SOLUTIONS OF THE FRACTIONAL 1-LAPLACIAN: EXISTENCE, ASYMPTOTICS AND FLATNESS RESULTS

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ABSTRACT. In this paper, we study the existence of solutions of the equation $(-\Delta)^s_1 u = f$ in a bounded open set with Lipschitz boundary $\Omega \subset \mathbb{R}^n$, vanishing on $\mathcal{C}\Omega$, for some given $s \in (0,1)$, and asymptotics as $p \to 1$ of solutions of $(-\Delta)^s_p u = f$. We obtain existence and convergence by comparing the $L^{\frac{n}{s}}$ norm of f to the sharp fractional Sobolev constant, or, when f is non-negative, the weighted fractional Cheegar constant to 1 – in this case, the results are sharp. We further prove that solutions are "flat" on sets of positive Lebesgue measure.

1. INTRODUCTION

In this note, we address some issues concerning minimizers and weak solutions of an equation related to the fractional 1-Laplacian. For a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, a fixed fractional parameter $s \in (0,1)$ and a given function f in $L^{\frac{n}{s}}(\Omega)$, we deal with the nonlocal problem

$$\begin{cases} (-\Delta)^s_1 u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega. \end{cases} \tag{1.1}$$

The fractional 1-Laplacian arises in the Euler-Lagrange equation related to functions of least $W^{s,1}$-energy and could be thought, roughly speaking, as

$$(-\Delta)^s_1 u(x) = \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|u(x)-u(y)|} \frac{dxdy}{|x-y|^{n+s}}, \tag{1.2}$$

by simply taking $p = 1$ in the definition of the fractional p-Laplacian. We recall that up to constants, the fractional p-Laplacian for some $p > 1$ is defined as

$$(-\Delta)^s_p u(x) := P.V. \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} \, dxdy.$$

Here, we continue the ongoing research started by the author and collaborators in [4, 5] regarding $(s,1)$-harmonic functions with non-vanishing boundary data, and minimizers of the related energy

$$\mathcal{E}^s_1(u,\Omega) = \frac{1}{2} \iint_{Q(\Omega)} \frac{|u(x)-u(y)|}{|x-y|^{n+s}} \, dxdy, \tag{1.3}$$

where $\varphi \colon \mathcal{C}\Omega \to \mathbb{R}$ is given and $Q(\Omega) := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$. Regarding minimizers of (1.3) – functions of least $W^{s,1}$-energy – we proved the fractional counterpart of some results from [1, 16, 22]. Precisely in [5], level sets of minimizers are proved to be nonlocal minimal surfaces

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and existence of minimizers is obtained, assuming that the exterior given data has "integrable tail". In [4] the sequence of minimizers of the energy

$$\mathcal{E}_p^s(u,\Omega) = \frac{1}{2p}\iint_{Q(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy, \tag{1.4}$$

as the parameter p tends towards 1, is proved to converge to a minimizer of (1.3).
Concerning weak solutions of (1.1), their definition was originally introduced in [19], for a L^2 right-hand side data. In [4], the existence of $(s,1)$-harmonic functions, together with their equivalence to minimizers is investigated. For more comprehensive details, interested readers can also refer to the survey [3], where these findings are sketched, and the classical problem is also discusses.

In this paper, also in view of these previous results, we wondered whether studying the asymptotics as $p \to 1$ of the p-problem could lead to the existence of weak solutions of (1.1). To be specific, our fractional setting is the following: for any $p \in [1, +\infty)$ we take

$$\mathcal{F}_p^s(u) := \frac{1}{2p}\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy - \int_\Omega fu\,dx.$$

For $p > 1$, minimizers of $\mathcal{F}_p^s$ are known to be weak solutions of

$$\begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega. \end{cases} \tag{1.5}$$

The study of existence of global minimizers for $\mathcal{F}_1^s$ seems to extend beyond direct methods of the calculus of variations. Notably, the energy does not always present a bound from below, and existence seems to depend, as in the classical case $s = 1$, on some characteristics of f and Ω. We approach the problem in two different manners. For any $f \in L^{\frac{n}{s}}(\Omega)$, we compare the norm of f to $(2S_{n,s})^{-1}$, where $S_{n,s}$ is the sharp fractional Sobolev constant, and prove existence and asymptotics as $p \to 1$ of the associated p-problem when the norm is not larger than such a constant, leaving open the case when the norm is larger. On the other hand, for $f \geqslant 0$, we are able to provide sharp existence and asymptotics results, by comparing to 1 the weighted fractional Cheegar constant.

To be more precise, asking that $f \in L^{\frac{n}{s}}(\Omega)$, let u_p denote the unique minimizer/weak solution of $\mathcal{F}_p^{s_p}$, vanishing on $\mathcal{C}\Omega$, for $p \in (1, c_{n,s})$, where $c_{n,s} > 1$ is such that

$$s_p := n + s - \frac{n}{p} \in (s, 1) \tag{1.6}$$

and $s_p p < 1$. If

$$\|f\|_{L^{\frac{n}{s}}(\Omega)} < (2S_{n,s})^{-1} \qquad \text{then} \qquad u_p \xrightarrow[p\to 1]{} u_1 = 0,$$

where $u_1 = 0$ is the unique minimzer of $\mathcal{F}_1^s$ and weak solution of (1.1). If

$$\|f\|_{L^{\frac{n}{s}}(\Omega)} = (2S_{n,s})^{-1}, \qquad \text{then} \qquad u_p \xrightarrow[p\to 1]{} u_1,$$

minimzer of $\mathcal{F}_1^s$ and weak solution of (1.1). We give examples that assure the reader that in this case, non-vanishing minimizers exist, and also that when $\|f\|_{L^{\frac{n}{s}}(\Omega)} > (2S_{n,s})^{-1}$ the energy may be unbounded from below, and global minimizer may not exist. This results are the content of Theorem 3.2.

In the case $f \geqslant 0$, we are able obtain sharp results, by appealing to another interesting problem, that of Cheegar sets. To be more precise, we take a non-negative $f \in L^{\frac{n}{\sigma}}(\Omega)$ for some $\sigma \in (0, s)$ and recall the definition of the fractional perimeter of the set $E \subset \mathbb{R}^n$, introduced in [9], as

$$\operatorname{Per}_s(E, \Omega) := \frac{1}{2} \iint_{Q(\Omega)} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+s}} dx\, dy, \tag{1.7}$$

where χ_E is the characteristic function of the set E, and we denote

$$|E|_f = \int_E f\, dx.$$

We point out that for $E \subset \Omega$, $\operatorname{Per}_s(E, \Omega) = \operatorname{Per}_s(E)$, where we write $\operatorname{Per}_s(E) = \operatorname{Per}_s(E, \mathbb{R}^n)$. We slightly modify the renown problem of the s-Cheegar constant (see [2,8]), by adding the weight f in the contribution to the volume. We call the (s, f)-Cheegar constant

$$h_s^f(\Omega) := \inf \left\{ \frac{\operatorname{Per}_s(A)}{|A|_f} \;\middle|\; A \subset \Omega,\, |A|_f > 0 \right\}, \tag{1.8}$$

and say that $\tilde{E} \subset \Omega$ is a (s, f)-Cheegar set if

$$h_s^f(\Omega) := \frac{\operatorname{Per}_s(\tilde{E})}{|\tilde{E}|_f}.$$

We obtain the following sharp existence and aysmptotic result. Let again u_p denote the unique minimizer of $\mathcal{F}_p^{s_p}$. If

$$h_s^f(\Omega) > 1 \qquad \text{then} \qquad u_p \xrightarrow[p \to 1]{} u_1 = 0,$$

where $u_1 = 0$ is the unique minimzer of $\mathcal{F}_1^s$ and weak solution of (1.1). If

$$h_s^f(\Omega) = 1, \qquad \text{then} \qquad u_p \xrightarrow[p \to 1]{} u_1,$$

minimzer of $\mathcal{F}_1^s$ and weak solution of (1.1). In this case, we provide an example showing non-uniquness of solutions. If

$$h_s^f(\Omega) > 1 \qquad \text{then} \qquad [u_p]_{W^{s_p,p}(\Omega)} \xrightarrow[p \to 1]{} +\infty,$$

and minimizers of $\mathcal{F}_1^s$ do not exist. We insert these findings in Theorem 4.8.

We compare our findings with those in the classical framewok. The asymptotic results of the type were first investigated in [17], where the author studied the behavior, as $p \to 1$, of weak solution u_p of the torsion problem for the p-Laplacian $-\Delta_p u = 1$ in Ω. The author observed that, if Ω is sufficiently small, then

$$u_p \xrightarrow[p \to 1]{} 0 \tag{1.9}$$

while for Ω large enough

$$u_p \xrightarrow[p \to 1]{} +\infty. \tag{1.10}$$

The research is continued in [11], where the authors considered a right hand side $f \in L^n(\Omega)$ and prove that when $\|f\|_{L^n(\Omega)} \leqslant 1/S_n$ with S_n being the sharp Sobolev constant, then (1.9) holds for weak solutions. Additionally, minimizers of the p-energy approach minimizers of the 1-energy, as $p \to 1$, even though the 1-minimizer might not vanish. We point out that these classical results rely on an explicit formula for solutions of the p-Laplace equation on the ball, which is not available in the fractional case. We also remark that the constant

$2S_{n,s}$ in our result is as sharp as in the classical case, given that the multiplicative term $1/2$ depends solely on the use of this constant in (1.4) introduced for symmetry purposes.

Regarding studying asymptotics by comparing the Cheegar constant $h(\Omega)$ to 1, this was done in the classical case in [6,7], only when f has the constant value 1. In these two papers, the authors emphasized the strong connection between the Cheegar constant and p-torsion functions, i.e. unique solutions of $(-\Delta)_p u = 1$ in Ω, showing that

$$\lim_{p\to 1} \|\phi_p\|^{1-p}_{L^1(\Omega)} = \lim_{p\to 1} \|\phi_p\|^{1-p}_{L^\infty(\Omega)} = h(\Omega), \tag{1.11}$$

and also that ϕ_p, renormalized by the L^1 norm, converges to a solution of the $(-\Delta)_1 u = 1$ as $p \to 1$. From this, comparing $h(\Omega)$ to 1, ϕ_p are proved to converge to either zero or $+\infty$, in the L^1, L^∞ norm.

We remark that with respect to the classical case, the nonlocal nature of our problem significantly complicates the situation. In particular, the uniform bound on the minimizers, which is needed to prove the convergence results, is not at all as straight forward as in the classical case, and a lot of work is necessary to deal with the nonlocal contribution to the energy.

To best of our knowledge, the results presented in this paper are completely new. We shortly describe what is known in the fractional case. The fractional Cheegar constant was introduced in [2], where the connection between a variational formulation for the torsion problem for the $(s,1)$-Laplacian and the eingenvalue problem $(-\Delta)^s_p u = \lambda |u|^{p-2} u$ is studied. In [8], an alternative characterization of the fractional Cheegar constant is provided, by studying the (s,p)-torsional problem $(-\Delta)^s_p u = 1$ and obtaining (1.11) in the fractional case. The difference with our results – besides our use of a general term f – is that we prove that the sequence of u_p, minimizers of $\mathcal{F}^{s_p}_p$, converges to minimizer and a weak solution of $\mathcal{F}^s_1$ when $h_s(\Omega) \geqslant 1$. Moreover, the approach we use is different from that of [8], which is based on results from [2]. Our approach, more similar to [6], allowed us to make full use of the fractional parameter s_p, which plays a significant role in our investigation. We point our furthermore that neither of [2,8] are interested in solutions of the problem (1.1), nor mention the existence of weak solutions.

To give some further input on weak solutions, we also discuss a so called "flatness" results. The difficulty of defining a weak solution of the fractional 1-Laplacian is evident looking at (1.2) – the quotient $(u(x) - u(y))/|u(x) - u(y)|$, i.e. the sign function of $(u(x) - u(y))$, does not have a meaning when $u(x) = u(y)$. The definition of a weak solution cannot be given by using purely an integro-differential equation, rather it is necessary, as done in [19], to require that there exists a multivalued function $\mathbf{z}\colon \mathbb{R}^{2n} \to [-1,1]$ equal to $\operatorname{sgn}(u(x) - u(y))$, where sgn is the generalized sign function. A concern of this paper is to prove that generally, such a definition cannot be simplified, since solutions are "flat" on sets of positive Lebesgue measure. Precisely, if u is a weak solution of (1.2) then the set $\{(x,y) \in \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2 \,|\, u(x) = u(y)\}$ has positive Lebesgue measure – see Theorem 6.1. We prove a similar result for minimizers, showing that $\{x \in \Omega \,|\, |u(x)| = \|u\|_{L^\infty(\Omega)}\}$ has positive Lebesgue measure, in Theorem 6.2. Such results are the fractional counterpart of [21], where the authors prove that a weak solution of the 1-Laplacian equation with $L^n(\Omega)$ right hand side and zero boundary data has a vanishing gradient on a set of positive Lebesgue measure.

We draw the reader's attention to the use of the particular fractional parameter s_p in (1.6), first introduced in [19]. Such a choice appears necessary for technical reasons – see Remark

3.3 – but is enforced by the fractional embedding $W^{s,1}(\Omega) \subset W^{s_p,p}(\Omega)$ (see [2, Lemma 2.6] or [5, Lemma 3.1]), in other words when sending $p \to 1$ in the (s,p)-fractional problem, in the limit the fractional parameter s has to "decrease" as well, and this ratio is described precisely by the choice of s_p. To further justify the use of s_p, in the Appendix we obtain that for all $u \in \mathcal{W}_0^{s_q,q}(\Omega)$ for some fixed $q > 1$ the pointwise limit holds

$$\lim_{p \to 1} \mathcal{E}_p^{s_p}(u) = \mathcal{E}_1^s(u). \tag{1.12}$$

A very interesting (in our opinion) issue is also discussed in Section 4: that of non-negative minimizers of $\mathcal{F}_1^s$ and sets $E \subset \Omega$ that minimize

$$\mathcal{P}_s(E) = \mathrm{Per}_s(E) - |E|_f.$$

Whether it is true that E is a minimal set if only if χ_E is a minimal function and weak solution of

$$(-\Delta)_1^s \chi_E = 1$$

is investigated, together with the equivalence that u is a non-negative minimizer of $\mathcal{F}_1^s$ if and only if any super-level set of u is a minimal set for $\mathcal{P}_s$.

In the rest of the paper, we proceed as follows. We use Section 2 for the setting and some very useful tools. Section 3 is dedicated to studying the limit case as $p \to 1$ when the $L^{\frac{n}{s}}$ norm of f is sufficiently small, and prove the existence of a minimizer and a weak solution. We also deal with the equivalence minimizer - weak solution, even in the absence of a bound on $\|f\|_{L^{\frac{n}{s}}}(\Omega)$. Section 4 contains the sharp results on existence and asymptotics by comparing the fractional Cheegar constant to 1 and we also discuss the relation to sets that minimize $\mathcal{P}_s$. We give examples of existence of non-trivial solutions and of non-existence when the $L^{\frac{n}{s}}$ norm of f is sufficiently large, coinciding with the case $h_s(f) \leqslant 1$. In Section 6 we discuss the "flatness" of weak solutions of problem (1.1) and of minimizers. The Appendix contains the proof of (1.12) and some basic knowledge on (s,p)-minimizer/weak solutions.

2. Setting of the problem, tools and remarks

Let $n \geqslant 1$, $0 < s < 1 \leqslant p < +\infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Since we mainly look at asymptotics as $p \to 1$, we will emphasize some useful properties when p is close enough to one, precisely when $sp < 1$.

We will use the notations

$$\omega_n = \mathcal{H}^{n-1}(\partial B_1), \qquad B_r = \{x \in \mathbb{R}^n \, : \, |x| < r\} \text{ for some } r > 0$$

and recall that $\mathcal{L}^n(B_1) := |B_1| = \omega_n / n$. We also denote

$$Q(\Omega) := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2.$$

We use the notation for the fractional Sobolev space $\mathcal{W}^{s,p}(\Omega)$,

$$\mathcal{W}^{s,p}(\Omega) := \left\{ u \colon \mathbb{R}^n \to \mathbb{R} \, \middle| \, u|_\Omega \in L^p(\Omega), \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dxdy < +\infty \right\},$$

and we denote

$$[u]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dxdy \right)^{\frac{1}{p}}, \qquad \|u\|_{W^{s,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + [u]_{W^{s,p}(\Omega)}^p \right)^{\frac{1}{p}}$$

the fractional (Gagliardo) (s,p)-seminorm, respectively norm. Of course, $\mathcal{W}^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$. We denote $\mathcal{W}_0^{s,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the $W^{s,p}(\Omega)$ norm. We point out that, when $sp < 1$, it also holds

$$\mathcal{W}_0^{s,p}(\Omega) = \left\{ u\colon \mathbb{R}^n \to \mathbb{R} \;\middle|\; u \in \mathcal{W}^{s,p}(\Omega), u = 0 \text{ on } \mathcal{C}\Omega \right\},$$

check [12, Proposition A.1].
What is more, when $sp < 1$ and $u = 0$ on $\mathcal{C}\Omega$, the norms $W^{s,p}(\Omega)$ and $W^{s,p}(\mathbb{R}^n)$ are equivalent, i.e. there exist $C_1, C_2 > 0$ (depending on n, s, p, Ω) such that

$$C_1 \|u\|_{W^{s,p}(\Omega)} \leqslant [u]_{W^{s,p}(\mathbb{R}^n)} \leqslant C_2 \|u\|_{W^{s,p}(\Omega)}. \tag{2.1}$$

To see this, we point out the very useful result in [12, Corollary A.3].

Proposition 2.1. *Let $1 \leqslant p < 1/s$. Then for every $u \in W^{s,p}(\Omega)$*

$$\int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}}\, dy \right) dx \leqslant C \|u\|^p_{W^{s,p}(\Omega)}, \tag{2.2}$$

where $C = C(n, s, p, \Omega) > 0$.

Due to (2.2),

$$[u]^p_{W^{s,p}(\mathbb{R}^n)} = [u]^p_{W^{s,p}(\Omega)} + 2 \int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}}\, dy \right) dx \leqslant C_2 \|u\|^p_{W^{s,p}(\Omega)},$$

while

$$\begin{aligned}
[u]^p_{W^{s,p}(\mathbb{R}^n)} &\geqslant [u]^p_{W^{s,p}(\Omega)} + 2 \int_\Omega |u(x)|^p \left(\int_{\mathcal{C}B_{\operatorname{diam}(\Omega)}(x)} \frac{dy}{|x-y|^{n+sp}} \right) dx \\
&= [u]^p_{W^{s,p}(\Omega)} + \|u\|^p_{L^p(\Omega)} \frac{\omega_n}{\operatorname{diam}(\Omega)^{sp} sp} \geqslant C_1 \|u\|^p_{W^{s,p}(\Omega)},
\end{aligned}$$

since for all $x \in \Omega$, it holds that $\mathcal{C}B_{\operatorname{diam}(\Omega)}(x) \subset \mathcal{C}\Omega$. Thanks to this, we have that for $sp < 1$,

$$\mathcal{W}_0^{s,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{[\cdot]_{W^{s,p}(\mathbb{R}^n)}}. \tag{2.3}$$

We recall now the Sobolev inequality with sharp constants, which we state in our case $sp < 1$. This follows directly from [14, Corollary 4.2] just by checking (2.3) .

Theorem 2.2. *Let $n \geqslant 1$, $0 < s < 1 \leqslant p < 1/s$. For all $u \in \mathcal{W}_0^{s,p}(\Omega)$ it holds that*

$$\|u\|^p_{L^{p^*}(\Omega)} \leqslant S_{n,s,p} [u]^p_{W^{s,p}(\mathbb{R}^n)},$$

where

$$S_{n,s,p} = \left(\frac{p}{p^*} \right)^{\frac{1}{p}} \left(\frac{n}{\omega_n} \right)^{\frac{s}{n}} C_{n,s,p}^{-\frac{1}{p}}, \tag{2.4}$$

with

$$C_{n,s,p} = 2 \int_0^1 r^{ps-1} |1-r|^{\frac{n-ps}{p}} \phi_{n,s,p}(r)\, dr,$$

and

$$\begin{aligned}
\phi_{n,s,p} &= |\mathbb{S}^{n-2}| \int_{-1}^1 \frac{(1-t^2)^{\frac{N-3}{2}}}{(1-2rt+r^2)^{\frac{n+ps}{2}}}\, dt, \quad N \geqslant 2 \\
\phi_{n,s,p} &= \left((1-r)^{-1-ps} - (1+r)^{-1-ps} \right), \quad N = 1.
\end{aligned}$$

As a consequence of Theorem 2.2, we get the fractional sharp isoperimetric inequality for sets $E \subset \Omega$ (see also [15] for more details), recalling (1.7).

Theorem 2.3. *Let $E \subset \Omega$ be a Borel set with finite Lebesgue measure. Then*

$$|E|^{\frac{n-s}{n}} \leqslant 2S_{n,s} \operatorname{Per}_s(E).$$

Notice also that

$$S_{n,s} = \frac{|B_1|^{\frac{n-s}{n}}}{2\operatorname{Per}_s(B_1)} = \frac{n-s}{n}\left(\frac{n}{\omega_n}\right)^{\frac{s}{n}} \frac{1}{C_{n,s,1}}. \tag{2.5}$$

We define now our notions of minimizers and weak solutions.

Definition 2.4. *Let $0 < s < 1 \leqslant p < +\infty$. We say that $u \in \mathcal{W}_0^{s,p}(\Omega)$ is an (s,p)-minimizer if*

$$\mathcal{F}_p^s(u) \leqslant \mathcal{F}_p^s(v)$$

for all $v \in \mathcal{W}_0^{s,p}(\Omega)$.

Furthermore, we recall the following.

Definition 2.5. *Let $0 < s < 1$ and $1 < p < +\infty$. We say that $u\colon \mathbb{R}^n \to \mathbb{R}$ is a weak solution of the (s,p)-problem (1.5) if $u \in \mathcal{W}_0^{s,p}(\Omega)$, and for all $w \in \mathcal{W}_0^{s,p}(\Omega)$*

$$\frac{1}{2}\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+sp}}\, dxdy = \int_\Omega fw\, dx.$$

For the fractional 1-Laplacian we give the following definition of weak solution – check also [4, 19].

Definition 2.6. *Let $0 < s < 1$. We say that a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ is a weak solution of the problem (1.1) if $u \in \mathcal{W}_0^{s,1}(\Omega)$ and if there exists*

$$\mathbf{z} \in L^\infty(\mathbb{R}^{2n}) \qquad \|\mathbf{z}\|_{L^\infty(\mathbb{R}^{2n})} \leqslant 1, \qquad \mathbf{z}(x,y) = -\mathbf{z}(y,x),$$

satisfying

$$\frac{1}{2}\iint_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)}{|x-y|^{n+s}}(w(x)-w(y))dx\,dy = \int_\Omega f(x)w(x)\,dx \quad \text{for all } \; w \in \mathcal{W}_0^{s,1}(\Omega), \tag{2.6}$$

and

$$\mathbf{z}(x,y) \in \operatorname{sgn}(u(x)-u(y)) \quad \text{for almost all } (x,y) \in Q(\Omega). \tag{2.7}$$

We remark that since $u = 0$ in $\mathcal{C}\Omega$, the contribution to the double integral in $(\mathcal{C}\Omega)^2$ is null, hence it is enough to require (2.7) to hold on $Q(\Omega)$. We further point out that as a consequence of Proposition 2.1, Definition 2.6 is well posed since

$$\left|\iint_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)}{|x-y|^{n+s}}(w(x)-w(y))dx\,dy\right| \leqslant \iint_{\mathbb{R}^{2n}} \frac{|w(x)-w(y)|}{|x-y|^{n+s}}dx\,dy$$
$$= [w]_{W^{s,1}(\Omega)} + 2\int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|w(x)|}{|x-y|^{n+s}}dy\right) dx \leqslant C\|w\|_{W^{s,1}(\Omega)} < +\infty,$$

for all $w \in W^{s,1}(\Omega)$.

3. Existence of minimizers and of weak solutions when $f \in L^{\frac{n}{s}}(\Omega)$

In this section we prove the existence of minimizers and weak solutions (and their equivalence) when the $L^{\frac{n}{s}}(\Omega)$ norm of f is sufficiently small. Contextually, we prove that as $p \to 1$, the sequence of (s_p, p)-minimizers converges to an $(s, 1)$-minimizer.

We point to the definition of s_p in (1.6) and prove first the following continuous embedding.

Proposition 3.1. *Let $u \in \mathcal{W}_0^{s_p,p}(\mathbb{R}^n)$. It holds that*

$$[u]_{W^{s,1}(\mathbb{R}^n)} \leqslant C_{n,s,\Omega}^{\frac{1-p}{p}} [u]_{W^{s_p,p}(\mathbb{R}^n)},$$

with $C_{n,s,\Omega} > 0$.

Proof. By employing Hölder, we have that

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+s_p p}}\, dxdy \geqslant [u]^p_{W^{s,1}(\Omega)} |\Omega|^{2(1-p)}. \tag{3.1}$$

Denote for all $x \in \Omega$, $B_\Omega(x) := B_{\mathrm{diam}(\Omega)}(x)$ and notice that $\mathcal{C}B_\Omega(x) \subset \mathcal{C}\Omega$. We have that

$$\int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(x)|^p}{|x-y|^{n+s_p p}}\, dy \right) dx = \int_\Omega \left(\int_{B_\Omega(x) \setminus \Omega} \frac{|u(x)|^p}{|x-y|^{n+s_p p}}\, dy \right) dx + \int_\Omega \left(\int_{\mathcal{C}B_\Omega(x)} \frac{|u(x)|^p}{|x-y|^{n+s_p p}}\, dy \right) dx.$$

Using two times Hölder, for a fixed $x \in \Omega$,

$$\int_{B_\Omega(x)\setminus\Omega} \frac{|u(x)|}{|x-y|^{n+s}}\, dy \leqslant \left(\int_{B_\Omega(x)\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{(n+s)p}}\, dy \right)^{\frac{1}{p}} |B_\Omega(x) \setminus \Omega|^{\frac{p-1}{p}},$$

and

$$\begin{aligned}
\int_\Omega \left(\int_{B_\Omega(x)\setminus\Omega} \frac{|u(x)|}{|x-y|^{n+s}}\, dy \right) dx &\leqslant \int_\Omega \left(\int_{B_\Omega(x)\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{(n+s)p}}\, dy \right)^{\frac{1}{p}} |B_\Omega(x) \setminus \Omega|^{\frac{p-1}{p}}\, dx \\
\leqslant \left[\int_\Omega \left(\int_{B_\Omega(x)\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{(n+s)p}}\, dy \right) dx \right]^{\frac{1}{p}} &\left(\int_\Omega |B_\Omega(x) \setminus \Omega|\, dx \right)^{\frac{p-1}{p}} \\
= \left[\int_\Omega \left(\int_{B_\Omega(x)\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{n+s_p p}}\, dy \right) dx \right]^{\frac{1}{p}} &M_\Omega^{\frac{p-1}{p}},
\end{aligned}$$

recalling $(n+s)p = n + s_p p$ and denoting

$$M_\Omega = \int_\Omega |B_\Omega(x) \setminus \Omega|\, dx.$$

Integrating and again by Hölder,

$$\begin{aligned}
\int_\Omega \left(\int_{\mathcal{C}B_\Omega(x)} \frac{|u(x)|^p}{|x-y|^{n+s_p p}}\, dy \right) dx &= \|u\|^p_{L^p(\Omega)} \frac{\omega_n}{s_p p (\mathrm{diam}(\Omega))^{s_p p}} \\
\geqslant \|u\|^p_{L^1(\Omega)} |\Omega|^{1-p} &\frac{\omega_n}{s_p p (\mathrm{diam}(\Omega))^{s_p p}} \\
= |\Omega|^{1-p} \left(\int_\Omega |u(x)| \int_{\mathcal{C}B_\Omega(x)} \frac{dy}{|x-y|^{n+s}} \right)^p &dx \frac{\omega_n}{s_p p (\mathrm{diam}(\Omega))^{s_p p}} \left(\frac{s\, \mathrm{diam}(\Omega)^s}{\omega_n} \right)^p \\
=: |\Omega|^{1-p} \gamma^p_{n,s,p} \left(\int_\Omega |u(x)| \int_{\mathcal{C}B_\Omega(x)} \frac{dy}{|x-y|^{n+s}} \right)^p &dx
\end{aligned}$$

where

$$\gamma_{n,s,p} = (\omega_n \operatorname{diam}(\Omega)^n)^{\frac{1-p}{p}} \frac{s}{(s_p p)^{\frac{1}{p}}}. \tag{3.2}$$

We use the notations

$$\begin{aligned} A &= [u]_{W^{s,1}(\Omega)}, \quad B = \int_\Omega \left(\int_{B_\Omega(x)\setminus\Omega} \frac{|u(x)|}{|x-y|^{n+s}}\, dy \right) dx, \\ C &= \|u\|_{L^1(\Omega)} \frac{\omega_n}{s \operatorname{diam}(\Omega)^s} = \int_\Omega |u_p(x)| \left(\int_{\mathcal{C}B_\Omega(x)} \frac{dy}{|x-y|^{n+s}} \right) dx, \end{aligned} \tag{3.3}$$

and summing up the inequalities above, we have that

$$\begin{aligned} \frac{1}{2}[u]_{W^{s_p,p}(\mathbb{R}^n)} &\geqslant \frac{1}{2} A^p |\Omega|^{2(1-p)} + B^p M_\Omega^{1-p} + C^p |\Omega|^{1-p} \gamma_{n,s,p}^p \\ &\geqslant \min\left\{|\Omega|^{2(1-p)}, |\Omega|^{1-p}, M_\Omega^{1-p}\right\} \left(\frac{1}{2} A^p + B^p + C^p \gamma_{n,s,p}^p \right) \\ &\geqslant 4^{1-p} \min\left\{|\Omega|^{2(1-p)}, |\Omega|^{1-p}, M_\Omega^{1-p}\right\} \left(2^{-\frac{1}{p}} A + B + C\gamma_{n,s,p} \right)^p \\ &\geqslant 4^{1-p} \min\left\{|\Omega|^{2(1-p)}, |\Omega|^{1-p}, M_\Omega^{1-p}\right\} 2^{-p} \left(2^{-\frac{1}{p}+1} A + 2B + 2C\gamma_{n,s,p} \right)^p \\ &\geqslant 4^{1-p} \min\left\{|\Omega|^{2(1-p)}, |\Omega|^{1-p}, M_\Omega^{1-p}\right\} 2^{-p} \left(A + 2B + 2C\gamma_{n,s,p} \right)^p \end{aligned}$$

recalling that $(a+b+c)^p \leqslant 4^{p-1}(a^p + b^p + c^p)$ for all $p > 1, a, b, c \geqslant 0$ and since $2^{-1/p+1} > 1$. Noticing that

$$[u]_{W^{s,1}(\mathbb{R}^n)} = A + 2B + 2C,$$

if

$$\gamma_{n,s,p} \geqslant 1,$$

then

$$[u]^p_{W^{s_p,p}(\mathbb{R}^n)} \geqslant 8^{1-p} \min\left\{|\Omega|^{2(1-p)}, |\Omega|^{1-p}, M_\Omega^{1-p}\right\} [u]^p_{W^{s,1}(\mathbb{R}^n)},$$

otherwise if

$$\gamma_{n,s,p} < 1,$$

then

$$\begin{aligned} [u]^p_{W^{s_p,p}(\mathbb{R}^n)} &\geqslant \gamma_{n,s,p}^p 8^{1-p} \min\left\{|\Omega|^{2(1-p)}, |\Omega|^{1-p}, M_\Omega^{1-p}\right\} [u]^p_{W^{s,1}(\mathbb{R}^n)} \\ &\geqslant s^{p-1} \left(e^{\frac{n}{s}+1} \right)^{1-p} (\omega_n \operatorname{diam}(\Omega)^s)^{1-p}\, 8^{1-p} \min\left\{|\Omega|^{2(1-p)}, |\Omega|^{1-p}, M_\Omega^{1-p}\right\} [u]^p_{W^{s,1}(\mathbb{R}^n)}, \end{aligned}$$

counting on the fact that

$$\frac{\log p}{p-1} \leqslant 1, \qquad \frac{\log \frac{s_p}{s}}{p-1} = \frac{\log\left(1 + \dfrac{n(p-1)}{sp}\right)}{p-1} \leqslant \frac{n}{s}.$$

We have reached our conclusion. □

The main result of the Section is the following.

Theorem 3.2. *Let $f \in L^{\frac{n}{s}}(\Omega)$ be such that*

$$\|f\|_{L^{\frac{n}{s}}(\Omega)} \leqslant \frac{1}{2S_{n,s}}, \tag{3.4}$$

where $S_{n,s}$ is given in (2.4). Let $\{u_p\}_p \in \mathcal{W}_0^{s_p,p}(\Omega)$ be a sequence of (s_p,p)-minimizers. Then, there exists $u_1 \in \mathcal{W}_0^{s,1}(\Omega)$ such that, up to a subsequence,

$$u_p \xrightarrow[p\to 1]{} u_1 \quad \text{in } L^1(\Omega), \ \text{ a.e. in } \mathbb{R}^n \ \text{ and weakly in } L^{\frac{n}{n-s}}(\Omega). \tag{3.5}$$

Furthermore, u_1 is an $(s,1)$-minimizer and weak solution of (1.2).
If

$$\|f\|_{L^{\frac{n}{s}}(\Omega)} < \frac{1}{2S_{n,s}},$$

$u_1 = 0$ is the unique $(s,1)$-minimizer and a weak solution of (1.2).
Furthermore, it holds that

$$\lim_{p\to 1} \mathcal{F}_p^{s_p}(u_p) = \mathcal{F}_1^s(u_1). \tag{3.6}$$

Proof. We articulate the proof in five parts. We first use Proposition 3.1 to get a uniform bound (in p) of the $W^{s,1}(\Omega)$ norm of the (s_p,p)-minimizer and obtain, by compactness, the existence of u_1 in the limit as $p \to 1$. We then focus on showing that u_1 is an $(s,1)$-minimizer and a weak solution of (1.1). We easily obtain the result that $u_1 = 0$ is the unique minimizer in the case of strict inequality. In the last part, we study the pointwise limit.

Part 1. Uniform bound on the (s_p,p)-minimizer.
We apply Corollary 8.2 and obtain that there exists a unique (s_p,p)-minimizer $u_p \in \mathcal{W}_0^{s_p,p}(\Omega)$. Comparing with the null function, we have that

$$\mathcal{F}_p^{s_p}(u_p) \leqslant \mathcal{F}_p^{s_p}(0) = 0,$$

hence using Proposition 3.1 and Sobolev inequality,

$$\frac{1}{2p}[u_p]^p_{W^{s,1}(\mathbb{R}^n)} \leqslant \frac{C^{1-p}_{n,s,\Omega}}{2p}[u_p]^p_{W^{s_p,p}(\mathbb{R}^n)} \leqslant C^{1-p}_{n,s,\Omega}\int_\Omega f u_p\,dx \leqslant C^{1-p}_{n,s,\Omega} S_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega)}[u_p]_{W^{s,1}(\mathbb{R}^n)}. \tag{3.7}$$

We obtain

$$[u_p]^{p-1}_{W^{s,1}(\mathbb{R}^n)} \leqslant C^{1-p}_{n,s,\Omega}\, p\left(2S_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega)}\right),$$

and

$$[u_p]_{W^{s,1}(\mathbb{R}^n)} \leqslant \left(2S_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega)}\right)^{\frac{1}{p-1}} \frac{e}{C_{n,s,\Omega}}. \tag{3.8}$$

Since $L^1(\Omega) \subset W^{s,1}(\Omega)$ compactly, there exists $\tilde u_1 \in W^{s,1}(\Omega)$ such that up to subsequences, $u_p \to \tilde u_1$ as $p \to 1$, in L^1 norm and almost everywhere in Ω.
Furthermore, $\|u_p\|_{L^{\frac{n}{n-s}}(\Omega)}$ is uniformly bounded (by the Sobolev inequality), hence up to subsequences

$$u_p \xrightarrow[p\to 1]{} u_1$$

weakly in $L^{\frac{n}{n-s}}(\Omega)$, i.e.

$$\lim_{p\to 1}\int_\Omega f u_p\,dx = \int_\Omega f u_1\,dx. \tag{3.9}$$

We let $u_1 = \tilde u_1$ in Ω and $u_1 = 0$ in $\mathcal{C}\Omega$ and get $u_1 \in \mathcal{W}_0^{s,1}(\Omega)$ with the desired properties.

Part 2. Existence of a minimizer
We prove now that u_1 is an $(s,1)$-minimizer of $\mathcal{F}_1^s(u)$. Indeed, let $v \in \mathcal{W}_0^{s,1}(\Omega)$ be a competitor

for u. By the density of $C_c^\infty(\Omega)$ in $\mathcal{W}^{s,1}(\Omega)$, we have that there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}}$ with $\psi_j\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\psi_j \in C_c^\infty(\Omega), \qquad \lim_{j\to+\infty} \|v-\psi_j\|_{W^{s,1}(\Omega)} = 0, \qquad \lim_{j\to+\infty}(v-\psi_j) = 0 \text{ a.e. in } \mathbb{R}^n.$$

Notice that

$$\lim_{j\to+\infty}[\psi_j]_{W^{s,1}(\Omega)} = [v]_{W^{s,1}(\Omega)}, \quad \lim_{j\to+\infty}\int_\Omega f\psi_j\,dx = \int_\Omega fv\,dx, \quad \lim_{j\to+\infty}\mathcal{F}_1^s(\psi_j) = \mathcal{F}_1^s(v), \quad (3.10)$$

given that

$$\begin{aligned}\left|\int_\Omega (f\psi_j - fv)\,dx\right| &\leqslant C\|f\|_{L^{\frac{n}{\sigma}}(\Omega)}\|\psi_j - v\|_{L^{\frac{n}{n-\sigma}}(\Omega)} \leqslant C\|f\|_{L^{\frac{n}{\sigma}}(\Omega)}\|\psi_j - v\|_{L^{\frac{n}{n-s}}(\Omega)} \\ &\leqslant C[\psi_j - v]_{W^{s,1}(\mathbb{R}^n)} \leqslant C\|\psi_j - v\|_{W^{s,1}(\mathbb{R}^n)}\end{aligned}$$

and

$$\left|[\psi_j]_{W^{s,1}(\mathbb{R}^n)} - [v]_{W^{s,1}(\mathbb{R}^n)}\right| \leqslant [\psi_j - v]_{W^{s,1}(\mathbb{R}^n)} \leqslant C\|\psi - v\|_{W^{s,1}(\Omega)}$$

by Hölder's, Sobolev's inequalities and (2.1). According to Theorem 7.1, for all $j\in\mathbb{N}$,

$$\lim_{p\to 1}\mathcal{E}_p^{s_p}(\psi_j) = \mathcal{E}_1^s(\psi_j). \quad (3.11)$$

Using, in order, Fatou's lemma coupled with (3.9), the minimality of u_p, and (3.11), we have the line of inequalities

$$\mathcal{F}_1^s(u_1) \leqslant \liminf_{p\to 1}\mathcal{F}_p^{s_p}(u_p) \leqslant \limsup_{p\to 1}\mathcal{F}_p^{s_p}(u_p) \leqslant \limsup_{p\to 1}\mathcal{F}_p^{s_p}(\psi_j) = \mathcal{F}_1^s(\psi_j). \quad (3.12)$$

Now from (3.10),

$$\mathcal{F}_1^s(u) - \mathcal{F}_1^s(v) \leqslant \lim_{j\to+\infty}\mathcal{F}_1^s(\psi_j) - \mathcal{F}_1^s(v) = 0, \quad (3.13)$$

and we obtain the desired conclusion that u_1 is a minimizer of $\mathcal{F}_1^s(u)$.

Part 3. Existence of a weak solution We follow here the proof of [19, Theorem 3.4] (see also [4, Theorem 1.6 (iii)]), leaving the details to these two references. We have thanks to Theorem 8.1 that u_p is a weak solution of (1.5), hence

$$\frac{1}{2}\int_{\mathbb{R}^{2n}}\frac{|u_p(x)-u_p(y)|^{p-2}(u_p(x)-u_p(y))(w(x)-w(y))}{|x-y|^{n+s_p p}}\,dxdy = \int_\Omega fw\,dx \quad (3.14)$$

for all $w\in\mathcal{W}_0^{s_p,p}(\Omega)$. We take a sequence $\{p_k\}_k$ with $p_k\to 1$ as $k\to+\infty$ and we define

$$C_{p_k,M} = \left\{(x,y)\in\mathbb{R}^{2n}\,\middle|\,\frac{|u_{p_k}(x)-u_{p_k}(y)|^{p_k-2}(u_{p_k}(x)-u_{p_k}(y))}{|x-y|^{n+s_{p_k}p_k}} > M\right\}$$

and, thanks to the uniform bound

$$\|u_{p_k}\|_{W^{s,1}(\Omega)} \leqslant c,$$

are able to show that there exists a subsequence p_k^M and multi-valued function $\mathbf{z}\colon\mathbb{R}^{2n}\to[-1,1]$ such that $\|\mathbf{z}\|_{L^\infty(\mathbb{R}^{2n})}\leqslant 1$ and

$$\lim_{M\to+\infty}\lim_{k\to+\infty}\int_{\mathbb{R}^{2n}}\frac{|u_{p_k^M}(x)-u_{p_k^M}(y)|^{p_k^M-2}(u_{p_k^M}(x)-u_{p_k^M}(y))(w(x)-w(y))}{|x-y|^{n+s_{p_k^M}p_k^M}}\chi_{\mathbb{R}^{2n}\setminus C_{p_k^M,M}}(x,y)\,dxdy$$
$$= \int_{\mathbb{R}^{2n}}\frac{\mathbf{z}(x,y)(w(x)-w(y))}{|x-y|^{n+s}}\,dxdy,$$

together with

$$\lim_{M\to+\infty}\lim_{k\to+\infty}\int_{\mathbb{R}^{2n}}\frac{|u_{p_k^M}(x)-u_{p_k^M}(y)|^{p_k^M-2}(u_{p_k^M}(x)-u_{p_k^M}(y))(w(x)-w(y))}{|x-y|^{n+s_{p_k^M}p_k^M}}\chi_{C_{p_k^M,M}}(x,y)\,dxdy=0,$$

for all $w\in\mathcal{W}_0^{s,1}(\Omega)$. Therefore, using (3.14)

$$\frac{1}{2}\int_{\mathbb{R}^{2n}}\frac{\mathbf{z}(x,y)(w(x)-w(y))}{|x-y|^{n+s}}\,dxdy=\int_\Omega fw\,dx,$$

for all $w\in\mathcal{W}_0^{s,1}(\Omega)$. At this point, it remains to see that $\mathbf{z}$ obtained in this way satisfies

$$\mathbf{z}(x,y)\in\operatorname{sgn}(u_1(x)-u_1(y)). \tag{3.15}$$

Indeed, taking $w=u_1$, we have that

$$\frac{1}{2}\int_{\mathbb{R}^{2n}}\frac{\mathbf{z}(x,y)(u_1(x)-u_1(y))}{|x-y|^{n+s}}\,dxdy=\int_\Omega fu_1\,dx\geqslant\frac{1}{2}\int_{\mathbb{R}^{2n}}\frac{|u_1(x)-u_1(y)|}{|x-y|^{n+s}}\,dxdy,$$

given that u_1 is an $(s,1)$-minimizer and comparing with the null function. This shows (3.15) and, according to Definition 2.6, concludes the proof that u_1 is a weak solution.

Part 4. Null minimizer/weak solution. Due to the strict inequality $\|f\|_{L^{\frac{n}{s}}(\Omega)}<(2S_{n,s})^{-1}$, from (3.8) by sending $p\to1$ we obtain that

$$u_p\longrightarrow u_1=0.$$

Notice furthermore that using the Sobolev inequality,

$$\mathcal{F}_1^s(v)\geqslant[v]_{W^{s,1}(\mathbb{R}^n)}\left(\frac{1}{2}-S_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega)}\right)>0,$$

for all $v\in\mathcal{W}_0^{s,1}(\Omega)$, $v\neq0$. Thus $u_1=0$ is the unique minimizer/weak solution.

Part 5. Pointwise limit. By the density of $C_c^\infty(\Omega)$ in $\mathcal{W}^{s,1}(\Omega)$, we have that there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}}$ with $\phi_j\colon\mathbb{R}^n\to\mathbb{R}$ such that

$$\phi_j\in C_c^\infty(\Omega),\qquad\lim_{j\to+\infty}\|u-\phi_j\|_{W^{s,1}(\Omega)}=0.$$

Notice that, as in (3.10),

$$\lim_{j\to+\infty}\mathcal{F}_1^s(\phi_j)=\mathcal{F}_1^s(u_1)$$

and reasoning as in (3.12), we get that

$$\mathcal{F}_1^s(u_1)\leqslant\liminf_{p\to1}\mathcal{F}_p^{s_p}(u_p)\leqslant\limsup_{p\to1}\mathcal{F}_p^{s_p}(u_p)\leqslant\lim_{j\to+\infty}\lim_{p\to1}\mathcal{F}_p^{s_p}(\phi_j)=\lim_{j\to+\infty}\mathcal{F}_1^s(\phi_j)=\mathcal{F}_1^s(u_1)$$

and the conclusion immediatly follows. □

Remark 3.3. We discuss shortly the technical reason for which the specific fractional parameter s_p in (1.6) is needed. As a matter of fact, with the methods here employed, one cannot hope to work with the (s,p)-energy instead of the (s_p,p)-energy. The underlying justification lies in the fact that instead of (3.1), one would obtain that the $W^{s,p}(\Omega)$ semi-norm of u_p is bounded from below by the $W^{\sigma,1}(\Omega)$ semi-norm of u_p for some $\sigma\in(0,s)$ (see, in this regard, for instance [5, Lemma 3.1]). On the other hand – unless one requires higher integrability on f, i.e. that $f\in L^{\frac{n}{\sigma}}(\Omega)$ – (3.7) would read

$$\frac{1}{2p}[u_p]^p_{W^{\sigma,1}(\mathbb{R}^n)}\leqslant CS_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega)}[u_p]_{W^{s,1}(\mathbb{R}^n)}.$$

This inequality however does not allow to continue – since the $(\sigma, 1)$ seminorm is smaller then the $(s, 1)$-seminorm, [13, Proposition 2.1], and not vice-versa.

Let us point out a more general result about the equivalence between mimimizers and weak solutions, as in [4, Theorem 1.6].

Proposition 3.4. *a) Let $u \in \mathcal{W}_0^{s,1}(\Omega)$ be a weak solution of (1.1). Then u is a minimizer of $\mathcal{F}_1^s(u)$.*
b) Suppose that here exists a weak solution of (1.1). Then any minimizer of $\mathcal{F}_1^s(u)$ is a weak solution of (1.1).

Proof. a) If u is weak solution, consider $v \in \mathcal{W}_0^{s,1}(\Omega)$ any competitor for u, and use $w = u - v$ in (2.6). Using that $\mathbf{z}(x,y)(u(x) - u(y)) = |u(x) - u(y)|$ and that $\mathbf{z}(x,y)(v(x) - v(y)) \leqslant |v(x) - v(y)|$, we have

$$\frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(u(x) - u(y) - v(x) + v(y))}{|x-y|^{n+s}}\,dxdy = \int_\Omega f(u-v)\,dx,$$

hence

$$\begin{aligned}\mathcal{F}_1^s(u) &= \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|}{|x-y|^{n+s}}\,dxdy - \int_\Omega fu\,dx = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(v(x) - v(y))}{|x-y|^{n+s}}\,dxdy - \int_\Omega fv\,dx \\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|}{|x-y|^{n+s}}\,dxdy - \int_\Omega fv\,dx = \mathcal{F}_1^s(v),\end{aligned}$$

and we have the thesis.
b) If there exists $\bar{u}$ a weak solution, then consider any $w \in \mathcal{W}_0^{s,1}(\Omega)$ and it holds that there exists $\mathbf{z} \in L^\infty(\mathbb{R}^{2n})$ such that $\mathbf{z}(x,y) \in \operatorname{sgn}(\bar{u}(x) - \bar{u}(y)$ and such that

$$0 = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(w(x) - w(y))}{|x-y|^{n+s}}\,dxdy - \int_\Omega fw\,dx. \tag{3.16}$$

We want to prove that $\mathbf{z}(x,y) \in \operatorname{sgn}(u(x) - u(y))$. Let us take $w = \bar{u} - u$, and notice that

$$\begin{aligned} 0 &= \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(\bar{u}(x) - \bar{u}(y))}{|x-y|^{n+s}}\,dxdy - \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(u(x) - u(y))}{|x-y|^{n+s}}\,dxdy - \int_\Omega f(\bar{u}-u)\,dx \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|\bar{u}(x) - \bar{u}(y)|}{|x-y|^{n+s}}\,dxdy - \int_\Omega f\bar{u}\,dx - \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|}{|x-y|^{n+s}}\,dxdy + \int_\Omega fu\,dx \\ &\geqslant 0.\end{aligned}$$

Since by using (3.16) first with $w = \bar{u}$ and then with $w = u$, we have that

$$\frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|}{|x-y|^{n+s}}\,dxdy = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(u(x) - u(y))}{|x-y|^{n+s}}\,dxdy,$$

the conclusion follows. □

Putting together Theorem 3.2 and Proposition 3.4, we have that if $\|f\|_{L^{\frac{n}{s}}(\Omega)} \leqslant (2S_{n,s})^{-1}$ then u is a minimizer if and only if u is a weak solution. The interesting case here remains $\|f\|_{L^{\frac{n}{s}}(\Omega)} = (2S_{n,s})^{-1}$, since for the strict inequality, the null function is the unique minimizer and weak solution. It is expected, in the equality case, for non-null minimizers to exist, and we give an example in the subsequent Section 4.

What is more, in general, it is not expected uniqueness of solutions to hold. We give an example of this in Section 4. However, we can say something more about the "uniqueness" of the multi-valued function $\mathbf{z}$. Notice that in the proof of Proposition 3.4, it comes up that if $\bar{u}$ is a weak solution and $\mathbf{z}$ is used to verify this solution, then the same $\mathbf{z}$ can be used to verify any other weak solution. This is a actually a general fact, true in the classical framework as well, see [18, Remark 2.9].

Corollary 3.5. *Let $u_1, u_2 \in \mathcal{W}_0^{s,1}(\Omega)$ be two weak solutions of (1.1), and let $\mathbf{z}_1, \mathbf{z}_2$ be as in (2.6), (2.7). Then almost everywhere in $Q(\Omega)$,*

$$\mathbf{z}_j(x,y) \in \operatorname{sgn}(u_i(x) - u_i(y))$$

for $i, j \in \{1, 2\}$.

Proof. We observe that, since for all $i \neq j \in \{1, 2\}$

$$\mathbf{z}_i(x,y)(u_i(x) - u_i(y)) = |u_i(x) - u_i(y)|, \quad \mathbf{z}_i(x,y)(u_j(x) - u_j(y)) \leqslant |u_j(x) - u_j(y)|,$$

we have that

$$\begin{aligned} \mathbf{z}_1(x,y)\,(u_1(x) - u_1(y) - (u_2(x) - u_2(y))) &= |u_1(x) - u_1(y)| - \mathbf{z}_1(x,y)((u_2(x) - u_2(y)) \\ &\geqslant |u_1(x) - u_1(y)| - |u_2(x) - u_2(y)| \\ \mathbf{z}_2(x,y)\,(u_1(x) - u_1(y) - (u_2(x) - u_2(y))) &= \mathbf{z}_2(x,y)((u_1(x) - u_1(y)) - |u_2(x) - u_2(y)| \\ &\leqslant |u_1(x) - u_1(y)| - |u_2(x) - u_2(y)|. \end{aligned}$$

By hypothesis, we have that for all $w \in \mathcal{W}_0^{s,1}(\Omega)$,

$$\iint_{\mathbb{R}^{2n})} \frac{\mathbf{z}_1(x,y)\,(w(x) - w(y))}{|x-y|^{n+s}} dx\,dy = \iint_{\mathbb{R}^{2n})} \frac{\mathbf{z}_2(x,y)\,(w(x) - w(y))}{|x-y|^{n+s}} dx\,dy.$$

Inserting the above inequalities for $w = u_1 - u_2$, we have that

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \frac{|u_1(x) - u_1(y)| - |u_2(x) - u_2(y)|}{|x-y|^{n+s}} dx\,dy \\ \leqslant &\iint_{\mathbb{R}^{2n})} \frac{\mathbf{z}_1(x,y)\,((u_1 - u_2)(x) - (u_1 - u_2)(y))}{|x-y|^{n+s}} dx\,dy \\ = &\iint_{\mathbb{R}^{2n})} \frac{\mathbf{z}_2(x,y)\,((u_1 - u_2)(x) - (u_1 - u_2)(y))}{|x-y|^{n+s}} dx\,dy \\ \leqslant &\iint_{\mathbb{R}^{2n}} \frac{|u_1(x) - u_1(y)| - |u_2(x) - u_2(y)|}{|x-y|^{n+s}} dx\,dy. \end{aligned}$$

It follows that almost everywhere in $Q(\Omega)$

$$|u_i(x) - u_i(y)| - \mathbf{z}_j(x,y)((u_i(x) - u_i(y)) = 0$$

for $i, j \in \{1, 2\}$, and we have achieved our conclusion. □

4. Necessary and sufficient conditions for existence when f is non-negative

We provide in this section necessary and sufficient conditions for the existence of non-negative minimizers of $\mathcal{F}_1^s$, when f is non-negative. We further provide a sharp result on the asymptotics as $p \to 1$ of solutions of (1.5). These sharp conditions are given in terms of the "weighted Cheegar constant".

We focus on the case

$$f \geqslant 0, \qquad f \in L^{\frac{n}{\sigma}}(\Omega) \tag{4.1}$$

for some $\sigma \in (0, s)$, and such that there is $r_o > 0$ and $x_o \in \Omega$ such that $B_{r_0} := B_{r_o}(x_o) \subset \Omega$ and

$$\int_{B_{r_o}} f(x)\, dx > 0. \tag{4.2}$$

Alternative conditions are $f \in L^\infty(\Omega)$ with $f > 0$ almost everywhere in Ω. Each of these alternatives admits the case $f = 1$, treated in [7] in the classical case. We remark that we are not able to cover the entire class $f \in L^{\frac{n}{s}}(\Omega)$ (see Remark 4.5). We point out that our result is new, even in the constant case $f = 1$.

We first notice that for $f \geqslant 0$, if $u \in \mathcal{W}^{s,1}_0(\Omega)$ then $\mathcal{F}^s_1(u_+) \leqslant \mathcal{F}^s_1(u)$ with $u_+ = \min\{u, 0\}$, that is, minimizers are non-negative. It is not restrictive then to look for non-negative minimizers of the energy, i.e.

$$\mathcal{F}^s_1(u) \leqslant \mathcal{F}^s_1(v) \qquad \text{for all } v \in \mathcal{W}^{s,1}_0(\Omega) \text{ such that } v \geqslant 0.$$

We observe a very important homogeneity feature of our energy.

Remark 4.1. If there exists $v \in \mathcal{W}^{s,1}_0(\Omega)$ such that $\mathcal{F}^s_1(v) < 0$, then for any $\lambda > 0$,

$$\lim_{\lambda\to+\infty} \mathcal{F}^s_1(\lambda v) = \lim_{\lambda\to+\infty} \lambda \mathcal{F}^s_1(v) = -\infty,$$

i.e. our energy is unbounded from below in the space $\mathcal{W}^{s.1}_0(\Omega)$, and a global minimizer does not exist. Therefore, to assure the existence of a minimizer, it has to hold that

$$\mathcal{F}^s_1(v) \geqslant 0 \qquad \text{for all } v \in \mathcal{W}^{s,1}_0(\Omega).$$

It follows that, if Ω is such that there exists a minimizer in $\mathcal{W}^{s,1}_0(\Omega)$ then

$$\mathcal{F}^s_1(u) = 0$$

by comparing with the null function. Moreover, if $u \neq 0$, then for all $\lambda \in \mathbb{R}$

$$\mathcal{F}^s_1(\lambda u) = \lambda \mathcal{F}^s_1(u) = 0,$$

and $\mathcal{F}^s_1$ has an infinite number of minimizers.

To obtain to the results in this section, we connect non-negative minimizers of $\mathcal{F}^s_1$ with sets $E \subset \Omega$ that minimize

$$\mathcal{P}_s(E) := \mathrm{Per}_s(E) - |E|_f,$$

where we recall the fractional perimeter in (1.7), that

$$|E|_f = \int_E f\, dx,$$

and that

$$\mathcal{F}^s_1(\chi_E) = \mathcal{P}_s(E). \tag{4.3}$$

We also point out that by Tonelli

$$\begin{aligned}\int_\Omega f(x)u(x)\, dx &= \int_\Omega f(x) \left(\int_{(0,+\infty)} \chi_{\{u(x)\geqslant t\}}(t)\, dt \right) dx = \int_{(0,+\infty)} \left(\int_\Omega f(x)\chi_{\{u\geqslant t\}}(x)\, dx \right) dt \\ &= \int_{(0,+\infty)} |\{u \geqslant t\}|_f\, dt,\end{aligned}$$

and that by the co-area formula [23], for $u \in \mathcal{W}_0^{s,1}(\Omega)$ it holds

$$\frac{1}{2}[u]_{W^{s,1}(\mathbb{R}^n)} = \int_{(0,+\infty)} \mathrm{Per}_s(\{u \geqslant t\})\, dt,$$

hence

$$\mathcal{F}_1^s(u) = \int_{(0,+\infty)} \mathcal{P}_s(\{u \geqslant t\})\, dt. \tag{4.4}$$

We make some notes on sets $E \subset \Omega$ of finite fractional perimeter that realize

$$\mathcal{P}_s(E) \leqslant \mathcal{P}_s(F) \quad \text{for all } F \subset \Omega,$$

proving at first existence for all $\Omega \subset \mathbb{R}^n$.

Proposition 4.2. *Let $f \geqslant 0$, $f \in L^1(\Omega)$. For any bounded open set $\Omega \subset \mathbb{R}^n$, there exists $E \subset \Omega$ minimizer of $\mathcal{P}_s$. Furthermore, E is of finite fractional perimeter.*

Proof. We proceed by direct methods. We have that

$$\mathcal{P}_s(E) \geqslant -\|f\|_{L^1(\Omega)},$$

so there exists a minimizing sequence

$$m := \inf_{F\subset\Omega} \mathcal{P}_s(F) = \lim_{k\to+\infty} \mathcal{P}_s(E_k)$$

and notice that $m + \|f\|_{L^1(\Omega)} \geqslant 0$. There exists some $\bar{k} > 0$ such that for all $k \geqslant \bar{k}$

$$\mathcal{P}_s(E_k) \leqslant m + 1,$$

thus

$$\mathrm{Per}_s(E_k) \leqslant m + 1 + \|f\|_{L^1(\Omega)} \qquad \text{and} \qquad \|\chi_{E_k}\|_{W^{s,1}(\Omega)} \leqslant m + 1 + \|f\|_{L^1(\Omega)}.$$

By compactness of $L^1(\Omega)$ in $W^{s,1}(\Omega)$ there exists some set $E \subset \Omega$ of finite perimeter such that up to subsequences

$$\chi_{E_k} \xrightarrow[k\to+\infty]{} \chi_E \text{ in } L^1(\Omega) \text{ and a.e. in } \mathbb{R}^n.$$

By Fatou's lemma and the dominated convergence theorem,

$$m \geqslant \mathcal{P}_s(E),$$

hence E is a minimizer of $\mathcal{P}_s$. Also, there is some $B_{r_o} \subset \Omega$, and

$$\mathrm{Per}_s(E) - |\Omega|_f \leqslant \mathrm{Per}_s(E) - |E|_f = \mathcal{P}_s(E) \leqslant \mathcal{P}_s(B_{r_o}) = \mathrm{Per}_s(B_{r_o}) - |B_{r_o}|_f \leqslant \mathrm{Per}_s(B_{r_o}) < +\infty.$$

This concludes the proof of the proposition. □

Notice that if there exists some set $E \subset \Omega$ that minimizes $\mathcal{P}_s$ such that $\mathcal{P}_s(E) < 0$ then from (4.3) and Remark 4.1, $\inf_{u\in\mathcal{W}_0^{s,1}(\Omega)} \mathcal{F}_1^s(u) = -\infty$. We focus on the connection between minimial sets of $\mathcal{P}_s$ and minimizers of $\mathcal{F}_1^s$.

Proposition 4.3. *Let $f \geqslant 0$, $f \in L^1(\Omega)$.*
(i) Let $E \subset \Omega$ be such that $\chi_E \in \mathcal{W}_0^{s,1}(\Omega)$ is a minimizer of $\mathcal{F}_1^s$, then E is a minimizer of $\mathcal{P}_s$.
(ii) Let $E \subset \Omega$ be a minimizer of $\mathcal{P}_s$ such that $\mathcal{P}_s(E) \geqslant 0$. Then χ_E is a non-negative minimizer of $\mathcal{F}_1^s$.

Proof. (i) For all $F \subset \Omega$ – and we assume without loss of generality that F has finite fractional perimeter –, we have

$$\mathcal{P}_s(E) = \mathcal{F}_1^s(\chi_E) \leqslant \mathcal{F}_1^s(\chi_F) = \mathcal{P}_s(F).$$

(ii) For any non-negative competitor $v \in \mathcal{W}_0^{s,1}(\Omega)$, we notice that by [5, Lemma 2.8], the set $\{v \geqslant t\}$ is of finite fractional perimeter. Furthermore, we have that

$$\{v \geqslant t\} \subset \Omega \qquad \text{for all } t \in (0, +\infty),$$

hence $\{v \geqslant t\}$ is a competitor for E for all $t \in (0, +\infty)$. By the minimality of E

$$0 \leqslant \mathcal{P}_s(E) \leqslant \mathcal{P}_s(\{v \geqslant t\} \qquad \text{for all } t \in (0, +\infty).$$

Now, if $t \in (0, 1]$ then $\{\chi_E \geqslant t\} = E$, while if $t \in (1, +\infty)$ then $\{\chi_E \geqslant t\} = \emptyset$, hence

$$\mathcal{P}_s(\{\chi_E \geqslant t\}) \leqslant \mathcal{P}_s(\{v \geqslant t\} \qquad \text{for all } t \in (0, +\infty).$$

Then by (4.4)

$$\mathcal{F}_1^s(\chi_E) = \int_{(0,+\infty)} \mathcal{P}_s(\{\chi_E \geqslant t\})\, dt \leqslant \int_{(0,+\infty)} \mathcal{P}_s(\{v \geqslant t\})\, dt = \mathcal{F}_1^s(v).$$

This concludes the proof. □

We recall the definition of the (s, f)-Cheegar constant in (1.8). From [2, Proposition 5.3] – with minor modifications – we have the existence of an (s, f)-Cheegar set. We insert the proof for completeness, and to clarify the role played by f in the proof of existence.

Proposition 4.4. *Let f satisfy (4.1) and (4.2). For any bounded open set $\Omega \subset \mathbb{R}^n$ there exists a (s, f)-Cheegar set $\tilde{E} \subset \Omega$ such that*

$$h_s^f(\Omega) := \frac{\mathrm{Per}_s(\tilde{E})}{|\tilde{E}|_f}.$$

Moreover, $h_s^f(\Omega), |\tilde{E}|_f, \mathrm{Per}_s(\tilde{E}) \in (0, +\infty)$.

Proof. Notice that for all $E \subset \Omega$,

$$|E|_f \leqslant |\Omega|_f \leqslant \|f\|_{L^{\frac{n}{\sigma}}(\Omega)} |\Omega|^{\frac{n-\sigma}{n}}.$$

From (4.2), there is $B_{r_o} \subset \Omega$, then $\mathrm{Per}_s(B_{r_o}) \in (0, +\infty)$ and $|B_{r_o}|_f \in (0, |\Omega|_f)$, hence

$$h_s^f(\Omega) < +\infty.$$

We let $\{E_k\}_k$ be a minimizing sequence, i.e.

$$\lim_{k \to +\infty} \frac{\mathrm{Per}_s(E_k)}{|E_k|_f} = h_s^f(\Omega) < +\infty,$$

with $|E_k|_f > 0$ for all $k \in \mathbb{N}$. Then there is some $\bar{k} \in \mathbb{N}$ such that for all $k \geqslant \bar{k}$,

$$\mathrm{Per}_s(E_k) \leqslant (h_s^f(\Omega) + 1)|E_k|_f < (h_s^f(\Omega) + 1)|\Omega|_f,$$

hence, applying (2.1),

$$\|\chi_{E_k}\|_{W^{s,1}(\Omega)} < C(h_s^f(\Omega) + 1)|\Omega|_f.$$

By the Sobolev inequality

$$|E_k|_f \leqslant \|f\|_{L^{\frac{n}{\sigma}}(\Omega)} |E_k|^{\frac{n-\sigma}{n}}, \tag{4.5}$$

and by the isoperimetric inequality we obtain

$$\left(\frac{|E_k|_f}{\|f\|_{L^{\frac{n}{\sigma}}(\Omega)}}\right)^{\frac{n-s}{n-\sigma}} \leqslant |E_k|^{\frac{n-s}{n}} \leqslant 2S_{n,s}\operatorname{Per}_s(E_k) \leqslant 2S_{n,s}(h_s^f(\Omega)+1)|E_k|_f.$$

This yields that

$$|E_k|_f \geqslant c_{n,s,\sigma,\Omega} > 0. \tag{4.6}$$

By compactness of $W^{s,1}(\Omega)$ in $L^1(\Omega)$, there is $\tilde{E} \subset \Omega$ such that $\chi_{E_k} \to \chi_{\tilde{E}}$ in $L^1(\Omega)$ and almost everywhere in $\mathbb{R}^n$. By (4.6) and the dominated convergence theorem we have that

$$\lim_{k\to+\infty} |E_k|_f = |\tilde{E}|_f > 0.$$

Also by Fatou's lemma

$$m \geqslant \mathcal{P}_s(\tilde{E})$$

and $\tilde{E}$ is the minimizing set. Notice also that by the isoperimetric inequality and the fact that $h_s^f(\Omega) < +\infty$ we get that $\operatorname{Per}_s(\tilde{E}) \in (0,+\infty)$, and finally that $h_s^f(\Omega) \in (0,+\infty)$. ☐

Remark 4.5. Notice the necessity of taking $f \in L^{\frac{n}{\sigma}}(\Omega)$ instead of $f \in L^{\frac{n}{s}}(\Omega)$ in (4.5). The larger class would not allow to obtain a uniform lower bound for $|E_k|_f$.

Remark 4.6. When $f \in L^\infty(\Omega)$ with $f > 0$ almost everywhere, one could use also the definition

$$h_s^f(\Omega) := \inf\left\{ \frac{\operatorname{Per}_s(A)}{|A|_f} \,\middle|\, A \subset \Omega,\ |A| > 0 \right\}.$$

It holds that $|E|_f > 0$ for all $|E| > 0$, while the uniform bound (4.6) follows by using

$$|E_k|_f \leqslant \|f\|_{L^\infty(\Omega)}|E_k|$$

instead of (4.5).

Notice furthermore that for all $E \subset \Omega$,

$$\operatorname{Per}_s(E) - h_s^f(\Omega)|E|_f \geqslant 0, \qquad \text{and} \qquad \operatorname{Per}_s(\tilde{E}) - h_s^f(\Omega)|\tilde{E}|_f = 0,$$

hence $\tilde{E}$ - the (s,f)-Cheegar set, minimizes $\operatorname{Per}_s(E) - h_s^f(\Omega)|E|_f$ among all sets contained in Ω.

Our sharp existence result if the following.

Theorem 4.7. *Let f satisfy* (4.1) *and* (4.2).
If $h_s^f(\Omega) > 1$ then the null function is the unique minimizer of $\mathcal{F}_1^s$.
If $h_s^f(\Omega) = 1$ there exists a non-negative minimizer of $\mathcal{F}_1^s$.
If $h_s^f(\Omega) < 1$ then

$$\inf_{u\in\mathcal{W}_0^{s,1}(\Omega)} \mathcal{F}_1^s(u) = -\infty.$$

Proof. If $h_s^f(\Omega) \geqslant 1$, then

$$\mathcal{P}_s(A) = \operatorname{Per}_s(A) - |A| \geqslant \operatorname{Per}_s(A) - h_s^f(\Omega)|A| \geqslant 0.$$

for all $A \subset \Omega$. Let E be a minimizer of $\mathcal{P}_s$, then $\mathcal{P}_s(E) \geqslant 0$. According to Proposition 4.3 (ii), the function χ_E is a minimizer of $\mathcal{F}_1^s$. Hence, if the inequality is strict, $E = \emptyset$ is the unique minimal set of $\mathcal{P}_s$, hence the unique minimizer of $\mathcal{F}_1^s$ is the null function, otherwise

any $\lambda\chi_E$ is a minimizer (see Remark 4.1).
If $h_s^f(\Omega) < 1$, then

$$\mathcal{F}_1^s(\chi_{\tilde{E}}) = \mathrm{Per}_s(\tilde{E}) - |\tilde{E}|_f < 0,$$

where $\tilde{E}$ is an (s, f)-Cheegar set. The conclusion follows from Remark 4.1. □

We complement the existence result with the study of the asymptotics. Notice that from Corollary 8.2, there is a unique minimizer of $\mathcal{F}_p^{s_p}$, since $n/\sigma > n/s$.

Theorem 4.8. *Let f satisfy (4.1), (4.2) and u_p be the unique minimizer of $\mathcal{F}_p^{s_p}$. If $h_s^f(\Omega) \geqslant 1$ then*

$$u_p \xrightarrow[p\to 1]{} u_1,$$

where u_1 is a minimizer of $\mathcal{F}_1^s$. Furthermore, u_1 is a weak solution of (1.1). If $h_s(\Omega) < 1$ then

$$\lim_{p\to 1} [u_p]_{W^{s_p,p}(\mathbb{R}^n)} = +\infty$$

Proof. We look first at the case $h_s^f(\Omega) \geqslant 1$. Notice that $u_p \in \mathcal{W}_0^{s_p,p}(\mathbb{R}^n)$ implies that $u_p \in \mathcal{W}_0^{s,1}(\mathbb{R}^n)$ (see (7.2)) hence for all $t \in (0, +\infty)$, $\{u_p \geqslant t\}$ is of finite fractional perimeter. By the definition of the Cheegar constant and the existence of a (s, f)-Cheegar set we have

$$\mathrm{Per}_s(\{u_p \geqslant t\}) \geqslant h_s^f(\Omega)|\{u_p \geqslant t\}|_f$$

and integrating, by the co-area formula (4.4), using that u_p minimizes $\mathcal{F}_p^{s_p}$ (so $\mathcal{F}_p^{s_p}(u_p) \leqslant 0$) and Proposition 3.1,

$$\begin{aligned}\frac{1}{2}[u_p]_{W^{s,1}(\mathbb{R}^n)} &\geqslant h_s^f(\Omega)\int_\Omega f u_p\,dx \geqslant h_s^f(\Omega)\frac{1}{2p}[u_p]^p_{W_{s_p,p}(\mathbb{R}^n)} \\ &\geqslant h_s^f(\Omega)\frac{1}{2p}[u_p]^p_{W_{s,1}(\mathbb{R}^n)} C^{p-1}_{n,s,\Omega}.\end{aligned}$$

Hence

$$[u_p]_{W^{s,1}(\mathbb{R}^n)} \leqslant \left(\frac{p}{h_s^f(\Omega)}\right)^{\frac{1}{p-1}} \frac{1}{C_{n,s,\Omega}} \leqslant \frac{e}{C_{n,s,\Omega}}, \tag{4.7}$$

and from (2.1),

$$\|u_p\|_{W^{s,1}(\mathbb{R}^n)} \leqslant C$$

with $C > 0$ independent of p. By compactness, and reasoning as in (3.12) and (3.13) we obtain that u_p converges to u_1, minimizer of $\mathcal{F}_1^s$. The fact that u_1 is also a weak solution follows as in Part 3 of the proof of Theorem 3.2. That $u_1 = 0$ when $h_s^f(\Omega) > 1$ is clear either from Theorem 4.7 or sending $p \to 1$ in (4.7). Observe also that we obtain

$$\limsup_{p\to 1} [u_p]^{p-1}_{W_{s_p,p}(\mathbb{R}^n)} \leqslant \frac{1}{h_s^f(\Omega)}. \tag{4.8}$$

We consider now the case $h_s^f(\Omega) < 1$. Denote $\tilde{E}$ a (s, f)-Cheegar set. We first notice, as in [2, Remark 5.4], that $\partial\tilde{E}$ has to touch the boundary $\partial\Omega$. Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \to 1^-$ as $n \to +\infty$ and let $E_k = t_k\tilde{E}$. Then $\overline{E_k} \subset \Omega$, $\chi_{E_k} \to \chi_E$ as $k \to +\infty$ and by the dominated convergence theorem

$$\lim_{k\to\infty} |E_k|_f = |\tilde{E}|_f.$$

Since $|\tilde{E}|_f \in (0, |\Omega|_f]$ from Proposition 4.4, for k large enough $|E_k|_f \in (0, |\Omega|_f + 1)$. We also have that

$$\frac{\mathrm{Per}_s(E_k)}{|E_k|_f} = t_k^{-s} \frac{\mathrm{Per}_s(\tilde{E})}{|\tilde{E}|_f} \xrightarrow[k\to+\infty]{} \frac{\mathrm{Per}_s(\tilde{E})}{|\tilde{E}|_f} = h_s^f(\Omega) \in (0, +\infty), \tag{4.9}$$

for k large enough also $\mathrm{Per}_s(E_k) \in (0, +\infty)$, thus $\chi_{E_k} \in \mathcal{W}_0^{s,1}(\Omega)$. By density, for any k large enough, there exists $\{\varphi_j^k\}_{j\in\mathbb{N}} \in C_0^\infty(\Omega)$ such that

$$\lim_{j\to+\infty} \|\varphi_j^k - \chi_{E_k}\|_{W^{s,1}(\Omega)} = 0.$$

It also follows that

$$\lim_{j\to+\infty} \int_\Omega f\varphi_j^k \, dx = |E_k|_f \in (0, |\Omega|_f + 1), \qquad \lim_{j\to+\infty} \frac{1}{2}[\varphi_j^k]_{W^{s,1}(\mathbb{R}^n)} = \mathrm{Per}_s(E_k) \in (0, +\infty), \tag{4.10}$$

reasoning as in (3.10). Hence for k, j large enough

$$\int_\Omega f\varphi_j^k \, dx \in (0, +\infty), \qquad [\varphi_j^k]_{W^{s,1}(\mathbb{R}^n)} \in (0, +\infty).$$

From Proposition 3.1 and the fact that $\varphi_j^k \in C_0^\infty(\Omega)$, also $[\varphi_j^k]_{W^{s_p,p}(\mathbb{R}^n)} \in (0, +\infty)$. We define, following an idea from [6, Lemma 1]

$$c_{p,j,k}^{p-1} = \left(2p - \frac{1}{j}\right) \frac{\displaystyle\int_\Omega f\varphi_j^k \, dx}{[\varphi_j^k]^p_{W^{s_p,p}(\mathbb{R}^n)}} \in (0, +\infty). \tag{4.11}$$

Now, let u_p be the minimizer of $\mathcal{F}_p^{s_p}$ and weak solution of (1.5). Then

$$\int_\Omega f u_p \, dx = \frac{1}{2}[u_p]^p_{W^{s_p,p}(\mathbb{R}^n)},$$

hence

$$\frac{1}{2p}[u_p]^p_{W^{s_p,p}(\mathbb{R}^n)} - \int_\Omega f u_p \, dx = \frac{1-p}{2p}[u_p]^p_{W^{s_p,p}(\mathbb{R}^n)} \leqslant \mathcal{F}_p^{s_p}(\phi) = \frac{1}{p}\left(\frac{1}{2}[\phi]^p_{W^{s_p,p}(\mathbb{R}^n)} - p\int_\Omega f\phi \, dx\right)$$

for all $\phi \in C_0^\infty(\Omega)$, by minimality. This yields that

$$[u_p]^p_{W^{s_p,p}(\mathbb{R}^n)} \geqslant \frac{1}{p-1}\left(2p\int_\Omega f\phi \, dx - [\phi]^p_{W^{s_p,p}(\mathbb{R}^n)}\right).$$

Let $\phi = c_{p,j,k}\varphi_j^k \in C_0^\infty(\Omega)$. Then

$$[u_p]^p_{W^{s_p,p}(\mathbb{R}^n)} \geqslant \frac{c_{p,j,k}}{p-1}\left(2p\int_\Omega f\varphi_j^k \, dx - c_{p_j}^{p-1}[\varphi_j^k]^p_{W^{s_p,p}(\mathbb{R}^n)}\right) = \frac{c_{p,j,k}}{j(p-1)}\int_\Omega f\varphi_{j,k} \, dx.$$

It follows that

$$[u_p]^{p-1}_{W^{s_p,p}(\mathbb{R}^n)} \geqslant c_{p,j,k}^{\frac{p-1}{p}}\left(\frac{1}{j}\int_\Omega f\varphi_{j,k} \, dx\right)^{\frac{p-1}{p}}\left(\frac{1}{p-1}\right)^{\frac{p-1}{p}},$$

so

$$\liminf_{p\to1}[u_p]^{p-1}_{W^{s_p,p}(\mathbb{R}^n)} \geqslant \lim_{p\to1} c_{p,j,k}^{p-1} = \left(2 - \frac{1}{j}\right)\frac{\displaystyle\int_\Omega f\varphi_j^k \, dx}{[\varphi_j^k]_{W^{s,1}(\mathbb{R}^n)}},$$

where we have used (7.5). We send first $j \to \infty$ and then $k \to \infty$, make use of (4.10) and (4.9) and get that

$$\liminf_{p\to 1} [u_p]^{p-1}_{W^{s_p,p}(\mathbb{R}^n)} \geqslant \frac{|\tilde{E}|_f}{\mathrm{Per}_s(\tilde{E})} = \frac{1}{h^f_s(\Omega)},$$

and together with (4.8),

$$\lim_{p\to 1} [u_p]^{p-1}_{W^{s_p,p}(\mathbb{R}^n)} = \frac{1}{h^f_s(\Omega)}.$$

For an arbitrary $\varepsilon \in (0, 1 - h^f_s(\Omega))$ there is some $\bar{p}$ close to 1 such that

$$[u_p]_{W^{s_p,p}(\mathbb{R}^n)} > \left(h^f_s(\Omega) + \varepsilon\right)^{\frac{1}{p-1}},$$

hence

$$\lim_{p\to 1} [u_p]_{W^{s_p,p}(\mathbb{R}^n)} = +\infty$$

and we conclude the proof. □

We remark that the limit to infinity obtained when $h^f_s(\Omega) < 1$ excludes that u_p might tend to a $W^{s,1}$ function.

Corollary 4.9. *Let f satisfy (4.1), (4.2) and u_p be the unique minimizer of $\mathcal{F}^{s_p}_p$. There does not exist $u_1 \in \mathcal{W}^{s,1}_0(\Omega)$ such that*

$$u_p \xrightarrow[p\to 1]{} u_1 \quad \text{in } L^1(\Omega) \quad \text{and weakly in } L^{\frac{n}{n-\sigma}}(\Omega).$$

Proof. Suppose by contradiction that such a function u_1 exists. Employing (3.6), and noticing that

$$\lim_{p\to 1} \int_\Omega f u_p \, dx = \int_\Omega f u_1 \, dx$$

we obtain

$$\lim_{p\to 1} [u_p]_{W^{s_p,p}(\Omega)} = [u_1]_{W^{s,1}(\Omega)}.$$

The contradiction immediately follows. □

Corollary 4.10. *Let f satisfy (4.1), (4.2) and*

$$h^f_s(\Omega) \geqslant 1.$$

Let $E \subset \Omega$ be a minimizer of $\mathcal{P}_s$ such that $\mathcal{P}_s(E) \geqslant 0$. Then in a weak sense

$$(-\Delta)^s_1 \chi_E = f.$$

Proof. In our hypothesis, Theorem 4.8 gives the existence of a weak solution of (1.1). Then any minimizer is also a weak solution according to Proposition 3.4. That χ_E is a weak solution follows from Proposition 4.3 ii). □

We provide contextually a characterization of the weighted fractional Cheegar constant, similar to that of [8].

Corollary 4.11. *Let f satisfy (4.1), (4.2) and u_p be the unique minimizer of $\mathcal{F}_p^{s_p}$. Then*

$$\lim_{p\to 1}[u_p]_{W^{s_p,p}(\mathbb{R}^n)}^{p-1} = \frac{1}{h_s^f(\Omega)}$$

and

$$\lim_{p\to 1}\left(\int_\Omega f u_p\,dx\right)^{p-1} = \frac{1}{h_s^f(\Omega)}.$$

Proof. The conclusion is obtined by noticing that u_p is also a weak solution, thus

$$\int_\Omega f u_p\,dx = \frac{1}{2}[u_p]^p_{W^{s_p,p}(\Omega)}.$$

□

As further observation, we have the following result, mimicking [5, Theorem 1.3] and saying basically that – under some additional assumptions – u is a minimizer of $\mathcal{F}_1^s$ if and only if every level set of u is a minimizer for $\mathcal{P}_s$.

Proposition 4.12. *(i) Let $h_s^f(\Omega) \geqslant 1$ and let $u \in \mathcal{W}_0^{s,1}(\Omega)$. If for almost all $t \in (0,+\infty)$, the set $\{u \geqslant t\}$ is a minimizer of $\mathcal{P}_s$ then u is a non-negative minimizer of $\mathcal{F}_1^s$.*
(ii) There exists a weak solution of $\mathcal{F}_1^s$. If $u \in \mathcal{W}_0^{s,1}(\Omega)$ is a non-negative minimizer of $\mathcal{F}_1^s$, the set $\{u \geqslant t\}$ is a minimizer of $\mathcal{P}_s$.

Proof. (i) Let

$$Z := \{t \in (0,+\infty) \,|\, \text{ the set } \{u \geqslant t\} \text{ is a minimizer of } \mathcal{P}_s\}$$

and it holds that $\mathcal{L}^1(Z) = 0$. We have that, for any $v \in \mathcal{W}_0^{s,1}(\Omega)$ and $t \in (0,+\infty)\setminus Z$,

$$0 \leqslant \mathcal{P}_s(\{u \geqslant t\}) \leqslant \mathcal{P}_s(\{v \geqslant t\}),$$

where the non-negativity follows from the existence of a minimizer of $\mathcal{F}_1^s$. By the coarea formula

$$\mathcal{F}_1^s(u) = \int_{(0,+\infty)} \mathcal{P}_s(\{u \geqslant t\})\,dt = \int_{(0,+\infty)\setminus Z} \mathcal{P}_s(\{u \geqslant t\})\,dt$$
$$\leqslant \int_{(0,+\infty)\setminus Z} \mathcal{P}_s(\{v \geqslant t\})\,dt = \mathcal{F}_1^s(v),$$

hence u is a minimizer.
(ii) According to Proposition 3.4, $u \in \mathcal{W}_0^{s,1}$ is also a weak solution. We use here an idea from [20]. Recalling that $\chi_{\{u\geqslant t\}} \in \mathcal{W}_0^{s,1}(\Omega)$ thanks to [5, Lemma 2.8] and to (4.3), and picking any $F \subset \Omega$ of finite fractional perimeter, hence $\chi_F \in \mathcal{W}_0^{s,1}(\Omega)$, we can use them as test functions in the definition of weak solution, i.e. it holds that

$$\int_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(\chi_F(x)-\chi_F(y))}{|x-y|^{n+s}}\,dxdy - \int_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(\chi_{\{u\geqslant t\}}(x)-\chi_{\{u\geqslant t\}}(y))}{|x-y|^{n+s}}\,dxdy$$
$$= \int_\Omega f(x)(\chi_F - \chi_{\{u\geqslant t\}})(x)\,dx.$$

Since

$$\mathbf{z}(x,y)(u(x)-u(y)) = |u(x)-u(y)|,$$

and

$$u(x)-u(y) = \int_0^{+\infty} \big(\chi_{\{u\geqslant t\}}(x) - \chi_{\{u\geqslant t\}}(x)\big)\,dt,$$

together with

$$|u(x) - u(y)| = \int_0^{+\infty} |\chi_{\{u\geqslant t\}}(x) - \chi_{\{u\geqslant t\}}(x)|dt,$$

then

$$\mathbf{z}(x,y) \left(\chi_{\{u\geqslant t\}}(x) - \chi_{\{u\geqslant t\}}(x)\right) = |\chi_{\{u\geqslant t\}}(x) - \chi_{\{u\geqslant t\}}(x)|$$

for almost all $t \in (0, +\infty)$. Also

$$\mathbf{z}(x,y)(\chi_F(x) - \chi_F(y)) \leqslant |\chi_F(x) - \chi_F(y)|,$$

and we obtain that

$$\int_\Omega f(x)(\chi_F - \chi_{\{u\geqslant t\}})(x)\,dx \leqslant \int_{\mathbb{R}^{2n}} \frac{|\chi_F(x) - \chi_F(y)|}{|x-y|^{n+s}}\,dxdy - \int_{\mathbb{R}^{2n}} \frac{|\chi_{\{u\geqslant t\}}(x) - \chi_{\{u\geqslant t\}}(y)|}{|x-y|^{n+s}}dxdy,$$

hence

$$\mathcal{P}_s(\{u \geqslant t\}) \leqslant \mathcal{P}_s(F),$$

and the proof is concluded. □

5. Examples of non-existence and non-uniqueness when $f = 1$

In this section we take a closer look at the torsion problem, i.e. (1.1) with $f = 1$ and denote

$$\mathcal{J}_s(u, \Omega) = \frac{1}{2}[u]_{W^{s,1}(\mathbb{R}^n)} - \int_\Omega u\,dx. \tag{5.1}$$

We give examples of non-uniqueness encompassing open questions from both Section 3 and 4. Precisely, when $h_s(\Omega) = 1$ coinciding with $|\Omega|^{\frac{s}{n}} = 1/(2S_{n,s})$, we give examples of non-uniqueness of minimizers. When $h_s(\Omega) > 1$ coinciding with $|\Omega|^{\frac{s}{n}} > 1/(2S_{n,s})$ we provide examples of non-existence. In both cases, the example is provided by considering a ball B_R large enough, and relying on the isoperimetric inequality (2.5) and on s-calibrable sets, that we define. A set Ω is said to be s-calibrable if it is the s-Cheegar set of itself. It is known , see [2, Remark 5.2], that the ball is such a set, i.e.

$$h_s(B_R) = \frac{\mathrm{Per}_s(B_R)}{|B_R|} = \inf_{A\subset B_R} \frac{\mathrm{Per}_s(A)}{|A|}.$$

Owing to the Faber-Krahn inequality (see [2, Proposition 5.5.]), it holds that

$$h_s(\Omega) \geqslant |\Omega|^{-\frac{s}{n}}|B_1|^{-\frac{s}{n}}h_s(B_1) = |\Omega|^{-\frac{s}{n}}\frac{1}{2S_{n,s}}, \tag{5.2}$$

hence $|\Omega|^{-\frac{s}{n}}\frac{1}{2S_{n,s}} \geqslant 1$ implies $h_s(\Omega) \geqslant 1$.
We were not able to prove – or disprove – if there exists Ω such that

$$h_s(\Omega) \geqslant 1, \qquad \text{and } |\Omega|^{\frac{s}{n}} > 1/(2S_{n,s}),$$

i.e., if minimizers exist but the bound of Theorem 3.2 does not hold.

In case of equality, we can give an example of a non-trivial minimizer, and at the same time, of non-uniqueness of minimizers.

Proposition 5.1. *Let $\Omega = B_R$ be such that*

$$h_s(B_R) = 1.$$

Then any $u \in \{\lambda\chi_{B_R} \,|\, \lambda \geqslant 0\}$ is a non-negative minimizer of $\mathcal{J}_s(\cdot, B_R)$.

Proof. If $h_s(B_R) = 1$ and since B_R is s-calibrable,

$$1 = \frac{\mathrm{Per}_s(B_R)}{|B_R|},$$

and this happens if

$$R = (2S_{n,s})^{-\frac{1}{s}} |B_1|^{-\frac{1}{n}} = h_s(B_1)^{\frac{1}{s}}. \tag{5.3}$$

We remark that in this case

$$|B_R|^{\frac{s}{n}} = \frac{1}{2S_{n,s}}.$$

Let $u = \chi_{B_R}$ be the characteristic function of B_R. Then

$$\mathcal{J}_s(\chi_{B_R}, B_R) = \mathcal{P}_s(B_R).$$

Rescaling by R and using (2.5), we have that

$$\begin{aligned} \mathcal{J}_s(\chi_{B_R}, B_R) &= R^{n-s}\,\mathrm{Per}_s(B_1) - |B_1| R^n = R^{n-s}\frac{1}{2S_{n,s}}|B_1|^{\frac{n-s}{n}} - R^n|B_1| \\ &= R^{n-s}\frac{1}{2S_{n,s}}|B_1|^{\frac{n-s}{n}}\left(1 - 2R^s S_{n,s}|B_1|^{\frac{s}{n}}\right) = 0, \end{aligned} \tag{5.4}$$

by hypothesis. We conclude the proof recalling Remark 4.1. □

On the other hand, we have the following.

Proposition 5.2. *Let $\Omega \supset B_R$ with*

$$h_s(B_R) > 1,$$

then

$$\inf_{u \in \mathcal{W}_0^{s,1}(\Omega)} \mathcal{J}_s(u, \Omega) = -\infty.$$

Proof. Notice that $B_R \subseteq \Omega$ with R from (5.3) implies that

$$|\Omega|^{\frac{s}{n}} > |B_R| > \frac{1}{2S_{n,s}}.$$

The computations in (5.4) yield

$$\mathcal{J}_s(\chi_{B_R}, \Omega) = \mathcal{J}_s(\chi_{B_R}, B_R) < 0$$

and we conclude by homogeneity, see Remark 4.1. □

We point out that in [17], the author studies $(-\Delta)_p u = 1$ on B_R, proving that if $R \leqslant n$ then (1.9) holds, and if $R > n$ then (1.10) is in place. Notice that $h(B_1) = n$, thus our results are sharp in the fractional case, and are the counterpart of those in [17].

We can draw from [2] some other very interesting aspects of sets $E \subset \Omega$ that minimize $\mathcal{P}_s$, precisely a regularity result following from [10]. One can prove that these minimizing sets are almost minimal for the perimeter (basically, a perturbation of the set in a small ball produces produced a term which is controlled by radius of the ball to a certain power). Such sets are proved to have C^1 boundary outside of a set of singular points $\Sigma \subset \Omega$ such that $\dim_{\mathcal{H}} \Sigma \leqslant n-2$. Precisely, the authors of [2, Proposition 6.4] prove that if $\tilde{E} \subset \Omega$ is s-Cheegar set, then $\partial E \cap \Omega$ is C^1 outside of a set of Hausdorff dimension at most $n-2$. The proof remains unchanged if one analyzes minimizers of $\mathcal{P}_s$ for $f = 1$.

Further on, if $x_0 \in \partial E \subset \Omega$ is a smooth point (i.e., there exists an interior and exterior tangent ball to ∂E at x_0) then an Euler-Lagrange equation holds pointwisely at that point. To be more precise, we define

$$H_s[E](x) = \lim_{\delta\to 0} \int_{\mathbb{R}^n\setminus B_\delta(x)} \frac{|\chi_{\mathcal{C}E}(y) - \chi_E(y)|}{|x-y|^{n+s}}\, dy$$

the fractional mean curvature of ∂E at $x \in \partial E$. In [2, Theorem 6.7] it is proved that if $E \subset \Omega$ is a minimizer of $2\,\mathrm{Per}_s(E) - h_s(\Omega)|E|$, then for any any smooth point $x \in \partial E$, it holds that

$$H_s[E](x) = h_s(\Omega),$$

(we remark that the signs are opposite to those in [2] since we consider the definition of the opposite sign for the mean curvature). In exactly the same way, it follows that if E minimizes $\mathcal{P}_s$ then for at all smooth points on ∂E it holds point-wisely that

$$H_s[E](x) = 1.$$

6. "Flatness" of weak solutions and of minmizers

In [21], the authors prove (using Stampacchia's truncation method) that a weak solution of the 1-Laplacian equation with $L^n(\Omega)$ right hand side data and zero boundary data has a vanishing gradient on a set of positive Lebesgue measure (and that the same holds for a right hand side with small norm in the Marcinkiewicz space or for a BV minimizer of the associated energy).

In this section, we prove similar results for the nonlocal problem, following the lines of the proofs in [21]. We believe that the results are interesting and worth a few words, in particular to overcome the difficulties arrising from the nonlocal character of the problem.

We emphasize that when $\|f\|_{L^{\frac{n}{s}}(\Omega)} < (2S_{n,s})^{-1}$, the unique minimizer is the null function and the flatness result is obvious. However, the "flatness" still holds, independently on the size of the norm of f.

We reiterate also that, as a consequence of the results in this section, in Definition 2.6, one cannot get rid of $\mathbf{z}(x,y)$, since in general weak solutions of (1.1) for which $\mathbf{z} \in \{-1,1\}$ up to sets of measure zero (hence, when $\mathbf{z}(x,y)$ is the classical sign function of $u(x)-u(y)$) do not exist. Precisely, we have the following result.

Theorem 6.1. *Let $f \in L^{\frac{n}{s}}(\Omega)$. Let $u \in \mathcal{W}_0^{s,1}(\Omega)$ be a weak solution of (1.1). Then the set*

$$\{(x,y) \in Q(\Omega)\, u(x) = u(y)\}$$

is of positive Lebesgue measure. In other words, there exists no function $u \in \mathcal{W}_0^{s,1}(\Omega)$ such that $u(x) \neq u(y)$ for almost any $(x,y) \in Q(\Omega), x \neq y$, and such that u is a weak solution of (1.1).

Proof of Theorem 6.1. We recall that $Q(\Omega) = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$. Suppose that such a u described in the theorem exists. Then

$$\frac{1}{2}\iint_{\mathbb{R}^{2n}} \frac{\mathbf{z}(x,y)(w(v)-w(y))}{|x-y|^{n+s}} dx\, dy = \int_\Omega fw, \qquad \text{for all} \quad w \in \mathcal{W}_0^{s,1}(\Omega).$$

Since $u(x) \neq u(y)$ for almost any $(x,y) \in Q(\Omega)$, then almost everywhere in $Q(\Omega)$ we have that $\mathbf{z}(x,y)$ is the classical sign function and

$$\mathbf{z}(x,y) = \begin{cases} 1, & \text{if } u(x) > u(y), \\ -1, & \text{if } u(x) < u(y). \end{cases} \tag{6.1}$$

We define for $k > 0$,

$$G_k(t) = \begin{cases} t+k, & t < -k \\ 0, & -k \leqslant t \leqslant k \\ t-k, & t > k. \end{cases} \tag{6.2}$$

Let us denote

$$\Omega_k^1 := \{x \in \Omega \mid u(x) < -k\}, \qquad \Omega_k^2 := \{x \in \Omega \mid u(x) > k\}, \qquad \Omega_k := \Omega_k^1 \cup \Omega_k^2. \tag{6.3}$$

Notice that $G_k(u) \in \mathcal{W}_0^{s,1}(\Omega)$, since $G_k(u) = 0$ almost everywhere in $\mathcal{C}\Omega_k = (\Omega \setminus \Omega_k) \cup \mathcal{C}\Omega$, and

$$\begin{aligned} &\int_\Omega \int_\Omega \frac{|G_k(u(x)) - G_k(u(y))|}{|x-y|^{n+s}} dx\, dy \\ &= \int_{\Omega_k} \int_{\Omega_k} \frac{|G_k(u(x)) - G_k(u(y))|}{|x-y|^{n+s}} dx\, dy + 2 \int_{\Omega_k} \int_{\Omega \setminus \Omega_k} \frac{|G_k(u(x)) - G_k(u(y))|}{|x-y|^{n+s}} dx\, dy. \end{aligned}$$

Now,

$$\begin{aligned} &\int_{\Omega_k} \int_{\Omega_k} \frac{|G_k(u(x)) - G_k(u(y))|}{|x-y|^{n+s}} dx\, dy \\ &\leqslant \int_{\Omega_k^1} \int_{\Omega_k^1} \frac{|u(x) - u(y)|}{|x-y|^{n+s}} dx\, dy + \int_{\Omega_k^2} \int_{\Omega_k^2} \frac{|u(x) - u(y)|}{|x-y|^{n+s}} dx\, dy \\ &\quad + 2 \int_{\Omega_k^1} \left(\int_{\Omega_k^2} \frac{|u(x) - u(y)| + 2k}{|x-y|^{n+s}}\, dy \right) dx \leqslant 4[u]_{W^{s,1}(\Omega)} + 2 \int_{\Omega_k^1} \left(\int_{\Omega_k^2} \frac{u(y) - u(x)}{|x-y|^{n+s}}\, dy \right) dx \\ &\leqslant 6[u]_{W^{s,1}(\Omega)}, \end{aligned}$$

noting that and $k < -u(x)$ on Ω_k^1 and $k < u(y)$ for $y \in \Omega_k^2$. On the other hand

$$\begin{aligned} &\int_{\Omega_k} \left(\int_{\Omega \setminus \Omega_k} \frac{|G_k(u(x)) - G_k(u(y))|}{|x-y|^{n+s}}\, dy \right) dx = 2 \int_{\Omega_k^1} \left(\int_{\Omega \setminus \Omega_k} \frac{-u(x) - k}{|x-y|^{n+s}}\, dy \right) dx \\ &+ 2 \int_{\Omega_k^2} \left(\int_{\Omega \setminus \Omega_k} \frac{u(x) - k}{|x-y|^{n+s}}\, dy \right) dx \leqslant 2 \int_{\Omega_k^1} \left(\int_{\Omega \setminus \Omega_k} \frac{-u(x) + u(y)}{|x-y|^{n+s}}\, dy \right) dx \\ &+ 2 \int_{\Omega_k^2} \left(\int_{\Omega \setminus \Omega_k} \frac{u(x) - u(y)}{|x-y|^{n+s}}\, dy \right) dx \leqslant 4[u]_{W^{s,1}(\Omega)}, \end{aligned}$$

noting that for $y \in \Omega \setminus \Omega_k$ we have that $-k \leqslant u(y) \leqslant k$. This concludes the proof that $G_k(u) \in \mathcal{W}_0^{s,1}(\Omega)$.

What is more,

$$[G_k(u)]_{W^{s,1}(\mathbb{R}^n)} = \int_{\Omega_k} \int_{\Omega_k} \frac{|G_k(u(x)) - G_k(u(y))|}{|x-y|^{n+s}} dx\, dy + 2 \int_{\Omega_k} \left(\int_{\mathcal{C}\Omega_k} \frac{|G_k(u(x))|}{|x-y|^{n+s}}\, dy \right) dx.$$

By Hölder's inequality and the fractional Sobolev inequality in Theorem 2.2,

$$\left|\int_\Omega f(x)G_k(u(x))\,dx\right| \leqslant \|f\|_{L^{\frac{n}{s}}(\Omega_k)}\|G_k(u)\|_{L^{\frac{n}{n-s}}(\Omega_k)} \leqslant S_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega_k)}[G_k(u)]_{W^{s,1}(\mathbb{R}^n)}. \tag{6.4}$$

We also point out that, since $\mathbf{z}(x,y) \in \operatorname{sgn}(u(x)-u(y))$

$$\mathbf{z}(x,y)\left(G_k(u(x)) - G_k(u(y))\right) = |G_k(u(x)) - G_k(u(y)|$$

almost everywhere on $\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega_k)^2$. Indeed, this is obvious if $(x,y) \in (\Omega_k^1 \times \Omega_k^1) \cup (\Omega_k^2 \times \Omega_k^2)$, while, for instance, if $x \in \Omega_k^1, y \in \Omega_k^2$, since $u(y) > k > -k > u(x)$ and $\mathbf{z}(x,y) = -1$, it holds that

$$\mathbf{z}(x,y)\left(G_k(u(x)) - G_k(u(y))\right) = u(y) - u(x) - 2k = \left|G_k(u(x)) - G_k(u(y))\right|.$$

The other cases can be settled with similar observations.
This, together with the use of $G_k(u)$ as a test function in (2.6), gives that

$$\begin{aligned}\frac{1}{2}\iint_{\mathbb{R}^{2n}} \frac{|G_k(u(x)) - G_k(u(y))|}{|x-y|^{n+s}}dx\,dy &= \frac{1}{2}\iint_{\mathbb{R}^{2n}} \mathbf{z}(x,y)\frac{G_k(u(x)) - G_k(u(y))}{|x-y|^{n+s}}dx\,dy \\ &= \int_\Omega f G_k(u)\,dx,\end{aligned}$$

hence by using the Sobolev inequality

$$\frac{1}{2}[G_k(u)]_{W^{s,1}(\mathbb{R}^n)} \leqslant S_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega_k)}[G_k(u)]_{W^{s,1}(\mathbb{R}^n)},$$

that is

$$\left(1 - 2S_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega_k)}\right)[G_k(u)]_{W^{s,1}(\mathbb{R}^n)} \leqslant 0. \tag{6.5}$$

Denote $T := \|u\|_{L^\infty(\Omega)} \in [0,+\infty]$, then we claim that

$$|\{|u| = T\}| = 0.$$

Indeed, if $T = +\infty$ it is obvious since u is summable. Otherwise, on the set $\{|u| = T\}$ we have that $u(x) = u(y)$, but this can hold only almost everywhere, according to our hypothesis. From this and the Lebesgue dominated convergence theorem, we deduce that that

$$\lim_{k \nearrow T}\|f\|_{L^{\frac{n}{s}}(\Omega\cap\{|u|>k\})} = \|f\|_{L^{\frac{n}{s}}(\Omega\cap\{|u|=T\})} = 0.$$

Consequently, for all $\varepsilon > 0$ there exists $\tilde{k} \in (0,T)$ such that

$$2S_{n,s}\|f\|_{L^{\frac{n}{s}}(\Omega\cap\{|u|>k\})} < \varepsilon,$$

for all $k \geqslant \tilde{k}$. In particular, from (6.5), we obtain

$$\|G_{\tilde{k}}(u)\|_{L^{\frac{n}{n-s}}(\Omega_{\tilde{k}})} \leqslant S_{n,s}[G_{\tilde{k}}(u)]_{W^{s,1}(\mathbb{R}^n)} \leqslant 0.$$

Hence $|u| \leqslant \tilde{k}$ almost everywhere in Ω, hence $T \leqslant \tilde{k}$. This is in contradiction with the choice of $\tilde{k}$ and concludes the proof. □

What is more, we are able to prove that minimizers of $\mathcal{F}_s^1$ are bounded in Ω, and that they reach the value of the L^∞ norm on a set with positive Lebesgue measure. More precisely, we have the following.

Theorem 6.2. *Let $f \in L^{\frac{n}{s}}(\Omega)$ and let $u \in \mathcal{W}_0^{1,s}(\Omega)$ be a minimizer of $\mathcal{F}_1^s$. Then $u \in L^\infty(\mathbb{R}^n)$ and the set of extremal points*

$$\left\{x \in \mathbb{R}^n \;\middle|\; |u(x)| = \|u\|_{L^\infty(\mathbb{R}^n)}\right\}$$

has positive Lebesgue measure.

Proof of Theorem 6.2. We define, for $k > 0$,

$$T_k(t) = \begin{cases} -k, & \text{if } t < -k \\ t, & \text{if } -k \leqslant t \leqslant k \\ k, & \text{if } t > k. \end{cases}$$

We recall the definition of $G_k(t)$, and remark that

$$t = T_k(t) + G_k(t), \qquad |t - \tau| = |T_k(t) - T_k(\tau)| + |G_k(t) - G_k(\tau)|. \tag{6.6}$$

We notice that $T_k(u) = 0$ in $\mathcal{C}\Omega$ and we claim that $T_k(u) \in \mathcal{W}_0^{s,1}(\Omega)$. Using the notations in (6.3), when $(x, y) \in \Omega_k^i \times \Omega_k^i$ for $i \in \{1, 2\}$, $T_k(u(x)) - T_k(u(y)) = 0$. Then

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|T_k(u(x)) - T_k(u(y))|}{|x-y|^{n+s}} \, dxdy \\
&= 2 \int_{\Omega_k^1} \left(\int_{\Omega_k^2} \frac{2k}{|x-y|^{n+s}} \, dy \right) dx + 2 \int_{\Omega_k^1} \left(\int_{\Omega \setminus \Omega_k} \frac{u(y) + k}{|x-y|^{n+s}} \, dy \right) dx \\
&\quad + 2 \int_{\Omega_k^2} \left(\int_{\Omega \setminus \Omega_k} \frac{-u(y) + k}{|x-y|^{n+s}} \, dy \right) dx + \int_{\Omega \setminus \Omega_k} \int_{\Omega \setminus \Omega_k} \frac{|u(x) - u(y)|}{|x-y|^{n+s}} \, dydx \\
&\leqslant 2 \int_{\Omega_k^1} \left(\int_{\Omega_k^2} \frac{u(y) - u(x)}{|x-y|^{n+s}} \, dy \right) dx + 2 \int_{\Omega_k^1} \left(\int_{\Omega \setminus \Omega_k} \frac{u(y) - u(x)}{|x-y|^{n+s}} \, dy \right) dx \\
&\quad + 2 \int_{\Omega_k^2} \left(\int_{\Omega \setminus \Omega_k} \frac{u(x) - u(y)}{|x-y|^{n+s}} \, dy \right) dx + \int_{\Omega \setminus \Omega_k} \int_{\Omega \setminus \Omega_k} \frac{|u(x) - u(y)|}{|x-y|^{n+s}} \, dydx \\
&\leqslant C[u]_{W^{s,1}(\Omega)},
\end{aligned}$$

since $-u > k$ in Ω_k^1, and $k < u$ in Ω_k^2. Therefore $T_k(u)$ is a competitor for u, and from the minimality of u,

$$\mathcal{F}_1^s(u) \leqslant \mathcal{F}_1^s(T_k(u)),$$

that is

$$\iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)| - |T_k(u(x)) - T_k(u(y))|}{|x-y|^{n+s}} dx\, dy \leqslant \int_\Omega f(x) \left(u(x) - T_k(u(x))\right) dx.$$

From (6.6) we obtain that

$$\iint_{Q(\Omega)} \frac{|G_k(u(x)) - G_k(u(y))|}{|x-y|^{n+s}} dx\, dy \leqslant \int_\Omega f(x) G_k(u(x)) \, dx.$$

This implies (6.5), and also that

$$\left(1 - 2S_{n,s} \|f\|_{L^{\frac{n}{s}}(\Omega_k)}\right) \|G_k(u)\|_{L^{\frac{n}{n-s}}(\Omega)} \leqslant 0.$$

Denoting $T := \|u\|_{L^\infty(\mathbb{R}^n)} \in [0,\infty]$, f we have that

$$\|G_k(u)\|^{\frac{n-s}{n}}_{L^{\frac{n}{n-s}}(\Omega)} = \int_{\Omega_k^1} |u(x)+k|^{\frac{n-s}{n}}\,dx + \int_{\Omega_k^2} |u(x)-k|^{\frac{n-s}{n}}\,dx > 0,$$

so for all $0 < k < T$

$$\|f\|_{L^{\frac{n}{s}}(\Omega_k)} \geqslant \frac{1}{2S_{n,s}}. \tag{6.7}$$

Since $f \in L^{\frac{n}{s}}(\Omega)$, and u is finite almost everywhere, it should hold that $\|f\|_{L^{\frac{n}{s}}(\Omega_k)}$ is small for k large enough. Suppose by contradiction that $T = +\infty$. Since $\Omega_k = \{x \in \Omega \,|\, |u(x)| > k\}$, and $|\{u = +\infty\}| = 0$, by Lebesgue's dominated convergence theorem

$$\lim_{k\to+\infty} \|f\|_{L^{\frac{n}{s}}(\Omega_k)} = \|f\|_{L^{\frac{n}{s}}(\{u=+\infty\})} = 0,$$

hence for any $\varepsilon > 0$ there is some $\tilde{k} > 0$ such that for all $k > \tilde{k}$

$$\|f\|_{L^{\frac{n}{s}}(\Omega_k)} < \varepsilon.$$

This gives a contradiction to (6.7) and implies that $T < \infty$. It follows that $u \in L^\infty(\mathbb{R}^n)$. Furthermore, letting $k \nearrow T$ in (6.7), we observe that

$$\lim_{k\nearrow T} \|f\|_{L^{\frac{n}{s}}(\Omega_k)} = \|f\|_{L^{\frac{n}{s}}(\{|u|=T\})} \geqslant \frac{1}{2S_{n,s}},$$

hence $\{|u| = \|u\|_{L^\infty(\mathbb{R}^n)}\}$ has positive Lebesgue measure, as stated. □

7. Appendix A. Pointwise limit

In this section, we are interested in the behavior of the limit as $p \to 1$ of the kinetic part of the energy, specifically what we have denote in (1.4) by $\mathcal{E}_p^s$. What we want to emphasize is that the fractional parameter s_p, used throughout this paper, arises naturally when looking at this pointwise limit. To broaden the scope of our statement, we allow for non-vanishing exterior data under specific additional conditions. These prerequisites are expressed in terms of the so-called nonlocal tail of u introduced in [5], precisely for $x \in \Omega$

$$\mathrm{Tail}^p_{s_p}(u,x) = \int_{\mathcal{C}\Omega} \frac{|u(y)|^p}{|x-y|^{(n+s)p}}\,dy,$$

basically encompassing the contribution of the exterior data to the energy.

Theorem 7.1. *Let $q \in (1, c_{n,s})$, where $c_{n,s}$ is such that $s_q \in (s,1)$ and $s_q q < 1$. Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be such that $u \in W^{s_q,q}(\Omega)$ and*

$$\mathrm{Tail}^1_s(u,\cdot),\ \mathrm{Tail}^q_{s_q}(u,\cdot) \in L^1(\Omega). \tag{7.1}$$

Then it holds that

$$\lim_{p\to1} \mathcal{E}_p^{s_p}(u,\Omega) = \mathcal{E}_1^s(u,\Omega).$$

Proof. Let $s_0 \in (s,1), \delta \in (0,1-s)$. Since p converges to 1, it is safe to assume that

$$p \leqslant \min\left\{q, \frac{n-\delta}{n+1-s}, \frac{n+s_0}{n+s}\right\}.$$

Notice that

$$(n+s)q = n + s_q q.$$

We remark that if $u \in \mathcal{W}^{s_q,q}(\Omega)$, then by the Hölder inequality we have that $u \in \mathcal{W}^{s_t,t}(\Omega)$ for all $t \in [1,q)$. Indeed

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^t}{|x-y|^{n+s_t t}}\, dxdy \leqslant [u]^t_{W^{s_q,q}(\Omega)} |\Omega|^{\frac{2(q-t)}{q}}, \qquad \text{and} \qquad \int_\Omega |u|^t\, dx \leqslant \|u\|^t_{L^q(\Omega)} |\Omega|^{\frac{q-t}{q}}. \tag{7.2}$$

Furthermore, (7.1) implies that $\mathrm{Tail}^t_{s_t}(u,\cdot) \in L^1(\Omega)$ for all $t \in (1,q)$. Indeed, there exists a unique $\tau \in (0,1)$ such that $t = \tau + (1-\tau)q$, and by Young inequality we have that for $a, b \geqslant 0$,

$$a^\tau b^{(1-\tau)q} \leqslant \tau a + (1-\tau) b^q \leqslant a + b^q,$$

which, for $a = b$ becomes

$$a^t \leqslant a + a^q \tag{7.3}$$

for all $t \in (1,q)$. This applied to $a = |g(y)|/|x-y|^{n+s}$, gives

$$\frac{|u(y)|^t}{|x-y|^{(n+s)t}} \leqslant \frac{|u(y)|}{|x-y|^{n+s}} + \frac{|u(y)|^q}{|x-y|^{(n+s)q}}, \tag{7.4}$$

hence integrating, we get that $\mathrm{Tail}^t_{s_t}(u,\cdot) \in L^1(\Omega)$ for all $t \in (1,q)$, as desired.

We prove the thesis of the theorem in two steps, first for smooth, compactly supported functions, and then conclude by density.

Step 1. We prove that

$$\lim_{p\to 1} \mathcal{E}^{s_p}_p(\psi,\Omega) = \mathcal{E}^s_1(\psi,\Omega), \tag{7.5}$$

for all $\psi \colon \mathbb{R}^n \to \mathbb{R}$, such that $\psi \in C^\infty_0(\Omega)$ and $\psi = u$ on $\mathcal{C}\Omega$.

To estimate the contribution in $\Omega\times\Omega$, we notice that if $|x-y| \geqslant 1$,

$$\frac{1}{p}\frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n+s_p p}} \leqslant \frac{2\|\psi\|^p_{L^\infty(\Omega)}}{|x-y|^{n+s_p p}} \leqslant \frac{C_1}{|x-y|^{n+s}},$$

with $C_1 = C_1(\|\psi\|_{L^\infty(\Omega)}) > 0$. On the other hand, when $|x-y| < 1$,

$$\frac{1}{p}\frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n+s_p p}} \leqslant \frac{\|\nabla\psi\|^p_{L^\infty(\Omega)}|x-y|^p}{|x-y|^{n+s_p p}} \leqslant \frac{C_2}{|x-y|^{n-\delta}},$$

with $C_2 = C_2(\|\nabla\psi\|_{L^\infty(\Omega)}) > 0$, recalling that $(n+s-1)p < n-\delta$ by choice of δ. Now,

$$\iint_{(\Omega\times\Omega)\cap\{|x-y|\geqslant 1\}} \frac{dx\,dy}{|x-y|^{n+s}} \leqslant \iint_{\Omega\times\Omega} dx\,dy = |\Omega|^2,$$

and

$$\iint_{(\Omega\times\Omega)\cap\{|x-y|<1\}} \frac{dx\,dy}{|x-y|^{n-\delta}} \leqslant \int_\Omega \left(\int_{B_1(x)} \frac{dy}{|x-y|^{n-\delta}} \right) dx \leqslant C(\Omega,\delta,n).$$

By the dominated convergence theorem we have that

$$\lim_{p\to 1} \frac{1}{p} \iint_{\Omega\times\Omega} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n+s_p p}}\, dxdy = \iint_{\Omega\times\Omega} \frac{|\psi(x)-\psi(y)|}{|x-y|^{n+s}}\, dxdy. \tag{7.6}$$

Furthermore, for the nonlocal contribution $(x,y) \in \Omega\times\mathcal{C}\Omega$,

$$\frac{1}{p}\frac{|\psi(x)-u(y)|^p}{|x-y|^{n+s_p p}} \leqslant \frac{2^{p-1}(|\psi(x)|^p + |u(y)|^p)}{|x-y|^{n+s_p p}} \leqslant \frac{C_3}{|x-y|^{n+s_p p}} + \frac{2|u(y)|^p}{|x-y|^{n+s_p p}},$$

with $C_3 = C_1(\|\psi\|_{L^\infty(\Omega)}) > 0$. When $|x-y| \geqslant 1$ then $|x-y|^{n+s_p p} \geqslant |x-y|^{n+s}$ and when $|x-y| \leqslant 1$ then $|x-y|^{n+s_p p} \geqslant |x-y|^{n+s_0}$ by the choice of s_0. Together with (7.4), we have then

$$\frac{1}{p}\frac{|\psi(x)-u(y)|^p}{|x-y|^{n+s_p p}} \leqslant \frac{C_3}{|x-y|^{n+\sigma}} + \frac{2|u(y)|}{|x-y|^{n+s}} + \frac{2|u(y)|^q}{|x-y|^{(n+s)q}}$$

with either $\sigma = s$ or $\sigma = s_0$, for the suitable cases $|x-y|$ less or greater than 1. Recalling (7.1) and that

$$\mathrm{Per}_\sigma(\Omega, \mathbb{R}^n) = \iint_{\Omega\times\mathcal{C}\Omega} \frac{dx\,dy}{|x-y|^{n+\sigma}} < +\infty,$$

the right hand side is an $L^1(\Omega\times\mathcal{C}\Omega)$ and the dominated convergence theorem gives that

$$\lim_{p\to 1}\frac{1}{p}\iint_{\Omega\times\mathcal{C}\Omega}\frac{|\psi(x)-u(y)|^p}{|x-y|^{n+s_p p}}\,dxdy = \iint_{\Omega\times\mathcal{C}\Omega}\frac{|\psi(x)-u(y)|}{|x-y|^{n+s}}\,dxdy.$$

This, together with (7.6), concludes (7.5).

Step 2. By the density of $C_c^\infty(\Omega)$ in $\mathcal{W}^{s_q,q}(\Omega)$, we have that for $u \in \mathcal{W}^{s_q,q}(\Omega)$, there exists $\psi_j \colon \Omega \to \mathbb{R}$ such that $\psi_j \in C_c^\infty(\Omega)$ and

$$\|\psi_j - u\|_{W^{s_q,q}(\Omega)} \longrightarrow 0, \qquad \text{as } j \to \infty.$$

Without changing notations, we take $\psi_j \colon \mathbb{R}^n \to \mathbb{R}$ such that $\psi_j = u$ on $\mathcal{C}\Omega$. From (7.2) it holds

$$\|\psi_j - u\|_{W^{s_t,t}(\Omega)} \longrightarrow 0, \qquad \text{as } j \to \infty.$$

for all $t \in [1,q]$. We have

$$\begin{aligned}\lim_{p\to 1}\left(\mathcal{E}_p^{s_p}(u,\Omega) - \mathcal{E}_1^s(u,\Omega)\right) =& \lim_{j\to+\infty}\lim_{p\to 1}\left(\mathcal{E}_p^{s_p}(u,\Omega) - \mathcal{E}_p^{s_p}(\psi_j,\Omega)\right)\\ &+ \lim_{j\to+\infty}\lim_{p\to 1}\left(\mathcal{E}_p^{s_p}(\psi_j,\Omega) - \mathcal{E}_1^{s}(\psi_j,\Omega)\right)\\ &+ \lim_{j\to+\infty}\left(\mathcal{E}_1^{s}(\psi_j,\Omega) - \mathcal{E}_1^{s}(u,\Omega)\right)\\ =& L_1 + L_2 + L_3.\end{aligned}$$

Notice that $L_2 = 0$ by Step 1, and $L_3 = 0$ since

$$|\mathcal{E}_1^s(\psi_j,\Omega) - \mathcal{E}_1^s 1(u,\Omega)| \leqslant \iint_{Q(\Omega)} \frac{|(\psi_j-u)(x)-(\psi_j-u)(y)|}{|x-y|^{n+s}}\,dydx \leqslant C_{n,s,\Omega}\|\psi_j-u\|_{W^{s,1}(\Omega)},$$

using also (2.2). To prove that also $L_1 = 0$, we proceed as follows. We use the inequalities $||a|^p - |b|^p| \leqslant p(|a|^{p-1}+|b|^{p-1})|a-b|$ and Hölder's to get that

$$\begin{aligned}&\left|(\mathcal{E}_p^{s_p}(u,\Omega) - (\mathcal{E}_p^{s_p}(\psi_j,\Omega)\right|\\ &\leqslant \iint_{Q(\Omega)} \frac{p(|u(x)-u(y)|^{p-1} + |\psi_j(x)-\psi_j(y)|^{p-1})|(u-\psi_j)(x)-(u-\psi_j)(y)|}{|x-y|^{(n+s)p}}\,dxdy\\ &\leqslant 2\left(\iint_{Q(\Omega)}\frac{|(u-\psi_j)(x)-(u-\psi_j)(y)|^p}{|x-y|^{(n+s)p}}\,dxdy\right)^{\frac{1}{p}}\\ &\qquad\left[\left(\iint_{Q(\Omega)}\frac{|u(x)-u(y)|^p}{|x-y|^{(n+s)p}}\right)^{\frac{p-1}{p}} + \left(\iint_{Q(\Omega)}\frac{|\psi_j(x)-\psi_j(y)|^p)}{|x-y|^{(n+s)p}}\right)^{\frac{p-1}{p}}\right]\\ &:= 2I(j,p)\left[J(p)+K(j,p)\right].\end{aligned}$$

Now, using (7.3) for $a = |(u-\psi_j)(x) - (u-\psi_j)(y)|/|x-y|^{(n+s)}$

$$\begin{aligned}
I(j,p)^p &\leqslant \iint_{Q(\Omega)} \frac{|(u-\psi_j)(x) - (u-\psi_j)(y)|}{|x-y|^{n+s}}\,dxdy + \iint_{Q(\Omega)} \frac{|(u-\psi_j)(x) - (u-\psi_j)(y)|^q}{|x-y|^{(n+s)q}}\,dxdy \\
&\leqslant [u-\psi_j]_{W^{s,1}(\Omega)} + [u-\psi_j]^q_{W^{s_q,q}(\Omega)} \\
&\quad + 2\iint_{\Omega\times\mathcal{C}\Omega} \frac{|(u-\psi_j)(x)|}{|x-y|^{n+s}}\,dxdy + 2\iint_{\Omega\times\mathcal{C}\Omega} \frac{|(u-\psi_j)(x)|^q}{|x-y|^{n+s_q q}}\,dxdy \\
&\leqslant C_{n,s,q,\Omega}\left(\|u-\psi_j\|_{W^{s,1}(\Omega)} + \|u-\psi_j\|^q_{W^{s_q,q}(\Omega)}\right),
\end{aligned}$$

by (2.2) and recalling that $u = \psi_j$ on $\mathcal{C}\Omega$. Renaming the constants and using (7.2), we get that

$$I(j,p) \leqslant C_{n,s,q,\Omega}\|u-\psi_j\|_{W^{s_q,q}(\Omega)},$$

from which follows that

$$\lim_{j\to+\infty}\lim_{p\to1} I(j,p) = 0.$$

On the other hand, again by (7.3) and (2.2),

$$\begin{aligned}
J(p)^{\frac{p}{p-1}} &\leqslant C_{n,s,q,\Omega}\Big(\|u\|_{W^{s,1}(\Omega)} + \|u\|^q_{W^{s_q,q}(\Omega)} \\
&\quad + \iint_{\Omega\times\mathcal{C}\Omega} \frac{|u(y)|}{|x-y|^{n+s}}dy\,dx + \iint_{\Omega\times\mathcal{C}\Omega} \frac{|u(y)|^q}{|x-y|^{(n+s)q}}dy\,dx\Big) \\
&\leqslant C_{n,s,q,\Omega}\left(\|u\|_{W^{s,1}(\Omega)} + \|u\|^q_{W^{s_q,q}(\Omega)} + \|\operatorname{Tail}_s(u,\cdot)\|_{L^1(\Omega)} + \|\operatorname{Tail}^q_{s_q}(u,\cdot)\|_{L^1(\Omega)}\right)
\end{aligned}$$

using (7.2) and renaming the constants. Thus $J(p)$ can be bounded from above, uniformly in p. Finally, in the same way,

$$K(j,p)^{\frac{p}{p-1}} \leqslant C_{n,s,q,\Omega}\left(\|\psi_j\|_{W^{s,1}(\Omega)} + \|\psi_j\|^q_{W^{s_q,q}(\Omega)} + \|\operatorname{Tail}_s(u,\mathcal{C}\Omega)\|_{L^1(\Omega)} + \|\operatorname{Tail}^q_{s_q}(u,\cdot)\|_{L^1(\Omega)}\right),$$

using that for j large enough,

$$\|\psi_j\|_{W^{s_t,t}(\Omega)} \leqslant \|u\|_{W^{s_t,t}(\Omega)} + \|u-\psi_j\|_{W^{s_t,t}(\Omega)} \leqslant \|u\|_{W^{s_t,t}(\Omega)} + 1$$

for all $t \in [1,q]$. Thus also $K(j,p)$ can be bounded from above uniformly in p. It follows that

$$\lim_{j\to+\infty}\lim_{p\to1} I(j,p)(J(p)+K(j,p)) = 0,$$

hence that $L_1 = 0$. This concludes the proof of the theorem. □

We remind the reader that according to [4, Lemma 2.3], (7.1) can be achieved with $u \in W^{s,q}(\mathcal{C}\Omega)$ – or can be sharpened, by requiring a combination of condition near the boundary of Ω, and far from the boundary. Notice also that when $\varphi = 0$, such a requirement is satisfied, and we can write the following corollary.

Corollary 7.2. *Let $q \in (1, c_{n,s})$, where $c_{n,s}$ is such that $s_q \in (s,1)$ and $s_q q < 1$. Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be such that $u \in \mathcal{W}^{s_q,q}_0(\Omega)$. Then*

$$\lim_{p\to1} \mathcal{F}^{s_p}_p(u) = \mathcal{F}^s_1(u).$$

8. Appendix B. The (s,p)-problem

For completeness of the exposition and for the reader's benefit, we prove in this appendix some basic facts about minimizers/weak solutions of the (s,p)-problem, that we have used in this note.

Theorem 8.1. *Let $\gamma \geqslant \frac{n}{s_p}$ and let $f \in L^\gamma(\Omega)$. There exists $u_p \in \mathcal{W}^{s,p}_0(\Omega)$, the unique (s,p)-minimizer and weak solution of (1.5).*

Proof. Recalling that

$$p_s^* = \frac{np}{n-sp},$$

notice that

$$\frac{n}{s_p} = \frac{np}{np-n+sp} = (p_s^*)',$$

the conjugate Sobolev exponent. We point out that the constants may change value from line to line, indicating a positive quantity, possibly depending on s, p, n, γ, Ω. Using the Hölder and Sobolev inequality, we get that

$$\left| \int_\Omega f u \, dx \right| \leqslant \|f\|_{L^{\frac{n}{s_p}}(\Omega)} \|u\|_{L^{p_s^*}(\Omega)} \leqslant S^{\frac{1}{p}}_{n,s,p} \|f\|_{L^\gamma(\Omega)} [u]_{W^{s,p}(\mathbb{R}^n)}. \tag{8.1}$$

Using the Young inequality, we further have for some fixed $\varepsilon \in (0,1/2)$,

$$\|f\|_{L^\gamma(\Omega)}[u]_{W^{s,p}(\mathbb{R}^n)} \leqslant \varepsilon \frac{[u]^p_{W^{s,p}(\mathbb{R}^n)}}{p} + \frac{p-1}{p\varepsilon^{\frac{1}{p-1}}} \left(\|f\|_{L^\gamma(\Omega)}\right)^{\frac{p}{p-1}} =: \varepsilon \frac{[u]^p_{W^{s,p}(\mathbb{R}^n)}}{p} + C_\varepsilon \|f\|^{\frac{p}{p-1}}_{L^\gamma(\Omega)}.$$

It follows that

$$\mathcal{F}^s_p(u,\Omega) \geqslant \frac{1}{2p}[u]^p_{W^{s,p}(\mathbb{R}^n)} - \int_\Omega f u \, dx = \frac{1}{p}\left(\frac{1}{2} - \varepsilon\right)[u]^p_{W^{s,p}(\mathbb{R}^n)} - C_\varepsilon \|f\|^{\frac{p}{p-1}}_{L^\gamma(\Omega)} \geqslant -C_\varepsilon \|f\|^{\frac{p}{p-1}}_{L^\gamma(\Omega)}.$$

Thus the energy is bounded from below, and it follows that there exists a minimizing sequence. Let $\{u_k\}_k \in \mathcal{W}^{s,p}_0(\Omega)$ be such that

$$\lim_{k\to\infty} \mathcal{F}^s_p(u_k,\Omega) = \inf_{u \in \mathcal{W}^{s,}(\Omega)} \mathcal{F}^s_p(u,\Omega) =: m \geqslant -C_\varepsilon \|f\|^{\frac{p}{p-1}}_{L^\gamma(\Omega)}.$$

There exists $\overline{k} \in \mathbb{N}$ such that for all $k > \overline{k}$

$$\mathcal{F}^s_p(u_k) < m+1.$$

By this and by (2.1), we obtain that

$$\|u_k\|^p_{W^{s,p}(\Omega)} \leqslant C_2 [u_k]^p_{W^{s,p}(\mathbb{R}^n)} \leqslant C\left(m+1+C_\varepsilon \|f\|^{\frac{p}{p-1}}_{L^\gamma(\Omega)}\right).$$

By compactness, there exists $u \in \mathcal{W}^{s,p}_0(\Omega)$ and a subsequence

$$u_{k_i} \to u \qquad \text{in } L^p(\Omega) \text{ and a.e. in } \mathbb{R}^n.$$

Further, $\|u_{k_i}\|_{L^{\frac{np}{n-sp}}(\Omega)}$ is uniformly bounded by the Sobolev inequality, hence up to taking a subsequence of u_{k_i} that we still call u_{k_i},

$$u_{k_i} \longrightarrow u$$

weakly in $L^{\frac{np}{n-sp}}(\Omega)$. By Fatou it holds that

$$\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy \leqslant \liminf_i \iint_{\mathbb{R}^{2n}} \frac{|u_{k_i}(x)-u_{k_i}(y)|^p}{|x-y|^{n+sp}}dxdy,$$

and this summed up with the weak convergence in $L^{\frac{np}{n-sp}}(\Omega)$, gives that

$$\mathcal{F}^s_p(u) \leqslant m,$$

hence u is a minimizer.
To see that the minimizer u is also a weak solution, i.e. that u solves the corresonding Euler-Lagrange equation, one takes in a standard way a perturbation of u with a test function $\varphi \in \mathcal{W}^{s,p}_0(\Omega)$ and deduces u is a weak solution by using that the first variation of the energy vanishes,

$$\frac{d}{dt}\mathcal{F}^s_p(u+t\varphi)\Big|_{t=0} = 0.$$

As customary, if u is a weak solution and v is any competitor for u, we consider as test function $w = u - v$, and the fact that u is a minimizer is obtained by using the Young inequality.
Finally, uniqueness follows by strict convexity of the energy $\mathcal{F}^s_p$. □

Since we need the existence of a (s_p, p)-minimizer, we clarify the following corollary.

Corollary 8.2. *Let $f \in L^{\frac{n}{s}}(\Omega)$. There exists $u_p \in \mathcal{W}^{s_p,p}_0(\Omega)$, the unique (s_p, p)-minimizer and weak solution of (1.5).*

Proof. The proof is immediate, noticing that

$$\gamma = \frac{n}{s} \geqslant \frac{n}{(s_p)_p} = \frac{n}{n+s_p-\frac{n}{p}} = \frac{np}{(2n+s)p-2n}$$

and applying Theorem 8.1. □

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