod_{q\geqslant 0}\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(M_q,\, N[-2q]\big).\end{aligned}$$

Therefore the fully faithfulness of Yoneda embedding yields that $M_\bullet \simeq \mathrm{triv}_\varepsilon\, \mathrm{oblv}_\varepsilon\, M_\bullet$, i.e., that $M_\bullet$ has trivial mixed structure. ☐

Remark 2.4.33. We believe that the statement of Corollary 2.4.31 can be refined in the following way: this is an equivalence of ∞-functors with values in mixed graded *cocommutative coalgebras*. We provide a sketch of the proof: since $\mathrm{triv}_\varepsilon$ is strongly monoidal and commutes also with limits, it becomes a strongly monoidal right adjoint also at the level of cocommutative coalgebras; hence, it admits a left adjoint which, after forgetting the cocommutative coalgebra structure, agrees with a left adjoint $\mathrm{triv}_\varepsilon^{\mathrm{L}}$ of $\mathrm{triv}_\varepsilon$. One can see, using Porism 1.5.45 and Lemma 1.2.5.1, that a model for such left adjoint is given by the ∞-functor sending a non-negatively mixed graded $\Bbbk$-module $M_\bullet$ to the non-negatively graded $\Bbbk$-module described in weight $p \geqslant 0$ by

$$\mathrm{triv}_\varepsilon^{\mathrm{L}} M_p \simeq \big|\sigma_{\geqslant 0} M_\bullet((-p))\big|^{\mathrm{t}}.$$

In particular, since $\mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}$ is the ind-nilpotent cofree cocommutative coalgebra ∞-functor, we have that providing a map of mixed graded cocommutative coalgebras $\mathrm{CE}_\varepsilon(\mathrm{triv}_{\mathrm{Lie}}\, M) \to \mathrm{triv}_\varepsilon\, \mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}(M[-1](1))$ is equivalent to providing a map of $\Bbbk$-modules

$$\mathrm{triv}_\varepsilon^{\mathrm{L}}(\mathrm{CE}_\varepsilon(\mathrm{triv}_{\mathrm{Lie}}\, M))_1[1] \simeq \big|\sigma_{\geqslant 0}(\mathrm{CE}_\varepsilon(\mathrm{triv}_{\mathrm{Lie}}\, M_\bullet)((1)))\big|^{\mathrm{t}}[1] \longrightarrow M.$$

Since $\mathfrak{g} \simeq \mathrm{triv}_{\mathrm{Lie}}\, M$ has a trivial Lie bracket, such totalization contains M as a direct summand. The map corresponding, by adjunction, to the projection on M yields the map which after forgetting the cocommutative coalgebra structure realizes the equivalence of Corollary 2.4.31.

2.5 The cohomological Chevalley-Eilenberg ∞-functor

In this section, we want to construct a *cohomological* variant of ∞-functor 2.4.23, i.e., an ∞-functor

$$\mathrm{CE}^{\varepsilon}\colon \mathrm{Lie}_{\Bbbk}^{\mathrm{op}} \longrightarrow \varepsilon\text{-}\mathrm{CAlg}_{\Bbbk}^{\mathrm{gr}}.$$

Luckily, for the most part the construction of our cohomological ∞-functor relies on the technical machinery developed in order to define the ∞-functor 2.4.23, so we only need to apply some minor modifications to our previous construction.

2.5.1. Let us recall the ∞-functor 2.4.19, sending a Lie algebra $\mathfrak{g}$ to $\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big)$ seen as a $\mathrm{U}(\mathfrak{g})(0)$-module in mixed graded $\Bbbk$-modules. We have the obvious "opposite" ∞-functor

$$(2.4.19)^{\mathrm{op}}\colon \mathrm{Lie}_{\Bbbk}^{\mathrm{op}} \longrightarrow \mathrm{LMod}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)^{\mathrm{op}} \tag{2.5.2}$$

which we can further compose with the inclusion

$$\mathrm{LMod}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)^{\mathrm{op}} \subseteq \mathrm{LMod}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)^{\mathrm{op}} \times_{\mathrm{Alg}(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}})} \mathrm{RMod}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big).$$

We can now observe that the ∞-category $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is enriched over $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ (this is a mixed graded analogue of [Lur17, Variant 7.2.1.24]). This can be shown in the following way: let T denote the relative tensor product described in 2.4.20. Then, since $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is equivalent to the ∞-category of $(\mathrm{U}(\mathfrak{g})(0), \Bbbk(0))$-bimodules, the relative tensor product

$$T\colon \varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}} \times \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$$

provides a (right) tensor structure of $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ over $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$, in the sense of [Lur17, Definition 4.2.1.19]. Such relative tensor product commutes with all colimits separably in each variable, since the forgetful ∞-functor

$$\mathrm{oblv}_{\mathrm{U}(\mathfrak{g})}\colon \varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

preserves all limits and colimits in virtue of [Lur17, Corollary 4.2.3.7], and the tensor product of mixed graded $\Bbbk$-modules commutes with all colimits separably in each variable as well. So, we can apply [Lur17, Proposition 4.2.1.33.(2)] to get that $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is enriched over $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$. In particular, we can consider the morphism object ∞-functor

$$\mathrm{Mor}_{\mathrm{U}(\mathfrak{g})(0)}^{\varepsilon\text{-}\mathrm{gr}}\colon \Big(\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}\Big)^{\mathrm{op}} \times \varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}.$$

Moreover, using [Lur17, Remark 4.2.1.31], we can promote the morphism object ∞-functor to a functorial assignation

$$\mathrm{Mor}^{\varepsilon\text{-gr}}\colon \left(\mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\right)\right)^{\mathrm{op}} \times_{\mathrm{Alg}\left(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\right)} \mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\right) \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}. \tag{2.5.3}$$

2.5.4. Thanks to this formalism, we get a "cohomological" (contravariant) analogue of the ∞-functor 2.4.22

$$\{\mathfrak{g}\} \hookrightarrow \mathrm{Lie}_{\Bbbk}^{\mathrm{op}} \xrightarrow{2.5.2} \left(\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}\right)^{\mathrm{op}} \xrightarrow{2.5.3} \mathrm{Fun}\left(\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}},\ \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\right). \tag{2.5.5}$$

Informally, the assignation 2.5.5 is defined by sending $\mathfrak{g}$ to the ∞-functor

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\ -) := \mathrm{Mor}_{\mathrm{U}(\mathfrak{g})(0)}^{\varepsilon\text{-gr}}\left(\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\left(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\right),\ -\right).$$

The above mixed graded $\Bbbk$-module must be understood as the mixed graded sub-$\Bbbk$-module of $\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}\left(\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\left(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\right),\ -\right)$ comprising of those morphisms which are also left $\mathrm{U}(\mathfrak{g})(0)$-linear, with mixed differential given by restricting the mixed differential described in 1.1.6.
As in the homological case, this cannot be promoted to a contravariant ∞-functor from Lie algebras to presheaves over $\mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\right)$ with values in mixed graded $\Bbbk$-modules; but since $\Bbbk(0)$ is obviously a left (trivial) $\mathrm{U}(\mathfrak{g})(0)$-module for any Lie algebra $\mathfrak{g}$ (as already observed in 2.4.21), we do have the contravariant ∞-functor

$$\mathrm{Lie}_{\Bbbk}^{\mathrm{op}} \xrightarrow{2.5.2} \mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\right)^{\mathrm{op}} \subseteq \mathrm{LMod}\left(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\right)^{\mathrm{op}} \times \{\Bbbk(0)\} \xrightarrow{\mathrm{Mor}^{\varepsilon\text{-gr}}} \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}. \tag{2.5.6}$$

Definition 2.5.7. The ∞-functor 2.5.6 is the *(mixed graded) cohomological Chevalley-Eilenberg* ∞-functor, and shall be denoted by $\mathrm{CE}^{\varepsilon}$.

Remark 2.5.8. For the remainder of this section, it will be useful to show explicitly the underlying graded structure of the mixed graded cohomological Chevalley-Eilenberg $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$. For any integer p, we have

$$\begin{aligned}
\mathrm{CE}^{p}(\mathfrak{g}) &\simeq \left(\mathrm{Mor}_{\mathrm{U}(\mathfrak{g})(0)}^{\varepsilon\text{-gr}}\left(\mathrm{U}^{\varepsilon\text{-gr}}(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})),\ \Bbbk(0)\right)\right)_{p}\\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathrm{U}(\mathfrak{g})}}\left(\mathrm{U}^{-p}(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})),\ \Bbbk\right)\\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathrm{U}(\mathfrak{g})}}\left(\mathrm{U}(\mathfrak{g})(0)\otimes_{\Bbbk}\mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1]),\ \Bbbk\right)\\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(\mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1]),\ \mathrm{Map}_{\mathrm{Mod}_{\mathrm{U}(\mathfrak{g})}}(\mathrm{U}(\mathfrak{g}),\ \Bbbk)\right)\\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(\mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1]),\ \Bbbk\right).
\end{aligned}$$

In particular, we have an equivalence of graded $\Bbbk$-modules

$$\mathrm{oblv}_{\varepsilon}\left(\mathrm{CE}^{\varepsilon}(\mathfrak{g})\right) \simeq \underline{\mathrm{Map}}_{\Bbbk}^{\mathrm{gr}}\left(\mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1](1)),\ \Bbbk(0)\right) =: \left(\mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1](1))\right)^{\vee},$$

where $\underline{\mathrm{Map}}_{\Bbbk}^{\mathrm{gr}}$ is the internal graded mapping $\Bbbk$-module ∞-functor for $\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$.

Just like in the homological case, the mixed graded cohomological Chevalley-Eilenberg $\Bbbk$-module is endowed with a much richer structure, namely it is a mixed graded commutative $\Bbbk$-algebra.

Proposition 2.5.9. *The mixed graded Chevalley-Eilenberg ∞-functor can be promoted to an ∞-functor*

$$\mathrm{CE}^{\varepsilon}\colon \mathrm{Lie}_{\Bbbk}^{\mathrm{op}} \longrightarrow \varepsilon\text{-}\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$$

where $\varepsilon\text{-}\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$ is the ∞-category of augmented commutative algebra objects in $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$.

We shall deduce Proposition 2.5.9 from the following results.

Lemma 2.5.10. *Let $\mathscr{C}^{\otimes}$ be a closed symmetric monoidal ∞-category with unit $\mathbb{1}$, and denote with $(-)^{\vee}\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ the ∞-functor given by internally mapping objects onto $\mathbb{1}$. Then, such ∞-functor can be lifted to $(-)^{\vee}\colon \mathrm{cCcAlg}(\mathscr{C})^{\mathrm{op}} \to \mathrm{CAlg}(\mathscr{C})$.*

Proof. (See also [Lur17, Remark 5.2.5.10].) Given two objects C and D of $\mathscr{C}$, the natural map

$$C^{\vee}\otimes D^{\vee}\otimes C\otimes D \simeq C^{\vee}\otimes C\otimes D^{\vee}\otimes D \xrightarrow{\mathrm{ev}_D} C^{\vee}\otimes \mathbb{1}\otimes C \simeq C^{\vee}\otimes C \xrightarrow{\mathrm{ev}_C} \mathbb{1}$$

yields a lax symmetric monoidal structure on $(-)^{\vee}$ by adjunction. Hence, we have

$$\mathrm{CAlg}(\mathscr{C}^{\mathrm{op}}) \simeq \mathrm{cCcAlg}(\mathscr{C})^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathscr{C})$$

as desired. □

Proposition 2.5.11. *The ∞-functor $\mathrm{CE}^{\varepsilon}\colon \mathrm{Lie}_{\Bbbk}^{\mathrm{op}} \to \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$ agrees with the composition*

$$\mathrm{Lie}_{\Bbbk} \xrightarrow{\mathrm{CE}_{\varepsilon}^{\mathrm{op}}} \left(\varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}\right)^{\mathrm{op}} \xrightarrow{\mathrm{oblv}^{\mathrm{op}}_{\mathrm{cCcAlg}}} \left(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\right)^{\mathrm{op}} \xrightarrow{(-)^{\vee}} \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}.$$

Proof. Given a Lie algebra $\mathfrak{g}$, we provide a morphism

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \longrightarrow \mathrm{CE}_{\varepsilon}(\mathfrak{g})^{\vee} := \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(\mathrm{CE}_{\varepsilon}(\mathfrak{g}),\ \Bbbk(0))$$

in the following way. By adjunction, we can restate our goal as providing a morphism

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}_{\varepsilon}(\mathfrak{g}) \longrightarrow \Bbbk(0).$$

Recall that $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ is the morphism object $\mathrm{Mor}^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)}\left(\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\left(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\right),\ \Bbbk(0)\right)$; in particular, by the very same definition of *morphism object* ([Lur17, Definition 4.2.1.28]), there exists a map

$$\mathrm{CE}^{\varepsilon} \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{U}^{\mathrm{gr}}_{\varepsilon}\left(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\right) \longrightarrow \Bbbk(0).$$

Let us notice now that the associativity of the relative tensor product ([Lur17, Proposition 4.4.3.14]) guarantees that we have equivalence of mixed graded $\Bbbk$-modules

$$\mathrm{CE}^{\varepsilon} \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{U}^{\mathrm{gr}}_{\varepsilon}\left(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\right) \simeq \mathrm{CE}^{\varepsilon} \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{U}(\mathfrak{g})(0) \otimes^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)} \mathrm{U}^{\mathrm{gr}}_{\varepsilon}\left(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\right),$$

hence we can pre-compose the inverse to such equivalence with the natural morphism

$$\mathrm{CE}^{\varepsilon} \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \Bbbk(0) \otimes_{\mathrm{U}(\mathfrak{g})(0)}^{\varepsilon\text{-gr}} \mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big) \longrightarrow \mathrm{CE}^{\varepsilon} \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{U}(\mathfrak{g})(0) \otimes_{\mathrm{U}(\mathfrak{g})(0)}^{\varepsilon\text{-gr}} \mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big).$$

Noting that the source of such morphism is precisely $\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}_{\varepsilon}(\mathfrak{g})$, we can compose everything accordingly and get a morphism

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}_{\varepsilon}(\mathfrak{g}) \longrightarrow \Bbbk(0)$$

hence by adjunction

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \longrightarrow \mathrm{CE}_{\varepsilon}(\mathfrak{g})^{\vee}.$$

Unraveling all the definition, using the description of $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ (Remark 2.4.25), the description of the mixed graded $\Bbbk$-linear dual (1.1.6) and the description of $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ (Remark 2.5.8), we obtain that the map $\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \longrightarrow \mathrm{CE}_{\varepsilon}(\mathfrak{g})^{\vee}$, at the level of the underlying graded $\Bbbk$-modules, is precisely the identity of $\mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1](1))^{\vee}$. □

Lemma 2.5.10 and Proposition 2.5.11 together obviously imply that, $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ being the dual of the underlying $\Bbbk$-module of a cocommutative coalgebra, is endowed with a commutative algebra structure. Hence Proposition 2.5.9 is proved.

Warning 2.5.12. In general, the $\Bbbk$-linear dual *does not* exchange cocommutative coalgebras and commutative algebras: it simply sends the former to the latter. The reason lies in the lax monoidal structure of the ∞-functor $(-)^{\vee}$, which is not oplax monoidal without some suitable finiteness assumptions on the algebra. For example, [Lur18, Corollary 3.2.5] states that in a symmetric monoidal ∞-category $\mathscr{C}$ the linear dual exchanges commutative algebras and cocommutative coalgebras in the full sub-∞-category of $\mathscr{C}$ spanned by fully dualizable objects in the sense of [Lur17, Section 4.6.1]. For our scope, this is a way too restrictive assumption: just consider the case of abelian Lie algebra $\mathrm{triv}_{\mathrm{Lie}}\,\Bbbk[-1]$ whose Chevalley-Eilenberg algebra is the graded $\Bbbk$-algebra $\Bbbk[t]$ where t sits in homological degree 2 and in weight -1. In particular $\mathrm{CE}^{\varepsilon}(\mathrm{triv}_{\mathrm{Lie}}\,\Bbbk[-1])$ is not fully dualizable: indeed, the tensor product $\Bbbk[t]\otimes_{\Bbbk}^{\varepsilon\text{-gr}}(\Bbbk[t])^{\vee}$ is equivalent in each weight p to a direct sum of countably many copies of $\Bbbk$ sitting in homological degree $-p$, hence there is no reasonable candidate for the coevaluation morphism $\Bbbk \to \Bbbk[t]\otimes_{\Bbbk}^{\varepsilon\text{-gr}} (\Bbbk[t])^{\vee}$. However, it is clear that $\mathrm{CE}^{\varepsilon}(\mathrm{triv}_{\mathrm{Lie}}\,\Bbbk[-1])$ and $\mathrm{CE}_{\varepsilon}(\mathrm{triv}_{\mathrm{Lie}}\,\Bbbk[1])$ should be one the dual of the other. So, for the mixed graded setting, we can refine [Lur18, Corollary 3.2.5] in the following way.

Proposition 2.5.13. *Let $\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}$ and $\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},+}$ be the full sub-∞-categories of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ defined in Notation 1.1.7. Then the mixed graded linear dual ∞-functor*

$$(-)^{\vee} := \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(-,\, \Bbbk(0))\colon \big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)^{\mathrm{op}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

induces equivalences

$$\mathrm{cCcAlg}\big(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}\big)^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{CAlg}\big(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},+}\big)$$

and

$$\mathrm{cCcAlg}\left(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},+}\right)^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{CAlg}\left(\varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}\right).$$

Remark 2.5.14. The class of mixed graded (co)commutative (co)algebras described in Proposition 2.5.13 is way larger than the class of fully dualizable mixed graded $\Bbbk$-modules, which we described explicitly in Proposition 1.1.8 as those mixed graded $\Bbbk$-modules which are perfect in each weight *and trivial outside at most finitely many weights*. Instead, our full sub-∞-category $\mathscr{C}$ allows us to consider (co)commutative (co)algebras whose underlying objects, while consisting of perfect $\Bbbk$-modules in each weight, are unbounded in weights $p \gg 0$ or $p \ll 0$, such as $\mathrm{CE}_{\varepsilon}(\mathfrak{g})$ and $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ for $\mathfrak{g}$ a Lie algebra which is perfect as a $\Bbbk$-module.

Proof of Proposition 2.5.13. Since the two statements are clearly dual one to the other, we can just prove the former. The main ingredient of the proof is the fact that the lax symmetric monoidal structure of $(-)^{\vee}$ is *strict* when restricted to such ∞-categories, hence the linear dual actually swaps cocommutative coalgebras and commutative algebras.
So, let $M_{\bullet}$ and $N_{\bullet}$ be two mixed graded $\Bbbk$-modules, both perfect in each weight and both bounded above. Then, we can describe each weight of the tensor product of their dual as

$$\begin{aligned}\left(\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(M_{\bullet},\,\Bbbk(0)) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(N_{\bullet},\,\Bbbk(0))\right)_p &\simeq \bigoplus_{i+j=p} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(M_{\bullet},\,\Bbbk(0))_i \otimes_{\Bbbk} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(N_{\bullet},\,\Bbbk(0))_j \\ &\simeq \bigoplus_{i+j=p} M_{-i}^{\vee} \otimes_{\Bbbk} N_{-j}^{\vee}.\end{aligned}$$

On the other hand, the dual of their tensor product is described in each weight by the formula

$$\begin{aligned}\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}\left(M_{\bullet} \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_{\bullet},\,\Bbbk(0)\right)_p &\simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(\bigoplus_{i+j=-p} M_i \otimes N_j,\,\Bbbk\right) \\ &\simeq \prod_{i+j=-p} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_i \otimes_{\Bbbk} N_j,\,\Bbbk\right) \\ &\simeq \prod_{i+j=-p} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(M_i,\,\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(N_j,\,\Bbbk)\right) \\ &\simeq \prod_{i+j=-p} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(M_i,\,\Bbbk) \otimes_{\Bbbk} \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(N_j,\,\Bbbk\right) \simeq \prod_{i+j=p} M_{-i}^{\vee} \otimes_{\Bbbk} N_{-j}^{\vee}.\end{aligned}$$

In these chains of equivalences, we only used the adjunction between tensor product and mapping spaces, and the fact that $M_{\bullet}$ is perfect in each weight. Since both $M_{\bullet}$ and $N_{\bullet}$ are bounded below, there are only finitely many admissible couples of integers (i,j), hence this product is actually a sum. In particular, the lax symmetric monoidal structure provided in Lemma 2.5.10 is strict, so $\Bbbk$-linear duality does swap commutative algebras and cocommutative coalgebras, and it is straightforward to show that if $A_{\bullet}$ is bounded below its dual is bounded above. (In general, if $M_{\bullet}$ is concentrated in weights $[m,n]$, for m and n ranging in $(-\infty,+\infty)$, its mixed graded $\Bbbk$-linear dual is concentrated in weights $[-n,-m]$.)
The fact that $\Bbbk$-linear duality is actually an equivalence is easily checked. The mixed graded

$\Bbbk$-linear duality is adjoint to itself: in particular, for any cocommutative coalgebra $C_\bullet$ as above we have a unit map

$$C_\bullet \longrightarrow \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}\big(\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(C_\bullet,\ \Bbbk(0)),\ \Bbbk(0)\big).$$

The ∞-functor is fully faithful if and only if this map is an equivalence, and since forgetting the cocommutative coalgebra structure is conservative, we can check this at the level of the underlying mixed graded $\Bbbk$-module. Using the usual formulas for the internal mixed graded mapping $\Bbbk$-module and the fact that $C_\bullet$ is perfect in each weight, one straightforwardly checks that this map is actually an equivalence on all weights, hence it is an equivalence. □

Remark 2.5.15. Let $\mathfrak{g}$ be a Lie algebra which is perfect as a $\Bbbk$-module. Then its Chevalley-Eilenberg algebra $\mathrm{CE}^\varepsilon(\mathfrak{g})$ is described, in each weight, by

$$\mathrm{CE}^p(\mathfrak{g}) \simeq \mathrm{Sym}_{\Bbbk}^p(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1](1))^\vee.$$

Being $\mathfrak{g}$ (and in particular $\mathfrak{g}[-1]$) perfect, every symmetric power $\mathrm{Sym}_{\Bbbk}^p(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1])$ is perfect, and moreover one has an equivalence

$$\mathrm{Sym}_{\Bbbk}^p(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1])^\vee \simeq \mathrm{Sym}_{\Bbbk}^p\big((\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})[-1])^\vee\big) \simeq \mathrm{Sym}_{\Bbbk}^p\big(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g})^\vee[1]\big).$$

In particular, under these assumptions, the underlying mixed graded $\Bbbk$-module of $\mathrm{CE}^\varepsilon(\mathfrak{g})$ lies in $\varepsilon\text{-}\mathrm{Perf}^{\mathrm{gr},\leqslant 0} \subseteq \varepsilon\text{-}\mathrm{Perf}_{\Bbbk}^{\mathrm{gr},-}$, hence its dual is a cocommutative coalgebra which is precisely $\mathrm{CE}_\varepsilon(\mathfrak{g})$. This fixes the drawback highlighted in Warning 2.5.12.

Proposition 2.5.11 straight-forwardly implies the following comparison result.

Proposition 2.5.16. *Given a Lie algebra $\mathfrak{g}$, the Tate realization $|\mathrm{CE}^\varepsilon(\mathfrak{g})|^{\mathrm{t}}$ agrees with the cohomological Chevalley-Eilenberg complex of [Lur11c, Construction 2.2.13].*

Proof. In virtue of Proposition 2.5.13, and using the fact that $\mathrm{CE}^\varepsilon(\mathfrak{g})$ is non-negatively graded, we have the following chain of equivalences of $\Bbbk$-modules.

$$\begin{aligned}
|\mathrm{CE}^\varepsilon(\mathfrak{g})|^{\mathrm{t}} &\simeq \big|\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(\mathrm{CE}_\varepsilon(\mathfrak{g}),\ \Bbbk(0))\big|^{\mathrm{t}}\\
&\simeq \big|\underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(\mathrm{CE}_\varepsilon(\mathfrak{g}),\ \Bbbk(0))\big|\\
&\simeq \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}\big(\Bbbk(0),\ \underline{\mathrm{Map}}_{\Bbbk}^{\varepsilon\text{-gr}}(\mathrm{CE}_\varepsilon(\mathfrak{g}),\ \Bbbk(0))\big)\\
&\simeq \mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\mathrm{CE}_\varepsilon(\mathfrak{g}),\ \Bbbk(0))
\end{aligned}$$

where the latter $\Bbbk$-module provides the $\mathrm{Mod}_{\Bbbk}$-enrichment of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$. On the other hand, the usual cohomological Chevalley-Eilenberg complex is the mapping $\Bbbk$-module $\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(\mathrm{CE}_\bullet,\ \Bbbk)$ ([Lur11c, Remark 2.2.14]), where $\mathrm{CE}_\bullet(\mathfrak{g})$ is the usual Chevalley-Eilenberg homology of $\mathfrak{g}$. Proposition 2.4.30 yields an equivalence $\mathrm{CE}_\bullet(\mathfrak{g}) \simeq |\mathrm{CE}_\varepsilon(\mathfrak{g})|^{\mathrm{t}}$, our claim would follow if we had an equivalence

$$\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}}(\mathrm{CE}^\varepsilon(\mathfrak{g}),\ \Bbbk(0)) \simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(|\mathrm{CE}_\varepsilon(\mathfrak{g})|^{\mathrm{t}},\ \Bbbk\big).$$

But this is true because of our computation of the explicit right adjoint to the Tate realization when restricted to non-negatively graded mixed $\Bbbk$-modules, in Porism 1.5.45. $\square$

Remark 2.5.17. Remark 2.5.8 and Proposition 2.5.16 suggest that our construction matches the explicit mixed graded structure on the cohomological Chevalley-Eilenberg algebra described in [CG19, Proposition A.3].

2.6 Representations and mixed graded modules

Just like (perfect) Lie algebras can be seen inside mixed graded commutative $\Bbbk$-algebras via the cohomological Chevalley-Eilenberg construction, the ∞-categories of representations over Lie algebras live naturally inside the ∞-category of mixed graded $\Bbbk$-modules, i.e., also the Chevalley-Eilenberg $\Bbbk$-modules with coefficients in some representation admit a mixed graded structure. Throughout all this section, we shall fix a Lie algebra $\mathfrak{g}$: while we expect our construction to be fully faithful only with some size constraints over $\mathfrak{g}$ (Conjecture 3.2.3), it can be carried out in all generality for any Lie algebra.

2.6.1. Let $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \simeq \mathrm{Rep}_{\mathfrak{g}}$ be the ∞-category of left $\mathrm{U}(\mathfrak{g})$-modules (hence, of representations of $\mathfrak{g}$), endowed with the Hopf symmetric monoidal structure of Construction 2.2.5. Since the trivial representation $\Bbbk$ is the unit for such symmetric monoidal structure, we have a natural equivalence

$$\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \simeq \mathrm{Mod}_{\Bbbk}\left(\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}^{\otimes_{\mathfrak{g}}}\right).$$

Recall moreover the ∞-functor 2.5.5

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$$

which sends a left $\mathrm{U}(\mathfrak{g})$-module to the mixed graded $\Bbbk$-module

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) := \mathrm{Mor}_{\mathrm{U}(\mathfrak{g})(0)}^{\varepsilon\text{-gr}}\left(\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\left(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\right),\, M(0)\right).$$

Again, looking at the weight p component, we have an equivalence of mixed graded $\Bbbk$-modules

$$\begin{aligned}\mathrm{CE}^{p}(\mathfrak{g};\, M) &:= \mathrm{Mor}_{\mathrm{U}(\mathfrak{g})(0)}^{\varepsilon\text{-gr}}\left(\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\left(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\right),\, M(0)\right)_{p}\\ &\simeq \mathrm{Mor}_{\mathrm{U}(\mathfrak{g})}\left(\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\left(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\right)_{-p},\, M\right)\\ &\simeq \mathrm{Mor}_{\mathrm{U}(\mathfrak{g})}\left(\mathrm{U}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]),\, M\right) \simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(\mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]),\, M\right).\end{aligned}$$

Again, the mixed differential is completely inherited from the action of $\mathrm{B}\mathbb{G}_{\mathrm{a},\Bbbk}$ on $\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\left(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\right)$.

Remark 2.6.2. When $\mathfrak{g}$ is perfect as a $\Bbbk$-module, we can write the weight p component of $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M)$ in another way. The discussion of 2.6.1 yields that

$$\mathrm{CE}^{p}(\mathfrak{g};\, M) \simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\left(\mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]),\, M\right).$$

But if $\mathfrak{g}$ is perfect as a $\Bbbk$-module, then so are $\mathfrak{g}[-1]$ and each symmetric power of $\mathfrak{g}[-1]$. In particular, we can write

$$\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(\mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]),\ M\big) \simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(\mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]),\ \Bbbk\big) \otimes_{\Bbbk} M$$

and since we are in characteristic 0, we can write

$$\mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}\big(\mathrm{Sym}_{\Bbbk}^{-p}(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1]),\ \Bbbk\big) \simeq \mathrm{Sym}_{\Bbbk}^{-p}\big(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}^{\vee}[1]\big),$$

where $(-)^{\vee}$ denotes the usual $\Bbbk$-linear dual of $\Bbbk$-modules. In particular, we have an equivalence of graded $\Bbbk$-modules

$$\mathrm{oblv}_{\varepsilon}\,\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \simeq \mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}\big(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}^{\vee}[1](-1)\big) \otimes_{\Bbbk}^{\mathrm{gr}} M(0).$$

Proposition 2.6.3. *The ∞-functor 2.5.5 can be promoted to an ∞-functor*

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\,-)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$$

where $\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$ is the ∞-category of modules for the commutative algebra $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ in $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$.

In order to prove Proposition 2.6.3, we shall need the following technical lemma.

Lemma 2.6.4. *Given two mixed graded left $\mathrm{U}(\mathfrak{g})$-modules $M_\bullet$ and $N_\bullet$, let $M_\bullet \otimes_{\mathfrak{g}} N_\bullet$ the mixed graded tensor product of Construction 2.2.5, i.e., the mixed graded tensor product of left modules for the mixed graded Hopf algebra $\mathrm{U}(\mathfrak{g})(0)$. For all mixed graded $\Bbbk$-modules $P_\bullet$, there is an equivalence of mixed graded left $\mathrm{U}(\mathfrak{g})$-modules*

$$\big(M_\bullet \otimes_{\mathfrak{g}} N_\bullet\big) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} P_\bullet \simeq M_\bullet \otimes_{\mathfrak{g}} \big(N_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} P_\bullet\big)$$

where $\otimes_{\Bbbk}^{\varepsilon\text{-gr}}$ denotes the right action of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ over $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ given by the relative tensor product over $\Bbbk(0)$ described in 2.5.1.

Proof. Recall that, in virtue of [Lur17, Remark 4.2.1.21], the fact that $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is right tensored over $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ is equivalent to the saying that $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is a right $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$-module in the ∞-category of (not necessarily small) ∞-categories Cat_{∞}. Since $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ is a symmetric monoidal ∞-category, hence an $\mathbb{E}_{\infty}$-algebra in the ∞-category Cat_{∞}, it follows that the $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is both a left and a right $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$-module in ∞-categories in a compatible way, i.e., it is actually a $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$-bimodule ([Lur17, Section 4.5]). This implies that our request is equivalent to saying that $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is a $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$-algebra object in Cat_{∞}. Since both $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ and $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ are $\mathbb{E}_{\infty}$-monoidal categories, a $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$-algebra structure on $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is equivalent to a symmetric monoidal ∞-functor $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \to \varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$. But now $\mathrm{U}(\mathfrak{g})(0)$ is an augmented associative $\Bbbk$-algebra, hence pullback along the augmentation provides an

∞-functor

$$F\colon \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \longrightarrow \varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}.$$

We are only left to prove that this ∞-functor is strongly monoidal, but this is a straight-forward computation. Indeed, given two mixed graded $\Bbbk$-modules $M_\bullet$ and $N_\bullet$, the underlying mixed graded $\Bbbk$-module of both $FM_\bullet \otimes_{\mathfrak{g}} FN_\bullet$ and $F\big(M_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet\big)$ is exactly $M_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet$. Moreover, the action of $\mathrm{U}(\mathfrak{g})$ is given, in both cases, by the augmentation over $\Bbbk(0)$, hence the identity of $M_\bullet \otimes_{\Bbbk}^{\varepsilon\text{-gr}} N_\bullet$ is a morphism of mixed graded left $\mathrm{U}(\mathfrak{g})$-modules which yields our claim. □

Proof of Proposition 2.6.3. In order to prove our assertion, we only need to show that $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)$ is lax monoidal with respect to the Hopf monoidal structure of Construction 2.2.5 and the mixed graded tensor product of $\Bbbk$-modules. Then our claim will follow from the fact that $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, \Bbbk)$ is, by definition, the underlying mixed graded $\Bbbk$-module of $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$ (see Definition 2.5.7 and Proposition 2.5.9).
Let M and N be two representations of $\mathfrak{g}$ seen as left $\mathrm{U}(\mathfrak{g})$-modules. By the definition of the ∞-functor 2.5.5, for any left $\mathrm{U}(\mathfrak{g})$-module M we have a natural evaluation map

$$\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) \longrightarrow M(0).$$

Tensoring over $\mathfrak{g}$ with $\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big)$ on the left and tensoring over $\Bbbk$ with $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, N)$ on the right, we obtain a map

$$\begin{aligned}\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big) \otimes_{\mathfrak{g}} \mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, N) \longrightarrow \\ \longrightarrow \mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big) \otimes_{\mathfrak{g}} M(0) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, N).\end{aligned} \tag{2.6.5}$$

Let us remark that in the above formula there is no associativity ambiguity, because $\varepsilon\text{-}\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})(0)}^{\mathrm{gr}}$ is a $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$-algebra in ∞-categories in virtue of Lemma 2.6.4. Since the tensor product of left $\mathrm{U}(\mathfrak{g})$-modules is symmetric monoidal, we can swap the first two factors in the target and then apply again the evaluation map:

$$M(0) \otimes_{\mathfrak{g}} \mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, N) \longrightarrow M(0) \otimes_{\mathfrak{g}} N(0) \tag{2.6.6}$$

Recall now that $\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big)$ is a mixed graded Hopf algebra, since it is the universal enveloping mixed graded algebra of the mixed graded Lie algebra $\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})$. Moreover, the comultiplication map

$$\mu\colon \mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big) \longrightarrow \mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big)$$

is a morphism of mixed graded left $\mathrm{U}(\mathfrak{g})$-modules, since it is a morphism of mixed graded left $\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big)$-modules (in virtue of Remark 2.2.6), and the action of $\mathrm{U}(\mathfrak{g})$ over $\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big)$ is given by the inclusion of $\mathrm{U}(\mathfrak{g})$ in weight 0. In particular, tensoring the comultiplication for $\mathrm{U}_{\varepsilon}^{\mathrm{gr}}\big(\mathrm{Cn}_{\varepsilon}^{\mathrm{gr}}(\mathfrak{g})\big)$ with the identity on $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) \otimes_{\Bbbk}^{\varepsilon\text{-gr}} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, N)$ and post-composing with 2.6.5 and

2.6.6, we obtain a map

$$\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, N) \longrightarrow M(0) \otimes_{\mathfrak{g}} N(0).$$

By adjunction, this is the same as a map

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, N) \longrightarrow \mathrm{Mor}^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)}\Big(\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big),\, \big(M \otimes_{\mathfrak{g}} N\big)(0)\Big) =: \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M \otimes_{\mathfrak{g}} N),$$

which testifies the lax monoidality of $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)$. □

Proposition 2.6.3 allows us to promote the ∞-functor $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)$ to a ∞-functor

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big). \tag{2.6.7}$$

Warning 2.6.8. The ∞-functor $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)$ is *never* strongly monoidal, not even if $\mathfrak{g}$ is perfect. In the latter case, we know by the discussion in Remark 2.6.2 that the underlying graded of $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M)$ is $\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}(\mathrm{oblv}_{\mathrm{Lie}}\, \mathfrak{g}^{\vee}[1](-1)) \otimes_{\Bbbk} M(0)$, so we can interpret the natural map

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, N) \longrightarrow \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M \otimes_{\mathfrak{g}} N)$$

simply as tensoring the natural multiplication of the commutative symmetric algebra over $\mathrm{oblv}_{\mathrm{Lie}}\, \mathfrak{g}^{\vee}[1]$, which is compatible with the Chevalley-Eilenberg mixed differential, with the identity of $M \otimes_{\Bbbk} N$. Again, such multiplication map is almost never an isomorphism.

Warning 2.6.9. Even if $\mathfrak{g}$ is perfect, hence there is an equivalence of graded $\Bbbk$-modules between $\mathrm{oblv}_{\varepsilon}\big(\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} M(0)\big) \simeq \mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}(\mathrm{oblv}_{\mathrm{Lie}}\, \mathfrak{g}^{\vee}[1](-1)) \otimes^{\mathrm{gr}}_{\Bbbk} M(0)$ and $\mathrm{oblv}_{\varepsilon}\, \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M)$, this equivalence is *not* an equivalence of mixed graded $\Bbbk$-modules. Indeed, one can construct a map

$$\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} \mathrm{CE}^{\varepsilon}(\mathfrak{g}) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} M(0) \longrightarrow \Bbbk(0) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} M \xrightarrow{\simeq} M(0)$$

but this is only a map of mixed graded $\Bbbk$-modules, and does not provide by adjunction a map $\mathrm{CE}^{\varepsilon}(\mathfrak{g}) \otimes^{\varepsilon\text{-gr}}_{\Bbbk} M(0) \to \mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M)$. This cannot be avoided: doing some checking with explicit mixed graded chain complexes of $\Bbbk$-modules, one can see that the mixed differential ε_0 on the domain is given by

$$\Bbbk \otimes_{\Bbbk} M \simeq M \xrightarrow{0} \mathfrak{g}^{\vee}[1] \otimes_{\Bbbk} M[-1]$$

while the one on the codomain is given by

$$\alpha\colon M \longrightarrow \mathfrak{g}^{\vee}[1] \otimes_{\Bbbk} M[-1] \simeq \mathrm{Map}_{\mathrm{Mod}_{\Bbbk}}(\mathfrak{g},\, M)$$

where α is the morphism that encodes the left action of $\mathfrak{g}$ on M.

We conclude this section with just a simple observation.

Proposition 2.6.10. *The Chevalley-Eilenberg ∞-functor*

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\,-)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\mathbb{k}}^{\mathrm{gr}}\big)$$

is an accessible ∞-functor which preserves all limits. If $\mathfrak{g}$ is perfect, it preserves also all colimits.

Proof. Recall that, if $\mathscr{C}$ is a monoidal ∞-category such that the tensor product preserves colimits separably in each variable, then colimits in the ∞-category of modules over some algebra of $\mathscr{C}$ are detected by the forgetful ∞-functor towards $\mathscr{C}$ ([Lur17, Corollary 4.2.3.5]). Moreover, both $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}$ and $\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\mathbb{k}}^{\mathrm{gr}}\big)$ are presentable ∞-categories, being the ∞-categories of modules over an algebra in some presentable stable ∞-category. In particular, our statement reduces to show that $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\,-)$ sends limits of left $\mathrm{U}(\mathfrak{g})$-modules to limits of mixed graded $\mathbb{k}$-modules. The inclusion

$$(-)(0)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \lhook\joinrel\longrightarrow \varepsilon\text{-}\mathrm{LMod}^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)}$$

preserves all limits and colimits, since they are computed weight-wise in the ∞-category of mixed graded $\mathbb{k}$-modules. So we are left to prove that

$$\mathrm{Mor}^{\varepsilon\text{-gr}}_{\mathrm{U}(\mathfrak{g})(0)}\Big(\mathrm{U}^{\mathrm{gr}}_{\varepsilon}\big(\mathrm{Cn}^{\mathrm{gr}}_{\varepsilon}(\mathfrak{g})\big),\,-\Big)\colon \varepsilon\text{-}\mathrm{LMod}^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)} \longrightarrow \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\mathbb{k}}$$

commutes with limits. This is clear, since the ∞-functor is constructed as the right adjoint to the right tensor action of $\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\mathbb{k}}$ over $\varepsilon\text{-}\mathrm{LMod}^{\mathrm{gr}}_{\mathrm{U}(\mathfrak{g})(0)}$.
Now, suppose that $\mathfrak{g}$ is perfect as a $\mathbb{k}$-module. Then by the equivalence

$$\mathrm{oblv}_{\varepsilon}\,\mathrm{CE}^{\varepsilon}(\mathfrak{g};\,M) \simeq \mathrm{Sym}^{\mathrm{gr}}_{\mathbb{k}}\big(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}^{\vee}[1](-1)\big) \otimes^{\mathrm{gr}}_{\mathbb{k}} M(0)$$

provided in Remark 2.6.2, we see that colimits of left $\mathrm{U}(\mathfrak{g})$-modules are preserved by $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\,-)$, since the tensor product commutes with them. □

Chapter 3

Conjectures and open problems

As already mentioned in Chapter 2, we conjecture that mixed graded $\Bbbk$-modules offer a framework in which homotopy Lie algebras and their representations can be fully recovered. In this last chapter, we shall carefully formalize our expectations, providing some conjectures, showing how they are related one to the other, and hinting at possible strategies to adopt in order to prove our assertions.

3.1 Lie algebras as mixed graded coalgebras

In this section, we shall describe how we expect that Lie algebras live fully faithfully inside the ∞-category of mixed graded cocommutative coalgebras. Our standing assumptions and notations are the same as the ones of Chapter 2.

Definition 3.1.1. We say that an augmented mixed graded cocommutative coalgebra $A_\bullet$ is *of Chevalley-Eilenberg type* (or *semi-free*) if its underlying graded (augmented) cocommutative coalgebra is equivalent to a symmetric (free) cocommutative coalgebra $\mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}(M[-1](1))$, for some $\Bbbk$-module M.

3.1.2. Mixed graded cocommutative coalgebras of Chevalley-Eilenberg type are naturally gathered in a full sub-∞-category $\varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}}$ of $\varepsilon\text{-}\mathrm{cCcAlg}_{\Bbbk//\Bbbk}^{\mathrm{gr}}$, which we can express via a pullback of ∞-categories in the following way.

$$\begin{tikzcd}
\varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \varepsilon\text{-}\mathrm{cCcAlg}_{\Bbbk//\Bbbk}^{\mathrm{gr}} \arrow[d, "\mathrm{oblv}_{\varepsilon}"] \\
\mathrm{Mod}_{\Bbbk} \arrow[r, "1.3.23"'] & \mathrm{cCcAlg}_{\Bbbk//\Bbbk}^{\mathrm{gr}}
\end{tikzcd} \tag{3.1.3}$$

The above description exhibits $\varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}}$ as a pullback of presentable ∞-categories: in fact, $\mathrm{Mod}_{\Bbbk}$ is obviously presentable ([Lur17, Corollary 4.2.3.7]) while $\mathrm{cCcAlg}_{\Bbbk//\Bbbk}^{\mathrm{gr}}$ and $\varepsilon\text{-}\mathrm{cCcAlg}_{\Bbbk//\Bbbk}^{\mathrm{gr}}$

are ∞-categories of (augmented) cocommutative coalgebra objects in some symmetric monoidal ∞-category whose tensor product preserves colimits separately in each variable. Hence, they are presentable in virtue of [Lur18, Proposition 3.1.3].
Moreover, it is clear from the description of $\mathrm{CE}_\varepsilon(\mathfrak{g})$ yielded in Remark 2.4.25 that the ∞-functor

$$\mathrm{CE}_\varepsilon\colon \mathrm{Lie}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$$

factors through $\varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk}$.

We characterize some main properties of the ∞-category $\varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk}$.

Lemma 3.1.4. *The ∞-category $\varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk}$ is a localization (in the sense of [Lur09, 5.2.7.2]) of the ∞-category $\varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$.*

Proof. Being a full presentable sub-∞-category of a presentable ∞-category, it is enough to show that the inclusion preserves all limits and κ-filtered colimits for some regular cardinal κ. The discussion in 1.3.12 allows us to reduce ourselves to check the claim at the level of the underlying graded coalgebras. But now the assertion is clear, since $\mathrm{oblv}_\varepsilon \circ \mathrm{CE}_\varepsilon$ is naturally equivalent to $\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}((-)[-1](1)) \circ \mathrm{oblv}_{\mathrm{Lie}}$, which is a composition of two right adjoints and as such preserves all limits and κ-filtered colimits for some regular cardinal κ. $\square$

3.1.5. Lemma 3.1.4 implies that we have a left adjoint

$$\mathrm{L}_{\mathrm{CE}}\colon \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk}$$

which, by usual abstract nonsense, is the identity when restricted to mixed graded cocommutative coalgebra of Chevalley-Eilenberg type. Moreover, a slight reworking of the proof of Proposition 2.4.26 shows that CE_ε actually preserves *all* limits: we just need to observe that limits of Lie algebras are sent to limits on the underlying $\Bbbk$-modules by the forgetful ∞-functor $\mathrm{oblv}_{\mathrm{Lie}}\colon \mathrm{Lie}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$, and then use again that $\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}$ is a right adjoint to get

$$\begin{aligned}\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}\left(\mathrm{oblv}_{\mathrm{Lie}}\left(\lim_{i\in I}\mathfrak{g}_i\right)[-1](1)\right) &\simeq \mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}\left(\left(\lim_i\left(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g}_i)\right)\right)[-1](1)\right)\\ &\simeq \mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}\left(\lim_i\left(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g}_i)[-1](1)\right)\right)\\ &\simeq \lim_i\left(\mathrm{Sym}^{\mathrm{gr}}_{\Bbbk}\left(\mathrm{oblv}_{\mathrm{Lie}}(\mathfrak{g}_i)[-1](1)\right)\right).\end{aligned}$$

In particular,

$$\mathrm{CE}_\varepsilon\colon \mathrm{Lie}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk}$$

admits a left adjoint as well, which we denote with

$$\mathfrak{L}\colon \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk} \longrightarrow \mathrm{Lie}_{\Bbbk}. \tag{3.1.6}$$

Conjecture 3.1.7. *The ∞-functor $\mathrm{CE}_\varepsilon\colon \mathrm{Lie}_{\Bbbk} \to \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$ is a fully faithful embedding. The essential image is the full sub-∞-category $\varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk}$ of cocommutative mixed graded coalgebras of Chevalley-Eilenberg type.*

As a direct consequence of Conjecture 3.1.7 and Proposition 2.5.13, we have the following cohomological analogue.

Conjecture 3.1.8. *The ∞-functor $\mathrm{CE}^\varepsilon\colon \mathrm{Lie}^{\mathrm{op}}_{\Bbbk} \to \varepsilon\text{-}\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk}$ restricts to a fully faithful embedding*

$$\mathrm{Lie}^{\mathrm{perf}}_{\Bbbk} \hookrightarrow \varepsilon\text{-}\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk}$$

where $\mathrm{Lie}^{\mathrm{perf}}_{\Bbbk} := \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Perf}_{\Bbbk})$ is the ∞-category of Lie algebras whose underlying $\Bbbk$-module is perfect.

Conjecture 3.1.7 and Conjecture 3.1.8 can be summarized as follows. Recall the definition of the ∞-categories $\varepsilon\text{-}\mathrm{Perf}^{\mathrm{gr},\geqslant 0}_{\Bbbk}$ and $\varepsilon\text{-}\mathrm{Perf}^{\mathrm{gr},\leqslant 0}_{\Bbbk}$ of Notation 1.1.7: they are closed under the mixed graded tensor product, hence we can consider both commutative algebras and cocommutative coalgebras in these ∞-categories. Let moreover $\mathrm{cLie}_{\Bbbk}$ be the ∞-category of Lie coalgebras in $\mathrm{Mod}_{\Bbbk}$: one has an ∞-functor

$$\mathrm{coChev}\colon\ \mathrm{cLie}_{\Bbbk} \longrightarrow \mathrm{CAlg}_{\Bbbk//\Bbbk}$$

which computes the Chevalley-Eilenberg cohomology of Lie coalgebras ([Ho21, Section 5]).

Conjecture 3.1.9. *There exists an ∞-functor $\mathrm{cC}^\varepsilon\colon\ \mathrm{cLie}_{\Bbbk} \to \varepsilon\text{-}\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk}$, such that its Tate realization agrees with* coChev, *which sits in a diagram of ∞-categories*

$$\begin{tikzcd}
\mathrm{Lie}_{\Bbbk} \arrow[rrr, hook, "\mathrm{CE}_\varepsilon"] & & & \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk} \arrow[ddd, bend left, "(-)^\vee"] \\
& \mathrm{Lie}^{\mathrm{perf}}_{\Bbbk} \arrow[ul, hook] \arrow[r, hook, "\mathrm{CE}_\varepsilon"] \arrow[d, leftrightarrow, "(-)^\vee"', "\wr"] & \mathrm{cCcAlg}\big(\varepsilon\text{-}\mathrm{Perf}^{\mathrm{gr},\geqslant 0}_{\Bbbk}\big)_{\Bbbk/} \arrow[ur, hook] \arrow[d, leftrightarrow, "(-)^\vee"', "\wr"] & \\
& \big(\mathrm{cLie}^{\mathrm{perf}}_{\Bbbk}\big)^{\mathrm{op}} \arrow[dl, hook] \arrow[r, hook, "(\mathrm{cC}^\varepsilon)^{\mathrm{op}}"'] & \mathrm{CAlg}\big(\varepsilon\text{-}\mathrm{Perf}^{\mathrm{gr},\leqslant 0}_{\Bbbk}\big)^{\mathrm{op}}_{/\Bbbk} \arrow[dr, hook] & \\
(\mathrm{cLie}_{\Bbbk})^{\mathrm{op}} \arrow[rrr, hook, "(\mathrm{cC}^\varepsilon)^{\mathrm{op}}"'] & & & \big(\varepsilon\text{-}\mathrm{CAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}\big)^{\mathrm{op}}
\end{tikzcd}$$

which commutes in every direction.

Remark 3.1.10. While the dual of a Lie coalgebra is always a Lie algebra, the converse is true only for perfect Lie algebras; hence, we cannot hope to have a well defined $\Bbbk$-linear dual ∞-functor $\mathrm{Lie}_{\Bbbk} \to \mathrm{cLie}_{\Bbbk}$. Moreover, the diagram in Conjecture 3.1.9 cannot commute in every direction if we add the $\Bbbk$-linear dual $\mathrm{cLie}_{\Bbbk} \to \mathrm{Lie}_{\Bbbk}$: if $\mathfrak{a}$ is an abelian (i.e., trivial) Lie coalgebra, its Chevalley-Eilenberg algebra is equivalent as a commutative algebra to $\mathrm{Sym}_{\Bbbk}(\mathfrak{a}[-1])$. In particular, if $\mathfrak{a} := \Bbbk[t]^{\mathrm{ab}}$, we have that its Chevalley-Eilenberg algebra is the trivial square-zero extension $\Bbbk\oplus\Bbbk[t][-1]$, while the Chevalley-Eilenberg algebra of its dual Lie algebra is $\Bbbk\oplus\Bbbk[\![t]\!]^{\vee}$.

3.1.11. Conjectures 3.1.7 and 3.1.8 provide a derived analogue of Proposition 2.3.6. Indeed, if $\mathfrak{g}$ is a discrete Lie algebra, then the homological mixed graded Chevalley-Eilenberg construction can be identified with the usual Grassmann graded coalgebra $\mathrm{Sym}^{\bullet}_{\Bbbk}(\mathfrak{g}[-1]) \cong \bigwedge^{\bullet}\mathfrak{g}$, endowed with the usual Chevalley-Eilenberg differential, considered as a honest chain complex in homological non-negative degrees in the heart of the mixed graded Postnikov t-structure. If moreover $\mathfrak{g}$ is perfect (hence, finite and projective), then the cohomological mixed graded Chevalley-Eilenberg construction sends $\mathfrak{g}$ to the Grassmann graded algebra $\mathrm{Sym}^{\bullet}_{\Bbbk}(\mathfrak{g}[1])^{\vee} \cong \bigwedge^{\bullet}\mathfrak{g}^{\vee}$, again with a Chevalley-Eilenberg differential, considered as a honest chain complex in homological non-positive degrees in the heart of the mixed graded Postnikov t-structure.

Remark 3.1.12. In [CPT+17, Proposition 3.6.2], the authors proved Conjecture 3.1.8, in terms of formal mixed graded commutative $\Bbbk$-algebras, for the case of discrete Lie algebras whose underlying $\Bbbk$-module is finite and projective.

The main obstruction to proving Conjecture 3.1.7 lies in the description of the left adjoint L_{CE} to the inclusion of mixed graded cocommutative coalgebras of Chevalley-Eilenberg type inside all coaugmented mixed graded cocommutative coalgebras. In the following, we provide some evidence that strongly suggests our conjectures to hold.

Lemma 3.1.13. *The natural ∞-functor* $\mathrm{Prim}\circ\mathrm{oblv}_{\varepsilon}\colon \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$ *in the diagram 3.1.3 preserves sifted colimits.*

Proof. The forgetful ∞-functor $\mathrm{oblv}_{\varepsilon}\colon \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$ factors through the essential image of the cofree ind-nilpotent cocommutative coalgebra on mixed graded $\Bbbk$-modules of the form $M[-1](1)$. Since such ∞-functor is fully faithful , the ∞-functor Prim can be identified with the composition of the forgetful ∞-functor $\mathrm{oblv}_{\varepsilon}$ and the inverse equivalence that takes a cofree ind-nilpotent cocommutative graded coalgebra to its primitive elements (i.e., the $\Bbbk$-module of elements in weight 1). But the composition

$$\mathrm{oblv}_{\mathrm{cCcAlg}}\circ\mathrm{coFree}_{\mathrm{CoComm}^{\mathrm{aug}}}\circ(-)(-1)\circ[-1]\colon \mathrm{Mod}_{\Bbbk}\longrightarrow\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$$

agrees with

$$\mathrm{oblv}_{\mathrm{CAlg}}\circ\mathrm{Free}_{\mathrm{Comm}^{\mathrm{aug}}}\circ(-)(-1)\circ[-1]\colon \mathrm{Mod}_{\Bbbk}\longrightarrow\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk},$$

because in characteristic 0 the graded cofree ind-nilpotent cocommutative coalgebra and the graded free commutative algebra on a $\Bbbk$-module are equivalent ([GR17b, Chapter 6, Section 4.2]). Now, $\mathrm{oblv}_{\mathrm{cCcAlg}}$ is conservative and commutes with all colimits, being a left adjoint. Moreover,

$$\mathrm{oblv}_{\mathrm{CAlg}}\circ\mathrm{Free}_{\mathrm{Comm}^{\mathrm{aug}}}\circ(-)(-1)\circ[-1]\colon \mathrm{Mod}_{\Bbbk}\to\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$$

commutes with sifted colimits, because it is a composition of ∞-functors which commute with sifted colimits (they are all left adjoint except for $\mathrm{oblv}_{\mathrm{CAlg}}$, which however commutes with sifted colimits in virtue of [GR17b, Chapter 6, Section 1.1.3]). Finally, the forgetful ∞-functor $\mathrm{oblv}_{\varepsilon}\colon \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk} \to \mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$ commutes with colimits, since the usual forgetful ∞-functor $\mathrm{oblv}_{\varepsilon}\colon \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$ commutes with colimits and colimits of coalgebras are computed in the underlying category. All these ingredients together imply that sifted colimits of mixed graded cocommutative coalgebras of Chevalley-Eilenberg type are again of Chevalley-Eilenberg type, hence our claim follows. □

Porism 3.1.14. Lemma 3.1.13 implies that the ∞-functor $\mathrm{Prim}\colon \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$ is a monadic ∞-functor: it commutes with all limits, it commutes with sifted colimits, and it is conservative, hence the conditions of Barr-Beck-Lurie's monadicity theorem ([Lur17, Theorem 4.7.0.3]) are satisfied. Arguing analogously to what we did in the proof of Lemma 3.1.4, we obtain then that the ∞-functor $\mathrm{CE}_{\varepsilon}\colon \mathrm{Lie}_{\Bbbk} \to \mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$ is monadic as well.
In particular, Conjecture 3.1.7 can be re-formulated as follows: the two monads over $\mathrm{Mod}_{\Bbbk}$ defined by $\mathrm{oblv}_{\mathrm{Lie}}\colon \mathrm{Lie}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$ and $\mathrm{Prim}\circ\mathrm{oblv}_{\varepsilon}\colon \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk} \to \mathrm{Mod}_{\Bbbk}$ are equivalent.

3.1.15. Recall ([GR14, Chapter 6, Proposition 1.6.4]) that the loop ∞-functor

$$\Omega_{\mathrm{Lie}}\colon \mathrm{Lie} \to \mathrm{Grp}(\mathrm{Lie}_{\Bbbk})$$

is an equivalence of ∞-categories. Moreover, the diagram

$$\begin{tikzcd}
\mathrm{Lie}_{\Bbbk} \arrow[rr, "\simeq", "\Omega_{\mathrm{Lie}}"'] \arrow[d, "\mathrm{oblv}_{\mathrm{Lie}}"'] & & \mathrm{Grp}(\mathrm{Lie}_{\Bbbk}) \arrow[d, "\mathrm{oblv}_{\mathrm{Grp}}"] \\
\mathrm{Mod}_{\Bbbk} \arrow[r, "\simeq", "{[-1]}"'] & \mathrm{Mod}_{\Bbbk} \arrow[r, "\mathrm{triv}_{\mathrm{Lie}}"'] & \mathrm{Lie}_{\Bbbk}
\end{tikzcd}$$

commutes ([GR14, Chapter 6, Proposition 1.7.2]).

A similar result applies also to mixed graded cocommutative coalgebras of Chevalley-Eilenberg type.

Lemma 3.1.16. *The loop ∞-functor*

$$\Omega_{\varepsilon}\colon \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk} \xrightarrow{\simeq} \mathrm{Grp}\left(\varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_{\Bbbk}\right)$$

provides an equivalence of ∞-categories, with inverse given by delooping.

Proof. This is, essentially, a re-writing of the proof of [GR17b, Proposition 1.6.2]. Since the delooping $\mathrm{B}_{\varepsilon}\colon \mathrm{Grp}\big(\mathrm{Chev}^{\mathrm{enh}}\big) \to \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$ is a left adjoint to the loop ∞-functor Ω_{ε}, we have a unit morphism

$$\mathrm{id}_{\mathrm{Grp}\left(\varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}\right)} \longrightarrow \Omega_{\varepsilon}\circ \mathrm{B}_{\varepsilon}$$

and a counit morphism
$$\mathsf{B}_\varepsilon \circ \Omega_\varepsilon \longrightarrow \mathrm{id}_{\varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}}.$$
In order to prove the equivalence, it suffice to show that these natural transformations are equivalences of ∞-functors. Since both forgetting the group structure and the mixed graded structure are conservative operations, it suffices to prove that

$$\mathrm{oblv}_\varepsilon \circ \mathrm{oblv}_{\mathrm{Grp}} \longrightarrow \mathrm{oblv}_\varepsilon \circ \mathrm{oblv}_{\mathrm{Grp}} \circ \Omega_\varepsilon \circ \mathsf{B}_\varepsilon$$

and
$$\mathrm{oblv}_\varepsilon \circ \mathsf{B}_\varepsilon \circ \Omega_\varepsilon \longrightarrow \mathrm{oblv}_\varepsilon$$
are natural equivalences of graded *cofree* cocommutative coalgebras. Now the ∞-functor

$$\mathrm{Prim}\colon \mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk} \longrightarrow \mathrm{Mod}_\Bbbk,$$

given by considering the $\Bbbk$-module $A_1[1]$ of a graded cocommutative coalgebra $A_\bullet$, reflects equivalences of graded cofree cocommutative coalgebras, since

$$\mathrm{Sym}^{\mathrm{gr}}_\Bbbk((-)[-1](1))\colon \mathrm{Mod}_\Bbbk \longrightarrow \mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$$

is fully faithful in virtue of Proposition 1.3.22. In particular, we are left to show that the natural transformations

$$\mathrm{Prim} \circ \mathrm{oblv}_\varepsilon \circ \mathrm{oblv}_{\mathrm{Grp}} \longrightarrow \mathrm{Prim} \circ \mathrm{oblv}_\varepsilon \circ \mathrm{oblv}_{\mathrm{Grp}} \circ \Omega_\varepsilon \circ \mathsf{B}_\varepsilon$$

and
$$\mathrm{Prim} \circ \mathrm{oblv}_\varepsilon \circ \mathsf{B}_\varepsilon \circ \Omega_\varepsilon \longrightarrow \mathrm{Prim} \circ \mathrm{oblv}_\varepsilon$$
are equivalences. Let us remark that B_ε is, by definition, a sifted colimit of mixed graded cocommutative Chevalley-Eilenberg coalgebras. Thanks to Lemma 3.1.13 we have that the following square

$$\begin{tikzcd}
\mathrm{Grp}\left(\varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_\Bbbk\right) \arrow[r, "\mathsf{B}_\varepsilon"] \arrow[d, "\mathrm{Prim}\circ\mathrm{oblv}_\varepsilon\circ\mathrm{oblv}_{\mathrm{Grp}}"'] & \varepsilon\text{-}\mathrm{CEcAlg}^{\mathrm{gr}}_\Bbbk \arrow[d, "\mathrm{Prim}\circ\mathrm{oblv}_\varepsilon"] \\
\mathrm{Mod}_\Bbbk \arrow[r, "\simeq", "{[1]}"'] & \mathrm{Mod}_\Bbbk
\end{tikzcd}$$

commutes, since Prim commutes with sifted colimits.
On the other hand, since $\mathrm{oblv}_\varepsilon\colon \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_\Bbbk \to \mathrm{Mod}^{\mathrm{gr}}_\Bbbk$ is strongly monoidal and preserves all colimits, hence geometric realizations, it preserves also relative tensor products. In particular,

$$\mathrm{oblv}_\varepsilon\colon \varepsilon\text{-}\mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk} \longrightarrow \mathrm{cCcAlg}^{\mathrm{gr}}_{\Bbbk//\Bbbk}$$

preserves pullbacks, since they are given by the relative tensor product: this follows from the fact that $\mathrm{cCcAlg}(\mathscr{C}) \simeq \mathrm{CAlg}(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$ and pushouts of commutative algebras are given by the relative tensor product, in virtue of [Lur17, Proposition 3.2.4.7]. Therefore, we have another commutative square

$$\begin{tikzcd}
\varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}} \arrow[r, "\Omega_\varepsilon"] \arrow[d, "\mathrm{Prim}\circ\mathrm{oblv}_\varepsilon"'] & \mathrm{Grp}\left(\varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}}\right) \arrow[d, "\mathrm{Prim}\circ\mathrm{oblv}_\varepsilon\circ\mathrm{oblv}_{\mathrm{Grp}}"] \\
\mathrm{Mod}_{\Bbbk} \arrow[r, "\simeq", "{[-1]}"'] & \mathrm{Mod}_{\Bbbk}.
\end{tikzcd}$$

Then our claim follows from the fact that looping and delooping are one inverse to the other in $\mathrm{Mod}_{\Bbbk}$. ☐

Lemma 3.1.16 states that the ∞-category $\varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}}$, which is the ∞-category of algebras for some monad in virtue of Porism 3.1.14, shares another important feature with the ∞-categories of algebras for the monad defined by an operad $\mathfrak{O}$. Namely, every group object admits an essentially unique delooping.

3.1.17. As already observed (3.1.5), $\mathrm{CE}_\varepsilon : \mathrm{Lie}_{\Bbbk} \longrightarrow \varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}}$ preserves limits and in particular products. Hence, it preserves group objects, and so lifts to an ∞-functor

$$\mathrm{CE}_\varepsilon : \mathrm{Grp}(\mathrm{Lie}_{\Bbbk}) \longrightarrow \mathrm{Grp}\left(\varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}}\right).$$

Moreover, being the Tate realization ∞-functor strongly monoidal when restricted to non-negatively graded $\Bbbk$-modules, it lifts to the ∞-category of cocommutative coalgebras and there it respects products, so we have also an ∞-functor landing in the ∞-category of cocommutative bialgebras

$$|-|^{\mathrm{t}} : \mathrm{Grp}\left(\varepsilon\text{-}\mathrm{CEcAlg}_{\Bbbk}^{\mathrm{gr}}\right) \longrightarrow \mathrm{Grp}\left(\mathrm{cCcAlg}_{\Bbbk//\Bbbk}.\right) \subseteq \mathrm{Alg}\left(\mathrm{cCcAlg}_{\Bbbk//\Bbbk}\right) =: \mathrm{cCBAlg}_{\Bbbk//\Bbbk}. \tag{3.1.18}$$

It is clear that Proposition 2.4.30 implies that the composition

$$|-|^{\mathrm{t}} \circ \mathrm{CE}_\varepsilon : \mathrm{Grp}(\mathrm{Lie}_{\Bbbk}) \longrightarrow \mathrm{Grp}\left(\mathrm{cCcAlg}_{\Bbbk//\Bbbk}\right)$$

is equivalent to the ∞-functor $\mathrm{Grp}\left(\mathrm{Chev}^{\mathrm{enh}}\right)$ of [GR17b, Chapter 6, Section 4.3], which in turn is equivalent to the fully faithful ∞-functor of the universal enveloping algebra with its Hopf structure ([GR17b, Chapter 6, Theorem 6.1.2]). We can restate Conjecture 3.1.7 in the following way.

Conjecture 3.1.19. *The ∞-functor 3.1.18 is fully faithful.*

The advantage of this perspective to the problem is that the Lie structure of any group Lie algebra is canonically trivial ([GR17b, Corollary 1.7.3]). This allows to reduce the proof of the fully faithfulness to trivial Lie algebras, which are sent to mixed graded modules with trivial mixed structure in virtue of Corollary 2.4.31. However, it is not guaranteed that $\mathfrak{L}$ sends mixed graded Chevalley-Eilenberg coalgebras with trivial mixed structure to abelian Lie algebras.

3.2 Representations as mixed graded modules

Analogously to the claim of Conjecture 3.1.7, we can expect representations of Lie algebras to fully faithfully embed into some precise ∞-category of mixed graded $\Bbbk$-modules, assuming a sufficiently well behaviour of our Lie algebra..
For this section, we shall fix a Lie algebra $\mathfrak{g}$, perfect as a $\Bbbk$-module. As already stated in Proposition 2.6.3, the ∞-functor $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \to \varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ factors through the ∞-category of mixed graded $\mathrm{CE}^{\varepsilon}$-modules $\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$. As a purely graded $\Bbbk$-module, $\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M)$ is equivalent to the graded $\Bbbk$-linear dual

$$\mathrm{oblv}_{\varepsilon}\,\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) \simeq \underline{\mathrm{Map}}_{\Bbbk}^{\mathrm{gr}}(\mathrm{oblv}_{\varepsilon}\,\mathrm{CE}_{\varepsilon}(\mathfrak{g}),\, M(0))$$

in virtue of the discussion of 2.6.1. Since $\mathfrak{g}$ is perfect, and $\mathrm{oblv}_{\varepsilon}\,\mathrm{CE}_{\varepsilon}(\mathfrak{g}) \simeq \mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}(\mathrm{oblv}_{\mathrm{Lie}}\,\mathfrak{g}[-1](1))$, we can equivalently write

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, M) \simeq \mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}\big(\mathfrak{g}^{\vee}[1](-1)\big) \otimes_{\Bbbk}^{\mathrm{gr}} M(0).$$

This motivates the following definition.

Definition 3.2.1. We say that a mixed graded $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$-module $M_{\bullet}$ is *constant* if, as a graded $\Bbbk$-module, is equivalent to

$$\mathrm{Sym}_{\Bbbk}^{\mathrm{gr}}\big(\mathfrak{g}^{\vee}[1](1)\big) \otimes_{\Bbbk}^{\mathrm{gr}} M(0)$$

for some $\Bbbk$-module M.

Constant mixed graded $\mathrm{CE}^{\varepsilon}(\mathfrak{g})$-modules are naturally gathered in a full sub-∞-category of $\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$, which we denote by $\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$.

Remark 3.2.2. It is clear, in virtue of the discussion above, that the Chevalley-Eilenberg ∞-functor $\mathrm{CE}^{\varepsilon}\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \to \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$ factors through $\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$.

The main conjecture concerning left $\mathrm{U}(\mathfrak{g})$-modules, for $\mathfrak{g}$ perfect as a $\Bbbk$-module, is the following.

Conjecture 3.2.3. *Under our assumptions, the Chevalley-Eilenberg ∞-functor*

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$$

is fully faithful.

3.2.4. Similarly to the case of Lie algebras and mixed graded cocommutative coalgebras, Proposition 2.6.10 implies that the Chevalley-Eilenberg ∞-functor

$$\mathrm{CE}_{\varepsilon}\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$$

admits both a left and right adjoint if $\mathfrak{g}$ is perfect. We suspect that it is the *right* adjoint, that usually does not exist without the finitenss assumption on $\mathfrak{g}$, that realizes the quasi-inverse of the equivalence. Indeed, if $A_{\bullet}$ is a mixed graded commutative $\Bbbk$-algebra with some size constraint (e.g., it is bounded in non-positive weights), we can always consider both the ∞-category of mixed graded $A_{\bullet}$-modules and the full sub-∞-category $\mathrm{Mod}_{A_{\bullet}}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$ of *constant* mixed graded $A_{\bullet}$-modules, that is the ∞-category spanned by mixed graded $A_{\bullet}$-modules such that the map

$$A_q \otimes_{A_0} M_0 \longrightarrow M_q$$

is an equivalence in each weight. This sub-∞-category is always a colocalization of $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$: it is a presentable ∞-category and, since the tensor product commutes with colimits separably in each variable (and they are computed as in $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$, i.e., weight-wise), it is also closed under colimits. In particular, in our case, the left adjoint should correspond to the colocalization ∞-functor

$$\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big) \longrightarrow \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$$

followed by the inverse equivalence $\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big) \simeq \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}$.

Remark 3.2.5. Given a mixed graded $\Bbbk$-algebra $A_{\bullet}$, the ∞-category of mixed graded $A_{\bullet}$-modules $\mathrm{Mod}_{A_{\bullet}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$ is endowed itself with a left complete Postnikov t-structure, where the connective part is spanned by those mixed graded $A_{\bullet}$-modules which are connective for the mixed graded Postnikov t-structure of Theorem 1.4.1. Similarly, one can define a t-structure on $\mathrm{Mod}_{A_{\bullet}}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$ whose connective part is spanned by those mixed graded $A_{\bullet}$-modules which are connective for the mixed graded Postnikov t-structure. If $\mathfrak{g}$ is perfect *and coconnective as a* $\Bbbk$*-module*, then it is clear that

$$\mathrm{CE}^{\varepsilon}(\mathfrak{g};\, -)\colon \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$$

is a t-exact ∞-functor, where on the source we consider the t-structure described in [Lur11c, Warning 3.5.9]. Indeed, if $\mathfrak{g}$ is coconnective, then its dual $\mathfrak{g}^{\vee}$ is connective and each symmetric power $\mathrm{Sym}_{\Bbbk}^{p}(\mathfrak{g}^{\vee}[1])$ is p-connective. Therefore, given a left $\mathrm{U}(\mathfrak{g})$-module M which is connective as a $\Bbbk$-module, we have

$$\mathrm{CE}^{-p}(\mathfrak{g};\, M) \simeq \mathrm{Sym}_{\Bbbk}^{p}\big(\mathrm{oblv}_{\mathrm{Lie}}\, \mathfrak{g}^{\vee}[1]\big) \otimes_{\Bbbk} M$$

which is again p-connective, since the t-structure on $\Bbbk$-modules is compatible with the monoidal structure ([Lur17, Lemma 7.1.3.10]).

Remark 3.2.6. It is clear that these t-structures on both $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}$ and $\mathrm{Mod}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}^{\mathrm{const}}\big(\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}\big)$

described in Remark 3.2.5 are left complete, because limits and connective objects are detected by the forgetful ∞-functors $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})} \to \mathrm{Mod}_{\Bbbk}$ and $\mathrm{Mod}^{\mathrm{const}}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big) \to \varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}$, and so one can reduce to check the assertion of Proposition 1.5.21 at the level of the underlying $\Bbbk$-modules and mixed graded $\Bbbk$-modules, respectively. In particular, if Conjecture 3.2.3 is true, the t-structures on $\mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}$ and $\mathrm{Mod}^{\mathrm{const}}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big)$ should correspond. This is linked to deformation theory of formal moduli problems. Namely, let X be the formal moduli problem that corresponds to $\mathfrak{g}$ under the equivalence between homotopy Lie algebras and formal moduli problems over fields of characteristic 0 ([Lur11c, Theorem 2.0.2]). In virtue of [Lur11c, Theorem 2.4.1], there exists a monoidal fully faithful embedding

$$\mathrm{QCoh}(X) \longhookrightarrow \mathrm{Rep}_{\mathfrak{g}} \simeq \mathrm{LMod}_{\mathrm{U}(\mathfrak{g})}$$

of the ∞-category of quasi-coherent sheaves over X in the ∞-category of representations of $\mathfrak{g}$, which preserves connective objects. Moreover, $\mathrm{Rep}_{\mathfrak{g}}$ can be described geometrically as the ∞-category of Ind-coherent sheaves $\mathrm{IndCoh}(X)$ ([Lur11c, Theorem 3.5.1]): under this equivalence the t-structure of $\mathrm{Rep}_{\mathfrak{g}}$ corresponds to the canonical t-structure of $\mathrm{IndCoh}(X)$ characterized by the property that the t-structure is compatible with filtered colimits, and the canonical inclusion $\mathrm{Coh}(X) \hookrightarrow \mathrm{IndCoh}(X)$ is t-exact ([Gai13, Section 1.2.1]). By Ind-extending the inclusion $\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X)$, one has an ∞-functor

$$\Phi_X : \mathrm{IndCoh}(X) \longrightarrow \mathrm{QCoh}(X) \tag{3.2.7}$$

which exhibits $\mathrm{QCoh}(X)$ with its t-structure described above as the left completion of the t-structure of $\mathrm{IndCoh}(X)$ ([Gai13, Proposition 1.3.4]). In particular, Conjecture 3.2.3 would allow to recover quasi-coherent modules over the formal moduli problem X associated to a perfect coconnective Lie algebra $\mathfrak{g}$ in the ∞-category $\mathrm{Mod}^{\mathrm{const}}_{\mathrm{CE}^{\varepsilon}(\mathfrak{g})}\big(\varepsilon\text{-}\mathrm{Mod}^{\mathrm{gr}}_{\Bbbk}\big)$.

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