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Research article

A C^∞ Nekhoroshev theorem†

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Abstract: We prove a C^∞ version of the Nekhoroshev's estimate on the stability times of the actions in close to integrable Hamiltonian systems. The proof we give is a variant of the original Nekhoroshev's proof and it consists in first conjugating, globally in the phase space, and up to a small remainder, the system to a normal form. Then we perform the geometric part of the proof in the normalized variables. As a result, we obtain a proof which is simpler than the usual ones.

Keywords: Nekhoroshev's theorem; Hamiltonian systems; quasi-integrable systems; normal form; perturbation theory

1. Introduction

In this paper we prove a C^∞ version of Nekhoroshev's Theorem for the stability times in a close to integrable Hamiltonian system. The main point is that our proof relies on an analytic and a geometrical construction which, although on the same line as the original Nekhoroshev's ones, are much simpler. We also obtain an intermediate result that we think could have some interest in itself (see Theorem 1.2).

To be definite and in order to avoid as much as possible technical complications, we study here a system of the form

$$\begin{aligned} H(p,q) &= H_0(p) + \varepsilon V(p,q)\,, \\ H_0(p) &= \sum_{j=1}^{d} \frac{p_j^2}{2}\,, \end{aligned} \tag{1.1}$$

with $V \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$. However, our technique is applicable also to the general case of perturbations of steep integrable systems, and with a perturbation which is not globally bounded in the momenta p.

The result we get is the following version of Nekhoroshev's Theorem.

Theorem 1.1. *Assume that $V \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ is globally bounded and fix a positive $b < \frac{1}{2}$. Then, for any positive M, there exists C_M, ε_M such that, if $0 < \varepsilon < \varepsilon_M$ then, corresponding to any initial datum, one has*

$$|p(t) - p(0)| \leq C_M \varepsilon^b , \tag{1.2}$$

$$|t| \leq \frac{C_M}{\varepsilon^M} . \tag{1.3}$$

We recall that in the analytic (or Gevrey) case the time of stability, cf. Eq (1.3), is known to be exponentially long.

Theorem 1.1 is not new, for example it is a direct consequence of Theorem 2.1 of [9]*. However, as far as we know our proof is new and the value of the exponent b that we get is better than those present in literature†.

Essentially two methods of proof of Nekhoroshev's theorem are known: the original one [3, 5, 12, 14, 19, 20, 23], and Lochak's one [15, 16] (see also [17, 18] and [1, 4] for infinite dimensional generalizations). We emphasize that Lochak's proof is much simpler than Nekhoroshev's one, but, at least in its original form, applies only to perturbations of quasiconvex integrable systems. The paper [9] is a generalization of Lochak's proof to the case of finite smoothness. Lochak's method was also extended to the steep case by using also ideas from the original proof by Nekhoroshev [21], see also [8, 10, 22].

Our proof is a variant of Nekhoroshev's original one which consists of two steps: the analytic part and the geometric part. Classically, in the analytic part one shows that in a region of the phase space where only some resonances are present one can conjugate, up to a small remainder, the system to a system in resonant normal form. In the geometrical part one collects all the information and shows that, if the regions are suitably constructed, then for any initial datum there exists a region in which it remains for the considered times, and this leads to Nekhoroshev's estimate.

In the classical approach the analysis of the geometrical part is slightly complicated by the fact that it has to be performed in the original coordinates, so that one has to take into account the effects of the coordinate transformation used to conjugate the system to normal form. The novelty of the present paper is that we use a canonical transformation which is globally defined and globally conjugates the system to a normal form which is different in each region of the phase space, depending on the resonances which are present in each region (a similar technique has been used for the first time in a probabilistic context in [7, 11]). This is obtained by splitting each Fourier coefficient of V, namely $\hat{V}_k(p)$, into a part localized in the region $|\omega(p) \cdot k| < \varepsilon^\delta$ (with a suitable δ) and a part localized in the nonresonant region. The part localized in the nonresonant region is then removed through the normalizing canonical transformation. Technically the localization is obtained simply by multiplying by a smooth cutoff function.

Then the geometrical part consists in making a decomposition of the phase space in regions which are invariant for the dynamics of the normalized system. This leads in particular to the conclusion that,

*Actually such a theorem is also stated in [18], where reference is made to a slightly different statement present in [17]

†In the analytic case any $b < 1/2$ is allowed at the price of worsening the estimate on the times, see for example [23]

in the dynamics of the normalized system, estimate (1.2) is valid for all times. This is the content of the following theorem, which, as far as we know, is new.

Theorem 1.2. *Fix a positive $b < \frac{1}{2}$, then, for any positive M, there exists C_M, ε_M such that, if $0 < \varepsilon < \varepsilon_M$, then there exists a canonical transformation $(p,q) = \mathcal{T}(\tilde{p},\tilde{q})$ and a (normal form) Hamiltonian H_Z, with the following properties*

1). $|p - \tilde{p}| \leq C_M \varepsilon^b$
2). $\|H \circ \mathcal{T} - H_Z\|_{C^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq C_M \varepsilon^M$,
3). Along the solutions of the Hamilton equations of H_Z, one has

$$|\tilde{p}(t) - \tilde{p}(0)| \leq C_M \varepsilon^b \ , \quad \forall t \in \mathbb{R} \ .$$

When adding the remainder, one gets the limitation (1.3) on the times.

Finally we remark that Lochak's proof applies to system (2.5), but we think that our approach to the geometric part of the proof is the main interest of the present paper, since it is suitable for generalizations to the steep case.

This paper originates from our research on the spectrum of Sturm Liouville operators in general tori [6], which lead to a quantum version of Nekhoroshev's theorem. When we were still lost on how to construct a quantum analogue of the geometric part, we had several very enlightening discussions with Antonio Giorgilli on the classical Nekhoroshev's theorem. At the end we realized that the quantum method we constructed had a classical counterpart which is the content of the present paper. It is a pleasure to dedicate this paper to Antonio Giorgilli in the occasion of his 70th birthday.

One of us, Dario Bambusi, would like to thank Antonio who introduced him to science and in particular to the study of Hamiltonian dynamics: His presence has always been fundamental and I would be a different person if I had not met him. Thank you!

2. Analytical part

2.1. Preliminaries and statement

In this subsection we present the tools we will use in order to deal with the C^∞ context.

Having fixed a parameter $0 < \delta < \frac{1}{2}$ and an interval $\mathcal{U} = [0, \varepsilon_0)$ with some positive ε_0, we give the following definition.

Definition 2.1. *A family of functions $\{f_\varepsilon\}_{\varepsilon \in \mathcal{U}}$, $f_\varepsilon \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$ will be said to be a symbol of order m if for all $\alpha, \beta \in \mathbb{N}^d$ there exists a positive constant $C_{\alpha,\beta}$ such that*

$$\sup_{\varepsilon \in \mathcal{U}} \sup_{p \in \mathbb{R}^d, q \in \mathbb{T}^d} \left| \frac{\partial_p^\alpha \partial_q^\beta f_\varepsilon(p,q)}{\varepsilon^m} \varepsilon^{|\alpha|\delta} \right| \leq C_{\alpha,\beta} \ . \tag{2.1}$$

In this case we will denote $f_\varepsilon \in \mathcal{P}^{m,\delta}$. We will often omit the index ε.

Remark 2.2. *The constants $C_{\alpha,\beta}$ encode the smoothness properties of f. This is particularly clear for ε independent functions. Indeed in this case the function is of class Gevrey s if and only if there exist constants R, C s.t.*

$$C_{\alpha,\beta} \leq C \frac{(\alpha!\beta!)^s}{R^{|\alpha|+|\beta|}} \ .$$

We think that keeping track of the dependence of $C_{\alpha,\beta}$ on α and β should allow to obtain exponentially long stability times for the Gevrey case. Similar ideas have been developed in [2].

Remark 2.3. *It is immediate to see that $f \in \mathcal{P}^{m,\delta}$ if and only if for all integers N_1 and N_2 there exists a positive constant $C^m_{N_1,N_2}$ such that*

$$\sup_{\varepsilon\in\mathcal{U}} \sup_{\substack{p\in\mathbb{R}^d,\ k\in\mathbb{Z}^d,\\ \alpha\in\mathbb{N}^d,\ |\alpha|=N_1}} \left|\partial_p^\alpha \hat{f}_k(\varepsilon,p)\right| |k|^{N_2} \varepsilon^{-(m-|\alpha|\delta)} \le C^m_{N_1,N_2}\,, \tag{2.2}$$

where

$$\hat{f}_k(\varepsilon,p) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f_\varepsilon(p,q)e^{-\mathrm{i}k\cdot q}, \quad \varepsilon\in\mathcal{U}, \quad p\in\mathbb{R}^d$$

are the Fourier coefficients of f.

Remark 2.4. *The space $\mathcal{P}^{m,\delta}$ endowed by the family of seminorms given by the constants $C^m_{N_1,N_2} = C^m_{N_1,N_2}(f)$ of equation (2.2) is a Fréchet space.*

Remark 2.5. *A direct computation shows that, if $f \in \mathcal{P}^{m_1,\delta}$ and $g \in \mathcal{P}^{m_2,\delta}$, then*

1). $f + g \in \mathcal{P}^{\min\{m_1,m_2\},\delta}$
2). $fg \in \mathcal{P}^{m_1+m_2,\delta}$
3). the Poisson bracket, $\{f,g\} \in \mathcal{P}^{m_1+m_2-\delta,\delta}$.

In the following, given a C^∞ function g, we will denote by X_g its Hamiltonian vector field and by Φ^t_g the flow it generates (which in our framework will always be globally defined).

In order to state the analytic Lemma, we start by defining what we mean by normal form of order N. From now of we fix the number N controlling the number of steps in the normal form procedure.

Furthermore, we will denote

$$a := 1 - 2\delta\,;$$

we fix a positive (small) $0 < \beta < 1$ and we define

$$K = K(\varepsilon) := \left[\frac{1}{\varepsilon^\beta}\right] + 1 \tag{2.3}$$

with the square bracket denoting the integer part. Eventually we will link β, δ, b, M and N.

Definition 2.6. *A family of functions $Z_\varepsilon : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ will be said to be in normal form if*

$$Z_\varepsilon(p,q) = \sum_{|k|\le K} \hat{Z}_k(\varepsilon,p)e^{\mathrm{i}k\cdot q}$$

with

$$\hat{Z}_k(\varepsilon,p) \ne 0 \Rightarrow |p\cdot k| \le \varepsilon^\delta\,, \quad \forall k\in\mathbb{Z}^d\backslash\{0\}\,, \tag{2.4}$$

Namely the k-th Fourier coefficient is supported in the resonant region $|p\cdot k| \le \varepsilon^\delta$.

Lemma 2.7. *(Normal Form Lemma) Consider the system*

$$H_\varepsilon := H_0(p) + P_\varepsilon(p,q) \tag{2.5}$$

with H_0 *as in* (1.1) *and* $P_\varepsilon \in \mathcal{P}^{1,\delta}$, *then there exists a canonical transformation* $\mathcal{T}$ *such that*

$$H_\varepsilon \circ \mathcal{T} = H_0 + \sum_{j=1}^{N} Z_j + \mathcal{R}^{(N)}\,, \tag{2.6}$$

with $Z_j \in \mathcal{P}^{1+a(j-1),\delta}$ *in normal form and* $\mathcal{R}^{(N)}$ *s.t.*

$$\sup_{\varepsilon\in\mathcal{U}}\ \sup_{p\in\mathbb{R}^d,q\in\mathbb{T}^d}\left|\frac{\partial_p^\alpha\partial_q^\beta\mathcal{R}^{(N)}(p,q)}{\varepsilon^{1+Na}}\right|\varepsilon^{|\alpha|\delta} \le C_{\alpha,\beta}\,, \tag{2.7}$$

$$\forall \alpha,\beta\in\mathbb{N}^d\times\mathbb{N}^d \quad \textit{with } |\alpha|+|\beta|\le 1\,. \tag{2.8}$$

Furthermore, given a symbol $f\in\mathcal{P}^{m,\delta}$, *define* $\mathcal{R}_f := f\circ\mathcal{T} - f$, *then one has*

$$\sup_{p\in\mathbb{R}^d,q\in\mathbb{T}^d}\left|\mathcal{R}_f(\varepsilon,p,q)\right| \le C\varepsilon^{m+1-2\delta}\,. \tag{2.9}$$

In the case $f = p_j$, $j = 1,...,d$, *one has*

$$\sup_{p\in\mathbb{R}^d,q\in\mathbb{T}^d}\left|\mathcal{R}_{p_j}(\varepsilon,p,q)\right| \le C\varepsilon^{1-\delta}\,. \tag{2.10}$$

Definition 2.8. *A function fulfilling Eqs* (2.7) *and* (2.8) *will be said to be a remainder of order N, or simply a remainder.*

The proof of Lemma 2.7 consists of a few steps: first we give a decomposition of an arbitrary symbol in a normal form part, a nonresonant part and a remainder, then we remove the nonresonant part of the perturbation and then we iterate. The canonical transformation used to remove the nonresonant part will be constructed using the Lie transform method, namely by using the time one flow of an auxiliary Hamiltonian. This requires the study of the Lie transform in our C^∞ context. We will also have to solve the cohomological equation in order to construct the auxiliary Hamiltonian. Finally we state and prove the iterative Lemma which is the last step of the proof of the Normal Form Lemma.

2.2. Cutoffs and splittings

Let us consider an even C^∞ function $\eta:\mathbb{R}\to\mathbb{R}^+$ such that $\eta(t)\equiv 1$ if $|t|\le\dfrac{1}{2}$ and $\eta(t)\equiv 0$ if $|t|\ge 1$. For all $k\in\mathbb{Z}^d$ such that $0\ne|k|\le K$, we define the cut-off function

$$\chi_k(p) = \eta\left(\frac{p\cdot k}{\varepsilon^\delta}\right),\quad p\in\mathbb{R}^d\,, \tag{2.11}$$

which is thus supported in $|p\cdot k|\le\varepsilon^\delta$. Consider a smooth family of functions $f_\varepsilon\in\mathcal{P}^{m,\delta}$, $f_\varepsilon(p,q)=\sum_{k\in\mathbb{Z}^d}\hat{f}_k(\varepsilon,p)e^{\mathrm{i}k\cdot q}$, we perform for f_ε the following decomposition:

$$f(p,q) = f^{(\mathrm{res})}(p,q) + f^{(\mathrm{nr})}(p,q) + f_K(p,q)\,, \tag{2.12}$$

where

$$
\begin{aligned}
f^{(\mathrm{res})}(p,q) &= \sum_{0<|k|\le K} \hat{f}_k(\varepsilon,p)\chi_k(p)e^{\mathrm{i}k\cdot q} + \hat{f}_0(\varepsilon,p),\\
f^{(\mathrm{nr})}(p,q) &= \sum_{0<|k|\le K} \hat{f}_k(\varepsilon,p)\,(1-\chi_k(p))\,e^{\mathrm{i}k\cdot q}\,,\\
f_K(p,q) &= \sum_{|k|>K} \hat{f}_k(\varepsilon,p)e^{\mathrm{i}k\cdot q}\,.
\end{aligned}
\tag{2.13}
$$

Remark 2.9. *If $f \in \mathcal{P}^{m,\delta}$ then $f^{(\mathrm{res})}, f^{(\mathrm{nr})} \in \mathcal{P}^{m,\delta}$. Furthermore $f^{(\mathrm{res})}$ is in normal form.*

Remark 2.10. *Since compactly supported analytic functions do not exist, this step would be impossible in an analytic context. On the contrary a Gevrey cutoff would be possible, so in principle this method is suitable also to deal with the Gevrey case.*

Lemma 2.11. *Let $f \in \mathcal{P}^{1,\delta}$, then $f_K \in \mathcal{P}^{1+Na,\delta}$, so in particular it is a remainder in the sense that it fulfills Eqs* (2.7) *and* (2.8).

Proof. This is related to the fact that the Fourier coefficients of a C^∞ function decrease faster than any power of $|k|^{-1}$. Formally we have to bound the following seminorms

$$
\begin{aligned}
C^{1+Na}_{N_1,N_2}(f_K) &= \sup_{\varepsilon} \sup_{p,|k|>K,|\alpha|=N_1} \left| \frac{\partial_p^\alpha \hat{f}_k(\varepsilon,p)|k|^{N_2}\varepsilon^{|\alpha|\delta}}{\varepsilon^{1+Na}} \right|\\
&\le \sup_{\varepsilon} \sup_{p,|k|>K,|\alpha|=N_1} \left| \frac{\partial_p^\alpha \hat{f}_k(\varepsilon,p)|k|^{N_2+N_3}\varepsilon^{|\alpha|\delta}}{\varepsilon^{1+Na}K^{N_3}} \right|
\end{aligned}
\tag{2.14}
$$

and, choosing $N_3 > Na/\beta$, one has $K^{N_3}\varepsilon^{Na} > 1$ and thus

$$
|(2.14)| \le \sup_{\varepsilon} \sup_{p,|k|>K,|\alpha|=N_1} \left| \frac{\partial_p^\alpha \hat{f}_k(\varepsilon,p)|k|^{N_2+N_3}\varepsilon^{|\alpha|\delta}}{\varepsilon} \right| = C^1_{N_1,N_2+N_3}(f)
$$

which is the thesis. □

2.3. Lie transform and cohomological equation

Definition 2.12. *Given a function $g \in \mathcal{P}^{m,\delta}$, with $m \ge 0$, the time one flow $\Phi_g^1 \equiv \Phi_g^t\big|_{t=1}$ will be called* Lie transform *generated by g.*

Given a function $f \in \mathcal{P}^{m_1,\delta}$, we study $f \circ \Phi_g^1$. To this end define the sequence $f_{(l)}$, by

$$
f_{(0)} := f\,, \quad f_{(l)} := \{f_{(l-1)}; g\} \equiv \left.\frac{d^l}{dt^l}\right|_{t=0} f \circ \Phi_g^t\,, \quad l \ge 1\,,
\tag{2.15}
$$

and remark that $f_{(l)} \in \mathcal{P}^{m_1+l(m-\delta),\delta}$ if $g \in \mathcal{P}^{m,\delta}$. We have the following lemma.

Lemma 2.13. *Let $g \in \mathcal{P}^{m_2,\delta}$ and $f \in \mathcal{P}^{m_1,\delta}$, with $m_2 \ge 1-\delta$ and $m_1 \ge 1$ then one has*

$$
f \circ \Phi_g^1 = \sum_{l=0}^{N} \frac{f_{(l)}}{l!} + \mathcal{R}^{(N)}_{Lie}\,,
\tag{2.16}
$$

with $\mathcal{R}^{(N)}_{Lie}$ a remainder, in the sense that it fulfills Eqs (2.7) *and* (2.8).

Proof. Use the formula for the remainder of the Taylor series (in time); this gives

$$f\circ\Phi_g^1=\sum_{l=0}^{N}\frac{f_{(l)}}{l!}+\frac{1}{N!}\int_0^1(1+s)^N f_{(N+1)}\circ\Phi_g^s ds\ .$$

Of course the integral term is $\mathcal{R}_{Lie}^{(N)}$. To estimate its supremum it is immediate. To estimate its first differential remark that

$$d(f_{(N+1)}\circ\Phi_g^s)=df_{(N+1)}(\Phi_g^s)\circ d\Phi_g^s\ .$$

Then, from the very definition of the flow one has that its differential fulfills

$$\frac{d}{dt}d\Phi_g^t=dX_g(\Phi_g^t)d\Phi_g^t\ ,$$

which is estimated by

$$\|\frac{d}{dt}d\Phi_g^t\|\le\varepsilon^{m_2-\delta}\|d\Phi_g^t\|\ ,$$

where we used the fact that g is a symbol. From this it follows that, provided ε is small enough one has $\|d\Phi_g^t\|\le 2$ for $|t|\le 1$.

From this and from the fact that $f_{(l+1)}$ is a symbol the thesis immediately follows. □

Concerning the cohomological equation we have the following simple lemma

Lemma 2.14. *Let $f\in\mathcal{P}^{m,\delta}$ and consider the cohomological equation*

$$\{H_0;g\}+f^{(\mathrm{nr})}=0\ . \tag{2.17}$$

It admits a solution $g\in\mathcal{P}^{m-\delta,\delta}$.

Proof. Expanding in Fourier series, the cohomological equation takes the form

$$\sum_j \mathrm{i}\frac{\partial H_0}{\partial p_j}\sum_{0<|k|\le K}\mathrm{i}k_j\hat{g}_k(p,\varepsilon)e^{\mathrm{i}k\cdot q}=-\sum_{0<|k|\le K}\hat{f}_k(p,\varepsilon)(1-\chi_k(p,\varepsilon))e^{\mathrm{i}k\cdot q}\ ,$$

whose solution is

$$\hat{g}_k(p,\varepsilon)=\frac{\hat{f}_k(p,\varepsilon)(1-\chi_k(p,\varepsilon))}{-\mathrm{i}p\cdot k}\ .$$

Since $1-\chi_k$ is supported in the region $|p\cdot k|\ge\varepsilon^\delta/2$, the result follows. □

2.4. Iterative Lemma

In this subsection we prove the following lemma.

Lemma 2.15. *Let $\ell<N$ be an integer, and let $H^{(\ell)}$ be of the form*

$$H^{(\ell)}=H_0+Z^{(\ell)}+f_\ell+\mathcal{R}_\ell^{(N)}\ , \tag{2.18}$$

with $Z^{(\ell)}\in\mathcal{P}^{1,\delta}$ in normal form, $f_\ell\in\mathcal{P}^{m_\ell,\delta}$ with

$$m_\ell:=1+\ell a$$

and $\mathcal{R}_\ell^{(N)}$ a remainder.

Then there exists a symbol $g_{\ell+1}\in\mathcal{P}^{m_\ell-\delta,\delta}$ which generates a Lie transform $\Phi^1_{g_{\ell+1}}$ with the property that $H^{(\ell+1)}:=H^{(\ell)}\circ\Phi^1_{g_{\ell+1}}$ fulfills the assumption of the lemma with $\ell+1$ in place of ℓ.

Proof. Decompose f_ℓ as in (2.12) and let $g_{\ell+1}$ be the solution of the cohomological Eq (2.17) with $f_\ell^{(\mathrm{nr})}$ in place of $f^{(\mathrm{nr})}$ and compute

$$
\begin{aligned}
H^{(\ell)} \circ \Phi^1_{g_{\ell+1}} &= H_0 + \{H_0; g_{\ell+1}\} && (2.19)\\
&+ H_0 \circ \Phi^1_{g_{\ell+1}} - (H_0 + \{H_0; g_{\ell+1}\}) && (2.20)\\
&+ f_\ell^{(\mathrm{nr})} + f_\ell^{(\mathrm{res})} + f_{\ell,K} && (2.21)\\
&+ f_\ell \circ \Phi^1_{g_{\ell+1}} - f_\ell && (2.22)\\
&+ Z^{(\ell)} \circ \Phi^1_{g_{\ell+1}} - Z^{(\ell)} && (2.23)\\
&+ Z^{(\ell)} + \mathcal{R}_\ell^{(N)} \circ \Phi^1_{g_{\ell+1}} . && (2.24)
\end{aligned}
$$

Exploiting Lemma 2.13, one can decompose the different lines as

$$
\begin{aligned}
(2.22) &= \sum_{l=1}^{N} f_{\ell,(l)} + \widetilde{\mathcal{R}}_1^{(N)} =: f^1_{\ell+1} + \widetilde{\mathcal{R}}_1^{(N)}\\
(2.23) &= \sum_{l=1}^{N} Z^{(\ell)}_{(l)} + \widetilde{\mathcal{R}}_2^{(N)} =: f^2_{\ell+1} + \widetilde{\mathcal{R}}_2^{(N)}
\end{aligned}
$$

with $f^1_{\ell+1} \in \mathcal{P}^{2m_\ell - 2\delta,\delta}$, $f^2_{\ell+1} \in \mathcal{P}^{m_\ell+1-2\delta,\delta}$ and $\widetilde{\mathcal{R}}_j^{(N)}$ remainders (actually of order higher than ε^{Na+1}).

Concerning (2.20), just remark that the sequence $H_{0,(l)}$ defining the Lie transform of H_0 (cf. (2.15)), can be generated computing $H_{0,(1)}$ from the cohomological equation, giving $H_{0,(1)} = \{H_0; g_{\ell+1}\} = -f_\ell^{(\mathrm{nr})} \in \mathcal{P}^{m_\ell,\delta}$. In this way one gets $H_{0,(l)} \in \mathcal{P}^{2(m_\ell-\delta),\delta}$ and also

$$
(2.20) = \sum_{l=2}^{N} H_{0,(l)} + \widetilde{\mathcal{R}}_3^{(N)} =: f^3_{\ell+1} + \widetilde{\mathcal{R}}_3^{(N)} .
$$

It follows that, defining $f_{\ell+1} := f^1_{\ell+1} + f^2_{\ell+1} + f^3_{\ell+1}$,

$$
\mathcal{R}^{(N)}_{\ell+1} := f_{\ell,K} + \widetilde{\mathcal{R}}_1^{(N)} + \widetilde{\mathcal{R}}_2^{(N)} + \widetilde{\mathcal{R}}_3^{(N)} + \mathcal{R}_\ell^{(N)} \circ \Phi^1_{g_{\ell+1}} , \quad Z^{(\ell+1)} := Z^{(\ell)} + f_\ell^{(\mathrm{res})}
$$

one has the thesis. □

3. Geometric part

3.1. Dynamics of a normal form Hamiltonian

In this section we define a partition of the action space $\mathbb{R}^d$ into blocks which are left invariant by the flow of a Hamiltonian which is in normal form, namely

$$
H_Z(p,q) := H_0(p) + Z(p,q) , \tag{3.1}
$$

with Z in normal form. As usual this partition will be labeled by the sub moduli of $\mathbb{Z}^d$ identifying the resonances present in each region.

Definition 3.1. *(Module and related notations.) A subgroup $M \subseteq \mathbb{Z}^d$ will be called a module if $\mathbb{Z}^d \cap span_{\mathbb{R}} M = M$. Given a module M, we will denote $M_{\mathbb{R}}$ the linear subspace of $\mathbb{R}^d$ generated by M. Furthermore, given a vector $p \in \mathbb{R}^d$ we will denote by p_M its orthogonal projection on $M_{\mathbb{R}}$.*

Remark 3.2. *From the Definition 2.6 of normal form it immediately follows that, if a point $p \in \mathbb{R}^d$ is such that*

$$|p \cdot k| \geq \varepsilon^{\delta} \quad \forall k \in \mathbb{Z}^d \backslash \{0\} \; s.t. \; |k| \leq K \,,$$

then

$$\{p, H_Z\} = 0 \,, \tag{3.2}$$

hence, in this region the action p is conserved along the motion of H_Z.

The first block we define is

$$E_0 = \left\{ p \in \mathbb{Z}^d \,\middle|\, |p \cdot k| \geq \varepsilon^{\delta - 2\beta} \quad \forall k \text{ s.t. } 0 < |k| \leq K \right\} , \tag{3.3}$$

where the correction 2β to the exponent has been inserted in order to separate E_0 from the regions where some resonances are present.

In order to define the other blocks, we introduce the following parameters:

$$\delta_1 = \delta - 2\beta, \tag{3.4}$$

$$\delta_{s+1} = \delta_s - \beta s \quad \forall s = 1, \dots d-1 \,, \tag{3.5}$$

$$C_1 = 1 \,, \tag{3.6}$$

$$C_{s+1} = 3s2^s C_s + 1 \quad \forall s = 1, \dots d-1 \,. \tag{3.7}$$

The next definition we give is meant to identify the points p which are in resonance with vectors of a given module $M \subseteq \mathbb{Z}^d$:

Definition 3.3 (Resonant zones)**.** *For any module $M \subseteq \mathbb{Z}^d$ of dimension $s \geq 1$ and for any (ordered) set $\{k_1, \dots, k_s\}$ of linearly independent vectors in M such that $|k_j| \leq K$ for all $j = 1, \dots, s$, we define*

$$Z_{k_1,\dots,k_s} = \left\{ p \in \mathbb{Z}^d \,\middle|\, |p \cdot k_j| < C_j \varepsilon^{\delta_j} \quad \forall j = 1, \dots, d \right\}$$

and

$$Z_M^{(s)} = \bigcup_{\substack{\{k_1,\dots,k_s\} \\ \text{lin. ind. in } M}} Z_{k_1,\dots,k_s} \,.$$

Remark that the definition of $Z_{k_1,\dots,k_s}$ depends on the order in which the vectors k_j are chosen. Thus the definition of resonant zone slightly differs from the analogous definition of resonant zone given in [13]. This is due to the fact that in the present construction we are interested in exhibiting a partition, and not only a covering, of the action space $\mathbb{R}^d$. In particular we have the following remark.

Remark 3.4. *The resonant regions $Z_M^{(s)}$ are not reciprocally disjoint; on the contrary, given an arbitrary module M of dimension $s \geq 2$, for any $s' < s$, the following inclusion holds*

$$Z_M^{(s)} \subseteq \bigcup_{M' : \dim M' = s'} Z_{M'}^{(s')} \,.$$

We now define the set composed by the points which are resonant with the vectors in a module M, but are non-resonant with the vectors $k \notin M$.

Definition 3.5 (Resonant blocks)**.** *Let M be a module of dimension s, we define the resonant block*

$$B_M^{(s)} = Z_M^{(s)} \setminus \left(\bigcup_{\substack{s'>s \\ \dim M'=s'}} Z_{M'}^{(s')} \right) .$$

We prove below that the resonant blocks $\{B_M^{(s)}\}_{M,s}$ are reciprocally disjoint; nevertheless, they are not suitable for the geometric part, since they are not left invariant by the dynamics associated to a normal form Hamiltonian. For such a reason, we need the following definition:

Definition 3.6 (Extended blocks and fast drift blocks)**.** *For any module M of dimension s, we define*

$$\widetilde{E}_M^{(s)} = \left\{ B_M^{(s)} + M_{\mathbb{R}} \right\} \cap Z_M^{(s)}$$

and the extended blocks

$$E_M^{(s)} = \widetilde{E}_M^{(s)} \setminus \left(\bigcup_{\substack{s'<s \\ \dim M'=s'}} E_{M'}^{(s')} \right) ,$$

where $A + B$ is the Minkowski sum between sets, namely $A + B = \{a + b \mid a \in A, b \in B\}$. Moreover, for all $p \in E_M^{(s)}$ we define the fast drift blocks

$$\Pi_M^{(s)}(p) = \{p + M_{\mathbb{R}}\} \cap Z_M^{(s)} \ .$$

With the above definitions, the following result holds true:

Theorem 3.7. *1). The blocks $E_0 \cup \{E_M^{(s)}\}_{s,M}$ are a partition of $\mathbb{R}^d$.*
2). Each block is invariant for the dynamics of a system in normal form.
3). Along such a dynamics, for any initial datum one has

$$|p(t) - p(0)| \leq 3d2^{d-1} C_d \varepsilon^{\delta-((d-1)(d+1)+2)\beta} . \tag{3.8}$$

Corollary 3.8. *Theorem 1.2 holds.*

Proof. Choosing $\delta = \frac{1}{4} + \frac{b}{2}, \beta = \left(\frac{1}{2} - b\right) \frac{1}{2(d^2+1)}$ and $N = \left[\frac{M-1}{a}\right] + 1$ one immediately gets the result. □

The proof of Theorem 3.7 will occupy the rest of this subsection. We start by stating a few geometric results.

Lemma 3.9. *(Lemma 5.7 of [13]) Let $1 \leq s \leq d$, and let $k_1, ..., k_s$ be linearly independent vectors in $\mathbb{R}^d$ satisfying $|k_j| \leq K$ for some positive K and for $1 \leq j \leq s$. Denote by $\mathrm{Vol}(k_1, ..., k_s)$ the s-dimensional volume of the parallelepiped with sides $k_1, ..., k_s$; let moreover $w \in \mathrm{Span}(k_1, ..., k_s)$ be any vector, and let*

$$\alpha := \max_j \left| w \cdot k_j \right| \ ,$$

then one has

$$\|w\| \leq \frac{sK^{s-1}\alpha}{\mathrm{Vol}(k_1, ..., k_s)} . \tag{3.9}$$

For the proof see [13].

Lemma 3.10. *(Extended blocks are close to blocks.) For all $p \in \widetilde{E}_M^{(s)}$ there exists a point $p' \in B_M^{(s)}$ such that*

$$|p - p'| \leq 2sC_sK^{s-1}\varepsilon^{\delta_s} . \tag{3.10}$$

Proof. By the very definition of $\widetilde{E}_M^{(s)}$, if $p \in \widetilde{E}_M^{(s)}$ then there exists a point $p' \in B_M^{(s)}$ such that $p - p' \in M$. In particular one has that

$$p - p' = p_M - p'_M . \tag{3.11}$$

Moreover, since $p \in \widetilde{E}_M^{(s)} \subseteq Z_M^{(s)}$, there exist $k_1, \ldots, k_s$ linearly independent vectors in M, with $|k_j| \leq K$, such that

$$|p_M \cdot k_j| = |p \cdot k_j| \leq C_j\varepsilon^{\delta_j} , \quad \forall j = 1, \ldots, s .$$

Hence, remarking that for s independent vectors with integer components $\mathrm{Vol}(k_1, ..., k_s) \geq 1$, Lemma 3.9 implies

$$|p_M| \leq sK^{s-1}C_s\varepsilon^{\delta_s} .$$

Analogously, since $p' \in B_M^{(s)} \subseteq Z_M^{(s)}$,

$$|p'_M| \leq sK^{s-1}C_s\varepsilon^{\delta_s} .$$

Thus (3.11) gives

$$|p - p'| \leq |p_M| + |p'_M| \leq 2sK^{s-1}C_s\varepsilon^{\delta_s} .$$

□

Lemma 3.10 enables us to deduce the following result.

Lemma 3.11. *(Non overlapping of resonances) Consider two arbitrary resonance moduli M and M' respectively of dimensions s and s'. If $s' \leq s$ and $M' \not\subseteq M$, then for all $p \in E_M^{(s)}$ one has that*

$$dist\left(\Pi_M^{(s)}(p),\ Z_{M'}^{(s')}\right) > sC_sK^{s-1}\varepsilon^{\delta_s} , \tag{3.12}$$

where $dist(A, B) = \inf\{|a - b| \mid a \in A,\ b \in B\}$ denotes the distance between two sets.

Proof. By contradiction, assume that (3.12) is not true. Then, by Lemma 3.10, one also has

$$\mathrm{dist}\left(B_M^{(s)},\ Z_{M'}^{(s')}\right) \leq 3sC_sK^{s-1}\varepsilon^{\delta_s} ,$$

It follows that there exist $p'' \in B_M^{(s)}(p)$ and $p' \in Z_{M'}^{(s')}$ such that

$$|p'' - p'| \leq 3sC_sK^{s-1}\varepsilon^{\delta_s} .$$

Since $p' \in Z_{M'}^{(s')}$ and $M' \not\subseteq M$, there exists $h \notin M$ such that $|h| \leq K$ and $|p' \cdot h| \leq C_s\varepsilon^{\delta_s}$. Compute now

$$\begin{aligned} |p'' \cdot h| &\leq |p'' - p'||h| + |p' \cdot h| \\ &\leq 3sK^{s-1}C_s\varepsilon^{\delta_s}K + C_s\varepsilon^{\delta_s} , \end{aligned}$$

thus, recalling that $K = [\varepsilon^{-\beta}] + 1$, one has that

$$|p'' \cdot h| < (3s2^s + 1)\, C_s\varepsilon^{\delta_s - \beta s} .$$

Due to definitions (3.4), it follows that

$$|p'' \cdot h| < C_{s+1}\varepsilon^{\delta_{s+1}} .$$

Hence $p'' \in Z^{(s+1)}_{M_h}$, where M_h is the resonance module generated by $M \cup \{h\}$, which is impossible, since $p'' \in B^{(s)}_M$ implies that it is not in any higher dimensional resonance zones. □

Lemma 3.12. *Fix an arbitrary module M of dimension s, for all $p \in E^{(s)}_M$,*

$$diam\ \left(\Pi^{(s)}_M(p)\right) \le 2sK^{s-1}C_s\varepsilon^{\delta_s} .$$

Proof. Arguing as in the proof of Lemma 3.10, if p' and p'' are two points in $\Pi^{(s)}_M(p)$, then

$$|p'_M|,\ |p''_M| \le sC_sK^{s-1}\varepsilon^{\delta_s} .$$

Hence, recalling that $p' - p'' \in M$, we deduce that

$$|p' - p''| \le 2sC_sK^{s-1}\varepsilon^{\delta_s}$$

□

Remark 3.13. *Recall that $K = [\varepsilon^{-\beta}] + 1$; then Lemma 3.12 implies that for any module M and for all $p \in E^{(s)}_M$*

$$diam\left(\Pi^{(s)}_M(p)\right) \le 3d2^{d-1}C_d\varepsilon^{\delta-((d-1)(d+1)+2)\beta} .$$

We are now in position to prove Theorem 3.7. Remark that its proof has also as a consequence the fact that, for any resonance modulus M,

$$\partial E^{(s)}_M \subseteq \left(\bigcup_{\substack{s'<s \\ M'\subset M}} \partial E^{(s')}_{M'}\right) \cup \overline{Z^{(s)}_M} ,$$

which shows that it is possible to move from the extended block $E^{(s)}_M$ only losing resonances (that is, entering a block $E^{(s')}_{M'}$ with $M' \subset M$), or remaining inside the same resonant zone $Z^{(s)}_M$. The latter option will be excluded by the dynamics, which ensures that the motion entirely evolves along planes parallel to M.

Proof of Theorem 3.7. Since each point $p \in \mathbb{R}^d$ belongs either to E_0 or to $Z^{(s)}_M$ for some M and s, it immediately follows from the definition of the extended blocks that $E_0 \cup \{E^{(s)}_M\}_{M,s}$ is a covering of $\mathbb{R}^d$. If $E^{(s)}_M$ and $E^{(s')}_{M'}$ are such that $s \neq s'$, then the two blocks are disjoint by their very definition; if $s = s'$ and $M' \neq M$, then their intersection is empty by Lemma 3.11, recalling that for all M' and s' one has $E^{(s')}_{M'} \subseteq Z^{(s')}_{M'}$. This proves Item 1.

We now prove the invariance of the extended blocks $\{E^{(s)}_M\}_{s,M}$ along the flow $\Phi^t_{H_Z}$, arguing by induction on their dimension s.

Inductive base: $s = 0$. As already observed in Remark 3.2, if $p(0) \in E_0$, then $p(t) \equiv p(0)\ \forall t \in \mathbb{R}$, hence the invariance of the block E_0 immediately follows.

Inductive step: Fix M of dimension $s \geq 1$ and a point $p \in E_M^{(s)}$. We are now going to prove that

$$p(t) \in \Pi_M^{(s)}(p)\,, \quad \forall t \in \mathbb{R}\,. \tag{3.13}$$

Suppose by contradiction that there exists a finite time $\bar{t} > 0$ such that

$$p(t) \in \Pi_M^{(s)}(p) \quad \forall t < \bar{t}\,, \qquad \text{and} \qquad p(\bar{t}) \notin \Pi_M^{(s)}(p)\,.$$

Then for such $\bar{t}$ one has that

$$p(\bar{t}) \in \{p + M_{\mathbb{R}}\}\ . \tag{3.14}$$

Indeed, for any normalized vector $\lambda \in \mathbb{R}^d$ such that $\lambda \perp M$, consider the quantity $I(t) = p(t) \cdot \lambda$. For all t such that $0 \leq t < \bar{t}$ one has

$$\begin{aligned} \dot{I}(t) &= \{I(t),\ H_Z(p(t), q(t))\} \\ &= \sum_{0<|k|\leq K} i\hat{Z}_k(\varepsilon, p(t))(k \cdot \lambda)e^{ik\cdot q(t)} = 0\,, \end{aligned}$$

due to the fact that $\hat{Z}_k(\varepsilon, p(t)) = 0$ if $k \notin M$, but $k \cdot \lambda = 0$ if $k \in M$. Hence

$$p(\bar{t}) \cdot \lambda = \lim_{t \to \bar{t}} p(t) \cdot \lambda = p \cdot \lambda\,,$$

from which (3.14) follows, given the arbitrariety of the vector $\lambda \in M^{\perp}$.
Recall now the definition of $\Pi_M^{(s)}(p) \equiv \{p + M_{\mathbb{R}}\} \cap Z_M^{(s)}$; since by eq. (3.14) $p(\bar{t}) \in \{p + M_{\mathbb{R}}\}$, it must be that

$$p(\bar{t}) \in \partial Z_M^{(s)}\,. \tag{3.15}$$

Since $E_0 \cup \{E_M^{(s)}\}_{s,M}$ is a partition of $\mathbb{R}^d$, there exists M' such that $p(t) \in E_{M'}^{(s')}$ with $s' = dim M'$ (possibly with $s' = 0$, if $p(\bar{t}) \in E_0$). We analyze all the possible configurations.

1). $p(\bar{t}) \in E_{M'}^{(s')}$ with $s' = s$: then, since by its definition $p(\bar{t}) \notin Z_M^{(s)}$, it must be $M' \neq M$. Thus

$$p(\bar{t}) \in \partial Z_M^{(s)} \cap Z_{M'}^{(s)}\,,$$

which is empty by Lemma 3.11. Hence this case is contradictory.

2). $p(\bar{t}) \in E_{M'}^{(s')}$ with $s' > s$. This leads again to a contradiction, since due to Remark 3.4 one would have

$$p(\bar{t}) \in E_{M'}^{(s')} \subseteq Z_{M'}^{(s')} \subseteq \left(\bigcup_{\substack{M'' \text{ of dim. } s \\ M'' \neq M}} Z_{M''}^{(s)} \right) \cup Z_M^{(s)}\,,$$

but $p(\bar{t}) \in \partial Z_M^{(s)}$ implies that neither $p(\bar{t}) \in Z_M^{(s)}$, nor $p(\bar{t}) \in Z_{M''}^{(s)}$ for any M'' of dimension s, with $M'' \neq M$, due to Lemma 3.11.

3). $p(\bar{t}) \in E_{M'}^{(s')}$, with $s' < s$. Due to the induction assumption, the blocks $E_{M'}^{(s')}$ of dimension $s' < s$ are invariant under the dynamics of H_Z, thus no orbit can enter or exit from it

Hence none of the above situations is possible, contradicting the assumption that $\bar{t} < \infty$. Since the same occurs for negative times, we can conclude that

$$p(t) \in \Pi_M^{(s)}(p) \quad \forall t \in \mathbb{R} .$$

Estimate (3.8) then follows from Remark 3.13. Moreover, since $p(t) \in \Pi_M^{(s)}(p) \subseteq \widetilde{E}_M^{(s)}$ and by inductive hypothesis each block $E_{M'}^{(s')}$ with $s' < s$ has been proven to be invariant under the flow of H_Z for all real times, it must be

$$p(t) \in \widetilde{E}_M^{(s)} \setminus \left(\bigcup_{\substack{s' \leq s \\ \dim M' = s'}} E_{M'}^{(s')} \right) = E_M^{(s)} , \quad \forall t \in \mathbb{R} .$$

□

3.2. Adding the perturbation

Here we come to study the dynamics of the Hamiltonian $H \circ \mathcal{T}$.

In the following, we will denote $(\overline{p}(t), \overline{q}(t)) = \Phi^t_{H \circ \mathcal{T}}(p, q)$, with $H \circ \mathcal{T}$ as in Lemma 2.7. Furthermore, in order to be able to study the dynamics of a point starting in an extended block, say $E_M^{(s)}$, we consider the following sets:

$$\left(\Pi_M^{(s)}(p)\right)_{\varepsilon^\delta} = \left\{ p' \in \mathbb{R}^d \mid \mathrm{dist}(p', \Pi_M^{(s)}(p)) < \varepsilon^\delta \right\} .$$

The result we obtain is the following:

Proposition 3.14. *For all N there exists a positive threshold ε_N such that, if $\varepsilon \leq \varepsilon_N$, then $\forall p \in \mathbb{R}^d$*

$$|\overline{p}(t) - p| \leq \varepsilon^{\delta - ((d-1)(d+1)+2)\beta} \quad \forall t \ s.\ t.\ |t| \leq \varepsilon^{-Na} . \tag{3.16}$$

Proof. Fix $p \in \mathbb{R}^d$, then, for any time t such that $|t| \leq \varepsilon^{-Na}$, one has that either $\overline{p}(t) \in E_0$, or $\overline{p}(t) \in \Pi_{M_t}^{(s_t)}(p_t)$, for some fast drift block identified by a suitable $M_t \subseteq \mathbb{Z}^d$ of dimension $s_t \geq 1$ and some (not unique) $p_t \in E_{M_t}^{(s_t)}$. Let $t_0 \in [-\varepsilon^{-Na}, \varepsilon^{-Na}]$ be such that M_{t_0} is of minimal dimension, namely such that

$$s_{t_0} = \dim M_{t_0} = \min_{|t| \leq \varepsilon^{-Na}} \dim M_t .$$

Of course, if there exists a time t_0 such that $p(t_0) \in E_0$, then $M_{t_0} = \{0\}$ and $\Pi_{M_{t_0}}^{(s_{t_0})}(p_{t_0}) = \{p_{t_0}\}$ and $\left(\Pi_{M_{t_0}}^{(s_{t_0})}(p_{t_0})\right)_{\varepsilon^\delta} \equiv B_{\varepsilon^\delta}(p_{t_0})$, namely the ball of center p_{t_0} and radius ε^δ.

We are going to prove that

$$p(t) \in \left(\Pi_{M_{t_0}}^{(s_{t_0})}(p_{t_0})\right)_{\varepsilon^\delta} \quad \forall |t| \leq \varepsilon^{-Na} \tag{3.17}$$

This is obtained arguing essentially as in the proof of Theorem 3.7.

Let $\bar{t} > 0$ be the exit time of $\overline{p}(t)$ from $\left(\Pi_{M_{t_0}}^{(s_{t_0})}(p_{t_0})\right)_{\varepsilon^\delta}$, namely the time s.t. $\forall t$ with $0 \leq t < \bar{t}$

$$\overline{p}(t + t_0) \in \left(\Pi_{M_{t_0}}^{(s_{t_0})}(p_{t_0})\right)_{\varepsilon^\delta} , \quad \text{and} \quad \overline{p}(\bar{t} + t_0) \notin \left(\Pi_{M_{t_0}}^{(s_{t_0})}(p_{t_0})\right)_{\varepsilon^\delta} .$$

We prove that $|\bar{t}+t_0| > \varepsilon^{-Na}$, from which (3.17) follows. Indeed, suppose by contradiction that $|\bar{t}+t_0| \le \varepsilon^{-Na}$. Then for any normalized vector $\lambda \in \mathbb{R}^d$ with $\lambda \perp M_{t_0}$, we consider the quantity

$$I(t) = \overline{p}(t+t_0)\cdot\lambda$$

due to Lemma 3.11, for $0 \le t < \bar{t}$,

$$\left|\dot{I}(t)\right| = |\{I(t),\ H\circ\mathcal{T}\}| = |\{I(t),\ R(t)\}| \le K_N\varepsilon^{1+Na-\delta} \tag{3.18}$$

where K_N is a constant bounding the r.h.s. of (2.7). Since $|t+t_0| < \varepsilon^{-Na}$, $|I(t)-I(0)| \le 2K_N\varepsilon^{1-\delta}$, thus, passing to the limit $t \to \bar{t}$, we obtain

$$|I(\bar{t}) - I(0)| \le 2K_N\varepsilon^{1-\delta},$$

which is strictly less than $\frac{\varepsilon^\delta}{2}$ if ε is small enough.

If $M_{t_0} = \{0\}$, this enables us to conclude that

$$\operatorname{dist}(\overline{p}(\bar{t}+t_0), p_{t_0}) < \frac{\varepsilon^\delta}{2},$$

which contradicts the definition of $\bar{t}$ as the time of exit from $B(p_{t_0}, \varepsilon^\delta)$.
Assume now $s_{t_0} \ge 1$, then (3.18) implies

$$\operatorname{dist}\left(\overline{p}(\bar{t}+t_0), \Pi^{s_{t_0}}_{M_{t_0}}(p_{t_0})\right) < \frac{\varepsilon^\delta}{2}. \tag{3.19}$$

Since

$$\Pi^{(s_{t_0})}_{M_{t_0}}(p_{t_0}) = \{p_{t_0} + M_{\mathbb{R}}\} \cap Z^{(s_{t_0})}_{M_{t_0}};$$

now, by the definition of $\bar{t}$, $\overline{p}(\bar{t}+t_0) \notin \left(\Pi^{(s_{t_0})}_{M_{t_0}}(p_{t_0})\right)_{\varepsilon^\delta}$, equation (3.19) implies that in particular $\overline{p}(\bar{t}+t_0) \notin Z^{(s_{t_0})}_{M_{t_0}}$. Then one argues as in the proof of Theorem 3.7 to deduce that, by Lemma 3.11, the point $\overline{p}(\bar{t}+t_0)$ cannot belong to any block $E^{(s)}_M$ with $s \ge s_{t_0}$. Thus it must be

$$\overline{p}(\bar{t}+t_0) \in E^{(s)}_M \quad \text{for some } s < s_{t_0},$$

which contradicts the minimality hypothesis on s_{t_0}. Hence, arguing analogously for negative times, we can deduce that (3.17) holds. Finally, recall that by Remark 3.13 this implies that

$$|\overline{p}(t) - p| \le 3d2^{d-1}C_d\varepsilon^{\delta-((d-1)(d+1)+2)\beta}.$$

□

Combining the estimate in Proposition 3.14 and estimate (2.10) on the size of the deformation induced on the action variables by the canonical transformation $\mathcal{T}$, we are finally able to deduce that for all $N \in \mathbb{N}$ and $\forall t \in \mathbb{R}$ such that $|t| \le \varepsilon^{-Na}$ there exists a positive constant K'_N such

$$\begin{aligned} |p(t)-p(0)| &\le |p(t)-\overline{p}(t)| + |\overline{p}(t)-\overline{p}(0)| + |\overline{p}(0)-p(0)| \\ &\le K'_N\varepsilon^{1-\delta} + 3d2^{d-1}C_d\varepsilon^{\delta-((d-1)(d+1)+2)\beta} + K'_N\varepsilon^{1-\delta} \\ &\le \left(2K'_N + 3d2^{d-1}C_d\right)\varepsilon^{\delta-((d-1)(d+1)+2)\beta}, \end{aligned}$$

which concludes the proof of Theorem 1.1.

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Conflict of interest

The authors declare no conflict of interest.

References

1. Bambusi D (1999) Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations. *Math Z* 230: 345–387.
2. Bounemoura A, Féjoz J (2017) Hamiltonian perturbation theory for ultra-differentiable functions. *Memoir AMS*, Arxiv: 1710.01156.
3. Benettin G, Gallavotti G (1986) Stability of motions near resonances in quasi-integrable Hamiltonian systems. *J Statist Phys* 44: 293–338.
4. Bambusi D, Giorgilli A (1993) Exponential stability of states close to resonance in infinite-dimensional Hamiltonian systems. *J Statist Phys* 71: 569–606.
5. Benettin G, Galgani L, Giorgilli A (1985) A proof of Nekhoroshev's theorem for the stability times in nearly integrable Hamiltonian systems. *Celestial Mech* 37: 1–25.
6. Bambusi D, Langella B, Montalto R (2019) On the spectrum of the Schrödinger operator on T^d: A normal form approach. *Commun Part Diff Eq* 45: 303–320.
7. Bambusi D, Maiocchi A, Turri L (2019) A large probability averaging theorem for the defocusing NLS. *Nonlinearity* 32: 3661–3694.
8. Bounemoura A, Niederman L (2012) Generic Nekhoroshev theory without small divisors. *Ann Inst Fourier* 62: 277–324.
9. Bounemoura A (2010) Nekhoroshev estimates for finitely differentiable quasi-convex Hamiltonians. *J Differ Equations* 249: 2905–2920.
10. Bounemoura A (2011) Effective stability for Gevrey and finitely differentiable prevalent Hamiltonians. *Commun Math Phys* 307: 157–183.
11. De Roeck W, Huveneers F (2015) Asymptotic localization of energy in nondisordered oscillator chains. *Commun Pure Appl Math* 68: 1532–1568.
12. Guzzo M, Chierchia L, Benettin G (2016) The steep Nekhoroshev's theorem. *Commun Math Phys* 342: 569–601.
13. Giorgilli A (2003) *Notes on Exponential Stability of Hamiltonian Systems*, Pisa: Centro di Ricerca Matematica "Ennio De Giorgi".
14. Giorgilli A, Zehnder E (1992) Exponential stability for time dependent potentials. *Z Angew Math Phys* 43: 827–855.
15. Lochak P, Neĭshtadt AI (1992) Estimates of stability time for nearly integrable systems with a quasiconvex Hamiltonian. *Chaos* 2: 495–499.

16. Lochak P (1992) Canonical perturbation theory: An approach based on joint approximations. *Uspekhi Mat Nauk* 47: 59–140.

17. Marco JP, Sauzin D (2003) Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems. *Publ Math Inst Hautes Études Sci* 96: 199–275.

18. Marco JP, Sauzin D (2004) Wandering domains and random walks in Gevrey near-integrable systems. *Ergodic Theory Dynam Systems* 24: 1619–1666.

19. Nekhoroshev NN (1977) An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. *Uspehi Mat Nauk* 32: 5–66.

20. Nekhoroshev NN (1979) An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. II. *Trudy Sem Petrovsk* 5: 5–50.

21. Niederman L (2004) Exponential stability for small perturbations of steep integrable Hamiltonian systems. *Ergodic Theory Dynam Systems* 24: 593–608.

22. Niederman L (2007) Prevalence of exponential stability among nearly integrable Hamiltonian systems. *Ergodic Theory Dynam Systems* 27: 905–928.

23. Pöschel J (1993) Nekhoroshev estimates for quasi-convex Hamiltonian systems. *Math Z* 213: 187–216.