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WHAT IS NORMAL?

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Abstract

On the basis of the definition of a normal modal system, it is common to distinguish between normal and non-normal modal operators. However, we argue that the standard definition of a normal operator in the literature leads to at least two problems. First, it induces a mismatch between the normality of a system and the attribution of normality to an operator of this system. Second, the standard definition is insensitive to different occurrences of the possibility operator in various normal and non-normal systems. We introduce a definition of normal operator which is not affected by either of these problems, and discuss its applications both to the monomodal and to the multimodal settings.

Keywords: Normal modal operator, Normality and non-normality, Necessity and possibility operators, Non-contingency and consistency operators, Multimodal system

1. Introduction

A thorough investigation of the nature of a logical system should be able to explain how some distinguishing feature of a system **S** is reflected in one of its operators. Intuitively, a basic desideratum is that there should be no mismatch between identifying a property of **S** and the attribution of such a property to an operator of **S**. For instance, the classicality of a system is defined by the satisfaction of the principles of classical propositional logic. This is reflected in the classicality of the operators of classical logic. In particular, the classicality of negation is defined by the satisfaction of the principles of classical propositional logic for negation, e.g., De Morgan's laws, double negation elimination, and others. In what follows, we investigate a particular instance of this concerning the property of normality in a modal system.

The distinction between normal and non-normal modal systems is a widely applied distinction at least since the introduction of Kripke semantics (see Kripke (1963)). This distinction highlights the crucial difference between stronger and weaker systems in terms of several principles. As Blackburn et al. (2001, p. 145) put it: "[normal modal logics] are the standard tool for capturing the notion of validity syntactically." From the definition of a normal system, it seems straightforward to derive a definition of a

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normal operator. However, such a definition is often omitted in the literature and this, as shown below, leads to an ambiguous use of the terms "normal" and "non-normal".

The remainder of this article is organised as follows. Section 2 provides the definition of normal modal system and makes explicit a standard definition of normal modal operator. In Section 3, we point out two problems arising from this definition: the mismatch and insensitivity problems. In Section 4, we discuss and dismiss some possible solutions to these problems. Section 5 provides a new definition of normal modal operator which keeps the positive features of the standard definition but gets rid of both problems, by bridging the gap between the normality of a system and of an operator. Finally, in Section 6, we extend our analysis to the case of multimodal systems.

2. From normal system to normal operator

A modal system is called normal if it validates axioms and inference rules that are widely accepted and are required of any logic whose semantics will be specified using relational frames. A *relational frame* is a tuple $F = \langle W, R \rangle$, where W is a set of worlds, and $R \subseteq W \times W$ is an accessibility relation. A *model* $\langle F, v \rangle$ consists of a frame F and a valuation function v that assigns truth values to each atomic sentence at each world in W. If no restriction on the accessibility relation is specified, we call the corresponding frame a K-*frame*. Syntactically, it is usual to define a normal modal system using the modality $\Box$, called the necessity operator, as follows.[1]

DEFINITION 1. Let $\mathbf{S} = (\mathcal{L}, \vdash)$ be a modal system, where $\mathcal{L}$ is its language and $\vdash$ is a consequence relation. Let $\phi_1, .., \phi_n, \psi$ be arbitrary formulas of the language of $\mathbf{S}$, i.e., $\phi_1, ..., \phi_n, \psi \in Form(\mathcal{L})$. $\mathbf{S}$ is called *normal* if it is closed under uniform substitution and the following rule:

$$(RK_\Box) \text{ if } \vdash (\phi_1 \wedge ... \wedge \phi_n) \to \psi, \text{ then } \vdash (\Box\phi_1 \wedge ... \wedge \Box\phi_n) \to \Box\psi, \text{ where } n \geq 0.$$

Otherwise, $\mathbf{S}$ is called non-normal.

According to this definition, the normality of $\mathbf{S}$ depends on a property encapsulated in $RK_\Box$ asserting that a proposition is necessary if it is implied by a conjunction of necessary propositions.

Different formulations of the definition of normal system can be found in the literature. For instance, a system is normal if it is closed under the K axiom, $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, and the necessitation rule, if $\vdash \phi$,

[1] See, for instance, Chellas (1980, p. 114). Chellas considers the definition of the possibility operator ($\Diamond\phi \Leftrightarrow \neg\Box\neg\phi$) as part of the definition of a normal modal system. However, this condition can be omitted because it is possible to not include $\Diamond$ as a primitive symbol of the language of the logic under consideration.

then $\vdash \Box\phi$. These alternative formulations are equivalent to Definition 1.[2] We choose closure under the rule $RK_{\Box}$ for its compactness, but our analysis can be extended to the other formulations.

Remark 1. If **S** is a normal system, it contains $\Box$ either as a primitive, or as a definable operator, such that $RK_{\Box}$ is a valid rule.

In the light of the Remark 1, systems which do not contain $\Box$, and which do not permit one to express the property encapsulated in $RK_{\Box}$ are non-normal. For instance, consider a modal system with a standard propositional language containing the only modality $\star$, which is interpreted on standard Kripke frames as follows:

$$\mathcal{M}, w \vDash \star\phi \text{ iff there exists } w' \text{ such that } R(w, w') \text{ and } \mathcal{M}, w' \vDash \phi,$$

or

$$\text{for all } w' \text{ if } R(w, w'), \text{ then } \mathcal{M}, w' \vDash \phi.$$

From the definition of $\star$, it is clear that for any formula ϕ, for any world w, $\mathcal{M}, w \vDash \star\phi$. Thus, the rule $RK_{\star}$, that is

$$\text{if} \vdash (\phi_1 \wedge \ldots \wedge \phi_n) \to \psi, \text{ then} \vdash (\star\phi_1 \wedge \ldots \wedge \star\phi_n) \to \star\psi, \text{ where } n \geq 0,$$

is a valid rule. However, $RK_{\star}$ does not encapsulate the same property as $RK_{\Box}$, and thus its validity does not lead to the normality of the system. Otherwise, any system could be trivially extended to the system containing $\star$, and, thus, any system would be normal.

On the basis of the distinction between normal and non-normal systems, it is common to distinguish between normal and non-normal modal operators. In what follows, we call an *n-ary operator* definable in any set of formulas *Form* any mapping $\otimes : Form^n \to Form$.[3] Our study will focus only on unary operators, even though it can be extended to the n-ary case.

It is commonly accepted that $\Box$ is a normal operator whenever it is interpreted in systems validating $RK_{\Box}$, and non-normal otherwise. By generalizing the case of $\Box$, it is usual to define a normal operator as follows.

Definition 2. A modal operator $\otimes$ is *normal* in **S** iff for all $\phi_1, \ldots, \phi_n, \psi \in Form(\mathcal{L})$

$$(RK_{\otimes}) \text{ if} \vdash (\phi_1 \wedge \ldots \wedge \phi_n) \to \psi, \text{ then} \vdash (\otimes\phi_1 \wedge \ldots \wedge \otimes\phi_n) \to \otimes\psi, \text{ where } n \geq 0.$$

Otherwise $\otimes$ is called non-normal.

This definition is based on a straightforward replacement of $\Box$ by $\otimes$ in $RK_{\Box}$, and is explicitly used by Humberstone & Williamson (1997). It permits

[2] See (Chellas 1980, theorem 4.3).
[3] The same definition is used, for instance, by Radev (1987, p. 295).

identifying $\Box$ as normal whenever it is interpreted on relational frames, and as non-normal whenever it is interpreted on minimal models of neighbourhood semantics.[4] In this case, the reason for $\Box$ to be normal or non-normal is the same as for a system to be normal or non-normal, i.e., it depends on the validity of $RK_{\Box}$.

Notice that the $\star$ operator defined above, can be considered as a normal operator in accordance with the Definition 2. Moreover, on the basis of Definition 1, it can be the only primitive modality of a non-normal system.

3. The mismatch and the insensitivity problems

In order to test Definition 2, let us apply it to two modal operators different from $\Box$, namely, non-contingency (Δ) and consistency ($\circ$).[5] These operators are defined via $\Box$ as follows: $\Delta\phi \Leftrightarrow \Box\phi \vee \Box\neg\phi$ and $\circ\phi \Leftrightarrow \phi \to \Box\phi$. The instantiation of both of these operators into the rule $RK_{\otimes}$ does not result in a valid principle whenever Δ and $\circ$ are interpreted on K-frames, as well as on other standard relational frames. Hence, following Definition 2, Δ and $\circ$ are non-normal operators whenever they are interpreted on these frames.

In accordance with Definition 2, Δ and $\circ$ should be called non-normal also on T-frames which are relational frames in which the accessibility relation R is reflexive, i.e., Rww for all $w \in W$. As is pointed out by Sergerberg (1982, p. 128), the $\Box$ operator is definable as $\Box\phi \Leftrightarrow \phi \wedge \Delta\phi$ only in systems validating the axiom T: $\Box\phi \to \phi$, which is a characteristic axiom of T-frames. Similarly to Δ, the definition of $\Box$ in terms of $\circ$ is possible only if the T axiom is valid: $\Box\phi \Leftrightarrow \phi \wedge \circ\phi$. Thus, whenever **S** is defined on T-frames and it contains Δ or $\circ$, the $\Box$ operator can be used as an abbreviation for $(\phi \wedge \Delta\phi)$ and $(\phi \wedge \circ\phi)$ in **S**, even if $\Box$ is not explicitly defined in **S**. This leads to a counter-intuitive situation: **S** is a normal system, in accordance with Definition 1, but its language contains only non-normal modal operators, Δ or $\circ$. The reason for this mismatch is that S has a property expressed via $RK_{\Box}$, but not the property expressed via $RK_{\otimes}$, where $\otimes$ is replaced by either Δ or $\circ$. If one uses Definition 2, the nature of the normality of **S** *qua* system would be different from the nature of the normality of Δ and $\circ$ *qua* operators. More generally, the normality of a system, defined by Definition 1, and the normality of an operator, defined by Definition 2, do not depend on the same property expressed via $RK_{\Box}$. Let us call this problematic situation the *mismatch problem*.

Another counter-intuitive feature of Definition 2 can be observed once we apply it to the analysis of the possibility operator $\Diamond$, usually defined as $\Diamond\phi$

[4] For details on these models see (Chellas 1980, chapter 7).

[5] See (Montgomery & Routley 1966) for non-contingency and (Marcos 2005) for consistency.

$\Leftrightarrow \neg\Box\neg\phi$. If such a definition holds, $\Diamond$ is a non-normal operator whenever it belongs to a logic defined on K-frames and other standard frames. This situation is already quite odd because it indicates a significant difference between $\Box$ and $\Diamond$, which are usually considered as dual operators. Moreover, in the literature on intuitionistic modal logic[6], it is common to refer to intuitionistic $\Diamond$ as a non-normal operator. There are various systems of intuitionistic modal logic, but the common point in all of them is that $\Diamond$ and $\Box$ are not interdefinable. In this context, this seems to be the intuitive reason to call $\Diamond$ non-normal, and not Definition 2. Indeed, this definition clearly does not distinguish between non-normal $\Diamond$ in standard normal systems and non-normal $\Diamond$ in intuitionistic modal systems. Let us call this indistinguishability between classical and intuitionistic $\Diamond$ the *insensitivity problem*.

4. Possible solutions?

One may argue that the mismatch problem can be solved by changing the definition of normal modal system. In this case, a system is normal with respect to an operator $\otimes$ if it is closed under $RK_{\otimes}$, and not only under $RK_{\Box}$. By taking into account this modification, a system characterised by K-frames is normal with respect to $\Box$, but it is non-normal with respect to Δ or $\circ$. However, this strategy would be *ad hoc*, as it changes the very meaning of normality captured by Definition 1. In the standard definition of normal modal system, $RK_{\Box}$ permits one to express the minimal condition for a system to be normal. The generalized rule $RK_{\otimes}$ does not necessarily express the same condition, because the definition of $\otimes$ may vary. Moreover, this modified definition does not solve the insensitivity problem as formulated above. Claiming that a system interpreted on standard K-frames and a system interpreted on intuitionistic K-frames are both non-normal with respect to $\Diamond$ does not permit one to distinguish between non-normal $\Diamond$ in standard normal systems and non-normal $\Diamond$ in intuitionistic modal systems. Finally, if one accepts this modified definition, a system characerized by K-frames should be considered as normal with respect to $\Box$, but non-normal with respect to $\Diamond$, which also diverges from the usual use of normality found in the literature.

One may also argue that the insensitivity problem is not a genuine problem. In particular, one may consider that $\Diamond$ is a non-normal operator in standard normal systems. However, in accordance with this consideration, one cannot distinguish $\Diamond$ interpreted on classical K-frames from $\Diamond$ interpreted on intuitionistic K-frames. This shows that the Definition 2 is useless in providing such a distinction. Moreover, even if one considers only classical K-frames, conceiving $\Diamond$ as a non-normal operator leads to a counterintuitive result. In particular, the system **K**, in which $\Box$ is the only primitive modality,

[6] See Simpson (1994) for a discussion of intuitionistic modal logic and references.

is a normal system containing a normal operator. Let us now replace all the instances of $\Box$ by $\neg\Diamond\neg$ in **K**. The resulting system containing $\Diamond$ as the only modality is equivalent to **K** containing $\Box$, and thus it is also normal. However, the Definition 2 does not explain why a system with a sole non-normal operator is normal.

5. A new definition of normal operator

To eliminate both the mismatch and the insensitivity problems, we propose a new definition of normal operator based on the existence of a *conservative translation* of $\Box$ in terms of some unary modality.[7]

DEFINITION 3. A translation t of a system $\mathbf{S}_1 = (\mathcal{L}_1, \vdash_{S1})$ into a system $\mathbf{S}_2 = (\mathcal{L}_2, \vdash_{S2})$ is said to be *conservative* if it is a function $\mathrm{t} : Form(\mathcal{L}_1) \to Form(\mathcal{L}_2)$ such that, for every subset $\Gamma \cup \{\phi\} \subseteq Form(\mathcal{L}_1)$:

$$\Gamma \vdash_{S1} \phi \text{ iff } \mathrm{t}(\Gamma) \vdash_{S2} \mathrm{t}(\phi).$$

Now we are in the position to formulate the conditions under which a modal operator is normal in a system.

DEFINITION 4. Let $\mathbf{S}_1$ be a system containing the operator $\otimes$, $\mathbf{S}_2$ be a normal modal system containing $\Box$ as the only primitive modality, and $\mathbf{S}_1$ and $\mathbf{S}_2$ share the same non-modal basis. A modal operator $\otimes$ is *normal* in $\mathbf{S}_1$ iff

(i) there exists a conservative translation t of $\mathbf{S}_2$ into $\mathbf{S}_1$ and

(ii) for any $\phi \in Form(\mathcal{L}_2)$, the formula $\mathrm{t}(\Box\phi)$ contains at least one occurrence of $\otimes$ and no other modal operator.

Otherwise $\otimes$ is called non-normal.

Intuitively, this definition states that an operator $\otimes$ is normal in $\mathbf{S}_1$ if it is possible to define a translation t of $\Box$ in terms of $\otimes$ such that it would satisfy $RK_\Box$, i.e., if $\vdash \mathrm{t}((\phi_1 \wedge ... \wedge \phi_n) \to \psi)$, then $\vdash \mathrm{t}((\Box\phi_1 \wedge ... \wedge \Box\phi_n) \to \Box\psi)$, where $n \geq 0$. Condition (i) presupposes that $\Box$ is weakly definable in $\mathbf{S}_1$ in the sense of Prawitz (1965). It should be noticed that the existence of a conservative translation of the normal system $\mathbf{S}_2$ into $\mathbf{S}_1$ will preserve the validity of a translation of $RK_\Box$ in $\mathbf{S}_1$. Condition (ii) guarantees that the translation of $\Box$ in $\mathbf{S}_1$ is defined only in terms of $\otimes$ and, if needed, of other non-modal operators.

Let us now consider the modalities we mentioned above in the light of the new definition of a normal operator. It is clear that $\Box$ defined in a normal

[7] For more details on conservative translations see (Feitosa & D'Ottaviano 2001).

system will be a normal operator because it is always possible to define a homophonic translation[8] from a normal system containing $\Box$ as a sole modality into a normal system containing $\Box$. The conservative translation of $RK_{\Box}$ would remain valid. Similarly, $\Box$ would be a non-normal operator whenever it is interpreted in a non-normal system because there would not be a conservative translation of a normal system containing $\Box$ into a non-normal one. The invalidity of $RK_{\Box}$ in a non-normal system guarantees the impossibility of defining a conservative translation. Hence, Definition 4 recovers the conditions for the normality of $\Box$ encapsulated in Definition 2.

Definition 4 allows one to get rid of the mismatch problem. Let us analyse Δ (the case of $\circ$ is similar). As noted before, $\Box$ is definable in terms of Δ only if the T axiom is valid. On K-frames, which do not validate the T axiom, it is not possible to define a conservative translation of a normal system containing $\Box$ in terms of a logic containing Δ. Thus, Δ is a non-normal operator if it is interpreted on K-frames. The reason is that Δ on K-frames does not permit expressing the property represented by $RK_{\Box}$, which is essential for a system to be normal. However, if Δ is defined in a system interpreted on T-frames, one can define a translation of normal $\Box$ in terms of Δ: $\mathrm{t}(\Box\phi) \Leftrightarrow \mathrm{t}(\phi) \wedge \Delta\mathrm{t}(\phi)$. The translation of non-modal formulas would be homophonic, as before. Thus, on T-frames, one can define a conservative translation of a normal modal system into the system containing Δ. In this case Δ would be considered a normal operator, as it permits one to express and validate the condition for the normality of a system. This can be generalized for an arbitrary operator $\otimes$:

OBSERVATION 1. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be as in Definition 4. In accordance with this definition, the normality of an arbitrary operator $\otimes$ depends on the expressibility of $RK_{\Box}$[9] and the validity of its translation.

Proof. Let $\mathbf{S}_1$ be a modal system containing the operator $\otimes$, $\mathbf{S}_2$ be a normal modal system containing $\Box$ as the only primitive modality, and $\mathbf{S}_1$ and $\mathbf{S}_2$ share the same non-modal basis. If $\otimes$ is normal according to Definition 4, then (i) there exists a conservative translation t of $\mathbf{S}_2$ into $\mathbf{S}_1$ and (ii) for any $\phi \in Form(\mathcal{L}_2)$, the formula $\mathrm{t}(\Box\phi)$ contains at least one occurrence of $\otimes$ and no other modal operator. The existence of a translation and the condition (ii) assure that $RK_{\Box}$ is expressible in the system $\mathbf{S}_1$ as follows: if $\vdash \mathrm{t}((\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \psi)$, then $\vdash \mathrm{t}((\Box\phi_1 \wedge \ldots \wedge \Box\phi_n) \rightarrow \Box\psi)$, where $n \geq 0$. The conservativity of this translation assures that $\mathbf{S}_1$ validates the translation of this rule, and thus the system has the property encapsulated in $RK_{\Box}$.

[8] For a definition of homophonic mapping and translation see (Epstein 2011, p. 405).
[9] By '$RK_{\Box}$ is expressible in $\mathbf{S}_1$' we mean that every concrete instance of $RK_{\Box}$ (i.e., every formula of $\mathbf{S}_2$ which can be instantiated in $RK_{\Box}$ in $\mathbf{S}_2$) is expressible in $\mathbf{S}_1$; that is, there exists a translation of $RK_{\Box}$ in terms of $\otimes$ in $\mathbf{S}_1$.

If $\otimes$ is not normal in accordance with Definition 4, then either (iii) there is no conservative translation t of $\mathbf{S}_2$ into $\mathbf{S}_1$, or (iv) for some $\phi \in Form(\mathcal{L}_2)$, the formula $\mathrm{t}(\Box\phi)$ either does not contain $\otimes$, or contains other modal operators. Let us consider the case (iii) under the assumption that, for any $\phi \in Form(\mathcal{L}_2)$, the formula $\mathrm{t}(\Box\phi)$ contains at least one occurrence of $\otimes$ and no other modal operator. Since $\mathbf{S}_1$ and $\mathbf{S}_2$ share the same non-modal basis, it is clear that one can define a homophonic conservative translation of the non-modal basis of $\mathbf{S}_2$ into the non-modal basis of $\mathbf{S}_1$. Thus, (iii) implies that there is no conservative translation for formulas containing $\Box$, which, in turn, means that there is no translation of $RK_\Box$ that is valid in $\mathbf{S}_2$. Let us now consider the case (iv) under the assumption that there is a conservative translation t of $\mathbf{S}_2$ into $\mathbf{S}_1$. The fact that there is no $\mathrm{t}(\Box\phi)$ such that it contains at least one occurrence of $\otimes$ and no other modal operator means that the propositional language extended with $\otimes$ is not expressive enough to represent the property encapsulated by $RK_\Box$.

This shows that the normality of an operator, defined by Definition 4, and the normality of a system, defined by Definition 1 depend on the same property expressed via $RK_\Box$, thus solving the mismatch problem in full generality.

Concerning the insensitivity problem, one can easily verify that $\Diamond$ interpreted in a classical normal modal system validating $\Diamond\phi \Leftrightarrow \neg\Box\neg\phi$ would be a normal operator. We just define a homophonic translation of all non-modal formulas of a normal system containing $\Box$ as a sole primitive modality. For the case of $\Box$, we define $\mathrm{t}(\Box\phi) \Leftrightarrow \neg\Diamond\neg\mathrm{t}(\phi)$. However, $\Diamond$ would be non-normal if it is defined in intuitionistic modal logic, because in an intuitionistic setting, $\Box$ is not definable in principle via $\Diamond$.

An interesting consequence of adopting this definition is that it sheds new light on some technical results bridging normal and non-normal systems. For instance, Demri (1997) shows that the language containing $\Box$ and the language containing Δ are equally expressive whenever they are interpreted on reflexive models. To prove this, the following conservative translation is used:

Definition 5. Let $\mathcal{L}^\Box$ be a standard propositional language containing the operator $\Box$, and $\mathcal{L}^\Delta$ a standard propositional language containing Δ. Define the following translation from formulas of $\mathcal{L}^\Box$ to formulas of $\mathcal{L}^\Delta$.

$$\mathrm{t}^\Delta(p) = p$$
$$\mathrm{t}^\Delta(\neg\phi) = \neg\mathrm{t}\Delta(\phi)$$
$$\mathrm{t}^\Delta(\phi \wedge \psi) = \mathrm{t}^\Delta(\phi) \wedge \mathrm{t}^\Delta(\psi)$$
$$\mathrm{t}^\Delta(\Box\phi) = \Delta\mathrm{t}^\Delta(\phi) \wedge \mathrm{t}^\Delta(\phi)$$

This means that a normal modal system with $\Box$ is equally expressive as a system containing Δ interpreted on reflexive models. In particular, both systems permit one to express and validate normality encapsulated in the rule $RK_\square$. Following Definition 2, $\square$ is normal, while Δ is non-normal, even if they both express and validate the same principles. This should be contrasted with Definition 4. By accepting this new definition, it is clear that Δ such that $t^\Delta(\square\phi) = \Delta t^\Delta(\phi) \wedge t^\Delta(\phi)$ is a normal operator. Thus, Definition 4 avoids the mismatch by considering both operators interpreted on reflexive models as normal. The same argument can be applied in the case of $\circ$.

6. The multimodal case

Gasquet & Herzig (1996) present various translations of non-normal modal systems into normal systems. Unfortunately, the authors do not make explicit the definitions of normal system and normal operators they are using. One may deduce that normal systems are identified with systems characterised by standard Kripke semantics, and that they use the same definition of normal operator used by Kracht & Wolter (1999) in the same line of research, which ultimately coincides with Definition 2.

Properly speaking, identifying normal systems with systems interpreted on Kripke frames is not equivalent to the conditions provided in Definition 1. For instance, systems which are interpreted on K-frames and contain Δ (or $\circ$) as the only primitive modality are non-normal, because these systems do not permit one to define $\square$ and thus $RK_\square$ is not valid in these systems. However, this observation does not affect neither the technical results presented in (Gasquet & Herzig 1996) nor its interpretation, because the authors consider normal multimodal systems containing only operators $\square_1$, ..., $\square_n$.

The following definition summarises the use of the term *normal multimodal system* both in (Gasquet & Herzig 1996) and in (Kracht & Wolter 1999)[10].

Definition 6. Let the language $\mathcal{L}_n$ of n-modal propositional system contain the classical propositional operators and $\square_1$, ..., $\square_n$, $n > 1$.

A *normal multimodal* system is a subset of $\mathcal{L}_n$ which contains all classical tautologies and in which each $\square_1$, ..., $\square_n$ is normal in the sense of Definition 2.

The general idea of Gasquet & Herzig (1996) is to provide translations of non-normal modal systems with non-normal $\square$ in multimodal systems containing only normal operators in the sense of Definition 2. Kracht & Wolter (1999), in turn, show that these multimodal systems can be translated into

[10] Kracht & Wolter (1999) use the term "polymodal system" instead of "multimodal system".

monomodal normal system, i.e., into a standard modal system with only one $\Box$. However, the use of Definition 2 leads to a misleading interpretation of the results provided in both articles. Moreover, it veils the distinct steps of translating a non-normal system with non-normal operators into a normal system with normal operators.

The standard Definition 6 used both by (Gasquet & Herzig 1996) and (Kracht & Wolter 1999) depends on the definition of a normal operator (Definition 2). However, as the mismatch problem indicates, if one uses Definition 2, the nature of the normality of a system can be different from the nature of the normality of the operators it contains. From this perspective, the nature of the normality of a system (Definition 1) can be different from the one of the multimodal system (Definition 6), whenever the normality of an operator is defined via Definition 2. In this sense, the mismatch problem is transmitted from the monomodal to the multimodal setting. As argued before, it seems natural to require that the normality of a system and the normality of an operator should rely on the *same* property. This requirement can be extended to the multimodal case: the normality of a monomodal system and the normality of a multimodal system should also rely on the *same* property. Moreover, consider a multimodal system containing the usual normal $\Box$, which is defined for *all* accessibility relations, and the non-normal (in the sense of Definition 2) operator $\Box_1$. Such a system will be normal in the sense of Definition 1 because of the presence of the normal $\Box$, but non-normal in the sense of Definition 6 because of the presence of $\Box_1$. This example can possibly be eliminated by adopting some *ad hoc* restrictions on the applicability of Definitions 1 and 6. However, this seems to indicate a gap between the definitions of normality in the case of monomodal and multimodal systems.

From our perspective, in accordance with the new Definition 4, none of the $\Box_1, ..., \Box_n$, as defined by Gasquet & Herzig (1996) and Kracht & Wolter (1999), is a normal operator. However, as it is shown by Kracht & Wolter (1999), it is possible to define a single $\Box$ in terms of $\Box_1, ..., \Box_n$, such that $RK_\Box$ is validated. Thus, even though the multimodal system contains only non-normal $\Box_1, ..., \Box_n$, the system has the property encapsulated in $RK_\Box$, and thus should be considered as normal. This suggests that Definition 6 should be reformulated in order to uniform the definitions of normal monomodal system and of normal multimodal system. Our proposal is as follows.

Definition 7. Let the language $\mathcal{L}_n$ of n-modal propositional logic contain the classical propositional operators and $\Box_1, ..., \Box_n$, $n > 1$. Let $\mathbf{S}_1$ be a multimodal system defined on $\mathcal{L}_n$, and $\mathbf{S}_2$ be a normal modal system containing $\Box$ as the only primitive modality.

The system $\mathbf{S}_1$ is a *normal* multimodal system iff there exists a conservative translation t of $\mathbf{S}_2$ into $\mathbf{S}_1$.

Definitions 1, 4 and 7 represent normality of a monomodal system, of an operator and of a multimodal system with respect to the same property encapsulated in the rule $RK_\Box$. These definitions permit one to reinterpret the results of Gasquet & Herzig (1996) and in Kracht & Wolter (1999). We will focus only on the translation of the minimal modal system interpreted on neighbourhood frames. The translations of further extensions can be dealt with similarly.

The minimal system can be translated into a multimodal system containing three distinct modal operators as follows:

$$\mathrm{t}(p) = p;$$
$$\mathrm{t}(\neg\phi) = \neg\mathrm{t}(\phi);$$
$$\mathrm{t}(\phi \wedge \psi) = \mathrm{t}(\phi) \wedge \mathrm{t}(\psi);$$
$$\mathrm{t}(\Box\phi) = \Diamond_1(\Box_2\mathrm{t}(\phi) \wedge \Box_3\neg\mathrm{t}(\phi)).$$

By the use of the old definitions, Gasquet & Herzig (1996) have interpreted this result as if a non-normal system with a non-normal operator is expressible in a normal multimodal system containing only normal operators. However, considering only the issue of normality, this interpretation does not differentiate the result on expressivity of non-normal system into multimodal system by Gasquet & Herzig (1996) from the one into monomodal system by Kracht & Wolter (1999): in both cases the translations result in a normal system with normal operators. Also, this interpretation does not highlight the role of $\Box_1$, $\Box_2$, $\Box_3$ in a normal multimodal system.

By applying Definitions 1, 4 and 7, the translation above suggests that a non-normal system, i.e., the minimal system, with a non-normal operator $\Box$ can be expressed in a normal multimodal system with non-normal operators $\Box_1$, $\Box_2$ and $\Box_3$. The multimodal system is normal because the union of $\Box_1$, $\Box_2$, $\Box_3$ provides a normal $\Box$, which can express $RK_\Box$. However, $\Box_1$, $\Box_2$ and $\Box_3$ taken separately are not sufficient to express $RK_\Box$ and thus they should be considered as non-normal. In this sense, each operator $\Box_1$, $\Box_2$ and $\Box_3$ can be seen as a non-normal counterpart of the normal $\Box$. From this perspective, the translation of Gasquet & Herzig (1996) of a non-normal system into a normal one requires an intermediate step of translating a non-normal operator into a combination of non-normal counterparts of a normal operator.

Kracht & Wolter (1999) provide the result on expressivity of a multimodal system in a monomodal one. This monomodal system is normal and it contains one normal (in the sense of Def. 4) operator. This passage from a normal system with non-normal operators to a normal system with a normal operator completes the work of Gasquet & Herzig (1996). By applying Definitions 1, 4 and 7, we have now all the elements to make explicit all the steps of translating a non-normal system with a non-normal operator into a normal system with a normal operator. First, the non-normal $\Box$ is expressed via non-normal counterparts of normal $\Box$. Second, these non-normal counterparts are rearranged in order to be expressed by one normal $\Box$.

7. Conclusion

We introduced a new definition of normal modal operator, which solves both the mismatch and the insensitivity problems. We showed the advantages of applying this definition in interpreting two different settings. First, it bridges normal and non-normal operators: the same operator can be normal or non-normal depending on the expressibility and validity of $RK_{\Box}$, and thus depending on the normality of the system. Secondly, it clarifies the case of multimodal systems: multiple modalities play the role of non-normal counterparts of the normal operator, which permits one to express a non-normal operator through a combination of these counterparts. These applications make clear that the conceptual interpretation of the technical results depends directly on the definition of normal operator one has in mind. On the one hand, the use of Definitions 2 and 6 is unable to explain how the expressivity of the language is transmitted from normal to non-normal systems and this leads to puzzling conclusions. On the other hand, Definitions 4 and 7 are more explicative: the normality of an operator and of a multimodal system now depends on the same property as the normality of a system, i.e., the property encapsulated in the rule $RK_{\Box}$, which is the fundamental condition for calling a system normal.

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