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BLOW-UP AND GLOBAL SOLUTIONS FOR SUBCRITICAL AND CRITICAL PARABOLIC EQUATIONS IN $\mathbb{R}^N$

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Abstract. We study the local well-posedness in the framework of the Sobolev space $H^1(\mathbb{R}^N)$, $N \geq 3$, for a semilinear parabolic equation with asymptotically polynomial nonlinearity up to the critical Sobolev growth. Then we establish the dichotomy between blow-up and global existence for solutions with small energy by means of variational methods and the so-called potential well argument.

1. Introduction

In this paper we consider the asymptotic behavior of solutions of a semilinear parabolic equation in $\mathbb{R}^N$ with asymptotically polynomial nonlinearity up to the critical growth. Note that the orbit of the solution may be noncompact due to the noncompactness of $\mathbb{R}^N$ (and the critical growth of the nonlinearity if this is the case), hence the structure of the omega-limit set of

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the orbit is quite different from that for the problem on bounded domains. More precisely, we consider the semilinear heat equation

$$\begin{cases}\partial_t u = \Delta u - u + f(u) & \text{in } \mathbb{R}^N \times (0,T),\\ u(x,0) = u_0(x) & x \in \mathbb{R}^N,\end{cases} \tag{1.1}$$

where $u_0 \in H^1(\mathbb{R}^N)$, $N \geq 3$, and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) = 0$ and satisfying the following growth conditions:

[label=(f_0)] there exist $C > 0$, $p \in (2, 2^*]$ with $2^* := 2N/(N-2)$, and $q \in (2, p)$ such that

$$|f(s_1) - f(s_2)| \leq C|s_1 - s_2| \left(|s_1|^{p-2} + |s_2|^{p-2} + |s_1|^{q-2} + |s_2|^{q-2}\right) \quad \text{for any } s_1, s_2 \in \mathbb{R};$$

there exists $\mu > 2$ such that $0 < F(s) := \displaystyle\int_0^s f(\tau)\, d\tau \leq \frac{1}{\mu} s f(s)$ for any $s \neq 0$; $\quad f'(s) - \dfrac{f(s)}{s} > 0$ for any $s \neq 0$.

In a similar way as for the pure polynomial nonlinearity, it is possible to prove local existence and uniqueness for (1.1) for any $u_0 \in H^1(\mathbb{R}^N)$.

Theorem 1.1. *Assume* $p \in (2, 2^*)$. *For any* $M > 0$ *there exists* $T = T(M) > 0$ *such that the Cauchy problem* (1.1) *with initial datum* $u_0 \in H^1(\mathbb{R}^N)$, $\|u_0\|_{H^1} \leq M$, *has a unique solution* $u \in \mathcal{C}([0,T], H^1(\mathbb{R}^N))$. *Assume* $p = 2^*$. *For any* $u_0 \in H^1(\mathbb{R}^N)$ *there exists* $T = T(u_0) > 0$ *such that the Cauchy problem* (1.1) *with initial datum* u_0 *has a unique solution* $u \in \mathcal{C}([0,T], H^1(\mathbb{R}^N))$.

Moreover, thanks to the smoothing effect of the heat kernel it is possible to prove that u is a classical solution; in fact, it belongs to the class

$$u \in L^\infty_{loc}((0,T], L^\infty(\mathbb{R}^N)) \cap C^1((0,T), L^2(\mathbb{R}^N)) \cap C^{1,2}((0,T) \times \mathbb{R}^N),$$

see [17], [2]. We define the maximal existence time T_m of the solution u as

$$T_m := \sup\{T > 0 \,:\, \text{the solution } u \text{ to (1.1) satisfies } u \in C([0,T], H^1(\mathbb{R}^N))\}.$$

If $T_m < +\infty$ then the L^∞-norm of the solution blows up, i.e. $\limsup_{t \to T_m^-} \|u(t)\|_{L^\infty} = +\infty$, see e.g. [3]. Moreover, in the subcritical case, $2 < p < 2^*$, if $T_m < +\infty$ then $\limsup_{t \to T_m^-} \|u(t)\|_{H^1} = +\infty$ (see Remark 2.4). Instead, in the critical case, $p = 2^*$, it is not true, in general, that $T_m < +\infty$ yields $\limsup_{t \to T_m^-} \|u(t)\|_{H^1} = +\infty$ (see [19]).

The aim of this paper is to give sufficient conditions to determine whether the solution blows up in finite time or it is global in time. Let

$$I(u) := \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbb{R}^N} F(u)\,dx \quad \text{with } \|u\|_{H^1} := \sqrt{\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2}$$

be the energy functional associated with (1.1), and let

$$d := \inf_{u \in N} I(u)$$

be the potential depth of I, where

$$N := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \ : \ J(u) = 0\} \quad \text{and} \quad J(u) := \|u\|_{H^1}^2 - \int_{\mathbb{R}^N} uf(u)\,dx$$

are the Nehari manifold and the Nehari functional, respectively. We consider the splitting of the d-sublevelset of I determined by the sets:

$$W := \{u \in H^1(\mathbb{R}^N) \ : \ I(u) < d,\ J(u) \geq 0\}, \quad V := \{u \in H^1(\mathbb{R}^N) \ : \ I(u) < d,\ J(u) < 0\}.$$

We prove that if the solution intersects the set V (unstable set) then it remains in V and it blows up in finite time, while if it enters the set W (stable set), it stays in W and it is global in time. The asymptotic analysis of solutions intersecting the unstable set V is not influenced by the value of the power p in the range $(2, 2^*]$ and the following result holds.

Theorem 1.2. *Assume $p \in (2, 2^*]$. Let $u \in C([0,T_m), H^1(\mathbb{R}^N))$ be the solution to* (1.1) *with $u_0 \in H^1(\mathbb{R}^N)$. If $u(t_0) \in V$ for some $t_0 \in [0, T_m)$, then $T_m < \infty$ and $u(t) \in V$ for $t \in [t_0, T_m)$.*

Conversely, the asymptotic analysis of solutions entering the stable set is much easier when the value of the power p is in the subcritical range $(2, 2^*)$. In particular, in order to tackle the critical case $p = 2^*$, we assume an additional growth condition as stated in the following result.

Theorem 1.3. *Let $u \in C([0,T_m), H^1(\mathbb{R}^N))$ be the solution to* (1.1) *with $u_0 \in H^1(\mathbb{R}^N)$. Assume $p \in (2, 2^*)$. If $u(t_0) \in W$ for some $t_0 \in [0, T_m)$, then $T_m = \infty$, $u(t) \in W$ for $t \in [t_0, \infty)$ and $\lim_{t\to\infty} \|u(t)\|_{H^1(\mathbb{R}^N)} = 0$. The same result holds for $p = 2^*$ under the additional condition*

$$(f_4) \qquad \lim_{s\to\pm\infty} \frac{f(s)}{s|s|^{p-2}} = \lim_{s\to\pm\infty} \frac{f(s)}{s|s|^{\frac{4}{N-2}}} = \beta > 0.$$

There is a significant amount of work devoted to the study of the asymptotic behavior of solutions to Cauchy problems for evolution equations.

Particularly, the method of potential-well which gives asymptotics of solutions in terms of V and W as is stated in Theorem 1.2 and Theorem 1.3 was introduced first for abstract equations with Lyapunov functionals, see e.g. [15] for the complete form and see also [20, 9] and references therein. This method was also applied to abstract parabolic equations by Payne and Sattinger [16] and to concrete parabolic equations in bounded domains with the nonlinearity of subcritical polynomial growth in e.g. [7, 8]. The parabolic equation in $\mathbb{R}^N$ with critical Sobolev exponent was treated by [10] in which we have to deal with the noncompactness coming from the unboundedness of $\mathbb{R}^N$ and the criticality of the nonlinearity. More recently, the asymptotically linear case was studied in [5] with similar methods. Potential-well type results for parabolic equations in $\mathbb{R}^2$ with exponential growth nonlinearities are given in [4, 18, 11, 12]. For analogous results in the framework of nonlinear Klein-Gordon equations, we refer to [13, 1, 6] and references therein.

While the case of general subcritical polynomial nonlinearities has been widely studied, up to our knowledge, the critical polynomial case has been analyzed only for the pure power nonlinearity $f(u) = |u|^{2^*-2}u$. The new aspect of our analysis is that it includes more general critical polynomial nonlinearities. An example of nonlinearity f that we consider is $f(s) = |s|^{p-2}s + |s|^{q-2}s$ with $p \in (2, 2^*]$ and $q \in (2, p)$. Even this specific case seems to be new.

For these nonlinearities, since the embedding

$$\mathcal{D}^{1,2}(\mathbb{R}^N) := \{\, u \in L^{2^*}(\mathbb{R}^N) \ : \ |\nabla u| \in L^2(\mathbb{R}^N) \,\} \hookrightarrow L^p(\mathbb{R}^N)$$

holds only if $p = 2^*$, the analysis of the energy functional cannot be performed in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ when the problem is subcritical, i.e. when $p < 2^*$. By adding the linear term $-u$ into the equation, we can unify the study of subcritical and critical problems in the framework of the Sobolev space $H^1(\mathbb{R}^N)$.

As in the statement of the main results above, in what follows we will tacitly assume that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) = 0$ such that (f_1), (f_2), and (f_3) hold. We will only specify the range of the power p when we will need to distinguish between the subcritical and the critical case. We will also emphasize the additional condition (f_4) when we will use it.

The plan of the paper is the following. Section 2 is devoted to the proof of Theorem 1.1. In Section 3 we collect some geometric properties of the unstable set V and the stable set W, and in Section 4 we derive some characteristics of blow-up and global solutions. The results in these Sections will be useful in the asymptotic analysis of solutions to (1.1). The proof of Theorem 1.2 and Theorem 1.3 can be found in Section 5 and 6, respectively.

2. TIME-LOCAL WELL-POSEDNESS OF THE PROBLEM

In this Section we will prove Theorem 1.1. It will follow from the two Propositions 2.2 and 2.3 establishing wellposedness in the subcritical and critical case, respectively. These Propositions hold for more general nonlinearities, namely for any function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and satisfying

(f_1^*) there exist $C > 0$ and $p \in (2, 2^*]$ such that

$$|f(s_1) - f(s_2)| \leq C|s_1 - s_2| \left(|s_1|^{p-2} + |s_2|^{p-2} + 1\right) \quad \text{for any } s_1,\ s_2 \in \mathbb{R}\,.$$

In the proofs of the following Propositions we will use the well-known smoothing effect of the heat kernel (see [2]). We denote by

$$e^{t\Delta}\varphi(x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|\cdot|^2}{4t}} * \varphi(x) \quad \text{for any } \varphi \in L^p(\mathbb{R}^N) \text{ with } 1 \leq p \leq +\infty.$$

Lemma 2.1. *Let $1 \leq p \leq q \leq +\infty$. Then there exists $C > 0$ such that it holds*

$$\|e^{t\Delta}\varphi\|_{L^q} \leq \frac{1}{t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}} \|\varphi\|_{L^p} \tag{2.1}$$

and

$$\|\nabla e^{t\Delta}\varphi\|_{L^q} \leq \frac{C}{t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}} \|\varphi\|_{L^p} \tag{2.2}$$

for all $t > 0$ and all $\varphi \in L^p(\mathbb{R}^N)$. Moreover, if $1 \leq p < q \leq +\infty$ then

$$\lim_{t\to 0} t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \|e^{t\Delta}\varphi\|_{L^q} = 0 \tag{2.3}$$

for all $\varphi \in L^p(\mathbb{R}^N)$.

Proposition 2.2. *Assume $p \in (2, 2^*)$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(0) = 0$ and (f_1^*). For any $M > 0$ there exists $T = T(M) > 0$ such that the Cauchy problem (1.1) with initial datum $\|u_0\|_{H^1} \leq M$ has a unique solution $u \in \mathcal{C}([0, T], H^1(\mathbb{R}^N))$.*

(3) *Proof.* We transform the equation to the integral form

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\Big(f(u(s)) - u(s)\Big)\, ds,$$

and we prove that there exists a unique solution of the integral equation in the space $X_T = \mathcal{C}([0,T], L^{2^*}(\mathbb{R}^N))$, normed with $\|u\|_{X_T} = \sup_{t\in(0,T)} \|u(t)\|_{L^{2^*}}$. Let us introduce the integral operator

$$L(u)(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\Big(f(u(s)) - u(s)\Big)\,ds,$$

and for $u_0 \neq 0$ choose $M = 2\|u_0\|_{L^{2^*}}$. Then we prove that, for $T = T(\|u_0\|_{L^{2^*}})$ small, L has a unique fixed point in the ball

$$B_M = \{u \in X_T :\ \|u\|_{X_T} = \sup_{t\in(0,T)} \|u\|_{L^{2^*}} \leq M\}.$$

Thanks to (f_1^*), the smoothing effect (2.1), and Hölder's inequality, for any $u,\ v \in B_M$ and $t \in (0,T)$, we have

$$\begin{aligned}
\|L(u)(t) - L(v)(t)\|_{L^{2^*}} &= \left\| \int_0^t e^{(t-s)\Delta} \Big[\Big(f(u(s)) - f(v(s)) \Big) + \Big(v(s) - u(s) \Big)\Big]\ ds \right\|_{L^{2^*}} \\
&\leq C \left\| \int_0^t e^{(t-s)\Delta}\ |u(s) - v(s)| \left(|u(s)|^{p-2} + |v(s)|^{p-2} + 1\right)\ ds \right\|_{L^{2^*}} \\
&\leq C \left(\int_0^t (t-s)^{-\beta} \|u(s) - v(s)\|_{L^{2^*}} (\|u(s)\|_{L^{2^*}}^{p-2} + \|v(s)\|_{L^{2^*}}^{p-2})\, ds + T\|u - v\|_{X_T} \right) \\
&\leq C \left(\int_0^t (t-s)^{-\beta}\, ds \|u - v\|_{X_T} (\|u\|_{X_T}^{p-2} + \|v\|_{X_T}^{p-2}) + T\|u - v\|_{X_T} \right) \\
&\leq C(T^{1-\beta}M^{p-2} + T)\|u - v\|_{X_T},
\end{aligned}$$

where $\beta = \frac{N(p-2)}{2\cdot 2^*}$ and $\beta < 1$ for $p < 2^*$. Using again (f_1^*) and $f(0) = 0$, the smoothing effect (2.1) and Hölder's inequality, in a similar way as above, for $u \in B_M$ and $t \in (0,T)$ we get

$$\begin{aligned}
\|L(u)(t)\|_{L^{2^*}} &\leq \|e^{t\Delta}u_0\|_{L^{2^*}} + \left\| \int_0^t e^{(t-s)\Delta}\Big(f(u(s)) - u(s)\Big)\, ds \right\|_{L^{2^*}} \\
&\leq \|u_0\|_{L^{2^*}} + C \left\| \int_0^t e^{(t-s)\Delta} |u(s)| (|u(s)|^{p-2} + 1)\, ds \right\|_{L^{2^*}} \\
&\leq \frac{M}{2} + C \left(\int_0^t (t-s)^{-\beta} \|u(s)\|_{L^{2^*}} \|u(s)\|_{L^{2^*}}^{p-2}\, ds + T\|u\|_{X_T} \right) \\
&\leq \frac{M}{2} + C \left(\|u\|_{X_T}^{p-1} \int_0^t (t-s)^{-\beta}\, ds + T\|u\|_{X_T} \right) \\
&\leq \frac{M}{2} + C(T^{1-\beta}M^{p-2} + T)M.
\end{aligned}$$

Therefore choosing $T = T(M)$ such that

$$C\left(T^{1-\beta}M^{p-2} + T\right) \le \frac{1}{2},$$

we can prove that L maps the space X_T into itself and it is a contraction. Therefore the sequence

$$u^{(0)} = e^{t\Delta}u_0, \quad u^{(j)} = L(u^{(j-1)}) \ \ \forall j \ge 1$$

converges in X_T to a fixed point u which is the unique solution in X_T of the integral equation.

Let us now prove that the solution u belongs to the space $Y_T = \mathcal{C}([0,T], H^1(\mathbb{R}^N))$ for some possibly smaller $T = T(\|u_0\|_{H^1})$. We denote by $\|u\|_{Y_T} = \sup_{t\in(0,T)} \|u(t)\|_{H^1}$ and we prove that there exists some $T = T(\|u_0\|_{H^1})$ such that

$$\|u^{(j)}\|_{Y_T} = \sup_{t\in(0,T)} \|u^{(j)}(t)\|_{H^1} \le 2\|u_0\|_{H^1}. \tag{2.4}$$

Thanks to the inequality (2.4), for any $t \in (0,T)$ there exists a subsequence of $u^{(j)}(t)$ that converges weakly to $u(t)$ in $H^1(\mathbb{R}^N)$ and

$$\|u(t)\|_{H^1} \le \liminf_j \|u^{(j)}(t)\|_{H^1} \le 2\|u_0\|_{H^1}.$$

This implies that $\|u\|_{Y_T} \le 2\|u_0\|_{H^1}$. Next, we prove (2.4). For any $t > 0$

$$\|u^{(0)}(t)\|_{H^1} = \|e^{t\Delta}u_0\|_{H^1} \le \|u_0\|_{H^1} \le 2\|u_0\|_{H^1}.$$

Let us consider $u^{(1)} = L(u^{(0)})$. We have

$$\begin{aligned}
\|u^{(1)}(t)\|_{H^1} &= \|L(u^{(0)}(t))\|_{H^1} \\
&\le \|e^{t\Delta}u_0\|_{H^1} + \left\|\int_0^t e^{(t-s)\Delta}\left(f(u^{(0)}(s)) - u^{(0)}(s)\right)\,ds\right\|_{L^2} + \\
&\quad + \left\|\int_0^t \nabla e^{(t-s)\Delta}\left(f(u^{(0)}(s)) - u^{(0)}(s)\right)\,ds\right\|_{L^2} \\
&= I + II + III.
\end{aligned}$$

For the first term on the right side of the inequality it holds

$$I = \|e^{t\Delta}u_0\|_{H^1} \le \|u_0\|_{H^1}.$$

In view of (f_1^*) and $f(0) = 0$, the smoothing effect (2.1) and Hölder's inequality, by choosing $r \in [2, 2^*]$ such that $1 \le \frac{r}{p-1} < 2$ and $\frac{N}{2}(\frac{p-1}{r} - \frac{1}{2}) < \frac{1}{2}$, we have the following estimate for the second term

$$\begin{aligned} II &= \left\| \int_0^t e^{(t-s)\Delta} \Big(f(u^{(0)}(s)) - u^{(0)}(s) \Big) \, ds \right\|_{L^2} \\ &\le C \int_0^t \left\| e^{(t-s)\Delta} \left(|u^{(0)}(s)|^{p-1} + |u^{(0)}(s)| \right) \right\|_{L^2} ds \\ &\le C \left(\int_0^t (t-s)^{-\frac{N}{2}(\frac{p-1}{r} - \frac{1}{2})} \|u^{(0)}(s)\|_{L^r}^{p-1} + \|u^{(0)}(s)\|_{L^2} \, ds \right) \\ &\le CT^{1-\frac{N}{2}(\frac{p-1}{r}-\frac{1}{2})} \|u_0\|_{H^1}^{p-1} + CT\|u_0\|_{H^1}. \end{aligned}$$

For the third term, using again the assumptions (f_1^*), $f(0)=0$, the gradient estimate for the heat operator (2.2), and Hölder's inequality, we get

$$\begin{aligned} III &= \left\| \int_0^t \nabla e^{(t-s)\Delta} \left(f(u^{(0)}(s)) - u^{(0)}(s) \right) \, ds \right\|_{L^2} \\ &\le C \left(\int_0^t (t-s)^{-\frac{N}{2}(\frac{p-1}{r}-\frac{1}{2})-\frac{1}{2}} \|u^{(0)}(s)\|_{L^r}^{p-1} + \frac{1}{\sqrt{t-s}} \|u^{(0)}(s)\|_{L^2} \, ds \right) \\ &\le CT^{\frac{1}{2}-\frac{N}{2}(\frac{p-1}{r}-\frac{1}{2})} \|u_0\|_{H^1}^{p-1} + CT^{\frac{1}{2}} \|u_0\|_{H^1}. \end{aligned}$$

Therefore we have for any $t \in (0,T)$

$$\|u^{(1)}(t)\|_{H^1} \le \|u_0\|_{H^1} + (\alpha(T, \|u_0\|_{H^1}) + \beta(T, \|u_0\|_{H^1}))\|u_0\|_{H^1},$$

where

$$\alpha(T, \|u_0\|_{H^1}) = CT^{1-\frac{N}{2}(\frac{p-1}{r}-\frac{1}{2})} \|u_0\|_{H^1}^{p-2} + CT,$$

and

$$\beta(T, \|u_0\|_{H^1}) = CT^{\frac{1}{2}-\frac{N}{2}(\frac{p-1}{r}-\frac{1}{2})} \|u_0\|_{H^1}^{p-2} + CT^{\frac{1}{2}}.$$

Choosing $T = T(\|u_0\|_{H^1})$ such that

$$\alpha(T, \|u_0\|_{H^1}) + \beta(T, \|u_0\|_{H^1}) < 1,$$

we obtain

$$\|u^{(1)}\|_{Y_T} = \sup_{t \in (0,T)} \|u^{(1)}(t)\|_{H^1} \le 2\|u_0\|_{H^1}.$$

By a recursive argument we establish the inequality (2.4) for any $u^{(j)}$, so we just proved the $u \in L^\infty((0,T), H^1(\mathbb{R}^N))$. Finally, $u \in Y_T$, since using similar estimates as in the proof of (2.4) it is possible to show that $u(t_n) \to u(t)$ in $H^1(\mathbb{R}^N)$ for any t, $t_n \in (0,T]$ with $t_n \to t$, and $u(t_n) \to u_0$ in $H^1(\mathbb{R}^N)$ as $t_n \to 0$. In the end, uniqueness in $C([0,T], H^1(\mathbb{R}^N))$ is a consequence of the fact that the integral operator L is a contraction on $X_T = C([0,T], L^{2^*}(\mathbb{R}^N))$.

□

Proposition 2.3. *Assume* $p = 2^*$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *satisfy* $f(0) = 0$ *and* (f_1^*). *For any* $u_0 \in H^1(\mathbb{R}^N)$ *there exists* $T = T(u_0)$ *such that the Cauchy problem* (1.1) *with initial datum* u_0 *has a unique solution* $u \in \mathcal{C}([0,T], H^1(\mathbb{R}^N))$.

Proof. In the critical case the strategy to prove existence is still a fixed point argument but in a different space. We consider the Banach space

$$Z_T = \{u \in L^\infty_{loc}((0,T), L^r(\mathbb{R}^N)) \ : \ \|u\|_{Z_T} = \sup_{t \in (0,T)} t^\alpha \|u(t)\|_{L^r} < +\infty\},$$

where $2^* < r < 2^*(2^*-1)$, $\alpha = \frac{N}{2}(\frac{1}{2^*} - \frac{1}{r})$. Let $B_\varepsilon = \{u \in Z_T : \ \|u\|_{Z_T} \le \varepsilon\}$, for $\varepsilon > 0$. Using the property (f_1^*) and $f(0) = 0$, the smoothing effect (2.1) and Hölder's inequality, for any $u \in B_\varepsilon$ we have

$$\begin{aligned}
t^\alpha \|L(u)(t)\|_{L^r} &\le t^\alpha \|e^{t\Delta} u_0\|_{L^r} + t^\alpha \left\| \int_0^t e^{(t-s)\Delta} \Big(f(u(s)) - u(s) \Big) \, ds \right\|_{L^r} \\
&\le t^\alpha \|e^{t\Delta} u_0\|_{L^r} + t^\alpha \left\| \int_0^t e^{(t-s)\Delta} |u(s)|(|u(s)|^{p-2} + 1) \, ds \right\|_{L^r} \\
&\le t^\alpha \|e^{t\Delta} u_0\|_{L^r} + C t^\alpha \int_0^t \Big((t-s)^{-\frac{N}{2}(\frac{2^*-2}{r})} s^{-\alpha(2^*-1)} s^{\alpha(2^*-1)} \|u(s)\|_{L^r}^{2^*-1} + s^{-\alpha} s^{\alpha} \|u(s)\|_{L^r} \Big) \, ds \\
&\le t^\alpha \|e^{t\Delta} u_0\|_{L^r} + C \left(t^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{2^*-2}{r})} s^{-\alpha(2^*-1)} \, ds \right) \|u\|_{Z_T}^{2^*-1} + CT \|u\|_{Z_T} \\
&\le \sup_{t \in (0,T)} t^\alpha \|e^{t\Delta} u_0\|_{L^r} + C \left(\|u\|_{Z_T}^{2^*-1} + T \|u\|_{Z_T} \right) \\
&\le \sup_{t \in (0,T)} t^\alpha \|e^{t\Delta} u_0\|_{L^r} + C \left(\varepsilon^{2^*-1} + T\varepsilon \right),
\end{aligned}$$

$\blacksquare$

and in a similar way for any $u,\ v \in B_\varepsilon$ we have (thanks to (f_1^*), the smoothing effect (2.1), and Hölder's inequality):

$$
\begin{aligned}
&t^\alpha \|L(u)(t) - L(v(t))\|_{L^r} \le t^\alpha \left\| \int_0^t e^{(t-s)\Delta} \Big(f(u(s)) - f(v(s)) + v(s) - u(s) \Big)\, ds \right\|_{L^r} \\
&\le C t^\alpha \left\| \int_0^t e^{(t-s)\Delta} |u(s) - v(s)| \left(|v(s)|^{p-2} + |u(s)|^{p-2} + 1 \right)\, ds \right\|_{L^r} \\
&\le C t^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{2^*-2}{r})} \|u(s) - v(s)\|_{L^r} \left(\|u(s)\|_{L^r}^{2^*-2} + \|v(s)\|_{L^r}^{2^*-2} \right) + \|u(s) - v(s)\|_{L^r}\, ds \\
&\le C \left(t^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{2^*-2}{r})} s^{-\alpha(2^*-1)}\, ds\ \|u - v\|_{Z_T} \left(\|u\|_{Z_T}^{2^*-2} + \|v\|_{Z_T}^{2^*-2} \right) + T \|u - v\|_{Z_T} \right) \\
&\le C \left(\|u\|_{Z_T}^{2^*-2} + \|v\|_{Z_T}^{2^*-2} + T \right) \|u - v\|_{Z_T} \\
&\le C \left(2\varepsilon^{2^*-2} + T \right) \|u - v\|_{Z_T},
\end{aligned}
$$

since $\alpha - \frac{N}{2}\left(\frac{2^*-2}{r}\right) - \alpha(2^*-1) + 1 = 0$. Choosing $\varepsilon > 0$ such that $C2\varepsilon^{2^*-2} \le \frac{1}{4}$, and $T > 0$ such that $CT \le \frac{1}{4}$, and possibly choosing T even smaller so that (2.3) enables us to control the size of $\sup_{t \in (0,T)} t^\alpha \|e^{t\Delta} u_0\|_{L^r} \le \frac{\varepsilon}{2}$, we get that L maps the ball B_ε into itself and it is a contraction. Therefore there exists a unique solution of the integral equation in B_ε.

Moreover, as for the subcritical case since $u_0 \in H^1(\mathbb{R}^N)$, this solution belongs also, for some possibly smaller T, to the space $\mathcal{C}([0,T], H^1(\mathbb{R}^N))$. Indeed, thanks to the previous estimates the sequence

$$
u^{(0)} = e^{t\Delta} u_0, \quad u^{(j)} = L(u^{(j-1)}) \ \ \forall j \ge 1
$$

converges in Z_T to a fixed point u which is the unique solution in Z_T to the integral equation. Then we can prove that there exists some possibly smaller T such that

$$
\sup_{t \in (0,T)} \|u^{(j)}(t)\|_{H^1} \le 2\|u_0\|_{H^1},
$$

and we can conclude as in the subcritical case that the solution u belongs to $\mathcal{C}([0,T], H^1(\mathbb{R}^N))$. For $u^{(0)} = e^{t\Delta} u_0$ since $u_0 \in H^1(\mathbb{R}^N)$ we have $\|u^{(0)}(t)\|_{H^1} \le 2\|u_0\|_{H^1}$ for any positive t. Let us consider $u^{(1)} = L(u^{(0)})$.

We have

$$\begin{aligned}\|u^{(1)}(t)\|_{H^1} &= \|L(u^{(0)}(t))\|_{H^1}\\ &\le \|e^{t\Delta}u_0\|_{H^1} + C\left\|\int_0^t e^{(t-s)\Delta}\Big(f(u^{(0)}(s)) - u^{(0)}(s)\Big)\,ds\right\|_{L^2} +\\ &+ C\left\|\int_0^t \nabla e^{(t-s)\Delta}\Big(f(u^{(0)}(s)) - u^{(0)}(s))\Big)\,ds\right\|_{L^2}\\ &= I + II + III.\end{aligned}$$

For the first term I it holds

$$I = \|e^{t\Delta}u_0\|_{H^1} \le \|u_0\|_{H^1}.$$

For the second, by using the property that $u^{(0)} \in Z_T$ we have

$$\begin{aligned}II &= C\left\|\int_0^t e^{(t-s)\Delta}\Big(f(u^{(0)}(s)) - u^{(0)}(s)\Big)\,ds\right\|_{L^2}\\ &\le C\int_0^t \left\|e^{(t-s)\Delta}|u^{(0)}(s)|(|u^{(0)}(s)|^{2^*-2}+1)\right\|_{L^2}\,ds\\ &\le C\left(\int_0^t (t-s)^{-\frac{N}{2}(\frac{2^*-1}{r}-\frac{1}{2})}\|u^{(0)}(s)\|_{L^r}^{2^*-1} + \|u^{(0)}(s)\|_{L^2}\,ds\right)\\ &\le CT^{\frac{1}{2}}\|u^{(0)}\|_{Z_T}^{2^*-1} + CT\|u_0\|_{L^2},\end{aligned}$$

since $\frac{N}{2}(\frac{2^*-1}{r}-\frac{1}{2}) < 1$, $\alpha(2^*-1) < 1$, $-\frac{N}{2}(\frac{2^*-1}{r}-\frac{1}{2}) - \alpha(2^*-1) + 1 = \frac{1}{2}$ and $\|u^{(0)}(s)\|_{L^2} = \|e^{s\Delta}u_0\|_{L^2} \le \|u_0\|_{L^2}$ for any $s \in (0,T)$. Finally, for the third term we get

$$\begin{aligned}III &= \left\|\int_0^t \nabla e^{(t-s)\Delta}\Big(f(u^{(0)}(s)) - u^{(0)}(s)\Big)\,ds\right\|_{L^2}\\ &\le C\left(\int_0^t (t-s)^{-\frac{N}{2}(\frac{2^*-1}{r}-\frac{1}{2})-\frac{1}{2}}\|u^{(0)}(s)\|_{L^r}^{2^*-1} + \frac{1}{\sqrt{t-s}}\|u^{(0)}(s)\|_{L^2}\,ds\right)\\ &\le C\|u^{(0)}\|_{Z_T}^{2^*-1} + CT^{\frac{1}{2}}\|u_0\|_{L^2},\end{aligned}$$

since $-\frac{N}{2}(\frac{2^*-1}{r}-\frac{1}{2}) - \alpha(2^*-1) + \frac{1}{2} = 0$. Therefore since $\|u^{(0)}(t)\|_{H^1} \le 2\|u_0\|_{H^1}$, for any $t > 0$, and $u^{(0)} = e^{t\Delta}u_0 \in B_\varepsilon$ (in view of our choice of T), we have for any $t \in (0,T)$

$$\begin{aligned}\|u^{(1)}(t)\|_{H^1} &\le \|u_0\|_{H^1} + C(T+T^{1/2})\|u_0\|_{L^2} + C(1+T^{\frac{1}{2}})\|u^{(0)}\|_{Z_T}^{2^*-1}\\ &\le \|u_0\|_{H^1} + C(T+T^{1/2})\|u_0\|_{L^2} + C(1+T^{\frac{1}{2}})\varepsilon^{2^*-1}.\end{aligned}$$

Choosing $T = T(u_0)$ small enough such that

$$C(T + T^{1/2})\|u_0\|_{L^2} + C(1 + T^{\frac{1}{2}})\varepsilon^{2^*-1} \le \|u_0\|_{H^1},$$

we get the desired inequality. By a recursive argument we establish the inequality (2.4) for any $u^{(j)}$. Finally, in a similar way as in [2] it is possible to prove the uniqueness of the solution in $X_T = C([0,T], L^{2^*}(\mathbb{R}^N))$ and so in $C([0,T], H^1(\mathbb{R}^N))$. □

Remark 2.4. We remark that in the sucritical case $p \in (2, 2^*)$ the existence time of the solution depends only on the size of the initial data in $H^1(\mathbb{R}^N)$. Therefore if $T_m < +\infty$ then $\lim_{t\to T_m} \|u(t)\|_{H^1} = +\infty$. We argue by contradiction. Let us assume that

$$\liminf_{t\to T_m} \|u(t)\|_{H^1} < +\infty.$$

Then there exists a sequence $t_n \to T_m$ such that the set of values $\|u(t_n)\|_{H^1}$ is bounded by a positive constant K. Therefore there exists some time $T = T(K)$ and a solution u on $[t_n, T + t_n]$ for any $u(t_n)$, as initial data. Thus if t_n is close to T_m the solution can be continued beyond T_m but this is impossible.

3. PROPERTIES OF THE SETS V AND W

Lemma 3.1. *The Nehari functional J is positive around the origin in $H^1(\mathbb{R}^N)$, and more precisely there exists $\varrho > 0$ such that*

$$J(u) > 0 \quad \text{for any } u \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ with } \|u\|_{H^1} \le \varrho. \tag{3.1}$$

Proof. Combining 1, 2, and the Sobolev inequality, we get

$$\int_{\mathbb{R}^N} uf(u)\,dx \le C\left(\|u\|_{L^p}^p + \|u\|_{L^q}^q\right) \le C\left(\frac{1}{S_p^{p/2}}\|u\|_{H^1}^p + \frac{1}{S_q^{q/2}}\|u\|_{H^1}^q\right) \quad \text{for any } u \in H^1(\mathbb{R}^N),$$

where

$$S_r := \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{H^1}^2}{\|u\|_{L^r}^2}, \quad r \in [2, 2^*].$$

Hence

$$J(u) \ge \|u\|_{H^1}^2 \left(1 - \frac{C}{S_p^{p/2}}\|u\|_{H^1}^{p-2} - \frac{C}{S_q^{q/2}}\|u\|_{H^1}^{q-2}\right) \quad \text{for any } u \in H^1(\mathbb{R}^N),$$

and since by assumption $p, q > 2$, we easily deduce that (3.1) holds, provided $\varrho > 0$ is sufficiently small. □

We point out that 2 implies that for any $s_0 > 0$ there exists $C_0 > 0$ such that

$$F(s) \geq C_0 |s|^\mu \quad \text{for any } |s| > s_0. \tag{3.2}$$

It is not difficult to see that the Nehari manifold N is *not* empty. In fact, let $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ be such that

$$|u| > 1 \quad \text{on } B_R$$

for some $R > 0$. Then, for any $\lambda \geq 1$, we have

$$\int_{\mathbb{R}^N} \lambda u f(\lambda u)\, dx \geq \int_{B_R} \lambda u f(\lambda u)\, dx \geq \mu \int_{B_R} F(\lambda u)\, dx \geq \mu C_0 \lambda^\mu \|u\|^\mu_{L^\mu(B_R)},$$

where we applied 2 and (3.2). In particular, for any $\lambda \geq 1$

$$J(\lambda u) \leq \lambda^\mu \left(\lambda^{2-\mu} \|u\|^2_{H^1} - \mu C_0 \|u\|^\mu_{L^\mu(B_R)} \right),$$

and we deduce that $J(\lambda u) < 0$ if $\lambda > 1$ is sufficiently large. On the other hand, Lemma 3.1 implies that $J(\lambda u) > 0$ if $\lambda > 0$ is sufficiently small, and by continuity, we find $\overline{\lambda} > 0$ such that $J(\overline{\lambda} u) = 0$.

Lemma 3.2. *The level d is* strictly *positive, i.e.* $d > 0$.

Proof. Let $\{u_k\}_k$ be a minimizing sequence for d, i.e. $u_k \in H^1(\mathbb{R}^N) \setminus \{0\}$, $J(u_k) = 0$ for any $k \geq 1$, and

$$\lim_{k \to +\infty} I(u_k) = d.$$

We have

$$I(u_k) = I(u_k) - \frac{1}{2} J(u_k) = \int_{\mathbb{R}^N} \left(\frac{1}{2} u_k f(u_k) - F(u_k) \right) dx \geq \left(\frac{\mu}{2} - 1 \right) \int_{\mathbb{R}^N} F(u_k)\, dx > 0, \tag{3.3}$$

and hence $d \geq 0$. We argue by contradiction assuming that $d = 0$. Then (3.3) yields

$$\lim_{k \to +\infty} \int_{\mathbb{R}^N} F(u_k)\, dx = 0.$$

Therefore

$$\lim_{k \to +\infty} \|u_k\|^2_{H^1} = \lim_{k \to +\infty} \left(2I(u_k) + 2 \int_{\mathbb{R}^N} F(u_k)\, dx \right) = 0,$$

and the positivity of J around the origin of $H^1(\mathbb{R}^N)$ expressed by Lemma 3.1 yields $J(u_k) > 0$ for any k sufficiently large, which is a contradiction. □

The following characterization of the level d will be crucial in the proof of the instability of the set V.

Lemma 3.3. *The level d can be characterized as*

$$d = \inf \left\{ H(u) \,:\, u \in H^1(\mathbb{R}^N) \setminus \{0\},\ J(u) \le 0 \right\}, \tag{3.4}$$

where

$$H(u) := I(u) - \frac{1}{2}J(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2}uf(u) - F(u) \right) dx \quad \textit{for any } u \in H^1(\mathbb{R}^N). \tag{3.5}$$

In particular, for any $\varepsilon > 0$

$$\inf \left\{ I(u) \,:\, u \in H^1(\mathbb{R}^N) \setminus \{0\},\ J(u) = -\varepsilon \right\} \ge d - \frac{\varepsilon}{2}. \tag{3.6}$$

Proof. To simplify notations, we denote by d' the infimum on the right hand side of (3.4). Clearly $d' \le d$ and, in order to prove that $d \le d'$, it is enough to show that

$$d \le H(u) \quad \text{for any } u \in H^1(\mathbb{R}^N) \text{ with } J(u) < 0. \tag{3.7}$$

To this aim, first we point out that if $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ then the function

$$h(\lambda) := H(\lambda u), \quad \lambda > 0,$$

is monotone increasing on $(0, +\infty)$. In fact, as a consequence of 3, we have

$$h'(\lambda) = \langle dH(\lambda u), u \rangle = \int_{\mathbb{R}^N} \frac{1}{2}\lambda u^2 \left(f'(\lambda u) - \frac{f(\lambda u)}{\lambda u} \right) dx > 0.$$

Next, let $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ be such that $J(u) < 0$. In view of the continuity of J in $H^1(\mathbb{R}^N)$ and its positivity near the origin of $H^1(\mathbb{R}^N)$ (see Lemma 3.1), there exists $\lambda \in (0,1)$ such that $J(\lambda u) = 0$. Hence

$$d \le I(\lambda u) = H(\lambda u) \le H(u),$$

and the proof of (3.7) is complete.

Finally, (3.6) easily follows from (3.4) by direct computations. □

In the following Lemma, we collect some useful properties to describe the sets V and W in the energy space $H^1(\mathbb{R}^N)$.

Lemma 3.4. *The following properties of the sets V and W hold.*

[label=(), leftmargin=+0.9cm] W contains a neighborhood of the origin in $H^1(\mathbb{R}^N)$. The energy I is positive on $W \setminus \{0\}$ and W is bounded in $H^1(\mathbb{R}^N)$. More precisely,

$$\|u\|_{H^1}^2 \le \frac{2\mu}{\mu - 2} d \quad \textit{for any } u \in W. \tag{3.8}$$

$0 \notin \overline{V}$. $\overline{W} \cap \overline{V} = \{u \in N \,:\, I(u) = d\}$.

(**3**) *Proof. 1* First, note that

$$W = \{u \in H^1(\mathbb{R}^N) \,:\, I(u) < d,\ J(u) > 0\} \cup \{0\}.$$

Moreover, the energy I is continuous in $H^1(\mathbb{R}^N)$ and $d > 0$ (see Lemma 3.2). Therefore, in order to conclude that W is a neighborhood of the origin in $H^1(\mathbb{R}^N)$, it is enough to recall that J is positive around the origin in $H^1(\mathbb{R}^N)$ (see Lemma 3.1).

2 If $u \in W \setminus \{0\}$ then $J(u) > 0$, i.e.

$$\int_{\mathbb{R}^N} uf(u)\,dx < \|u\|^2_{H^1},$$

and in view of 2 we get

$$\int_{\mathbb{R}^N} F(u)\,dx < \frac{1}{\mu}\|u\|^2_{H^1}.$$

Therefore

$$I(u) > \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2_{H^1} \quad \text{for any } u \in W \setminus \{0\},$$

and since $\mu > 2$, we deduce that I is positive on $W \setminus \{0\}$. Moreover, recalling that if $u \in W$ we also have $I(u) < d$, the above inequality yields (3.8).

3 Since W is a neighborhood of the origin in $H^1(\mathbb{R}^N)$ and $W \cap V = \varnothing$, if we assume $0 \in \overline{V}$ then we easily reach a contradiction. Hence $0 \notin \overline{V}$.

4 If $u \in \overline{W} \cap \overline{V}$ then $J(u) = 0$ and $I(u) \leq d$. Since $0 \notin \overline{V}$, in particular $u \neq 0$, and hence $u \in N$. Moreover, from the definition of d, we deduce that $I(u) = d$. On the other hand, if $u \in N$ and $I(u) = d$ then $u \neq 0$ and $J(u) = 0$. Therefore, $u \in \overline{W} \cap \overline{V}$. □

4. BLOW-UP AND GLOBAL SOLUTIONS

Let us fix $u_0 \in H^1(\mathbb{R}^N)$, and let $u \in \mathcal{C}\left([0, T_m); H^1(\mathbb{R}^N)\right)$ be the maximal solution to (1.1). It is well known that the energy is decreasing along the solution $u(t)$, more precisely

$$\|\partial_t u(t)\|^2_{L^2} = -\frac{d}{dt} I\left(u(t)\right) \quad \text{for any } t \in (0, T_m). \tag{4.1}$$

Moreover, the following identities hold:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} = -J\left(u(t)\right) \quad \text{for any } t \in (0, T_m), \tag{4.2}$$

and if $\varphi \in H^1(\mathbb{R}^N)$ then

$$\left|\left\langle dI\left(u(t)\right), \varphi\right\rangle\right| \leq \|\partial_t u(t)\|_{L^2}\|\varphi\|_{L^2} \quad \text{for any } t \in (0, T_m). \tag{4.3}$$

In view of the concavity method developed by Levine in [14], it is also well known that if the energy of the solution becomes negative then the solution blows up in finite time. More precisely,

Lemma 4.1 ([14])**.** *If* $u(t_0) \neq 0$ *and* $I\left(u(t_0)\right) \leq 0$ *at some* $t_0 \in [0, T_m)$ *then* $T_m < +\infty$.

We omit the proof, since it can be obtained by well known arguments.

Lemma 4.2. *If* $T_m = +\infty$ *then*

$$\exists \lim_{t \to +\infty} I\left(u(t)\right) \in [0, +\infty), \tag{4.4}$$

and there exists a sequence $\{t_k\}_k$ *satisfying*

$$\lim_{k \to +\infty} t_k = +\infty \qquad \textit{and} \qquad \lim_{k \to +\infty} J\left(u(t_k)\right) = 0.$$

Proof. If $T_m = +\infty$ then Lemma 4.1 implies $I\left(u(t)\right) > 0$ for any $t \in [0, +\infty)$. Therefore from the monotonicity of the energy (4.1), we deduce that (4.4) holds, and

$$\limsup_{t \to +\infty} \frac{d}{dt} I\left(u(t)\right) = 0.$$

This ensures the existence of a sequence $\{t_k\}_k$ such that

$$\lim_{k \to +\infty} t_k = +\infty \quad \text{and} \quad \lim_{k \to +\infty} \|\partial_t u(t_k)\|_{L^2} = 0,$$

where the second identity follows again from (4.1).

Let

$$\varepsilon_k := \|\partial_t u(t_k)\|_{L^2} \to 0 \quad \text{as } k \to +\infty,$$

so that we can rewrite (4.3) as

$$|J\left(u(t_k)\right)| \leq \varepsilon_k \|u(t_k)\|_{H^1} \quad \text{for any } k \geq 1, \tag{4.5}$$

and the proof is complete if we show that $\{u(t_k)\}_k$ is bounded in $H^1(\mathbb{R}^N)$.

Since $\{I\left(u(t_k)\right)\}_k$ is bounded, there exists $C > 0$ such that for any $k \geq 1$

$$\frac{1}{2}\|u(t_k)\|_{H^1}^2 \leq C + \int_{\mathbb{R}^N} F\left(u(t_k)\right)\, dx.$$

Moreover, combining 2 with (4.5), we get for any $k \geq 1$

$$\int_{\mathbb{R}^N} F\left(u(t_k)\right)\, dx \leq \frac{1}{\mu}\int_{\mathbb{R}^N} u(t_k) f\left(u(t_k)\right)\, dx < \frac{1}{\mu}\|u(t_k)\|_{H^1}^2 + \varepsilon_k \|u(t_k)\|_{H^1}.$$

In conclusion, we have for any $k \geq 1$

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u(t_k)\|_{H^1}^2 \leq C + \varepsilon_k \|u(t_k)\|_{H^1},$$

and since $\mu > 2$, the sequence $\{u(t_k)\}_k$ must be bounded in $H^1(\mathbb{R}^N)$. □

5. Invariance and instability of the set V

In this section we prove Theorem 1.2, namely if $u(t_0) \in V$ at some $t_0 \in [0, T_m)$ then $u(t) \in V$ for any $t \in [t_0, T_m)$ and $T_m < +\infty$. Since the space variable ranges in the whole of $\mathbb{R}^N$, the argument developed by Payne and Sattinger in [16] for *bounded* domains does *not* work in our setting. However a similar conclusion holds.

Proof of Theorem 1.2. It is enough to prove the existence of $\varepsilon > 0$ such that

$$J(u(t)) < -\varepsilon \quad \text{for any } t \in [t_0, T_m). \tag{5.1}$$

In fact, combining (5.1) with the monotonicity of the energy (4.1), we can conclude that the set V is invariant under the flow associated with (1.1), i.e. $u(t) \in V$ for any $t \in [t_0, T_m)$. Moreover, if we argue by contradiction assuming that $T_m = +\infty$ then Lemma 4.2 ensures the existence of a sequence $\{t_k\}_k$ such that

$$\lim_{k\to+\infty} t_k = +\infty \quad \text{and} \quad \lim_{k\to+\infty} J(u(t_k)) = 0,$$

which contradicts (5.1).

In order to prove (5.1), let $\varepsilon > 0$ be such that

$$\min\{d - I(u(t_0)), -J(u(t_0))\} > \varepsilon,$$

and let

$$d_\varepsilon := \inf\left\{I(u) \ : \ u \in H^1(\mathbb{R}^N)\setminus\{0\},\ J(u) = -\varepsilon\right\}.$$

Then (3.6) yields

$$d_\varepsilon \geq d - \frac{\varepsilon}{2} > I(u(t_0)),$$

and using the monotonicity of the energy, we get

$$d_\varepsilon > I(u(t)) \quad \text{for any } t \in [t_0, T_m). \tag{5.2}$$

Assume that $J(u(t_1)) = -\varepsilon$ at some $t_1 \in (t_0, T_m)$, then

$$d_\varepsilon \leq I(u(t_1)),$$

which contradicts (5.2), and in view of the continuity of J along the solution, we deduce that (5.1) holds. □

6. INVARIANCE AND STABILITY OF THE SET W

In this section we prove Theorem 1.3 and we split its proof as follows. First, in Proposition 6.1 we establish the invariance of the set W both in the subcritical and critical case. Next, with Proposition 6.3 we complete the proof of Theorem 1.3 in the subcritical case. The proof of Theorem 1.3 in the critical case is more involved and the additional assumption (f_4) comes into play:

- Proposition 6.6 ensures that solutions entering W are global and bounded in $L^\infty((\tau,+\infty)\times\mathbb{R}^N)$ for any $\tau>0$ and its proof is based on a comparison with a suitable limiting stationary problem (see Proposition 6.5);
- finally Proposition 6.8 guarantees the vanishing of the H^1-norm of solutions entering W.

Proposition 6.1. *If $u(t_0)\in W$ at some $t_0\in[0,T_m)$ then $u(t)\in W$ for any $t\in[t_0,T_m)$.*

Proof. If $u(t_0)\equiv 0$ then by uniqueness $u(t)\equiv 0$ for any $t\geq t_0$, and the proof is complete. Therefore, from now on, we will assume that $u(t_0)\in W\setminus\{0\}$. By the monotonicity of the energy,

$$I\left(u(t)\right)\leq I\left(u(t_0)\right)<d \quad \text{for any } t\in[t_0,T_m), \tag{6.1}$$

and the proof is complete if we show that $J\left(u(t)\right)\geq 0$ for any $t\in(t_0,T_m]$.

If not then there exists $t_1\in(t_0,T_m)$ such that

$$J\left(u(t_1)\right)<0. \tag{6.2}$$

By continuity, and recalling that $J\left(u(t_0)\right)>0$, there exists $t_2\in(t_0,t_1)$ such that $J\left(u(t_2)\right)=0$, and either $u(t_2)\equiv 0$ or $d\leq I\left(u(t_2)\right)$. The latter case contradicts (6.1). Therefore, $u(t_2)\equiv 0$ and again by uniqueness $u(t)\equiv 0$ for any $t\geq t_2$ which contradicts (6.2). □

6.1. Stability of W in the energy subcritical case.

Next, we consider (1.1) in the *energy subcritical* case, i.e. $p\in(2,2^*)$. In this case, in order to analyze the asymptotic behavior of solutions in W, we will use the following convergence result.

Lemma 6.2. *Assume $p\in(2,2^*)$. If $\{u_k\}_k\subset W$ is such that*

$$i)\qquad \lim_{k\to+\infty} I(u_k)=c\in[0,d).$$

$$ii)\qquad \lim_{k\to+\infty} J(u_k)=0,$$

then up to a subsequence

$$\lim_{k\to+\infty} \|u_k\|_{H^1} = 0 \quad \textit{and} \quad c = 0.$$

Proof. If we show that up to a subsequence

$$\lim_{k\to+\infty} \int_{\mathbb{R}^N} u_k f(u_k)\,dx = 0 \quad \text{and} \quad \lim_{k\to+\infty} \int_{\mathbb{R}^N} F(u_k)\,dx = 0, \tag{6.3}$$

then

$$\lim_{k\to+\infty} \|u_k\|^2_{H^1} = \lim_{k\to+\infty} \left(J(u_k) + \int_{\mathbb{R}^N} u_k f(u_k)\,dx \right) = 0,$$

from which we deduce that

$$c = \lim_{k\to+\infty} \left(\frac{1}{2}\|u_k\|^2_{H^1} - \int_{\mathbb{R}^N} F(u_k)\,dx \right) = 0,$$

and the proof is complete.

From Lemma 3.4.*2*, we know that the sequence $\{u_k\}_k$ is bounded in $H^1(\mathbb{R}^N)$. Since Schwarz symmetrization does not increase the H^1-norm, also the symmetrized sequence $\{u_k^*\}_k$ is bounded in $H^1_{rad}(\mathbb{R}^N)$. Hence, up to a subsequence, we have

$$u_k^* \rightharpoonup V \text{ in } H^1(\mathbb{R}^N), \quad u_k^* \to V \text{ in } L^p(\mathbb{R}^N), \quad u_k^* \to V \text{ in } L^q(\mathbb{R}^N), \quad \text{and} \quad u_k^* \to V \text{ a.e. in } \mathbb{R}^N,$$

where the strong convergence in $L^p(\mathbb{R}^N)$ and in $L^q(\mathbb{R}^N)$ follows by the assumption on p and q, indeed the embedding $H^1_{rad}(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ is *compact* if $r \in (2, 2^*)$. Therefore, the Lebesgue dominated convergence theorem ensures that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^N} u_k^* f(u_k^*)\,dx = \int_{\mathbb{R}^N} V f(V)\,dx \quad \text{and} \quad \lim_{k\to+\infty} \int_{\mathbb{R}^N} F(u_k^*)\,dx = \int_{\mathbb{R}^N} F(V)\,dx.$$

In particular, we have

$$\lim_{k\to+\infty} H(u_k^*) = H(V),$$

where H is the functional defined by (3.5).

It is easy to see that $J(V) \le 0$. In fact, Schwarz symmetrization preserves the nonlinear parts and hence $J(u_k^*) \le J(u_k)$, therefore

$$J(V) \le \liminf_{k\to+\infty} J(u_k^*) \le \liminf_{k\to+\infty} J(u_k) = 0.$$

Therefore, $V = 0$. If not then

$$d \le H(V) = \lim_{k\to+\infty} H(u_k^*) = \lim_{k\to+\infty} H(u_k) = \lim_{k\to+\infty} \left(I(u_k) - \frac{1}{2} J(u_k) \right) < d,$$

and we reach a contradiction.

In conclusion,

$$\lim_{k\to+\infty}\int_{\mathbb{R}^N} u_k^* f(u_k^*)\,dx = 0 \quad \text{and} \quad \lim_{k\to+\infty}\int_{\mathbb{R}^N} F(u_k^*)\,dx = 0,$$

and this yields (6.3). □

Proposition 6.3. *Assume* $p \in (2, 2^*)$. *If* $u(t_0) \in W$ *at some* $t_0 \in [0, T_m)$ *then* $T_m = +\infty$ *and*

$$\lim_{t\to+\infty} \|u(t)\|_{H^1} = 0.$$

Proof. Without loss of generality, we assume that $u(t_0) \in W\backslash\{0\}$. Since W is invariant under the flow associated with (1.1), $u(t) \in W$ for any $t \in [t_0, T_m)$. In particular,

$$I\left(u(t)\right) \leq I\left(u(t_0)\right) < d \quad \text{and} \quad J\left(u(t)\right) \geq 0 \quad \text{for any } t \in [t_0, T_m), \tag{6.4}$$

and recalling that W is bounded in $H^1(\mathbb{R}^N)$ (see Lemma 3.4.*2*), we also have

$$\|u(t)\|_{H^1} \leq C \quad \text{for any } t \in [t_0, T_m). \tag{6.5}$$

Since $p \in (2, 2^*)$, this is enough to conclude that $T_m = +\infty$. Therefore, Lemma 4.2 ensures the existence of a sequence $\{t_k\}_k$ such that

$$\lim_{k\to+\infty} t_k = +\infty, \quad \lim_{k\to+\infty} I\left(u(t_k)\right) = c \in [0, d), \quad \lim_{k\to+\infty} J\left(u(t_k)\right) = 0,$$

in particular the fact that $c < d$ follows from the first inequality in (6.4). Moreover, $\{u(t_k)\}_k \subset W$ for any k sufficiently large, and hence we can apply the convergence Lemma 6.2 to conclude that up to a subsequence

$$\lim_{k\to+\infty} \|u(t_k)\|_{H^1} = 0 \quad \text{and} \quad c = 0.$$

In view of (4.2) and the second inequality in (6.4), the function $t \in [t_0, +\infty) \mapsto \|u(t)\|_{L^2}$ is monotone non-increasing, and hence

$$\exists \lim_{t\to+\infty} \|u(t)\|_{L^2}.$$

Since $\|u(t_k)\|_{L^2} \to 0$ as $k \to +\infty$, we deduce that

$$\lim_{t\to+\infty} \|u(t)\|_{L^2} = 0.$$

Applying the Gagliardo-Nirenberg inequality and (6.5), we can estimate for any $t \geq t_0$

$$\|u(t)\|_{L^p}^p \leq C_{p,N} \|\nabla u(t)\|_{L^2}^{N\left(\frac{p}{2}-1\right)} \|u(t)\|_{L^2}^{pN\left(\frac{1}{p}-\frac{1}{2^*}\right)} \leq \tilde{C}_{p,N} \|u(t)\|_{L^2}^{pN\left(\frac{1}{p}-\frac{1}{2^*}\right)}, \tag{6.6}$$

and hence

$$\lim_{t\to+\infty} \|u(t)\|_{L^p} = 0.$$

Therefore

$$\lim_{t\to+\infty} \int_{\mathbb{R}^N} F\left(u(t)\right)\, dx = 0$$

and

$$\lim_{t\to+\infty} \|u(t)\|_{H^1}^2 = \lim_{t\to+\infty} \left(2I\left(u(t)\right) + 2\int_{\mathbb{R}^N} F\left(u(t)\right)\, dx\right) = 2c = 0.$$

□

6.2. **Stability of W in the energy critical case.** In the *energy critical case* $p = 2^*$, we assume the additional growth condition (f_4), namely

$$\lim_{s\to\pm\infty} \frac{f(s)}{s|s|^{p-2}} = \lim_{s\to\pm\infty} \frac{f(s)}{s|s|^{\frac{4}{N-2}}} = \beta > 0.$$

The limiting (homogeneous) stationary equation

$$-\Delta U = \beta U|U|^{\frac{4}{N-2}} \quad \text{in } \mathbb{R}^N \tag{6.7}$$

will play a role in the study of the stability of the W, and we denote by

$$I_\infty(U) := \frac{1}{2}\|\nabla U\|_{L^2}^2 - \frac{\beta}{2^*}\|U\|_{L^{2^*}}^{2^*}, \qquad U \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{6.8}$$

the energy functional associated with (6.7). Let

$$d_\infty := \inf_{U\in N_\infty} I_\infty(U), \tag{6.9}$$

where

$$N_\infty := \{U \in \mathcal{D}^{1,2}(\mathbb{R}^N)\backslash\{0\} : J_\infty(U) = 0\} \quad \text{and} \quad J_\infty(U) := \|\nabla U\|_{L^2}^2 - \beta\|U\|_{L^{2^*}}^{2^*}. \tag{6.10}$$

Lemma 6.4. *The level d_∞ can be characterized as*

$$d_\infty = \inf\left\{H_\infty(U)\ :\ U \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\},\ J_\infty(U) < 0\right\},$$

where

$$H_\infty(U) := I_\infty(U) - \frac{1}{2}J_\infty(U) = \frac{\beta}{N}\|U\|_{L^{2^*}}^{2^*} \quad \textit{for any } U \in \mathcal{D}^{1,2}(U). \tag{6.11}$$

Proof. Let

$$d'_\infty := \inf\left\{H_\infty(U) \,:\, U \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\},\ J_\infty(U) < 0\right\}.$$

In order to prove that $d_\infty = d'_\infty$, it is useful to point out that for any $U \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}$ there exists $\overline{\lambda} = \overline{\lambda}(U) > 0$ such that

$$J_\infty(\lambda U)\begin{cases} > 0 & \text{if } 0 < \lambda < \overline{\lambda}, \\ = 0 & \text{if } \lambda = \overline{\lambda}, \\ < 0 & \text{if } \lambda > \overline{\lambda}. \end{cases}$$

If $U \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}$ satisfies $J_\infty(U) = 0$ then $\overline{\lambda} = 1$ and

$$d'_\infty \le H_\infty(\lambda U) \quad \text{for any } \lambda > 1.$$

Therefore

$$d'_\infty \le \lim_{\lambda\to 1^+} H_\infty(\lambda U) = \lim_{\lambda\to 1^+}\frac{\beta}{N}\lambda^{2^*}\|U\|^{2^*}_{L^{2^*}} = H_\infty(U) = I_\infty(U) - \frac{1}{2}J_\infty(U) = I_\infty(U),$$

and we deduce that $d'_\infty \le d_\infty$. On the other hand, if $U \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}$ satisfies $J_\infty(U) < 0$ then $\overline{\lambda} \in (0,1)$ and

$$d_\infty \le I_\infty(\overline{\lambda}U) = H_\infty(\overline{\lambda}U) = \frac{\beta}{N}\overline{\lambda}^{2^*}\|U\|^{2^*}_{L^{2^*}} < \frac{\beta}{N}\|U\|^{2^*}_{L^{2^*}} = H_\infty(U),$$

and we deduce that $d_\infty \le d'_\infty$. Therefore $d_\infty = d'_\infty$. □

With the help of the above characterization of the level d_∞ and the analogous characterization of d expressed by (3.4), we deduce the following inequality between d and d_∞.

Proposition 6.5. *Assume $p = 2^*$ and assume that (f_4) holds. Then $d \le d_\infty$.*

Proof. It is enough to prove that

$$d \le H_\infty(U) \quad \text{for any } U \in \mathcal{C}_0^\infty(\mathbb{R}^N)\setminus\{0\} \text{ with } J_\infty(U) < 0. \tag{6.12}$$

To this aim, we point out that for any $U \in \mathcal{C}_0^\infty(\mathbb{R}^N)\setminus\{0\}$ we have

$$\lim_{\lambda\to+\infty} I(u_\lambda) = I_\infty(U) \quad \text{and} \quad \lim_{\lambda\to+\infty} J(u_\lambda) = J_\infty(U), \tag{6.13}$$

where

$$u_\lambda(x) = \lambda^{\frac{N-2}{2}}U(\lambda x), \quad \lambda > 0, \quad x \in \mathbb{R}^N.$$

We postpone the proof of (6.13), and first we show that it is possible to deduce (6.12) with the help of (6.13). Let $U \in \mathcal{C}_0^\infty(\mathbb{R}^N)\setminus\{0\}$ be such that $J_\infty(U) < 0$ then (6.13) in particular yields

$$J(u_\lambda) < 0 \quad \text{for any } \lambda > \overline{\lambda} \text{ with } \overline{\lambda} > 0 \text{ sufficiently large,}$$

and hence

$$d \le H(u_\lambda) \quad \text{for any } \lambda > \overline{\lambda}.$$

Clearly (6.13) also yields

$$\lim_{\lambda \to +\infty} H(u_\lambda) = H_\infty(U),$$

and we reach the conclusion $d \le H_\infty(U)$.

Next we complete the proof showing that (6.13) holds. Since

$$\|u_\lambda\|_{H^1}^2 = \|\nabla U\|_{L^2}^2 + \frac{1}{\lambda^2}\|U\|_{L^2}^2 \quad \text{for any } \lambda > 0,$$

we have only to prove that

$$\lim_{\lambda \to +\infty} \int_{\mathbb{R}^N} u_\lambda f(u_\lambda)\, dx = \beta \|U\|_{L^{2^*}}^{2^*} \quad \text{and} \quad \lim_{\lambda \to +\infty} \int_{\mathbb{R}^N} F(u_\lambda)\, dx = \frac{\beta}{2^*}\|U\|_{L^{2^*}}^{2^*}.$$

By means of the change of variable $y = \lambda x$, we can rewrite

$$\int_{\mathbb{R}^N} u_\lambda f(u_\lambda)\, dx = \int_{\mathbb{R}^N} \lambda^{-\frac{N+2}{2}} U f(\lambda^{\frac{N-2}{2}} U)\, dy \quad \text{and} \quad \int_{\mathbb{R}^N} F(u_\lambda)\, dx = \int_{\mathbb{R}^N} \lambda^{-N} F(\lambda^{\frac{N-2}{2}} U)\, dy,$$

and if we set

$$g_\lambda := \lambda^{-\frac{N+2}{2}} U f(\lambda^{\frac{N-2}{2}} U) \quad \text{and} \quad G_\lambda := \lambda^{-N} F(\lambda^{\frac{N-2}{2}} U),$$

then we have only to prove that

$$\lim_{\lambda \to +\infty} \int_{\mathbb{R}^N} g_\lambda(y)\, dy = \beta \|U\|_{L^{2^*}}^{2^*} \quad \text{and} \quad \lim_{\lambda \to +\infty} \int_{\mathbb{R}^N} G_\lambda(y)\, dy = \frac{\beta}{2^*}\|U\|_{L^{2^*}}^{2^*}.$$

In view of (f_1), we can estimate

$$0 \le g_\lambda \le C \left(|U|^{2^*} + |U|^q \right) \in L^1(\mathbb{R}^N).$$

Moreover, for any $y \in \mathbb{R}^N$

$$\lim_{\lambda \to +\infty} g_\lambda(y) = \beta |U(y)|^{2^*}.$$

In fact, if $U(y) = 0$ then $g_\lambda(y) = 0$ and there is nothing to prove. On the other hand, if $U(y) \neq 0$ then $\lambda^{\frac{N-2}{2}} U(y) \to \infty$ as $\lambda \to \infty$, and in view of (f_4) we get

$$\lim_{\lambda \to +\infty} g_\lambda(y) = \lim_{\lambda \to +\infty} \frac{f\left(\lambda^{\frac{N-2}{2}} U(y)\right)}{\lambda^{\frac{N-2}{2}} U(y) \left|\lambda^{\frac{N-2}{2}} U(y)\right|^{\frac{4}{N-2}}} \cdot |U(y)|^{2^*} = \beta |U(y)|^{2^*}.$$

Hence as a consequence of the Lebesgue dominated convergence theorem we get

$$\lim_{\lambda\to+\infty}\int_{\mathbb{R}^N} g_\lambda(y)\,dy=\beta\|U\|_{L^{2^*}}^{2^*}.$$

Finally, in view of 2, we can estimate

$$0\le G_\lambda\le\frac{\lambda^{-N}}{\mu}\cdot\lambda^{\frac{N-2}{2}}U\cdot f(\lambda^{\frac{N-2}{2}}U)=\frac{1}{\mu}g_\lambda\le\frac{C}{\mu}\left(|U|^{2^*}+|U|^q\right)\in L^1(\mathbb{R}^N).$$

Moreover, l'Hôpital's rule and (f_4) yield

$$\lim_{s\to\pm\infty}\frac{F(s)}{|s|^{2^*}}=\lim_{s\to\pm\infty}\frac{f(s)}{2^*\cdot s|s|^{\frac{4}{N-2}}}=\frac{\beta}{2^*}.$$

Arguing as before, we get for any $y\in\mathbb{R}^N$

$$\lim_{\lambda\to+\infty}G_\lambda(y)=\lim_{\lambda\to+\infty}\frac{F\left(\lambda^{\frac{N-2}{2}}U(y)\right)}{\left|\lambda^{\frac{N-2}{2}}U(y)\right|^{2^*}}\cdot|U(y)|^{2^*}=\frac{\beta}{2^*}|U(y)|^{2^*}$$

and, again as a consequence of the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{\lambda\to+\infty}\int_{\mathbb{R}^N}G_\lambda(y)\,dy=\frac{\beta}{2^*}\|U\|_{L^{2^*}}^{2^*}.$$

□

The above Proposition 6.5 will enable us to prove that, also in the *energy critical* case $p=2^*$, if the solution u to (1.1) enters the set W then u is a *global* solution.

Proposition 6.6. *Assume $p=2^*$ and assume that (f_4) holds. If $u(t_0)\in W$ at some $t_0\in[0,T_m)$ then $T_m=+\infty$ and $\limsup_{t\to+\infty}\|u(t)\|_{L^\infty}<+\infty$.*

Proof. In view of the invariance of W under the flow associated with (1.1), without loss of generality, we assume that $u(t)\in W\setminus\{0\}$ for any $t\in[t_0,T_m)$ and $t_0>0$. Then the boundedness of W yields

$$\sup_{t\in[t_0,T_m)}\|u(t)\|_{H^1}\le\sqrt{\frac{2\mu}{\mu-2}d}. \tag{6.14}$$

Moreover, the energy functional is bounded from above and positive in $W\setminus\{0\}$, and hence the monotonicity of the energy along the solution yields

$$\exists\lim_{t\to T_m^-}I\left(u(t)\right)\in[0,d). \tag{6.15}$$

It is enough to show that

$$\limsup_{t\to T_m^-} \|u(t)\|_{L^\infty} < +\infty. \tag{6.16}$$

Indeed, (6.16) and the blow-up alternative yields $T_m = +\infty$.

We argue by contradiction assuming

$$\limsup_{t\to T_m^-} \|u(t)\|_{L^\infty} = +\infty,$$

and hence we find a sequence $\{t_k\}_k \subset (t_0, T_m)$ satisfying

$$\lim_{k\to+\infty} t_k = T_m, \quad \lim_{k\to+\infty} \|u(t_k)\|_{L^\infty} = +\infty, \quad \text{and} \quad \sup_{t\in[t_0,t_k]} \|u(t)\|_{L^\infty} = \|u(t_k)\|_{L^\infty}.$$

Construction of a rescaled sequence. Let $x_k \in \mathbb{R}^N$, be such that

$$\frac{1}{2}\|u(t_k)\|_{L^\infty} \leq |u(x_k, t_k)|,$$

and consider the rescaled functions

$$u(x,t) = \lambda_k^{\frac{N-2}{2}} u_k(y,s), \quad \text{where} \quad \lambda_k > 0, \quad y = \lambda_k(x-x_k), \quad \text{and} \quad s = \lambda_k^2(t-t_k).$$

By construction u_k solves

$$\partial_s u_k = \Delta_y u_k - \frac{1}{\lambda_k^2} u_k + \frac{1}{\lambda_k^{\frac{N+2}{2}}} f\left(\lambda_k^{\frac{N-2}{2}} u_k\right) \quad \text{in } \mathbb{R}^N \times \left[\lambda_k^2(t_0-t_k), \lambda_k^2(T_m-t_k)\right).$$

Next, we choose λ_k so that

$$|u_k(0,0)| = 1, \tag{6.17}$$

in other words

$$\lambda_k := |u(x_k,t_k)|^{\frac{2}{N-2}} \to +\infty \quad \text{as } k \to +\infty.$$

Moreover, with this choice of λ_k, we have

$$\sup_{s\in[\lambda_k^2(t_0-t_k),0]} \|u_k(s)\|_{L^\infty} = \lambda_k^{-\frac{N-2}{2}} \sup_{t\in[t_0,t_k]} \|u(t)\|_{L^\infty} = \lambda_k^{-\frac{N-2}{2}} \|u(t_k)\|_{L^\infty} \leq 2\lambda_k^{-\frac{N-2}{2}} |u(x_k,t_k)| = 2.$$

Locally uniform convergence of the rescaled sequence. For any k, there exists $T_k > 0$ such that u_k can be extended on $[0, T_k]$ as the unique $L^\infty([0,T_k], L^\infty(\mathbb{R}^N))$ solution of the Cauchy problem

$$\begin{cases} \partial_s v = \Delta_y v - \dfrac{1}{\lambda_k^2} v + \dfrac{1}{\lambda_k^{\frac{N+2}{2}}} f\left(\lambda_k^{\frac{N-2}{2}} v\right) & \text{in } \mathbb{R}^N \times [0, T_k], \\ v(x,0) = u_k(0) & x \in \mathbb{R}^N. \end{cases} \tag{6.18}$$

Exploiting the integral representation formula for (6.18), we can write for any $s \in [0, T_k]$

$$u_k(s) = e^{s\Delta} u_k(0) + \int_0^s e^{(s-\sigma)\Delta} \left[\frac{1}{\lambda_k^{\frac{N+2}{2}}} f\left(\lambda_k^{\frac{N-2}{2}} u_k(\sigma) \right) - \frac{1}{\lambda_k^2} u_k(\sigma) \right] d\sigma,$$

and we can estimate

$$\|u_k(s)\|_{L^\infty} \leq \|u_k(0)\|_{L^\infty} + s \left[\frac{1}{\lambda_k^{\frac{N+2}{2}}} \sup_{\sigma\in[0,s]} \left\| f\left(\lambda_k^{\frac{N-2}{2}} u_k(\sigma) \right) \right\|_{L^\infty} + \frac{1}{\lambda_k^2} \sup_{\sigma\in[0,s]} \|u_k(\sigma)\|_{L^\infty} \right]$$

$$\leq \|u_k(0)\|_{L^\infty} + s \left[C \left(\sup_{\sigma\in[0,s]} \|u_k(\sigma)\|_{L^\infty}^{\frac{N+2}{N-2}} + \sup_{\sigma\in[0,s]} \frac{1}{\lambda_k^{\frac{2N-(N-2)q}{2}}} \|u_k(\sigma)\|_{L^\infty}^{q-1} \right) + \frac{1}{\lambda_k^2} \sup_{\sigma\in[0,s]} \|u_k(\sigma)\|_{L^\infty} \right],$$

where we applied standard $L^\infty - L^\infty$-estimates and 1.

Let $s \in [0, T_k]$ and let

$$M_{k,\infty}(s) := \sup_{\sigma\in[0,s]} \|u_k(\sigma)\|_{L^\infty},$$

so that

$$M_{k,\infty}(s) \leq \|u_k(0)\|_{L^\infty} + s \left[C \left([M_{k,\infty}(s)]^{\frac{N+2}{N-2}} + \frac{1}{\lambda_k^{\frac{2N-(N-2)q}{2}}} [M_{k,\infty}(s)]^{q-1} \right) + \frac{1}{\lambda_k^2} M_{k,\infty}(s) \right]$$

and

$$\left(1 - \frac{T_k}{\lambda_k^2}\right) M_{k,\infty}(s) \leq \|u_k(0)\|_{L^\infty} + sC \left([M_{k,\infty}(s)]^{\frac{N+2}{N-2}} + \frac{1}{\lambda_k^{\frac{2N-(N-2)q}{2}}} [M_{k,\infty}(s)]^{q-1} \right).$$

We are going to show that T_k *cannot* tend to 0 as $k \to +\infty$. To this aim we assume, without loss of generality, that the sequence $\{T_k\}_k$ is bounded from above, so that

$$\lim_{k\to+\infty} \frac{T_k}{\lambda_k^2} = 0,$$

and we claim that there exists $\delta > 0$ such that for any k sufficiently large $T_k \geq \delta$ and

$$M_{k,\infty}(s) \leq 2\|u_k(0)\|_{L^\infty} \quad \text{for any } s \in [0, \delta].$$

Indeed, if $M_{k,\infty}(s) = 2\|u_k(0)\|_{L^\infty}$ at some $s = s(k) \in [0, T_k]$ then

$$\left(1 - \frac{2T_k}{\lambda_k^2}\right)\|u_k(0)\|_{L^\infty} \le \tilde{C}s\left(\|u_k(0)\|_{L^\infty}^{\frac{N+2}{N-2}} + \frac{1}{\lambda_k^{\frac{2N-(N-2)q}{2}}}\|u_k(0)\|_{L^\infty}^{q-1}\right), \qquad \tilde{C} := C2^{\frac{N+2}{N-2}} > 0$$

and

$$s \ge \left(1 - \frac{2T_k}{\lambda_k^2}\right)\cdot\tilde{C}^{-1}\frac{\|u_k(0)\|_{L^\infty}^{2-q}}{\|u_k(0)\|_{L^\infty}^{2^*-q} + \frac{1}{\lambda_k^{\frac{2N-(N-2)q}{2}}}} \ge \left(1 - \frac{2T_k}{\lambda_k^2}\right)\cdot\tilde{C}^{-1}\cdot\frac{2^{2-q}}{2^{2^*-q}+1},$$

indeed $\frac{2N-(N-2)q}{2} > 0$. Summarizing, we have the existence of $\delta > 0$ such that for any k sufficiently large $T_k \ge \delta$ and

$$\sup_{s\in[-1,\delta]} \|u_k(s)\|_{L^\infty} \le \max\{2, 2\|u_k(0)\|_{L^\infty}\} \le 4.$$

By standard parabolic L^q-regularity theory (see e.g. [17, Remark 48.3]), we have that $\{u_k\}_k$ is a bounded sequence in $W^{2,1;q}_{loc}(\mathbb{R}^N \times [-1,\delta])$ for any $q \in (1, +\infty)$. If $a \in (0,1)$, and q is sufficiently large so that $q > N+2$ and $1 - \frac{N+2}{q} > 0$, then

$$W^{2,1;q}(D) \hookrightarrow BUC^{a,\frac{a}{2}}(D) \quad \text{with } D \subset \mathbb{R}^N \times [-1,\delta] \text{ compact},$$

(see e.g. [17, Equation (1.2)]), and hence the sequence $\{u_k\}_k$ is also bounded in $BUC^{a,\frac{a}{2}}(D)$. Finally, by standard parabolic regularity theory (see e.g. [17, Remark 48.3]), we conclude that $\{u_k\}_k$ is a bounded sequence in $BUC^{2+a,1+\frac{a}{2}}(D)$ and

$$u_k \to U \quad \text{in } BUC^{2+a,1+\frac{a}{2}}_{loc}(\mathbb{R}^N \times [-1,\delta]). \tag{6.19}$$

In particular, from (6.17), we deduce that

$$|U(0,0)| = 1$$

and U is *nontrivial.*

The limiting function U is time independent. It is easy to see that

$$\lim_{k\to+\infty}\int_{-1}^{\delta}\|\partial_s u_k(s)\|_{L^2}^2\,ds = 0. \tag{6.20}$$

In fact, by construction

$$\|\partial_s u_k(s)\|_{L^2} = \lambda_k^{-1}\left\|\partial_t u\left(t_k + \frac{s}{\lambda_k^2}\right)\right\|_{L^2},$$

and hence

$$\begin{aligned}\int_{-1}^{\delta}\|\partial_s u_k(s)\|_{L^2}^2\,ds &= \int_{t_k-\frac{1}{\lambda_k^2}}^{t_k+\frac{\delta}{\lambda_k^2}}\|\partial_t u(t)\|_{L^2}^2\,dt\\ &= -\int_{t_k-\frac{1}{\lambda_k^2}}^{t_k+\frac{\delta}{\lambda_k^2}}\frac{d}{dt}I\left(u(t)\right)\,dt = I\left(u\left(t_k-\frac{1}{\lambda_k^2}\right)\right)-I\left(u\left(t_k+\frac{\delta}{\lambda_k^2}\right)\right).\end{aligned}$$

In order to obtain (6.20), it is enough to recall (6.15) which yields

$$\lim_{k\to+\infty}\left[I\left(u\left(t_k-\frac{1}{\lambda_k^2}\right)\right)-I\left(u\left(t_k+\frac{\delta}{\lambda_k^2}\right)\right)\right]=0.$$

Let s_1, $s_2\in[-1,\delta]$ and let $D\subset\mathbb{R}^N$ be *any* compact set. We can estimate

$$\|U(s_1)-U(s_2)\|_{L^2(D)}\le\|U(s_1)-u_k(s_1)\|_{L^2(D)}+\|u_k(s_1)-u_k(s_2)\|_{L^2(D)}+\|u_k(s_2)-U(s_2)\|_{L^2(D)}$$

and

$$\begin{aligned}0\le\|U(s_1)-U(s_2)\|_{L^2(D)}\le\lim_{k\to+\infty}\|u_k(s_1)-u_k(s_2)\|_{L^2(D)}&=\lim_{k\to+\infty}\left\|\int_{s_1}^{s_2}\partial_s u_k(s)\,ds\right\|_{L^2(D)}\\ &\le\lim_{k\to+\infty}\sqrt{|s_2-s_1|}\left(\int_{s_1}^{s_2}\|\partial_s u_k(s)\|_{L^2}^2\,ds\right)^{\frac{1}{2}}=0.\end{aligned}$$

In conclusion, $U(s_1)=U(s_2)$ a.e. in $\mathbb{R}^N$.

Limiting equation for U. From (6.20), we deduce that

$$\|\partial_s u_k(s)\|_{L^2}^2\to 0\quad\text{in } L^1([-1,\delta])$$

and up to a subsequence still denoted by $\{u_k\}_k$

$$\|\partial_s u_k(s)\|_{L^2}\to 0\quad\text{for a.e. } s\in[-1,\delta].$$

Therefore

$$\partial_s u_k(s)\to 0\quad\text{in } L^2(\mathbb{R}^N)\quad\text{for a.e. } s\in[-1,\delta]$$

and

$$\Delta_y u_k-\frac{1}{\lambda_k^2}u_k+\frac{1}{\lambda_k^{\frac{N+2}{2}}}f\left(\lambda_k^{\frac{N-2}{2}}u_k\right)\to 0\quad\text{in } L^2(\mathbb{R}^N)\quad\text{for a.e. } s\in[-1,\delta].$$

In particular, if we fix *any* $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ then for a.e. $s\in[-1,\delta]$ we have

$$\lim_{k\to+\infty}\int_{\mathbb{R}^N}\left[\Delta_y u_k(s)-\frac{1}{\lambda_k^2}u_k(s)+\frac{1}{\lambda_k^{\frac{N+2}{2}}}f\left(\lambda_k^{\frac{N-2}{2}}u_k(s)\right)\right]\varphi\,dy=0.$$

In what follows, let us fix $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and $s \in [-1,\delta]$. Using (6.14), we can estimate

$$\|u_k(s)\|_{L^2} = \lambda_k \|u(t)\|_{L^2} \leq \lambda_k \sup_{t\in[t_0,T_m)} \|u(t)\|_{H^1} \leq \lambda_k \sqrt{\frac{2\mu}{\mu-2}d} \tag{6.21}$$

and

$$\|\nabla_y u_k(s)\|_{L^2} = \|\nabla_x u(t)\|_{L^2} \leq \sup_{t\in[t_0,T_m)} \|u(t)\|_{H^1} \leq \sqrt{\frac{2\mu}{\mu-2}d}. \tag{6.22}$$

From (6.21), we deduce that

$$\left|\frac{1}{\lambda_k^2}\int_{\mathbb{R}^N} u_k(s)\varphi\,dy\right| \leq \frac{1}{\lambda_k^2}\|u_k(s)\|_{L^2}\|\varphi\|_{L^2} \leq \frac{1}{\lambda_k}\|\varphi\|_{L^2}\sqrt{\frac{2\mu}{\mu-2}d},$$

and hence

$$\lim_{k\to+\infty} \frac{1}{\lambda_k^2}\int_{\mathbb{R}^N} u_k(s)\varphi\,dy = 0.$$

In view of (6.22), we have that $u_k(s) \rightharpoonup V(s)$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Combining this information with (6.19), we deduce that $V(s) = U$ a.e. in $\mathbb{R}^N$. In other words, $u_k(s) \rightharpoonup U$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and

$$\lim_{k\to+\infty}\int_{\mathbb{R}^N} \Delta_y u_k(s)\varphi\,dy = \lim_{k\to+\infty} -\int_{\mathbb{R}^N} \nabla_y u_k(s)\cdot\nabla_y\varphi\,dy = -\int_{\mathbb{R}^N} \nabla_y U\cdot\nabla_y\varphi\,dy.$$

If we show that

$$\lim_{k\to+\infty} \frac{1}{\lambda_k^{\frac{N+2}{2}}}\int_{\mathbb{R}^N} f\left(\lambda_k^{\frac{N-2}{2}}u_k(s)\right)\varphi\,dy = \beta\int_{\mathbb{R}^N} U|U|^{\frac{4}{N-2}}\varphi\,dy \tag{6.23}$$

then $U \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a *nontrivial* weak solution to

$$-\Delta_y U = \beta U|U|^{\frac{4}{N-2}} \quad \text{in } \mathbb{R}^N, \tag{6.24}$$

and by standard elliptic regularity theory, U is a *nontrivial* classical solution for the above *limiting* equation.

Convergence of the nonlinear part: proof of (6.23). Fix $s \in [-1,\delta]$, and set

$$g_k = g_k(s) := \frac{1}{\lambda_k^{\frac{N+2}{2}}} f\left(\lambda_k^{\frac{N-2}{2}}u_k(s)\right) \quad \text{and} \quad g := \beta U|U|^{\frac{4}{N-2}}.$$

We are going to prove that for any compact set $D \subset \mathbb{R}^N$

$$\lim_{k\to+\infty}\int_D |g_k - g|\,dy = 0,$$

this yields $g_k \to g$ in $L^1_{loc}(\mathbb{R}^N)$, and the proof of (6.23) is complete.

Let $D \subset \mathbb{R}^N$ be a compact set. From the local uniform convergence (6.19), we deduce that for any $\varepsilon > 0$ there exists $\overline{k} \geq 1$ such that, for any $y \in D$ and any $k \geq \overline{k}$, we have

$$U(y) - \frac{\varepsilon}{2} \leq u_k(y,s) \leq U(y) + \frac{\varepsilon}{2} \quad \text{and} \quad |U(y)| - \frac{\varepsilon}{2} \leq |u_k(y,s)| \leq |U(y)| + \frac{\varepsilon}{2}.$$

Let $\varepsilon > 0$ be arbitrarily fixed, there exists $\overline{k} \geq 1$ such that the following inclusions hold for any $k \geq \overline{k}$:

$$D \cap \{U < -\varepsilon\} \subseteq D \cap \left\{u_k(s) < -\frac{\varepsilon}{2}\right\}, \quad D \cap \{U > \varepsilon\} \subseteq D \cap \left\{u_k(s) > \frac{\varepsilon}{2}\right\},$$

$$\text{and} \quad D \cap \{|U| \leq \varepsilon\} \subseteq D \cap \left\{|u_k(s)| \leq \frac{3}{2}\varepsilon\right\}.$$

Therefore, for any $k \geq \overline{k}$, we can estimate

$$\begin{aligned}\int_D |g_k - g|\, dy \leq \int_{D\cap\{|U|\leq\varepsilon\}} |g|\, dy + \int_{D\cap\{|u_k(s)|\leq 3\varepsilon/2\}} |g_k|\, dy \\ + \int_{D\cap\{|U|>\varepsilon\}} |g_k - g|\, dy.\end{aligned}$$

Clearly

$$\int_{D\cap\{|U|\leq\varepsilon\}} |g|\, dy \leq \beta |D| \varepsilon^{\frac{N+2}{N-2}},$$

and using 1

$$\begin{aligned}\int_{D\cap\{|u_k(s)|\leq 3\varepsilon/2\}} |g_k|\, dy &\leq C \int_{D\cap\{|u_k(s)|\leq 3\varepsilon/2\}} \left(|u_k(s)|^{\frac{N+2}{N-2}} + |u_k(s)|^{q-1}\right) dy \\ &\leq C|D| \left[\left(\frac{3}{2}\varepsilon\right)^{\frac{N+2}{N-2}} + \left(\frac{3}{2}\varepsilon\right)^{q-1}\right].\end{aligned}$$

Summarizing, for any $k \geq \overline{k}$, we have

$$\int_D |g_k - g|\, dy \leq |D| \left[\left(\beta + C\left(\frac{3}{2}\right)^{\frac{N+2}{N-2}}\right)\varepsilon^{\frac{N+2}{N-2}} + C\left(\frac{3}{2}\varepsilon\right)^{q-1}\right] + \int_{D\cap\{|U|>\varepsilon\}} |g_k - g|\, dy,$$

and it remains to show that

$$\lim_{k\to+\infty} \int_{D\cap\{|U|>\varepsilon\}} |g_k - g|\, dy = 0. \tag{6.25}$$

We deduce (6.25) as a consequence of the Lebesgue dominated convergence theorem. In fact, for any $k \geq \overline{k}$ and any $y \in D \cap \{|U| > \varepsilon\}$ we have:

- either $y \in D \cap \{U < -\varepsilon\}$ or $y \in D \cap \{U > \varepsilon\}$, and in view of the following inclusions
$$D \cap \{U < -\varepsilon\} \subseteq D \cap \Big\{u_k(s) < -\frac{\varepsilon}{2}\Big\} \quad \text{and} \quad D \cap \{U > \varepsilon\} \subseteq D \cap \Big\{u_k(s) > \frac{\varepsilon}{2}\Big\},$$
we can apply (f_4) to get
$$\begin{aligned}\lim_{k\to+\infty} g_k(y) &= \lim_{k\to+\infty} \frac{f\left(\lambda_k^{\frac{N-2}{2}} u_k(y,s)\right)}{\lambda_k^{\frac{N-2}{2}} u_k(y,s) \left|\lambda_k^{\frac{N-2}{2}} u_k(y,s)\right|^{\frac{N+2}{N-2}}} \cdot u_k(y,s)|u_k(y,s)|^{\frac{4}{N-2}} \\ &= \beta U(y)|U(y)|^{\frac{4}{N-2}} = g(y);\end{aligned}$$
- moreover, we can estimate
$$|u_k(y,s)| \le |U(y)| + \frac{\varepsilon}{2} < 2|U(y)|$$
and using (f_4) we get
$$|g_k(y) - g(y)| \le C\left[(2|U(y)|)^{\frac{N+2}{N-2}} + (2|U(y)|)^{q-1}\right] + \beta|U(y)|^{\frac{N+2}{N-2}}.$$

Finally, we recall that D is compact and, in view of (6.19), $U \in L^\infty(D)$, which yields
$$|U|^{\frac{N+2}{N-2}},\ |U|^{q-1} \in L^1(D).$$

Energy of U and a contradiction. We point out that the additional assumption (f_4) will be crucial also in this final step of the proof. Since $U \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ solves (6.24), we have $J_\infty(U) = 0$ and hence
$$d_\infty \le I_\infty(U) = H_\infty(U),$$
see (6.8)–(6.11) for the definitions of the level d_∞ and the functionals J_∞, I_∞, and H_∞.

On the other hand using Proposition 6.5, the monotonicity of the energy along the solution and the invariance of W, we get
$$d_\infty \ge d > I(u(t_0)) \ge \liminf_{k\to+\infty} I\left(u\left(t_k + \frac{s}{\lambda_k^2}\right)\right) \ge \liminf_{k\to+\infty} H\left(u\left(t_k + \frac{s}{\lambda_k^2}\right)\right),$$
where the functional H is defined by (3.5). Therefore in order to reach a contradiction it is enough to prove that
$$H_\infty(U) \le \liminf_{k\to+\infty} H\left(u\left(t_k + \frac{s}{\lambda_k^2}\right)\right). \tag{6.26}$$

Since

$$H\left(u\left(t_k+\frac{s}{\lambda_k^2}\right)\right)=\int_{\mathbb{R}^N}h\left(u\left(t_k+\frac{s}{\lambda_k^2}\right)\right)dx,\quad \text{where } h(\sigma)=\frac{1}{2}\sigma f(\sigma)-F(\sigma)\geq 0,$$

by means of the change of variable $x=x_k+\frac{y}{\lambda_k}$ we can rewrite

$$H\left(u\left(t_k+\frac{s}{\lambda_k^2}\right)\right)=\int_{\mathbb{R}^N}\lambda_k^{-N}h\left(\lambda_k^{\frac{N-2}{2}}u_k(y,s)\right)dy,$$

and the desired estimate (6.26) can be deduced from Fatou's Lemma provided

$$\lim_{k\to+\infty}\lambda_k^{-\frac{N+2}{2}}u_k(y,s)\cdot f\left(\lambda_k^{\frac{N-2}{2}}u_k(y,s)\right)=\beta|U(y)|^{2^*} \tag{6.27}$$

and

$$\lim_{k\to+\infty}\lambda_k^{-N}F\left(\lambda_k^{\frac{N-2}{2}}u_k(y,s)\right)=\frac{\beta}{2^*}|U(y)|^{2^*}. \tag{6.28}$$

Next, we show that indeed (6.27) and (6.28) hold, and in this way the proof is completed.

For any fixed $y\in\mathbb{R}^N$, we have $u_k(y,s)\to U(y)$ as $k\to+\infty$. If $U(y)=0$ then in view of 1 we can estimate

$$\begin{aligned}0&\leq\lambda_k^{-\frac{N+2}{2}}u_k(y,s)\cdot f\left(\lambda_k^{\frac{N-2}{2}}u_k(y,s)\right)\\&\leq C\left(|u_k(y,s)|^{2^*}+\frac{1}{\lambda_k^{N-\frac{q(N-2)}{2}}}|u_k(y,s)|^q\right)\to 0\quad\text{as } k\to+\infty,\end{aligned}$$

and in view of 2

$$\begin{aligned}0&\leq\lambda_k^{-N}F\left(\lambda_k^{\frac{N-2}{2}}u_k(y,s)\right)\\&\leq\frac{1}{\mu}\cdot\lambda_k^{-\frac{N+2}{2}}u_k(y,s)\cdot f\left(\lambda_k^{\frac{N-2}{2}}u_k(y,s)\right)\\&\leq\frac{C}{\mu}\left(|u_k(y,s)|^{2^*}+\frac{1}{\lambda_k^{N-\frac{q(N-2)}{2}}}|u_k(y,s)|^q\right)\to 0\quad\text{as } k\to+\infty.\end{aligned}$$

On the other hand if $U(y) \neq 0$ then $\lambda_k^{\frac{N-2}{2}} u_k(y,s) \to \infty$, and in view of (f_4) we get

$$\lim_{k\to+\infty} \lambda_k^{-\frac{N+2}{2}} u_k(y,s) \cdot f\left(\lambda_k^{\frac{N-2}{2}} u_k(y,s) \right) = \lim_{k\to+\infty} \frac{f\left(\lambda_k^{\frac{N-2}{2}} u_k(y,s) \right)}{\lambda_k^{\frac{N-2}{2}} u_k(y,s) \cdot |\, \lambda_k^{\frac{N-2}{2}} u_k(y,s) \,|^{\frac{4}{N-2}}} \cdot |u_k(y,s)|^{2^*}$$

$$= \beta |U(y)|^{2^*}.$$

Therefore the proof of (6.27) is complete. Since l'Hôpital's rule and (f_4) yield

$$\lim_{s\to\pm\infty} \frac{F(s)}{|s|^{2^*}} = \lim_{s\to\pm\infty} \frac{f(s)}{2^* \cdot s|s|^{\frac{4}{N-2}}} = \frac{\beta}{2^*},$$

arguing as before we can conclude that (6.28) holds also. □

The convergence result expressed by Lemma 6.2 can be obtained also in the critical case $p = 2^*$ at least for sequences of functions which are bounded in $L^\infty(\mathbb{R}^N)$.

Lemma 6.7. *Assume $p = 2^*$. If $\{u_k\}_k \subset W$ is such that*

i) $\displaystyle\lim_{k\to+\infty} I(u_k) = c \in [0,d)$,

ii) $\displaystyle\lim_{k\to+\infty} J(u_k) = 0$,

iii) $u_k \in L^\infty(\mathbb{R}^N)$ *for any* $k \geq 1$ *and* $\displaystyle\sup_k \|u_k\|_{L^\infty} < +\infty$,

then up to a subsequence

$$\lim_{k\to+\infty} \|u_k\|_{H^1} = 0 \quad \textit{and} \quad c = 0.$$

Proof. As in the proof of Lemma 6.2, it is enough to show that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^N} u_k f(u_k) dx = 0 \;\text{ and }\; \lim_{k\to+\infty} \int_{\mathbb{R}^N} F(u_k) dx = 0. \tag{6.29}$$

By assumption $\{u_k\}_k$ is bounded both in $H^1(\mathbb{R}^N)$ and in $L^\infty(\mathbb{R}^N)$, and hence also the symmetrized sequence $\{u_k^*\}_k$ is bounded both in $H^1_{rad}(\mathbb{R}^N)$ and in $L^\infty(\mathbb{R}^N)$. Therefore up to a subsequence

$$u_k^* \rightharpoonup V \text{ in } H^1(\mathbb{R}^N), \quad u_k^* \to V \text{ in } L^r(\mathbb{R}^N) \text{ for any } r \in (2, 2^*), \quad \text{and} \quad u_k^* \to V \text{ a.e. in } \mathbb{R}^N.$$

Since the sequence $\{u_k^*\}$ is bounded in $L^\infty(\mathbb{R}^N)$, if $r \in (2, 2^*)$ then we have for some positive constant M_r

$$|u_k^*|^{2^*} = |u_k^*|^{2^*-r} |u_k^*|^r \leq M_r |u_k^*|^r$$

and the generalized Lebesgue dominated convergence Theorem ensures

$$u_k^* \to V \quad \text{in } L^{2^*}(\mathbb{R}^N).$$

Summarizing

$$u_k^* \rightharpoonup V \text{ in } H^1(\mathbb{R}^N), \quad u_k^* \to V \text{ in } L^q(\mathbb{R}^N), \quad u_k^* \to V \text{ in } L^{2^*}(\mathbb{R}^N), \quad \text{and} \quad u_k^* \to V \text{ a.e. in } \mathbb{R}^N,$$

and the proof of (6.29) can be completed following the same lines of the proof of Lemma 6.2.

□

We are now in position to end the proof of Theorem 1.3 by establishing the vanishing of the H^1-norm of the solution as time goes to infinity.

Proposition 6.8. *Assume $p = 2^*$ and assume that (f_4) holds. If $u(t_0) \in W$ at some $t_0 \in [0, T_m)$ then the global bounded solution u is such that* $\limsup_{t\to+\infty} \|u(t)\|_{H^1} = 0$.

Proof. Thanks to Proposition 6.6 if $u(t_0) \in W$ at some $t_0 \in [0, T_m)$ then $T_m = \infty$ and

$$\limsup_{t\to+\infty} \|u(t)\|_{L^\infty} < +\infty. \tag{6.30}$$

Since $T_m = +\infty$, Lemma 4.2 ensures the existence of a sequence $\{t_k\}_k$ such that

$$\lim_{k\to+\infty} t_k = +\infty, \quad \lim_{k\to+\infty} I(u(t_k)) = c \in [0, d), \quad \lim_{k\to+\infty} J(u(t_k)) = 0.$$

Moreover $\{u(t_k)\}_k \subset W$ for any k sufficiently large and (6.30) yields

$$\sup_k \|u(t_k)\|_{L^\infty} < +\infty.$$

Therefore we can apply the convergence Lemma 6.7 and conclude that up to a subsequence

$$\lim_{k\to+\infty} \|u(t_k)\|_{H^1} = 0 \qquad \text{and} \qquad c = 0.$$

Then the proof follows the same lines of the proof of Lemma 6.3, we have only to justify how to replace the estimate (6.6) that we performed with the Gagliardo-Nirenberg inequality. This is a consequence of the fact that the solution $u \in L^\infty((\tau, +\infty) \times \mathbb{R}^N)$ for any $\tau > 0$. Indeed, for any $r \geq 2$ there exists some positive constant C such that for any $t \geq \tau$ we have

$$\|u(t)\|_{L^r}^r = \int_{\mathbb{R}^N} |u(t)|^{r-2}|u(t)|^2 dx \leq \left(\sup_{t \geq \tau} \|u(t)\|_{L^\infty}^{r-2} \right) \|u(t)\|_{L^2}^2 \leq C\|u(t)\|_{L^2}^2.$$

□

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