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# On curves of $\mathbb{P}^n$ with extremal Hartshorne–Rao module in positive degrees

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**Abstract**

In this paper, we study the curves $C$ in $\mathbb{P}^n$, of degree $d$ and genus $g$, with extremal Rao function in positive degrees, and non degenerate general hyperplane section. We describe their total ideal and various properties of the generators of the ideal. Moreover, we characterize these curves as intersection of two aCM curves of maximal genus whose union in aCM of maximal genus, and we completely describe the Rao function of these curves in negative degrees, too.


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In [7], Martin-Deschamps and Perrin established an optimal upper bound for the Rao function of curves (i.e. for closed locally CM subschemes of pure dimension 1) of $\mathbb{P}^3$. The non-aCM curves which achieve equality for the bound are called extremal curves.

In [3], Ellia geometrically characterized the extremal projective curves of degree $d$ as curves which contain a plane curve of degree $d-1$, and whose residual with respect to that plane is a line. Moreover, in [8], Martin-Deschamps and Perrin proved that the Hartshorne–Rao module of an extremal curve is Koszul, i.e., it is isomorphic to $k[x_0,\dots,x_3]/(l_1,\dots,l_4)$, where $(l_1,\dots,l_4)$ is a regular sequence of length 4 of homogeneous polynomials.

The extremal curves are minimal in their class of biliaison and for this reason they are important for the study of the other curves of their class and for the study of the corresponding Hilbert schemes. For example, in [8] Martin-Deschamps and Perrin

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proved that these curves form an irreducible component of the Hilbert scheme $H_{d,g}$ which is non-reduced, under some assumptions on $d$ and $g$.

A first generalization of these results for curves in any projective $n$-space is the sharp bound on the Rao function of curves with non-degenerate general hyperplane section, obtained by Chiarli et al. [1, Theorem 2.1].

In this paper, we study a larger family of curves $C \subset \mathbb{P}^n$ than the extremal ones. In fact we describe the curves, with non-degenerate general hyperplane section, that achieve equality for the bound on $h^1(\mathscr{I}_C(j))$ only for $j \geq 0$. We call these curves "quasi extremal" curves and we give their generic initial ideal, that is to say a Gröbner basis of their ideal, their minimal free resolution and we characterize them as intersection of two aCM curves of maximal genus whose union is an aCM curve of maximal genus, too. We observe also that for $n = 3$ a curve $C$ is quasi extremal if, and only if, $C$ is extremal. Moreover, we completely determine the admissible "tails" for the Hartshorne–Rao module of quasi-extremal curves, i.e. the possible functions $h^1(\mathscr{I}_C(j))$ for $j < 0$.

Let us describe the content of this paper in some more detail.

In Section 1, we define a quasi-extremal curve $C \subset \mathbb{P}^n$, of degree $d$ and genus $g$, and we prove that its ideal is of the form $I_C = I_Z + (G_1, G_2)$ where $G_1$ and $G_2$ are two polynomials of degree $d + n - 3$ and $d + n - 2 + a$, respectively, with

$$a = \binom{d-n+1}{2} - g,$$

and $I_Z$ is generated by $(n+1)(n-2)/2$ quadrics.

We also prove that the scheme $Z$, described by the ideal $I_Z$, contains a plane $\pi$. Moreover, the scheme-theoretical union of $C$ and $\pi$ is $Z$ and the scheme-theoretical intersection of $C$ and $\pi$ contains a plane curve of degree $d - n + 2$. Finally, we give conditions to "glue" a plane curve $C'$ and a scheme $Z$ as above to obtain a quasi-extremal curve.

In Section 2, we prove that if $C$ is a quasi-extremal curve given by the ideal $I_C$ as above, then $I_i = I_Z + (G_i)$, for $i = 1, 2$, describes an aCM curve $D_i$ of maximal genus, and $D_1 \cap D_2 = C$, scheme-theoretically. Moreover, we obtain that the scheme-theoretical union $D$ of $D_1$ and $D_2$ is an aCM curve of maximal genus, too. We also prove that if $D_1$ and $D_2$ are two aCM curves of maximal genus such that $D_1 \cup D_2$ is an aCM curve of maximal genus and suitable degree then $D_1 \cap D_2$ is a quasi-extremal curve.

The aim of Section 3 is twofold. The first one is to relate the module $H^1_*(\mathscr{I}_C)$ to $H^1_*(\mathscr{I}_{D_1/D})$ and $H^1_*(\mathscr{I}_{D_2/D})$, while the second and more significant one is to describe the possible "tails" of the Rao functions of quasi-extremal curves. In this direction, we prove that if $C$ is quasi-extremal, then $h^1(\mathscr{I}_C(j)) \geq (n-2)j + a$, for $j < 0$, and we deduce that, in $\mathbb{P}^3$, a curve $C$ is quasi-extremal if, and only if, it is extremal.

The main result of this section is the following:

**Theorem 3.2.** *A numerical function is the Rao function of a quasi-extremal curve if, and only if, its graph is a polygonal whose edges have an increasing slope, not exceeding* $n - 2$ *and passing through* $(0, a)$.

The proof is based on the observation that no other Rao function is possible and on the construction of quasi-extremal curves with every described Rao function, whatever considered $\mathbb{P}^n$.

Moreover, in Section 0, for the convenience of the reader, we recall the definition of generic initial ideal and some well-known results about it, because it is one of the main tools of the paper. See [2, Chapter 15; 4] for a wider look at this topic.

## 0. Preliminaries and notation

Let be $R = k[x_0, \dots, x_n]$, and $\mathbb{P}^n = \mathrm{Proj}(R)$, where $k$ is an algebraically closed field of any characteristic $p \geq 0$, but $p \neq 2$.

We order the terms $\underline{x}^A$ with respect to the reverse lexicographical order (rlex). For any $F \in R$, homogeneous of degree $\deg(F)$, we denote $\mathrm{in}(F)$ the largest monomial with non-zero coefficient in its expression, and $F = \mathrm{in}(F) + \mathrm{res}(F)$. Furthermore, for a homogeneous ideal $I$, the initial ideal $\mathrm{in}(I)$ of $I$ is the monomial ideal generated by $\mathrm{in}(F)$, for every $F \in I$.

One of the main feature of the initial ideal is the following well-known result.

**Proposition 0.1.** *Let $I \subseteq R$ be a homogeneous ideal. Then, for every $j$*

$$\dim(I)_j = \dim(\mathrm{in}(I))_j.$$

Now, we recall the definition of generic initial ideal.

**Theorem 0.2** (Green [4, Theorem 1.27]). *For any homogeneous ideal I and any multiplicative monomial order $\sigma$, there exists a non-empty Zariski open subset $\mathcal{U} \subseteq GL(n,k)$ such that the initial ideal $\mathrm{in}_\sigma(g(I))$ is constant and Borel-fixed, for each $g \in \mathcal{U}$.*

**Definition 0.3.** The monomial ideal $\mathrm{in}_\sigma(g(I))$, with $g \in \mathcal{U}$ is called the generic initial ideal $\mathrm{gin}_\sigma(I)$ of $I$, and we say that $I$ is in generic coordinates.

Now, we recall some properties of generic initial ideals we use in the following.

**Proposition 0.4.** *Let I be a homogeneous ideal, and let $\sigma$ be rlex. Then,*

(1) *a Borel-fixed monomial ideal is saturated if, and only if, $x_n$ divides no generator of the ideal;*

(2) *I is saturated if, and only if, $\mathrm{gin}(I)$ is saturated.*

**Proof.** [4, Corollary 2.10, Theorem 2.24]. □

Let $X \subseteq \mathbb{P}^n$ be a closed subscheme. Then, $I_x$ is the saturated ideal of $X$, while $\mathcal{I}_X$ is the sheaf of $X$.

A curve $C \subset \mathbb{P}^n$ is a locally Cohen–Macaulay (loc. CM) closed subscheme of pure dimension 1. The genus $g$ of $C$ is the arithmetical genus $p_a(C)$. Moreover, we suppose that $C$ is not arithmetically Cohen–Macaulay (aCM).

## 1. First properties of quasi-extremal curves

Let $C \subseteq \mathbb{P}^n$ be a curve of degree $d$ and genus $g$, and let

$$a = \binom{d-n+1}{2} - g$$

be the upper bound for the dimension of $H^1(\mathscr{I}_C(j))$, according to Chiarli et al. [1, Theorem 2.1].

Now, we can state the main definition of the paper.

**Definition 1.1.** Let $C \subset \mathbb{P}^n$ be a curve of degree $d \geq n+1$ and genus $g$. The curve $C$ is quasi-extremal if the general hyperplane section of $C$ is non-degenerate, and

$$h^1(\mathscr{I}_C(j)) = \begin{cases} \binom{d-n+1}{2} - g & \text{if } 0 \leq j \leq d-n+1, \\ \binom{d-n+2}{2} - j - g + 1 & \text{if } d-n+1 \leq j \leq \binom{d-n+2}{2} - g, \\ 0 & \text{if } j \geq \binom{d-n+2}{2} - g. \end{cases}$$

Moreover, $C$ is extremal if $C$ is quasi-extremal and

$$h^1(\mathscr{I}_C(j)) = \begin{cases} 0 & \text{if } j \leq g - \binom{d-n+1}{2}, \\ \binom{d-n+1}{2} + j - g & \text{if } g - \binom{d-n+1}{2} \leq j \leq 0. \end{cases}$$

We observe that if $C$ is quasi-extremal, then $h^1(\mathscr{I}_C(j))$ is as large as possible for $j \geq 0$, as shown in [1], and we have no hypothesis on $h^1(\mathscr{I}_C(j))$ for $j < 0$. If $C$ is extremal, then its Hartshorne–Rao module is as large as possible in every degree.

**Remark 1.2.** If $C$ is quasi-extremal, then, from the proof of Chiarli et al. [1, Theorem 2.1], and from the cohomological sequence associated to the general hyperplane section sequence, $0 \to \mathscr{I}_C(-1) \to \mathscr{I}_C \to \mathscr{I}_{C\cap H/H} \to 0$, we have

(i) the general hyperplane section of $C$ has the following cohomology:

$$h^1(\mathscr{I}_{C\cap H|H}(j)) = \begin{cases} d & \text{if } j \leq -1, \\ d-1 & \text{if } j = 0, \\ d-n-j+1 & \text{if } 1 \leq j \leq d-n+1, \\ 0 & \text{if } j \geq d-n+1; \end{cases}$$

(ii) $\ker(H^2(\mathscr{I}_C(j-1)) \to H^2(\mathscr{I}_C(j))) \simeq H^1(\mathscr{I}_{C\cap H/H}(j))\ \forall j \geq 1$;

(iii) $\operatorname{coker}(H^0(\mathscr{I}_C(j-1)) \to H^0(\mathscr{I}_C(j))) \simeq H^0(\mathscr{I}_{C\cap H/H}(j))\ \forall j \leq d-n+1$;

(iv) the curve $C$ is non-degenerate, and so $H^0(\mathscr{I}_C(1)) = 0$. Then, $H^0(\mathscr{I}_C(2)) \simeq H^0(\mathscr{I}_{C\cap H/H}(2))$, by (iii);

(v) the function $h^0(\mathscr{I}_C(j))$ is equal to

$$h^0(\mathscr{I}_C(j)) = \begin{cases} 0 & \text{if } j \le 1, \\ \binom{j+n}{n} - \dfrac{j^2+(2n-1)j+2}{2} & \text{if } 1 \le j \le d-n+1, \\ \binom{j+n}{n} - (d+1)j - 1 + \binom{d-n+2}{n} & \text{if } d-n+1 \le j \le \binom{d-n+2}{2} - g, \\ \binom{j+n}{n} - dj - 1 + g & \text{if } j \ge \binom{d-n+2}{2} - g + 1. \end{cases}$$

Now, we want to describe the total ideal of a quasi-extremal curve (see [8, Propositions 0.5, 0.6] for the case $n=3$).

**Proposition 1.3.** *Let $C$ be a quasi-extremal curve, then the generic initial ideal* $\mathrm{gin}(I_C)$ *of $C$ is generated by $x_ix_j$, for $0 \le i \le n-3$, $0 \le j \le n-2$, $i \le j$, and by $x_{n-2}^{d-n+3}$, $x_{n-2}^{d-n+2}x_{n-1}^a$.*

**Proof.** We suppose that the ideal $I_C$ is in generic coordinates. Then, $H = V(x_n)$ is a general hyperplane with respect to $C$ (see Proposition 0.4).

Set $\Gamma = (C \cap H)$ as a subscheme of $H$. Moreover, we identify $H = \mathrm{Proj}(k[x_0, \dots, x_{n-1}])$.

Now, we want to compute the generic initial ideal $\mathrm{gin}(I_\Gamma)$.

At first, let us suppose that $d \ge n+2$.

By Remark 1.2(i), we have

$$h^0(\mathscr{I}_\Gamma(j)) = \binom{j+n-1}{n-1} - d + \max(d-n-j+1, 0)$$

for $j \ge 1$.

In particular,

$$h^0(\mathscr{I}_\Gamma(2)) = \binom{n+1}{2} - (n+1).$$

But $x_{n-1}$ divides no generator of $\mathrm{gin}(I_\Gamma)$, because $I_\Gamma$ is a saturated ideal (see Proposition 0.4). Then, $(\mathrm{gin}(I_\Gamma))_2$ is spanned by all the degree 2 terms larger than $x_{n-2}^2$ in rlex ordering, because $\mathrm{gin}(I_\Gamma)$ is Borel-fixed. In fact, if $x_{n-2}^2 \in (\mathrm{gin}(I_\Gamma))_2$, every term $x_ix_j$, $0 \le i \le j \le n-2$, is in $(\mathrm{gin}(I_\Gamma))_2$, because they can be obtained with a sequence of elementary moves [1] [4, Definition 1.24], and so $\dim(\mathrm{gin}(I_\Gamma))_2 > h^0(\mathscr{I}_\Gamma(2))$, which is not possible.

By Chiarli et al. [1, Proposition 3.3] $I_\Gamma$ has a new minimal generator $F$ of degree $d-n+2$. $I_\Gamma$ being a saturated ideal, the initial term of $F$ is $x_{n-2}^{d-n+2}$.

Then, $\mathrm{gin}(I_\Gamma) = (x_0^2, x_0x_1, x_1^2, \dots, x_{n-3}x_{n-2}, x_{n-2}^{d-n+2})$. In fact, it is not possible to have other elements in a Gröbner basis of $I_\Gamma$, because $I_\Gamma$ is saturated.

Now, let us suppose that $d = n+1$.

[1] Here, we need $\mathrm{char}(k) \ne 2$ (see [2, Theorem 15.23]).

The scheme $\Gamma$ is a non-degenerate subscheme of $H$, and so $h^0(\mathscr{I}_\Gamma(1)) = 0$, i.e. $h^1(\mathscr{I}_\Gamma(1)) = 1$.

Then, for $j \geq 2$, $h^1(\mathscr{I}_\Gamma(j)) = 0$, and

$$h^0(\mathscr{I}_\Gamma(j)) = \binom{j+n-1}{n-1} - (n+1).$$

Also in this case,

$$h^0(\mathscr{I}_\Gamma(2)) = \binom{n+1}{2} - (n+1),$$

and so $(\mathrm{gin}(I_\Gamma))_2$ is spanned by all the terms larger than $x_{n-2}^2$, in relax ordering.

If we compute $\dim(\mathrm{gin}(I_\Gamma))_3$, we have

$$\dim(\mathrm{gin}(I_\Gamma))_3 = \binom{n+1}{2} - (n-1) - 1 \neq h^0(\mathscr{I}_\Gamma(3))$$

and so we have that a Gröbner basis of $I_\Gamma$ contains a new generator of degree $3 = d-n+2$. As in the previous case, the initial term of this generator is $x_{n-2}^3$. It is not possible to have other elements in a Gröbner basis of $I_\Gamma$, because $I_\Gamma$ is saturated. Then, $\mathrm{gin}(I_\Gamma) = (x_0^2, x_0x_1, x_1^2, \ldots, x_{n-3}x_{n-2}, x_{n-2}^3)$.

Now, we can compute $\mathrm{gin}(I_C)$ (see [9, Proposition 4.9] for a relation between $\mathrm{gin}(I_C)$ and $\mathrm{gin}(I_\Gamma)$).

By Remark 1.2(iv), we have

$$H^0(\mathscr{I}_C(2)) \simeq H^0(\mathscr{I}_\Gamma(2))$$

and so $H^0(\mathscr{I}_C(2))$ is generated by liftings of the generators of $H^0(\mathscr{I}_\Gamma(2))$. But $H = V(x_n)$ and then, if $F$ is the lifting of $\bar{F}$, we have that $F = \bar{F} + x_nG$. Hence, the initial term of $F$ is equal to the one of $\bar{F}$. Then, $\mathrm{gin}(I_C)_2$ is generated by $x_0^2, x_0x_1, x_1^2, \ldots, x_{n-3}x_{n-2}$.

Let $J_1$ be the monomial ideal of $k[x_0, \ldots, x_n]$ generated by $(\mathrm{gin}(I_C))_2$. We have

$$h^0(\mathscr{J}_1(j)) = \binom{j+n}{n} - \frac{j^2 + (2n-1)j + 2}{2} \quad \text{for } j \geq 1.$$

Then, $h^0(\mathscr{J}_1(j)) = h^0(\mathscr{I}_C(j))$ for $1 \leq j \leq d-n+2$, and so no other generator of degree less than $d-n+2$ is needed in the Gröbner basis of $I_C$ (see Remark 1.2(v) and Proposition 0.1).

If we compare $h^0(\mathscr{J}_1(d-n+3))$ and $h^0(\mathscr{I}_C(d-n+3))$ we have that $h^0(\mathscr{I}_C(d-n+3)) - h^0(\mathscr{J}_C(d-n+3)) = 1$. Then, there is a new element of the Gröbner basis of degree $d-n+3$. The initial term of this element is $x_{n-2}^{d-n+3}$. In fact, from the cohomology sequence associated to the general hyperplane section sequence, we deduce that this element is a lifting of $FL$, where $F$ is the last generator of $I_\Gamma$ and $L$ is a general linear form.

Let $J_2$ be the monomial ideal of $k[x_0,\dots,x_n]$ generated by $J_1+(x_{n-2}^{d-n+3})$. We have

$$h^0(\mathscr{J}_2(j)) = \binom{j+n}{n} - (d+1)j - 1 + \binom{d-n+2}{2} \quad \text{for } j \geq d-n+2.$$

Then, $h^0(\mathscr{J}_2(j)) = h^0(\mathscr{I}_C(j))$ for

$$d-n+2 \leq j \leq \binom{d-n+2}{2} - g.$$

Set

$$h = \binom{d-n+2}{2} - g + 1 = d-n+2+a.$$

Then, $h^0(\mathscr{I}_C(h)) - h^0(\mathscr{J}_2(h)) = 1$, and so there is a new element in the Gröbner basis of $I_C$ of degree $h$. Its initial term is $x_{n-2}^{d-n+2}x_{n-1}^a$ because this element is a lifting of a multiple of the last generator of $I_\Gamma$.

The monomial ideal generated by $J_2+(x_{n-2}^{d-n+2}x_{n-1}^a)$ has the same Hilbert function of $I_C$ and so we have the claim. $\square$

**Remark.** If $C$ is an arithmetically Cohen–Macaulay curve of degree $d \geq n+1$, then $\mathrm{gin}(I_C) = (x_0^2,\dots,x_{n-3}x_{n-2},x_{n-2}^{d-n+2})$. In fact, $I_C$ is generated by liftings of the generators of $I_\Gamma$.

*Notation*: From now on, we suppose that $I_C$ has a reduced Gröbner basis of the form $\{Q_{0,0},\dots,Q_{n-3,n-2},G_1,G_2\}$, where in $(Q_{i,j}) = x_ix_j$, in $(G_1) = x_{n-2}^{d-n+3}$, and in $(G_2) = x_{n-2}^{d-n+2}x_{n-1}^a$.

Now, we can compute a free resolution of $\mathscr{I}_C$.

**Corollary 1.4.** *Let $C \subseteq \mathbb{P}^n$ be a quasi extremal non aCM curve. Then, a free resolution of $\mathscr{I}_C$ is*

$$0 \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \cdots \to \mathscr{F}_1 \to \mathscr{I}_C \to 0,$$

*where* $\mathscr{F}_i(\mathscr{O}(-1-i))^{\alpha_i} \oplus (\mathscr{O}(-d+n-2-i))^{\beta_i} \oplus (\mathscr{O}(-d+n-1-a-i))^{\gamma_i}$, *and* $\alpha_i = \binom{n-1}{i}(n-2) - \binom{n-2}{i+1}$, $\beta_i = \binom{n-2}{i-1}, \gamma_i = \binom{n-1}{i-1}$.

**Proof.** The terms of a free resolution of $I_C$ can be obtained by the ones of $\mathrm{gin}(I_C)$, up to cancellation of adjacent terms. The minimal free resolution of $\mathrm{gin}(I_C)$ is combinatorial in its nature and it is given by the Eliahou–Kervaire Theorem [4, Theorem 1.31], and so we have

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to \mathrm{gin}(I_C) \to 0,$$

where $F_i = R^{\alpha_i}(-1-i) \oplus R^{\beta_i}(-d+n-2-i) \oplus R^{\gamma_i}(-d+n-1-a-i)$ and $\alpha_i,\beta_i,\gamma_i$ are the right ones, and the claim holds. $\square$

**Remark.** The free modules $R^{\alpha_i}(-1-i)$ of the resolution correspond to the degree 2 generators of $I_C$.

In Section 2, we will show that this resolution is minimal.

**Lemma 1.5.** *Let $Z$ be a scheme whose total ideal admits a minimal free resolution of the following type*:

$$0 \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to I_Z \to 0,$$

*where* $F_i = R^{\alpha_i}(-1-i)$ *and* $\alpha_i = \binom{n-1}{i}(n-2) - \binom{n-2}{i+1}$. *Then,*
(1) $\dim Z = 2$ *and* $\deg Z = 1$. *Then, the top dimensional part of $Z$ is a plane* $\pi$;
(2) $H^1_*(\mathscr{I}_Z) = 0; H^2(\mathscr{I}_Z(j)) = 0$ *for* $j \geq 0$.

**Proof.** We can compute the Hilbert polynomial of $Z$ from the resolution, and we have

$$p_Z(z) = \frac{z^2 + (2n-1)z + 2}{2}.$$

Then, $Z$ has dimension 2 and degree 1, and so $Z$ contains a plane $\pi$. To obtain the second part of the claim, we apply Serre's Duality Theorem [5, Chapter III, Theorem 7.1]. □

**Proposition 1.6.** *Let $C$ be a quasi-extremal curve of degree $d$ and genus $g$. Then, $C$ contains a plane curve of degree* $d-n+2$.

**Proof.** Let $Z$ be the scheme whose total ideal is generated by $(I_C)_2$. We have that the degree 2 forms which generate $(I_C)_2$ are a Gröbner basis because their S-polynomials reduce to zero modulo themselves. In fact, their S-polynomials have degree 3, while the other elements of the Gröbner basis have degree larger than $d-n+3 \geq 4$. Moreover, $Z$ is in generic coordinates, and so it is saturated because no generator can be divided by $x_n$ (see Proposition 0.4). We can compute a minimal free resolution of $Z$ as in Corollary 1.4, and so we have that $Z$ satisfies the hypotheses of Lemma 1.5. In particular, $Z$ contains a plane $\pi$. Let us suppose that the total ideal $I_\pi$ is generated by $n-2$ linear forms $L_0, \ldots, L_{n-3}$ such that the initial term of $L_i$ is $x_i$, for each $i$.

Now, we want to to prove that $I_C \cap I_\pi = I_Z$.

$I_Z \subseteq I_\pi$ because $\pi$ is the top dimensional part of $Z$, Then, $I_Z \subseteq I_C = I_\pi$.

Conversely, let $F$ be a form in $I_C \cap I_\pi$. Then, $F$ reduces to zero modulo the chosen Gröbner basis of $I_C$, that is to say

$$F = \sum \alpha_{i,j} Q_{i,j} + \beta G_1 + \gamma G_2,$$

where $\beta$ and $\gamma$ satisfy the following conditions:

- the initial terms in$(\beta)$ and in$(\gamma)$ cannot be divided by $x_0, \ldots, x_{n-3}$;
- $\mathrm{in}(\beta) x_{n-2}^{d-n+3} > \mathrm{in}(\gamma) x_{n-2}^{d-n+2} x_{n-1}^a$.

But $F \in I_\pi$ and so $\beta G_1 + \gamma G_2 \in I_\pi$. Then $\beta = \gamma = 0$ by reduction argument, and so we have that $F \in I_Z$.

To prove that $C$ contains a plane curve of degree $d-n+2$, it is enough to consider the exact sequence of saturated ideals.

$$0 \to I_Z \to I_C \oplus I_\pi \to I_{\pi\cap C} \to 0$$

which comes from the exact sequence of sheaves $0 \to \mathscr{I}_Z \to \mathscr{I}_C \oplus \mathscr{I}_\pi \to \mathscr{I}_{\pi\cap C} \to 0$, because $H^1_*(\mathscr{I}_Z)=0$ (see Lemma 1.5).

In fact, using the additvity of their Hilbert polynomials, and recalling that $p_Z(z)$ was computed in the proof of Lemma 1.5, we get

$$p_{\pi\cap C}(z) = dz+1-g+\binom{z+2}{2} - \frac{z^2+(2n-1)z+2}{2} = (d-n+2)z+1-g$$

and hence we have the claim. □

**Remark 1.7.** $I_{C\cap\pi}$ is minimally generated by $\{L_0,\dots,L_{n-3},\tilde{G}_1,\tilde{G}_2\}$, where $\tilde{G}_1$ and $\tilde{G}_2$ are the normal forms of $G_1$ and $G_2$ modulo $I_\pi$. Proposition 1.6 shows that $\tilde{G}_1$ and $\tilde{G}_2$ have a non-trivial greatest common divisor $\tilde{F}$ of degree $d-n+2$. Then, we set $\tilde{G}_1=H_1\tilde{F}$ and $\tilde{G}_2=H_2\tilde{F}$, where $H_1$ is linear and $\mathrm{in}(H_1)=x_{n-2}$, while $H_2$ is a degree $a$ form and $\mathrm{in}(H_2)=x^a_{n-1}$.

Now, we describe another property of the last two generators of $I_C$.

**Proposition 1.8.** *Let $C$ be a quasi-extremal curve and let $Z$ be the scheme whose total ideal is generated by $(I_C)_2$. Then*
(1) *$LG_i \in I_Z$ for every $L\in I_\pi$ and for $i=1,2$;*
(2) *$H_2G_1 - H_1G_2 \in I_Z$.*

**Proof.** At first, we want to compute a projective presentation of $I_{C\cap\pi}$ of the form $0\to K\to P\to I_{C\cap\pi}\to 0$.

We recall that $I_{C\cap\pi}=(L_0,\dots,L_{n-3},\tilde{G}_1,\tilde{G}_2)$, and so $P=R^{n-2}(-1)\oplus R(-d+n-3)\oplus R(-d+n-2-a)$ and $K$ is the first syzygy module.

Using notation as in the proof of Green [4, Theorem 1.31], the $R$-module $K$ is generated by the following $n$-tuples:

$$e_j^{x_i} = -L_je^{x_i}+L_ie^{x_j}, \quad 0\le j\le i\le n-3,$$
$$e_j^{x^{d-n+3}_{n-2}} = -L_je^{x^{d-n+3}_{n-2}}+\tilde{G}_1e^{x_j}, \quad 0\le j\le n-3,$$
$$e_j^{x^{d-n+2}_{n-2}x^a_{n-1}} = -L_je^{x^{d-n+2}_{n-2}x^a_{n-1}}+\tilde{G}_2e^{x_j}, \quad 0\le j\le n-3,$$
$$e_{n-2}^{x^{d-n+2}_{n-2}x^a_{n-1}} = -H_1e^{x^{d-n+2}_{n-2}x^a_{n-1}}+H_2e^{x^{d-n+3}_{n-2}},$$

where $e^{x_j}$, $e^{x^{d-n+3}_{n-2}}$ and $e^{x^{d-n+3}_{n-2}x^a_{n-1}}$ are the generators of $P$ corresponding to $L_j,\tilde{G}_1$, and $\tilde{G}_2$, respectively.

The sequence $0 \to I_Z \to I_C \oplus I_\pi \to I_{C\cap\pi} \to 0$ is an extension of $I_{C\cap\pi}$ by $I_Z$, and we want to explicitly compute a representative of this extension in $\mathrm{Hom}_R(K, I_Z)$, (see [6] for more details). To this aim, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \stackrel{\mu}{\longrightarrow} & P & \stackrel{\varepsilon}{\longrightarrow} & I_{C\cap\pi} & \longrightarrow & 0 \\ & & \psi\downarrow & & \varphi\downarrow & & \| & & \\ 0 & \longrightarrow & I_Z & \stackrel{\alpha}{\longrightarrow} & I_C \oplus I_\pi & \stackrel{\beta}{\longrightarrow} & I_{C\cap\pi} & \longrightarrow & 0 \end{array},$$

where $\alpha(f)=(f,f)$, $\beta(f,g)=f-g$ and $\varphi,\psi$ to be computed.

By the commutativity of the right square, we have that $\varepsilon(e^{x_i}) = L_i = \beta(\varphi(e^{x_i}))$, for $i=0,\ldots,n-3$. But $H^0(\mathscr{I}_C(1))=0$, $\beta$ is a degree 0 morphism and so $\varphi(e^{x_i})=(0,-L_i)$. Analogously, we have that $\varphi(e^{x_{n-2}^{d-n+3}})=(G_1, G_1-\tilde{G}_1)$ and $\varphi(e^{x_{n-2}^{d-n+2}x_{n-1}^a})=(G_2, G_2-\tilde{G}_2)$.

Now looking, at the left square, we have that $\psi(e_j^{x_i}) = 0$, $\psi(e_i^{x_{n-2}^{d-n-3}}) = -L_iG_1$, $\psi(e_i^{x_{n-2}^{d-n-3}x_{n-1}^a}) = -L_iG_2$, $\psi(e_{n-2}^{x_{n-2}^{d-n+2}x_{n-1}^a}) = H_2G_1 - H_1G_2$, and so we have the claim. □

At this point, we give a partial inversion of the previous description.

**Proposition 1.9.** *Let $I_1$ be an ideal satisfying the hypotheses of Lemma* 1.5. *Let $I_2$ be an ideal generated by $L_0,\ldots,L_{n-3}$, $\tilde{G}_1,\tilde{G}_2$, where the $L_i$'s generate the plane contained in the scheme defined by $I_1$, $\tilde{G}_i = H_i\tilde{F}$, and* $\deg(H_1)=1$, $\deg(\tilde{F})\geq 3$.

*If there exist polynomials $P_{1,j}$ of degree* $\deg(\tilde{G}_1)-1$ *and $P_{2,j}$ of degree* $\deg(\tilde{G}_2)-1$, $j=0,\ldots,n-3$, *such that the two polynomials $G_i = \tilde{G}_i + \sum_j L_jP_{i,j}$ with $i=1,2$, satisfy conditions* (1), (2) *of Proposition* 1.8, *then the ideal $I = I_1 + (G_1,G_2)$ defines a closed subscheme $C$ with Hilbert polynomial*

$$p_C(z) = (\deg(G_1)+n-3)z + 1 - \left(\binom{\deg(G_1)-2}{2} - \deg(H_2)\right),$$

*and the cohomology of a quasi extremal curve.*

**Proof.** With the above assumptions, the ideal $I$ has a Gröbner basis $\mathscr{F} = \{Q_{0,0},\ldots, Q_{n-3,n-2}, G_1, G_2\}$ where $Q_{i,j}$ are the degree 2 generators of $I_1$ (they are a Gröbner basis of $I_1$). As in the proof of Proposition 1.6, we have that $I_1 = I \cap (L_0,\ldots,L_{n-3})$. Then, the short sequence

$$0 \to I_1 \to I \oplus (L_0,\ldots,L_{n-3}) \to I_2 \to 0$$

is exact, and so we have that $p_C(z)=dz+1-g$, where $C$ is the scheme defined by $I$, and

$$d = \deg(G_1)+n-3, \quad g = \binom{\deg(G_1)-2}{2} - \deg(H_2).$$

Moreover, if we consider the associated cohomology sequence, we have $H^1(\mathscr{I}(j)) \simeq H^1(\mathscr{I}_2(j))$ for $j \geq 0$, by Lemma 1.5(2), and so we have the claim by the following Lemma 1.10. □

**Lemma 1.10.** *Let $P \subset \mathbb{P}^2$ be a closed subscheme of dimension* 1 *whose total ideal is generated by $H_1F$ and $H_2F$, with* $\deg(H_1) = 1$, *and* $\deg(H_2) = a$. *Then,*

$$h^1(\mathscr{I}_P(j)) = \begin{cases} a & \text{if } j \le \deg(P) - 1, \\ \deg(P) - 1 + a - j & \text{if } \deg(P) - 1 \le j \le \deg(P) - 1 + a, \\ 0 & \text{if } j \ge \deg(P) - 1 + a. \end{cases}$$

**Proof.** Let $I^{CM} = (F)$ be the total ideal of the largest locally CM curve contained in $P$.

Then we have an exact sequence of sheaves

$$0 \to \mathscr{I}_P \to \mathscr{I}_P^{CM} \to \mathfrak{G} \to 0.$$

The Hilbert polynomial of $\mathfrak{G}$ has degree 0, and so $\mathfrak{G}$ is supported on a zero-dimensional scheme. Moreover, $h^0(\mathfrak{G}(j)) = a$ for all $j$.

From the exact long cohomological sequence, we obtain that

$$h^1(\mathscr{I}_P(j)) = h^0(\mathfrak{G}(j)) - h^0(\mathscr{I}_P^{CM}(j)) + h^0(\mathscr{I}_P(j)).$$

But

$$h^0(\mathscr{I}_P^{CM}(j)) = \binom{j - \deg(P) + 2}{2},$$

while

$$h^0(\mathscr{I}_P(j)) = \binom{j - \deg(P) + 1}{2} + \binom{j - \deg(P) + 1 - a}{2}$$

as one can easily compute using a free resolution of $I_P$. Then the claim follows. □

Examples of curves with different Rao modules will be given in Section 3.

**Remark 1.11** (*Computation of the polynomials $P_{i,j}$*)**.** Without loss of generality, we can suppose that $L_i = x_i$, for $i = 0, \dots, n-3$, $H_1 = x_{n-2}$ and $H_2 \in k[x_{n-1}, x_n]$. In this situation, we can suppose $P_{i,j} \in k[x_{n-1}, x_n]$.

Conditions (1) and (2) of Proposition 1.8 can be written in the following form:

$$\begin{cases} H_2 \sum_j x_j P_{1,j} + \sum_j P_{2,j} \operatorname{res}(Q_{j,n-2}) = 0, \\ H_2 \overline{(x_i\tilde{F})} - \sum_j P_{2,j} \operatorname{res}(Q_{i,j}) = 0, \quad i = 0, \dots, n-3, \\ \overline{(x_i\tilde{G}_1)} - \sum_j P_{1,j} \operatorname{res}(Q_{i,j}) = 0, \end{cases}$$

where $\overline{(x_i\tilde{F})}$ and $\overline{(x_i\tilde{G}_1)}$ are the normal forms of the corresponding polynomials modulo $I_1$.

We can observe that the first equation allows to compute $P_{1,i}$ using the polynomials $P_{2,j}$'s for every $i$.

**Remark 1.12.** The hypotheses of Proposition 1.9 on the ideal $I_2$ can be modified supposing to know the polynomial $\tilde{F}$, only. Then, the choice of $H_1$ and $H_2$ has to be done according to the system of Remark 1.11.

In the reduced case, the ideal generated by $H_1$, $H_2$ defines, in the plane $\pi$, the intersection points of the dimension 1 part of $Z$ with $\pi$, out of the curve defined by $\tilde{F}$.

## 2. A geometrical description of quasi-extremal curves

In this section, we characterize the quasi-extremal curves of degree $d$ as scheme-theoretical intersection of two aCM curves of degree $d+1$ and $d+a$, respectively, both of maximal genus, whose union is an aCM curve of maximal genus and degree $d+a+1$.

**Theorem 2.1.** *Let $C \subseteq \mathbb{P}^n$ be a closed subscheme of dimension* 1, *of degree $d$ and genus $g$. Then*:

*$C$ is a quasi-extremal curve if, and only if, there exist two aCM curves $D_1$ and $D_2$ of maximal genus and of degrees $d+1$ and $d+a$ respectively, such that their scheme-theoretical union $D = D_1 \cup D_2$ is an aCM curve of maximal genus and of degree $d+a+1$, and their scheme-theoretical intersection is a curve $C$.*

Theorem 2.1 follows from Propositions 2.2 and 2.5, below.

**Proposition 2.2.** *Let $C$ be a quasi-extremal curve of degree $d$ and genus $g$. Then, $C$ is naturally contained in two aCM curves $D_1$ and $D_2$, of maximal genus, and degrees $d+1$ and $d+a$, respectively. Moreover, $C = D_1 \cap D_2$ and $D_1 \cup D_2 = D$, where $D$ is an aCM curve of maximal genus and of degree $d+a+1$.*

**Proof.** For the convenience of the reader, we divide the proof in two steps.

*Step* 1: The curves $D_1$ and $D_2$. The two ideals $I_1 = (Q_{0,0}, \dots, Q_{n-3,n-2}, G_1)$ and $I_2 = (Q_{0,0}, \dots, Q_{n-3,n-2}, G_2)$ describe the two required curves, as we shall prove. Of course $I_C = I_1 + I_2$.

In the proof of Proposition 1.6, we proved that $I_C \cap I_\pi = I_Z$, where $Z$ is the scheme whose total ideal is generated by $(I_C)_2$. Then, we have that $I_i \cap I_\pi = I_Z$, and it is easy to prove that $I_i + I_\pi = (L_0, \dots, L_{n-3}, \tilde{G}_i)$, for $i = 1, 2$.

Then, using the exact sequences

$$(*) \quad 0 \to I_Z \to I_i \oplus I_\pi \to I_i + I_\pi \to 0,$$

we can compute the Hilbert polynomial of the scheme $D_i$ and we obtain

$$p_{D_1}(z) = (d+1)z + 1 - \binom{(d+1) - n + 1}{2}$$

and

$$p_{D_2}(z) = (d+a)z + 1 - \binom{(d+a)-n+1}{2}.$$

Moreover, if we consider the long exact cohomological sequence associated to $(*)$ we easily deduce that $H^1_*(\mathscr{I}_1) = H^1_*(\mathscr{I}_2) = H^1_*(\mathscr{I}_Z) = 0$ by Lemma 1.5(2).

The general hyperplane section of the curve $D_i$ contains the one of $C$ because $D_i \supset C$ and so it is non-degenerate.

*Step* 2: The curve $D$. First of all, we prove that $I_{D_1\cup D_2} = I_1 \cap I_2 = (Q_{0,0},\dots,Q_{n-3,n-2}, H_2G_1) = (Q_{0,0},\dots,Q_{n-3,n-2}, H_1G_2)$. In fact, we observe that both the last equalities hold and $I_1 \cap I_2 \supseteq (Q_{0,0},\dots,Q_{n-3,n-2}, H_2G_1)$, by Proposition 1.8(2).

Conversely, if $P \in I_1 \cap I_2$, then there exist $G'$, $G'' \in R$ such that $P - G'G_1 \in I_Z$ and $P - G''G_2 \in I_Z$. But $I_Z \subset I_\pi$ and so, if we consider the class of their difference modulo $I_\pi$, we have that $H_1\tilde{F}\tilde{G}' - H_2\tilde{F}\tilde{G}'' = 0 \,(\mathrm{mod}\, I_\pi)$ and then $H_1\tilde{G}' - H_2\tilde{G}'' = 0 \,(\mathrm{mod}\, I_\pi)$.

Hence, we have that $G'' = H_1\bar{G}'' + \Gamma''$, and $G' = H_2\bar{G}' + \Gamma'$, where $\Gamma'$, $\Gamma'' \in I_\pi$. By Proposition 1.8(1), we have that $P - G_1H_2\bar{G}' \in I_Z$, and $P - G_2H_1\bar{G}'' \in I_Z$, and so the first claim follows.

Now, we prove that $D$ is an aCM curve of maximal genus and degree $d+a+1$, considering the exact sequence of sheaves

$$0 \to \mathscr{I}_{D_1\cup D_2} \to \mathscr{I}_1 \oplus \mathscr{I}_2 \to \mathscr{I}_1 + \mathscr{I}_2 = \mathscr{I}_C \to 0.$$

It is trivial to prove that the morphism $H^0_*(\mathscr{I}_1) \oplus H^0_*(\mathscr{I}_2) \to H^0_*(\mathscr{I}_C)$ is surjective, and the exact sequence $0 \to H^1_*(\mathscr{I}_{D_1\cup D_2}) \to H^1_*(\mathscr{I}_1) \oplus H^1_*(\mathscr{I}_2) = 0$ shows that $D$ is aCM.

The additivity of the Hilbert polynomials shows that $D_1 \cup D_2$ has maximal genus and degree $d+a+1$. □

**Remark 2.3** (*Geometrical description of reduced quasi extremal curves*)**.** Using [1, Theorem 3.10] we can say that a reduced quasi-extremal curve $C$ of degree $d \geq n+2$ is the union of a plane curve $C'$ of degree $d-n+2$ and a "residual" part which is a disjoint union of curves $D_1,\dots,D_h$ described in [1, Lemma 3.8] such that $D_i \cap \pi$ is one reduced point lying on $C'$ for $i = 1,\dots,s$, and $D_i \cap \pi$ is a reduced point lying on a fixed line, for $i = s+1,\dots,h$, where $s \in \{0,\dots,h\}$ is a suitable integer.

As a consequence of Proposition 2.2, we obtain

**Corollary 2.4.** *The free resolution of $I_C$ given in Corollary* 1.4 *is minimal.*

**Proof.** The resolution of $I_{D_1}$ [1, Proposition 3.4] injects in the one of $I_C$ and so no cancellation occurs in the given resolution (as follows by the Cancellation Principle [4, Corollary 1.21]). □

**Proposition 2.5.** *Let $D_1$, $D_2 \subseteq \mathbb{P}^n$ be two aCM curves of maximal genus and of degrees $d+1$ and $d+a$, respectively. Suppose that their scheme-theoretical union* *is an aCM curve* $D = D_1 \cup D_2$, *of maximal genus and of degree* $d + a + 1$. *Then the scheme-theoretical intersection* $C = D_1 \cap D_2$ *is a closed subscheme with Hilbert polynomial* $p_C(z) = dz + 1 - g$, *where*

$$g = \binom{d-n+1}{2} - a,$$

*and* $h^1(\mathscr{I}_C(j))$ *is the Rao function of a quasi-extremal curve of degree* $d$ *and genus* $g$.

**Proof.** We can always suppose $I_{D_i} = I_{Z_i} + (G_i)$ where $G_i$ is a polynomial of degree $d-n+3$ for $i=1$, and $d-n+2+a$ for $i=2$, and $I_D = I_Z + (G)$, with $\deg(G) = d-n+3+a$ (see [1, Proposition 3.4] for a description of the ideal of an aCM curve of maximal genus). Moreover, the ideals $I_{Z_i}$ and $I_Z$ are generated by $(n+1)(n-2)/2$ quadrics.

Using the equality $I_D = I_{D_1} \cap I_{D_2}$, we have that $I_Z = I_{Z_1} = I_{Z_2}$. Moreover, if $\pi$ is the top dimensional part of $Z$, $D_1 \cap \pi$, $D_2 \cap \pi$, and $D \cap \pi$ are plane curves of degrees $d-n+3$, $d-n+2+a$, and $d-n+3+a$, respectively. Then, the ideal $(\tilde{G}_1) \cap (\tilde{G}_2)$ is a principal ideal generated by $\tilde{G}$, where $\tilde{P}$ is the reduction of $P$ modulo $I_\pi$. In fact, $(\tilde{G}_1) \cap (\tilde{G}_2) \supseteq (\tilde{G})$, and $\tilde{G}_1$ cannot divide $\tilde{G}_2$ (otherwise, $I_{D_1} \supseteq I_{D_2}$ and this is not so because $\deg(D) \neq \deg(D_2)$). Then, $(\tilde{G}_1) \cap (\tilde{G}_2) = (\tilde{G})$ by degree argument. Hence, the forms $\tilde{G}_i$, $i = 1, 2$, have a common factor, i.e. $\tilde{G}_1 = H_1\tilde{F}$, $\tilde{G}_2 = H_2\tilde{F}$, where $\deg(H_1) = 1$, $\deg(H_2) = a$. We have also that $G - H_jG_i \in I_{D_i} \cap I_\pi = I_Z$, $i \neq j$, and so $H_2G_1 - H_1G_2 \in I_Z$. But $LG_i \in I_Z$ for each $L \in I_\pi$. Then, the quadrics which generate $I_Z$ plus $G_1$, $G_2$ are a Gröbner basis of $I_C$, and so $I_C \cap I_\pi = I_Z$.

Then, from the exact sequence

$$0 \to I_Z \to I_C \oplus I_\pi \to I_{C\cap\pi} \to 0,$$

we deduce that the Hilbert polynomial of $C$ is $p_C(z) = dz + 1 - g$ where

$$g = \binom{d-n+1}{2} - a.$$

Moreover, considering the long cohomological sequence, we have that $H^1(\mathscr{I}_C(j)) = H^1(\mathscr{I}_{C\cap\pi}(j))$ for $j \geq 0$, by Lemma 1.5(2).

But $C \cap \pi$ is a closed subscheme of dimension 1 which satisfies the hypotheses of Lemma 1.10 and then the cohomology of $C$ satisfies the claim. □

**Remark 2.6.** If the scheme-theoretical intersection $C$ of $D_1$ and $D_2$ is a curve then $C$ is a quasi-extremal curve.

## 3. Rao functions of quasi-extremal curves

In Section 2, we naturally associated three aCM curves $D_1$, $D_2$ and $D = D_1 \cup D_2$, to each quasi-extremal curve $C$. Now, we want to study the relation among the Rao module of $C$ and the Rao modules of the sheaves $\mathscr{I}_{D_i/D}$ and $\mathscr{I}_{D_i\cap\pi/D\cap\pi}$.

**Proposition 3.1.** *With the same notation as above, we have the following isomorphims*:

$$H^1_*(\mathscr{I}_C) \simeq H^1_*(\mathscr{I}_{D_1/D}) \cap H^1_*(\mathscr{I}_{D_2/D})$$

$$\simeq H^1_*(\mathscr{I}_{D_1\cap\pi/D\cap\pi}) \cap H^1_*(\mathscr{I}_{D_2\cap\pi/D\cap\pi}),$$

*where we look at the first three vector spaces as subvector spaces of* $H^2_*(\mathscr{I}_D)$ *and the last two as subvector spaces of* $H^2_*(\mathscr{I}_{D\cap\pi})$.

**Proof.** We consider the exact sequences

$$0 \longrightarrow \mathscr{I}_D \xrightarrow{\varphi=(\varphi_1,\varphi_2)} \mathscr{I}_{D_1} \oplus \mathscr{I}_{D_2} \to \mathscr{I}_C \to 0,$$

$$0 \longrightarrow \mathscr{I}_D \xrightarrow{\varphi_i} \mathscr{I}_{D_i} \longrightarrow \mathscr{I}_{D_i/D} \longrightarrow 0$$

for $i = 1, 2$.

From the first sequence we deduce the cohomological sequence

$$0 \longrightarrow H^1_*(\mathscr{I}_C) \longrightarrow H^2_*(\mathscr{I}_D) \xrightarrow{\varphi=(\varphi_1,\varphi_2)} H^2_*(\mathscr{I}_{D_1}) \oplus H^2_*(\mathscr{I}_{D_2}) \longrightarrow \cdots$$

which shows that $H^1_*(\mathscr{I}_C) \simeq \ker(\varphi) \simeq \ker(\varphi_1) \cap \ker(\varphi_2)$.

But for $i = 1, 2$, the other short sequences give the long cohomological sequences

$$0 \longrightarrow H^1_*(\mathscr{I}_{D_i/D}) \longrightarrow H^2_*(\mathscr{I}_D) \xrightarrow{\varphi_i} H^2_*(\mathscr{I}_{D_i}) \longrightarrow \cdots$$

and these show that $\ker(\varphi_i) \simeq H^1_*(\mathscr{I}_{D_i/D})$, proving the first isomorphism.

For the last isomorphism we need the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{I}_Z & \to & \mathscr{I}_D \oplus \mathscr{I}_\pi & \to & \mathscr{I}_{D\cap\pi} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{I}_Z & \to & \mathscr{I}_{D_i} \oplus \mathscr{I}_\pi & \to & \mathscr{I}_{D_i\cap\pi} & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \mathscr{I}_{D_i/D} & \to & \mathscr{I}_{D_i\cap\pi/D\cap\pi} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The last line of the diagram gives the isomorphism $\mathscr{I}_{D_i/D} \simeq \mathscr{I}_{D_i\cap\pi/D\cap\pi}$.

Moreover, from the cohomological diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & H^1_*(\mathscr{I}_{D_i/D}) & \simeq & H^1_*(\mathscr{I}_{D_i\cap\pi/D\cap\pi}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^2_*(\mathscr{I}_Z) & \to & H^2_*(\mathscr{I}_D) & \xrightarrow{\alpha} & H^2_*(\mathscr{I}_{D\cap\pi}), & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^2_*(\mathscr{I}_Z) & \to & H^2_*(\mathscr{I}_{D_i}) & \to & H^2_*(\mathscr{I}_{D_i\cap\pi}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we deduce that the morphism $\alpha/\mathscr{I}_{D_i/D}$ is injective and then we can read $H^1_*(\mathscr{I}_{D_i/D})$ and their intersection into the vector space $H^2_*(\mathscr{I}_{D\cap\pi})$, using their isomorphic images $H^1_*(\mathscr{I}_{D_{i\cap\pi/D\cap\pi}})$. This fact proves the second isomorphism. □

The last proposition gives the conditions for a quasi-extremal curve to be extremal, too.

Now, we shall describe all the possible Rao functions of a quasi-extremal curve $C$, that is to say, the functions $h^1(\mathscr{I}_C(j))$, for $j \le 0$.

**Theorem 3.2.** *A numerical function is the Rao function of a quasi-extremal curve if, and only if, its graph in negative degrees in a polygonal whose edges have an increasing slope, not exceeding $n-2$ and passing through $(0,a)$.*

**Proof.** We have that $h^1(\mathscr{I}_C(j)) = \dim_k (R/J_n)_{d-n+a-j}$, where $J_n$ is the ideal generated by the entries of the last map $\varphi_n$ of a minimal free resolution of $I_C$, by Corollary 1.4 and Serre's Duality Theorem [5, Chapter III, Theorem 7.1].

The last map of a minimal free resolution of $I_C$ is

$$\varphi_n : R(-d-1-a) \to R^{n-1}(-d-a) \oplus R(-d-1) \oplus R^{n-2}(-n).$$

To have the right function for $j \ge 0$, we have that the $n-1$ linear forms are independent, and that the degree $a$ form is independent from the previous ones. Then we can suppose that $\mathrm{Ext}^{n-1}(\mathscr{I}_C, \mathscr{O}_{\mathbb{P}^n}) \simeq (k[x_{n-1},x_n]/(F_0,F_1,\dots,F_{n-2}))(d+a+1)$ where $\deg F_0 = a$, $\mathrm{in}(F_0) = x_{n-1}^a$ and $\deg F_i = d-n+1+a$, for $i = 1,\dots,n-2$.

Moreover, one among the $n-2$ forms of degree $d-n+1+a$, modulo $F_0$, is equal to $x_n^{d-n+1+a}$, because $h^1(\mathscr{I}_C(j)) = 0$ if $j \ll 0$.

By direct computation,

$$\rho(j) = \dim_k \left( \frac{k[x_{n-1},x_n]}{(F_0,\dots,F_{n-2})} \right)_{d+a-n-j}$$

is one of the described functions.

To end the proof of Theorem 3.2, we prove the following Lemma 3.3, in which we exhibit examples of curves with every possible Rao function.

**Lemma 3.3.** *Let $I$ be the ideal generated by $(x_0^2, x_0x_1, x_1^2, \dots, x_{n-3}x_{n-2}, x_{n-2}^2F, x_{n-2}FH_2 + x_0P_0 + x_1P_1 + \cdots + x_{n-3}P_{n-3})$, where*

(1) $F \in k[x_{n-2},x_{n-1},x_n]$, $\mathrm{in}(F) = x_{n-2}^{\deg(F)}$ *and* $\deg(F) \ge 2$;
(2) $H_2 \in k[x_{n-1},x_n]$, $\mathrm{in}(H_2) = x_{n-1}^a$ *and* $\deg(H_2) = a \ge 1$;
(3) $P_i \in k[x_{n-1},x_n]$, *and the normal form of at least one of the $P_i$'s, modulo $H_2$, is* $x_n^{a+\deg(F)}$.

*Then $I$ defines a quasi-extremal curve $C$ of degree $d = n-1+\deg(F)$ and genus*

$$g = \binom{d-n+1}{2} - a$$

*and* $h^1(\mathscr{I}_C(j)) = \dim_k (k[x_{n-1},x_n]/(H_2,P_0,\dots,P_{n-3}))_{d-n+a-j}$.

**Proof.** Let $Z$ be the scheme defined by $I_Z=(x_0^2,x_0x_1,x_1^2,\dots,x_{n-3}x_{n-2})$, and let $D_1$ and $D_2$ be the schemes defined by $I_1=I_Z+(x_{n-2}^2F)$ and $I_2=I_Z+(x_{n-2}FH_2+\sum_{i=0}^{n-3}x_iP_i)$. It is evident that both $I_1$ and $I_2$ are generated by Gröbner basis with respect to rlex, and so $\mathrm{in}(I_1)$ and $\mathrm{in}(I_2)$ are saturated. Then, $I_1$ and $I_2$ are saturated by Green [4, Corollary 1.12].

Let $D$ be the scheme defined by $I_3=I_1\cap I_2$. At first, we show that $I_3=I_Z+(x_{n-2}^2FH_2)$. It is evident that $I_Z+(x_{n-2}^2FH_2)\subseteq I_1\cap I_2$.

Conversely, let $G\in I_1\cap I_2$. Then,

$$G=\sum\alpha_{ij}^1x_ix_j+\beta_1x_{n-2}^2F$$

and

$$G=\sum\alpha_{ij}^2x_ix_j+\beta_2\left(x_{n-2}FH_2+\sum x_iP_i\right),$$

where the initial terms of $\beta_1$ and $\beta_2$ cannot be divided by $x_0,\dots,x_{n-3}$.

Then $\beta_2(x_{n-2}FH_2+\sum x_iP_i)-\beta_1x_{n-2}^2F\in I_Z$. By the choice of the initial terms of $\beta_1$ and $\beta_2$, we have that $\mathrm{in}(\beta_2x_{n-2}FH_2)=\mathrm{in}(\beta_1x_{n-2}^2F)$ and so we can divide it by $x_{n-2}^{2+\deg(F)}x_{n-1}^a$. In particular, $x_{n-2}$ divides $\mathrm{in}(\beta_2)$, and so $\mathrm{in}(\beta_2)\sum x_iP_i\in I_Z$.

But, $\beta_i=\mathrm{in}(\beta_i)+\mathrm{res}(\beta_i),\ i=1,2,$ and so

$$\mathrm{res}(\beta_2)\left(x_{n-2}FH_2+\sum x_iP_i\right)-\mathrm{res}(\beta_1)x_{n-2}^2F\in I_Z.$$

By iterating the argument, we have that $x_{n-2}$ divides $\beta_2$, and so $G\in I_Z+(x_{n-2}^2FH_2)$.

$I_3$ is generated by a Gröbner basis. Then, $I_3$ is saturated because $\mathrm{in}(I_3)$ is.

Now, we want to compute the Hilbert polynomials of Lemma 1.5. In fact, the resolution of $I_Z$ was implicitly computed in Corollary 1.4. In particular, the plane contained in $Z$ is $\pi=V(x_0,\dots,x_{n-3})$.

Using arguments like in the proof of Proposition 1.6, we have that $I_i\cap I_\pi=I_Z,\ i=1,2,3$. Then, the sequences

$$0\to I_Z\to I_i\oplus I_\pi\to(I_i+I_\pi)\to 0,\quad i=1,2,3$$

are exact, and from them we obtain the Hilbert polynomials of $D_1,D_2$, and $D$:

$$p_{D_1}(z)=(\deg(F)+n)z+1-\binom{\deg(F)+1}{2},$$

$$p_{D_2}(z)=(\deg(F)+n+a-1)z+1-\binom{\deg(F)+a}{2},$$

$$p_D(z)=(\deg(F)+n+a)z+1-\binom{\deg(F)+a+1}{2}.$$

Now, we compute the minimal free resolutions of $I_1,I_2$, and $I_3$. To this aim we use the same formalism as in [4, Theorem 1.31].

We observe that the three ideals differ with respect to the last generator, and so their free resolutions have a common part, which is the one corresponding to the scheme $Z$

(see Lemma 1.5). Now, we can describe a basis for the $p$th syzygy module of each ideal.

- Common generators (basis of the $p$th syzygy module of $I_Z$):

$$e^{x_h x_k}_{i_p\dots i_1} = \sum_{j=0}^{p-1} (-1)^{j+1} x_{i_{p-j}}\, e^{x_h x_k}_{\dots\, \hat{i}_{p-j}\dots}$$
$$+(-1)^{p+1} x_k \begin{cases} e^{x_h x_{i_1}}_{\dots i_2\ \hat{i}_1} & \text{if } h \le i_1, \\ \sum_{j=1}^{p+1} (-1)^{j+1} e^{x_{i_1} x_k}_{\dots\ \hat{i}_j \dots} & \text{if } h > i_1, \end{cases}$$

where $0 \le i_p < i_{p-1} < \cdots < I_1 < k$.

- Generators of the basis of $I_1$, only:

$$s^{x_{n-2}^{2+\deg F}}_{i_p\dots i_1} = \sum_{j=0}^{p-1} (-1)^{j+1} s^{x_{n-2}^{2+\deg F}}_{\dots\ \hat{i}_{p-j}\dots} + (-1)^p x_{n-2} F \sum_{j=1}^{p} (-1)^j e^{x_{i_j} x_{n-2}}_{\dots\ \hat{i}_j \dots}.$$

- Generators of the basis of $I_2$, only:

$$y^{x_{n-2}^{1+\deg F} x_{n-1}^a}_{i_p\dots i_1} = \sum_{j=0}^{p-1} (-1)^{j+1} y^{x_{n-2}^{1+\deg F} x_{n-1}^a}_{\dots\ \hat{i}_{p-j}\dots} + (-1)^p F H_2 \sum_{j=1}^{p} (-1)^j e^{x_{i_j} x_{n-2}}_{\dots\ \hat{i}_j \dots}$$
$$+(-1)^{p+1} \begin{cases} \sum_{j=0}^{n-3} P_j e^{x_j x_{i_1}}_{\dots i_2\ \hat{i}_1} & \text{if } n-3 \le i_1, \\ \sum_{j=0}^{i_1} P_j e^{x_j x_{i_1}}_{\dots i_2\ \hat{i}_1} \sum_{j=i_1+1}^{n-3} P_j \sum_{\lambda=1}^{p} (-1)^{\lambda+1} e^{x_{i_\lambda} x_j}_{\dots\ \hat{i}_\lambda \dots} & \text{if } n-3 \le i_1. \end{cases}$$

- Generators of the basis of $I_3$, only:

$$z^{x_{n-2}^{2+\deg F} x_{n-1}^a}_{i_p\dots i_1} = \sum_{j=0}^{p-1} (-1)^{j+1} z^{x_{n-2}^{2+\deg F} x_{n-1}^a}_{\dots\ \hat{i}_{p-j}\dots} + (-1)^p x_{n-2} F H_2 \sum_{j=1}^{p} (-1)^j e^{x_{i_j} x_{n-2}}_{\dots\ \hat{i}_j \dots}.$$

The correctness of the previous expressions can be proved by induction on $p$, without difficulty, but with a very large amount of computations.

In particular, the last free module of each resolution occurs at the $(n-1)$th level. Then, $D_1, D_2$, and $D$ are aCM curves of maximal genus, and degrees $\deg(F) + n$, $\deg(F) + n + a - 1$, and $\deg(F) + n + a$, respectively.

By Proposition 2.5, $I$ is the total ideal of a closed subscheme $C$ of dimension 1, and $h^1(\mathscr{I}(j))$ is the Rao function of a quasi-extremal curve of degree $d = \deg(F) + n - 1$, and genus

$$g = \binom{d-n+1}{2} - a.$$

To end the proof, we compute the minimal free resolution of $I$, and $h^1(\mathscr{I}(j))$ for $j \le 0$, from this resolution.

The resolution of $I$ can be computed by mapping cone from

$$0 \to I_3 \to I_1 \oplus I_2 \to I \to 0.$$

In particular, we are interested in the last map $\varphi_n$ of the resolution. The map $\varphi_n$ is the following:

$$\varphi_n : R(-d-1-a) \to R(-d-1) \oplus R(-d-a) \oplus R^{n-2}(-n) \oplus R^{n-2}(-d-a)$$

and $\varphi_n = \alpha^{(1)}_{n-1,1} \oplus \alpha^{(2)}_{n-1} \oplus \delta^{D}_{n-1,1}$, where

(1) $\alpha^{(1)}_{n-1} : F^{D}_{n-1} \to F^{D_1}_{n-1}$ is the identity on the part corresponding to $Z$ and the product by $H_2$ on the other part, as can be easily obtained by $\alpha^{(1)}_0 : I_D \to I_{D_1}$, and so $\alpha^{(1)}_{n-1,1} : R(-d-1-a) \to R(-d-1)$ is the multiplication by $H_2$;

(2) $\alpha^{(2)}_{n-1} : F^{D}_{n-1} \to F^{D_2}_{n-1}$ is the identity on the part corresponding to $Z$, but, restricted to $R(-d-1-a)$, it is equal to

$$\alpha^{(2)}_{n-1}(1) = x_{n-2}\, y_{0,\dots,n-3}^{x_{n-2}^{1+\deg F} x_{n-1}^{a}} - \sum_{j=0}^{n-3} P_j e_{0,\dots n-3}^{x_j x_{n-2}}$$

and extended as homomorphism of $R$-modules, because of the map $\alpha^{(2)}_0 : I_D \to I_{D_2}$;

(3) $\delta^{D}_{n-1} : F^{D}_{n-1} \to F^{D}_{n-2}$ is given by the expression of $z_{0,\dots,n-3}^{x_{n-2}^{2+\deg F} x_{n-1}^{a}}$, and so, considered from $R(-d-1-a)$ to $R^{n-2}(-d-a)$, we have that

$$\delta^{D}_{n-1,1}(1) = \sum_{j=0}^{n-3} (-1)^{j+1} x_j z_{\dots\hat{j}\dots}^{x_{n-2}^{2+\deg F} x_{n-1}^{a}}$$

and extended as homomorphism of $R$-modules.

Then, the entries of $\varphi_n$, up to their signs, are

$$x_0, x_1, \dots, x_{n-3}, x_{n-2}, H_2, P_0, \dots, P_{n-3}.$$

Using the hypothesis on the normal form of at least one of the $P_i$'s, we have that $h^1(\mathscr{I}(j)) = 0$ for $j \ll 0$. Then, $C$ is a curve and we have the claim. $\square$

**Remark.** (1) The slope of the polygonal in Theorem 3.2 changes when the spans of two forms $F_i, F_j$ have non-empty intersection in $k[x_{n-1}, x_n]$.

(2) In Lemma 3.3, without the assumption on the normal form on one of the $P_i$'s, we can only say that $C$ is a closed subscheme of dimension 1 with the expected degree, genus and cohomology in positive degrees.

Theorem 3.2 gives, in particular, a lower bound for the dimension of the graded pieces of Rao module of a quasi-extremal curve, in negative degrees.

**Corollary 3.4.** *If $C$ is quasi-extremal then $h^1(\mathscr{I}_C(j)) \geq (n-2)j + a$, for $j \leq 0$.*

**Corollary 3.5.** *For $n = 3$, a curve $C$ is quasi-extremal if and only if it is extremal.*

In this case, the previous description completes the one given in [3].

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