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Abstract	In this paper we investigate the finite model property (FMP) for varieties of BL-algebras. In particular, we provide a full classification of the FMP for those varieties of BL-algebras which are generated by a finite class of chains with finitely-many components.

Keywords (separated by '-')	BL-algebras - Hoops - Finite model property - Lattices of varieties

Finite Model Property and Varieties of BL-Algebras

Stefano Aguzzoli and Matteo Bianchi

Abstract In this paper we investigate the finite model property (FMP) for varieties of BL-algebras. In particular, we provide a full classification of the FMP for those varieties of BL-algebras which are generated by a finite class of chains with finitely-many components.

Keywords BL-algebras · Hoops · Finite model property · Lattices of varieties

1 Introduction

BL-algebras have been introduced by P. Hájek in [12] as the algebraic semantics of Basic Logic BL, the logic of all continuous t-norms and their residua ([7]). BL and its axiomatic extensions are all algebraizable in the sense of Blok and Pigozzi [4]. In [2] a full classification of the structure of BL-chains, in terms of ordinal sums of Wajsberg hoops, has been provided.

A variety $\mathbb{L}$ of BL-algebras has the *finite model property* (FMP), whenever it is generated by its finite chains. Similarly, an axiomatic extension L of BL has the FMP whenever it is complete w.r.t. the class of finite L-chains: it is well known that if L has the FMP, then it is decidable [11]. So, the FMP plays a relevant role in the computational aspect of an axiomatic extension of BL. It is well known that the variety $\mathbb{BL}$ of BL-algebras has the FMP. For subvarieties of $\mathbb{BL}$ the situation is more complicated. Indeed, for the case of MV-algebras it is easy to check, using the Komori classification (see [6]), that the only varieties having the FMP are the ones generated by a finite set of finite MV-chains, and the variety of MV-algebras itself.

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However the lattice of varieties of BL-algebras is much larger and less understood: in particular there is no known analogous of the Komori classification for subvarieties of BL-algebras.

In this paper we provide a full classification of the FMP for those varieties of BL-algebras which are generated by a finite class of chains with finitely-many components. In Theorem 6 we provide a result concerning the general case, but completing the classification for the FMP remains an open problem.

The paper is structured as follows. After some basic background in Sects. 2, 3 is devoted to the study of the FMP. Our main result is the complete classification of the FMP for those varieties of BL-algebras which are generated by a finite class of chains with finitely-many components. In Sect. 4 we discuss open problems and future works.

2 BL-Algebras and Ordinal Sums

We assume that the reader is acquainted with many-valued logics in Hájek's sense, and with their algebraic semantics. We refer to [9, 12] for any unexplained notion. We recall that BL is the logic, on the language $\{\&, \wedge, \vee, \to, \neg, \bot, \top\}$, of all left-continuous t-norms and their residua, and that its associated algebraic semantics in the sense of Blok and Pigozzi [4] is the variety $\mathbb{BL}$ of *BL-algebras*, that is, prelinear, divisible, commutative, bounded, integral, residuated lattices [9]. Derived connectives are negation $\neg\varphi \stackrel{\text{def}}{=} \varphi \to \bot$, top element $\top \stackrel{\text{def}}{=} \neg\bot$, lattice disjunction $\varphi \vee \psi \stackrel{\text{def}}{=} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$. In a BL-algebra $\mathcal{A} = (A, *, \Rightarrow, \sqcap, \sqcup, \sim, 0, 1)$ the connectives $\&, \to, \wedge, \vee, \neg, \underline{\vee}, \bot, \top$ are interpreted, respectively, by $*, \Rightarrow, \sqcap, \sqcup, \sim, \oplus, 0, 1$. Totally ordered BL-algebras are called BL-chains. In every chain $\sqcap = \min$ and $\sqcup = \max$.

A logic L is the extension of BL via a set of axioms $\{\varphi_i\}_{i \in I}$ if and only if $\mathbb{L}$ is the subvariety of BL-algebras satisfying $\{\bar{\varphi}_i = 1\}_{i \in I}$, where $\bar{\varphi}_i$ is obtained from φ_i by replacing the connectives with the corresponding operations, and every propositional variable in φ with an individual variable.

Given a BL-chain $\mathcal{A}$, and an equation $e = 1$, the notation $\mathcal{A} \models e = 1$ ($\mathcal{A} \not\models e = 1$) indicates that $\mathcal{A}$ satisfies (does not satisfy) $e = 1$. The variety $\mathbb{MV}$ of MV-algebras is axiomatized as $\mathbb{BL}$ plus $x = \sim\sim x$.

We assume that the reader is acquainted with some basic notions of universal algebra, and we refer to [5] for more details. If K is a class of BL-chains, by $\mathbf{H}(K)$, $\mathbf{S}(K)$, $\mathbf{P}(K)$, $\mathbf{I}(K)$, $\mathbf{P}_u(K)$ we denote, respectively, the classes of all homomorphic images, subalgebras, direct products, isomorphic algebras and ultraproducts of members of K. If $\mathcal{A}$ is a BL-chain, by $\mathbf{V}(\mathcal{A})$ we denote the variety generated by $\mathcal{A}$, i.e. $\mathbf{HSP}(\mathcal{A})$ [5]. Similarly, if K is a class of BL-chains, then $\mathbf{V}(K)$ indicates the variety generated by them. For example $\mathbf{V}(\mathbf{2}) = \mathbb{B}$, where $\mathbf{2}$ is the two-element Boolean algebra.

Given a variety $\mathbb{L}$ of BL-algebras, by $\mathcal{L}(\mathbb{L})$ we denote its lattice of subvarieties, ordered by inclusion. If $\{\mathbb{L}_i\}_{i\in I}$ is a family of varieties of BL-algebras, by $\bigvee_{i\in I}\mathbb{L}_i$ we denote the join, in $\mathcal{L}(\mathbb{L})$, of all these varieties.

Given a variety $\mathbb{L}$ of BL-algebras, by $Ch(\mathbb{L})$ we denote the class of all chains in $\mathbb{L}$. Every variety of BL-algebras is generated by its chains, i.e. $\mathbb{L} = \mathbf{V}(Ch(\mathbb{L}))$. We have the following result.

Lemma 1 ([3]) *Let* $\mathbb{L}, \mathbb{M} \in \mathcal{L}(\mathbb{BL})$. *Then* $Ch(\mathbb{L} \vee \mathbb{M}) = Ch(\mathbb{L}) \cup Ch(\mathbb{M})$.

We assume that the reader is familiar with Wajsberg hoops, and with the ordinal sum construction. Here we recall only basic notions and some notation: for details we refer the reader to [1, 2]. The variety $\mathbb{WH}$ of Wajsberg hoops coincides with the 0-free subreducts of MV-algebras. The variety $\mathbb{CH}$ of cancellative hoops is axiomatized as $\mathbb{WH}$ plus $x \Rightarrow (x * y) = y$.

A bounded Wajberg hoop is an algebra $\mathcal{A} = (A, *, \Rightarrow, 0, 1)$ such that $(A, *, \Rightarrow, 1)$ is a Wajsberg hoop, and $0 \leq x$ for all $x \in A$. An unbounded hoop is a hoop without minimum.

It is well known that bounded Wajsberg hoops are term-equivalent to MV-algebras. The class of totally ordered cancellative hoops coincide with the class of totally ordered unbounded Wajsberg hoops.

Let $(I, \leq)$ be a linearly ordered set with minimum 0, and let $\{\mathcal{A}_i : i \in I\}$ be a family of totally ordered Wajsberg hoops. By $\bigoplus_{i\in I}\mathcal{A}_i$ we denote the ordinal sum of this family of Wajsberg hoops, which are called components of the ordinal sum. Every BL-chain is canonically representable as an ordinal sum of hoops.

Theorem 1 ([2]) *For every BL-chain* $\mathcal{A}$ *there are a unique totally ordered set* $(I, \leq)$ *and a unique class* $\{\mathcal{A}_i \mid i \in I\}$ *of non-singleton totally ordered Wajsberg hoops whose first component* $\mathcal{A}_0$ *is bounded, such that* $\mathcal{A} \cong \bigoplus_{i\in I}\mathcal{A}_i$.

The radical of a totally ordered Wajsberg hoop (resp. MV-chain) $\mathcal{A}$, is the intersection of all maximal filters of $\mathcal{A}$, and will be denoted by $Rad(\mathcal{A})$. Let $\mathcal{A}$ be an MV-chain (resp. a totally ordered Wajsberg hoop). We say that $\mathcal{A}$ has a finite rank if $\mathcal{A}/Rad(\mathcal{A}) \simeq \mathbf{L}_k$ (resp. $\mathcal{A}/Rad(\mathcal{A}) \simeq \mathbf{L}'_k$, where $\mathbf{L}'_k$ is the 0-free reduct of $\mathbf{L}_k$), for some k, and we write $rank(\mathcal{A}) = k$. $\mathcal{A}$ has infinite rank if $\mathcal{A}/Rad(\mathcal{A})$ is an infinite MV-chain (resp. infinite totally ordered Wajsberg hoop with minimum).[1]

Let $\mathbf{R}, \mathbf{Q}$ be additive lattice-ordered abelian groups over, respectively, real, rational and integer numbers. Let $\mathbb{Z}$ be the set of all integers. For $k \geq 2$, let $\mathbf{Q}_k$ be the lattice ordered abelian subgroup of $\mathbf{Q}$, with carrier $\{\frac{a}{k-1} : a \in \mathbb{Z}\}$.

We define the following MV-chains, via Mundici's functor Γ: see [6] for details. $[0,1]_{Ł} \overset{\text{def}}{=} \Gamma(\mathbf{R}, 1)$, $\mathbb{Q}_{Ł} \overset{\text{def}}{=} \Gamma(\mathbf{Q}, 1)$ and, for $n \geq 2$, $\mathbf{L}_n \overset{\text{def}}{=} \Gamma(\mathbf{Q}_n, 1)$.

Given an MV-chain of finite rank $\mathcal{A}$ we define $d(\mathcal{A}) \overset{\text{def}}{=} \max\{z : \mathbf{L}_z \hookrightarrow \mathcal{A}\}$.

Given a Wajsberg hoop or a BL-algebra $\mathcal{A}$, we define $Si(\mathcal{A})$ as the class of subdirectly irreducible algebras of $\mathbf{V}(\mathcal{A})$. As every variety of BL-algebras is congruence distributive, by Jónsson Lemma we have $Si(\mathcal{A}) \subseteq \mathbf{HSP}_u(\mathcal{A})$.

[1] Note that every non-trivial totally ordered cancellative hoop $\mathcal{A}$ does not have rank, since $\mathcal{A}/Rad(\mathcal{A})$ is an infinite cancellative hoop.

Proposition 1 ([2, 3]) *Let $\mathcal{A}_0$ be an MV-chain, and let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be a family of totally ordered Wajsberg hoops. Then $Si(\bigoplus_{i=0}^{k} \mathcal{A}_i)$ is equal to:*

$$Si(\mathcal{A}_0) \cup (\mathbf{ISP}_u(\mathcal{A}_0) \oplus Si(\mathcal{A}_1)) \cup \cdots \cup \left(\bigoplus_{i=0}^{k-1} \mathbf{ISP}_u(\mathcal{A}_i) \oplus Si(\mathcal{A}_k) \right).$$

Let $\mathcal{A}$ be an MV-chain with infinite rank (a totally ordered Wajsberg hoop with infinite rank). We define $F(\mathcal{A}) \stackrel{\text{def}}{=} \{n : L_n \hookrightarrow \mathcal{A}\}$, and by $SF(\mathcal{A})$ we denote the subalgebra of $\mathbb{Q}_Ł$ generated by $\{\mathbf{L}_n : n \in F(\mathcal{A})\}$ (if $\mathcal{A}$ is a totally ordered Wajsberg hoop with infinite rank, then we modify the definitions by taking the 0-free reducts of $\mathbb{Q}_Ł$ and $\mathbf{L}_n$).

Proposition 2 ([3]) *Let $\mathcal{A}$ be an MV-chain with infinite rank. Then:*

- *$F(\mathcal{A})$ is infinite if and only if $SF(\mathcal{A})$ is infinite.*
- *If $F(\mathcal{A})$ is infinite, then $\mathbf{ISP}_u(\mathcal{A}) = \mathbf{ISP}_u(SF(\mathcal{A}))$.*

3 FMP for Varieties of BL-Algebras

In this section we classify the FMP for those varieties of BL-algebras generated by a finite set of BL-chains with finitely-many components.

Definition 1 A variety $\mathbb{L}$ of BL-algebras has the *finite model property*, FMP, whenever $\mathbb{L}$ is generated by its finite chains.

We begin with the case of varieties generated by one BL-chain with finitely many components.

Theorem 2 *Let $\mathbb{L}$ be a variety of BL-algebras generated by a BL-chain $\mathcal{A} = \mathcal{A} \simeq \bigoplus_{i=0}^{k} \mathcal{A}_i$ such that:*

- *if $i < k$, then $\mathcal{A}_i$ is a finite totally ordered Wajsberg-hoop or a totally ordered Wajsberg hoop with infinite rank, such that $F(\mathcal{A}_i)$ is infinite.*
- *$\mathcal{A}_k$ is a finite totally ordered Wajsberg-hoop or a totally ordered Wajsberg hoop with infinite rank.*

Then $\mathbb{L}$ has the FMP.

Proof Let $\mathbb{L}$ be a variety of BL-algebras satisfying the theorem hypothesis. If no $\mathcal{A}_i$ is infinite, then $\mathcal{A}$ is finite, and by [8, Proposition 4.18] $\mathbb{L}$ has the FMP.

Assume now that there is at least one infinite $\mathcal{A}_i$. We construct the BL-chain $\mathcal{B} = \bigoplus_{i=0}^{k} \mathcal{B}_i$ as follows:

- For every $i < k$, $\mathcal{B}_i = \mathcal{A}_i$ if $\mathcal{A}_i$ is finite, otherwise $\mathcal{B}_i = SF(\mathcal{A}_i)$.
- $\mathcal{B}_k = \mathcal{A}_k$ if $\mathcal{A}_k$ is finite, otherwise $\mathcal{B}_k = \mathbb{Q}_Ł$.

By [2, Theorem 7.9], Propositions 1, 2 and a direct inspection we have $Si(\mathcal{A}) = Si(\mathcal{B})$, and hence $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{A}) = \mathbb{L}$. Suppose that an equation $e(x_1, \ldots, x_n) = 1$ fails in $\mathcal{B}$. Then there are $a_1, \ldots, a_n \in \mathcal{B}$ such that $e(a_1, \ldots, a_n) < 1$. As every $\mathcal{B}_i$ is a subalgebra of $\mathbb{Q}_Ł$, the subalgebra of $\mathcal{B}$ generated by $a_1, \ldots, a_n$ is a finite BL-chain $\mathcal{C}$, and clearly $e(x_1, \ldots, x_n) = 1$ fails also in $\mathcal{C}$. Whence $\mathbb{L}$ has the FMP.

Theorem 3 *Let $\mathbb{L}$ be a variety of BL-algebras generated by a BL-chain $\mathcal{A} \simeq \bigoplus_{i=0}^{k} \mathcal{A}_i$ having at least one $\mathcal{A}_i$ such that:*

(1) $\mathcal{A}_i$ is a cancellative hoop or
(2) $i < k$, and $\mathcal{A}_i$ is a totally ordered Wajsberg hoop with infinite rank such that $F(A_i)$ is finite or
(3) $\mathcal{A}_i$ is a non-simple totally ordered Wajsberg hoop with finite rank.

Then $\mathbb{L}$ does not have the FMP.

Proof Let $\mathbb{L}$ be a variety of BL-algebras generated by a BL-chain $\mathcal{A} \simeq \bigoplus_{i=0}^{k} \mathcal{A}_i$.

(1) Assume first that there is an $\mathcal{A}_i$ being a cancellative hoop. Then we must have $i > 0$. By [2, Theorem 7.9] every subdirectly irreducible algebra in $\mathbb{L}$, with $k+1$ components, is such that one of them is an infinite cancellative hoop. Since every finite chain is subdirectly irreducible, it follows that every finite chain in $\mathbb{L}$ must have at most k components. By [2, Lemma 4.2] the FMP fails to hold, for $\mathbb{L}$.

(2) Suppose that $i < k$, and $\mathcal{A}_i$ is a totally ordered Wajsberg hoop with infinite rank such that $F(\mathcal{A}_i)$ is finite. For every $a \in F(\mathcal{A}_i)$, let $\mathcal{D}_a$ be the BL-chain obtained from $\mathcal{A}$ by replacing $\mathcal{A}_i$ with $\mathbf{L}_a$. By Lemma 1 we have that the class of chains in $\bigvee_{a \in F(\mathcal{A}_i)} \mathbf{V}(\mathcal{D}_a)$ coincides with the class of chains in $\bigcup_{a \in F(\mathcal{A}_i)} \mathbf{V}(\mathcal{D}_a)$. By [2, Theorem 7.9] a direct inspection shows that the class of finite chains in $\mathbf{V}(\mathcal{A})$ (which are all subdirectly irreducible) coincides with the class of finite chains in $\bigcup_{a \in F(\mathcal{A}_i)} \mathbf{V}(\mathcal{D}_a)$, and hence with the ones in $\bigvee_{a \in F(\mathcal{A}_i)} \mathbf{V}(\mathcal{D}_a)$. So, if $\mathbf{V}(\mathcal{A})$ has the FMP, then $\mathbf{V}(\mathcal{A}) = \bigvee_{a \in F(\mathcal{A}_i)} \mathbf{V}(\mathcal{D}_a)$: we now show that this is not possible. Let $\mathcal{E}$ be the chain obtained from $\mathcal{A}$ by replacing $\mathcal{A}_i$ with a totally ordered infinite cancellative hoop, and $\mathcal{A}_k$ with a (non-trivial) chain in $Si(\mathcal{A}_k)$. Clearly $\mathcal{E}$ is subdirectly irreducible, and by [2, Theorem 7.9], $\mathcal{E} \in Si(\mathcal{A}) \subsetneq \mathbf{V}(\mathcal{A})$. However $\mathcal{E} \notin Si(\bigvee_{a \in F(\mathcal{A}_i)} \mathbf{V}(\mathcal{D}_a))$. Indeed, by [2, Theorem 7.9] every chain $\mathcal{F} = \mathcal{F}_0 \oplus \cdots \oplus \mathcal{F}_k \in Si(\bigvee_{a \in F(\mathcal{A}_i)} \mathbf{V}(\mathcal{D}_a))$ is such that $\mathcal{F}_i$ is a finite chain. Then we conclude that $\mathbb{L} = \mathbf{V}(\mathcal{A}) \neq \bigvee_{a \in F(\mathcal{A}_i)} \mathbf{V}(\mathcal{D}_a)$, and hence $\mathbb{L}$ cannot have the FMP.

(3) Suppose that $\mathcal{A}_i$ is a non-simple totally ordered Wajsberg hoop with finite rank, say n. We have two cases. If $i < k$, then the proof strategy is almost identical to the case 2), *mutatis mutandis*, since the set $\{n : \mathbf{L}'_n \hookrightarrow \mathcal{A}_i\}$ ($\{n : \mathbf{L}_n \hookrightarrow \mathcal{A}_i\}$, if $i = 0$) is finite.
Assume $i = k$. Let $\mathcal{B} = \bigoplus_{i=0}^{k-1} \mathcal{A}_i \oplus \mathbf{L}_n$, and $\mathcal{C} = \bigoplus_{i=0}^{k-1} \mathcal{A}_i \oplus \mathcal{D}$, where $\mathcal{D}$ is a subdirectly irreducible totally ordered cancellative hoop. Then $\mathcal{C}$ is subdirectly irreducible. Since $\mathcal{A}_i$ has rank n, by [2, Theorem 7.9] we have that the class of finite chains (which are all subdirectly irreducible) in $\mathbf{V}(\mathcal{A})$ coincides with the one in $\mathbf{V}(\mathcal{B})$. So, if $\mathbf{V}(\mathcal{A})$ has the FMP, then $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{A})$. However this is

not possible, since by [2, Theorem 7.9] the chain $\mathcal{C} \in \mathbf{V}(\mathcal{A})$, whilst $\mathcal{C} \notin \mathbf{V}(\mathcal{B})$. Indeed, by [2, Theorem 7.9] every chain $\mathcal{F} = \mathcal{F}_0 \oplus \cdots \oplus \mathcal{F}_k \in Si(\mathcal{B})$ is such that $\mathcal{F}_k$ is a finite chain.

Whence $\mathbb{L} = \mathbf{V}(\mathcal{A})$ does not have the FMP. The proof is settled.

Theorem 4 *Let $\mathcal{A} = \bigoplus_{i=0}^{k} \mathcal{A}_i$ be a BL-chain. Then $\mathbf{V}(\mathcal{A})$ has the FMP if and only if $\mathcal{A}$ satisfies the conditions of Theorem 2.*

Proof Immediate by Theorems 2 and 3.

Proposition 3 *Let $\mathbb{L}_1, \ldots, \mathbb{L}_k$ be a family of single-chain generated varieties of BL-algebras such that $\mathbb{L}_i \nsubseteq \mathbb{L}_j$, for every $1 \leq i \neq j \leq k$. Then $\bigvee_{i=1}^{k} \mathbb{L}_i$ has the FMP if and only if $\mathbb{L}_i$ has the FMP, for every $i \in \{1, \ldots, k\}$.*

Proof Let $\mathbb{L}_1, \ldots, \mathbb{L}_k$ be a family of single-chain generated varieties of BL-algebras such that $\mathbb{L}_i \nsubseteq \mathbb{L}_j$, for every $1 \leq i \neq j \leq k$. If every $\mathbb{L}_i$ has the FMP, by Lemma 1 we conclude that $\bigvee_{i=1}^{k} \mathbb{L}_i$ has the FMP. Suppose now that for some $h \in \{1, \ldots, k\}$, $\mathbb{L}_h$ does not have the FMP. For every $i \in \{1, \ldots, k\}$, let us call $\mathbb{F}_i$ the variety generated by all the finite chains of $\mathbb{L}_i$. By Lemma 1 we have that the variety generated by the finite chains of $\bigvee_{i=1}^{k} \mathbb{L}_i$ is $\bigvee_{i=1}^{k} \mathbb{F}_i$, and clearly $\bigvee_{i=1}^{k} \mathbb{F}_i \subseteq \bigvee_{i=1}^{k} \mathbb{L}_i$. By hypothesis there is a chain $\mathcal{A}$ such that $\mathbf{V}(\mathcal{A}) = \mathbb{L}_h$. As $\mathbb{F}_h \subsetneq \mathbb{L}_h$, we have $\mathcal{A} \notin \mathbb{F}_h$, and since $\mathbb{L}_h \nsubseteq \mathbb{L}_j$, for every $h \neq j$, we have that $\mathcal{A} \notin \mathbb{F}_j$. Then by Lemma 1 $\mathcal{A} \notin \bigvee_{i=1}^{k} \mathbb{F}_i$.

So we have $\bigvee_{i=1}^{k} \mathbb{F}_i \subsetneq \bigvee_{i=1}^{k} \mathbb{L}_i$. This implies that $\bigvee_{i=1}^{k} \mathbb{L}_i$ does not have the FMP, and the proof is settled.

We can now state our main result.

Theorem 5 *Let $\mathbb{L}$ be a variety of BL-algebras. If $\mathbb{L} = \mathbf{V}(S)$, where S is a finite set of BL-chains with finitely-many components, then $\mathbb{L}$ has the FMP if and only if every chain in S satisfies the conditions of Theorem 4.*

Proof Immediate by Proposition 3 and Theorem 4.

The classification of the FMP for general case remains an open problem. Nevertheless, we have the following theorem.

Theorem 6 *Let $\mathbb{L}$ be a variety of BL-algebras. Then $\mathbb{L}$ has the FMP if and only if there exists $C \subseteq Ch(\mathbb{L})$ such that:*

(1) $\mathbf{V}(C) = \mathbb{L}$,

(2) For every $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i \in C$, and every finite subset $\{0\} \subseteq J \subseteq I$, $\bigoplus_{j \in J} \mathcal{A}_j$ satisfies the hypothesis of Theorem 2.

Proof $\Leftarrow$ Let $\mathbb{L}$ be a variety of BL-algebras such that there exists $C \subseteq Ch(\mathbb{L})$ satisfying (1) and (2). To prove the FMP we show that if a formula φ fails in C, then it fails in some finite chain in $\mathbb{L}$. Suppose that $C \not\models \varphi$, and let $x_1 \ldots, x_k$ be the variables of φ. Then there is a chain $\mathcal{A} \in C$ and an $\mathcal{A}$-evaluation v such that $v(\varphi) < 1$. Let $\mathcal{B}$ be the subalgebra of $\mathcal{A}$ generated by $A_{\sigma(1)} \cup \cdots \cup A_{\sigma(k)}$, where

$\sigma : \{1, \dots k\} \to I$ is such that $\sigma(i) = j$ if and only if $v(x_i) \in A_j$. Clearly $\mathcal{B}$ has at most $k+1$ components, and by 2) $\mathcal{B}$ satisfies the hypothesis of Theorem 2. Then $\mathbf{V}(\mathcal{B})$ has the FMP. Clearly $\mathcal{B} \not\models \varphi$, and hence we can find a finite BL-chain $\mathcal{D} \in \mathbf{V}(\mathcal{B}) \subseteq \mathbb{L}$ such that $\mathcal{D} \not\models \varphi$. The proof is settled.

$\Rightarrow$ Let $\mathbb{L}$ be a variety of BL-algebras having the FMP, and let F be the class of all the finite chains in $\mathbb{L}$. By the FMP $\mathbf{V}(F) = \mathbb{L}$, and an easy check shows that each member of F satisfies the hypothesis of Theorem 2. So $F \subseteq Ch(\mathbb{L})$ satisfies (1) and (2), and the proof is settled.

4 Conclusions

In this paper we provided an analysis of the FMP for the varieties generated by a finite set of BL-chains with finitely-many components.

The general case is way more complicated, and remains an open problem. One of the issues is the lacking of a general description for the structure of the subdirectly irreducible members, for those varieties generated by BL-chains with infinitely-many components.

An analogous investigation of the structure of BL-chains in terms of their ordinal sum decomposition may throw new light on the study of the amalgamation property (AP) for varieties of BL-algebras.

Both the FMP and the AP are formulated in algebraic terms, for varieties of BL-algebras, but they are also related with logical properties. Specifically, whereas the FMP for a variety $\mathbb{L}$ of BL-algebras implies the decidability of the logic L, the AP for $\mathbb{L}$ is equivalent to the deductive interpolation property for L. For the case of MV-algebras there is a complete classification for the AP, provided by di Nola and Lettieri [10]. A variety of MV-algebras has the AP if and only if it is generated by one MV-chain. This is not true for the case of BL-algebras: the variety generated by the four element Gödel chain is a counterexample. At the moment we have a partial classification of the AP, for the varieties of BL-algebras which are generated by one BL-chain with finitely-many components. Future work will be devoted to this topic.

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Delete	/ through single character, rule or underline or through all characters to be deleted	or
Substitute character or substitute part of one or more word(s)	/ through letter or through characters	new character / or new characters /
Change to italics	under matter to be changed	
Change to capitals	under matter to be changed	
Change to small capitals	under matter to be changed	
Change to bold type	under matter to be changed	
Change to bold italic	under matter to be changed	
Change to lower case	Encircle matter to be changed	
Change italic to upright type	(As above)	
Change bold to non-bold type	(As above)	
Insert 'superior' character	/ through character or ⋏ where required	or under character e.g. or
Insert 'inferior' character	(As above)	⋏ over character e.g.
Insert full stop	(As above)	⊙
Insert comma	(As above)	,
Insert single quotation marks	(As above)	or and/or or
Insert double quotation marks	(As above)	or and/or or
Insert hyphen	(As above)	
Start new paragraph		
No new paragraph		
Transpose		
Close up	linking characters	
Insert or substitute space between characters or words	/ through character or ⋏ where required	
Reduce space between characters or words	| between characters or words affected	