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Poisson Search

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ABSTRACT

We present a model of the job market where the number of workers and companies is uncertain, representing the job search activity as a Poisson game. We allow for heterogeneity of workers and companies and show that in equilibrium more productive types choose higher terms of trade. The Poisson search model gives rise to multiple, possibly inefficient equilibria.

1. Introduction

The process of filling vacancies for companies and finding jobs for workers is often haphazard. The search literature models job market uncertainty assuming that the matching process of companies and workers is stochastic. In this paper, we model job market uncertainty assuming that the number of market participants is stochastic, representing the job search activity as a game with population uncertainty. We assume that the number of market participants is a Poisson random variable, as in the Poisson game literature à la Myerson (1998). Poisson games are simultaneous move games, with several unique properties that improve the tractability of a finite environment.

We begin with a centralized model of the job market in which a number of market places distinguished by the wage that is being offered is established exogenously by a third-party market maker.[1] The workers and companies can choose independently and simultaneously which market place to visit. The job search activity is represented as a Poisson game with a finite but unknown number of – possibly heterogeneous – workers and companies. Our focus is on the undominated Nash equilibria of such a game.

We show that, at any such equilibrium, the workers and companies of higher types choose market places with higher wages. This is an immediate consequence of the environmental equivalence property of Poisson games, whereby the posterior distribution of a player who has been selected to play the game over the number of other players in the game is the same as the prior distribution of an outsider over the number of all players. This property implies directly that the ranking of market places by the different types of workers and companies is driven by the wage offered in each market place.

Next, we prove that if there is only one market place, then, the equilibrium exists and is unique. When there exist equilibria with several market places, they coexist with the equilibria with each single market place. In some equilibria, there may be a failure of separation of traders with different productivities, in others there may be a failure of coordination among traders. The environmental equivalence mimics the effect of competition, but strategic elements survive leading to the coexistence of multiple – possibly inefficient – equilibria.

Finally, we move to a decentralized version of the model. Consistent with the nature of Poisson games, we assume that, independently and simultaneously, the companies announce wage offers, the workers announce desired wages, and, if at least one company and one worker announce the same wage, then, a market emerges. We show that there is a one-to-one relationship between equilibria of the centralized model with the market places organized by the market maker and equilibria of the decentralized model with simultaneous wage announcements by the companies and workers. Hence, all the properties derived in the previous centralized model carry over to the decentralized one.

The job market has been represented in two main ways in the search literature. In the random search model, trade is coordinated by a stochastic process that matches bilaterally a continuum of companies and workers who, then, bargain in pairs over the division of the surplus.[2] Typically, the model exhibits inefficient equilibria, as the search externality built into the matching process fails to be internalized through ex-post bargaining.[3] In the directed search model, first a continuum of companies post wages and then a continuum of workers direct their search to the best offers, with random matching

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[1] This is inspired by Moen (1997). However, the market maker has no direct role in our setting.

[2] A classic introduction to the random search model is *Equilibrium Unemployment Theory* by Pissarides (2000).

[3] Except possibly in non-generic cases in which the ex-post division of the surplus offsets exactly the search externalities (see Hosios, 1990).

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if several workers apply for the same job offer.[4] Typically, the model exhibits efficient equilibria, thanks to the working of competition in a continuum economy.[5]

Finite versions of directed search models have also been developed.[6] Peters (2000) has shown that the equilibrium outcomes of a directed search model with a finite number of agents converge to those of the corresponding competitive economy as the number of agents grows.[7] In finite settings, coordination failures due to strategic complementarity become a source of frictions. Galenianos and Kircher (2012) have pointed out that it may happen that more productive firms end up posting lower offers than less productive ones due to the discontinuities that arise from the competition among price posting companies. This inefficiency disappears approaching a large economy, where the discontinuities are smoothed out.

Despite the presence of a finite number of agents, this is not the source of inefficiency in our setting where the forces of competition are at work through environmental equivalence, which implies immediately that more productive agents obtain better terms of trade. However, coordination failures may still survive at equilibrium compromising efficiency, even when the number of agents becomes large. Unlike the directed search framework, our setting allows for cases in which, alongside efficient equilibria, there exist other equilibria in which more productive types fail to separate themselves from less productive ones, inducing a mismatch of workers and companies, so that more productive agents find a counterpart with less than the maximal feasible probability. This phenomenon may persist even in a large economy. Moreover, even without productivity differences, coordination failures may survive when multiple market places are active. We note that the multiplicity of possibly inefficient equilibria is reminiscent of the view of the labor market originally advocated by Diamond (1982).

The Poisson game structure has become standard in economics to model large *but finite* settings. It has been applied, so far, mostly to voting games, as shown by Myerson (2000, 2002). The idea is that in mass elections the voters ignore the exact size and identities of the electorate. Applications to other economic environments with a large number of agents have been flourishing in the past years.[8] In the macroeconomic literature, Poisson games have been studied in connection with strategic complementarity by Makris (2008). Related to our setting, the Poisson structure has been introduced in search theory by Norman (2016), whose model embodies the standard directed search element by assuming, first, a Poisson game among homogeneous sellers and, then, a Poisson game among homogeneous buyers who observe the realizations of the sellers and the posted price. In Norman (2016), the Poisson framework is used to endogenize the restriction to symmetric equilibria often made in the literature and to handle the sellers' entry problem. Equilibrium uniqueness is also established, like in finite models that rely on symmetric continuation strategies (e.g. Kim and Camera, 2014). Not surprisingly, the coordination problem embedded in our simultaneous move setting may induce a multiplicity of equilibria even when agents are homogeneous.

[4] In some version of the model, there are market makers. The classic references are Montgomery (1991), Moen (1997), Mortensen and Wright (2002).

[5] There are exceptions when workers can make multiple applications, as shown in Albrecht et al. (2006).

[6] See, e.g., Peters (1991, 2000), Burdett et al. (2001), Galenianos and Kircher (2009). Wright et al. (2021) provide a survey of directed search models.

[7] Convergence to competitive equilibrium relies on an appropriate selection criterion for equilibria of the subgame, which is satisfied by symmetric equilibria.

[8] See, for instance, Makris (2009) for public good provision; Ritzberger (2009) and De Sinopoli et al. (2023) for, respectively, Bertrand and Cournot competition; Lauermann and Speit (2023) for auctions.

Let us stress that our model is framed in a job market setting for concreteness, but can be applied to a variety of situations of frictional trade where the number and identities of the market participants are not common knowledge (as, for example, goods markets, marriage markets, or general partnership formation situations).

In sum, population uncertainty captures in a simple way the anonymity and noise that seem to be typical of large search markets, while keeping the environment finite, so that the impact of individual behavior on aggregate outcomes remains non-negligible and is fully taken into account. At the same time, the environmental equivalence property of Poisson games captures the effect of competition in a model where mismatch inefficiencies due to the non-negligible role of individuals survive even within a large population.

The paper proceeds as follows. The model is described in Section 2, the main assumptions are discussed in Section 3 and in Appendix B. Section 4 derives the equilibria and contains the existence results. Section 5 explores the efficiency properties of equilibria. Section 6 presents the model with wage announcements. Section 7 concludes. Appendix A outlines the basic structure of Poisson games, Appendices C and D contain proofs omitted from Section 4.

2. The model

We consider an economy where the total number of agents is a Poisson random variable with mean n.[9] Agents divide in employers and workers and can be of different types. There are I types of employers who differ in their productivity levels, and J types of workers who differ in their unemployment incomes. The set of types is $\mathcal{T} = \{1, \dots, I, I+1, \dots, I+J\}$. With probability r_i a randomly sampled agent is an employer with productivity y_i (i.e. she is of type i), where $i = 1, \dots, I$, while with probability r_j she is a worker with unemployment income z_j (i.e. she is of type j), where $j = I+1, \dots, I+J$. We order types so that $y_i < y_{i'}$ for $i < i'$ and $z_j < z_{j'}$ for $j < j'$. We let

$$r_e = \sum_{i=1}^{I} r_i$$

denote the probability that a randomly sampled agent is an employer, and

$$r_w = \sum_{j=I+1}^{I+J} r_j = 1 - r_e$$

denote the probability that a random agent is a worker.

We assume that there are K given market places arranged exogenously by a third-party market maker, each one characterized by a wage. That is, all the jobs in the same market pay the same wage. We let $w_k \in \mathbb{R}_+$ be the wage paid in market k and we order markets so that $w_k \leq w_{k'}$ for $k < k'$, where $k, k' = 1, \dots, K$.

Each employer chooses whether to open a vacancy in one of the market places, sustaining a sunk cost $c \geq 0$, or not. We denote employers' action set with

$$A_e = \{1, \dots, K, O\}$$

(with $A_i = A_e$ for every i), where action k corresponds to entering market k and action O to not entering any market. Each worker chooses whether to search for a job in one of the market places or to remain unemployed. We denote workers' action set with

$$A_w = \{\bar{1}, \dots, \bar{K}, \bar{O}\}$$

(with $A_j = A_w$ for every j), where action $\bar{k}$ corresponds to entering market k and action $\bar{O}$ to not searching for a job. The set of actions is $A = A_e \cup A_w$.

[9] Appendix A presents the basic structure of Poisson games and the properties that are relevant for our analysis (we refer to Myerson, 1998).

For every $k = 1, \dots, K$ we define $x_k = (x(k), x(\bar{k}))$, where $x(k)$ and $x(\bar{k})$ are respectively the realized number of employers and the realized number of workers who enter market k. The number of matches between employers and workers in market k is determined by the matching function $f(x(k), x(\bar{k})) : \mathbb{Z}_+^2 \to \mathbb{Z}_+$. An employer fills her vacancy in market k with probability

$$q(x(k), x(\bar{k})) = \frac{f(x(k), x(\bar{k}))}{x(k)},$$

while a worker finds a job in k with probability

$$s(x(k), x(\bar{k})) = \frac{f(x(k), x(\bar{k}))}{x(\bar{k})}.$$

An agent's payoff depends on her type, her action, and the realized number of other agents who choose each action, which is summarized by the action profile $x \in \mathbb{Z}_+^{|A|}$. For every $x \in \mathbb{Z}_+^{|A|}$, $i = 1, \dots, I$, $j = I+1, \dots, I+J$, $k = 1, \dots, K$ and $\bar{k} = \bar{1}, \dots, \bar{K}$, the payoffs obtained by an employer of type i are given by

$$u_i(O, x) = 0,$$
$$u_i(k, x) = q(x(k)+1, x(\bar{k}))(y_i - w_k) - c,$$

while the payoffs obtained by a worker of type j are given by

$$u_j(\bar{O}, x) = z_j,$$
$$\begin{aligned} u_j(\bar{k}, x) &= s(x(k), x(\bar{k})+1) w_k + (1 - s(x(k), x(\bar{k})+1)) z_j \\ &= z_j + s(x(k), x(\bar{k})+1)(w_k - z_j). \end{aligned}$$

Note that the payoff for an agent of entering a given market depends only on the number of other agents who also enter that market.[10]

A strategy function (or, simply, strategy) maps each type $t \in \mathcal{T}$ to the corresponding set of mixed actions $\Delta(A_t)$. Strategy σ induces the average behavior $\tau(\sigma) \in \Delta(A)$ which is defined by $\tau(\sigma)(a) = \sum_{t \in \mathcal{T}} r_t \sigma_t(a)$ for each $a \in A$. When agents play according to σ, $\tau(\sigma)(a)$ is the probability that a randomly sampled agent chooses action a.[11]

When the population's average behavior is summarized by τ, the probability that in market k there are exactly $x(k)$ employers and $x(\bar{k})$ workers is equal to

$$P(x_k \mid \tau) = e^{-n(\tau(k)+\tau(\bar{k}))} \frac{[n\tau(k)]^{x(k)}}{x(k)!} \frac{[n\tau(\bar{k})]^{x(\bar{k})}}{x(\bar{k})!}.$$

Then, the probability for an employer to fill her vacancy if she enters market k is given by

$$Q(\tau(k), \tau(\bar{k})) = \sum_{x_k \in \mathbb{Z}_+^2} P(x_k \mid \tau) q(x(k)+1, x(\bar{k})),$$

and her expected payoff if she is of type i is

$$U_i(k, \tau) = \sum_{x_k \in \mathbb{Z}_+^2} P(x_k \mid \tau) u_i(k, x) = Q(\tau(k), \tau(\bar{k}))(y_i - w_k) - c.$$

Similarly, the probability for a worker to find a job in market k is given by

$$S(\tau(k), \tau(\bar{k})) = \sum_{x_k \in \mathbb{Z}_+^2} P(x_k \mid \tau) s(x(k), x(\bar{k})+1),$$

and her expected payoff if she is of type j is

$$U_j(\bar{k}, \tau) = \sum_{x_k \in \mathbb{Z}_+^2} P(x_k \mid \tau) u_j(\bar{k}, x) = z_j + S(\tau(k), \tau(\bar{k}))(w_k - z_j).$$

[10] Workers' heterogeneity may also lie in their productivities rather than in their unemployment incomes. In particular, the results of our analysis remain valid as long as more productive workers are better off than less productive ones in case they do not find a counterpart, and the gain of matching with a higher productivity worker is larger the higher is a firm's productivity.

[11] As explained in Appendix A, τ is a sufficient statistic for the analysis of agents' optimal behavior. Hence, we will often avoid to specify its dependence on σ.

To simplify notation, we will sometimes use Q_k and S_k instead of $Q(\tau(k), \tau(\bar{k}))$ and $S(\tau(k), \tau(\bar{k}))$.

Definition 1. The strategy function $\sigma \in \Sigma$ is a *Nash equilibrium* if $U_t(\sigma_t, \tau(\sigma)) \geq U_t(\sigma_t', \tau(\sigma))$ for all $t \in \mathcal{T}$, $\sigma_t' \in \Delta(A_t)$.

We say that the average behavior τ is an *equilibrium behavior* (or, simply, *equilibrium*) if it is induced by a Nash equilibrium.

The analysis will focus on undominated Nash equilibria. We recall the standard concepts of dominated action and dominated strategy. An action is dominated if there is another action that gives higher utility for every possible average behavior of the population, and strictly higher utility for at least one. A strategy is dominated if it prescribes a dominated action for some type.

Definition 2. Action σ_t is *dominated* by action σ_t' for agents of type t if $U_t(\sigma_t, \tau) \leq U_t(\sigma_t', \tau)$ for every τ and $U_t(\sigma_t, \tau') < U_t(\sigma_t', \tau')$ for some τ'.

Definition 3. The strategy function σ is *dominated* if there is a $t \in \mathcal{T}$ such that σ_t is a dominated action for players of type t.

3. Assumptions

Matching function

We make the following natural assumptions on the matching function $f(x(k), x(\bar{k}))$:

(A1) $f(x(k), x(\bar{k})) \leq \min\{x(k), x(\bar{k})\}$,
(A2) $f(x(k), x(\bar{k}))$ is not identically equal to zero,
(A3) $f(x(k), x(\bar{k}))$ is non-decreasing in both arguments.

Moreover, we make the standard assumption that

(A4) $f(x(k), x(\bar{k}))$ is homogeneous of degree one.

In our discrete setting, this implies that the matching function is necessarily the min function, i.e. $f(x(k), x(\bar{k})) = \min\{x(k), x(\bar{k})\}$. To see this note that, by (A1), we have $f(0,0) = 0$. Then, note that $f(1,1) = 0$ would contradict (A2), as (A4) would imply that $f(n,n) = 0$ for every n and then (A1) and (A3) would imply that $f(n,n') = 0$ for every $n \neq n'$. Thus, we have $f(1,1) = 1$, $f(n,n) = n$ for every n by (A4), and then $f(n,n') = \min\{n,n'\}$ for every $n \neq n'$ by (A1) and (A3).

The assumption (A4) can be substituted by some sensible properties of the probabilities $q(x(k), x(\bar{k}))$ and $s(x(k), x(\bar{k}))$, specifically, the fact that q is non-increasing in $x(k)$ and non-decreasing in $x(\bar{k})$, and that s is non-decreasing in $x(k)$ and non-increasing in $x(\bar{k})$. These properties readily imply (A3). In Appendix B, we show that if we assume these properties in addition to (A1) and (A2) then the matching function can be either the min function $f(x(k), x(\bar{k})) = \min\{x(k), x(\bar{k})\}$ or the min function with a threshold $f(x(k), x(\bar{k})) = \min\{x(k), x(\bar{k}), \hat{x}\}$ for some $\hat{x} \in \mathbb{Z}_+ \backslash \{0\}$. We employ the min function for simplicity, but all the qualitative results remain valid for any strictly positive threshold.

Remark. Alternatively to a deterministic matching function, we could employ a probabilistic matching function in the vein of Burdett et al. (2001).[12] Assume that, once each worker has entered in a given market k, applies to all firms that are there with the same probability, as in the equilibrium considered in that analysis. In this case, given the realizations $x(k)$ and $x(\bar{k})$ of the numbers of employers and workers in the market, the total number of matches is a random variable with expected value

$$\tilde{f}(x(k), x(\bar{k})) = x(k) \left[1 - \left(1 - \frac{1}{x(k)} \right)^{x(\bar{k})} \right].$$

[12] We thank two anonymous referees for having suggested this.

The probability for an employer to fill her open vacancy in the market is

$$\tilde{q}(x(k), x(\bar{k})) = \frac{\tilde{f}(x(k), x(\bar{k}))}{x(k)},$$

while the probability for a worker to find a job in the market is

$$\tilde{s}(x(k), x(\bar{k})) = \frac{\tilde{f}(x(k), x(\bar{k}))}{x(\bar{k})}.$$

The matching function $\tilde{f}$ is non-decreasing in both arguments. Moreover, $\tilde{q}$ is non-increasing in $x(k)$ and non-decreasing in $x(\bar{k})$, while $\tilde{s}$ is non-decreasing in $x(k)$ and non-increasing in $x(\bar{k})$.[13] This implies that all our qualitative results remain valid if we use the function $\tilde{f}$ instead of f.

Utilities

We assume that

$$w_1 > z_{I+J},$$

that is, the lowest paid wage is larger than the highest unemployment income for workers. This simplifying assumption guarantees that, in every undominated strategy, every worker searches for a job in some market.

Also, we assume that employers' highest productivity level y_I is such that

$$Q(0, r_w)(y_I - w_k) - c > 0$$

for every $k = 1, \ldots, K$. This implies that at least the highest productivity employers are willing to enter a market with strictly positive probability when all the workers are there, ensuring the existence of undominated equilibria in which matching occurs with positive probability in some market.

4. Nash equilibria

We begin this section with an illustrative example.

Example 1. Consider an economy with two market places paying wages $w_1 = 10$ and $w_2 = 12$, two types of employers with productivity levels $y_1 = 15$ and $y_2 = 20$, and two types of workers with unemployment incomes $z_3 = 2$ and $z_4 = 6$. Let $r_1 = r_2 = \frac{1}{4}$, $r_3 = \frac{1}{6}$, $r_4 = \frac{1}{3}$, and $c = 1$.

For computational purposes, we consider sufficiently large values of n. Note that, if n is large enough, for every average behavior τ and action a the number of agents choosing a is sufficiently close to $n\tau(a)$ with probability sufficiently close to 1. Consider the strategy such that every agent enters market 1. For n sufficiently large, the probability for each agent to be matched in that market is sufficiently close to 1, while the probability to be matched in market 2 is 0. Since, for each type, the payoff of entering market 1 is also larger than the payoff of staying out, the strategy under consideration is a Nash equilibrium. Similarly, for sufficiently large n, there exists an equilibrium where every agent enters market 2.

In addition to these two equilibria in which all the agents enter one of the markets, we can show that there exists an equilibrium where agents enter both markets. Consider the strategy such that employers and workers of low type enter market 1 while employers and workers of high type enter market 2. For n sufficiently large, the probability for an employer to be matched in market 1 (resp. 2) is sufficiently close to $\frac{2}{3}$ (resp. 1), while the probability for a worker to be matched in market 1 (resp. 2) is sufficiently close to 1 (resp. $\frac{3}{4}$). Thus, we have

$$Q_1(y_1 - w_1) > Q_2(y_1 - w_2)$$

and

$$Q_1(y_2 - w_1) < Q_2(y_2 - w_2),$$

since

$$\frac{3}{5} < \frac{Q_1}{Q_2} < \frac{4}{5}.$$

Hence, every employer of low type strictly prefers to enter market 1 rather than market 2 and every employer of high type strictly prefers to enter market 2 rather than market 1. Moreover, we have

$$S_1(w_1 - z_3) > S_2(w_2 - z_3)$$

and

$$S_1(w_1 - z_4) < S_2(w_2 - z_4),$$

since

$$\frac{5}{4} < \frac{S_1}{S_2} < \frac{3}{2}.$$

That is, every worker of low type strictly prefers to enter market 1 rather than market 2 and every worker of high type strictly prefers to enter market 2 rather than market 1. It can be easily verified that every agent prefers to enter some market rather than not. It follows that the strategy under consideration is a Nash equilibrium. In this equilibrium, higher types enter the market paying the higher wage. This ordering of types in equilibrium turns out to hold in general.

We now turn to the general analysis of the equilibria of the model. It is clear that the strategy function that prescribes every agent to not enter any market is a Nash equilibrium. The assumption that $w_1 > z_{I+J}$ implies that this autarkic equilibrium is dominated.

Lemma 1. *The average behavior τ such that $\tau(O) = r_e$ and $\tau(\bar{O}) = r_w$ is a dominated equilibrium.*

We focus on the undominated equilibria in which at least one market place is active.[14] We say that market k is *active* given the average behavior τ if at least some employers and some workers enter k with positive probability, that is, $\tau(k)\tau(\bar{k}) > 0$. Environmental equivalence implies that, given an average behavior τ and a market k, the probability to fill the vacancy in k is the same for every employer and the probability to find a job in k is the same for every worker, independently of their type. This determines an ordering of types in equilibrium that is in line with the findings of Moen (1997) with a continuum of traders, where the environment is equivalent for every agent because each agent is individually insignificant.

Consider an equilibrium behavior τ such that markets k and k' are both active, with $w_k < w_{k'}$. Since some employers choose to enter market k' in equilibrium, the higher wage they have to pay must be compensated by a larger probability of being matched in that market. Analogously, since some workers enter market k in equilibrium, the probability for them to find a job in that market must be higher than in market k'.[15] We can therefore state the following.

Lemma 2. *Let markets k and k' be active given the equilibrium τ, with $w_k < w_{k'}$. Then $Q_k < Q_{k'}$ and $S_k > S_{k'}$.*

[13] See the discussion in Burdett et al. (2001).

[14] There may be undominated equilibria in which workers randomize over entering different markets while employers choose to not enter any market. However, these equilibria are equivalent to the autarkic equilibrium and we exclude them from the analysis.

[15] Clearly, if two markets k and k' pay the same wage and are both active in equilibrium then, for every agent, the probability to be matched must be the same in k and in k'.

We can show that, in equilibrium, employers and workers of higher types choose markets that pay higher wages, as emerged from Example 1. Types are not necessarily well-separated as in that example, but some types may be indifferent between entering different markets (or staying out) and play a mixed action, the other types being ordered accordingly.[16]

The next lemma shows that, given an equilibrium behavior and two active markets, if an employer prefers to enter the market that pays the higher wage then all the employers with larger productivities also do so (and the larger the productivity, the larger the gain of choosing that market rather than the other). Likewise, if an employer prefers to enter the market that pays the lower wage, then the same is true for all the employers with smaller productivities (and the smaller the productivity, the larger the gain of choosing that market rather than the other).[17] Recall that $y_i < y_{i'}$ for $i < i'$.

Lemma 3. *Let markets k and k' be active given the equilibrium τ, with $w_k < w_{k'}$, and let $i < i'$. If $U_i(k', \tau) \geq U_i(k, \tau)$ then $U_{i'}(k', \tau) > U_{i'}(k, \tau)$, while if $U_{i'}(k, \tau) \geq U_{i'}(k', \tau)$ then $U_i(k, \tau) > U_i(k', \tau)$.*

Proof. We can prove that

$$U_{i'}(k', \tau) - U_{i'}(k, \tau) > U_i(k', \tau) - U_i(k, \tau),$$

which implies both results. The above inequality is equivalent to

$$Q_{k'}(y_{i'} - w_{k'}) - Q_k(y_{i'} - w_k) > Q_{k'}(y_i - w_{k'}) - Q_k(y_i - w_k).$$

Rearranging terms, we have

$$Q_{k'} > Q_k,$$

which is satisfied by Lemma 2. □

Analogously, given an equilibrium behavior and two active markets, if a worker prefers to enter the market that pays the larger wage then all the workers of higher types will do so (and the higher the type, the higher the gain of choosing that market rather than the other). On the other hand, if a worker prefers to enter the market that pays the lower wage then the same holds for every worker with lower unemployment income (and the lower the type, the lower the gain of choosing that market rather than the other). The proof of this result resembles that of Lemma 3, so we skip it. Recall that $z_j < z_{j'}$ for every $j < j'$.

Lemma 4. *Let markets k and k' be active given the equilibrium τ, with $w_k < w_{k'}$, and let $j < j'$. If $U_j(\bar{k}', \tau) \geq U_j(\bar{k}, \tau)$ then $U_{j'}(\bar{k}', \tau) > U_{j'}(\bar{k}, \tau)$, while if $U_{j'}(\bar{k}, \tau) \geq U_{j'}(\bar{k}', \tau)$ then $U_j(\bar{k}, \tau) > U_j(\bar{k}', \tau)$.*

Given that equilibria are such that agents are ordered according to their types, some market is active, and nobody chooses to enter markets where the probability of being matched is zero, it is convenient to conduct the equilibrium analysis fixing the number of active markets.

To this end, it is useful to examine the behavior of the functions Q and S as their arguments vary. First, note that both functions are continuous in their arguments. Consider then the extreme cases. The probability that an employer fills her vacancy in market k when no other employer is expected to enter there is equal to

$$\begin{aligned} Q(0, \tau(\bar{k})) &= \sum_{x(k)=0}^{\infty} \mathbf{P}(x(k) \mid 0) \sum_{x(\bar{k})=0}^{\infty} \mathbf{P}(x(\bar{k}) \mid n\tau(\bar{k})) q(x(k)+1, x(\bar{k})) \\ &= \sum_{x(\bar{k})=0}^{\infty} \mathbf{P}(x(\bar{k}) \mid n\tau(\bar{k})) q(1, x(\bar{k})) = \sum_{x(\bar{k})=1}^{\infty} \mathbf{P}(x(\bar{k}) \mid n\tau(\bar{k})) = 1 - e^{-n\tau(\bar{k})}, \end{aligned}$$

that is the probability that at least one worker enters market k given $\tau(\bar{k}) \geq 0$. On the other hand, if no worker enters market k then an employer will not fill her vacancy in that market, independently of other employers' behavior, i.e.

$$\begin{aligned} Q(\tau(k), 0) &= \sum_{x(k)=0}^{\infty} \mathbf{P}(x(k) \mid n\tau(k)) \sum_{x(\bar{k})=0}^{\infty} \mathbf{P}(x(\bar{k}) \mid 0) q(x(k)+1, x(\bar{k})) \\ &= \sum_{x(k)=0}^{\infty} \mathbf{P}(x(k) \mid n\tau(k)) q(x(k)+1, 0) = 0 \end{aligned}$$

for every $\tau(k) \geq 0$. Similar expressions define workers' probabilities of finding a job in market k. In particular, when no other worker is expected to enter that market, a worker will find a job with probability

$$S(\tau(k), 0) = 1 - e^{-n\tau(k)}$$

given $\tau(k) \geq 0$, while if no employer opens a vacancy in market k then a worker will not find a job in that market, i.e.

$$S(0, \tau(\bar{k})) = 0$$

for every $\tau(\bar{k}) \geq 0$.

Finally, the properties of the Poisson distribution imply that the probability for an agent to be matched in market k strictly increases with the probability that agents on the opposite side of the market enter k and strictly decreases with the probability that agents on the same side of the market choose k.

Lemma 5. *$Q(\tau(k), \tau(\bar{k}))$ is strictly decreasing in $\tau(k)$ and strictly increasing in $\tau(\bar{k})$, while $S(\tau(k), \tau(\bar{k}))$ is strictly increasing in $\tau(k)$ and strictly decreasing in $\tau(\bar{k})$.*

Proof. See Appendix C. □

We can now examine the equilibria in which only one market place is active and then consider equilibria in which matching between employers and workers occurs with positive probability in a couple of market places. This analysis will provide some economic insights and reveal characteristics of the equilibria that hold in general.

4.1. One active market

Let market k be the unique market that is active in equilibrium. No agent enters any other market with positive probability, since the probability of being matched there is zero. In fact, every undominated strategy prescribes every worker to choose k with probability 1 and, in equilibrium, at least some employers choose market k with positive probability given the assumption on the highest productivity level. By Lemma 3, there exists a type i^* that separates employers entering the market and employers staying out. In particular, either there is a type i^* who is indifferent between entering market k and staying out, all types $i > i^*$ entering k while all types $i < i^*$ staying out, or there is a type i^* such that all types $i \geq i^*$ enter market k while all types $i < i^*$ stay out. To characterize equilibria, we need to identify type i^*.

To this end, consider the average behavior τ such that $\tau(\bar{k}) = r_w$ and $\tau(k) = r_e - r_1$, that is, all the workers enter market k as well as all and only the employers with productivity larger than y_1. The expected payoff of entering k for the employers with the lowest productivity level y_1 is equal to

$$Q(r_e - r_1, r_w)(y_1 - w_k) - c. \tag{4.1}$$

If this payoff is strictly positive, in equilibrium type 1 employers enter k with positive probability. In particular, in case their payoff

$$Q(r_e, r_w)(y_1 - w_k) - c$$

[16] For instance, if we modify Example 1 letting $r_3 = r_4 = 1/4$, we obtain the equilibrium σ with $\sigma_1(1) = 1$, $\sigma_2(1) = 0$, $\sigma_3(\bar{1}) = 3/4$, $\sigma_4(\bar{1}) = 0$, where firms' types are well separated while workers' types are not, as low type workers randomize over the two markets.

[17] Of course, given any population behavior, if an employer prefers to enter some market rather than not then all the employers of higher types also do so, while if she prefers to not enter any market then the same is true for all the employers of lower types.

when every other employer enters the market is positive, they enter market k with probability 1.[18] In case it is strictly negative, they enter k with probability $\alpha \in (0,1)$ such that

$$Q(r_e - (1-\alpha)r_1, r_w)(y_1 - w_k) - c = 0.$$

Since the function Q is continuous and strictly decreasing in the first argument, α exists and is unique. In both cases, all the employers of higher types also enter market k with probability 1. Thus, in the first case we have the equilibrium τ^* with $\tau^*(k) = r_e$ and $\tau^*(\bar{k}) = r_w$, while in the second case we have $\tau^*(k) = r_e - (1-\alpha)r_1$, $\tau^*(O) = (1-\alpha)r_1$, and $\tau^*(\bar{k}) = r_w$.

If (4.1) is negative, then in equilibrium all employers of type 1 do not enter the market, and the incentives of employers with productivity y_2 must be examined. In particular, if

$$Q(r_e - r_1, r_w)(y_2 - w_k) - c \geq 0$$

then they enter market k with probability 1, and the same is true for all the employers of higher types. In this case we have the equilibrium τ^* such that $\tau^*(k) = r_e - r_1$, $\tau^*(O) = r_1$ and $\tau^*(\bar{k}) = r_w$. If

$$Q(r_e - r_1, r_w)(y_2 - w_k) - c < 0$$

and

$$Q(r_e - r_1 - r_2, r_w)(y_2 - w_k) - c > 0,$$

then type 2 employers enter market k with some probability $\beta \in (0,1)$ (where β is derived analogously to the above α, exists and is unique by Lemma 5), while all the employers of higher types enter k with probability 1. In this case we have $\tau^*(k) = r_e - r_1 - (1-\beta)r_2$, $\tau^*(O) = r_1 + (1-\beta)r_2$, and $\tau^*(\bar{k}) = r_w$. If

$$Q(r_e - r_1 - r_2, r_w)(y_2 - w_k) - c \leq 0,$$

then in equilibrium type 2 employers do not enter the market, and incentives of employers with productivity y_3 must be examined, and so on.

This analysis can be done for each market place. It shows that an equilibrium where a given market is the unique active one exists given the assumption that firms are productive enough.[19] The next proposition establishes the uniqueness of such an equilibrium.

Proposition 1. *For each market, there exists a unique equilibrium such that that market is the only active one.*

Proof. Suppose that there are two equilibria τ_1^* and τ_2^* such that market k is the unique active market. We have $\tau_1^*(\bar{k}) = \tau_2^*(\bar{k}) = r_w$. Let x be the probability that a randomly sampled agent is an employer who enters market k in the first equilibrium, i.e. $x = \tau_1^*(k)$. By Lemma 3, there is a type i_1^* such that either

$$Q(x, r_w)(y_i - w_k) - c < 0 \quad \text{for } i < i_1^*,$$
$$Q(x, r_w)(y_i - w_k) - c > 0 \quad \text{for } i \geq i_1^*,$$

where $x = \sum_{i=i_1^*}^{I} r_i$, or

$$Q(x, r_w)(y_{i_1^*} - w_k) - c = 0,$$

where $x = \alpha r_{i_1^*} + \sum_{i=i_1^*+1}^{I} r_i$ for some $\alpha \in [0,1]$. Likewise, let $y \neq x$ be the probability that an agent is an employer who enters market k in the second equilibrium, i.e. $y = \tau_2^*(k)$, and let i_2^* be the type such that either

$$Q(y, r_w)(y_i - w_k) - c < 0 \quad \text{for } i < i_2^*,$$
$$Q(y, r_w)(y_i - w_k) - c > 0 \quad \text{for } i \geq i_2^*,$$

where $y = \sum_{i=i_2^*}^{I} r_i$, or

$$Q(y, r_w)(y_{i_2^*} - w_k) - c = 0,$$

where $y = \beta r_{i_2^*} + \sum_{i=i_2^*+1}^{I} r_i$ for some $\beta \in [0,1]$.

Let $y > x$. By Lemma 5, we have $Q(y, r_w) < Q(x, r_w)$. Hence, the above conditions relative to the two equilibria hold simultaneously only if $i_1^* < i_2^*$. But this implies $y < x$, leading to a contradiction. A similar argument applies to the case $y < x$. Therefore, the result follows. □

4.2. Two active markets

We now consider equilibria in which two market places are active, namely k and k', with $w_k < w_{k'}$. In these equilibria every worker enters one of the markets, while some employers may choose to stay out. Thus, let s be the probability that a randomly selected agent is a worker who enters market k, i.e. $\tau(\bar{k}) = s$ and $\tau(\bar{k}') = r_w - s$, let x be the probability that an agent is an employer who enters market k and let y be the probability she is an employer who enters k', i.e. $\tau(k) = x$, $\tau(k') = y$, and $\tau(O) = r_e - x - y$.

We begin extending to this case the previous procedure to derive equilibria, and then we explore their existence.

Workers

Given the behavior of employers, we can determine the type j^* that, in equilibrium, separates workers who enter market k and workers who enter market k' and, consequently, the equilibrium behavior of workers.

To this end, fix x and y and consider the workers with the largest unemployment income z_{I+J}. Since markets k and k' are both active in equilibrium, either these workers are indifferent between the two markets or they strictly prefer to enter k' rather than k. They are indifferent if two conditions hold. First, their payoff of entering market k must be larger than that of entering market k' when all and only the workers of highest type enter k', i.e.

$$S(x, r_w - r_{I+J})(w_k - z_{I+J}) \geq S(y, r_{I+J})(w_{k'} - z_{I+J}).$$

Second, their payoff of entering k must be strictly smaller than that of entering k' when no other worker is in k', i.e.

$$S(x, r_w)(w_k - z_{I+J}) < S(y, 0)(w_{k'} - z_{I+J}),$$

which is the necessary condition for market k' to be active. In this case, the workers of highest type enter market k' with probability $\alpha \in (0,1]$ such that

$$S(x, r_w - \alpha r_{I+J})(w_k - z_{I+J}) = S(y, \alpha r_{I+J})(w_{k'} - z_{I+J}),$$

while all the workers of lower types enter market k, so $s = r_w - \alpha r_{I+J}$.[20]

On the other hand, if

$$S(x, r_w - r_{I+J})(w_k - z_{I+J}) < S(y, r_{I+J})(w_{k'} - z_{I+J})$$

then in equilibrium the workers of highest type enter market k' with probability 1, and the incentives of the workers of lower types need to be examined to determine type j^*.

Employers

Given the behavior of workers s, the equilibrium behavior of employers is described by two types. The type i^* that separates employers who do not enter any market and employers who enter one of them,

[18] Note that, by Lemma 5, in this case the payoff of entering k is the minimum given $\tau(\bar{k}) = r_w$.

[19] The same is true in Moen (1997).

[20] Note that α exists and is unique by Lemma 5.

and the type l^* that separates employers who enter market k' and employers who either enter market k or stay out.[21]

To this end, fix s and y. If $y > r_e - r_1$, then the employers of lowest type must be indifferent between the two markets while all the employers of higher types enter market k' with probability 1, so we have $i^* = l^* = 1$. In particular, it can be either

$$Q(r_e - y, s)(y_1 - w_k) = Q(y, r_w - s)(y_1 - w_{k'}) \geq c,$$

so that every employer enters some market and $x = r_e - y$, or

$$Q(\alpha r_1, s)(y_1 - w_k) = Q(y, r_w - s)(y_1 - w_{k'}) = c$$

for some $\alpha \in (0,1)$ with $\alpha r_1 < r_e - y$, so that employers of type 1 randomize over not entering any market, entering market k and entering market k', and $x = \alpha r_1$.

Let $y \leq r_e - r_1$. If the expected payoff for employers of type 1 of entering market k when no employer stays out is positive, i.e.

$$Q(r_e - y, s)(y_1 - w_k) - c \geq 0,$$

then they enter market k with probability 1, so $i^* = 1$ and $x = r_e - y$. If that payoff is strictly negative but their expected payoff of entering market k when all and only type 1 employers stay out is positive, i.e.

$$Q(r_e - r_1 - y, s)(y_1 - w_k) - c \geq 0,$$

then they are indifferent between staying out and entering market k. In this case they stay out with probability $\beta \in (0,1]$ such that

$$Q(r_e - \beta r_1 - y, s)(y_1 - w_k) - c = 0,$$

so we have $i^* = 1$ and $x = r_e - \beta r_1 - y$.[22] On the other hand, if

$$Q(r_e - r_1 - y, s)(y_1 - w_k) - c < 0,$$

then type 1 employers do not enter any market in equilibrium, and one has to consider employers of higher types to determine i^*.

Now, let s and x be fixed, and consider the employers with highest productivity level y_I. Note that for market k' to be active it must be

$$Q(x, s)(y_I - w_k) < Q(0, r_w - s)(y_I - w_{k'}).$$

If $x > r_e - r_I$, then employers of type I must be indifferent between entering market k and entering market k', and all the employers of lower types either enter market k or stay out, so we have $l^* = I$. In particular, the employers of highest type enter market k' with probability $\gamma \in (0,1)$ such that

$$Q(x, s)(y_I - w_k) = Q(\gamma r_I, r_w - s)(y_I - w_{k'}), \tag{4.2}$$

so we have $y = \gamma r_I$.

Let $x \leq r_e - r_I$. If employers of type I are better off entering market k rather than market k' when all and only the employers of their same type enter k', that is, if

$$Q(x, s)(y_I - w_k) \geq Q(r_I, r_w - s)(y_I - w_{k'}),$$

then, as before, they must be indifferent between the two markets. So we have $l^* = I$ and $y = \gamma r_I$, where $\gamma \in (0,1]$ solves (4.2).[23] On the other hand, if

$$Q(x, s)(y_I - w_k) < Q(r_I, r_w - s)(y_I - w_{k'})$$

then employers of type I enter market k' with probability 1, and the incentives of the employers of lower types need to be explored to determine l^*.

Equilibrium

The above analysis leads to three equilibrium conditions. One condition summarizes workers' equilibrium behavior, which is characterized by the type j^* that separates those entering market k and those entering market k'. Such a condition is either of the form

$$S(x, s)(w_k - z_{j^*}) = S(y, r_w - s)(w_{k'} - z_{j^*}) \tag{4.3a}$$

or of the form

$$\begin{aligned} &S(x, s)(w_k - z_j) > S(y, r_w - s)(w_{k'} - z_j) \quad \text{for } j < j^* \\ &S(x, s)(w_k - z_j) < S(y, r_w - s)(w_{k'} - z_j) \quad \text{for } j \geq j^*, \end{aligned} \tag{4.3b}$$

where $s = \sum_{j=I+1}^{j^*} r_j - \alpha r_{j^*}$ for some $\alpha \in [0,1]$.

Then, we have two conditions that describe employers' equilibrium behavior. One condition identifies the type i^* who separates employers not entering any market and employers entering market k, and can be either of the form

$$Q(x, s)(y_{i^*} - w_k) - c = 0 \tag{4.4a}$$

or of the form

$$\begin{aligned} &Q(x, s)(y_i - w_k) - c < 0 \quad \text{for } i < i^* \\ &Q(x, s)(y_i - w_k) - c > 0 \quad \text{for } i \geq i^*, \end{aligned} \tag{4.4b}$$

while the other condition identifies the type l^* who separates employers entering market k and employers entering market k', and can be either of the form

$$Q(x, s)(y_{l^*} - w_k) = Q(y, r_w - s)(y_{l^*} - w_{k'}) \tag{4.5a}$$

or of the form

$$\begin{aligned} &Q(x, s)(y_i - w_k) > Q(y, r_w - s)(y_i - w_{k'}) \quad \text{for } i < l^* \\ &Q(x, s)(y_i - w_k) < Q(y, r_w - s)(y_i - w_{k'}) \quad \text{for } i \geq l^*, \end{aligned} \tag{4.5b}$$

where

$$x = \sum_{i=i^*}^{l^*} r_i - \beta r_{i^*} - \gamma r_{l^*} \quad \text{and} \quad y = \gamma r_{l^*} + \sum_{i=l^*+1}^{I} r_i$$

for some $\beta \in [0,1]$ and $\gamma \in [0,1]$.[24]

Conditions (4.3a) and (4.3b) implicitly determine s as a continuous function of x and y. Lemma 5 implies that $s(x,y)$ is weakly increasing in x and weakly decreasing in y. As intuition suggests, the higher is the probability that employers enter one of the markets, the higher is the probability that workers will also choose that market. Weakness derives from the fact that, when the separating type of workers is not indifferent between the two markets so that the relevant condition is (4.3b), a small change in employers' behavior does not induce a change in workers' optimal choices. On the other hand, when the separating type is indifferent between the two markets, then s must adjust if x or y vary, in order to maintain the equality in (4.3a). Similarly, conditions (4.4a) and (4.4b) implicitly determine x as a continuous and weakly increasing function of s.[25] Intuitively, if the probability that workers enter market k increases, then the probability that employers choose that market also increases. Finally, given x, conditions (4.5a) and (4.5b)

[21] It is clear that i^* and l^* can coincide, either when there is a type who is indifferent between staying out and entering k (and possibly also entering k'), all lower types staying out while all higher types entering k', or when there is a type who is indifferent between the two markets, all lower types staying out while all higher types entering k'.

[22] Note that β exists and is unique by Lemma 5.

[23] Note that γ exists and is unique by Lemma 5.

[24] Recall that we are considering $w_k < w'_k$. If $w_k = w'_k$, we have $S(x,s) = S(y, r_w - s)$ and $Q(x,s) = Q(y, r_w - s)$. Every worker will be indifferent between the two markets and strictly prefer entering some market rather than not. As regards employers, there will be types indifferent between the two markets (who may strictly prefer entering a market rather than staying out or may be indifferent) and, possibly, types who strictly prefer to not enter any market.

[25] As before, when the separating type of employers is indifferent between entering k and not entering any market, if s increases then x increases so that the probability $Q(x,s)$ remains constant, while if the separating type strictly prefers to enter market k then a small increase in s does not induce a change in x, so $Q(x,s)$ increases.

implicitly determine y as a continuous and weakly decreasing function of s. In fact, the lower is the probability that workers enter market k', the lower will be the probability that employers choose that market.

An equilibrium is given by values of s, x and y for which the opportune conditions are simultaneously satisfied. Hence, it is the solution to a system of 3 equations (or 2 inequalities in place of some of them) and 3 unknowns. Such a solution does not always exist. We now illustrate an example in which there is no equilibrium where two given market places are both active, even if the two equilibria where each of the markets is the unique active one exist.

Example 2. The example is constructed as follows. We have an economy with two market places, one type of employers, and one type of workers. The probability of being an employer is close to 0 while the probability of being a worker is close to 1. In this case, the probability for an employer to be matched in a market when all the workers are there is close to the maximum. The wages paid in the two markets are close, and the difference between employers' productivity and each of these wages is slightly larger than the cost of opening the vacancy. When all the workers enter one of the markets, the probability for an employer to be matched in that market is sufficiently high to overcome the cost, implying the existence of the two equilibria in which each market is the unique active one. However, when workers are spread between the two markets, the probability for an employer to be matched in the market that workers choose with lower probability is too low to overcome the cost, implying that an equilibrium in which both markets are active does not exist.

Formally, let the expected number of agents be $n = 4$, let $y_1 = 3$, $z_2 = 1$, and $c = 0.9$. To simplify computations, let $w_1 = w_2 = 2$, $r_e = 0$, and $r_w = 1$. The results of this degenerate case will remain valid in a neighborhood of these parameters values given the continuity of the equilibrium conditions, which will involve strict inequalities. Consider the equilibrium in which only one market is active, namely market 1. Workers enter it with probability 1, and so do employers because

$$Q(r_e, r_w)(y_1 - w_1) = Q(0, 1) = 1 - e^{-4} \approx 0.981 > 0.9 = c.$$

This implies that the two equilibria in which each market is the only active one exist, and are such that all the agents enter the market.

Now, suppose that both markets are active in equilibrium. Workers divide between the two markets, entering market 1 with probability $\sigma_2(\bar{1})$ and market 2 with probability $1 - \sigma_2(\bar{1})$. Since $r_w = 1$, we have $\tau(\bar{1}) = \sigma_2(\bar{1})$. Let $\sigma_2(\bar{1}) \leq \frac{1}{2}$. Then

$$Q(0, \tau(\bar{1})) = 1 - e^{-4\tau(\bar{1})} < 1 - e^{-2} \approx 0.865 < 0.9 = c.$$

It follows that an employer strictly prefers to not open her vacancy rather than to open it in market 1, hence an equilibrium in which both markets are active does not exist.

The non-existence of equilibrium in this example is due to the fact that the cost for employers of opening the vacancy is relatively too high. We can prove that an equilibrium in which a given couple of market places are active exists as long as that cost is sufficiently small relative to the other parameters.

To this end, consider the simplest case with one type of employers and one type of workers, and let $c = 0$. In an equilibrium where two markets are active, each agent must be indifferent between entering one or the other, and nobody chooses to stay out. In this case, the equilibrium conditions are given by two equalities. Consider the condition relative to employers. It implicitly defines employers' behavior, which is described in this case by one variable, as a continuous and strictly increasing function of the behavior of the workers. Note that, if workers enter one market with probability 0, then employers strictly prefer to enter the other market rather than that one. By continuity, employers' indifference condition cannot be satisfied when the probability that workers enter one of the markets is in a neighborhood of 0. By Lemma 5, there exists a (unique) minimum value of that probability

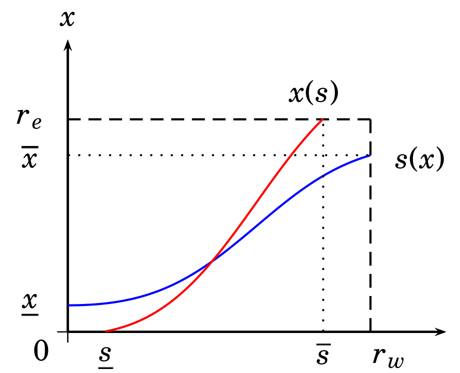

Fig. 1. Existence of equilibria with two active markets when $I = J = 1$.

such that employers' equilibrium condition is satisfied. For that value, an employer is indifferent between the two markets when no other employer is in the one that workers choose with the lowest probability. An analogous reasoning applies to the indifference condition relative to workers. It follows that the functions defining employers' and workers' equilibrium behavior intersect at least once, as shown in Fig. 1 (where x and s are the probabilities that an agent is, respectively, an employer and a worker entering the same market). This implies that at least an equilibrium exists. Since in equilibrium employers strictly prefer to enter some market rather than not, existence holds for sufficiently small, but strictly positive values of c.

This result can be extended to the general – but more twisted – case with I types of employers and J types of workers. We therefore have the following proposition, the proof of which is provided in Appendix D.

Proposition 2. *For each couple of markets, if c is sufficiently small then there exists an equilibrium such that those markets are the only active ones.*

To summarize, in an economy with K market places there exist at least K equilibria. For each market, indeed, there exists a unique equilibrium in which only that market is active, given the mild assumption that the economy is productive enough. As regards strategic stability considerations, all these equilibria satisfy every basic strategic principle one can hope for, being stable sets as singletons.[26] Equilibria in which a given pair of markets are active exist under some additional conditions on the parameters.

In general, equilibria can be derived from a simple and intuitive procedure, are characterized by clean semianalytical equalities and inequalities (see Meroni and Pimienta, 2017) and present the major feature that higher types meet in markets that pay higher wages. As discussed in the next section, this does not necessarily induce efficiency.

5. Efficiency

In this section we examine the question of efficiency. In our setting efficiency does not always obtain. In fact, according to the characteristics of the economy, it may require specific markets to be active in equilibrium.

[26] It is not difficult to see that each equilibrium is a stable set as defined in De Sinopoli et al. (2014). In broad terms, a stable set of a Poisson game is a minimal subset of Nash equilibria such that every close-by game obtained through perturbations of the average behavior has a Nash equilibrium close to the stable set. The equilibria under analysis prescribe strict best responses for all types except at most one, i.e. the type of employers who is indifferent between entering the market and staying out. However, whenever the best response of this type is not strict, any perturbation of the average behavior in the definition of stable set can be compensated by this type's mixed action.

When employers' productivities are sufficiently close, as well as workers' unemployment incomes, efficiency essentially requires that, for any realization of the population, the total number of matches should be maximized. This implies that only equilibria with a unique active market can be efficient, as equilibria with multiple active markets suffer from the misallocations of agents due to coordination failures among them.

To see this, consider the simplest case with one type of employers and one type of workers, and let $c = 0$. If one market place is active in equilibrium, all the agents drawn from the Poisson distribution to participate in the job market enter such a market with probability 1 and the number of matches is maximal, being the minimum between the number of employers and the number of workers. If more than one market place is active in equilibrium, the agents drawn to participate in the job market randomize over the active markets and combinations inducing less than the maximum number of possible matches occur with positive probability.[27] It is clear that if the cost c is sufficiently small then all the equilibria with a unique active market are efficient, since they all attract every agent. However, if the cost is substantial, the higher is the wage paid in a market the lower is the probability that firms enter that market for the same behavior of the workers and, consequently, the lower could be the resulting number of matches. For this reason, an equilibrium with a unique active market paying a higher wage could be less efficient than an equilibrium with a unique active market paying a lower one.

When types are sufficiently heterogeneous, not only the number of matches but also the types of agents that are matched become relevant for welfare considerations.

In this case, equilibria with a unique active market are not necessarily efficient, even if the cost c is negligible. In fact, for a given behavior of the workers, the higher is the wage that a market pays the fewer are the employers it attracts, but the higher is the productivity of such employers. Thus, an equilibrium with a unique active market paying a higher wage could dominate in welfare terms an equilibrium with a unique active market paying a lower one. Furthermore, efficiency may even require multiple markets to be active, as this could significantly increase the matching probabilities of types bearing on aggregate welfare more, thanks to a better allocation of types among markets. However, the inefficient equilibria with a unique active market would necessarily coexist given Proposition 1.[28]

The next example presents an economy where an equilibrium with two active markets exists and is more efficient than the equilibria in which each of the markets is the only active one.

Example 3. Consider an economy with two market places paying wages $w_1 = 8$ and $w_2 = 14$, two types of employers with productivity levels $y_1 = 15$ and $y_2 = 80$, and two types of workers with unemployment incomes $z_3 = 1$ and $z_4 = 5$. Let $r_1 = \frac{3}{7}$, $r_2 = \frac{1}{7}$, $r_3 = \frac{1}{7}$, $r_4 = \frac{2}{7}$, and let $c = 0$ for simplicity. We can show that, if n is sufficiently large, there exists an equilibrium τ^* in which employers and workers of low type enter market 1 while employers and workers of high type enter market 2. That is, $\tau^*(1) = \frac{3}{7}$, $\tau^*(\bar{1}) = \frac{1}{7}$, $\tau^*(2) = \frac{1}{7}$, and $\tau^*(\bar{2}) = \frac{2}{7}$. Moreover, we can show that the expected total welfare induced by this equilibrium is larger than that induced by the two equilibria in which either only market 1 or only market 2 is active.

Given τ^*, if n is sufficiently large then the probability a worker finds a job in market 1 (resp. market 2) is sufficiently close to 1 (resp. $\frac{1}{2}$), while the probability an employer fills her vacancy in market 1 (resp. market 2) is sufficiently close to $\frac{1}{3}$ (resp. 1). It follows that

$$S_1(w_1 - z_3) > S_2(w_2 - z_3),$$
$$S_1(w_1 - z_4) < S_2(w_2 - z_4),$$

since

$$\frac{13}{7} < \frac{S_1}{S_2} < 3,$$

and that

$$Q_1(y_1 - w_1) > Q_2(y_1 - w_2),$$
$$Q_1(y_2 - w_1) < Q_2(y_2 - w_2),$$

since

$$\frac{1}{7} < \frac{Q_1}{Q_2} < \frac{11}{12}.$$

That is, agents of low type strictly prefer to enter market 1 rather than market 2, while agents of high type strictly prefer to enter market 2 rather than market 1.

Besides this equilibrium τ^* where both markets are active, we have the two equilibria in which each of the two markets is the only active one. In particular, given that $y_1 > w_2$ and $w_1 > z_4$, these equilibria are such that every agent enters the market with probability 1.

We can now derive the total welfare induced by each of the above equilibria. Every match between an employer and a worker generates the corresponding productivity, as the wage is a mere transfer from one party to the other. Every worker that is not matched contributes to the total welfare with her unemployment income. Thus, for n large enough, the expected total welfare induced by each equilibrium with a unique active market can be approximated by

$$\frac{3}{7}n\left(\frac{3}{4}y_1 + \frac{1}{4}y_2\right) = \frac{375}{28}n,$$

while the expected total welfare induced by the equilibrium where both markets are active is approximated by

$$\frac{1}{7}ny_1 + \frac{1}{7}ny_2 + \frac{1}{7}nz_4 = \frac{380}{28}n + \frac{5}{7}n = \frac{100}{7}n.$$

6. Wage announcements

In this section, following Moen (1997), we show that the equilibria of the model in which wages are established by the market maker can be realized if one assumes that wages are announced by the agents.

Take a Poisson game with expected number of players n, I types of employers and J types of workers whose characteristics are as defined in Section 2. Differently from before, assume that agents can directly announce the desired wage or decide to not announce any wage. That is, the action sets are $A_e = A_w = \mathbb{R}_+ \cup \{O\}$, where action O corresponds to not announcing any wage. A market is formed when at least one employer and one worker post the same wage. As before, let the number of matches in a market be equal to the minimum between the number of employers and the number of workers who announce the corresponding wage. Let the agents' payoffs be as defined in Section 2, taking appropriately into account the change in their action sets.

Consider an equilibrium of the original model in which the set of active markets is $\tilde{K}$. Recall that no agent chooses with positive probability a market that is not active. In the model with wage announcements, the equivalent strategy combination is such that each type entering market $k \in \tilde{K}$ (resp. not entering any market) with some positive probability posts the corresponding wage w_k (resp. does not announce any wage) with the same probability. Then, an agent who expects opponents to play according to this combination has no incentive to deviate and choose a different wage $w_{k'}$ with $k' \in \tilde{K}$, because the original profile is an equilibrium. Moreover, no agent has incentive to announce a wage that no market in $\tilde{K}$ pays, because the probability of being matched in

[27] With a continuum of agents this form of inefficiency cannot occur, as probabilities and shares coincide. In our Poisson model, if n is large enough then it occurs with probability close to 0.

[28] With heterogeneous agents, efficiency has been attained in different models that feature continuum economies and allow firms to commit to richer mechanisms (see, e.g., Shi, 2006; Eeckhout and Kircher, 2010; Jerez, 2014; Cai et al., 2023).

that case would be zero. It follows that every equilibrium of the original model is an equilibrium also in the model with wage announcements.

Take now an equilibrium of the model with wage announcements with finite support $\tilde{W} \subset \mathbb{R}_+$, and consider the original model in which the market maker has established the corresponding set of markets; that is, for every wage $w \in \tilde{W}$ there is a market that pays w. The equivalent strategy combination in that model is such that each type posting salary w (resp. not posting any salary) with some positive probability enters the market that pays w (resp. stays out from any market) with the same probability. It is clear that, under such a strategy combination, no agent has incentive to deviate to any other action because the profile under consideration is an equilibrium. On the other hand, every equilibrium of the model with wage announcements with infinite support is equivalent to the autarkic equilibrium of the original model, since no matching occurs.[29] We can conclude that every equilibrium of the decentralized model with wage announcements is also induced by the original model.

Remark.

Since Poisson games are defined by Myerson (1998) as simultaneous games, in which players do not observe the realization of their opponents but have only some probabilistic information about their number and characteristics, we consider simultaneous announcements by employers and workers. The labor literature has considered both simultaneous and sequential wage determination. In particular, the directed search literature has analyzed the case of wage announcements assuming a two-stage process, in which first employers announce the wage offers and then workers apply for a job after having observed those offers. In the Poisson framework, this would correspond to a structure with, first, a Poisson game in which employers announce wages and, then, another Poisson game in which workers choose one of those wages knowing employers' realization and choices (as in Norman, 2016). However, this would be inconsistent with our setting in which the agents' strategies do not depend on the precise realization of opponents' behavior but only on their expectation of this behavior.

7. Conclusion

This paper has presented a new model of the job market based on the Poisson representation of a game with population uncertainty. The game is finite and the Poisson structure allows to analyze the equilibria in a fairly simple way and to obtain clean analytical solutions. The undominated equilibria have properties that place the model in between the random and directed search model typically used in the labor literature. The forces of competition are at work through the environmental equivalence property of Poisson games but strategic effects matter, giving rise to multiple, possibly inefficient equilibria even when the economy becomes large.

CRediT authorship contribution statement

Francesco De Sinopoli: Writing – review & editing, Writing – original draft, Funding acquisition, Formal analysis, Conceptualization. **Leo Ferraris:** Writing – review & editing, Writing – original draft, Funding acquisition, Formal analysis, Conceptualization. **Claudia Meroni:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Poisson games

In this appendix we outline the basic structure of Poisson games and the properties that are relevant for our analysis.[30]

A Poisson game is given by a tuple $\left(n, \mathcal{T}, r, A, (A_t)_{t\in\mathcal{T}}, u\right)$. The number of players is distributed according to a Poisson random variable with parameter n. Hence, the probability that there are m players in the game is equal to

$$\mathbf{P}(m \mid n) = \frac{e^{-n} n^m}{m!}.$$

The set $\mathcal{T} = \{1, \dots, T\}$ is the set of player types. The probability that a randomly selected player is of each type is given by the vector $r = (r_1, \dots, r_T) \in \Delta(\mathcal{T})$. That is, a player is of type $t \in \mathcal{T}$ with probability r_t. A type profile $y \in \mathbb{Z}_+^T$ is a vector that specifies for each type $t \in \mathcal{T}$ the number of players y_t of that type. The finite set of actions is A. The set of actions that are available to players of type t is $A_t \subseteq A$. An action profile $x \in Z(A) = \mathbb{Z}_+^{|A|}$ specifies for each action $a \in A$ the number of players $x(a)$ that choose that action. Players' preferences are summarized by the vector $u = (u_1, \dots, u_T)$, where $u_t : A_t \times Z(A) \to \mathbb{R}$ for every $t \in \mathcal{T}$. We interpret $u_t(a, x)$ as the payoff obtained by a type t player when she chooses action a and the realization resulting from the rest of the population's behavior is the action profile $x \in Z(A)$.

The set of mixed actions for players of type t is $\Delta(A_t)$. A strategy function σ is an element of $\Sigma = \left\{\sigma \in \Delta(A)^T : \sigma_t \in \Delta(A_t) \text{ for all } t\right\}$, mapping each type to the corresponding set of mixed actions. The average behavior induced by the strategy function σ is given by $\tau(\sigma) \in \Delta(A)$ and it is defined by $\tau(\sigma)(a) = \sum_{t\in\mathcal{T}} r_t \sigma_t(a)$. Let $\tau(\Sigma) = \{\tilde{\tau} \in \Delta(A) : \tilde{\tau} = \tau(\sigma) \text{ for some } \sigma \in \Sigma\}$. It is clear that the same average behavior can be induced by different strategy functions. However, τ is a sufficient statistic for the analysis of agents' optimal behavior, as an agent's payoff depends only on the number of other agents who choose each action, independently of their specific types.

A Poisson game is characterized by the following properties.

Decomposition property. Let each player be independently assigned some characteristic in a set S according to some given probability distribution $(\theta(s))_{s\in S}$. Let $w(s)$ denote the number of players who have characteristic s. For every $s \in S$, the random variables $w(s)$ are mutually independent, and each $w(s)$ has a Poisson distribution with mean $n\theta(s)$.

Independent actions property. For every strategy function σ and action a, the random variables $x(a)$ are independent random variables.

[29] Note that if some employers and some workers randomize with atomless probability distributions over a continuum of wages, the probability for an agent to be matched if she announces a wage in that continuum is zero. Since employers sustain a positive cost of posting, they strictly prefer to not announce any wage rather than to announce a wage in the continuum, while workers are indifferent between the two choices. Thus, any equilibrium with infinite support is such that at least some workers randomize with atomless probability distribution over a continuum of wages, while employers do not announce any wage, so every agent is matched with probability zero.

[30] We refer to Myerson (1998).

Environmental equivalence property. Let $\mathcal{P}(y \mid t)$ be the conditional probability that a player of type t assigns to the event that the other players in the game have type profile $y = (y_1, \dots, y_T)$. We have

$$\mathcal{P}(y \mid t) = \prod_{t \in \mathcal{T}} \left(e^{-nr_t} \frac{(nr_t)^{y_t}}{y_t!} \right)$$

for every $y \in \mathbb{Z}_+^T$ and $t \in \mathcal{T}$. That is, a player of any type assesses the same probability distribution for the type profile of the other players as an external observer assesses for the type profile of the whole game.

The decomposition and independent actions properties imply that, when the population's aggregate behavior is summarized by $\tau \in \tau(\Sigma)$, the number of players who choose action a is a Poisson random variable with mean $n\tau(a)$ and is independent of the number of players who choose any other action. Then, the probability that the action profile $x \in Z(A)$ is realized is equal to

$$P(x \mid \tau) = \prod_{a \in A} \left(e^{-n\tau(a)} \frac{(n\tau(a))^{x(a)}}{x(a)!} \right).$$

Environmental equivalence implies that $P(x \mid \tau)$ is also the probability that each player assigns to the event that the action profile resulting from the rest of the population's behavior is x. Hence, the expected payoff to a player of type t who plays $a \in A_t$ is given by

$$U_t(a, \tau) = \sum_{x \in Z(A)} P(x \mid \tau) u_t(a, x).$$

A Nash equilibrium of a Poisson game is a description of behavior for the entire population that is consistent with the players' utility maximizing actions given that they use such a description to form their beliefs about the population's expected behavior.

Definition 4. The strategy function $\sigma \in \Sigma$ is a *Nash equilibrium* if

$$U_t(\sigma_t, \tau(\sigma)) \geq U_t(\sigma'_t, \tau(\sigma)) \qquad \text{for all } t \in \mathcal{T}, \sigma'_t \in \Delta(A_t).$$

Appendix B. Matching function

In this appendix we show that the min function and the min function with a threshold are the only matching functions that satisfy the natural assumptions that

(A1) $f(x(k), x(\bar{k})) \leq \min\{x(k), x(\bar{k})\}$ and
(A2) $f(x(k), x(\bar{k}))$ is not identically equal to zero,

in addition to the reasonable assumptions that

(A3) $q(x(k), x(\bar{k}))$ is non-increasing in $x(k)$ and non-decreasing in $x(\bar{k})$, and
(A4) $s(x(k), x(\bar{k}))$ is non-increasing in $x(\bar{k})$ and non-decreasing in $x(k)$.

To simplify notation, let $x(k) = x$ and $x(\bar{k}) = \bar{x}$. First, note that $q(x, \bar{x})$ is not defined if $x = 0$ while $s(x, \bar{x})$ is not defined if $\bar{x} = 0$. By (A1), we have $f(n, 0) = f(0, n) = 0$ for every $n \in \mathbb{Z}_+$.

Let $x, \bar{x} \in \mathbb{Z}_+ \backslash \{0\}$. Since q is non-decreasing in $\bar{x}$ we have that f must be non-decreasing in $\bar{x}$, and since s is non-decreasing in x we have that f must be non-decreasing in x. Now, consider the function q. For q to be non-increasing in x it must be that, for every x and $\bar{x}$,

$$q(x + 1, \bar{x}) - q(x, \bar{x}) \leq 0,$$

which is equivalent to

$$x f(x + 1, \bar{x}) - x f(x, \bar{x}) - f(x, \bar{x}) \leq 0. \tag{B.1}$$

Suppose that $f(x + 1, \bar{x}) > f(x, \bar{x}) + 1$ for some x and $\bar{x}$. Then the lhs of (B.1) would be strictly greater than $x - f(x, \bar{x})$, which is non-negative by (A1), contradicting (B.1). So it must be $f(x + 1, \bar{x}) \leq f(x, \bar{x}) + 1$ for every x and $\bar{x}$. Given that f is non-decreasing in x, we have either $f(x + 1, \bar{x}) = f(x, \bar{x})$ or $f(x + 1, \bar{x}) = f(x, \bar{x}) + 1$. Consider $x \leq \bar{x}$ and suppose that $f(x, \bar{x}) < x$. If $f(x + 1, \bar{x}) = f(x, \bar{x}) + 1$ then the lhs of (B.1) would be equal to $x - f(x, \bar{x})$, again contradicting (B.1), so it must be $f(x + 1, \bar{x}) = f(x, \bar{x})$.

An analogous reasoning applied to s implies that, for every x and $\bar{x}$, it must be either $f(x, \bar{x}+1) = f(x, \bar{x})$ or $f(x, \bar{x}+1) = f(x, \bar{x})+1$. Moreover, if $\bar{x} \leq x$, whenever $f(x, \bar{x}) < \bar{x}$ it must be $f(x, \bar{x}+1) = f(x, \bar{x})$. Combining this with the above result we have that, whenever $f(x, \bar{x}) < \min\{x, \bar{x}\}$ for some x and $\bar{x}$, it must be

$$f(n, \bar{n}) = f(x, \bar{x})$$

for every $n \geq x$ and $\bar{n} \geq \bar{x}$.

It can be easily seen that the matching function $f(x, \bar{x}) = \min\{x, \bar{x}\}$ satisfies all the assumptions. On the other hand, the above discussion implies that, among the functions f such that $f(x, \bar{x}) < \min\{x, \bar{x}\}$ for some x and $\bar{x}$, all the assumptions can be satisfied only by the functions of type $f(x, \bar{x}) = \min\{x, \bar{x}, \hat{x}\}$ for some $\hat{x} \in \mathbb{Z}_+ \backslash \{0\}$.

Appendix C. Proof of Lemma 5

Lemma 5. *$Q(\tau(k), \tau(\bar{k}))$ is strictly decreasing in $\tau(k)$ and strictly increasing in $\tau(\bar{k})$, while $S(\tau(k), \tau(\bar{k}))$ is strictly increasing in $\tau(k)$ and strictly decreasing in $\tau(\bar{k})$.*

Proof. We can show that $Q(\tau(k), \tau(\bar{k}))$ is strictly decreasing in $\tau(k)$ by showing that, for every $\tau'(k) > \tau(k)$,

$$Q(\tau'(k), \tau(\bar{k})) - Q(\tau(k), \tau(\bar{k})) < 0. \tag{C.1}$$

The above inequality is equivalent to

$$\sum_{x(k) \in \mathbb{Z}_+} \left[\mathbf{P}(x(k) \mid n\tau'(k)) - \mathbf{P}(x(k) \mid n\tau(k)) \right] \sum_{x(\bar{k}) \in \mathbb{Z}_+} \mathbf{P}(x(\bar{k}) \mid n\tau(\bar{k})) q(x(k) + 1, x(\bar{k})) < 0.$$

The probability distribution over $\mathbb{Z}_+$ induced by $n\tau'(k)$ first order stochastically dominates the one induced by $n\tau(k)$, that is,

$$\sum_{x(k)=0}^{\bar{x}} \mathbf{P}(x(k) \mid n\tau'(k)) < \sum_{x(k)=0}^{\bar{x}} \mathbf{P}(x(k) \mid n\tau(k))$$

for every $\bar{x} \in \mathbb{Z}_+$.[31] Since $q(x(k) + 1, x(\bar{k}))$ is non-increasing in $x(k)$, inequality (C.1) follows. Likewise, $Q(\tau(k), \tau(\bar{k}))$ is strictly increasing in $\tau(\bar{k})$ because, for every $\tau'(\bar{k}) > \tau(\bar{k})$,

$$\begin{aligned} & Q(\tau(k), \tau'(\bar{k})) - Q(\tau(k), \tau(\bar{k})) = \\ & = \sum_{x(\bar{k}) \in \mathbb{Z}_+} \left[\mathbf{P}(x(\bar{k}) \mid n\tau'(\bar{k})) - \mathbf{P}(x(\bar{k}) \mid n\tau(\bar{k})) \right] \\ & \quad \times \sum_{x(k) \in \mathbb{Z}_+} \mathbf{P}(x(k) \mid n\tau(k)) q(x(k) + 1, x(\bar{k})) > 0, \end{aligned}$$

given that the probability distribution over $\mathbb{Z}_+$ induced by $n\tau'(\bar{k})$ first order stochastically dominates the one induced by $n\tau(\bar{k})$, and $q(x(k) + 1, x(\bar{k}))$ is non-decreasing in $x(\bar{k})$.

In an analogous way it can be shown that $S(\tau(k), \tau(\bar{k}))$ is strictly increasing in $\tau(k)$ and strictly decreasing in $\tau(\bar{k})$, given that s is non-decreasing in $x(k)$ and non-increasing in $x(\bar{k})$. □

Appendix D. Proof of Proposition 2

Proposition 2. *For each couple of markets, if c is sufficiently small then there exists an equilibrium such that those markets are the only active ones.*

[31] In fact, the decomposition property of Poisson games implies that the number of agents choosing k is a Poisson random variable, and $\sum_{k=0}^{\bar{k}} \mathbf{P}(k \mid n') < \sum_{k=0}^{\bar{k}} \mathbf{P}(k \mid n)$ for every $n' > n$ and $\bar{k} \in \mathbb{Z}_+$.

Proof. Let $c = 0$ and consider an equilibrium with two active markets, namely k and k'. All the agents enter some market in equilibrium. Let s and x be the probabilities that a random agent is, respectively, a worker and an employer entering market k. We can derive the equilibrium behavior of workers as a function of x, which is implicitly given by

$$S(x,s)(w_k - z_{j^*}) = S(r_e - x, r_w - s)(w_{k'} - z_{j^*})$$

and

$$S(x,s)(w_k - z_j) > S(r_e - x, r_w - s)(w_{k'} - z_j) \quad \text{for } j < j^*$$
$$S(x,s)(w_k - z_j) < S(r_e - x, r_w - s)(w_{k'} - z_j) \quad \text{for } j \geq j^*,$$

where j^* is the type that separates workers entering k and workers entering k', and $s = \sum_{j=I+1}^{j^*} r_j - \alpha r_{j^*}$ for some $\alpha \in [0,1]$. Analogously, we can derive the equilibrium behavior of employers as a function of s, which is implicitly given by

$$Q(x,s)(y_{i^*} - w_k) = Q(r_e - x, r_w - s)(y_{i^*} - w_{k'}) \tag{D.2a}$$

and

$$Q(x,s)(y_i - w_k) > Q(r_e - x, r_w - s)(y_i - w_{k'}) \quad \text{for } i < i^*$$
$$Q(x,s)(y_i - w_k) < Q(r_e - x, r_w - s)(y_i - w_{k'}) \quad \text{for } i \geq i^*, \tag{D.2b}$$

where i^* is the type that separates employers entering k and employers entering k', and $x = \sum_{i=1}^{i^*} r_i - \beta r_{i^*}$ for some $\beta \in [0,1]$.

We can show that the resulting functions $s(x)$ and $x(s)$ are both continuous and weakly increasing. Consider $x(s)$. If $s = 0$ we have

$$Q(x,0)(y_i - w_k) < Q(r_e - x, r_w)(y_i - w_{k'})$$

for every i, since $Q(x,0) = 0$ for every x. That is, no employer chooses to enter market k if all the workers choose k'. Given the strict inequalities, this holds true for strictly positive values of s sufficiently close to 0. In particular, since we have

$$Q(0,0)(y_1 - w_k) < Q(r_e, r_w)(y_1 - w_{k'})$$

and

$$Q(0, r_w)(y_1 - w_k) > Q(r_e, 0)(y_1 - w_{k'}),$$

by Lemma 5 there exists a value $s_1 > 0$ such that

$$Q(0, s_1)(y_1 - w_k) = Q(r_e, r_w - s_1)(y_1 - w_{k'}).$$

As s increases above s_1, x increases above 0 in order to keep condition (D.2a) satisfied for $i^* = 1$. That is, employers of type 1 begin to enter market k with strictly positive probability and both markets become active. Such probability increases continuously up to 1, in correspondence to the value $s_2 > s_1$ such that

$$Q(r_1, s_2)(y_1 - w_k) = Q(r_e - r_1, r_w - s_2)(y_1 - w_{k'}),$$

and we have $x(s_2) = r_1$. By Lemma 5, $s_2 < r_w$ because

$$Q(r_1, r_w)(y_1 - w_k) > Q(r_e - r_1, 0)(y_1 - w_{k'}).$$

Obviously, if $I = 1$ then $x(s_2) = r_e$. If $I > 1$, as s increases above s_2 the relevant condition becomes (D.2b) with $i^* = 2$, and we have $x(s) = r_1$ for all the values of s up to the value $s_3 < r_w$ such that

$$Q(r_1, s_3)(y_2 - w_k) = Q(r_e - r_1, r_w - s_3)(y_2 - w_{k'}).$$

As s increases above s_3, also employers of type 2 begin to enter market k with strictly positive probability, and so on, up to the value s_{2I} such that

$$Q(r_e, s_{2I})(y_I - w_k) = Q(0, r_w - s_{2I})(y_I - w_{k'}),$$

for which no employer enters market k'. By Lemma 5, $s_{2I} < r_w$ because

$$Q(r_e, r_w)(y_I - w_k) > Q(0,0)(y_I - w_{k'}).$$

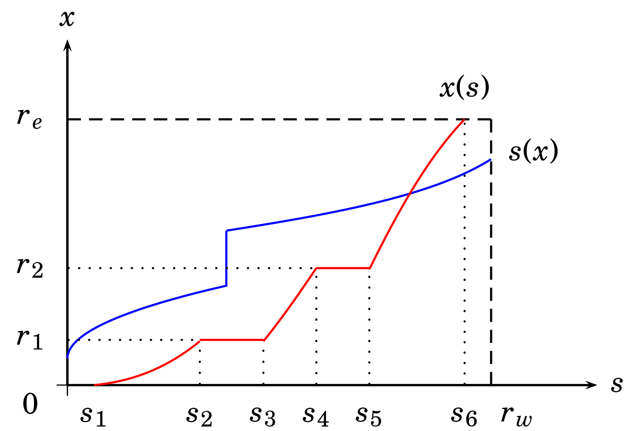

Fig. 2. Existence of equilibria with two active markets.

An analogous argument applies to $s(x)$. It implies that the functions $x(s)$ and $s(x)$ intersect at least once (as shown in Fig. 2 for the case $I = 3$, $J = 2$), so at least one equilibrium exists when $c = 0$. Since the probability for the employers to be matched in each market is strictly positive, they strictly prefer to enter some market rather than not. It follows that an equilibrium exists for strictly positive values of c sufficiently close to 0.

□

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