

A symmetry result in \mathbb{R}^2 for global minimizers of a general type of nonlocal energy

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ABSTRACT. In this paper, we are interested in a general type of nonlocal energy, defined on a ball $B_R \subset \mathbb{R}^n$ for some $R > 0$ as

$$\mathcal{E}(u, B_R) = \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} F(u(x) - u(y), x - y) dx dy + \int_{B_R} W(u) dx.$$

We prove that in \mathbb{R}^2 , under suitable assumptions on the functions F and W , bounded continuous global energy minimizers are one-dimensional. This proves a De Giorgi conjecture for minimizers in dimension two, for a general type of nonlocal energy.

1. INTRODUCTION

In this paper we deal with a general type of nonlocal energy. Let

$$F: \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \rightarrow [0, +\infty), \quad W: [-1, 1] \rightarrow [0, +\infty)$$

be two functions and let $R > 0$. Denoting as usual

$$B_R = \{x \in \mathbb{R}^n \mid |x| < R\}, \quad CB_R = \mathbb{R}^n \setminus B_R,$$

we consider for any $|u| \leq 1$,

$$\mathcal{E}(u, B_R) := \mathcal{K}_R(u) + \int_{B_R} W(u) dx, \tag{1.1}$$

with

$$\mathcal{K}_R(u) := \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} F(u(x) - u(y), x - y) dx dy. \tag{1.2}$$

Under suitable assumptions on F and W , we prove that for $n = 2$ continuous functions $u: \mathbb{R}^n \rightarrow [-1, 1]$, minimizing the energy $\mathcal{E}(\cdot, B_R)$ for any $R > 0$, are one-dimensional. We say that u is one-dimensional if every level set of u is a hyperplane, or in other terms, if there exists $u_0: \mathbb{R} \rightarrow [-1, 1]$ such that

$$u(x) = u_0(x \cdot \omega) \quad \text{for some } \omega \in \partial B_1.$$

This type of energy naturally arises in a phase transition problem, which leads to the well-known stationary Allen-Cahn equation

$$(-\Delta)u = u - u^3 \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

The Italian mathematician Ennio De Giorgi conjectured in 1978 that any smooth, bounded solution of this equation which is monotone in one direction is one-dimensional, at least if

1991 *Mathematics Subject Classification*. Primary: 35J40, 35R11. Secondary: 35B08.

Key words and phrases. Nonlocal energy, one dimensional solutions, Allen-Cahn.

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I sincerely thank Enrico Valdinoci and Luca Lombardini for their very useful suggestions.

$n \leq 8$. The interested reader can check [38] for a very nice survey on phase transitions, minimal surfaces, the Bernstein problem, since the connection between these problems is the reason why the dimension eight comes into play. For a further very nice reference, see [14]. This De Giorgi conjecture has received much attention in the last decades, and has been completely settled for $n \leq 3$, see [2, 3, 28]. The case $4 \leq n \leq 8$ with the additional assumption that

$$\lim_{x_n \rightarrow \pm\infty} u(x', x_n) = \pm 1, \quad \text{for any } x' \in \mathbb{R}^{n-1} \quad (1.4)$$

was proved in [37]. On the other hand, an example showing that the De Giorgi conjecture does not hold in higher dimensions (i.e. for $n \geq 9$) can be found in [18].

A model that accounts for long range interactions is given by the nonlocal, fractional counterpart of the Allen-Cahn equation

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n, \quad (1.5)$$

with $s \in (0, 1)$. The operator $(-\Delta)^s$ denotes the fractional Laplacian defined as

$$(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy, \quad \text{with } C_{n,s} > 0.$$

An analogue of the De Giorgi conjecture for any smooth, bounded, monotone solution of the fractional Allen-Cahn equation has first been proved for $n = 2, s = 1/2$ is [10]. In the case $n = 2$, for any $s \in (0, 1)$, the result is proved in [9, 43]. When $n = 3$, the papers [7, 8] contain the proof for $s \in [1/2, 1)$, [21, 22] for $s \in (0, 1/2)$, and [40] for a general $s \in (0, 1)$. For $n = 4$ and $s = 1/2$ a proof of the conjecture is given in [27]. On the other hand, for $4 \leq n \leq 8$ and $s \in [1/2, 1)$ the conjecture is proved with the additional assumption (1.4) in [36, 39]. For s in this range, only a counterexample for $n = 9$ is missing to complete the picture.

One way to tackle the De Giorgi conjecture is to study global minimizers of the Ginzburg-Landau energy functional and understand whether they are one-dimensional. For the Allen-Cahn equation (1.3), the related energy in some ball $B_R \subset \mathbb{R}^n$ is given by

$$\mathcal{E}(u, B_R) = \int_{B_R} \frac{1}{2} |\nabla u|^2 + W(u) dx,$$

with W being the double-well potential

$$W(u) = \frac{(u^2 - 1)^2}{4}.$$

Actually, the potential term W can denote any function with a double-well structure, that is

$$\begin{aligned} W: [-1, 1] &\rightarrow [0, +\infty), & W &\in C^2([-1, 1]), & W &> 0 \quad \text{in } (-1, 1), \\ W(\pm 1) &= W'(\pm 1) = 0, & W''(\pm 1) &> 0. \end{aligned} \quad (1.6)$$

A local minimizer u of the energy $\mathcal{E}(\cdot, B_R)$ is such that $\mathcal{E}(u, B_R) \leq \mathcal{E}(v, B_R)$ for any $v = u$ on ∂B_R . A global minimizer is a local minimizer for any $R > 0$. It turns out that global minimizers of the Ginzburg-Landau energy (with W as in (1.6)) are one-dimensional for $n \leq 7$, see [37] or [38, Theorem 10.1]. In fact, Savin proves the conjecture for global minimizers, and uses the additional assumption (1.4) to go from global minimizers to solutions.

The nonlocal energy related to problem (1.5) is

$$\mathcal{E}^s(u, B_R) = \frac{1}{2} \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{B_R} W(u) dx, \quad (1.7)$$

with W satisfying (1.6). Here, a local minimizer u of the energy $\mathcal{E}^s(\cdot, B_R)$ is such that

$$\mathcal{E}^s(u, B_R) \leq \mathcal{E}^s(v, B_R) \quad \text{for any } v = u \quad \text{in } \mathcal{C}B_R,$$

and a global minimizer is a local minimizer in any ball. That nonlocal minimizers are one-dimensional is proved for $n \leq 7$ and $s \in [1/2, 1)$ in [36, 39, Theorems 1.2], and the conjecture for solutions is settled (as in the classical case) by using the additional assumption (1.4).

In other references [22, 27, 40], the authors prove the conjecture with different techniques (for critical points of the energy, or for stable solutions). In [7, 8, 9, 10, 21, 43], again various techniques are employed, but all rely on the use of the harmonic extension for the fractional Laplacian. However, there is not an “extension procedure” for general nonlocal operators, hence such methods are specific to the fractional Laplacian case. On the other hand, in [6, Theorem 4.2.1], the present author with Valdinoci carry out the proof of the conjecture for minimizers for $n = 2$ in the nonlocal setting, thus without the harmonic extension. This allows to develop the technique therein introduced, and take the nonlocal energy in (1.7) to a much more general form.

We thus prove the conjecture for global minimizers for $n = 2$, for the general nonlocal energy given in (1.1). As a matter of fact, the results here introduced find as an immediate application the study of the energy related to (1.7). Furthermore, the result applies also to more engaging equations, involving for instance the fractional p -Laplacian, or the mean curvature equation (as we see in Section 6).

We mention for the Allen-Cahn equation with general kernels the papers [17] and [35]. While reviewing this paper, we learned about the result reached in [35]. There, the one-dimensional property of stable solutions is proved in \mathbb{R}^2 for the operator

$$\mathcal{L}u(x) = P.V. \int_{\mathbb{R}^n} (u(x) - u(x+y))K(y) dy$$

under some assumptions on the kernel K , and by using a Liouville theorem approach.

We organize the rest of the paper as follows. Section 2 contains the main result and the assumptions on the function F . In Section 3 we deal with the existence of minimizers of the nonlocal general energy (1.1) in a suitable functional setting. We discuss also some form of a strong comparison principle (i.e. if two ordered minimizers coincide on a small ball, then they coincide in the whole space). In Section 4 we introduce some energy estimates, which will contribute to the proof of the main result (Theorem 1) in Section 5. We give two examples of functions F that satisfy our assumptions in Section 6. As a matter of fact, in this last section we obtain that continuous bounded minimizers of the energy related to the fractional p -Laplacian and to the fractional mean curvature are one-dimensional in \mathbb{R}^2 . In other words, we prove the De Giorgi conjecture for minimizers in dimension two (also) for the fractional p -Laplacian and the mean curvature.

2. MAIN RESULT AND ASSUMPTIONS ON F

We fix some $s \in (0, 1)$ and $p \in [1, +\infty)$. We consider $F : \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \rightarrow [0, +\infty)$ and denote by $t \in \mathbb{R}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$ its variables.

Assumptions. In this paper, F satisfies the following on its domain of definition.

- symmetry

$$F(t, x) = F(-t, x) = F(-t, -x), \quad (2.1)$$

- monotonicity in t

$$F(t_1, x) \leq F(t_2, x) \quad \text{for any } |t_1| \leq |t_2|, \quad (2.2)$$

- monotonicity in x

$$F(t, x_1) \leq F(t, x_2) \quad \text{for any } |x_1| \geq |x_2|, \quad (2.3)$$

- scaling in x

$$F(t, \alpha x) \leq \alpha^{-n-sp-1} F(t, x) \quad \text{for any } \alpha \in (0, 1], \quad (2.4)$$

- integrability: there exist $c_*, c^* > 0$ such that

$$c_* \left(\frac{|t|^p}{|x|^{n+sp}} - \frac{1}{|x|^{n+sp-p}} \right) \leq F(t, x) \leq c^* \frac{|t|^p}{|x|^{n+sp}}, \quad (2.5)$$

- smoothness in x

$$F(t, \cdot) \in C^2(\mathbb{R}^n \setminus \{0\}), \quad (2.6)$$

- growth of the partial derivative in x : there exists $c_1 > 0$ such that

$$\left| \partial_{x_i} F(t, x) \right| \leq c_1 \frac{F(t, x)}{|x|} \quad \text{for any } i = 1, 2, \dots, n, \quad (2.7)$$

- growth of the second order partial derivative in x : there exists $c_2 > 0$ such that

$$\left| \partial_{x_i}^2 F(t, x) \right| \leq c_2 \frac{F(t, x)}{|x|^2} \quad \text{for any } i = 1, 2, \dots, n, \quad (2.8)$$

- smoothness in t

$$F(\cdot, x) \in C^1(\mathbb{R}) \quad \text{for a.e. } x \in \mathbb{R}^n, \quad (2.9)$$

- growth of the derivative in t : there exists $c_3 > 0$ such that

$$\left| \partial_t F(t, x) \right| \leq c_3 \frac{|t|^{p-1}}{|x|^{n+sp}}, \quad (2.10)$$

- strict monotonicity of $\partial_t F(t, x)$:

$$\partial_t F(T, x) > \partial_t F(\tau, x), \quad \text{whenever } T > \tau, \quad \text{for any } x \in \mathbb{R}^n \setminus \{0\}. \quad (2.11)$$

Let

$$W: [-1, 1] \rightarrow [0, +\infty), \quad W \in C^1([-1, 1]), \quad W(\pm 1) = W'(\pm 1) = 0. \quad (2.12)$$

When W satisfies (2.12), we say that $u: \mathbb{R}^n \rightarrow [-1, 1]$ is a minimizer for $\mathcal{E}(\cdot, B_R)$ given in (1.1) if $\mathcal{E}(u, B_R) < \infty$ and if it minimizes $\mathcal{E}(\cdot, B_R)$ among all admissible competitors, i.e.

$$\mathcal{E}(u, B_R) \leq \mathcal{E}(v, B_R) \quad \text{for any } v \text{ such that } |v| \leq 1 \quad \text{and} \quad v = u \quad \text{in } \mathcal{C}B_R.$$

We say that $u: \mathbb{R}^n \rightarrow [-1, 1]$ is a global minimizer for \mathcal{E} if it is a minimizer for $\mathcal{E}(\cdot, B_R)$ for any $R > 0$. The main result of the paper is the following.

Theorem 1. *Let $u: \mathbb{R}^n \rightarrow [-1, 1]$ be a continuous global minimizer of the energy (1.1). Then under the assumptions (2.1) to (2.12), u is one-dimensional.*

Notice that the assumptions on F give a generalization of the energy in (1.7), related to the fractional Laplacian. Moreover, we prove in the last Section 6, they are all natural conditions when we consider a nonlocal energy like the one related to the fractional p -Laplacian or the fractional mean curvature operator.

3. EXISTENCE AND COMPARISON OF MINIMIZERS

Let $p \in [1, +\infty)$, $s \in (0, 1)$, $R > 0$ and let $u: \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function. We consider $F: \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \rightarrow [0, +\infty)$ to be such that it satisfies at least (2.1) and (2.5) (other assumptions will be mentioned, when needed). Furthermore, let

$$W: \mathbb{R} \rightarrow [0, +\infty), \quad W \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R}) \quad (3.1)$$

and let $\mathcal{E}(\cdot, B_R)$ be defined by the formula (1.1).

We begin by describing the functional framework for the existence (for further reference, check [20]). Let

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) \mid [u]_{W^{s,p}(\Omega)} < \infty\}$$

where

$$[u]_{W^{s,p}(\Omega)} = \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}$$

is the Gagliardo semi-norm. Also, we denote

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + [u]_{W^{s,p}(\Omega)}^p \right)^{\frac{1}{p}}.$$

We define

$$\mathcal{X}_R := \{\varphi: \mathbb{R}^n \rightarrow \mathbb{R} \mid \varphi \in L^\infty(\mathbb{R}^n) \cap W^{s,p}(B_{2R})\}$$

and denote

$$[u]_{R,\varphi} := \left(\int_{B_R} \left(\int_{B_{2R} \setminus B_R} \frac{|u(x) - \varphi(y)|^p}{|x - y|^{n+sp}} dy \right) dx \right)^{\frac{1}{p}}.$$

For $\varphi \in \mathcal{X}_R$, let

$$\mathcal{W}_{R,\varphi}^{s,p} := \{u: \mathbb{R}^n \rightarrow \mathbb{R} \mid u \in W^{s,p}(B_R), [u]_{R,\varphi} < \infty \text{ and } u = \varphi \text{ on } \mathcal{C}B_R\}. \quad (3.2)$$

For $u \in \mathcal{W}_{R,\varphi}^{s,p}$, when we say that u is a minimizer for $\mathcal{E}(\cdot, B_R)$ it is implied that u is a minimizer with respect to the fixed exterior data φ . For the sake of precision, we recall that a measurable function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is a minimizer for \mathcal{E} in B_R if $\mathcal{E}(u, B_R) < \infty$ and

$$\mathcal{E}(u, B_R) \leq \mathcal{E}(v, B_R) \quad \text{for any } v = u \text{ in } \mathcal{C}B_R.$$

For any two sets $A, B \subset \mathbb{R}^n$, we define

$$u(A, B) := \int_A \int_B F(u(x) - u(y), x - y) dx dy,$$

and recall from (1.2) that

$$\mathcal{K}_R(u) = u(B_R, B_R) + u(B_R, \mathcal{C}B_R) + u(\mathcal{C}B_R, B_R).$$

We have the next useful result.

Proposition 2. *If $\varphi \in \mathcal{X}_R$ and $u \in \mathcal{W}_{R,\varphi}^{s,p}$, then there exists a positive constant C depending on $n, s, p, R, \|W\|_{L^\infty(\mathbb{R})}, \|\varphi\|_{L^\infty(\mathbb{R}^n)}$ such that*

$$\mathcal{E}(u, B_R) \leq C \left(\|u\|_{W^{s,p}(B_R)}^p + [u]_{R,\varphi}^p + 1 \right).$$

Moreover, it holds that

$$\mathcal{K}_R(u) = u(B_R, B_R) + 2u(B_R, \mathcal{C}B_R). \quad (3.3)$$

Proof. By the right hand side of (2.5) we have that

$$\begin{aligned} & u(B_R, B_R) + 2u(B_R, B_{2R} \setminus B_R) \\ & \leq c^* \int_{B_R} \int_{B_R} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy + 2c^* \int_{B_R} \int_{B_{2R} \setminus B_R} \frac{|u(x) - \varphi(y)|^p}{|x - y|^{n+sp}} dx dy \\ & \leq c^* \left([u]_{W^{s,p}(B_R)}^p + 2[u]_{R,\varphi}^p \right). \end{aligned}$$

When $x \in B_R, y \in \mathcal{C}B_{2R}$, we have that $|x - y| \geq |y|/2$, hence

$$\begin{aligned} u(B_R, \mathcal{C}B_{2R}) & \leq c^* \int_{B_R} \int_{\mathcal{C}B_{2R}} \frac{|u(x) - \varphi(y)|^p}{|x - y|^{n+sp}} dx dy \\ & \leq 2^{p-1} c^* \left(\int_{B_R} |u(x)|^p \int_{\mathcal{C}B_{2R}} \frac{dx dy}{|x - y|^{n+sp}} + \int_{B_R} \int_{\mathcal{C}B_{2R}} \frac{|\varphi(y)|^p}{|x - y|^{n+sp}} dx dy \right) \\ & \leq 2^{p-1+n+sp} c^* \left(\|u\|_{L^p(B_R)}^p + \|\varphi\|_{L^\infty(\mathcal{C}B_{2R})}^p |B_R| \right) \int_{\mathcal{C}B_{2R}} |y|^{-n-sp} dy \\ & \leq C_{n,s,p,R} \left(\|u\|_{L^p(B_R)}^p + \|\varphi\|_{L^\infty(\mathcal{C}B_{2R})}^p \right). \end{aligned}$$

Therefore we obtain

$$\mathcal{K}_R(u) \leq C_{n,s,p,R} \left(\|u\|_{W^{s,p}(B_R)}^p + [u]_{R,\varphi}^p + \|\varphi\|_{L^\infty(\mathcal{C}B_{2R})}^p \right). \quad (3.4)$$

It is enough then to notice that

$$\int_{B_R} W(u) dx \leq C_{n,R} \|W\|_{L^\infty(\mathbb{R})}$$

to conclude the first statement of the Proposition.

On the other hand, changing variables, using Fubini to change the order of integration and applying (2.1), we obtain

$$\begin{aligned} & u(\mathcal{C}B_R, B_R) \\ & = \int_{\mathcal{C}B_R} \left(\int_{B_R} F(u(x) - u(y), x - y) dy \right) dx = \int_{\mathcal{C}B_R} \left(\int_{B_R} F(u(y) - u(x), y - x) dx \right) dy \\ & = \int_{B_R} \left(\int_{\mathcal{C}B_R} F(u(x) - u(y), x - y) dy \right) dx = u(B_R, \mathcal{C}B_R), \end{aligned}$$

from which (3.3) immediately follows. \square

We give in the next proposition some a priori properties of the minimizers of the energy.

Proposition 3. *If $\varphi \in \mathcal{X}_R$ and u is a minimizer of $\mathcal{E}(\cdot, B_R)$ with $u = \varphi$ in $\mathcal{C}B_R$, then*

(1) *there exists $C = C_{n,s,p,R} > 0$ such that*

$$\mathcal{E}(u, B_R) \leq C \left(\|\varphi\|_{W^{s,p}(B_{2R})}^p + \|\varphi\|_{L^\infty(\mathcal{C}B_{2R})}^p + \|W\|_{L^\infty(\mathbb{R})} \right),$$

(2) *$u \in \mathcal{W}_{R,\varphi}^{s,p}$ and furthermore there exists $c = c_{n,s,p,R} > 0$ such that*

$$\|u\|_{L^p(B_R)} \leq c(1 + [u]_{W^{s,p}(B_R)}^p). \quad (3.5)$$

Proof. We can use φ as a competitor for u . Using (3.4) for φ (notice that $\varphi \in \mathcal{W}_{R,\varphi}^{s,p}$), we obtain

$$\mathcal{K}_R(\varphi) \leq C_{n,s,p,R} \left(\|\varphi\|_{W^{s,p}(B_{2R})}^p + \|\varphi\|_{L^\infty(\mathcal{C}B_{2R})}^p \right).$$

Given the minimality of u , we get that

$$\mathcal{E}(u, B_R) \leq \mathcal{E}(\varphi, B_R) \leq C_{n,s,p,R} \left(\|\varphi\|_{W^{s,p}(B_{2R})}^p + \|\varphi\|_{L^\infty(\mathcal{C}B_{2R})}^p + \|W\|_{L^\infty(\mathbb{R})} \right).$$

This proves point (1) of the proposition. By a change of variables, we obtain the bound

$$\int_{B_R} \int_{B_{2R}} \frac{1}{|x-y|^{n+sp-p}} dx dy \leq |B_R| \int_{B_{3R}} \frac{1}{|z|^{n+sp-p}} dz = C(n, s, p, R). \quad (3.6)$$

According to the left hand side of (2.5), we have

$$\begin{aligned} u(B_R, B_R) &\geq c_* \left(\int_{B_R} \int_{B_R} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dx dy - \int_{B_R} \int_{B_R} \frac{1}{|x-y|^{n+sp-p}} dx dy \right) \\ &= c_* \left([u]_{W^{s,p}(B_R)}^p - C_{n,s,p,R} \right). \end{aligned}$$

In the same way, we get that

$$u(B_R, B_{2R} \setminus B_R) \geq c_* \left([u]_{R,\varphi}^p - C_{n,s,p,R} \right).$$

Since $\mathcal{E}(u, B_R)$ is bounded, it holds that

$$[u]_{W^{s,p}(B_R)}^p + [u]_{R,\varphi}^p \leq \mathcal{E}(u, B_R) + C_{n,s,p,R} < C(n, s, p, R, \|\varphi\|_{L^\infty(\mathbb{R}^n)}, \|W\|_{L^\infty(\mathbb{R})}). \quad (3.7)$$

Using Proposition 14 we have that

$$\|u\|_{L^p(B_R)}^p \leq C_{n,p,s,R} \left([u]_{R,\varphi}^p + \|\varphi\|_{L^p(B_{2R} \setminus B_R)}^p \right).$$

This implies that $u \in L^p(B_R)$, hence by (3.7), we get that $u \in \mathcal{W}_{R,\varphi}^{s,p}$. The bound on the L^p norm (3.5) follows from Proposition 15. \square

Remark 4. Let us note that there are some cases in which the request

$$[u]_{R,\varphi} < \infty$$

can be avoided. For $sp < 1$, we can take

$$\mathcal{X}_R = \{ \varphi: \mathcal{C}B_R \rightarrow \mathbb{R} \mid \varphi \in L^\infty(\mathcal{C}B_R) \}.$$

In this case, we define

$$\mathcal{W}_{R,\varphi}^{s,p} := \{ u: \mathbb{R}^n \rightarrow \mathbb{R} \mid u|_{B_R} \in W^{s,p}(B_R) \text{ and } u = \varphi \text{ on } \mathcal{C}B_R \}.$$

Indeed, for $sp < 1$, one can use the fractional Hardy inequality, thanks to [34, Theorem D.1.4, Corollary D.1.5] and get that

$$\begin{aligned} \int_{B_R} \int_{B_{2R} \setminus B_R} \frac{|u(x)|^p}{|x-y|^{n+sp}} dx dy &\leq \int_{B_R} \int_{\mathcal{C}B_{d_R(x)}(x)} \frac{|u(x)|^p}{|x-y|^{n+sp}} dx dy \\ &\leq \int_{B_R} \frac{|u(x)|^p}{d_R(x)^{sp}} dx \leq C(n, s, p, R) \|u\|_{W^{s,p}(B_R)}^p, \end{aligned}$$

where $d_R(x) = \text{dist}(x, \partial B_R)$.

Just as a remark, the fractional Hardy inequality holds also for $sp > 1$, see [26, Theorem 1.1, (17)] for any $u \in C_c(B_R)$). Nevertheless, in this case one looks for minimizers in $W_0^{s,p}(B_R)$, a space which is too restrictive for our purposes.

Furthermore (check [34, Lemma 4.5.10], or the forthcoming paper [16])

$$\int_{B_R} \int_{B_{2R} \setminus B_R} \frac{|\varphi(y)|^p}{|x-y|^{n+sp}} dx dy \leq \|\varphi\|_{L^\infty(\mathcal{C}B_R)}^p \text{Per}_{sp}(B_R) < \infty.$$

This follows since the sp -perimeter is finite for sets with Lipschitz boundary (see [12]). Then

$$[u]_{R,\varphi}^p = \int_{B_R} \int_{B_{2R} \setminus B_R} \frac{|u(x) - \varphi(y)|^p}{|x - y|^{n+sp}} dx dy \leq C(n, s, p, R) \left(\|\varphi\|_{L^\infty(\mathcal{C}B_R)} + \|u\|_{W^{s,p}(B_R)}^p \right).$$

We also notice that, in order to obtain the estimates in Proposition 3, one can consider

$$\tilde{\varphi} = \begin{cases} \varphi, & \text{in } \mathcal{C}B_R \\ 0, & \text{in } B_R. \end{cases}$$

We prove now the existence of minimizers of the energy.

Theorem 5 (Existence). *Let F be lower semi-continuous in the first variable, and such that it satisfies (2.1), (2.5), and let W be such that it satisfies (3.1). If $\varphi \in \mathcal{X}_R$, there exists a minimizer $u \in \mathcal{W}_{R,\varphi}^{s,p}$ of $\mathcal{E}(\cdot, B_R)$.*

Proof. Since $F, W \geq 0$, we have that $\mathcal{E}(v, B_R) \geq 0$ for any $v \in \mathcal{W}_{R,\varphi}^{s,p}$. Then there exists $\{u_k\} \in \mathcal{W}_{R,\varphi}^{s,p}$ a minimizing sequence, i.e.

$$\liminf_{k \rightarrow \infty} \mathcal{E}(u_k, B_R) = \inf \{ \mathcal{E}(v, B_R) \mid v \in \mathcal{W}_{R,\varphi}^{s,p} \}.$$

There is $\bar{k} > 0$ such that for all $k \geq \bar{k}$ there exists $M > 0$ such that

$$\mathcal{E}(u_k, B_R) \leq M,$$

so in particular by (3.7) we have that

$$[u_k]_{W^{s,p}(B_R)} \leq C_1, \quad [u_k]_{R,\varphi} \leq C_2,$$

with $C_1, C_2 > 0$ depending on n, s, p, R, M . Also, by (3.5), we have that

$$\|u_k\|_{L^p(B_R)} < C(n, s, p, R) ([u]_{W^{s,p}(B_R)} + 1),$$

therefore for all $k \geq \bar{k}$ there is $\tilde{M} > 0$ such that

$$\|u_k\|_{W^{s,p}(B_R)} < \tilde{M}.$$

By compactness (see e.g. Theorem 7.1 in [20]), there exists a subsequence, which we still call $\{u_k\}$, such that

$$\|u_k - u\|_{L^p(B_R)} \longrightarrow 0, \quad \text{and} \quad u_k \longrightarrow u \quad \text{a.e. in } \mathbb{R}^n$$

for some $u \in W^{s,p}(B_R)$. Also, $u \in \mathcal{W}_{R,\varphi}^{s,p}$, by Fatou and the uniform bound on $[u_k]_{R,\varphi}$. Using Fatou's Theorem, the lower semi-continuity of F in the first variable and (3.1) we have that

$$\begin{aligned} & \inf \{ \mathcal{E}(v, B_R) \mid v \in \mathcal{W}_{R,\varphi}^{s,p} \} \\ &= \liminf_{k \rightarrow \infty} \left(\iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}B_R)^2} F(u_k(x) - u_k(y), x - y) dx dy + \int_{B_R} W(u_k) dx \right) \\ &\geq \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}B_R)^2} \liminf_{k \rightarrow \infty} F(u_k(x) - u_k(y), x - y) dx dy + \int_{B_R} W(u) dx \\ &\geq \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}B_R)^2} F(u(x) - u(y), x - y) dx dy + \int_{B_R} W(u) dx \\ &= \mathcal{E}(u, B_R). \end{aligned}$$

Hence u is a minimizer and this concludes the proof of the theorem. \square

We make now an observation on the Euler-Lagrange equation related to the energy \mathcal{E} .

Proposition 6. *Let F satisfy (2.1), (2.5), (2.9), (2.10) and W satisfy (3.1). If $\varphi \in \mathcal{X}_R$ and $u \in \mathcal{W}_{R,\varphi}^{s,p}$, then*

$$\begin{aligned} & \left. \frac{d}{d\varepsilon} \mathcal{E}(u + \varepsilon\phi, B_R) \right|_{\varepsilon=0} \\ &= \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}B_R)^2} \partial_t F(u(x) - u(y), x - y) (\phi(x) - \phi(y)) dx dy + \int_{B_R} W'(u(x)) \phi(x) dx \end{aligned}$$

for any $\phi \in C_c^\infty(B_R)$.

Proof. We give a sketch of the proof. First of all, notice that if $\phi \in C_c^\infty(B_R)$, for any $\varepsilon > 0$ we have that $u + \varepsilon\phi \in \mathcal{W}_{R,\varphi}^{s,p}$. By Proposition 2 it follows that both $\mathcal{E}(u, B_R)$ and $\mathcal{E}(u + \varepsilon\phi, B_R)$ are finite.

Since $F(\cdot, x) \in C^1(\mathbb{R})$ by the mean value theorem there is $\tau_\varepsilon(x, y)$ satisfying $|\tau_\varepsilon(x, y)| \leq \varepsilon$ such that

$$\begin{aligned} & \frac{F(u(x) - u(y) + \varepsilon(\phi(x) - \phi(y)), x - y) - F(u(x) - u(y), x - y)}{\varepsilon} \\ &= \partial_t F(u(x) - u(y) + \tau_\varepsilon(\phi(x) - \phi(y)), x - y) (\phi(x) - \phi(y)). \end{aligned}$$

The assumption (2.10) and the Hölder inequality lead to

$$\left| \partial_t F(u(x) - u(y) + \tau_\varepsilon(\phi(x) - \phi(y)), x - y) (\phi(x) - \phi(y)) \right| \leq F(x, y),$$

for some $F \in L^1(\mathbb{R}^{2n} \setminus (\mathcal{C}B_R)^2)$. It is enough to use the Dominated Convergence Theorem to conclude the proposition. \square

Furthermore, we prove some form of a strong comparison principle for minimizers.

Theorem 7. *Let F satisfy (2.1), (2.5), (2.9), (2.10) and (2.11) and let W satisfy (3.1). If $\varphi_1, \varphi_2 \in \mathcal{X}_R$ and $u_1 \in \mathcal{W}_{R,\varphi_1}^{s,p}$, $u_2 \in \mathcal{W}_{R,\varphi_2}^{s,p}$ are two minimizers of $\mathcal{E}(\cdot, B_R)$, such that*

$$\begin{aligned} & u_1, u_2 \in L^\infty(B_R), \\ & u_1 \geq u_2 \quad \text{in } \mathbb{R}^n \\ & u_1 = u_2 \quad \text{in } B_\delta(\bar{x}) \subset\subset B_R \end{aligned}$$

for some $\delta > 0$, $\bar{x} \in B_R$, then $u_1 = u_2$ almost everywhere in \mathbb{R}^n .

Proof. According to Proposition 6 we have that

$$\begin{aligned} & \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}B_R)^2} \left(\partial_t F(u_2(x) - u_2(y), x - y) - \partial_t F(u_1(x) - u_1(y), x - y) \right) (\phi(x) - \phi(y)) dx dy \\ & + \int_{B_R} (W'(u_2(x)) - W'(u_1(x))) \phi(x) dx = 0 \end{aligned}$$

for any $\phi \in C_c^\infty(B_R)$. In particular this equality holds for any

$$\phi \in C_c^\infty(B_{\frac{\delta}{2}}(\bar{x})), \quad \phi \geq 0.$$

Since $\phi(x) = 0$ on $\mathcal{C}B_{\frac{\delta}{2}}(\bar{x})$ and $u_1(x) = u_2(x)$ in $B_\delta(\bar{x})$, contributions come only from interactions between $B_{\frac{\delta}{2}}(\bar{x})$ and $\mathcal{C}B_\delta(\bar{x})$. So, using also (2.1), we are left with

$$\begin{aligned} 0 &= \int_{B_{\frac{\delta}{2}}(\bar{x})} \left(\int_{\mathcal{C}B_\delta(\bar{x})} \left(\partial_t F(u_1(x) - u_2(y), x - y) - \partial_t F(u_1(x) - u_1(y), x - y) \right) dy \right) \phi(x) dx \\ &+ \int_{\mathcal{C}B_\delta(\bar{x})} \left(\int_{B_{\frac{\delta}{2}}(\bar{x})} \left(\partial_t F(u_2(x) - u_1(y), x - y) - \partial_t F(u_1(x) - u_1(y), x - y) \right) (-\phi(y)) dy \right) dx \\ &= 2 \int_{B_{\frac{\delta}{2}}(\bar{x})} \left(\int_{\mathcal{C}B_\delta(\bar{x})} \left(\partial_t F(u_1(x) - u_2(y), x - y) - \partial_t F(u_1(x) - u_1(y), x - y) \right) dy \right) \phi(x) dx. \end{aligned}$$

Let

$$A_\delta := \{y \in \mathcal{C}B_\delta(\bar{x}) \mid u_1(y) > u_2(y)\}$$

and we argue by contradiction, supposing that

$$|A_\delta| \neq 0. \quad (3.8)$$

When $y \in \mathcal{C}A_\delta$, by hypothesis $u_1(y) = u_2(y)$, hence

$$0 = \int_{B_{\frac{\delta}{2}}(\bar{x})} \left(\int_{A_\delta} \left(\partial_t F(u_1(x) - u_2(y), x - y) - \partial_t F(u_1(x) - u_1(y), x - y) \right) dy \right) \phi(x) dx.$$

Denoting for $x \in B_{\frac{\delta}{2}}(\bar{x}), y \in A_\delta$,

$$h(x, y) := \partial_t F(u_1(x) - u_2(y), x - y) - \partial_t F(u_1(x) - u_1(y), x - y),$$

by (2.11) we have that on A_δ

$$h(x, y) > 0. \quad (3.9)$$

Defining $g : B_{\frac{\delta}{2}}(\bar{x}) \rightarrow \mathbb{R}_+$ as

$$g(x) := \int_{A_\delta} h(x, y) dy$$

we get that for any $\phi \in C_c^\infty(B_{\frac{\delta}{2}}(\bar{x}), [0, +\infty))$

$$0 = \int_{B_{\frac{\delta}{2}}(\bar{x})} g(x) \phi(x) dx.$$

It follows that

$$g(x) = 0 \quad \text{for almost any } x \in B_{\frac{\delta}{2}}(\bar{x}),$$

which by (3.8) and (3.9) gives a contradiction. It follows that that $|A_\delta| = 0$, hence $u_1 = u_2$ almost anywhere in $\mathcal{C}B_\delta(\bar{x})$.

We conclude by noticing that, by (2.10), g is well defined. Indeed

$$\begin{aligned} &\int_{A_\delta} |\partial_t F(u_1(x) - u_2(y), x - y)| dy \leq c_3 \int_{A_\delta} \frac{|u_1(x) - u_2(y)|^{p-1}}{|x - y|^{n+sp}} dy \\ &\leq 2^{p-2} c_3 \left(\|u_1\|_{L^\infty(B_R)}^{p-1} \int_{\mathcal{C}B_\delta(\bar{x})} |x - y|^{-n-sp} dy + \|u_2\|_{L^\infty(B_R)}^{p-1} \int_{B_R \setminus B_\delta(\bar{x})} |x - y|^{-n-sp} dy \right. \\ &\quad \left. + \|\varphi_2\|_{L^\infty(\mathcal{C}B_R)}^{p-1} \int_{\mathcal{C}B_R} |x - y|^{-n-sp} dy \right). \end{aligned}$$

We have that $|y - x| \geq |y - \bar{x}| - |x - \bar{x}| \geq |y - \bar{x}|/2$, hence

$$\int_{A_\delta} |\partial_t F(u_1(x) - u_2(y), x - y)| dy \leq C_{n,s,p,\delta} \left(\|u_1\|_{L^\infty(B_R)}^{p-1} + \|u_2\|_{L^\infty(B_R)}^{p-1} + \|\varphi_2\|_{L^\infty(\mathcal{C}B_R)}^{p-1} \right),$$

and this concludes the proof. \square

4. PRELIMINARY ENERGY ESTIMATES

The preliminary results in this Section hold in any dimension, however the main result works with our techniques only in dimension two. In fact, this depends on a Taylor expansion of order two, that we do in the next Lemma.

Lemma 8. *Let F satisfy (2.1) to (2.8), W satisfy (3.1) and let $\varphi \in C_c^\infty(B_1)$. Also, for any $R > 1$ and $y \in \mathbb{R}^n$, let*

$$\Psi_{R,\pm}(y) := y \pm \varphi\left(\frac{y}{R}\right) e_1 \quad \text{and} \quad u_{R,\pm}(x) = u(\Psi_{R,\pm}^{-1}(x)).$$

Then for large R the maps $\Psi_{R,\pm}$ are diffeomorphisms on \mathbb{R}^n and

$$\mathcal{E}(u_{R,+}, B_R) + \mathcal{E}(u_{R,-}, B_R) - 2\mathcal{E}(u, B_R) \leq \frac{C}{R^2} \mathcal{E}(u, B_R).$$

Proof. From here on, we denote for simplicity

$$u = u(y), \quad \bar{u} = u(\bar{y}), \quad \varphi = \varphi\left(\frac{y}{R}\right), \quad \bar{\varphi} = \varphi\left(\frac{\bar{y}}{R}\right).$$

Notice that

$$|\varphi - \bar{\varphi}| \leq \frac{\|\varphi\|_{C^1(\mathbb{R}^n)}}{R} |y - \bar{y}| \tag{4.1}$$

and that for any $\delta \in [-1, 1]$

$$|y - \bar{y} + \delta e_1(\varphi - \bar{\varphi})| \geq \left(1 - \frac{2\|\varphi\|_{C^1(\mathbb{R}^n)}}{R}\right)^{\frac{1}{2}} |y - \bar{y}|. \tag{4.2}$$

Indeed

$$|y - \bar{y} + \delta e_1(\varphi - \bar{\varphi})|^2 = |y - \bar{y}|^2 + \delta^2(\varphi - \bar{\varphi})^2 + 2\delta(y_1 - \bar{y}_1)(\varphi - \bar{\varphi}) \geq |y - \bar{y}|^2 - 2|\delta||y_1 - \bar{y}_1||\varphi - \bar{\varphi}|.$$

Using (4.1) we have that

$$|\delta||y_1 - \bar{y}_1||\varphi - \bar{\varphi}| \leq \frac{\|\varphi\|_{C^1(\mathbb{R}^n)}}{R} |y - \bar{y}|^2,$$

hence (4.2) is proved.

Now, checking Lemma 4.3 in [6], one sees that $\Psi_{R,\pm}$ are diffeomorphisms for large R , and that the change of variables

$$x := \Psi_{R,\pm}(y), \quad \bar{x} = \Psi_{R,\pm}(\bar{y})$$

gives

$$dx = 1 \pm \frac{1}{R} \partial_{x_1} \varphi\left(\frac{y}{R}\right) + \mathcal{O}\left(\frac{1}{R^2}\right) dy$$

and

$$dx d\bar{x} = 1 \pm \frac{1}{R} \partial_{x_1} \varphi \pm \frac{1}{R} \partial_{x_1} \bar{\varphi} + \mathcal{O}\left(\frac{1}{R^2}\right) dy d\bar{y}. \tag{4.3}$$

With this change of variables, we have that

$$\begin{aligned} F(u_{R,\pm}(x) - u_{R,\pm}(\bar{x}), x - \bar{x}) &= F(u(\Psi_{R,\pm}^{-1}(x)) - u(\Psi_{R,\pm}^{-1}(\bar{x})), x - \bar{x}) \\ &= F(u(y) - u(\bar{y}), y - \bar{y} + e_1(\pm\varphi \mp \bar{\varphi})). \end{aligned}$$

Notice that $\Psi_{R,\pm}^{-1}(B_R) = B_R$ and $\Psi_{R,\pm}^{-1}(\mathcal{C}B_R) = \mathcal{C}B_R$. Changing variables we have that

$$\begin{aligned} &\iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}B_R)^2} F(u_{R,\pm}(x) - u_{R,\pm}(\bar{x}), x - \bar{x}) dx d\bar{x} \\ &= \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}B_R)^2} F(u - \bar{u}, y - \bar{y} \pm e_1(\varphi - \bar{\varphi})) \left(1 \pm \frac{1}{R} \partial_{x_1} \varphi \pm \frac{1}{R} \partial_{x_1} \bar{\varphi} + \mathcal{O}\left(\frac{1}{R^2}\right)\right) dy d\bar{y}. \end{aligned} \quad (4.4)$$

Thanks to (4.2), (2.3) and (2.4), for any R large enough and any $\delta \in [-1, 1]$ we have the estimate

$$\begin{aligned} F(u - \bar{u}, y - \bar{y} \pm \delta e_1(\varphi - \bar{\varphi})) &\leq F\left(u - \bar{u}, \left(1 - \frac{2\|\varphi\|_{C^1(\mathbb{R}^n)}}{R}\right)^{\frac{1}{2}} (y - \bar{y})\right) \\ &\leq \left(1 - \frac{2\|\varphi\|_{C^1(\mathbb{R}^n)}}{R}\right)^{\frac{-n-sp-1}{2}} F(u - \bar{u}, y - \bar{y}). \end{aligned} \quad (4.5)$$

We define the function

$$g: \mathbb{R} \rightarrow \mathbb{R}_+, \quad g(h) := F(u - \bar{u}, y - \bar{y} + h e_1(\varphi - \bar{\varphi}))$$

and we have that

$$g(0) = F(u - \bar{u}, y - \bar{y}), \quad g(\pm 1) = F(u - \bar{u}, y - \bar{y} \pm e_1(\varphi - \bar{\varphi})). \quad (4.6)$$

Also, we take the derivatives

$$\begin{aligned} g'(h) &= \partial_{x_1} F(u - \bar{u}, y - \bar{y} + h e_1(\varphi - \bar{\varphi}))(\varphi - \bar{\varphi}), \\ g''(h) &= \partial_{x_1}^2 F(u - \bar{u}, y - \bar{y} + h e_1(\varphi - \bar{\varphi}))(\varphi - \bar{\varphi})^2. \end{aligned}$$

Using (2.7), (4.1), (4.2) and (4.5) we obtain

$$\begin{aligned} |g'(h)| &\leq c_1 |F(u - \bar{u}, y - \bar{y} + h e_1(\varphi - \bar{\varphi}))| \frac{|\varphi - \bar{\varphi}|}{|y - \bar{y} + h e_1(\varphi - \bar{\varphi})|} \\ &\leq c_1 \left(1 - \frac{2\|\varphi\|_{C^1(\mathbb{R}^n)}}{R}\right)^{\frac{-n-sp-2}{2}} \frac{\|\varphi\|_{C^1(\mathbb{R}^n)}}{R} F(u - \bar{u}, y - \bar{y}), \end{aligned}$$

hence

$$|g'(h)| \leq g(0) \mathcal{O}\left(\frac{1}{R}\right). \quad (4.7)$$

In the same way, using (2.8) we get that

$$|g''(h)| \leq g(0) \mathcal{O}\left(\frac{1}{R^2}\right). \quad (4.8)$$

By (2.6) since $g \in C^2(\mathbb{R})$ with a Taylor expansion we have

$$g(h) = g(0) + g'(\delta)h$$

for some $\delta = \delta(h) \in (0, h)$, hence

$$g(1) = g(0) + g'(\delta_+), \quad g(-1) = g(0) - g'(\delta_-), \quad \text{for some } \delta_+ \in (0, 1), \delta_- \in (-1, 0).$$

Moreover, there exists $\tilde{\delta} \in (\delta_-, \delta_+)$ such that

$$g'(\delta_+) - g'(\delta_-) = g''(\tilde{\delta})(\delta_+ - \delta_-). \quad (4.9)$$

So with this Taylor expansions and formula (4.4) we obtain

$$\begin{aligned} \mathcal{K}_R(u_{R,+}) + \mathcal{K}_R(u_{R,-}) &= \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} g(1) \left(1 + \frac{1}{R} \partial_{x_1} \varphi + \frac{1}{R} \partial_{x_1} \bar{\varphi} + \mathcal{O}\left(\frac{1}{R^2}\right) \right) \\ &\quad + g(-1) \left(1 - \frac{1}{R} \partial_{x_1} \varphi - \frac{1}{R} \partial_{x_1} \bar{\varphi} + \mathcal{O}\left(\frac{1}{R^2}\right) \right) dy d\bar{y} \\ &= \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} g(0) \left(2 + \mathcal{O}\left(\frac{1}{R^2}\right) \right) dy d\bar{y} \\ &\quad + \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} (g'(\delta_+) - g'(\delta_-)) \left(1 + \mathcal{O}\left(\frac{1}{R^2}\right) \right) dy d\bar{y} \\ &\quad + \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} \frac{1}{R} (g'(\delta_+) + g'(\delta_-)) (\partial_{x_1} \varphi + \partial_{x_1} \bar{\varphi}) dy d\bar{y} \\ &= \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} g(0) \left(2 + \mathcal{O}\left(\frac{1}{R^2}\right) \right) + T_1(y, \bar{y}) + T_2(y, \bar{y}) dy d\bar{y}. \end{aligned} \quad (4.10)$$

In order to have an estimate on T_1 , we use (4.9) and get that

$$\begin{aligned} T_1(y, \bar{y}) &\leq |g'(\delta_+) - g'(\delta_-)| \left(1 + \mathcal{O}\left(\frac{1}{R^2}\right) \right) dy d\bar{y} \\ &\leq |g''(\tilde{\delta})| \left(2 + \mathcal{O}\left(\frac{1}{R^2}\right) \right) dy d\bar{y}, \end{aligned}$$

where we have used that $\delta_+ - \delta_- \leq 2$. By (4.8) we obtain

$$T_1(y, \bar{y}) \leq g(0) \mathcal{O}\left(\frac{1}{R^2}\right).$$

On the other hand

$$T_2(y, \bar{y}) \leq \frac{2\|\varphi\|_{C^1(\mathbb{R}^n)}}{R} (|g'(\delta_+)| + |g'(\delta_-)|),$$

which by (4.7) leads to

$$T_2(y, \bar{y}) \leq g(0) \mathcal{O}\left(\frac{1}{R^2}\right).$$

Therefore in (4.10) we have that

$$\mathcal{K}_R(u_{R,+}) + \mathcal{K}_R(u_{R,-}) \leq \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} g(0) \left(2 + \mathcal{O}\left(\frac{1}{R^2}\right) \right) dy d\bar{y} = \mathcal{K}_R(u) \left(2 + \mathcal{O}\left(\frac{1}{R^2}\right) \right).$$

For the potential energy, the computation easily follows. It suffices to apply the change of variables (4) and to recall that $\Psi_{R,\pm}^{-1}(B_R) = B_R$. We get

$$\begin{aligned} & \int_{B_R} W(u_{R,+}(x)) dx + \int_{B_R} W(u_{R,-}(x)) dx \\ &= \int_{B_R} W(u(\Psi_{R,+}^{-1}(x))) dx + \int_{B_R} W(u(\Psi_{R,-}^{-1}(x))) dx \\ &= \int_{B_R} W(u(y)) \left(1 + \frac{1}{R} \partial_{x_1} \varphi\left(\frac{y}{R}\right) + \mathcal{O}\left(\frac{1}{R^2}\right)\right) dy \\ &\quad + \int_{B_R} W(u(y)) \left(1 - \frac{1}{R} \partial_{x_1} \varphi\left(\frac{y}{R}\right) + \mathcal{O}\left(\frac{1}{R^2}\right)\right) dy \\ &= \left(2 + \mathcal{O}\left(\frac{1}{R^2}\right)\right) \int_{B_R} W(u(y)) dy. \end{aligned}$$

This concludes the proof of Lemma 8. \square

We give now the following uniform bound on large balls of the energy of the minimizers. This result is an adaptation of Theorem 1.3 in [41] and it works in any dimension.

Theorem 9. *Let F satisfy (2.1), (2.2) and (2.5) and W satisfy (2.12). If u is a minimizer in B_{R+2} for a large R , such that $|u| \leq 1$, then*

$$\mathcal{E}(u, B_R) \leq \begin{cases} CR^{n-1} & \text{if } s \in \left(\frac{1}{p}, 1\right), \\ CR^{n-1} \log R & \text{if } s = \frac{1}{p}, \\ CR^{n-sp} & \text{if } s \in \left(0, \frac{1}{p}\right), \end{cases}$$

for some positive constant C depending on n, s and W .

This type of energy estimates for the fractional Laplacian are proved in [41, Theorem 1.3] (see also [6, Theorem 4.1.2]). We give a sketch of the proof following [41], pointing out which assumptions on F make the proof work in our case.

Proof of Theorem 9. As a first step, one introduces the auxiliary functions

$$\begin{aligned} \psi(x) &:= -1 + 2 \min\{|x| - R - 1, 1\}, & v(x) &:= \min\{u(x), \psi(x)\}, \\ d(x) &:= \max\{(R + 1 - |x|), 1\}. \end{aligned}$$

Notice that $|\psi|, |v| \leq 1$. For $|x - y| \leq d(x)$ we have that

$$|\psi(x) - \psi(y)| \leq \frac{2|x - y|}{d(x)}. \quad (4.11)$$

Moreover, one obtains the estimate

$$\int_{B_{R+2}} d(x)^{-sp} dx \leq \begin{cases} CR^{n-1} & \text{if } s \in \left(\frac{1}{p}, 1\right), \\ CR^{n-1} \log R & \text{if } s = \frac{1}{p}, \\ CR^{n-sp} & \text{if } s \in \left(0, \frac{1}{p}\right). \end{cases} \quad (4.12)$$

Also, by (2.5) and (4.11) we get that

$$\begin{aligned} & \int_{\mathbb{R}^n} F(\psi(x) - \psi(y), x - y) dy \leq c^* \int_{\mathbb{R}^n} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{n+sp}} dy \\ & \leq c^* \int_{|x-y| \leq d(x)} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{n+sp}} dy + c^* \int_{|x-y| \geq d(x)} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{n+sp}} dy \\ & \leq c^* d(x)^{-p} \int_{|x-y| \leq d(x)} |x - y|^{p-n-sp} dy + c^* \int_{|x-y| \geq d(x)} |x - y|^{-n-sp} dy \leq cd(x)^{-sp}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{E}(\psi, B_{R+2}) & \leq \int_{B_{R+2}} \left(\int_{\mathbb{R}^n} F(\psi(x) - \psi(y), x - y) dy \right) dx + \int_{B_{R+2}} W(\psi) dx \\ & \leq c \int_{B_{R+2}} d(x)^{-sp} dx + \int_{B_{R+2}} W(\psi) dx. \end{aligned}$$

Moreover $W(-1) = 0$ and $\psi = -1$ on B_{R+1} , so

$$\int_{B_{R+2}} W(\psi) dx = \int_{B_{R+2} \setminus B_{R+1}} W(\psi) dx \leq CR^{n-1}.$$

With this, we obtain the bound

$$\mathcal{E}(\psi, B_{R+2}) \leq \begin{cases} CR^{n-1} & \text{if } s \in (\frac{1}{p}, 1), \\ CR^{n-1} \log R & \text{if } s = \frac{1}{p}, \\ CR^{n-sp} & \text{if } s \in (0, \frac{1}{p}), \end{cases} \quad (4.13)$$

where $C = C(n, s, p) > 0$.

Letting

$$A := \{v = \psi\},$$

we notice that $B_{R+1} \subseteq A \subseteq B_{R+2}$ and that for $x \in A, y \in \mathcal{C}A$

$$|v(x) - v(y)| \leq \max \{|u(x) - u(y)|, |\psi(x) - \psi(y)|\}.$$

Then by (2.2) we have that

$$F(v(x) - v(y), x - y) \leq F(u(x) - u(y), x - y) + F(\psi(x) - \psi(y), x - y).$$

Integrating on $A \times \mathcal{C}A$ we get that

$$v(A, \mathcal{C}A) \leq u(A, \mathcal{C}A) + \psi(A, \mathcal{C}A).$$

We recall that u is a minimizer in B_{R+2} , and $u = v$ outside of B_{R+2} (and outside of A), so

$$0 \leq \mathcal{E}(v, B_{R+2}) - \mathcal{E}(u, B_{R+2}) = \mathcal{E}(v, A) - \mathcal{E}(u, A).$$

Since $v = \psi$ on A , it follows that

$$u(A, A) + \int_A W(u) dx \leq \mathcal{E}(\psi, A)$$

and, given that $B_{R+1} \subseteq A \subseteq B_{R+2}$,

$$u(B_{R+1}, B_{R+1}) + \int_{B_{R+1}} W(u) dx \leq \mathcal{E}(\psi, B_{R+2}).$$

Also, one has that

$$u(B_R, \mathcal{C}B_{R+1}) \leq C \int_{B_{R+2}} d(x)^{-sp} dx \leq \begin{cases} CR^{n-1} & \text{if } s \in (\frac{1}{p}, 1), \\ CR^{n-1} \log R & \text{if } s = \frac{1}{p}, \\ CR^{n-sp} & \text{if } s \in (0, \frac{1}{p}). \end{cases}$$

Using this together with the estimate (4.13), we obtain the claim of Theorem 9. \square

We have the following very useful lemma.

Lemma 10. *Let F be convex in the first variable and such that it satisfies (2.1), and let W be such that it satisfies (3.1). Let Ω be a measurable set and $u, v: \mathbb{R}^n \rightarrow \mathbb{R}$ be two measurable functions. Let*

$$m := \min\{u, v\}, \quad M := \max\{u, v\},$$

then

$$\mathcal{E}(m, \Omega) + \mathcal{E}(M, \Omega) \leq \mathcal{E}(u, \Omega) + \mathcal{E}(v, \Omega).$$

We omit the proof of this known result, a complete proof is given e.g. in [34, Lemma 4.5.15] (and the forthcoming paper [16]), while a general abstract version in [29].

Proposition 11. *If W satisfies (2.12), let*

$$\tilde{W} := \begin{cases} W, & \text{in } [-1, 1] \\ 0, & \text{in } \mathbb{R} \setminus [-1, 1], \end{cases}$$

$$\mathcal{E}(u, B_R) := \mathcal{K}_R(u) + \int_{B_R} W(u) dx, \quad \text{and} \quad \tilde{\mathcal{E}}(u, B_R) := \mathcal{K}_R(u) + \int_{B_R} \tilde{W}(u) dx.$$

For any measurable function $\tilde{u}: \mathbb{R}^n \rightarrow \mathbb{R}$, denote also

$$u := \max\{\min\{\tilde{u}, 1\}, -1\}.$$

a) *It holds that $\mathcal{E}(u, B_R) = \tilde{\mathcal{E}}(u, B_R) \leq \tilde{\mathcal{E}}(\tilde{u}, B_R)$.*

Furthermore, if $\varphi \in \mathcal{X}_R$ is such that $|\varphi| \leq 1$ and

b) *if $u \in \mathcal{W}_{R,\varphi}^{s,p}$ is a minimizer for $\mathcal{E}(\cdot, B_R)$, then u is a minimizer for $\tilde{\mathcal{E}}(\cdot, B_R)$;*

c) *if $\tilde{u} \in \mathcal{W}_{R,\varphi}^{s,p}$ is a minimizer for $\tilde{\mathcal{E}}(\cdot, B_R)$, then u is a minimizer for $\mathcal{E}(\cdot, B_R)$.*

Proof. Notice at first that \tilde{W} satisfies (3.1). Then by Lemma 10 and using the notations therein we have that

$$\max\{\tilde{\mathcal{E}}(m, B_R), \tilde{\mathcal{E}}(M, B_R)\} \leq \tilde{\mathcal{E}}(u, B_R) + \tilde{\mathcal{E}}(v, B_R).$$

given that $F, W \geq 0$ (hence $\tilde{\mathcal{E}}(w, B_R) \geq 0$ for any measurable function w). Moreover, since $W(\pm 1) = 0$ we get that

$$\tilde{\mathcal{E}}(\pm 1, B_R) = 0.$$

Therefore denoting

$$\bar{u} = \min\{\tilde{u}, 1\},$$

we obtain

$$\tilde{\mathcal{E}}(u, B_R) \leq \tilde{\mathcal{E}}(\bar{u}, B_R) \leq \tilde{\mathcal{E}}(\tilde{u}, B_R),$$

which concludes a).

For b), let $\tilde{w}: \mathbb{R}^n \rightarrow \mathbb{R}$ be any competitor for u , hence such that $\tilde{w} = \varphi$ on $\mathcal{C}B_R$. Denoting

$$w := \max\{\min\{\tilde{w}, 1\}, -1\},$$

by the minimality of u and according to a) we have that

$$\tilde{\mathcal{E}}(u, B_R) = \mathcal{E}(u, B_R) \leq \mathcal{E}(w, B_R) = \tilde{\mathcal{E}}(w, B_R) \leq \tilde{\mathcal{E}}(\tilde{w}, B_R).$$

On the other hand, let $w: \mathbb{R}^n \rightarrow [-1, 1]$ be any admissible competitor for u , thus such $w = \varphi$ on $\mathcal{C}B_R$. From the minimality of \tilde{u} and according to a) we have that

$$\mathcal{E}(u, B_R) = \tilde{\mathcal{E}}(u, B_R) \leq \tilde{\mathcal{E}}(\tilde{u}, B_R) \leq \tilde{\mathcal{E}}(w, B_R) = \mathcal{E}(w, B_R),$$

thus c) is proved. \square

5. PROOF OF THE MAIN RESULT

The proof of the main result follows the step of [6, Theorem 4.2.1]. We underline the main ideas from [6], and focus on the new computations needed for the type of energy here introduced.

Proof of Theorem 1. We organize the proof in four steps.

Step 1. A geometrical consideration

In order to prove that u is one dimensional, one has to prove that the level sets of u are hyperplanes. It is thus enough to prove that u is monotone in any direction. Using this, one has that the level sets are both convex and concave, thus flat.

Step 2. Energy estimates

Let $R > 8$ and $\varphi \in C_c^\infty(B_1)$ such that $\varphi = 1$ in $B_{1/2}$, and let $e = (1, 0)$. With the notations of Lemma 8 we notice that

$$u_{R,\pm}(y) = u(y) \quad \text{for } y \in \mathcal{C}B_R \quad (5.1)$$

$$u_{R,+}(y) = u(y - e) \quad \text{for } y \in B_{R/2}, \quad (5.2)$$

and

$$|u_{R,\pm}| \leq 1.$$

We use the notations of Proposition 11 for $\tilde{W}, \tilde{\mathcal{E}}$. Since u is a minimizer for \mathcal{E} in B_R and $u_{R,-}$ is a competitor, thanks to Lemma 8 (applied to $\tilde{\mathcal{E}}$) and to Proposition 11 a), we have that

$$\mathcal{E}(u_{R,+}, B_R) - \mathcal{E}(u, B_R) \leq \mathcal{E}(u_{R,+}, B_R) + \mathcal{E}(u_{R,-}, B_R) - 2\mathcal{E}(u, B_R) \leq \frac{C}{R^2} \mathcal{E}(u, B_R).$$

From Theorem 9 applied for $n = 2$ it follows that

$$\lim_{R \rightarrow \infty} (\mathcal{E}(u_{R,+}, B_R) - \mathcal{E}(u, B_R)) = 0. \quad (5.3)$$

We remark that this is the crucial point where we require $n = 2$.

Step 3. Monotonicity

Suppose by contradiction that u is not monotone in any direction. So, denoting $e = (1, 0)$, up to translation and dilation, we suppose that

$$u(0) > u(e), \quad \text{and} \quad u(0) > u(-e).$$

For R large enough, we denote

$$v_R(x) := \min\{u(x), u_{R,+}(x)\}, \quad w_R(x) := \max\{u(x), u_{R,+}(x)\}$$

and remark that v_R, w_R are continuous and that $|v_R|, |w_R| \leq 1$. So $v_R = w_R = u$ on $\mathcal{C}B_R$ and since u is a minimizer

$$\mathcal{E}(w_R, B_R) \geq \mathcal{E}(u, B_R).$$

Moreover, by Lemma 10 (applied to $\tilde{\mathcal{E}}$) and Proposition 11 a), we have that

$$\mathcal{E}(v_R, B_R) + \mathcal{E}(w_R, B_R) \leq \mathcal{E}(u, B_R) + \mathcal{E}(u_{R,+}, B_R),$$

therefore

$$\mathcal{E}(v_R, B_R) \leq \mathcal{E}(u_{R,+}, B_R). \quad (5.4)$$

Since $u(0) = u_{R,+}(-e)$ and $u(-e) = u_{R,+}(0)$, using the continuity of the two functions u and $u_{R,+}$, we obtain that

$$\begin{aligned} v_R &< u \quad \text{in a neighborhood of } 0 \\ v_R &= u \quad \text{in a neighborhood of } -e. \end{aligned}$$

This implies that v_R is not identically nor u , nor $u_{R,+}$.

We remark now that v_R is not a minimizer of $\mathcal{E}(\cdot, B_2)$. Indeed, u is a global minimizer, hence $u, v_R \in W^{s,p}(B_R)$ for any R , so $u, v_R \in \mathcal{X}_2$. Moreover,

$$\begin{aligned} |u|, |v_R| &\leq 1 \quad \text{in } \mathbb{R}^2, \\ v_R &\leq u \quad \text{in } \mathbb{R}^2, \\ v_R &= u \quad \text{in } B_\delta(-e) \quad \text{for some } \delta > 0. \end{aligned}$$

If also v_R is a minimizer for $\mathcal{E}(\cdot, B_2)$, then from Proposition 11 b), u, v_R are minimizers for $\tilde{\mathcal{E}}(\cdot, B_2)$. Theorem 7 (applied to $\tilde{\mathcal{E}}$) implies that $v_R = u$ on B_2 , which gives a contradiction.

According to Theorem 5 and to Proposition 11 c), there exists v_R^* a minimizer of $\mathcal{E}(\cdot, B_2)$ such that $v_R^* = v_R$ on $\mathcal{C}B_2$ and $|v_R^*| \leq 1$. Let

$$\delta_R := \mathcal{E}(v_R, B_2) - \mathcal{E}(v_R^*, B_2) \geq 0.$$

We prove that there exists an universal constant $c > 0$ such that $\lim_{R \rightarrow \infty} \delta_R \geq c$. For this, we define as in Theorem 4.2.1 in [6]

$$\tilde{u}(x) = u(x - e), \quad m(x) = \min\{u(x), \tilde{u}(x)\}, \quad |\tilde{u}|, |m| \leq 1,$$

and observe that m is not a minimizer for $\mathcal{E}(\cdot, B_2)$. Indeed, $u, m \in W^{s,p}(B_R)$ for any R , so $u, m \in \mathcal{X}_2$. Moreover,

$$\begin{aligned} |u|, |m| &\leq 1 \quad \text{in } \mathbb{R}^2, \\ m &\leq u \quad \text{in } \mathbb{R}^2, \end{aligned}$$

and

$$\begin{aligned} m &< u \quad \text{in a neighborhood of } 0 \\ m &= u \quad \text{in a neighborhood of } e. \end{aligned}$$

Using also Proposition 11 b), if m were a minimizer for $\mathcal{E}(\cdot, B_2)$, we would obtain a contradiction by the comparison principle in Theorem 7. According to Theorem 5 and Proposition 11 c), there exists z a minimizer of $\mathcal{E}(\cdot, B_2)$ such that $m = z$ in $\mathcal{C}B_2$ and $|z| \leq 1$. Furthermore, there exists some $c > 0$ (independent of R) such that

$$\mathcal{E}(m, B_2) - \mathcal{E}(z, B_2) := c.$$

Let

$$z_R(x) := \psi(x)z(x) + (1 - \psi(x))v_R(x),$$

with $\psi \in C_c^\infty(\mathbb{R}^n, [0, 1])$ a cut-off function such that

$$\psi(x) = \begin{cases} 1, & x \in B_{R/4} \\ 0, & x \in \mathcal{C}B_{R/2}, \end{cases}$$

and notice that $|z_R| \leq 1$. It holds that

$$\begin{aligned} m &= v_R, \quad z = z_R && \text{in } B_2 \\ m &= v_R = z = z_R = v_R^* && \text{in } B_{\frac{R}{2}} \setminus B_2 \\ v_R &= z_R = v_R^*, \quad m = z && \text{in } B_R \setminus B_{\frac{R}{2}} \\ m &= z, \quad u = v_R = v_R^* = z_R && \text{in } CB_R. \end{aligned}$$

With this in mind, we get that

$$\begin{aligned} c &= \mathcal{E}(m, B_2) - \tilde{\mathcal{E}}(z, B_2) \\ &= \mathcal{E}(m, B_2) - \mathcal{E}(v_R, B_2) + \delta_R + \tilde{\mathcal{E}}(v_R^*, B_2) - \tilde{\mathcal{E}}(z_R, B_2) + \tilde{\mathcal{E}}(z_R, B_2) - \tilde{\mathcal{E}}(z, B_2) \\ &\leq \mathcal{E}(m, B_2) - \mathcal{E}(v_R, B_2) + \mathcal{E}(z_R, B_2) - \mathcal{E}(z, B_2) + \delta_R, \end{aligned} \quad (5.5)$$

since z_R is a competitor for v_R^* in B_2 . Now, for $x \in B_2, y \in CB_{R/2}$ we have $|x - y| \geq |y|/2$, hence

$$\begin{aligned} &\mathcal{E}(m, B_2) - \mathcal{E}(v_R, B_2) \\ &= 2 \int_{B_2} \left(\int_{CB_{\frac{R}{2}}} F(m(x) - m(y), x - y) - F(m(x) - v_R(y), x - y) dy \right) dx \\ &\leq 4 \int_{B_2} \int_{CB_{\frac{R}{2}}} F\left(2, \frac{y}{2}\right) < CR^{-sp}, \end{aligned}$$

given (2.5) and (2.3). The same bound holds for $\mathcal{E}(z_R, B_2) - \mathcal{E}(z, B_2)$. It follows in (5.5) that

$$c \leq CR^{-sp} + \delta_R.$$

Sending $R \rightarrow \infty$ we get that

$$\lim_{R \rightarrow \infty} \delta_R \geq c > 0.$$

Finally, by minimality,

$$\mathcal{E}(u, B_R) \leq \mathcal{E}(v_R^*, B_R) = \mathcal{E}(v_R, B_R) - \delta_R \leq \mathcal{E}(u_{R,+}, B_R) - \delta_R,$$

according to (5.4). Sending $R \rightarrow \infty$

$$c \leq \lim_{R \rightarrow \infty} (\mathcal{E}(u_{R,+}, B_R) - \mathcal{E}(u, B_R)),$$

which contradicts (5.3). This concludes the proof of Theorem 1. \square

6. SOME EXAMPLES

We give in this section some examples of problems to which our results can be applied. We give two example of functions F , that agree with the requirements (2.1) to (2.11). As we see here, the context considered is general enough to be applied to the energy related to the fractional Laplacian, the fractional p -Laplacian and to the nonlocal mean curvature.

We consider in this section W as in (2.12).

Example 6.1. We consider

$$F(t, x) = \frac{1}{p} \frac{|t|^p}{|x|^{n+sp}} \quad (6.1)$$

for $p > 1$ and $s \in (0, 1)$. The nonlocal energy that we study is

$$\mathcal{E}(u, B_R) = \frac{1}{p} \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy + \int_{B_R} W(u) dx. \quad (6.2)$$

We note that the associated equation is given by

$$(-\Delta)_p^s u + W'(u) = 0,$$

with $(-\Delta)_p^s$ being the fractional p-Laplacian, defined as

$$(-\Delta)_p^s u(x) = P.V. \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{n+sp}} dy.$$

Notice that for $p = 2$, we obtain the fractional Laplacian. The interested reader can see [6, 20, 42] and references therein for the fractional Laplacian, [19, 31, 32, 33] and references therein for the fractional p-Laplacian, or more general fractional operators.

It is not hard to verify that F given in (6.1) satisfies (2.1) to (2.6) and (2.9) to (2.11). Also, (2.7) and (2.8) follow after simple computations and hold for

$$c_1 = n + sp, \quad c_2 = (n + sp)(n + sp + 1).$$

It follows that for $n = 2$ bounded global minimizers of (6.2) are one-dimensional, more precisely we have the following.

Corollary 12. *Let $u: \mathbb{R}^n \rightarrow [-1, 1]$ be a continuous global minimizer of the energy (6.2), with W satisfying (2.12). Then u is one-dimensional.*

We remark that [15, Corollary 3.3] gives that a global minimizer of (6.2) is continuous.

Example 6.2. We consider the function related to the fractional mean curvature equation. Nonlocal minimal surfaces were introduced in [12] as boundaries of sets that minimize a nonlocal operator, namely the fractional perimeter. The first variation of the fractional perimeter operator is the nonlocal mean curvature, defined as the weighted average of the characteristic function, with respect to a singular kernel (the interested reader can check [1, 5, 23] and other references therein). For smooth hypersurfaces that are globally graphs, i.e. taking ∂E as a graph in the e_n direction defined by a smooth function u , the nonlocal mean curvature is given by

$$\mathcal{I}_s[E](x, x_n) = 2P.V. \int_{\mathbb{R}^n} \frac{dy}{|x - y|^{n+s}} \int_0^{\frac{u(x)-u(y)}{|x-y|}} \frac{d\rho}{(1 + \rho^2)^{\frac{n+s+1}{2}}}$$

for $s \in (0, 1)$ and $(x, x_n) \in \partial E$, with $u(x) = x_n$ and taking $\nabla u(x) = 0$. See e.g. [13], the Appendix in [5] for the proof of this formula. Also, the interested reader may see [34, Chapter 4] (and the forthcoming paper [16]) for the the mean curvature (and related nonparametric minimal surfaces).

We use the notations

$$g(\rho) = \frac{1}{(1 + \rho^2)^{\frac{n+s+1}{2}}}, \quad G(\tau) = \int_0^\tau g(\rho) d\rho, \quad \mathcal{G}(t) = \int_0^t G(\tau) d\tau.$$

Notice that

$$G'(\tau) = g(\tau), \quad \mathcal{G}'(t) = G(t).$$

With this, the nonlocal energy to study is

$$\mathcal{E}(u, B_R) = \iint_{\mathbb{R}^{2n} \setminus (CB_R)^2} \frac{dx dy}{|x - y|^{n+s-1}} \mathcal{G} \left(\frac{u(x) - u(y)}{|x - y|} \right) + \int_{B_R} W(u) dx, \quad (6.3)$$

and the relative equation is

$$P.V. \int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n+s}} G\left(\frac{u(x)-u(y)}{|x-y|}\right) + W'(u) = 0,$$

see also Theorem 1.10 in [24] for other applications. So, we let

$$F(t, x) = \frac{1}{|x|^{n+s-1}} \mathcal{G}\left(\frac{t}{|x|}\right) \quad (6.4)$$

and prove that the requirements (2.1) to (2.11) hold for $s \in (0, 1)$ and $p = 1$. We notice that g, \mathcal{G} are even, G is odd and

$$0 < g(\rho) \leq 1, \quad |G(t)| < \int_{-\infty}^{\infty} g(\rho) d\rho < C, \quad 0 \leq \mathcal{G}(t) \leq C|t|. \quad (6.5)$$

Also, the following chain of inequalities holds

$$a^2 g(a) \leq aG(a) \leq 2\mathcal{G}(a), \quad \text{for any } a \geq 0. \quad (6.6)$$

Indeed, since g is decreasing we have that

$$G(a) = \int_0^a g(\rho) d\rho \geq \int_0^a g(a) d\rho = ag(a).$$

Also, denoting $f(a) = aG(a) - 2\mathcal{G}(a)$ we have that

$$f'(a) = ag(a) - G(a) \leq 0.$$

So f is decreasing and $f(a) \leq f(0) = 0$ for $a \geq 0$. This proves the inequalities in (6.6). It is easy to check that the assumptions (2.1), (2.2), (2.3) and (2.6) hold for F as in (6.4). Also for some $\beta > 1$, since $g(\beta\rho) \leq g(\rho)$, we have that

$$\mathcal{G}(\beta t) = \beta^2 \int_0^t d\tau \int_0^\tau g(\beta\rho) d\rho \leq \beta^2 \mathcal{G}(t).$$

Thus for some $\alpha \in (0, 1)$

$$F(t, \alpha x) \leq \alpha^{-n-s-1} \frac{1}{|x|^{n+s-1}} \mathcal{G}\left(\frac{t}{|x|}\right),$$

and we get (2.4). Using (6.5) we obtain

$$F(t, x) = \frac{1}{|x|^{n+s-1}} \mathcal{G}\left(\frac{t}{|x|}\right) \leq c \frac{|t|}{|x|^{n+s}},$$

that is the right hand side of (2.5) for $p = 1$. The left hand side follows using the bound in [34, Lemma 4.2.1] (and the forthcoming paper [16]), that is

$$\mathcal{G}(\tau) \geq c_*(|\tau| - 1).$$

By computing the derivative with respect to x_1 of F , we get that

$$|\partial_{x_1} F(t, x)| \leq (n+s-1) \frac{F(t, x)}{|x|} + \frac{1}{|x|^{n+s}} \left| \frac{t}{|x|} G\left(\frac{t}{|x|}\right) \right| \leq c_1 \frac{F(t, x)}{|x|}$$

thanks to (6.6). Moreover

$$|\partial_{x_1}^2 F(t, x)| \leq C_1 \frac{F(t, x)}{|x|^2} + \frac{1}{|x|^{n+s+1}} \left| \frac{t}{|x|} G\left(\frac{t}{|x|}\right) \right| + \frac{1}{|x|^{n+s+1}} \frac{t^2}{|x|^2} g\left(\frac{t}{|x|}\right) \leq c_2 \frac{F(t, x)}{|x|^2}$$

again by using (6.6). So (2.7) and (2.8) are satisfied. We see also that

$$|\partial_t F(t, x)| \leq \frac{1}{|x|^{n+s}} \left| G\left(\frac{t}{|x|}\right) \right| \leq \frac{C}{|x|^{n+s}},$$

where (6.6) and (6.5) were used. Assumptions (2.9) and (2.10) follow. Moreover, it is obvious from the definition of \mathcal{G} that

$$\partial_t^2 F(t, x) = \frac{1}{|x|^{n+s+1}} g\left(\frac{t}{|x|}\right) > 0.$$

From this (2.11) is straightforward.

Theorem 1 then says that in \mathbb{R}^2 , global minimizers of the energy (6.3) are one-dimensional. To our knowledge, this is a new result in the literature. The precise result goes as follows.

Corollary 13. *Let $u: \mathbb{R}^n \rightarrow [-1, 1]$ be a continuous global minimizer of the energy (6.3) with W satisfying (2.12). Then u is one-dimensional.*

We remark that, up to our knowledge, the continuity of minimizers of the energy (6.3) is not known. Even for the classical case the problem is quite a delicate one, the interested reader can consult e.g. [4, 11, 30, 44].

APPENDIX A. SOME KNOWN RESULTS

Proposition 14. *Let $\Omega \subset \mathcal{O} \subset \mathbb{R}^n$ be bounded, open sets such that $|\mathcal{O} \setminus \Omega| > 0$ and let $u: \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function. Then*

$$\|u\|_{L^p(\Omega)}^p \leq \frac{2^{p-1}}{|\mathcal{O} \setminus \Omega|} \left(d_{\mathcal{O}}^{n+sp} \int_{\Omega} \int_{\mathcal{O} \setminus \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy + |\Omega| \|u\|_{L^p(\mathcal{O} \setminus \Omega)}^p \right),$$

with $d_{\mathcal{O}} = \text{diam}(\mathcal{O})$.

Proof. We have that

$$\begin{aligned} |u(x)|^p &= |u(x) - u(y) + u(y)|^p \\ &= \frac{1}{|\mathcal{O} \setminus \Omega|} \int_{\mathcal{O} \setminus \Omega} |u(x) - u(y) + u(y)|^p dy \\ &\leq \frac{2^{p-1}}{|\mathcal{O} \setminus \Omega|} \left(\int_{\mathcal{O} \setminus \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} |x - y|^{n+sp} + |u(y)|^p dy \right) \\ &\leq \frac{2^{p-1}}{|\mathcal{O} \setminus \Omega|} \left(d_{\mathcal{O}}^{n+sp} \int_{\mathcal{O} \setminus \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dy + \int_{\mathcal{O} \setminus \Omega} |u(y)|^p dy \right). \end{aligned}$$

The conclusion follows by integrating on Ω . \square

We recall also a fractional Poincaré inequality (see [25, Proposition 2.1] for the proof).

Proposition 15 (A fractional Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$ be bounded, open set and let $u: \mathbb{R}^n \rightarrow \mathbb{R}$ be in $L^1(\Omega)$. Then*

$$\|u - u_{\Omega}\|_{L^p(\Omega)} \leq \left(\frac{d_{\Omega}^{n+sp}}{|\Omega|} \right)^{\frac{1}{p}} [u]_{W^{s,p}(\Omega)},$$

where

$$u_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx \quad \text{and} \quad d_{\Omega} = \text{diam}(\Omega).$$

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