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Normal form methods for some nonlinear Hamiltonian PDEs in higher dimension

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Abstract

The qualitative behavior of the solutions for Hamiltonian finite dimensional systems has been widely studied through normal form and KAM (Kolmogorov - Arnold - Moser) theory. Starting from the late '80s, such methods have been extended to systems with infinitely many degrees of freedom and PDEs: nowadays a satisfactory theory exists for PDEs in 1 space dimension. The case of higher dimensional manifolds has been tackled by Bourgain, who proved some KAM-type results for the non-linear wave equation (NLW) and the non-linear Schrödinger equation (NLS) on the torus; such results have been subsequently extended to some equations on Lie groups. Concerning normal form methods, only equations on Zoll manifolds or tori have been treated.

In this thesis, we extend the normal form results to a more general class of higher dimensional manifolds, which includes some examples that have never been treated and also all the known examples. Among new manifolds, we mention surfaces of revolution and compact Lie groups. The result that we prove is known as "almost global existence". Namely, we prove that, given an H^s initial datum (with $s \gg 1$) of size ϵ, the solution remains small in H^s norm for times of order ϵ^{-r}, for any $r \geq 3$. We emphasize that our result also bounds the energy flow among different modes of the linearized equations, thus preventing the possible insurgence of turbulence phenomena.

We first present the class of manifolds we deal with, namely manifolds in which the geodesic flow is globally integrable. Recently Bambusi and Langella constructed a Fourier expansion adapted to these manifolds: here we show how to use it to develop normal form theory for nonlinear Hamiltonian systems. To this end, essentially, two tools are required: (i) an estimate of the "Fourier" coefficients of the product of two eigenfunctions of the Laplacian; (ii) a partition of the "Fourier lattice" in suitable dyadic clusters. The understanding of the structure of (i) is the main contribution of the present thesis. Once this has been obtained, we introduce and study (following

Delort-Szeftel) a class of polynomials that enters the normal form procedure. Such a study is the main technical part of the thesis. Concerning (ii), here we prove that the partition constructed by Bambusi-Langella, proposed with a different purpose, actually enjoys the dyadic properties needed to develop perturbation theory.

Riassunto della Tesi

Il problema dello studio del comportamento qualitativo delle soluzioni per i sistemi hamiltoniani finito-dimensionali è stato ampiamente studiato attraverso la teoria della forma normale e la teoria KAM (Kolmogorov-Arnold-Moser). A partire dalla fine degli anni '80, tali metodi sono stati estesi a sistemi con un numero infinito di gradi di libertà ed equazioni differenziali alle derivate parziali (PDE): attualmente, esiste una teoria soddisfacente per le PDE in una dimensione spaziale. Il caso di varietà di dimensioni superiori è stato affrontato da Bourgain, il quale ha dimostrato alcuni risultati di tipo KAM per le equazioni delle onde non lineari (NLW) e le equazioni di Schrödinger non lineari (NLS) sul toro; tali risultati sono stati successivamente estesi ad alcune equazioni su gruppi di Lie. Per quanto riguarda i metodi di forma normale, sono state trattate solo equazioni su varietà Zoll o tori.

In questa tesi estendiamo i risultati di forma normale a una classe più generale di varietà di dimensione maggiore di uno, che include alcuni esempi mai trattati e tutti gli esempi conosciuti. Tra queste nuove varietà, menzioniamo le superfici di rivoluzione e i gruppi di Lie compatti. I risultati che dimostriamo sono noti come "almost-global existence" in letteratura. In altre parole, dimostriamo che, dato un dato iniziale in H^s (con $s \gg 1$) di dimensioni ϵ, la soluzione rimane piccola nella norma H^s per tempi dell'ordine di ϵ^{-r}, per ogni $r \geq 3$. Sottolineiamo che il nostro risultato limita il trasferimento di energia tra diversi modi delle equazioni linearizzate, impedendo così la possibile insorgenza di fenomeni di turbolenza.

Nella tesi presentiamo innanzitutto la classe di varietà che trattiamo, ovvero varietà in cui il flusso geodetico è globalmente integrabile. Recentemente, Bambusi e Langella hanno costruito una serie di Fourier adattata a queste varietà: qui mostriamo come utilizzarla per sviluppare la teoria della forma normale per sistemi hamiltoniani non lineari. A questo scopo, sono necessari essenzialmente due strumenti: (i) una stima dei "coefficienti di Fourier" del prodotto di due autofunzioni del laplaciano; (ii) una suddivisione del "reticolo di Fourier" in opportuni cluster di tipo diadico. La comprensione della struttura di (i) costituisce il contributo principale della presente tesi. Una volta ottenuto ciò, introduciamo e studiamo (seguendo Delort-Szeftel) una classe di polinomi che entra nella procedura della forma normale. Tale studio rappresenta la parte tecnicamente più complicata della tesi. Per quanto riguarda (ii), qui dimostriamo che la partizione costruita da Bambusi-Langella, proposta con uno scopo diverso, gode effettivamente delle proprietà diadiche necessarie per sviluppare la teoria delle perturbazioni.

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Contents

Introduction

The long-time behavior of the solutions to the equations of motion arising from finite-dimensional quasi-integrable Hamiltonian dynamical systems has been well understood in the last sixty years. Starting from the celebrated KAM (Kolmogorov - Arnold - Moser) theorem, a whole theory for quasi-periodic solutions has been developed. Essentially, under suitable non-degeneracy conditions, a full measure set of quasi-periodic solutions, lying on invariant tori, persists under small perturbations of an integrable system. Besides that, the Nekhoreshev theorem establishes an exponential time of stability for *all* the solutions of a generic quasi-integrable Hamiltonian system. Since the introduction of Hamiltonian partial differential equations, i.e., Hamiltonian systems with infinitely many degrees of freedom, similar issues have been investigated for those equations. However, long-time results for Hamiltonian partial differential equations on compact domains are considerably more involved, since the classical results of Hamiltonian perturbation theory do not hold for systems with infinitely many degrees of freedom [14].

The first positive results in the extension of KAM theory to Hamiltonian PDEs were presented in the late 80s [39], [78]. Starting from those results, a quite satisfactory theory concerning the persistence of periodic and quasi-periodic solutions has been developed for equations defined on a one-dimensional domain [27, 77, 87, 89]. A quite general method to treat quasi-linear (i.e., fully non-linear) perturbations has also been developed [1, 2]. See also [8] for an important early result on the extension of KAM methods for PDE's. The problem of the persistence of almost-periodic solutions, i.e., solutions that lie on fully dimensional tori, then in full analogy with classical KAM theory, is completely open and only partial results are known [38].
Regarding higher-dimensional domains, nowadays, many results about the persistence and possibly stability of quasi-periodic solutions are known, but a complete comprehension is still quite far. A first breakthrough was given by Bourgain dealing with the nonlinear wave equation and nonlinear Schrödinger equation on the d-dimensional torus [30, 32], and by Eliasson and Kuksin with quite different techniques [53]. Later

on, a remarkable result for a completely resonant nonlinear Schr"odinger equation was obtained [88]. Without trying to be exhaustive, we quote also [19, 20, 25, 3, 85] and the literature therein.

All of those results pertain to a special class of solutions, namely periodic or quasi-periodic solutions. A relevant problem concerns the existence and stability of general solutions of both linear and non-linear Hamiltonian PDEs, for a time that goes beyond the local existence theory. We recall that the theory of local well-posedness for general PDEs is nowadays well established [79].

This thesis deals exactly with this problem and provides some results about almost-global existence and stability in high Sobolev spaces of small solutions of some semi-linear Hamiltonian PDEs on quite general Riemannian manifolds. Following [12], we introduce a notion of integrable systems, which we call *globally integrable quantum systems* since they appear as the quantization of classical integrable systems. For this kind of system, we prove the long-time existence of small solutions and we control the growth of Sobolev norm, for a time of order $\epsilon^{-M}, \forall M \in \mathbb{N}$, where ϵ is the size of the initial datum. Consequently, we apply this abstract theorem to three concrete examples. Namely, we consider the semilinear beam equation and the semilinear Schr"odinger equation with convolution potential; moreover, applying our abstract theorem, we can prove Sobolev stability for the ground state of a nonlinear Schr"odinger. Before coming to a more detailed description of our work, we remark that its interest may be twofold. On the one hand, it proposes a general setting, including a quite general notion of quantum integrable system, that covers a large number of different examples; on the other hand, it extends the results of almost-global existence to a quite general class of high-dimensional domains.

The first breakthrough results about the long-time analysis of "generic" solutions were obtained by Bourgain in the nineties. First, the author prove that a solution of the NLS equation, corresponding to a small initial condition norm, stays ϵ-close to an unperturbed solution for a time of order ϵ^{-M} for any $M \in \mathbb{N}$ [28]. Moreover, considering linear perturbations of Schrödinger equations, he proved that the growth of the Sobolev norm of the solutions is controlled by $\|\phi(t)\| \leq Ct^{\epsilon}\|\phi_0\|$, for any ϵ small [29]. The result is improved for quasi-periodic potentials [31]. Later, a key result was obtained by Bambusi, who applied Birkhoff normal form techniques to prove almost global existence for a class of semilinear equations [4, 9]. Delort and Szeftel extended the research towards non-linear equations on high-dimensional domains different from tori [47, 48]; in particular, they were able to perform one step of Birkhoff normal form for a Klein-Gordon equation on Zoll manifolds, proving a time of existence of solutions that goes just beyond local existence theory. The result was then extended by performing Birkhoff normal form at any order and thus proving the almost global existence of solutions [6]. Extensions for quasi-linear equations and, on the other hand, for more general higher-dimensional domains need further analysis. This thesis furnishes a contribution in the second direction. We refer to Chapter 1 for a more detailed review of the relevant literature in this field.

I.1 Main results

The object of this Thesis is the study of the long-time behavior of small solutions of a class of Hamiltonian PDEs of the form

$$\mathrm{i}\dot{u} = H_L u + \nabla_{\bar{u}} P(u, \bar{u}), \;\; u \in H^s(M; \mathbb{C}) \tag{I.1.1}$$

with H_L a globally integrable quantum system that fulfills certain assumptions and P a generic, bounded, nonlinear perturbation. Within this class fall the non-linear Schrödinger (NLS) equation with convolution potential, the beam equation, and the plane wave stability problem for NLS. For these systems, we prove the almost global existence of small solutions. Essentially, it amounts to say that, given an initial datum u_0 of size ϵ in the Sobolev space H^s, the H^s norm of its related solution remains of order ϵ for a time of order ϵ^{-N}, for any $N > 0$. As a byproduct, one deduces that the time of existence and uniqueness of small, regular solutions goes far beyond the time guaranteed by local existence theory.

We begin the informal presentation of our abstract results describing the notion of globally integrable quantum system [12]. Since the celebrated Liouville theorem, it is well known that a classical d-dimensional Hamiltonian system is *integrable* if its Hamiltonian function can be written in terms of d variables, usually called the *actions* of the system. In our context, we call *quantum actions* a list of first order pseudodifferential operators $I_1, \ldots, I_d$, defined on a Riemannian manifold M; in particular, in analogy with classical integrable systems, we shall require that they commute mutually, $[I_i, I_j] = 0, \forall i, j$ and that their joint spectrum lays on a lattice, namely

$$\mathbb{Z}^d + \kappa \supset \Lambda = \left\{ a = (a^1, \ldots, a^d) \in \mathbb{R}^d \colon \exists \psi_a \in L^2(M) \text{ s.t. } I_j \psi_a = a^j \psi_a \right\}$$

for some $\kappa \in (0,1)$. This lattice plays a key role since it replaces the standard Fourier lattice on $\mathbb{T}^d$. We say that H is the Hamiltonian of a *quantum globally integrable* if it can be written in terms of those quantum actions. Namely, if there exists a smooth function $h \in C^\infty(\mathbb{R}^d, \mathbb{R})$ such that

$$H = h(I_1, \dots, I_d)\,.$$

Example I.1.1. *An easy example of a global integrable quantum system is the Laplacian on the torus* $\mathbf{T^d}$*. For that, the actions are given by* $I_j = D_j := -i\partial_j$ *so that* $h(\xi) = |\xi|^2$ *and*

$$-\Delta = h_L(I_1, \dots, I_d) = \sum_{j=1}^{d} \partial_j^2.$$

We remark that quantum integrable systems were already present in the literature, even with a slightly different approach: in particular, our definition of quantum integrable systems does not require nondegeneracy of the actions. We refer to [92, 93] for further reading.

The proof of our main theorem is based on a Birkhoff normal form procedure and so a key issue is represented by non-resonance conditions on the linear frequencies $\omega_a = h(a_1, \dots, a_d)$. We will come back to this crucial problem later on, in the discussion of the result (see also Chapter 1 for a more extended treatment of small divisor problems). Typically, in higher dimension domains, only extremely mild non-resonance conditions are available: this is the case for our problem. In particular, in the proof we will assume that there exists a set of indexes W, that we call *resonant*, such that the corresponding linear frequencies ω_a fulfill the following condition.

Assumption I.1. *For any* $r \geq 3$*, there are constants* $\gamma > 0, \tau > 0$ *such that the following holds for any* $(a_1, \dots, a_r) \in \Lambda^r \setminus W$

$$\left| \sum_{j=1}^{r} \sigma_j \omega_{a_j} \right| \geq \frac{\gamma}{(\max_{j=1,\dots,r} |a_j|)^\tau}$$

with $\sigma_j = \pm 1$.

In the applications we will take profit of a suitable external or internal parameter for tuning the frequencies, proving that nonresonance conditions are fulfilled in a full-measure set.

Under proper, but rather generic, conditions on the function h (namely, steepness and homogeneity), we will prove a clusterization property of the set $\{a, \omega_a\}_a$; this fact follows directly from the abstract construction in [12], so, in principle, it applies to any globally integrable system that fulfills these generic conditions. The same clusterization was already proved for many of the concrete examples that we can deal with; it was proved by Bourgain for the eigenvalues of the Laplacian on a square torus [29, 32] and then extended to irrational tori [23]. It was proved also for surfaces of revolution [42]. We stress that this clusterization of the frequencies is a key breakthrough that allows to deal with higher dimensional problems [7]. The precise hypothesis we assume is the following.

Assumption I.2. *There exists a partition*

$$\Lambda = \bigcup_{\alpha \in \mathfrak{A}} \Omega_\alpha$$

with the following properties.

i. Each Ω_α is dyadic, namely there exists a constant C, independent of α, such that

$$\sup_{a \in \Omega_\alpha} |a| \leq C \inf_{a \in \Omega_\alpha} |a|.$$

ii. There exist $\delta > 0$ such that, if $a \in \Omega_\alpha$ and $b \in \Omega_\beta$ with $\alpha \neq \beta$, then

$$|a - b| + |\omega_a - \omega_b| \geq C_\delta(|a|^\delta + |b|^\delta).$$

A mild assumption on the dispersive relation of the frequencies is needed in the statement of the main theorem. However, in practice, we shall restrict to operators with eigenvalues with super-linear asymptotic ($\beta > 1$) since there exist no examples of linear or sublinear asymptotics ($\beta \leq 1$) for which Bourgain partition holds. This limitation plays a major role since it prevents the application of our method to problems characterized by linear or sub-linear dispersive relation, for example, wave equations.

Assumption I.3. *There exist constants $C_1, C_2 > 0$ and β, with $\beta > 0$, s.t.*

$$C_1 |a|^\beta \leq |\omega_a| \leq C_2 |a|^\beta .$$

Eventually, we will assume that the non-linearity P is a rather generic bounded functional; we require it to have a zero of order at least 3 in the origin and to take real values when evaluated on real functions u.

Assumption P. *P is a functional defined on a neighborhood of the origin of $H^s(M) \times H^s(M)$ for some positive $s > d/2$, that has a zero of third order at the origin and has the structure*

$$P(u, \bar{u}) = \left(\int_M F(N(u, \bar{u}), u(x), \bar{u}(x), x) dx \right) ,$$

where

$$N(u, \bar{u}) := \int_M u(x)\bar{u}(x) dx ,$$

and $F \in C^\infty(\mathcal{U} \times \mathcal{U} \times \mathcal{U} \times M; \mathbb{C})$ is a smooth function and $\mathcal{U} \subset \mathbb{C}$ an open neighbourhood of the origin.

Under these assumptions, our main abstract theorem is the following.

Theorem I.1.2. *Consider the Hamiltonian system* (I.1.1) *that fulfills the assumptions I.1,I.2,I.3,P. Then, for any integer $r \geq 3$, there exists $s_r \in \mathbb{N}$ such that, for any $s \geq s_r$, there are constants $\epsilon_0 > 0$, $c > 0$ and $C > 0$ for which the following holds: if the initial datum $u_0 \in H^s(M, \mathbb{C})$ fulfills*

$$\epsilon := \|u_0\|_s < \epsilon_0,$$

then the Cauchy problem has a unique solution $u \in \mathcal{C}^0((-T_\epsilon, T_\epsilon), \mathcal{H}^s(M, \mathbb{C}))$ with $T_\epsilon > c\epsilon^{-r}$. Moreover, one has

$$\|u(t)\|_s \leq C\epsilon, \quad \forall t \in (-T_\epsilon, T_\epsilon) .$$

As outlined before, we apply our abstract theorem to some non-linear Hamiltonian PDEs. In particular, we will consider equations on Riemannian manifolds M on which the Laplacian-Beltrami Δ_g operator is a globally integrable quantum system, according to our definition. These are manifolds with integrable (in the classical sense) geodesic flow [92]. To be concrete, we consider:

i) any flat torus $\mathbb{T}^d_g$. Results of this type on flat tori are well-known nowadays for linear equations [42, 23, 13] and for non-linear systems [7]. We remark that our result extends the one in [7] since it applies to *all* flat tori.

ii) surfaces of revolution, on which the actions are constructed in [94]. Results on the growth of Sobolev norm on surfaces of revolution were known in the literature [42, 12], but only for linear systems. To the best of our knowledge, our theorem is the first result of almost global existence and growth of the Sobolev norm for non-linear equations on generic rotation invariant surfaces.

iii) Lie groups and homogeneous spaces, on which the actions are constructed explicitly in [12]. The existence of quasi-periodic solutions for both Schrödinger and wave equations on Lie groups has also already been proved [26, 20].

Before introducing the main ideas of the proof, we present quickly the results that follow as an application of our abstract theorem. The first result pertains to a nonlinear Schrödinger equation with convolution potential:

$$\begin{cases} i\partial_t \psi = -\Delta\psi + V * \psi + f(x, |\psi|^2)\psi, \quad x \in M \\ \psi(0) = \psi_0 \end{cases} \tag{I.1.2}$$

where the nonlinearity f is a smooth function in a neighborhood of the origin and has a zero of two in $u = 0$. The Fourier multipliers V belong to the set

$$\mathcal{V}_n := \left\{ V = \{V_a\}_{a \in \Lambda} \ : \ V_a \in \mathbb{R} \ , |V_a| \langle a \rangle^n \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\} \ ,$$

The precise statement is the following.

Theorem I.1.3. *There exists a set $\mathcal{V}^{(res)} \subset \mathcal{V}_n$ of zero measure, s.t., if $V \in \mathcal{V} \setminus \mathcal{V}^{(res)}$ the following holds. For any $r \in \mathbb{N}$, there exists $s_r > d/2$ such that for any $s > s_r$ there is $\epsilon_s > 0$ and $C > 0$ such that if the initial datum for* (I.1.2) *belongs to H^s and fulfills $\epsilon := \|\psi_0\|_s < \epsilon_s$ then*

$$\|\psi(t)\|_s \leq C\epsilon \,, \quad \textit{for all} \quad |t| \leq C\epsilon^{-r} \,.$$

This system has attracted a lot of research in the last years since the presence of the Fourier multiplier allows direct control of small divisors, but it contains all the typical difficulties in considering higher dimensional domains [66, 55, 17].

The second application concerns the beam equation

$$\psi_{tt} + \Delta^2\psi + m\psi = -\partial_\psi F(x, \psi) \ , \tag{I.1.3}$$

with $F \in C^\infty(M)$ in a neighborhood of the origin and with a zero of order at least 2 at $\psi = 0$. Here m is a real positive parameter, usually called *mass*. The precise statement of the result is analogous to the one in Theorem I.1.3.

Theorem I.1.4. *There exists a set of zero measure $\mathcal{M}^{(res)} \subset \mathbb{R}^+$ such that if $m \in \mathbb{R}^+ \setminus \mathcal{M}^{(res)}$ then for all $r \in \mathbb{N}$ there exist $s_r > d/2$ such that the following holds. For* *any $s > s_r$ there exist ϵ_{rs}, c, C such that if the initial datum for* (I.1.3) *fulfills*

$$\epsilon := \left\|(\psi_0, \dot\psi_0)\right\|_s := \|\psi_0\|_{H^{s+2}} + \left\|\dot\psi_0\right\|_{H^s} < \epsilon_{sr} \ ,$$

then the corresponding solution satisfies

$$\left\|\left(\psi(t), \dot\psi(t)\right)\right\|_s \leq C\epsilon\,, \quad \textit{for} \quad |t| \leq c\epsilon^{-r}\,.$$

The beam equation is widely studied in the literature for both its mathematical interest and its physical interpretation. We recall for example [58, 59, 86] for a first KAM approach to the problem. In [75] the author proves that the solutions remain stable for a time of order $\epsilon^{-\frac{5}{4}n}$, n being the order of the semi-linear perturbation, thus a time slightly longer than the one assured by the local theory. This result was improved up to a time of order ϵ^{-3n} and extended to irrational tori of any dimension [16]. Our result improves both of them since it proves almost-global existence, hence existence up to $T = \epsilon^{-M}, \forall M \geq 0$. Both of those results are based on a finite number of steps of Birkhoff normal form; our technique allows us to perform Birkhoff normal form at any fixed order.

Our third application regards the stability in Sobolev norm of the ground state solution of the NLS equation. Given a NLS equation of the form

$$i\dot\psi = -\Delta\psi + f\left(|\psi|^2\right)\psi \ , \tag{I.1.4}$$

with $f \in C^\infty(\mathcal{U};\mathbb{R})$, $\mathcal{U} \subset \mathbb{R}$ an open neighbourhood of the origin, f having a zero of order at least one at the origin, it is well-known that it has a solution given by a plane wave of the form $\psi_*(t) = \sqrt{p_0}e^{-i\nu t}$ provided that $\nu = f(p_0)$. We will prove that this solution is stable in high Sobolev regularity for almost any value of the L^2 norm of the initial datum. Denoteing with $\bar\lambda$ the lowest non vanishing eigenvalue of $-\Delta$, we will prove the following result.

Theorem I.1.5. *Assume there exists $\bar p_0 > 0$ such that $\bar\lambda + 2f(p_0) > 0$ for any $p_0 \in (0, \bar p_0]$. Then there exists a zero measure set $\mathcal{P}$ such that if $p_0 \in (0, \bar p_0] \setminus \mathcal{P}$ then for any $r \in \mathbb{N}$ there exists s_r for which the following holds. For any $s \geq s_r$, there exists constants ϵ_0 and C such that if the initial datum ψ_0 fulfills*

$$\|\psi_0\|_0^2 = p_0, \qquad \inf_{\alpha\in\mathbb{T}} \left\|\psi_0 - \sqrt{p_0}e^{-i\alpha}\right\|_s = \epsilon \leq \epsilon_0\,,$$

then the corresponding solution fulfills

$$\inf_{\alpha\in\mathbb{T}} \left\|\psi(t) - \sqrt{p_0}e^{-i\alpha}\right\|_s \leq C\epsilon \qquad \forall\, |t| \leq C\epsilon^{-r}$$

with $\psi(0) = \psi_0$.

Note that in the *defocusing* case (i.e. f is a positive function) there is no restriction in the L^2 norm of the initial datum since the condition $\bar{\lambda}+2f(p_0) > 0$ is automatically satisfied. Our result partially extends the known one on the standard torus $\mathbb{T}^d$, where the same problem, for any plane waves, is considered [54]. We remark that in our approach we are not allowed to consider generic plane waves, since in general the eigenfunctions of the Laplacian, on the manifolds we consider, do not have constant modulus. For flat tori, on which the eigenvalues are complex exponential of modulus 1, the same result for any plane wave can indeed be proven.

I.2 Ideas of the proof

In the last part of this introduction, we sketch the strategy of the proof of our results. We begin discussing the proof of Theorem I.1.2, essentially based on a Birkhoff normal form iteration, trying to underline the main technical steps and the novelty of our approach. Two major issues are to be considered when dealing with high-dimensional domains. Firstly, one has to deal with bad spectral properties, namely the presence of arbitrarily small gaps between linear frequencies, that make non-resonance conditions quite mild and hard to handle. On the other hand, the eigenfunctions present bad *localization* properties; for that reason, a key ingredient in our scheme is represented by a multilinear estimate of the product eigenfunctions (see Theorem 3.2.1), that leads to a suitable tame estimate of the vector fields associated to polynomials belonging to a certain class, that we will denote as *polynomials with localized coefficients*.

Given a function $u \in H^s$, we consider its spectral decomposition $u = \sum \Pi_a u$; consequently, one can write a polynomial P of degree r in terms of a convenient multilinear, symmetric function

$$P(u) = \sum \tilde{P}(\Pi_{a_1}u, \dots, \Pi_{a_r}u)\,. \tag{I.2.1}$$

We introduce the class of *polynomial with localized coefficients* [5, 47], whose elements fulfill an estimate of the form (here we are assuming for simplicity that $|a_1| \geq |a_2| \geq$

$\cdots \geq |a_r|)$

$$\left|\widetilde{P}(\Pi_{a_1}u, ..., \Pi_{a_r}u)\right| \leq C_N \frac{|a_3|^{\nu+N}}{|a_1-a_2|+|a_3|)^N}\|\Pi_{a_1}u\|_0 \tag{I.2.2}$$

for any $N>0$ and for some fixed $\nu>0$, independent from N. In Lemma 4.2.5, we prove that the vector field associated with a polynomial in this class satisfies a tame estimate of the form

$$\|X_P(u)\|_s \leq C\|u\|_s^{r-1}\|u\|_{s_0},$$

that allows linking the order of a polynomial with the smallness condition of the initial condition. Namely, we introduce a suitable norm that controls the homogeneity of a polynomial so that, if P has degree R, we have

$$\sup_{\|u\|_s\leq R}\|X_P(u)\|_s \sim R^r,\ R \ll 1\,.$$

Moreover, this class is closed with respect to Poisson brackets, so that the structure persists along the normal form iteration. This is the content of Lemma 4.2.9.

In Theorem 5.2.2, we prove that indeed the non-linear perturbation we consider (see Hypothesis P) belongs to this class. As outlined before, this is the consequence of an estimate of the product of the eigenfunctions of the form

$$\left|\int \psi_{a_1}\psi_{a_2}\psi_{a_3}\right| \leq C_N \frac{a_3^{\nu+N}}{(|a_1-a_2|+|a_3|)^N}\,, \tag{I.2.3}$$

proved in Theorem 3.2.1. We remark that a similar estimate was already proved and exploited [5, 47]. In our work, this result is the direct consequence of the geometrical structure carried out by globally integrable quantum systems, and, in particular, it relies directly on the existence of the lattice Λ. Since we label indexes according to Λ, computations are more involved with respect to the case of a one-dimensional lattice [5, 47] and a more extended analysis is implemented. The pseudodifferential nature of the quantum actions is also a key ingredient, as can be inferred from the proof. Already in [39] Craig and Wayne recognized the localization of eigenfunctions on the exponentials basis as a fundamental function in the normal form approach to Hamiltonian systems. We point out that (I.2.3) replaces the *localization* of the product of the eigenfunctions controlling the fast off-diagonal decay of their product. We remark that similar bilinear estimates of the eigenfunctions of the Laplacian were a key ingredient in proving global well-posedness for NLS on three-dimensional spheres, even by means of completely different methods [34].

In the Birkhoff normal form approach, at each step of the iteration, one looks for a canonical transformation induced by a polynomial G, that is the sum of some multilinear, homogeneous functions G_r of degree r; each of them solves a homological equation of the form

$$\left(\sum_{j=1}^{q}\omega_{a_j}-\sum_{j=q+1}^{r}\omega_{a_j}\right)\tilde{G}(\Pi_{a_1},\dots,\Pi_{a_r})=\tilde{P}(\Pi_{a_1},\dots,\Pi_{a_r})$$

for each choice of non resonant indexes $(a_1,\dots,a_r)$. To do that, the nonresonant condition in the Assumption I.1 is not enough, since it induces a loss of derivatives at each step, involving the largest index $\max\{a_1,\dots,a_r\}$. To avoid this loss, we need a so-called second-order Melnikov condition, that is typically automatically satisfied in one-dimensional domains; namely, we prove an estimate that entails the third largest index. At this end (see Lemma 5.3.16), we exploit the clusterization of the frequencies. We begin splitting the system in high and low mode with respect to a fixed cut-off $K \gg 1$.

- For the system in *low modes* ($|a_j| \leq K$), we proceed as usual in Birkhoff normal form theory for finite dimensional systems: we divide the set of frequencies in separated blocks Σ_n, we identify the so-called *superactions*

  $$I_n=\sum_{a\in E_n}\|\Pi_a u\|_0^2$$

  and we proceed to remove terms that produce the flow of energy between them, until a fixed order, since they coincide with our definition of nonresonant indexes (see Def. 5.1.3).

- Monomials that involve three or more large indexes ($\max_3(a_1,\dots,a_j) > K$) can be included in the remainder since they have small vector field (see Lemma 4.2.13), as already pointed out in [9].

- Monomials with exactly one index larger then K are non-resonant (see Lemma 5.3.15).

- Monomials with exactly two indexes larger then K are treated exploiting the partition described in Assumption I.2; if $a, b \notin \Omega_\alpha$ then either two indexes ($|a-b| \geq K^\delta$) are far apart or their related frequencies ($|\omega_a-\omega_b| \geq K^\delta$) are non resonant. In the first case, we take profit of the fast off-diagonal decay (I.2.2) and we prove that the contribution to the vector field is small (see Lemma 4.2.14); in the other case, the term is nonresonant and it is removed in the normal form iteration (see Lemma 5.3.16). Therefore, again we have proved that, in the normal form dynamics, the transfer of energy can occur only within each "dyadic" block Ω. Exploiting dydicity, from the conservation of L^2 norm we deduce the conservation of H^s norm.

At the end of the Birkhoff normal form iteration, the initial Hamiltonian is conjugated to $H^{(r)} = H_0 + Z + R$, with $\|X_R\|_s \sim R^{r+1}$, for any order $r \geq 3$ fixed at the beginning of the iterative procedure. Since the dynamics induced by the normal form Z conserve the H^s for any time, a standard bootstrap argument provides a control in the blow-up of the solutions for time of order ϵ^{-r}.

In Chapter 6 we conclude the proof of Theorems I.1.3, I.1.4 and I.1.5, applying the abstract theorem after simple algebraic manipulations to put the equations in the suitable form (I.1.1). The verification of Assumptions I.2 and I.3 follows directly from the geometry of globally integrable quantum systems. In fact, in Chapter 3 we prove that, under suitable but generic assumptions on the Hamiltonian H_L (essentially, we ask h_L to be a steep function, see Def. 3.2.5), the frequencies $\omega_{aa\in\Lambda}$ fulfill the Bourgain's partition property. Some examples of those manifolds (tori, Lie groups, surface of revolution) have been discussed beforehand in the Introduction. The verification of the nonresonance condition in the Assumption I.2 is rather standard. For each system, we deduce it by modulation of a given parameter: respectively, the Fourier multipliers in the convolution potential, the mass, and the L^2 norm of the plane wave. For the beam equation and the stability of ground state we exploit a sub-analytical method [47].

The Thesis is organized as follows. Chapter 1 contains a brief review of some known results and literature about the long-time analysis of the solutions of Hamiltonian PDEs. To be coherent we the results presented in this Thesis, we mostly treat equations on compact manifolds. In Chapter 2 we describe pseudodifferential calculus on Riemannian manifolds; in particular, we prove Lemma 2.3.11, a key technical result that allows us to prove the estimate of the product of the eigenfunctions of a quantum global integrable system. In Chapter 3 we introduce the notion of globally integrable quantum system and we conclude the estimate on the product of the eigenfunctions. The proof of the clusterization of the frequencies is also outlined. In Chapter 4 we study the property of polynomials with localized coefficients. In Chapter 5 we state our main abstract result and we develop the Birkhoff normal form iteration. Finally, in Chapter 6 we prove that it applies to the concrete examples.

Chapter 1

Long-time existence and growth of Sobolev norm

In this Chapter, we discuss some known results about the long-time behavior and the growth of Sobolev norm of the solutions of Hamiltonian PDEs. The interest in such problems is twofold, for their mathematical and physical aspects. On the one hand, these results extend the theory of local-wellposedness employing a variety of techniques; on the other hand, the growth of Sobolev norm is an important phenomenon in weak turbulence theory, since it describes the flow of energy towards higher internal frequencies of a system [37]. Along this Chapter, we revise some important results, with a focus on the ones obtained with perturbative normal form techniques, trying to highlight some key issues, in particular in considering higher dimensional domains. We start discussing some known results on linear systems; then, we take into account semilinear, i.e. equations with nonlinear bounded perturbations, and quasi-linear systems, i.e. equations with nonlinear unbounded perturbations.

1.1 Growth of Sobolev norm in linear systems

In the last thirty years, linear systems have attracted a lot of research interest. In fact, besides the interest *per sé*, in many cases, their comprehension represents a key step toward problems concerning nonlinear equations. As already pointed out in the Introduction, a seminal result regards the Schrödinger operator

$$iu_t + \Delta u + V(x,t)u = 0 \tag{1.1.1}$$

on $\mathbb{T}^d$, with V a real potential, periodic in x [29]. In particular, Bourgain proved a t^ϵ bound for the H^s norm of the solution. Precisely, the result is the following.

Theorem 1.1.1 (Bourgain, [29]). *Consider* (1.1.1) *in arbitrary dimension d with periodic boundary conditions. Assume V is bounded, smooth in x and t, and periodic in x. Then, denoting with $S(t)$ the associated flow map, for all $s < \infty$, $\epsilon > 0$,*

$$\|S(t)u(0)\|_s \leq Ct^\epsilon \qquad \text{for } t \to +\infty\,.$$

The proof relies on some spectral properties: the localization of the eigenfunctions of the Laplacian on the torus and a partition of the spectrum in clusters whose distance is increasing, as already discussed in I.2. Bourgain extended the result to quasi-periodic in time potential [31], proving a logarithmic-in-time bound of the form

$$\|S(t)u(0)\|_s \leq C\log(t)^{C_s} \qquad \text{for } t \to +\infty\,.$$

Later on, Delort considered a similar problem on a more general class of compact manifolds, that contains for example Zoll manifolds and surface of revolution, on which the eigenvalues fulfill a similar clusterization [42]. Although the result on the bound of the growth is the same, the proof is rather different, since it follows from an abstract result based on symbolic calculus. Such an abstract result was then exploited to prove that the t^ϵ bound extends to any arbitrary flat torus [23].
All of these results concern *bounded* perturbations. A first abstract result that involves an *unbounded* Schrödinger operator of the form $i\psi_t = L(t)\psi$ was given in [82]. In particular, the authors recognized some properties on L under which the equation is globally well-posed, with a polynomial growth of Sobolev norm. Remarkably, they prove that, if the spectrum L fulfills a further hypothesis very similar to the one assumed by Bourgain, therein called of *increasing gaps*, the control on the growth of Sobolev can be improved up a time of order t^ϵ. The result is optimal [91].
A key breakthrough was achieved by combining these techniques with pseudodifferential calculus [11] (see also [84]. Through an iterative normal form procedure essentially based on pseudodifferential calculus, an abstract unbounded Schrödinger operator is conjugated to a pure point operator with a smoothing perturbation and a bound of order t^ϵ may be proved. The same pseudodifferential smoothing procedure, altogether with a careful analysis of the geography of the resonances inspired by classical Nekhoroshev theorem, was then exploited to prove the result to hold on flat tori of any dimension [13]. Eventually, we recall the result in [12], to which, as pointed out in the Introduction, this Thesis owes much. In that work, the authors proved a t^ϵ upper bound for the solutions of the equations of motion associated with a wide general

linear Schrödinger operator on compact Riemannian manifolds, of the form

$$iu_t = (H_0 + V(t))\, u$$

with H_0 a globally integrable quantum system (see I.1 above for a definition of these systems). This abstract theorem is then applied to many concrete examples, that extend the known results of Delort and Bourgain to *unbounded* perturbations; in advance, the growth of Sobolev norm on Lie groups is firstly considered. In particular, we remark on the role of the spectral lattice in the proof: as discussed in this Thesis, it has been revealed to be a key breakthrough, altogether with Bourgain's clusterization, that allows us to consider general higher dimensional domains.

1.2 Non linear systems

1.2.1 A general approach with Birkhoff normal form

Since the first seminal results of Bourgain [31, 32] and Colliander, Keel, Staffilani, Takaoka and Tao [37], a large variety of tools have been developed to study the asymptotic behavior of nonlinear Hamiltonian PDE's. Among them, the approach through infinite dimension Birkhoff normal techniques has resulted to be particularly effective. The first important result was obtained for the semilinear wave equation [4, 9]

$$u_{tt} - \Delta u + \mu u + f(u) = 0\,. \tag{1.2.1}$$

on a one-dimensional domain with periodic boundary conditions. In particular, the following theorem of almost global existence is proved.

Theorem 1.2.1 ([9]). *Let $r \geq 2$. For almost every μ there exists s_r such that, for any $s > s_r$, for ϵ small enough*

$$\|(u(0), \dot{u}(0))\|_{H^s \times H^{s-1}} \leq \epsilon \implies \|(u(t), \dot{u}(t))\|_{H^s \times H^{s-1}} \leq 2\epsilon$$

for $t \leq \epsilon^{-r}$.

The crucial tool in the proof is a nonresonance condition that involves the linear frequencies of the system, namely $\omega_j = \sqrt{j^2 + \mu}$ with $j \in \mathbb{Z}$, that reads:

$$\left|\omega_{j_1} + \cdots + \omega_{j_r} - \omega_{k_1} - \cdots - \omega_{k_q}\right| \geq \frac{\gamma}{\max_3(\mathbf{j}, \mathbf{k})^\tau} \tag{1.2.2}$$

for any $|\mathbf{j}| = (|j_1|, \ldots, |j_r|) \neq (|k_1, \ldots, k_q|) = |\mathbf{k}|$. Here $\max_n(v)$ denote the n^{th} largest

element in the vector v. This condition allows, using a Birkhoff normal form procedure, to remove from the Hamiltonian all the monomials involving at most two large Fourier modes. In fact, at each step of the iteration, the solution of the homological equation induces a bounded canonical equation. Then, a trivial, but crucial observation allows neglecting terms with three or more large indexes since their vector is small [4]. Later on, such a nonresonance condition was recognized in several systems and almost global existence has been proved, for example, for plane waves in NLS [54], for one-dimensional Schrödinger operators and d-dimensional Schrödinger operator with convolution potential on tori [5, 9], for the quantum harmonic oscillator [64].
A similar nonresonance condition was also exploited by Delort and Szeftel, proving a partial result of long-time existence for a Klein-Gordon equation on spheres and Zoll manifolds [47, 48]. Later on, some of those results were extended to almost global existence through Birkhoff normal form [6]. Despite the structure of the linear frequencies being very similar to the one in one-dimensional domains, the results in [47, 48] (see also [46]) are quite relevant for our work. In fact, they are based on a crucial multi-linear estimate of the decay of the product of the eigenfunctions of the linear operator, that inspired the results presented in this Thesis, in particular those treated in Chapter 3.

Regarding quasi-linear perturbations, we quote for instance the breakthrough achieved combining normal form methods and para-differential calculus in the context of Klein-Gordon equation on a one-dimensional sphere [41]. The procedure is indeed rather general; later on, similar paradifferential techniques were exploited to prove almost global existence for even solutions of the capillarity-gravity water waves equation on a one-dimensional domain [45]. This result has been fully extended to any solution of that equation, employing a Darboux iterative procedure that corrects the non-symplectic nature of paradifferential linearization [24].

1.2.2 Towards general higher dimensional domains

The rather strict nonresonance condition discussed in (1.2.2) typically fails in higher dimensional domain, even in the simple case of the Laplacian on the irrational torus. In fact, assuming $|j_1|, |k_1| \geq \max_3(\mathbf{j}, \mathbf{k})$, in dimension $d \geq 2$ the difference $|j_1|^2 - |k_1|^2$ describes a dense set, that prevents condition (1.2.2) to be verified for many values of the tuning parameter. Instead, in many cases a first-order Melnikov condition, that involves the second-largest index, can be proved:

$$\left|\omega_{j_1} + \cdots + \omega_{j_r} - \omega_{k_1} - \cdots - \omega_{k_q}\right| \geq \frac{\gamma}{\max_2(\mathbf{j}, \mathbf{k})^\tau}\,.$$

This condition produces a loss of derivatives at each step of the iteration, preventing Birkhoff normal form from being applied straightforwardly.
This is the case for example for the existence of small solutions of the Klein-Gordon equation on $\mathbb{T}^d$, proven for a time just slightly beyond the one provided by local theory [43]. With a similar nonresonance condition, a much longer time existence can be proved, but with a loss of derivative in the initial datum [15]. The precise result is the following.

Theorem 1.2.2 ([15])**.** *Consider the non linear wave equation* (1.2.1)*, with $x \in \mathbb{T}^d$, $d \geq 2$. For any $r \geq 2$, $s \geq 2$, there exists $s_* = s_*(r,s)$ and ϵ small enough such that*

$$\|(u(0), \dot{u}(0))\|_{H^{s_*} \times H^{s_*-1}} \leq \epsilon \implies \|(u(t), \dot{u}(t))\|_{H^s \times H^{s-1}} \leq 2\epsilon$$

for any $t \leq \epsilon^{-r}$.

The strategy adopted in the proof is based on a decomposition in high and low modes according to a threshold that depends on the size of the perturbation, $N_\epsilon \sim \epsilon^{-\frac{r}{s-s_0}}$. Then, the loss of derivatives produced by small divisors is tackled employing a *pseudodifferential* commutator estimate, that provokes the loss $s_* > s$. As already discussed in the Introduction, combining a similar decomposition with Bourgain's decomposition of the frequencies, almost global existence without any loss in regularity has been proved for some semilinear equations on flat tori of any dimension [7].
Many other long-time existence results have been proved in the last few years. We recall the ones for semilinear beam equation [75, 16] and for an hydrodynamic system on irrational tori [56]. Regarding quasi-linear equations, we quote for example [55] in which time of existence of order $t \sim \epsilon^{-8/3}$, thus strictly beyond the local existence, is proved for the solutions of the Klein-Gordon equation on tori. The quadratic lifespan of solutions is shown also for a derivative Schrödinger equation [57], and water waves [21]. In particular, the water wave equation in higher dimensional domain has been widely studied in the last few years, see for instance [22, 74, 76] and literature therein.

Almost any result about long-time existence and Sobolev stability pertains to very regular solutions (i.e. solutions in H^s with $s \gg 1$) but, interestingly this condition appeared not to be required, at least in some numerical computations [36, 35]. Indeed, this observation has been confirmed analytically in [17] for a Schrödinger operator with a "well-prepared" convolution potential on $\mathbb{T}^d$. In particular, the authors can prove almost global existence for any initial datum $u_0 \in H^s$ with $s_0 \geq d/2$. Although the equation is rather artificial, the technique is quite interesting, since it relies essentially only on Birkhoff normal form and a novel analysis of small divisors, but it applies to low regular solutions. We recall also [18] for other results of long-time existence in low regularity.

1.3 Growth of Sobolev norm: lower bounds

Until now, we discussed results about the upper bounds of the growth of Sobolev norm. A natural, opposite problem is to exhibit solutions whose Sobolev norm grows arbitrarily in time; this amounts to prove that there exists a mechanism, often referred to as *forward energy cascade* in weak turbulence theory, that allows the energy to migrate towards higher internal frequencies of the system. Initially, the question was posed by Bourgain for the NLS equation.

Question (Bourgain, [33]). *Are there solutions $u(t)$ of the cubic nonlinear defocusing Schrödinger equation*

$$iu_t - \nabla u = -|u|^2 u, \qquad x \in \mathbb{T}^2$$

such that, for some $s > 1$,

$$\mathrm{lim\,sup_{t\to\infty}\|u(t)\|_{H^s}} = +\infty$$

In the last thirty years, much effort has been made to understand this problem, but, despite the known results, a lack of general comprehension persists.
Considering linear Schrödinger operators of the form

$$i\psi_t = K_0\psi + V(t)\psi\,. \tag{1.3.1}$$

Maspero proposed the notion of *transporter* [81]; essentially, a potential $V(t)$ is a transporter if there exists a solution with unbounded growth of Sobolev norm. Adopting this terminology, the first example of a transporter was constructed by Bourgain [31] and then extended to the harmonic oscillator [44]. Later on, examples of universal transporter (i.e. potential that produces unbounded growth of Sobolev norm for *any* solution) were presented for the harmonic potential, the half-wave equation on $\mathbb{T}$ and the Dirac equation on Zoll manifolds [10, 80, 71]. All of those examples have in common a quite rigid spectral structure since they present constant spectral gaps.

Concerning nonlinear systems, the situation is even more involved. In 2010, Colliander, Keel, Staffilani, Takaoka, Tao proved the following remarkable theorem, that proposes a partial solution for the Bourgain conjecture [37].

Theorem 1.3.1 (Colliander, Keel, Staffilani, Takaoka, Tao [37])**.** *Let $s > 1$, $C \gg 1$ and $\mu \ll 1$. Then there exists a solution of the cubic NLS on $\mathbb{T}^2$ and $T > 0$ such that*

$$\|u(0)\|_{H^s} < \mu\,, \qquad \|u(T)\|_{H^s} > C\,.$$

This solution is constructed following closely a completely resonant model obtained by restricting the Fourier support on a certain lattice $\Lambda \subset \mathbb{Z}^2$. Many further related results about NLS on $\mathbb{T}^d$ are known [69, 72, 65, 67]. All of those results rely on the particularly rich structure of the resonances provided by square tori. In fact, the irrationality of the torus may mitigate the exchange of energy between the degrees of freedom [90], and thus the strategy adopted in [37] does not apply on irrational tori. Moreover, on irrational tori, the polynomial-in-time upper bounds for the growth of Sobolev norms is slower with respect to the rational case [49, 50]. However, Guardia and Giuliani proved the existence of solutions of the cubic NLS undergoing arbitrarily large, but finite, growth even on many irrational tori [61]. The proof, still inspired by the one in [37], is based on the approximation of a *quasi-resonant* model obtained with a partial normal form reduction of the system, so avoiding the small divisor problems typical of irrational tori.
Examples of Sobolev instability results for systems different from NLS or in different domains are also present in the literature. See for example [70, 63, 62, 40] and the literature therein. We conclude this section by recalling an interesting work that adopts a quite different point of view [60]. In that work, a Sobolev instability is obtained employing the completely non-resonant mechanism of Arnold diffusion on an a-priori unstable infinite Hamiltonian lattice; it should be remarkable to prove a similar mechanism occurring also in a proper PDEs (i.e. a-priori stable) context.

Chapter 2

Pseudodifferential operators on Riemannian manifolds

In the first part of this Chapter, we recall some well-known definitions and results about pseudo-differential operators on Riemannian manifolds [73, 83]. In the second part, inspired by similar results we prove a key estimate concerning commutators of pseudodifferential operators [47, 48]. We will use this abstract estimate to prove a suitable decay of the product of eigenfunctions in the proof of our main result. All along this Section, M will denote a generic compact Riemannian manifold.

2.1 Local symbols

Definition 2.1.1 (Global symbols)**.** *For any $m \in \mathbb{R}$, a function $a \in C^\infty(\mathbb{R}^n)$ is said to be a (global) symbol of order m if for any $\alpha, \beta \in \mathbb{N}^n$ there exists a constant $C_{\alpha,\beta}$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|} \qquad x \in \mathbb{R}^n, \xi \in \mathbb{R}^n \,.$$

We will denote with $S^m(\mathbb{R}^n)$ the set of symbols of order m.

Definition 2.1.2. *For $a \in S^m(\mathbb{R}^n)$, we define its seminorms as*

$$|a|_\nu^m := \sup_{x\in\mathbb{R}^n} \sup_{\xi\in\mathbb{R}^n} \sup_{|\alpha|+|\beta|\leq\nu} \langle \xi \rangle^{-m+|\beta|} \left| \partial_\xi^\beta \partial_x^\alpha a(x,\xi) \right|$$

Definition 2.1.3 (Local symbols)**.** *Given an open subset $X \subset \mathbb{R}^n$, for every $m \in \mathbb{R}$ we define*

$$S^m(X) \coloneqq \{a \in C^\infty(X) \colon \phi(x)a(x,\xi) \in S^m, \, \forall \phi \in C_0^\infty(X)\} \,.$$

In other words, for any compact $K \subset X$, there are constants $C_{\alpha\beta,K}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta,K}(1+|\xi|)^{m-|\beta|} \qquad x \in K, \xi \in \mathbb{R}^n\,.$$

Here $C_0^\infty(X)$ is the set of smooth functions with compact support in $\mathbb{R}^n$.

Given $a \in S^m(X)$, the operator

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} a(x,\xi)\hat{u}(\xi) d\xi \tag{2.1.1}$$

restricts to a well defined operator $A\colon C_0^\infty(X) \to C^\infty(X)$. In particular, it ensures all the properties of pseudodifferential operators defined on the whole $\mathbb{R}^n$, since $u \in C_0^\infty(X)$ implies that $u(x)a(x,\xi) \in S^m$. As usual, we will denote with $A = Op(a)$ the pseudodifferential operator associated with $a \in S^m(X)$ and with $OPS^m(X)$ the set of pseudodifferential operators defined on the open subset X.

Remark 2.1.4. *The Schwartz kernel of the operator* (2.1.1) *is given by*

$$K(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)} a(x,\xi) d\xi \tag{2.1.2}$$

We will indicate with K_A the kernel associated with A. It is well known that $K_A \in C^\infty(X \times X)$ if and only if A is a smoothing operator.

The precise relation with global operators is clarified by the following proposition.

Proposition 2.1.5. *Let X be an open subset of $\mathbb{R}^n$. Given $A\colon C_0^\infty(X) \to C^\infty(X)$ linear and continuous, if for any $\phi, \psi \in C_0^\infty(X)$ the operator $u \to \phi A \psi u$ is in $OPS^m(\mathbb{R}^n)$, then there exists $a \in S^m(X)$ such that*

$$A = Op(a) + A_0$$

with A_0 with smooth kernel, namely $K_{A_0} \in C^\infty(X \times X)$.

Proof. Let $\{\psi_i\}_{i\in\mathcal{I}}$ a locally finite partition of unity in X. Then, we denote $A_{jk}u = \psi_j A \psi_k u$, with $a_{jk} \in S$ the correspondent symbols and we define

$$a(x,\xi) = {\sum}' a_{jk}(x,\xi) := \sum_{\mathrm{supp}(\psi_j)\cap\mathrm{supp}(\psi_k)\neq\emptyset} a_{jk}(x,\xi)\,.$$

By definition, this is a locally finite sum and so $a \in S^m(X)$. The kernel of $A - Op(a)$

is then given by

$$\sum_{\operatorname{supp}(\phi_j)\cap\operatorname{supp}(\psi_k)\ =\emptyset} \psi_j(x)K_A(x,y)\psi_k(y) \coloneqq \sum'' \psi_j(x)K_A(x,y)\psi_k(y)\,.$$

Since a kernel of the form (2.1.2) is smooth outside the diagonal $\Delta = \{(x,x)\} \subset X \times X$, we have the thesis. $\square$

In the definition of pseudodifferential operators on manifolds, we will need the following result, which essentially states the local coordinate invariance of symbols.

Lemma 2.1.6. *Let $X_1, X_2 \subset \mathbb{R}^n$ open and $\phi\colon X_1 \to X_2$ and $G\colon X_1 \to GL(n)$ smooth maps. Then, if $(x,\xi) \in X_1 \times \mathbb{R}^n$ implies $(\phi(x), G(x)\xi) \in X_2 \times \mathbb{R}^n$, then $a_2 \in S^m(X_2)$ implies*

$$a_1(x,\xi) \coloneqq a_2(\phi(x), G(x)\xi) \in S^m(X_1)\,.$$

2.2 Pseudodifferential operators on manifolds

Given two manifolds M, N, a map $f\colon M \to N$ and a map $\rho\colon N \to \mathbb{R}$ we denote with $f^*\rho\colon M \to \mathbb{R}$ the standard pullback, namely

$$f^*\rho(x) = \rho(f(x)), \quad \forall x \in M\,.$$

We define pseudodifferential operators on a compact manifold [83].

Definition 2.2.1. *Let M be a compact Riemannian manifold. For $m \in \mathbb{R}$, a linear operator $A : C_0^\infty(M) \to C^\infty(M)$ is said to be pseudodifferential of order m if the following hold*

1. *if $\psi_1, \psi_2 \in C^\infty(M)$ have disjoint support, then the kernel of the operator $\psi_1 A \psi_2$ belongs to $C^\infty(M \times M)$, namely there exists $K \in C^\infty(M \times M)$ such that $\psi_1 A \psi_2 u = \int_M K(x,y)u(y)$ for any $u \in C_0^\infty(M)$;*

2. *for any coordinate system $\varphi : M \supset \Omega \to \mathbb{R}^d$, if $\psi_1, \psi_2 \in C_0^\infty(M)$ has support in Ω, then the restricted operator $\psi_1 A \psi_2$ is the pull back of a pseudodifferential operator of order m. Namely there exists $B \in OPS^m(\varphi(\Omega))$ s.t. $\forall u \in C_0^\infty(M)$ one has*

$$\psi_1 A \psi_2 u|_\Omega = \psi_1 \varphi^* \left(B\left(\left(\varphi^{-1}\right)^* \psi_2 u\right)\right) \tag{2.2.1}$$

We denote with $\Psi^m(M)$ the set of pseudodifferential operators on M of order m.

Before introducing the seminorms of pseudodifferential operators defined on a manifold we introduce an equivalent definition [83]. For that, we need a couple of lemmas.

Lemma 2.2.2. *If $A\colon C_0^\infty(M) \to C^\infty(M)$ has smooth kernel K_A, then it is smoothing. Namely, it belongs to $\Psi^{-\infty}(M) \coloneqq \bigcap_m \Psi^m(M)$.*

Proof. If ψ_1, ψ_2 ha disjoint support, clearly the kernel $\psi_1 K_A \psi_2$ is smooth, so the first condition in Def. 2.2.1 is satisfied. The second condition is satisfied as well, since for any $\psi_1, \psi_2 \in C_0^\infty$ the operator B in (2.2.1) has kernel $K_B = (\varphi^{-1} \times \varphi^{-1})^* K_A$, thus it has compact support. This concludes the proof since any smoothing operator with a compactly supported kernel is a pseudodifferential operator on Euclidean spaces. □

Lemma 2.2.3. *Let $\varphi_i\colon \mathcal{U}_i \mapsto \mathbb{R}^n$ a local coordinate system on M and $B \in OPS^m(\varphi_i(\mathcal{U}_i))$, then the operator*

$$Au(x) = \varphi_i^* B \left(\varphi_i^{-1}\right)^* u(x), \qquad x \in \mathcal{U}_i$$

is a pseudodifferential operator on M, namely $A \in \Psi^m(M)$.

Proof. Before proving the lemma, we remark that the kernel of A is given by

$$K_A(x,y) = K_B\left(\varphi_i(x), \varphi_i(y)\right), \forall x, y \in \mathcal{U}_i$$

In fact, we have, for any $x \in \mathcal{U}_i$ and writing $\varphi_i(y) = t$,

$$\begin{aligned} \int_{\mathcal{U}_i} K_A(x,y)\, u(y) dy = Au(x) = \\ = \varphi_i^* B \left(\varphi_i^{-1}\right)^* u(x) = \varphi_i^* B\left(u \circ \varphi^{-1}\right) = \\ = B\left(u \circ \varphi_i^{-1}\right)(\varphi_i(x)) = \int_{\varphi_i(\mathcal{U}_i)} K_B\left(\varphi_i(x), t\right) u\left(\varphi_i^{-1}(t)\right) dt = \\ = \int_{\mathcal{U}_i} K_B\left(\varphi_i(x), \varphi_i(y)\right) u(y) dy\,. \end{aligned}$$

By similar computations, one prove that, for any $\phi, \psi \in C^\infty(M)$, the kernel of $\phi A \psi$ is given by

$$K_{\phi A \psi}(x,y) = \phi(x) K_B\left(\varphi_i(x), \varphi_i(y)\right) \psi(y).$$

If ϕ, ψ have disjoint support, then $K_{\phi A \psi}$ is smooth since K_B is smooth outside the diagonal and φ_i is a diffeomorphism. So the first condition in Def. 2.1.1 is satisfied.

The second condition is a consequence of the local coordinates invariance stated in Lemma 2.1.6. In fact, fix $j \neq i$ and a chart φ_j such that $\mathcal{U}_i \cap \mathcal{U}_i \neq \emptyset$. Then we have that, for any $\psi \in C^\infty(M)$ with $\mathrm{supp}(\psi) \subset \mathcal{U}_j$,

$$\psi A \psi u = \varphi_j^* \left(B' \left((\varphi_j^{-1}) \right) \psi u \right)$$

with $B' = \left(\varphi_j^{-1} \circ \varphi_i\right)^* B \left(\varphi_i^{-1} \circ \varphi_j\right)^* \in OPS^m \left(\varphi_j(\mathcal{U}_j)\right)$ and then the second condition of Def. 2.1.1 is satisfied. □

Let $\{(\mathcal{U}_i;\varphi_i)\}_{i\in I}$ the atlas of a given manifold M, it is well known that there exists a partition of unity $\{\psi_i\}_{i\in I}$ subordinate to the atlas, such that

$$\mathrm{supp}(\psi_i) \subset \mathcal{U}_i, \quad \forall i \in I\,.$$

Remark 2.2.4. *One can define a partition of unit, subordinate to a given atlas, such that $\forall i,j \in I$ one has*

$$supp(\psi_j) \cap supp(\psi_i) \neq \emptyset \implies \{\exists k \in I \colon supp(\psi_j) \cap supp(\psi_i) \subset \mathcal{U}_k\}\ . \tag{2.2.2}$$

The following proposition gives a characterization of pseudodifferential operators on M.

Proposition 2.2.5. *Given an atlas of M, let $\{\psi_i\}$ a partition of unity subordinate to it that moreover satisfies* (2.2.2)*. For each i,j such that $supp(\psi_j) \cap supp(\psi_i) \neq \emptyset$, let $\varphi_{ij} \colon \mathcal{U}_{ij} \mapsto \mathbb{R}^n$ a local system of coordinates in an open neighborhood of $supp(\psi_j) \cap supp(\psi_i)$. Then the operator $A\colon C_0^\infty(M) \mapsto C^\infty(M)$ is pseudodifferential if and only if, for any i,j, $\psi_i A \psi_j$ satisfies:*

i. if $supp(\psi_j) \cap supp(\psi_i) = \emptyset$, then $\psi_i A \psi_j$ has smooth kernel, i.e. is smoothing;

ii. if $supp(\psi_j) \cap supp(\psi_i) \neq \emptyset$, then $\psi_i A \psi_j$ has the form

$$\varphi_{ij}^* A_{ij} \left(\varphi_{ij}^{-1}\right)^*$$

for some $A_{ij} \in OPS\left(\varphi_{ij}(\mathcal{U}_{ij})\right)$ with kernel supported in $\varphi_{ij}\left(\mathcal{U}_{ij}\right) \times \varphi_{ij}\left(\mathcal{U}_{ij}\right)$.

Proof. If A is pseudodifferential, then conditions i. and ii. follow directly from Def. 2.2.1. On the other hand, let A be an operator that satisfies i. and ii. If $\phi_1, \phi_2 \in C^\infty(M)$, then we write

$$\phi_1 A \phi_2 = \sum_{i,j\colon \mathrm{supp}(\psi_j)\cap\mathrm{supp}(\psi_i)=\emptyset} \phi_1\psi_i A \psi_j \phi_2 + \sum_{i,j\colon \mathrm{supp}(\psi_j)\cap\mathrm{supp}(\psi_i)\neq\emptyset} \phi_1\psi_i A \psi_j \phi_2 \tag{2.2.3}$$

and we conclude the proof by applying Lemma 2.2.2 and Lemma 2.2.3 to each element respectively in the first and the second sum. □

In particular, we deduce that the composition rule for pseudodifferential operators directly follows from the composition rule of local pseudodifferential operators.

Lemma 2.2.6. *For any $A \in \Psi^m(M)$, $B \in \Psi^{m'}(M)$, $m, m' \in \mathbb{R}$, we have $A \circ B \in \Psi^{m+m'}(M)$*

In the last part of this Section, we discuss the notion of the seminorms of a pseudodifferential operator defined on M. From now on we fix a finite atlas and the partition of unity $\{\psi_i\}$ that satisfies the property (2.2.2). Given $P \in \Psi^m(M)$, we can write

$$P = \sum_{i,j} \psi_i P \psi_j = \sum{}' \psi_i P \psi_j + \sum{}'' \psi_i P \psi_j \tag{2.2.4}$$

where $\sum'$ indicates the sum over the i, j s.t. $\text{supp}(\psi_j) \cap \text{supp}(\psi_i) \neq \emptyset$, while $\sum''$ indicates the sum over the i, j s.t. $\text{supp}(\psi_j) \cap \text{supp}(\psi_i) = \emptyset$. Following Prop. 2.2.5, each term in $\sum'$ is represented in a local coordinate system by a pseudodifferential operator P_{ij} defined on an open set in $\mathbb{R}^d$ and $S := \sum'' \psi_i P \psi_j$ is a smoothing operator, i.e. with smoothing kernel.

Definition 2.2.7. *Fix a partition of unity $\{\psi_i\}$ that satisfies* (2.2.2)*. Let $P \in \Psi^m(M)$ and write $P = \sum' \psi_i P \psi_j + S$ as in* (2.2.4)*. For any $\nu \in \mathbb{R}$, we define the seminorm of P as*

$$|P|^m_\nu := \max \left\{ \sup_{ij} |P_{ij}|^m_\nu, \sup_{|\alpha| \le \nu} |K_S|_\alpha \right\}$$

where $|P_{ij}|^m_\nu$ are the seminorms of the local representation of each term $\psi_i P \psi_j$ and $|K_S|_\alpha$ are the usual C^∞ seminorms of the smooth kernel of S, defined by $|K_S|_\alpha = \sup_{x,y \in M} |D^\alpha K_s(x,y)|$.

Remark 2.2.8. *The definition above is well-posed on a compact manifold M since the C^∞ seminorms of K_S are bounded.*

Remark 2.2.9. *The definition of the seminorm depends on the choice of the partition of unity. For that reason, we assume to fix a partition of unity once and for all.*

Remark 2.2.10. *Given a function $u \in C^\infty(M)$, the seminorms of the multiplication operator $v \mapsto uv$ are controlled by*

$$|u|_s := \sum_{|\alpha| \le s} \sup_i \sup_{x \in \varphi_i(\mathcal{U}_i)} \left| \partial_x^\alpha ((\varphi_i^{-1})^* u)(x) \right| \ .$$

2.3 An estimate of commutators

In the following Subsections, we give a key result on the estimate of the commutators of pseudodifferential operators [47]. We start proving general results for local symbols and then we will conclude the proof in the case of global operators defined on M. For simplicity, we state the local results for $X = \mathbb{R}^n$. We deeply thank Jean-Marc Delort for the private notes shared with us.

2.3.1 Local estimates

We want to prove the following Lemma. For any couple of operators A, P, we use the notation $Ad_P(A) = [A, P]$.

Lemma 2.3.1. *There is $\nu > 0$ s.t. for any $A \in OPS^0(\mathbb{R}^n)$ and $P_1, ..., P_N \in OPS^{\,1}(\mathbb{R}^d)$ one has*

$$\|Ad_{P_N}...Ad_{P_1}A\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq C_N\, |a|^0_{\nu+N} \ . \tag{2.3.1}$$

for some constant $C_N = C_N(P_1, ..., P_N)$.

We start by defining "paradifferential symbols".

Definition 2.3.2. *For $\delta \in (0,1)$, we define the class of paradifferential symbols $\Sigma^m(\delta)$ as the set of the symbols $a \in S^m(\mathbb{R}^n)$ s.t.*

$$supp(\mathcal{F}a(\eta,\xi)) \subset \{(\eta,\xi) \ : \ |\eta| \leq \delta\langle\xi\rangle\}$$

Here $\mathcal{F}a(\eta,\xi) = \hat{a}(\eta,\xi)$ denotes the Fourier transform of a with respect to the first variable, namely

$$\hat{a}(\eta,\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}}^n e^{-i\eta\cdot x} a(x,\xi)dx\,.$$

We denote $\Sigma^m := \bigcup_\delta \Sigma^m(\delta)$.

The following lemma shows that a paradifferential operator of order m is bounded from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$. It follows directly from Calderon-Villancourt theorem for pseudodifferential operators.

Lemma 2.3.3. *There exists $\nu > 0$, depending on the dimension d, such that $\forall m \in \mathbb{R}$ and $\forall a \in \Sigma^m$, the paradifferential operator A is bounded from $H^s(\mathbb{R}^d)$ to $H^{s-m}(\mathbb{R}^d)$. Moreover,*

$$\|A\|_{\mathcal{B}(H^s,H^{s-m})} \leq C|a|^m_\nu\,.$$

We can approximate pseudodifferential operators with symbols with compact support in x using paradifferential symbols, up to a smoothing remainder. Consider a cut-off function $\chi \in \mathcal{C}_0^\infty$ such that $\chi(s) = 1$ for $|s| < \bar{s} < 1$ and $\chi(s) = 0$ for $|s| > 1$, and define, $\forall \delta \in (0, 1)$,

$$\chi_\delta(s) \coloneqq \chi\left(\frac{s}{\delta}\right).$$

Lemma 2.3.4. *Let $a \in S^m(\mathbb{R}^n)$, compactly supported in x, and define $a_\delta(x, \xi)$ by*

$$\hat{a}_\delta(\eta, \xi) \coloneqq \chi_\delta\left(\frac{\eta}{\langle\xi\rangle}\right)\hat{a}(\eta, \xi)$$

with $\langle\xi\rangle \coloneqq (1+|\xi|^2)^{1/2}$ Then $a_\delta \in \Sigma^m(\delta)$ and $A = A_\delta + R$ with R smoothing. Moreover

$$\|R\|_{\mathcal{B}(H^{s_1}, H^{s_2})} \leq C|a|^m_{(s_2)_+ + (m-s_1)_+ + \nu}$$

for some fixed $\nu = \nu(n)$ and some $C >$. Moreover $|a_\delta|^m_l \leq |a|^m_l$ $\forall l \in \mathbb{R}$.

Proof. One checks immediately that by definition

$$a_\delta(x, \xi) = \int_{\mathbb{R}^n} \left(\mathcal{F}^{-1}\chi_\delta\right)(\langle\xi\rangle(x-y))\, \langle\xi\rangle^\delta a(y, \xi) dy$$

and then $a_\delta \in \Sigma^n(\delta)$. Moreover, one can write explicitly

$$Ru(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}(a - a_\delta)(x, \xi)\hat{u}(\xi)d\xi$$

and thus

$$\widehat{(Ru)}(\eta) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{a}(\eta-\xi, \xi)(1-\chi_\delta)\left(\frac{\eta-\xi}{\langle\xi\rangle}\right)\hat{u}(\xi)d\xi\,.$$

Denoting $q_+ = \max(q, 0)$, we can write, for any $s_1, s_2 \in \mathbb{R}$,

$$\langle\eta\rangle^{s_2}\widehat{(Ru)}(\eta) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} b(\xi, \eta)(1-\chi_\delta)\left(\frac{\eta-\xi}{\langle\xi\rangle}\right)\left[\frac{\langle\eta\rangle^{s_2}\langle\xi\rangle^{m-s_1}}{\langle\xi-\eta\rangle^{(s_2)_+ + (m-s_1)_+}}\right]\langle\xi\rangle^{s_1}\hat{u}(\xi)d\xi \tag{2.3.2}$$

with $b(x, \xi) = \hat{a}(\eta-\xi, \xi)\langle\xi\rangle^{-m}\langle\xi-\eta\rangle^{(s_2)_+ + (m-s_1)_+}$. First, note that on the support of $(1-\chi_\delta)\left(\frac{\eta-\xi}{\langle\xi\rangle}\right)$ one has $|\eta-\xi| \geq c(|\xi| + |\eta| + 1)$ for some positive constant and thus the square bracket in (2.3.2) is bounded by a constant. Then, since a is compactly supported in x, one has also $|\hat{a}(\eta-\xi, \xi)\langle\xi\rangle^{-m}| \leq C|a|^m_\rho\langle\xi-\eta\rangle^{-\rho}$ for ρ large enough. Choosing $\rho = (s_2)_+ + (m-s_1)_+ + \nu$, with ν large enough depending on the dimension, we can conclude that

$$\|\langle D\rangle^{s_2} Ru\|_{L^2} \le c|a|_\rho^m \cdot \|\langle D\rangle^{s_1} u\|_{L^2}\,.$$

□

In the next Lemma, we describe calculus for paradifferential symbols.

Lemma 2.3.5. *There exists $\nu = \nu(n) > 0$ such that, for any $a \in \Sigma^m(\delta), b \in \Sigma^{m'}(\delta)$ with $\delta < \frac{1}{3}$ and any $N \ge 1$, there are symbols*

$$c_j \in \Sigma^{m+m'-j}(3\delta), \quad \forall j = 0, \dots, N-1,$$
$$r_N \in \Sigma^{m+m'-N}(3\delta)$$

such that

$$Op(a)\,Op(b) = \sum_{j=0}^{N-1} Op(c_j) + Op(r_N).$$

In particular $c_0 = ab$ and for any $l \in \mathbb{N}$,

$$\begin{aligned} |c_j|_l^{m+m'-j} &\le C|a|_{l+j}^m |b|_{l+j}^{m'} \\ |r_N|_l^{m+m'-N} &\le C|a|_{l+N+\nu}^m |b|_{l+N+\nu}^{m'}. \end{aligned} \tag{2.3.3}$$

for some positive constant C.

Proof. We start noting that $\mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(c)$ with

$$c(x,\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta} a(x,\eta-\xi) b(x-y,\xi)\, dy\, d\eta\,. \tag{2.3.4}$$

By the assumptions on a and b, we can deduce that the integrand in

$$\hat{c}(\zeta,\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{a}(\zeta+\eta,\xi-\eta)\hat{b}(-\eta,\xi)\, d\eta\,.$$

is supported in the intersection of $|\zeta+\eta| \le \delta\langle\xi-\eta\rangle$ and $|\eta| \le \delta\langle\xi\rangle$. Thus, the support of $\hat{c}$ is contained in $|\eta| \le 3\delta\langle\xi\rangle$, namely, c is a paradifferential symbol. Moreover, we can insert a cut-off function $\widetilde{\chi}_\delta\left(\frac{\eta}{\langle\xi\rangle}\right)$ in the integral (2.3.4), with $\widetilde{\chi}_\delta \in C_0^\infty$ and $\widetilde{\chi}_\delta \equiv 1$ in a neighborhood of zero. Inserting the Taylor expansion

$$a(x,\xi-\eta) = \sum_{|\alpha|\le M-1} \frac{(-\eta)^\alpha}{\alpha!}\partial_\xi^\alpha a(x,\xi) + M\sum_{|\alpha|=M}\frac{(-\eta)^\alpha}{M!}\int_0^1 \partial_\xi^\alpha a(x,\xi-t\eta)dt$$

in (2.3.4) and recalling that we can add the function $\widetilde{\chi}_\delta$, we get terms of the form

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-iy\cdot\eta}\widetilde{\chi}_\delta\left(\frac{\eta}{\langle\xi\rangle}\right)\frac{(-\eta)^\alpha}{\alpha!}b(x-y,\xi)d\eta\, dy\partial_\xi^\alpha a(x,\xi) = \\ = \frac{1}{\alpha!}D_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi)$$

Thus, we can define $c_j(x,\xi) = \sum_{|\alpha|=j}\frac{1}{\alpha!}D_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi)$. It remains to deal with the remainder. Its symbol is given by

$$r_M(x,\xi) = M\sum_{|\alpha|=M}\frac{1}{(2\pi)^n}\int\int_0^1 \frac{(-\eta)^\alpha}{M!}e^{-iy\cdot\eta}\widetilde{\chi}_\delta\left(\frac{\eta}{\langle\xi\rangle}\right)\partial_\xi^\alpha a(x,\xi-t\eta)b(x-y,\xi)dy\, d\eta\, dt = \\ = M\sum_{|\alpha|=M}\frac{(-)^{|\alpha|}}{(2\pi)^n}\int\int_0^1 e^{-iy\cdot\eta}\widetilde{\chi}_\delta\left(\frac{\eta}{\langle\xi\rangle}\right)\partial_\xi^\alpha a(x,\xi-t\eta)\left((i\partial_x)^\alpha b\right)(x-y,\xi)dy\, d\eta\, dt$$

Integrating by parts ν times (with ν depending on the dimension) in y and η, there appear factors $\langle\eta\rangle^{-d-1}\langle y\rangle^{-d-1}$, so that the integral is absolutely convergent. This shows $r_N \in \Sigma^{m+m'-N}(3\delta)$ with estimates (2.3.3). □

The next corollary follows writing explicitly $[A,B] = AB - BA$ and applying Lemma 2.3.5.

Corollary 2.3.6. *There exists $\nu = \nu(n) > 0$ such that, for any $a \in \Sigma^m(\delta), b \in \Sigma^{m'}(\delta)$ with $\delta < \frac{1}{3}$ and any $N \geq 1$ there are symbols*

$$c_j \in \Sigma^{m+m'-j}(3\delta), \quad j = 1,\dots,N-1, \\ r_N \in \Sigma^{m+m'-N}(3\delta)$$

such that

$$[A,B] = \sum_{j=1}^{N-1} Op(c_j) + Op(r_N),$$

and, $\forall l \in \mathbb{N}$,

$$|c_j|_l^{m+m'-j} \leq C|a|_{l+j}^m|b|_{l+j}^{m'} \\ |r_n|_l^{m+m'-N} \leq C|a|_{l+N+\nu}^m|b|_{l+N+\nu}^{m'}.$$

It follows that $[A,B] \in \Sigma^{m+m'-1}(3\delta)$.

In the next Lemma and Corollary, we deduce from Lemma 2.3.4 that one can prove Lemma 2.3.1 restricting the analysis on paradifferential symbols.

Remark 2.3.7. *In the following, we will use a compact notation: for any $k \geq 1$, given $J = (j_1, ..., j_k)$ with $j_i \in \{1, 2, \dots, N\}$ for any $1 \leq i \leq k$ and $j_i > j_l$ if $i > l$, we will write*

$$P^J := P_{j_k} P_{j_{k-1}} ... P_{j_1} \qquad \bar{P}^J := P_{j_1} P_{j_2} ... P_{j_k} \ .$$

Lemma 2.3.8. *Let $P_1, \dots, P_N \in OPS^1(\mathbb{R}^n)$, $A \in OPS^0(\mathbb{R}^n)$ and denote $P_{i,\delta} = (P_i)_\delta, \forall i = 1, \dots, N$ and A_δ the approximated operators defined in Lemma 2.3.4. Then, one has*

$$\left\| Ad_{P_{N,\delta}} ... Ad_{P_{1,\delta}} A_\delta - Ad_{P_N} ... Ad_{P_1} A \right\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq C_N \left| a \right|^0_{\nu+N} \tag{2.3.5}$$

for $C_N = C_N(P_1, ..., P_N)$

Proof. The difference in (2.3.5) is a sum of terms of the form

$$Ad_{\tilde{P}_N} ... Ad_{\tilde{P}_1} \tilde{A}$$

where at least one of $\tilde{P}_i$ or $\tilde{A}$ is equal to, respectively, $P_i - P_{i,\delta}$ or $A - A_\delta$. Thus, we consider terms of the form $P^J (A - A_\delta) \bar{P}^K$, for some $J = (j_1, \dots, j_{N_1})$ and $K = (k_1, \dots, k_{N_2})$ with $N = N_1 + N_2$.
We claim that, for any $u \in L^2$,

$$\left\| P^J (A - A_\delta) \bar{P}^K u \right\|_{L^2} \leq C |a|^0_{\nu+N}.$$

In fact, following Lemmas 2.3.3 and 2.3.4, we have

$$\begin{aligned} \left\| P^J (A - A_\delta) \bar{P}^K u \right\|_{L^2} &\leq C \left\| (A - A_\delta) \bar{P}^K u \right\|_{H^{s-N_1}} \leq \\ &\leq C |a|^0_{N+\nu} \left\| \bar{P}^K u \right\|_{H^{-N_2}} \leq C |a|^0_{N+\nu} \|u\|_{L^2} \end{aligned}$$

with $C = C(P_1, \dots, P_N)$.
Similarly, one proves that terms in which at least one $P_i - P_{i,\delta}$ satisfy estimate (2.3.1), and thus the thesis follows. □

Corollary 2.3.9. *Let $P_1, \dots, P_N \in OPS^1(\mathbb{R}^n)$, $A \in OPS^0(\mathbb{R}^n)$ and denote $P_{i,\delta} = (P_i)_\delta, \forall i = 1, \dots, N$ and A_δ the approximated operators. If*

$$\left\| Ad_{P_{N,\delta}} ... Ad_{P_{1,\delta}} A_\delta \right\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq C_N \left| a \right|^0_{\nu+N}$$

then the thesis of Lemma 2.3.1 holds true.

In the next lemma, we prove the key estimate that allows us to prove (2.3.5). We will write $A \lesssim B$ if there exists a constant, depending only on $P_1, \ldots, P_N$, such that $A \leq C \cdot B$.

Lemma 2.3.10. *There exists $\nu > 0$ with the following property: for any $k = 1, ..., N$ one has*

$$Ad_{P_{k,\delta}} ... Ad_{P_{1,\delta}} A_\delta = \sum_{j=0}^{N-k-1} Op(a_j^k) + R_k \ , \tag{2.3.6}$$

where $a_j^k \in \Sigma^{-j}$ with, $\forall 1 \leq j \leq N-k$,

$$\left| a_j^k \right|_l^{-j} \lesssim |a|_{l+k+j}^0 \ , \quad \forall l \in \mathbb{N} \ , \tag{2.3.7}$$

R_k are operators of the form

$$R_k = \sum_{|J_1|+|J_2| \leq k-1} P_\delta^{J_1} Op(r_{J_1,J_2}^k) \bar{P}_\delta^{J_2} \tag{2.3.8}$$

with $r_{J_1,J_2}^k \in \Sigma^{-N+k-|J_1|-|J_2|}$ fulfilling

$$\left| r_{J_1 J_2}^k \right|_l^{-N+k-|J_1|-|J_2|} \lesssim |a|_{l+N+\nu}^0 \ , \quad \forall l \in \mathbb{N}. \tag{2.3.9}$$

Proof. From Lemma 2.3.6, we have that, $\forall b \in \Sigma^m, p \in \Sigma^1$ and for all $M \in \mathbb{N}$,

$$[P, B] = \sum_{i=0}^{M-1} Op(c_i) + Op(r_M) \tag{2.3.10}$$

with $c_i \in \Sigma^{m-i}$, $r_M \in \Sigma^{m-M}$ and

$$|c_i|_l^{m-i} \lesssim |b|_{l+i+1}^m,$$
$$|r_M|_l^{m-M} \lesssim |b|_{l+M+1+\nu}^m, \qquad \forall l \in \mathbb{N}$$

for some ν that depends on the dimension.
We prove the result inductively on k.
For $k = 1$, we apply (2.3.10) with $M = N-1$, $B = A_\delta$ and we get

$$[P_{1,\delta}, A_\delta] = \sum_{j=0}^{N-2} Op(c_j) + Op(r_{N-1})$$

which is the thesis denoting $a_j^1 = c_j$ and $r_{0,0}^1 = r_{N-1}$ (since $k = 1$, must be $|j_1| =$

$|j_2| = 0)$. In fact, we have

$$|a_j^1|_l^{-j} \lesssim |a|_{l+j+1}^0$$
$$|r_{0,0}^1|_l^{-N+1} \lesssim |a|_{l+N+\nu}^0.$$

Suppose the thesis is true for k, that is

$$Ad_{P_{k,\delta}} ... Ad_{P_{1,\delta}} A_\delta = \sum_{j=0}^{N-k-1} Op(a_j^k) + R_k$$

with

$$R_k = \sum_{|J_1|+|J_2| \leq k-1} P_\delta^{J_1} Op(r_{J_1,J_2}^k) \bar{P}_\delta^{J_2}$$

and

$$\begin{aligned} |a_j^k|_l^{-j} &\lesssim |a|_{l+k+j}^0 \\ |r_{|J_1,J_2}^k|_l^{-N+k-|J_1|-|J_2|} &\lesssim |a|_{l+N+\nu}^0, \qquad \forall l \in \mathbb{N}. \end{aligned} \tag{2.3.11}$$

Then, we compute

$$\begin{aligned} Ad_{P_{k+1,\delta}} Ad_{P_{k,\delta}} ... Ad_{P_{1,\delta}} A_\delta &= [P_{k+1.\delta}, \sum_{j=0}^{N-k-1} Op(a_j^k)] + [P_{k+1,\delta}, R_k] = \\ &= \sum_{j=0}^{N-k-1} [P_{k+1,\delta}, Op(a_j^k)] + [P_{k+1,\delta}, R_k] \end{aligned}$$

For each term term $[P_{k+1,\delta}, Op(a_j^k)]$, we apply (2.3.10) with $M = N - k - j - 1$ and $B = Op(a_j^k)$. So, we can write

$$[P_{k+1,\delta}, Op(a_j^k)] = \sum_{i=0}^{N-k-l-2} Op(c_i^j) + Op(r_M^j) \tag{2.3.12}$$

where $c_i^j \in \Sigma^{-i-j}$ and $r_M^j \in \Sigma^{-M-j}$, and moreover

$$\begin{aligned} |c_j^i|_l^{-i-j} &\lesssim |a_j^k|_{l+1+i}^{-j} \underset{(2.3.11)}{\lesssim} |a|_{l+i+j+(k+1)}^0 \\ |r_M^j|_l^{-M-j} = |r_M^j|_l^{-N+(k+1)} &\lesssim |a_j^k|_{l+N-k-j+\nu}^{-j} \underset{(2.3.11)}{\lesssim} |a|_{l+N+\nu}^0\,. \end{aligned} \tag{2.3.13}$$

Collecting the terms $[P_{k+1,\delta}, Op(a_j^k)]$, we get

$$Ad_{P_{k+1,\delta}}Ad_{P_{k,\delta}}...Ad_{P_{1,\delta}}A_\delta = \sum_{j=0}^{N-k-1}[P_{k+1,\delta}, Op(a_j^k)] + [P_{k+1,\delta}, R_k] =$$
$$= \sum_{j=0}^{N-k-1}\left(\sum_{i=0}^{N-k-j-2} Op(c_i^j) + Op(r_M^j)\right) + [P_{k+1,\delta}, R_k]$$

with c_i^j defined in (2.3.12).
We define

$$a_j^{k+1} := \sum_{i=0}^{N-k-j-2} c_i^{j-i}$$

with the convention that $c_i^z = 0$ if $z < 0$ and we get (notice that each a_j^{k+1} is the sum of terms with the same order)

$$|c_i^{j-i}|_l^{-i-(j-i)} \underset{(2.3.1)}{\lesssim} |a|^0_{l+i+j-i+(k+1)} = |a|^0_{l+j+(k+1)} \tag{2.3.14}$$

and then, since $j = i - (i-j)$,

$$|a_j^{k+1}|_l^{-j} \leq \sum_{i=0}^{N-k-j-2} |c_i^{j-i}|_l^{-j} \underset{(2.3.14)}{\lesssim} \sum_{i=0}^{N-k-j-2} |a|^0_{l+j+(k+1)} \leq C|a|^0_{l+j+(k+1)}\,.$$

Moreover, noticing that the term corresponding to $j = N-k-1$ in the sum (2.3.1) is equal to zero since $N-k-j-2 = N-K-N+K+1-2 = -1$, we can write

$$\sum_{j=0}^{N-k-1}\sum_{i=0}^{N-k-j-2} Op(c_i^j) = \sum_{j=0}^{N-k-2} a_j^{k+1}$$

and we get the first part of the thesis.
We pass to consider the terms $Op(r_M^j)$. Notice that, from $M = N-k-j-1$, we have $-M-j = -N+k+j+1-j$. Then $r_M^j \in \Sigma^{-N+(k+1)}$, $\forall j = 0, \dots, N-k-1$ and we can consider $\sum_{j=0}^{N-k-1} Op(r_M^j)$ as a term in R_{k+1}, with $|J_1| = |J_2| = 0$. The estimate

$$|r_M^j|_l^{-N+(k+1)} \lesssim |a|^0_{l+N+\nu}$$

follows from (2.3.1). It remains to consider the term $[P_{k+1,\delta}, R_k]$. For $P_{k+1,\delta}R_k$, we have, denoting $\bar{J} = (J_1, 1)$,

$$P_{k+1,\delta} R_k = \sum_{|J_1|+|J_2|\leq k-1} P_\delta^{J_1} Op(r^k_{J_1,J_2}) \bar{P}_\delta^{J_2} =$$
$$= \sum_{|\bar{J}|+|J_2|\leq k} \bar{P}_\delta^{J} Op(r^{k+1}_{\bar{J},J_2}) \bar{P}_\delta^{J_2}$$

with $r^{k+1}_{\bar{J},J_2} := r^k_{J_1,J_2}$ and then

$$|r^{k+1}_{\bar{J},J_2}|_l^{-N+(k+1)-|\bar{J}|-|J_2|} \underset{(|\bar{J}|=|J_1|+1)}{=} |r^k_{J_1,J_2}|_l^{-N+k-j_1-j_2} \underset{(2.3.11)}{\lesssim} |a|^0_{l+N+\nu}.$$

We reason similarly for $R_k P_{k+1,\delta}$ and we get the thesis. □

Proof of Lemma 2.3.1 . Applying Lemma 2.3.10 with $K = N$, we get

$$Ad_{P_{N,\delta}} ... Ad_{P_{1,\delta}} A_\delta = R_N = \sum_{|J_1|+|J_2|\leq N-1} P_\delta^{J_1} Op(r^N_{J_1,J_2}) \bar{P}_\delta^{J_2},$$

with $|r^N_{J_1,J_2}|_l^{-|J_1|-|J_2|} \lesssim |a|^0_{l+N+\nu}$. Applying Lemma 2.3.3 we get, for any $u \in L^2(\mathbb{R}^n)$,

$$\left\| P_\delta^{J_1} Op(r^N_{J_1,J_2}) \bar{P}_\delta^{J_2} u \right\|_{L^2} \lesssim C_N |r|_l^{-|J_1|-|J_2|} \lesssim C_N |a|^0_{l+N+\nu} .$$

From Lemma 2.3.8, we get the thesis.

□

2.3.2 Global estimate

In this Section, we prove a global version of Lemma 2.3.1, in the special case of an operator that is simply the multiplication with a function $f \in C^\infty(M)$. We still indicate with f this operator and we use the notation

$$Ad^0_P(f) := f \quad \text{and} \quad Ad^k_G(P) := [Ad^{k-1}_P(f), P] \text{ for } k \geq 1$$

Lemma 2.3.11. *Let M be a Riemannian manifold endowed by an atlas as above. There exists ν s.t. for any $P \in \Psi^1(M)$ and any $N \in \mathbb{N}$ there exist constants $C_N = C_N(P)$ with the property that for any $f \in C^\infty(M)$ one has*

$$\left\| Ad^N_P f \right\|_{\mathcal{B}(L^2(M))} \leq C_N \, |f|_{N+\nu} \ . \tag{2.3.15}$$

Proof. Consider a partition of unity with the property (2.2.2) and decompose P according to (2.2.3)

$$P = \sum{}' \varphi_{ij}^* P_{ij} (\varphi_{ij}^{-1})^* + S = \sum{}' \widetilde{P}_{ij} + S$$

where φ_{ij} are local coordinates, P_{ij} are local pseudodifferential operators and S is smoothing. Let's deal firstly, for simplicity, with the case $N = 1$. We have

$$Ad_P f = [P, f] = \sum{}' [\varphi_{ij}^* P_{ij} (\varphi_{ij}^{-1})^*, f] + [S, f] \, .$$

The estimate for the term involving S is obvious since it is smoothing, namely we have

$$\|[S, f]\|_{\mathcal{B}(L^2(M))} \leq C_S |f|_\nu^0$$

for some ν depending on the dimension. For the other term we have, for any $u \in C^\infty(M)$,

$$[\varphi_{ij}^* P_{ij} (\varphi_{ij}^{-1})^*, f] u = [P_{ij}, (\varphi_{ij}^{-1})^* f] (\varphi_{ij}^{-1})^* u,$$

then we can take profit of (2.3.1) applied to each P_{ij} and we get the thesis.
For $N > 1$, we need to control terms that involve Ad_S and terms that do not involve S. A term that involves S is a linear combination of terms of the form

$$\widetilde{P}_{i_1 j_1} ... \widetilde{P}_{i_l j_l} f \widetilde{P}_{i_{l+1} j_{l+1}} ... \widetilde{P}_{i_{l+k} j_{l+k}} S u \ ,$$

with $l + k \leq N_1$. Thus the norm of this object is estimated by

$$\begin{aligned} \left\| P_{i_1 j_1} ... P_{i_l j_l} f P_{i_{l+1} j_{l+1}} ... P_{i_{l+k} j_{l+k}} S u \right\|_{\mathcal{B}(L^2)} &\leq \| P_{i_1 j_1} ... P_{i_l j_l} \|_{\mathcal{B}\left(H^{\nu_1 + l}, L^2\right)} \|f\|_{\nu_1 + l} \\ &\quad \times \left\| P_{i_{l+1} j_{l+1}} ... P_{i_{l+k} j_{l+k}} \right\|_{\mathcal{B}\left(H^{\nu_1 + l + k}, H^{\nu_1 + l}\right)} \|Su\|_{H^{\nu_1 + l + k}} \\ &\leq C(P) |bf|_{\nu_1 + N - 1} \|u\|_{L^2}. \end{aligned}$$

For terms that do not involve S, we have that they contain operators localized on the same chart. Then, reasoning as in the case $N = 1$, we are lead to consider terms of the form

$$Ad_{P_{i_1 j_1}} \dots Ad_{P_{i_N j_N}} (\varphi^{-1})^* f.$$

Note that the local coordinate φ does not depend on $i_l j_l$, since we are considering elements of the partition with mutual nonempty support. Then we can apply again (2.3.1) and get the thesis. $\square$

2.4 Sobolev spaces

In this Section we introduce Sobolev spaces on compact manifolds in a slightly different way with respect to more classical formulations [83]. Essentially, we define the Sobolev spaces on M so that the boundedness of pseudodifferential operators is trivial, and then we check that we recover the standard definition, using elliptic operators and proving that those spaces are Hilbert spaces. To do this, we exploit Fredholm property of elliptic operators, which we briefly recall later on.

Theorem 2.4.1. *Any $A \in \Psi^0(M)$ extends by continuity to a bounded operator*

$$A\colon L^2(M) \mapsto L^2(M)\,.$$

Proof. Following (2.2.4), for any partition of unity $\{\psi_i\}$ subordinate to the atlas of M, we write

$$A = \sum_{i,j} \psi_i A \psi_j = \sum{}' \psi_i A \psi_j + S$$

so that $\psi_i A \psi_j$ is represented in local coordinates by a pseudodifferential operator on $\mathbb{R}^n$ and S is smoothing. Then the result follows from the L^2 boundedness of pseudodifferential operators on $\mathbb{R}^n$. □

We define Sobolev spaces as imposing boundedness of any pseudodifferential operator of any order.

Definition 2.4.2. *For any $s \in \mathbb{R}$, set*

$$H^s(M) := \left\{u \in C^\infty(M)\colon Au \in L^2(M) \quad \forall A \in \Psi^s(M)\right\}\,. \tag{2.4.1}$$

Remark 2.4.3. *For any $A \in \Psi^s(M)$ we have*

$$u \in L^2(M) \implies Au \in H^{-s}(M)\,. \tag{2.4.2}$$

In fact, from the composition rule of pseudodifferential operators, $BA \in \Psi^0(M)$ for any $B \in \Psi^{-s}$, and then from Theorem 2.4.1 it follows that $BAu \in L^2$. Then $Au \in H^{-s}(M)$ by our definition of Sobolev spaces.

To recover the usual definition of Sobolev spaces, we need to recall the notion of parametrix for elliptic operators.

Definition 2.4.4. *We say that $A \in \Psi^m(M)$ is elliptic if there exists $B \in \Psi^{-m}(M)$ such that $AB - Id \in \Psi^{-\infty}(M)$, namely if it is invertible modulo a smooth remainder with an inverse of order -m.*

Remark 2.4.5. *For any manifold M, there exist elliptic operators of any order s, for example the operator P_s with local symbols $(1+|\xi|^2)^{\frac{s}{2}}$.*

The ellipticity of an operator implies also the existence of a left-inverse, that is the left parametrix. The following lemma is proved in [83], using the theory of principal symbols.

Lemma 2.4.6. *For any elliptic $P \in \Psi^m(M)$ there exists a left parametrix, namely an operator $Q \in \Psi^{-s}(M)$ such that*

$$QP - \mathrm{Id} \in \Psi^{-\infty}(\mathrm{M})$$

Theorem 2.4.7. *For any $m \in \mathbb{R}$, every $A \in \Psi^m(M)$ extends by continuity to a bounded operator*

$$A\colon H^s(M) \mapsto H^{s-m}(M)\,.$$

Moreover, if $A \in \Psi^m(M)$ is elliptic then

$$Au \in H^s(M) \implies u \in H^{s+m}(M)\,. \tag{2.4.3}$$

Proof. Let $P_s \in \Psi^s(M)$ elliptic and $Q_s \in \Psi^{-s}(M)$ its parametrix. Then for any $A \in \Psi^m(M)$ we can write

$$A = A \circ Id = AQ_sP_s + G$$

for some G smoothing. Then, since by composition rule $AQ_s \in \Psi^{m-s}(M)$, the proof follows from (2.4.1) and (2.4.2). In fact, we observe that

$$H^s(M) \underset{P_s}{\longrightarrow} L^2(M) \underset{AQ_s}{\longrightarrow} H^{s-m}(M)\,.$$

Let now $A \in \Psi^m(M)$ elliptic and suppose $Au \in H^s(M)$. Then, denoting with Q the parametrix of A, we have again

$$QA = Id + G$$

for some smoothing operator G and then $u \in H^{s+m}$ easily follows. □

In particular, from (2.4.3) it follows that in the definition of Sobolev spaces, we can reduce to consider any elliptic operator.

Remark 2.4.8. *Definition* (2.4.1) *is equivalent to*

$$H^s(M) := \{u \in C^\infty(M) \colon Pu \in L^s(M)\}$$

for some elliptic operator $P \in \Psi^s(M)$.

The last equivalent definition allows us to prove that $H^s(M)$ are indeed Hilbert spaces. To do this, we exploit the Fredholm property of elliptic operators. See [51, 52] for further readings on Fredholm theory.

Definition 2.4.9. *Given two Banach spaces* X, Y*, a bounded operator* $A \colon X \to Y$ *is Fredholm if* $ker(A)$ *and* $coker(A) = Y/Im(A)$ *are finite-dimensional. In particular, we define*

$$\mathrm{ind(A)} = \dim(\mathrm{ker(A)}) - \dim(\mathrm{coker(A)})$$

Remark 2.4.10. *Denoting* A^* *the adjoint of* A*, one has*

$$\mathrm{ind(A^*)} = -\mathrm{ind(A)}\,. \tag{2.4.4}$$

Before proving that $H^s(M)$ is a Hilbert space, we recall the following well-known result.

Lemma 2.4.11. *An elliptic operator* $A \in \Psi^m(M)$*,* $A \colon C^\infty(M) \mapsto C^\infty(M)$ *is Fredholm.*

Theorem 2.4.12. *Fix* $s > 0$*. For any compact manifold* M*, there exists an elliptic operator* $B_s \in \Psi^s(M)$ *such that*

$$H^s(M) = \{u \in C^\infty(M) \colon B_s u \in L^s \in\}\,,$$

with $\|u\|_{H^s} = \|B_s u\|_{L^2}$*, is a Hilbert space.*

Proof. Let $P \in \Psi^{s/2}(M)$ elliptic and define

$$B_s = P^* P + Id \in \Psi^s(M)$$

where P^* denotes the adjoint operator. To complete the proof, it remains to prove that $\|\cdot\|_{H^s}$ is indeed a norm induced by a complete inner product. To do this, we prove that B_s is an isomorphism on $C^\infty(M)$.

First, note that B_s is elliptic, and then Fredholm and it is selfadjoint by definition. Let $u \in C^\infty(M)$ and suppose $u \in \ker(B_s)$. Then we have,

$$B_s u = 0 \implies \langle Pu, Pu \rangle + \|u\|_{L^2}^2 = 0$$

that implies $u = 0$. Since B_s is self-adjoint, the same is true for B_s^*. Then from (2.4.4) it follows that $\dim(\mathrm{coker(B_s)}) = 0$, thus it is invertible. □

Remark 2.4.13. *For any $s > 0$, the negative Sobolev space $H^{-s}(M)$ is defined by duality through the L^2 pairing $H^{-s}(M) \times H^s(M) \to \mathbb{C}$.*

Chapter 3

Globally integrable quantum systems

In this Chapter, we recall the notion of *globally integrable quantum system*, introduced in [12]. Informally, it is a linear operator that can be written as a function of some first order pseudodifferential operators, thereby called the *quantum actions*. Similar systems, usually denoted as *toric* integrable systems, were widely studied in the literature [92, 93]. Then we prove a couple of properties of quantum integrable systems: the localization of the eigenfunctions of the actions and the separation of their joint spectrum[1]. In Chapter 6, we will provide some examples of manifolds on which the Laplacian is a globally integrable quantum system on which we will concentrate later on in the work.

3.1 Globally integrable quantum systems

Let (M, g) be a compact Riemannian manifold of dimension n, denote by $\Psi^m(M)$ the space of pseudodifferential operators.

Definition 3.1.1 (System of Quantum Actions). *Let* $\{I_j\}_{j=1,\dots,d}$ *be* d *selfadjoint pseudo-differential operators of order 1, fulfilling*

i. $I_j \in \Psi^1(M)$, *for any* $j = 1, \dots, d$*;*

ii. $[I_i, I_j] = 0$, *for any* $i, j = 1, \dots, d$*;*

iii. there exists a constant $c_1 > 0$ *such that* $c_1\sqrt{1-\Delta_g} \leq \sqrt{1+\sum_{j=1}^d I_j^2}$.

We refer to $(I_1, ..., I_d)$ *as the* quantum actions.

[1]see Def. 3.1.2

Definition 3.1.2. *The joint spectrum* Λ *of the operators* I_j *is defined as the set of* $a = (a^1, \ldots, a^d) \in \mathbb{R}^d$ *s.t. there exists* $\psi_a \in L^2(M)$ *with* $\psi_a \neq 0$ *and*

$$I_j \psi_a = a^j \psi_a \ , \quad \forall j = 1, ..., d \ .$$

Definition 3.1.3. *A linear selfadjoint operator* H_L *will be said to be the Hamiltonian of a* globally integrable quantum system *if there exists a function* $h_L \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ *such that*

$$H_L = h_L(I_1, \ldots, I_d)$$

where the operator function is spectrally defined.

Remark 3.1.4. *Systems fulfilling Definition 3.1.3 with the further property that the multiplicity of common eigenvalues* $(a_1, ..., a_d)$ *of the actions is 1 were called toric integrable quantum systems [92].*

Remark 3.1.5. *In the definition of global integrable quantum systems, we do not require the number of actions* d *to be the dimension of the manifold.*

If $H_L = h_L(I)$ is the Hamiltonian of a globally integrable quantum system, then its eigenvalues are

$$\omega_a := h_L(a^1, ..., a^d) \equiv h_L(a) \ , \quad a \equiv (a^1, ..., a^d) \in \Lambda \ .$$

We denote with $\Sigma = \{\omega_a\}_{a \in \Lambda}$ the spectrum of H_L and we introduce the spectral projectors and the relative orthogonal decomposition of functions.

Definition 3.1.6. *For* $a \equiv (a^1, \ldots, a^d) \in \Lambda$*,*

$$\Pi_a \coloneqq \Pi_{a^1} \ldots \Pi_{a^d}$$

where Π_{a^j} *is the orthogonal projector on the eigenspace of* I_j *with eigenvalue* a^j*. Given* $u \in L^2(M, \mathbb{C})$*, we will consider its spectral decomposition*

$$u = \sum_{a \in \Lambda} \Pi_a u \, .$$

Remark 3.1.7. *By spectrally defined, we mean that one has*

$$H_L \psi_a = h_L(a) \psi_a$$

for any $a \in \Lambda$. Then, for a general $u \in L^2(M)$, we have

$$H_L u = H_L \sum_{a \in \Lambda} \Pi_a u = \sum_{a \in \Lambda} \omega_a \Pi_a u \,.$$

Example 3.1.8. *An easy example of a global integrable quantum system is the Laplacian defined on the torus $\mathbb{T}^d$. In fact, given the family of actions $I_j = -i\partial_j$ and the function $h_L\colon \mathbb{R}^d \mapsto \mathbb{R}$, $h_L(\xi) = |\xi|^2$, one has*

$$h_L(I_1, \ldots, I_d) = \sum_{j=1}^{d} \partial_j^2 .$$

3.1.1 Projectors and Sobolev spaces

We begin introducing an order based on the value of the frequencies. This order is equivalent to the one given by the length of the vectors in Λ (see Remark 3.1.10). We prefer to use the one based on the frequencies since it is more suitable in the analysis of the normal form dynamics.

Definition 3.1.9. *Assume there exist constants C, C' and β, with $\beta > 1$, s.t.*

$$\left|\omega_a - C|a|^\beta\right| \leq C' \,.$$

For any $a \in \Lambda$, we define

$$[\![a]\!] := \omega_a^{\frac{1}{\beta}} \,. \tag{3.1.1}$$

Remark 3.1.10. *There exist two constants $C_1, C_2 > 0$ s.t. for any $0 \neq a \in \Lambda$*

$$C_1|a| \leq [\![a]\!] \leq C_2|a| \,.$$

where, for a vector $a \in \Lambda$, we denote $|a| := \sqrt{\sum_{j=1}^d (a^j)^2}$ its Euclidean norm.

Definition 3.1.11. *Given a multi-index $\boldsymbol{a} = (a_1, \ldots, a_r)$, we denote by τ_{ord} the permutation of $(1, ..., r)$ with the property that*

$$[\![a_{\tau_{ord}(j)}]\!] \geq [\![a_{\tau_{ord}(j+1)}]\!] \ , \forall j = 1, ..., r-1 \ . \tag{3.1.2}$$

Definition 3.1.12. *Given a multi-index $\boldsymbol{a} = (a_1, \ldots, a_r)$, we denote*

$$\mu(\boldsymbol{a}) := [\![a_{\tau_{ord}(3)}]\!], \tag{3.1.3}$$

$$S(\boldsymbol{a}) := \mu(\boldsymbol{a}) + |a_{\tau_{ord}(1)} - a_{\tau_{ord}(2)}|. \tag{3.1.4}$$

Before stating the main result of this subsection, we define Sobolev spaces on M.

Definition 3.1.13. *For any $s \geq 0$, the space $\mathcal{H}^s := \mathcal{H}^s(M)$ is the space of the functions $u \in L^2(M, \mathbb{C})$ s.t.*

$$\|u\|_s^2 := \sum_{a\in\Lambda} (1 + |a|)^{2s} \|\Pi_a u\|_{L^2}^2 < \infty \, .$$

For $s < 0$, $\mathcal{H}^s$ is the completion of L^2 in the norm (3.1.13). *In view of Remark 3.1.10, the norm* (3.1.13) *is equivalent to*

$$\|u\|_s^2 = \sum_{a\in\Lambda} (1 + [\![a]\!])^{2s} \|\Pi_a u\|_{L^2}^2$$

Remark 3.1.14. *By (i.) and (iii.) of Def. 3.1.1, for any s the norm* (3.1.13) *is equivalent to the standard Sobolev norm*

$$\left\| (Id - \Delta)^{\frac{s}{2}} \, u \right\|_{L^2(M,\mathbb{C})} ,$$

Then, the spaces $\mathcal{H}^s$ are equivalent to the standard Sobolev spaces $H^s(M, \mathbb{C})$.

3.2 Spectral properties

In this Section, we study two important properties of quantum integrable systems. Firstly, we prove a localization property for the product of the eigenfunctions; this result will inspire later the definition of a class of polynomials (see Def. 4.1.8 and properties below). Then, assuming that the function h_L fulfills some generic conditions, namely steepness and homogeneity, we will deduce, following the results in [12], that the frequencies $\{\omega_a\}_{a\in\Lambda}$ satisfy the Bourgain's clusterization property.

3.2.1 Localization of the eigenfunctions

In this subsection, we will prove the following result.

Theorem 3.2.1. *There exists $\nu > 0$, depending on k and the dimension d, and $\forall N \in \mathbb{N}$ there exists a constant C_N such that, $\forall u_1, \ldots, u_k \in L^2(M)$,*

$$\left| \int_M \Pi_{a_1} u_1 \ldots \Pi_{a_k} u_k \right| \leq C_N \frac{\mu(a_1, \ldots, a_k)^{N+\nu}}{S(a_1, \ldots, a_k)^N} \prod_{l=1}^{k} \|\Pi_{a_l} u_l\|_0 \tag{3.2.1}$$

for any $(a_1, \ldots, a_k) \in \Lambda^k$, where Π_{a_i} are the projectors defined in (3.1.6).

The strategy of the proof is very similar to one of the corresponding results in [48, 6, 46]; the difference being that the indexes run over the set Λ related to the quantum actions. We recall the main steps and we write the proof of the new lemmas. First, define, for any $N \in \mathbb{N}$,

$$Ad_P^N(B) = [Ad_P^{N-1}(B), P]\,, \qquad Ad_P^0(B) = B\,.$$

Recalling Remark 2.2.10, we can restate Lemma 2.3.11 exploiting the Sobolev norm of the function f.

Lemma 3.2.2. *There exists ν s.t. for any $P \in \Psi^1(M)$ and any $N \in \mathbb{N}$ there exist constants $C_N = C_N(P)$ with the property that for any $f \in C^\infty(M)$ one has*

$$\left\| Ad_P^N f \right\|_{\mathcal{B}(L^2(M))} \le C_N \|f\|_{N+\nu}\,.$$

Given $a = (a^1, \dots, a^d) \in \Lambda$, we denote with $l^*(a)$ the index for which $|a^l|$ is maximum. Namely,

$$l^*(a) := \operatorname*{argmax}_{l=1,\dots,d} \left|a^l\right|\,. \tag{3.2.2}$$

Lemma 3.2.3. *Let $B \in \mathcal{B}(L^2(M))$ and $a, b \in \Lambda$. For any $N \ge 0$ and $u_1, u_2 \in \mathcal{H}^\infty$ we have*

$$|\langle B\Pi_a u_1, \Pi_b u_2\rangle| \le C_N \frac{\left\|ad_{I_{l^*}}^N(B)\right\|_{\mathcal{B}(L^2(M))}}{|a-b|^N} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0\,.$$

with $l^* = l^*(a-b)$.

Proof. First we claim that, for any $l = 1, \dots, d$,

$$|\langle B\Pi_a u_1, \Pi_b u_2\rangle| \le C_N \frac{\left\|ad_{I_l}^N(B)\Pi_a u_1\right\|_0}{|a^l - b^l|^N} \|\Pi_b u_2\|_0\,. \tag{3.2.3}$$

For N = 1, recalling that the actions I_j are selfadjoint, we have

$$\begin{aligned}
\langle Ad_{I_l}(B)\Pi_a u_1, \Pi_b u_2\rangle &= |\langle BI_l\Pi_a u_1, \Pi_b u_2\rangle - \langle I_l B\Pi_a u_1, \Pi_b u_2\rangle| \\
&= |\langle BI_l\Pi_a u_1, \Pi_b u_2\rangle - \langle B\Pi_a u_1, I_l\Pi_b u_2\rangle| \\
&= \left|a^l\langle B\Pi_a u_1, \Pi_b u_2\rangle - b^l\langle B\Pi_a u_1, \Pi_b u_2\rangle\right| \\
&= \left|a^l - b^l\right| |\langle B\Pi_a u_1, \Pi_b u_2\rangle|\,.
\end{aligned}$$

Then, by induction on N, and replying the case N=1 with B replaced by $ad_{I_l}^{N-1}(B)$, we get

$$\langle Ad_{I_l}^N(B)\Pi_a u_1, \Pi_b u_2\rangle = \langle Ad_{I_l}\left(Ad_{I_l}^{N-1}(B)\right)\Pi_a u_1, \Pi_b u_2\rangle$$
$$= |a^l - b^l|\langle Ad_{I_l}^{N-1}(B)\Pi_a u_1, \Pi_b u_2\rangle = |a^l - b^l|^N\langle B\Pi_a u_1, \Pi_b u_2\rangle\,.$$

This means that

$$|\langle B\Pi_a u_1, \Pi_b u_2\rangle| = \frac{|\langle ad_{I_l}^N(B)\Pi_a u_1, \Pi_b u_2\rangle|}{|a^l - b^l|^N}\,.$$

Then, by Cauchy-Schwartz inequality, it follows that the claim (3.2.3) holds. Choosing $l = l^*(a-b)$ defined in (3.2.2) and exploiting the trivial inequality

$$|a| \leq \sqrt{d}\,|a^{l^*}|\,,$$

we get the thesis. □

Proof of Theorem 3.2.1. We first prove the theorem for $k = 3$. Without loss of generality, we can assume $[\![a]\!] \geq [\![b]\!] \geq [\![c]\!]$. We distinguish two cases.

If $[\![c]\!] \leq 2|a-b|$, we apply Lemma 3.2.2 with $f = \Pi_c u_3$ and $P = I_{l^*(a-b)}$ and we get

$$\left\|Ad_{I_{l^*}}^N\,(\Pi_c u_3)\right\|_{\mathcal{B}(L^2(M))} \leq C_N\|\Pi_c u_3\|_{N+\nu} \leq C_N[\![c]\!]^{N+\nu}\|\Pi_c u_3\|_0\,.$$

To get the thesis we apply Lemma 3.2.3. We have

$$\left|\int \Pi_a u_1 \Pi_b u_2 \Pi_c u_3\right| \leq \frac{\left\|Ad_{I_{l^*}}^N\,(\Pi_c u_3)\right\|_{\mathcal{B}(L^2(M))}}{|a-b|^N}\|\Pi_a u_1\|_0\|\Pi_b u_2\|_0$$
$$\leq C_N\frac{[\![c]\!]^{N+\nu}}{|a-b|^N}\|\Pi_a u_1\|_0\|\Pi_b u_2\|_0\|\Pi_c u_3\|_0\,,$$

for some constant C_N depending on I_j and N. Finally, we observe that

$$\frac{[\![c]\!]^{N+\nu}}{|a-b|^N} \leq 3^N\frac{[\![c]\!]^{N+\nu}}{([\![c]\!] + |a-b|)^N}\,,$$

since $[\![c]\!] \leq 2|a-b|$.
If $[\![c]\!] > 2|a-b|$, then

$$\frac{2}{3} \leq \frac{\mu(a,b,c)}{S(a,b,c)} \leq 1\,. \tag{3.2.4}$$

In this case, we exploit the Sobolev embedding $H^{d/2} \hookrightarrow L^\infty$ and we get

$$
\begin{aligned}
\left| \int \Pi_a u_1 \Pi_b u_2 \Pi_c u_3 \right| &\le \|\Pi_c u_3\|_{L^\infty} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \\
&\le C_{s_0} \|\Pi_c u_3\|_{s_0} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \\
&\le C_{s_0} [\![c]\!]^{s_0} \|\Pi_c u_3\|_0 \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \\
&\overset{(3.2.4)}{\le} C_N \frac{\mu(a,b,c)^{N+\nu}}{S(a,b,c)^N} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \|\Pi_c u_3\|_0 \,,
\end{aligned}
$$

with $\nu = s_0$.

We consider now the general case concerning k eigenfunctions, assuming for simplicity that $[\![a_1]\!] \ge [\![a_2]\!] \ge [\![a_3]\!]$ are the largest indexes (with respect to the order induced by $[\![\cdot]\!]$).

If $[\![a_3]\!] \le 2|a_1 - a_2|$, we reason as in the first case, with $f = \Pi_{a_3} u_3 \dots \Pi_{a_k} u_k$ the multiplication operator. In particular, we bound

$$
\begin{aligned}
\|\Pi_{a_3} u_3 \dots \Pi_{a_k} u_k\|_{N+\nu} &\le \sum_{l=3}^{k} \Big(\|\Pi_{a_l} u_l\|_{N+\nu} \prod_{n \ne l} \|\Pi_{a_n} u_n\|_{s_0} \Big) \\
&\le \sum_{l=3}^{k} \Big([\![a_l]\!]^{N+\nu} \|\Pi_{a_l} u_l\|_0 \prod_{n \ne l} |a_n|^{s_0} \|\Pi_{a_n} u_n\|_0 \Big) \\
&\le C_N [\![a_3]\!]^{N+\nu} \|\Pi_{a_3} u_3\|_0 \prod_{l=3}^{k} [\![a_l]\!]^{s_0} \|\Pi_{a_l} u_l\|_0 \\
&\le C_N \mu(a_1, \dots, a_k)^{N+\nu+s_0} \prod_{l=3}^{k} \|\Pi_{a_l} u_l\|_0 \,.
\end{aligned}
$$

Then we get the thesis applying again Lemmas 3.2.2 and 3.2.3, with ν replaced by $\nu + s_0$. If $[\![a_3]\!] > 2|a_1 - a_2|$, again we take profit of Sobolev embedding. Namely, we compute

$$
\begin{aligned}
\left| \int \Pi_{a_1} u_1 \Pi_{a_2} u_2 \dots \Pi_{a_k} u_k \right| &\le \Big(\prod_{l=3}^{k} \|\Pi_{a_l} u_l\|_\infty \Big) \|\Pi_{a_1} u_1\|_0 \|\Pi_{a_2} u_2\|_0 \\
&\le \Big(\prod_{l=3}^{k} [\![a_l]\!]^{s_0} \|\Pi_{a_l} u_l\|_{s_0} \Big) \|\Pi_{a_1} u_1\|_0 \|\Pi_{a_2} u_2\|_0 \\
&\le [\![a_3]\!]^{(k-2)s_0} \prod_{l=1}^{k} \|\Pi_{a_l} u_l\|_0 \,,
\end{aligned}
$$

and we observe that (3.2.4) holds as above. □

3.2.2 Separation of the eigenvalues

In this subsection, we prove that the eigenvalues of a global integrable quantum system fulfill a notable partition, as long as the Hamiltonian function is steep and homogeneous.

Definition 3.2.4. *A function $h_L \in C^\infty(\mathbb{R}^d;\mathbb{R})$ is said to be homogeneous of degree* $\mathtt{d}$ *at infinity if there exists an open ball $\mathcal{B}_r \in \mathbb{R}^d$, centered at the origin, such that,*

$$h_L(\lambda a) = \lambda^{\mathtt{d}} h_L(a), \qquad \forall \lambda > 0$$

for any $a \in \mathbb{R}^d \setminus \mathcal{B}_r$;

We recall the definition of steepness [68].

Definition 3.2.5 (Steepness)**.** *Let $\mathcal{U} \subset \mathbb{R}^d$ be a bounded connected open set with nonempty interior. A function $h_L \in C^1(\mathcal{U})$, is said to be steep in $\mathcal{U}$ with steepness radius* $\mathtt{r}$*, steepness indices $\alpha_1, \dots, \alpha_{d-1}$ and (strictly positive) steepness coefficients $\mathtt{B}_1, \dots, \mathtt{B}_{d-1}$, if its gradient $\upsilon(a) := \frac{\partial h_L}{\partial a}(a)$ satisfies the following estimates: $\inf_{a\in\mathcal{U}} \|\upsilon(a)\| > 0$ and for any $a \in \mathcal{U}$ and for any s dimensional linear subspace $M \subset \mathbb{R}^d$ orthogonal to $\upsilon(a)$, with $1 \le s \le d-1$, one has*

$$\max_{0\le\eta\le\xi} \min_{u\in M:\|u\|=1} \|\Pi_M \upsilon(a+\eta u)\| \ge \mathtt{B}_s \xi^{\alpha_s} \quad \forall \xi \in (0, \mathtt{r}] \ ,$$

where Π_M is the orthogonal projector on M; the quantities u and η are also subject to the limitation $a + \eta u \in \mathcal{U}$.

Remark 3.2.6. *It is well known that steepness is generic. Examples of steep functions are given by convex or quasiconvex functions.*

The proof of the following crucial result is a consequence of the construction in [12]. We report here the proof for the sake of completeness.

Theorem 3.2.7. *Let $H_L = h_L(I_1, \dots, I_d)$ a globally integrable quantum systems and assume that h_L is homogeneous at infinity (see Def. 3.2.4) and steep (see Def. 3.2.5). Then there exists a partition*

$$\Lambda = \bigcup_{\alpha\in\mathfrak{A}} \Omega_\alpha$$

with the following properties.

i. Each Ω_α is dyadic, namely there exists a constant C, independent of α, such that

$$\sup_{a\in\Omega_\alpha} |a| \le C \inf_{a\in\Omega_\alpha} |a|.$$

ii. There exist $\delta > 0$ such that, if $a \in \Omega_\alpha$ and $b \in \Omega_\beta$ with $\alpha \neq \beta$, then

$$|a-b| + |\omega_a - \omega_b| \ge C_\delta(|a|^\delta + |b|^\delta). \tag{3.2.5}$$

Proof. We consider the partition

$$\Lambda = \bigcup_{s=1}^{d} \bigcup_{\substack{M \text{ s.t.} \\ \dim(M)=s}} \bigcup_{j\in\mathcal{J}_M} E^{(s)}_{M,j}$$

as in Definition 8.26 of [12]. We take $\mathtt{R} > 0$ large enough such that Theorem 8.28 of [12] is satisfied and we modify the partition as follows:

$$\begin{aligned} \widetilde{E}_{\mathtt{R}} &:= \bigcup \{E^{(s)}_{M,j} \mid \exists a \in E^{(s)}_{M,j} \quad \text{s.t.} |a| \le \mathtt{R}\} \\ \widetilde{E}^{(s)}_{M,j} &= E^{(s)}_{M,j} \quad \text{if} \quad |a| \ge \mathtt{R} \quad \forall a \in E^{(s)}_{M,j}\,. \end{aligned}$$

We are going to show that such a new partition satisfies properties (i) and (ii).
Proof of Property (i). By Theorem 8.28, the blocks $E^{(s)}_{M,j}$ are dyadic, thus (i) holds for the blocks $\widetilde{E}^{(s)}_{M,j}$ with $C = 2$. Let us consider $\widetilde{E}_{\mathtt{R}}$: for any $a \in \widetilde{E}_{\mathtt{R}}$ one has

$$|a| \le 2 \min_{b\in E^{(s)}_{M,j}} |b| \le 2\mathtt{R}\,,$$

where $E^{(s)}_{M,j}$ is the block in the original partition $\{E^{(s)}_{M,j}\}_{s,M,j}$ to which the lattice point a belongs. This immediately implies

$$\sup_{a\in\widetilde{E}_{\mathtt{R}}} |a| \le 2\mathtt{R} \le 2\gamma^{-1}\mathtt{R} \min_{a\in\widetilde{E}_{\mathtt{R}}} |a|\,,$$

if $\gamma := \min\{|b| \mid b \in \Lambda \setminus \{0\}\}$. Then (i) holds with $C := \max\{2, 2\gamma^{-1}\mathtt{R}\}$ for all the elements of the new partition.
Proof of Property (ii). We show that there exist μ and C_μ such that, if $a, b \in \Lambda$ are such that (3.2.5) is violated, namely

$$|a-b| + |\omega_a - \omega_b| < C_\mu \left(|a| + |b|\right)^\mu\ , \tag{3.2.6}$$

then a and b must belong to the same block. If both $|a| \leq \mathtt{R}$ and $|b| \leq \mathtt{R}$, then a and b belong to the same block since they are both contained into $\widetilde{E}_{\mathtt{R}}$. Thus let us suppose that $|a| \geq \mathtt{R}$. First , we observe that there exists $c_\mu > 0$ such that, if $C_\mu \leq c_\mu$, then (3.2.6) implies

$$|a-b| \leq |a|^\mu . \tag{3.2.7}$$

Indeed, one has $|b|^\mu \leq 2^\mu(|a|^\mu + |b-a|^\mu)$, which implies

$$|a-b| \leq C_\mu(1+2^\mu)|a|^\mu + C_\mu 2^\mu |a-b|^\mu .$$

Now, if $|a-b| \leq (2^{\mu+1}C_\mu)^{\frac{1}{\mu-1}}$, one has

$$|a-b| \leq (2^{\mu+1}C_\mu)^{\frac{1}{\mu-1}} \leq (2^{\mu+1}C_\mu)^{\frac{1}{\mu-1}}|a|^\mu , \tag{3.2.8}$$

while if $|a-b| \geq (2^{\mu+1}C_\mu)^{\frac{1}{\mu-1}}$ one has

$$\frac{|a-b|}{2} \leq |a-b| - C_\mu 2^\mu |a-b|^\mu \leq C_\mu(1+2^\mu)|a|^\mu . \tag{3.2.9}$$

Combining (3.2.8) and (3.2.9) and reducing the value of C_μ, one gets (3.2.7).
We proceed showing that, if (3.2.6) holds with C_μ is small enough, one also has

$$|\varpi(a)\cdot(b-a)| \leq |a|^\delta |a-b| , \tag{3.2.10}$$

where $\varpi(a) := \partial_a h_0(a)$ and h_0 is the function whose existence is assumed in 3.1.1. Indeed, let us suppose by contradiction that (3.2.10) does not hold true: then, since $|a-b| \leq |a|^\mu$, $\forall t \in [0,1]$ one has

$$\begin{aligned}|\varpi(a+tb)\cdot(b-a)| &\geq |\varpi(a)\cdot(b-a)| - |\varpi(a+tb)-\varpi(a)||b-a| \\ &\geq |\varpi(a)\cdot(b-a)| - \sup_{\substack{a' \text{ s.t.} \\ |a-a'|\leq|b-a|}} |\partial_a^2 h_0(a')||b-a|^2 \\ &\geq |a|^\delta|a-b| - C_{h_0} \sup_{\substack{a' \text{ s.t.} \\ |a-a'|\leq|b-a|}} |a'|^{M-2}|b-a|^2 ,\end{aligned}$$

where in the last passage we have used the fact that h_0 is a homogeneous function of degree M. Then $\forall t$ one has

$$\begin{aligned}|\varpi(a+tb)\cdot(b-a)| &\geq |a|^\delta|b-a| - C_{h_0}(|a|+|b-a|)^{M-2}|b-a|^2 \\ &\geq |a|^\delta - 2C_{h_0}|a|^{M-2+2\mu} \geq \frac{|a|^\delta}{2} ,\end{aligned} \tag{3.2.11}$$

provided $\mathtt{R}$ is large enough. But then on the one hand (3.2.11) gives

$$|\omega_a - \omega_b| = \left| \int_0^1 \varpi(a + t(b-a)) \cdot (b-a) \, dt \right| \geq \frac{|a|^\delta}{2} \,, \tag{3.2.12}$$

while on the other hand (3.2.6) gives

$$|\omega_a - \omega_b| \leq C_\mu |a|^\mu + C_\mu |b|^\mu \leq C^\mu (2^\mu + 1)|a|^\mu + 2^\mu C_\mu |b-a|^\mu \leq 2C^\mu (2^\mu + 1)|a|^\mu \,. \tag{3.2.13}$$

Since estimates (3.2.13) and (3.2.12) are not compatible for $\mathtt{R}$ large enough, since $\delta > \mu$, one gets a contradiction. Thus we conclude that (3.2.10) holds true and, combining estimates (3.2.7) and (3.2.10), that

$$|\varpi(a) \cdot (b-a)| \leq |a|^\delta |a-b| \,, \quad |a-b| \leq |a|^\mu \,, \quad |a| \geq \mathtt{R} \,, \tag{3.2.14}$$

namely a is resonant with $b - a$ according to Definition 6.3 of [12]. Then the proof follows exactly with the same passages used to prove Lemma 8.39 of [12]. □

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Chapter 4

Functions with localized coefficients

In this Chapter we describe the class of *polynomials with localized coefficients* [48, 5], generalizing this notion for polynomial components labeled by vectors $a \in \Lambda$. In particular, we will study these polynomials in a Hamiltonian setting, since later on we exploit them in the context of Hamiltonian equations.

4.1 Functional settings: the Hamiltonian structure and polynomials with localized coefficients

4.1.1 Phase space

Recalling the definition of the Sobolev spaces $\mathcal{H}^s$ (see Def. 3.1.13), we introduce the spaces

$$\mathcal{H}^{-\infty} := \bigcup_s \mathcal{H}^s, \qquad \mathcal{H}^{\infty} := \bigcap_s \mathcal{H}^s .$$

In the following, we will work on the complex extension of the phase space.

Definition 4.1.1. *We define $\Lambda_e := \Lambda \times \{\pm 1\}$ and we denote the index as $A \equiv (a, \sigma) \in \Lambda_e$.*

Definition 4.1.2. *For $s \in \mathbb{R}$, define $\mathcal{H}^s_e := \mathcal{H}^s_e(M) := \mathcal{H}^s(M) \times \mathcal{H}^s(M)$. For $u \equiv (u_+, u_-) \in \mathcal{H}^s_e$ we will use the norm*

$$\|u\|^2_{\mathcal{H}^s_e(M)} \equiv \|(u_+, u_-)\|^2_{\mathcal{H}^s_e(M)} := \|u_+\|^2_s + \|u_-\|^2_s .$$

Correspondingly, for $u \in \mathcal{H}^s_e$ and $A \equiv (a, \sigma) \in \Lambda_e$, we define

$$\Pi_A u = \Pi_{(a,\sigma)}(u_+, u_-) := \Pi_a u_\sigma ,$$

where Π_a is given in Definition 3.1.6. For $u \in \mathcal{H}^s_e$ one has

$$u = \sum_{A \in \Lambda_e} \Pi_A u\,.$$

Given an element in $\mathcal{H}^s_e$, we define the involution that we use in order to identify the subspace of "real functions", on which $u_+ = \overline{u}_-$.

Definition 4.1.3. *Let $u \equiv (u_+, u_-) \in \mathcal{H}^s_e$ for some $s \in \mathbb{R}$, we define*

$$I(u) := (\overline{u_-}, \overline{u_+})$$

with the bar denoting the complex conjugate. If $I(u) = u$ we will say that u is real.

Correspondingly, it is useful to define, for $A \in \Lambda_e$,

$$\bar{A} \equiv \overline{(a, \sigma)} := (a, -\sigma)\,,$$

so that one has

$$\Pi_A\left(I(u)\right) = \overline{\Pi_{\bar{A}} u}\,.$$

Definition 4.1.4. *A function $F \in C^\infty(\mathcal{O}; \mathbb{C})$, with $\mathcal{O} \subset \mathcal{H}_{s_0}$ an open neighborhood of the origin will be said to be real for real u if $F(u) \in \mathbb{R}$ whenever $u = I(u)$ (see Def. 4.1.3).*

Definition 4.1.5. *We will denote the ball centered in the origin of $\mathcal{H}^s_e$ of radius R by*

$$\mathcal{B}^s_R := \{u \in \mathcal{H}^s_e \colon \|u\|_s < R\}\,.$$

4.1.2 Hamiltonian structure

Given a function $H \in C^\infty(\mathcal{O}, \mathbb{C})$ for some open set $\mathcal{O} \subset \mathcal{H}^s_e$ and for some s, we define the corresponding Hamiltonian vector field by

$$\dot{u}_+ = i\boldsymbol{\nabla}_{u_-} H, \qquad \dot{u}_- = -i\boldsymbol{\nabla}_{u_+} H$$

where $\boldsymbol{\nabla}_{u_\pm}$ is the L^2-gradient with respect to $u_\pm$. Namely, it is defined by the following identity:

$$\mathrm{d}_{u_+} H h_+ = \langle \boldsymbol{\nabla}_{u_+} H, h_+ \rangle, \qquad \forall h_+ \in \mathcal{H}^\infty(M)\,,$$

and similarly for $\boldsymbol{\nabla}_{u_-}$. We will denote the vector field associated to H as

$$X_H := \left(i\boldsymbol{\nabla}_{u_-}H, -i\boldsymbol{\nabla}_{u_+}H\right). \tag{4.1.1}$$

The Poisson brackets of two functions are defined as usual as follows.

Definition 4.1.6. *Given two functions $f, g \in \mathcal{C}^\infty(\mathcal{O}, \mathbb{C})$, with $\mathcal{O}$ as above, we define their Poisson brackets by*

$$\{f, g\}(u) := df(u)X_g(u).$$

4.1.3 Polynomial with localized coefficients

We recall that, given a polynomial functions P of degree r, there exists a unique r-linear symmetric function $\widetilde{P}$ such that

$$P(u) = \widetilde{P}(u, ..., u). \tag{4.1.2}$$

Then, we can write

$$P(z) = \sum_{A_1,...,A_r} \widetilde{P}(\Pi_{A_1}u, ..., \Pi_{A_r}u)\ . \tag{4.1.3}$$

We define a property of localization for monomials

$$\widetilde{P}(\Pi_{A_1}u, ..., \Pi_{A_r}u)$$

as functions of the indexes $A_1, ..., A_r$. To do that, we redefine the functions S and μ, introduced in Def. 3.1.12, as functions of the extended indexes $A \in \Lambda_e$.

Definition 4.1.7. *For any $\Lambda_e \ni A = (a, \sigma)$, we denote*

$$[\![A]\!] := [\![a]\!] = \omega_a^{\frac{1}{\beta}} \tag{4.1.4}$$

Coherently, for any $r \geq 3$ and $\boldsymbol{A} = (A_1, \ldots, A_r) \in \Lambda_e^r$ we denote (see Def. 3.1.12)

$$\begin{aligned} \mu(\boldsymbol{A}) &:= \mu(\boldsymbol{a}) = [\![a_{\tau_{ord}(3)}]\!] \\ S(\boldsymbol{A}) &:= S(\boldsymbol{a}) = \mu(\boldsymbol{a}) + |a_{\tau_{ord}(1)} - a_{\tau_{ord}(2)}| \end{aligned} \tag{4.1.5}$$

We are now in the position to state the definition of polynomials with localized coefficients.

Definition 4.1.8 (Polynomial with localized coefficients)**.** *(i) Let $\nu \in [0, +\infty)$, $N \geq 1$. We denote by $L_r^{\nu,N}$ the class of the polynomials F homogeneous of degree r, such* *that there exists* C_N *s.t.*

$$\left|\widetilde{F}(\Pi_{A_1}u_1, ..., \Pi_{A_r}u_r)\right| \le C_N \frac{\mu(\mathbf{A})^{\nu+N}}{S(\mathbf{A})^N} \|\Pi_{A_1}u_1\|_0 ... \|\Pi_{A_r}u_r\|_0 \, , \qquad (4.1.6)$$
$$\forall\, u_1, \dots, u_r \in \mathcal{H}_e^\infty \; , \quad \forall \mathbf{A} \in \Lambda_e^r \, .$$

The smallest possible constant C_N *such that* (4.1.6) *holds defines a norm in* $L_r^{\nu,N}$*, precisely*

$$\|F\|^{\nu,N} := \sup_{\|u_1\|_0=1,\dots,\|u_r\|_0=1} \sup_{A_1,\dots,A_r} |\tilde{F}(\Pi_{A_1}u_1, \dots, \Pi_{A_r}u_r)| \frac{S(\boldsymbol{A})^N}{\mu(\boldsymbol{A})^{N+\nu}} \, .$$

(*ii*) *We say that a polynomial* F *has* localized coefficients *if there exists* $\nu \in [0, +\infty)$ *and* N_0*, such that, for any* $N \ge N_0$ *one has* $F \in L_r^{\nu,N}$*. In this case we write* $F \in L_r := \cup_{N_0} \cup_{\nu \ge 0} \cap_{N \ge N_0} L_r^{\nu,N}$.

We introduce also a norm that depends on the homogeneity of polynomials. This norm will turn out to be appropriate in the normal form iteration.

Definition 4.1.9. *Given* $F \in L_r^{\nu,N}$ *a homogeneous polynomial of degree* r *with localized coefficients, we define*

$$\|F\|_R^{\nu,N} := \|F\|^{\nu,N} R^r. \qquad (4.1.7)$$

For non-homogeneous polynomials, we use the following notation.

Definition 4.1.10. *For* $r < \bar{r}$*, we define the space*

$$L_{r,\bar{r}}^{\nu,N} := \bigoplus_{l=r}^{\bar{r}} L_l^{\nu,N} \, .$$

Definition 4.1.11. *For* $F \in L_r^{\nu,N}$ *we define*

$$\|F\|_R^{\nu,N} := \|F\|^{\nu,N} R^r \, .$$

For $F \in L_{r,\bar{r}}^{\nu,N}$*, so that* $F = \sum_{l=r}^{\bar{r}} F_l$ *with* $F_l \in L_l^{\nu,N}$*, we define*

$$\|F\|_R^{\nu,N} := \sum_{l=r}^{\bar{r}} \|F_l\|_R^{\nu,N} \, .$$

We conclude by defining functions with localized coefficients.

Definition 4.1.12 (Function with localized coefficients)**.** *Let* $s_0 > 0$ *and consider a function* $F \in C^\infty(\mathcal{O}; \mathbb{C})$*, with* $\mathcal{O} \subset \mathcal{H}_e^{s_0}$ *an open neighborhood of the origin.*

F is said to have localized coefficients if both the following properties hold:

- *all the monomials of the Taylor expansion of F at the origin have localized coefficients.*
- *For any $s > 0$ large enough there exists an open neighborhood of the origin $\mathcal{O}_s \subset \mathcal{H}^s_e$ s.t. X_F belongs to $\mathcal{C}^\infty(\mathcal{O}_s, \mathcal{H}^s_e)$.*

4.2 Properties of polynomials with localized coefficients

In this Section, we state and prove the main properties of polynomials with localized coefficients. Throughout this section, we write $x \lesssim y$ if there exists a constant C, independent of the relevant parameters, such that $x \leq Cy$. If we need to specify that the constant depends on a parameter s, we will write $x \lesssim_s y$. Moreover, we will write $x \sim y$ if $x \lesssim y$ and $y \lesssim x$

4.2.1 Localized vector fields and tame estimate

Given a polynomial map $X : \mathcal{H}^\infty \to \mathcal{H}^{-\infty}$ of degree r, we denote with $\widetilde{X}$ the unique r-linear symmetric function such that

$$X(u) = \widetilde{X}(u, ..., u). \tag{4.2.1}$$

Definition 4.2.1. *Let $X : \mathcal{H}^\infty \to \mathcal{H}^{-\infty}$ be a polynomial map of degree r and let $\widetilde{X}$ be the associated multi-linear form. We will say that X has localized coefficients if there exists $\nu \in [0, +\infty)$ such that $\forall N \geq 1$ there exists C_N s.t. such that*

$$\begin{aligned} \|\Pi_B \widetilde{X}(\Pi_{A_1} u_1, ..., \Pi_{A_r} u_r)\|_0 \leq C_N \frac{\mu(B, \mathbf{A})^{\nu+N}}{S(B, \mathbf{A})^N} \|\Pi_{A_1} u_1\|_0 \dots \|\Pi_{A_r} u_r\|_0 \ , \\ \forall\, u_1, \dots, u_r \in \mathcal{H}^\infty_e \ , \quad \forall (B, \mathbf{A}) \in \Lambda_e \times \Lambda^r_e \, , \end{aligned} \tag{4.2.2}$$

where we denoted by $(B, \mathbf{A})$ the multi-index $(B, A_1, ..., A_r)$.
The smallest possible constant C_N defines a seminorm, namely

$$\|X\|^{\nu,N} = \sup_{\|u_1\|_0 = 1, \dots, \|u_r\|_0 = 1} \ \sup_{A_1, \dots, A_r, B} \left\| \tilde{X}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r) \right\|_0 \frac{S(\boldsymbol{A}, B)^N}{\mu(\boldsymbol{A}, B)^{N+\nu}}. \tag{4.2.3}$$

We also define

$$\|X\|^{\nu,N}_R := \|X\|^{\nu,N} R^r \tag{4.2.4}$$

and we call $M^{\nu,N}_r$ the space of r-homogeneous polynomial maps for which (4.2.2) *holds.*

It is easy to see that if a polynomial function has localized coefficients, then its

Hamiltonian vector field is a map with localized coefficients.

Lemma 4.2.2. *Let $P \in L^{\nu,N}_{r+1}$, then $X_P \in M^{\nu,N}_r$. Furthermore,*

$$\|X_P\|^{\nu,N} \leq r\|P\|^{\nu,N} \ , \tag{4.2.5}$$

and therefore

$$\|X_P\|^{\nu,N}_R \leq \frac{r}{R}\|P\|^{\nu,N}_R \, . \tag{4.2.6}$$

Proof. For $B \in \Lambda_e$ and $\mathbf{A} = (A_1, \dots, A_r) \in \Lambda_e^r$, we want to bound

$$\left\|\Pi_B \widetilde{X}_P \left(\Pi_{A_1} u, \dots, \Pi_{A_r} u\right)\right\|_0 .$$

Suppose for simplicity that $B = (b, +)$, the case $B = (b, -)$ being totally analogous. We compute, exploiting self-adjointness of Π_B and the definition of the L^2-gradient (4.1.1),

$$\begin{aligned}
&\left\|\Pi_B \widetilde{X}_P \left(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r\right)\right\|_0 \\
&\quad = \sup_{h\in\mathcal{H}^\infty, \|h\|=1} \left|\langle \Pi_B \widetilde{X}_P(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r), h\rangle_e\right| \\
&\quad = \sup_{h\in\mathcal{H}^\infty, \|h\|=1} \left|\langle \widetilde{X}_p(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r), \Pi_B h\rangle_e\right| \\
&\quad \leq r \sup_{h\in\mathcal{H}^\infty, \|h\|=1} \left|\widetilde{P}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r, \Pi_B h)\right| \\
&\quad = r\|P\|^{\nu,N}_{r+1} \frac{\mu(\mathbf{A}, B)^{N+\nu}}{S(\mathbf{A}, B)^N} \|\Pi_{A_1} u_1\| \dots \|\Pi_{A_r} u_r\| \|\Pi_B h\|.
\end{aligned}$$

From that, we deduce (4.2.5). □

Before proving the tame estimate, we need a simple technical result.

Lemma 4.2.3. *Fix $\nu > 0$. For $s > \nu + \frac{d}{2}$, we have*

$$\sum_{A\in\Lambda_e} (1 + [\![a]\!])^\nu \|\Pi_A u\|_0 \lesssim_s \|u\|_s. \tag{4.2.7}$$

Proof. We compute

$$\begin{aligned}
&\sum_{A\in\Lambda_e} (1 + [\![a]\!])^\nu \|\Pi_A u\|_0 \leq \sum_{A\in\Lambda_e} \frac{(1 + [\![a]\!])^s}{(1 + [\![a]\!])^{s-\nu}} \|\Pi_A u\|_0 \leq \\
&\leq \sqrt{\sum_{A\in\Lambda_e} \frac{1}{(1 + [\![a]\!])^{2(s-\nu)}}} \sqrt{\sum_{A\in\Lambda_e} (1 + [\![a]\!])^{2s} \|\Pi_A u\|_0} \lesssim C_s \|u\|_s
\end{aligned} \tag{4.2.8}$$

since the first sum converges for $2(s - \nu) > d$, that is $s > \nu + \frac{d}{2}$. □

Remark 4.2.4. *In the following computations, we will repeatedly compare $|a|$ and $[\![a]\!]$, taking profit of Lemma 3.1.1. In particular, we notice that for any constant $0 < K_2 < 1$ small enough there exists $K_1 > 0$ large enough such that*

$$[\![a]\!] \geq K_1 [\![b]\!] \implies |a-b| \geq K_2 [\![a]\!] . \tag{4.2.9}$$

In fact we have, defining C_1, C_2 as in Lemma 3.1.1,

$$|a-b| \geq |a| - |b| \geq \frac{1}{C_2}[\![a]\!] - \frac{1}{C_1}[\![b]\!] \geq \left(\frac{1}{C_2} - \frac{1}{C_1 K_1}\right)[\![a]\!] \tag{4.2.10}$$

that is the wanted estimate for K_1 large enough. Moreover, we point out that there exists a constant C such that

$$[\![a]\!] \geq [\![b]\!] \implies |a-b| \leq C[\![a]\!] . \tag{4.2.11}$$

Theorem 4.2.5 (Tame estimate)**.** *Let $X \in \mathcal{M}_r^{\nu,N}$ and fix $s > \frac{3}{2}d + \nu$. If $N > d + s$, for any $s_0 \in (\frac{3}{2}d + \nu, s)$, one has*

$$\begin{aligned} \left\|\widetilde{X}(u_1, \dots, u_r)\right\|_s \lesssim_{s,s_0,N} \|X\|^{\nu,N} \sum_{j=1}^{r} \|u_j\|_s \prod_{k \neq j} \|u_k\|_{s_0} \\ \forall u_1, \dots, u_r \in \mathcal{H}_e^\infty . \end{aligned} \tag{4.2.12}$$

Proof. We have

$$\left\|\tilde{X}(u_1, \dots, u_r)\right\|_s^2 \leq \sum_{B \in \Lambda} (1 + [\![B]\!])^{2s} \left\| \sum_{A_1, \dots A_r} \Pi_B \tilde{X}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r) \right\|_0^2 .$$

Exploiting Def. 4.2.1, the argument of the sum in B of the equality above is controlled by the square of

$$\|X\|^{\nu,N} \sum_{A_1, \dots A_r} (1 + [\![b]\!])^s \frac{\mu(\mathbf{A}, b)^{N+\nu}}{S(\mathbf{A}, b)^N} \|\Pi_{A_1} u_1\|_0 \dots \|\Pi_{A_r} u_r\|_0 .$$

By symmetry, we can consider the case $[\![a_1]\!] \geq [\![a_2]\!] \geq \dots \geq \|a_r\|$. By definition of S and μ, we have

$$\begin{cases} [\![b]\!] \geq [\![a_2]\!] \implies S(\mathbf{A}, b) = |a_1 - b| + [\![a_2]\!] \geq |a_1 - b| \\ [\![b]\!] \leq [\![a_2]\!] \implies S(\mathbf{A}, b) \geq |a_1 - a_2| + [\![b]\!] . \end{cases}$$

We have, $\forall\kappa > d$,

$$\sum_{B\in\Lambda_e} S^{-\kappa} \le \sum_{[\![b]\!]\ge[\![a_2]\!]} S^{-\kappa} + \sum_{[\![b]\!]<[\![a_2]\!]} S^{-\kappa} \lesssim \tag{4.2.13}$$

$$\lesssim \sum_{B\in\Lambda_e} \frac{1}{(|a_1-b|+1)^\kappa} + \sum_{B\in\Lambda_e} \frac{1}{(|a_1-a_2|+[\![b]\!])^\kappa} \lesssim \tag{4.2.14}$$

$$\lesssim \sum_{B'\in\Lambda_e} \frac{1}{(|b'|+1)^\kappa} + \sum_{B\in\Lambda_e} \frac{1}{(|a_1-a_2|+|b|)^\kappa} < \infty \tag{4.2.15}$$

where $B' = (b', \sigma_b)$, with $b' = b - a_1$. With a similar calculation, we obtain also

$$\sum_{A_1\in\Lambda_e} S^{-\kappa} < \infty\,.$$

By Cauchy-Schwarz inequality, we estimate

$$\sum_{A_1,\dots A_r} (1+[\![b]\!])^s \frac{\mu(\mathbf{A},b)^{N+\nu}}{S(\mathbf{A},b)^N} \|\Pi_{A_1}u_1\|_0 \dots \|\Pi_{A_r}u_r\|_0 \le \tag{4.2.16}$$

$$\le \left(\sum_{A_1,\dots A_r} (1+[\![b]\!])^{2s} \frac{\mu(\mathbf{A},b)^{2N+\nu-\kappa}}{S(\mathbf{A},b)^{2N-\kappa}} \|\Pi_{A_1}u_1\|_0^2 \|\Pi_{A_2}u_2\|_0 \dots \|\Pi_{A_r}u_r\|_0 \right)^{\frac{1}{2}} \cdot \tag{4.2.17}$$

$$\cdot \left(\sum_{A_1,\dots A_r} \frac{\mu(\mathbf{A},b)^{\nu+\kappa}}{S(\mathbf{A},b)^{\kappa}} \|\Pi_{A_2}u_2\|_0 \dots \|\Pi_{A_r}u_r\|_0 \right)^{\frac{1}{2}} . \tag{4.2.18}$$

Exploiting $\mu(\mathbf{A},b) \le [\![a_2]\!]$, the second term is estimated by

$$\left(\sum_{A_1,\dots A_r} \frac{\mu(\mathbf{A},b)^{\nu+\kappa}}{S(\mathbf{A},b)^{\kappa}} \|\Pi_{A_2}u_2\|_0 \dots \|\Pi_{A_r}u_r\|_0 \right)^{\frac{1}{2}}$$

$$\le \left(\sum_{A_1} S(\mathbf{A},b)^{-\kappa} \sum_{A_2,\dots,A_r} [\![a_2]\!]^{\nu+\kappa} \|\Pi_{A_2}u_2\|_0 \dots \|\Pi_{A_r}u_r\|_0 \right)^{\frac{1}{2}} \lesssim \prod_{l=2}^r \|u_l\|_{s_0}^{\frac{1}{2}}$$

for each $s_0 > \nu + \frac{d}{2} + \kappa$.

Consider now the first term (4.2.17). We claim that

$$\frac{\mu(\mathbf{A},b)}{S(\mathbf{A},b)} (1+[\![b]\!]) \lesssim 1 + [\![a_1]\!]\ . \tag{4.2.19}$$

Indeed, (4.2.19) is trivial for $1+[\![b]\!] \lesssim (1+[\![a_1]\!])$, since $\frac{\mu}{S} < 1$ by definition. On the other end, if $1+[\![b]\!] \gtrsim (1+[\![a_1]\!])$, we have, from (4.2.9),

$$S(A,b) \geq |b-a_1| \gtrsim [\![b]\!]$$

and then

$$\frac{\mu(\mathbf{A},b)}{S(\mathbf{A},b)}(1+[\![b]\!]) \lesssim \mu(\mathbf{A},b) \lesssim 1+[\![a_1]\!]$$

that is (4.2.19). Then we can control the first term, provided that $N > s+\kappa$, with

$$\begin{aligned}
&\sum_B \left(\sum_{A_1,\dots A_r} (1+[\![b]\!])^{2s} \frac{\mu(\mathbf{A},b)^{2N+\nu-\kappa}}{S(\mathbf{A},b)^{2N-\kappa}} \|\Pi_{A_1}u_1\|_0^2 \|\Pi_{A_2}u_2\|_0 \dots \|\Pi_{A_r}u_r\|_0 \right)^{\frac{1}{2}} \\
&\lesssim \sum_B \left(\sum_{A_1,\dots A_r} (1+[\![a_1]\!])^{2s} \frac{\mu(\mathbf{A},b)^{\kappa+\nu}}{S(\mathbf{A},b)^{\kappa}} \|\Pi_{A_1}u_1\|_0^2 \|\Pi_{A_2}u_2\|_0 \dots \|\Pi_{A_r}u_r\|_0 \right)^{\frac{1}{2}} \\
&\lesssim \sum_B \left(\sum_{A_1} \frac{(1+[\![a_1]\!])^{2s}}{S(\mathbf{A},b)^{\kappa}} \|\Pi_{A_1}u_1\|_0^2 \sum_{A_2,\dots,A_r} [\![a_2]\!]^{\nu+\kappa} \|\Pi_{A_2}u_2\|_0 \dots \|\Pi_{A_r}u_r\|_0 \right)^{\frac{1}{2}} \\
&\lesssim \|u_1\|_s \prod_{l=2}^r \|u_l\|_{s_0}^{\frac{1}{2}},
\end{aligned}$$

with $s_0 \geq \kappa+\nu+\frac{d}{2}$. Summing over all the possible choices of the biggest index, we obtain the sum in the thesis. □

Corollary 4.2.6. *Let P be a polynomial function with localized coefficients, then* (4.2.12) *holds for its Hamiltonian vector field.*

Corollary 4.2.7. *Let $X \in \mathcal{M}_r^{\nu,N}$ and let $s > \frac{3}{2}d+\nu$. If $N > d+s$, for any $s_0 \in (\frac{3}{2}d+\nu, s)$, one has*

$$\sup_{\|u\|_s \leq R} \|X(u)\|_s \lesssim_{s,s_0,N} \|X\|_R^{\nu,N} . \tag{4.2.20}$$

4.2.2 Poisson brackets

Our class is closed with respect to Poisson brackets.

Lemma 4.2.8. *Given a polynomial P and a polynomial map X, we have*

$$dP\,X(u) = \eta\left[\tilde{P}(u,\dots,u,X(u))\right] \tag{4.2.21}$$

where we denote with $\eta[\tilde{P}]$ *the symmetrization, namely*

$$\eta[\tilde{P}(u,\dots,u,h)] \coloneqq \tilde{P}(h,u,\dots,u) + \tilde{P}(u,h,\dots,u) + \cdots + \tilde{P}(u,\dots,u,h) \tag{4.2.22}$$

for any $h \in \mathcal{H}_e^\infty$.
Moreover if P *has degree* $p+1$ *and* X *has degree* q, *the multilinear polynomial associated to* $dP\,X$ *is given by*

$$\widetilde{dP\,X}(u_1,\dots,u_{p+q}) = \eta\left[\tilde{P}\left(u_1,\dots,u_p,\tilde{X}(u_{p+1},\dots,u_{p+q})\right)\right]. \tag{4.2.23}$$

Proof. The thesis follows from the definition of differential. In fact, we have, $\forall h \in \mathcal{H}_e^\infty$,

$$\mathrm{d}P(h) = \eta[\tilde{P}(u,\dots,u,h)], \tag{4.2.24}$$

since we can compute

$$P(u+h) - P(u) = \tilde{P}(u+h,\dots,u+h) - \tilde{P}(u,\dots,u) = \tag{4.2.25}$$
$$= \eta\left[\tilde{P}(u,\dots,u,h)\right] + o(h)\,. \tag{4.2.26}$$

Then (4.2.23) follows from the definition of a multilinear map associated with a polynomial. □

Lemma 4.2.9 (Poisson brackets)**.** *Given* $P \in L_{r_1+1}^{\nu_1,N}$ *and* $X \in M_{r_2}^{\nu_2,N}$, *we have*

$$dP\,X \in L^{\nu',N'}\,,$$

with $N' = N - d - 1 - \max\{\nu_1,\nu_2\}$ *and* $\nu' = \nu_1 + \nu_2 + d + 1$. *Moreover,*

$$\|dP\,X\|^{\nu'\,N'} \lesssim \|P\|^{\nu_1,N}\|X\|^{\nu_2,N}\,.$$

Proof. Let $u_1,\dots,u_p,u_{p+1},\dots,u_{p+q} \in \mathcal{H}_e^\infty$ and $\mathbf{A} = (A_1,\dots,A_{p+q}) \in \Lambda_e^{p+q}$. By

Lemma 4.2.8, we have

$$\begin{aligned}
&\left|\widetilde{dP\,X}\left(\Pi_{A_1}u_1,\dots,\Pi_{A_{p+q}}u_{p+q}\right)\right| \\
&\quad= \left|\eta\left[\widetilde{P}\left(\Pi_{A_1}u_1,\dots,\Pi_{A_p}u_p,\widetilde{X}(\Pi_{A_{p+1}}u_{p+1},\dots,\Pi_{A_{p+q}}u_{p+q})\right)\right]\right| \\
&\quad\le (p+1)\sum_{B\in\Lambda_e}\left|\widetilde{P}\left(\Pi_{A_1}u_1,\dots,\Pi_{A_p}u_p,\Pi_B\widetilde{X}(\Pi_{A_{p+1}}u_{p+1},\dots,\Pi_{A_{p+q}}u_{p+q})\right)\right| \\
&\quad\le (p+1)\|P\|^{\nu_1,N}\times \\
&\qquad\quad\times\sum_{B\in\Lambda_e}\frac{\mu(A_1,\dots,A_p,b)^{N+\nu_1}}{S(A_1,\dots,A_p)^N}\left\|\Pi_B\widetilde{X}(\Pi_{A_{p+1}}u_{p+1},\dots,\Pi_{A_{p+q}}u_{p+q})\right\|_0 \\
&\quad\le (p+1)\|P\|^{\nu_1,N}\|X\|^{\nu_2,N}\times \\
&\qquad\quad\times\sum_{B\in\Lambda_e}\frac{\mu(A_1,\dots,A_p,b)^{N+\nu_1}}{S(A_1,\dots,A_p)^N}\frac{\mu(A_{p+1},\dots,A_{p+q},b)^{N+\nu_2}}{S(A_1,\dots,A_p)^N},
\end{aligned}$$

since P, X have localized coefficients. That is, we need to prove the following estimate

$$\begin{aligned}
\sum_{B\in\Lambda_e}\frac{\mu(A_1,\dots,A_p,b)^{N+\nu_1}}{S(A_1,\dots,A_p,b)^N}&\frac{\mu(A_{p+1},\dots,A_{p+q},b)^{N+\nu_2}}{S(A_{p+1},\dots,A_{p+q},b)^N} \\
&\lesssim\frac{\mu(A_1,\dots,A_{p+q})^{N'+\nu'}}{S(A_1,\dots,A_{p+q})^{N'}}\,.
\end{aligned}\tag{4.2.27}$$

By symmetry, we can assume the following relations:

$$[\![a_1]\!]\ge\cdots\ge[\![a_p]\!],\quad [\![a_{p+1}]\!]\ge\cdots\ge[\![a_{p+q}]\!]\,,\qquad [\![a_{p+1}]\!]\le[\![a_1]\!]\,.$$

Case 1 Assume $[\![a_1]\!]\ge[\![a_{p+1}]\!]\ge[\![a_2]\!]$.
In this case, we have

$$\begin{aligned}
\mu(A_1,\dots,A_{p+q})&=\max([\![a_2]\!],[\![a_{p+2}]\!])\,,\\
S(A_1,\dots,A_{p+q})&=|a_1-a_{p+1}|+\mu(A_1,\dots,A_{p+q})\,,
\end{aligned}$$

and

$$\begin{aligned}
\mu(A_1,\dots,A_p,b)&\le\mu(A_1,\dots,A_{p+q})\,,\\
\mu(A_{p+1},\dots,A_{p+q},b)&\le\mu(A_1,\dots,A_{p+q})\,.
\end{aligned}\tag{4.2.28}$$

Case 1.i For $[\![b]\!]>\max([\![a_2]\!],[\![a_{p+2}]\!])$, we have

$$\begin{aligned}
S(A_1,\dots,A_p,b)&=|a_1-b|+[\![a_2]\!]\,,\\
S(A_{p+1},\dots,A_{p+q},b)&=|a_{p+1}-b|+[\![a_{p+2}]\!]\,.\\
|a_1-a_{p+1}|=|a_1-b-a_{p+1}+b|&\le|a_1-b|+|a_{p+1}-b|\,.
\end{aligned}\tag{4.2.29}$$

From (4.2.28) and (4.2.29), we deduce

$$\begin{aligned}\frac{S(A_1,\dots,A_{p+q})}{\mu(A_1,\dots,A_{p+q})} &= 1+\frac{|a_1-a_{p+1}|}{\mu(A_1,\dots,A_{p+q})}\\ &\le 1+\frac{|a_1-b|}{\mu(A_1,\dots,A_p,b)}+\frac{|a_{p+1}-b|}{\mu(A_{p+1},\dots,A_{p+q},b)}\\ &\le \frac{S(A_1,\dots,A_p,b)}{\mu(A_1,\dots,A_p,b)}+\frac{S(A_{p+1},\dots,A_{p+q},b)}{\mu(A_{p+1},\dots,A_{p+q},b)}\\ &\le 2\max\left\{\frac{S(A_1,\dots,A_p,b)}{\mu(A_1,\dots,A_p,b)},\frac{S(A_{p+1},\dots,A_{p+q},b)}{\mu(A_{p+1},\dots,A_{p+q},b)}\right\}.\end{aligned} \tag{4.2.30}$$

Let us define

$$L_1=\left\{b\in\Lambda\ :\ [\![b]\!]>\max([\![a_2]\!],[\![a_{p+2}]\!]):\frac{S(A_1,\dots,A_p,b)}{\mu(A_1,\dots,A_p,b)}\ge\frac{S(A_{p+1},\dots,A_{p+q},b)}{\mu(A_{p+1},\dots,A_{p+q},b)}\right\}.$$

Depending on the value of $\mathbf{A}$ one could have $L_1=\emptyset$, but this is irrelevant for the following. If $b\in L_1$, estimate (4.2.30) implies

$$\frac{\mu(A_1,\dots,A_p,b)}{S(A_1,\dots,A_p,b)}\le 2\frac{\mu(A_1,\dots,A_{p+q})}{S(A_1,\dots,A_{p+q})}. \tag{4.2.31}$$

We observe moreover that

$$\frac{\mu(A_{p+1},\dots,A_{p+q},b)^{N+\nu_2}}{S(A_{p+1},\dots,A_{p+q},b)^N}\le\mu(A_{p+1},\dots,A_{p+q},b)^{\nu_2}. \tag{4.2.32}$$

Then, using 4.2.28, 4.2.31 and 4.2.32, we have

$$\begin{aligned}&\sum_{B\in L_1}\frac{\mu(A_1,\dots,A_p,b)^{N+\nu_1}}{S(A_1,\dots,A_p,b)^N}\frac{\mu(A_{p+1},\dots,A_{p+q},b)^{N+\nu_2}}{S(A_{p+1},\dots,A_{p+q},b)^N}\\ &\overset{(4.2.28)}{\lesssim}\ \mu(A_1,\dots,A_{p+q})^{\nu_1+\nu_2}\times\\ &\qquad\times\sum_{B\in L_1}\frac{\mu(A_1,\dots,A_p,b)^{N-d-1}}{S(A_1,\dots,A_p,b)^{N-d-1}}\frac{\mu(A_1,\dots,A_p,b)^{d+1}}{S(A_1,\dots,A_p,b)^{d+1}}\\ &\lesssim\mu(A_1,\dots,A_{p+q})^{\nu_1+\nu_2}\sum_{b\in L_1}\frac{\mu(A_1,\dots,A_{p+q})^{N-d-1}}{S(A_1,\dots,A_{p+q})^{N-d-1}}\frac{\mu(A_1,\dots,A_{p+q})^{d+1}}{S(A_1,...,A_p,b)^{d+1}},\\ &\overset{(4.2.31),(4.2.28)}{\lesssim}\frac{\mu(A_1,\dots,A_{p+q})^{N+\nu_1+\nu_2}}{S(A_1,\dots,A_{p+q})^{N-d-1}}\sum_{B\in L_1}\frac{1}{(|a_1-b|+1)^{d+1}},\end{aligned}$$

that is (4.2.27), with $N'=N-d-1$ and $\nu'=\nu_1+\nu_2+d+1$. The case $b\in L_1^c$ is analogous.

Case 1.ii For $[\![b]\!]\le\max([\![a_2]\!],[\![a_{p+2}]\!])$, we remark that the index of the sum in (4.2.27) runs over a set with cardinality controlled by $\mu(A_1,\dots,A_{p+q})$. That is, it suffices to prove that

$$\frac{\mu(A_1,\dots,A_p,b)^{N+\nu_1}}{S(A_1,\dots,A_p,b)^N}\frac{\mu(A_{p+1},\dots,A_{p+q},b)^{N+\nu_2}}{S(A_{p+1},\dots,A_{p+q},b)^N}\frac{S(A_1,\dots,A_{p+q})^{N'}}{\mu(A_1,\dots,A_{p+q})^{N'+\nu'}} \tag{4.2.33}$$

is controlled by a constant independent of b.

Take $0<K_2<1$ and let K_1 and C be as in (4.2.9) If $[\![a_1]\!]\geq K_1\max([\![a_2]\!],[\![a_{p+2}]\!])$, we get,

$$\begin{aligned}(4.2.33) \leq{}& \frac{\max([\![b]\!],[\![a_3]\!])^{N+\nu_1}}{(K_2[\![a_1]\!])^N}\max([\![b]\!],[\![a_{p+3}]\!])^{\nu_2}\times\\ &\times\frac{(C[\![a_1]\!]+\max([\![a_2]\!],[\![a_{p+2}]\!]))^{N'}}{\max([\![a_2]\!],[\![a_{p+2}]\!])^{N'+\nu'}}\\ \lesssim{}& \frac{\max([\![a_2]\!],[\![a_{p+2}]\!])^{N+\nu_1+\nu_2}}{[\![a_1]\!]^N}\frac{([\![a_1]\!])^{N'}}{\max([\![a_2]\!],[\![a_{p+2}]\!])^{N'+\nu'}}\lesssim 1\,,\end{aligned} \tag{4.2.34}$$

choosing $N'=N$ and $\nu'=\nu_1+\nu_2$.
If $[\![a_1]\!]\leq K_1\max([\![a_2]\!],[\![a_{p+2}]\!])$, we get instead

$$\begin{aligned}(4.2.33) \leq{}& \max([\![b]\!],[\![a_3]\!])^{\nu_1}\max([\![b]\!],[\![a_{p+3}]\!])^{\nu_2}\times\\ &\times\frac{(C[\![a_1]\!]+\max([\![a_2]\!],[\![a_{p+2}]\!])^{N'}}{\max([\![a_2]\!],[\![a_{p+2}]\!])^{N'+\nu'}}\\ \lesssim{}& \max([\![a_2]\!],[\![a_{p+2}]\!])^{\nu_1+\nu_2}\frac{(\max([\![a_2]\!],[\![a_{p+2}]\!])^{N'}}{\max([\![a_2]\!],[\![a_{p+2}]\!])^{N'+\nu'}}\lesssim 1\,,\end{aligned} \tag{4.2.35}$$

for $\nu'=\nu_1+\nu_2$. This proves the claim in the case 1.ii.

Case 2 Assume $[\![a_1]\!]\geq[\![a_2]\!]\geq[\![a_{p+1}]\!]$. In this case, we have

$$\begin{aligned}\mu(A_1,\dots,A_{p+q})&=\max([\![a_3]\!],[\![a_{p+1}]\!])\,,\\ S(A_1,\dots,A_{p+q})&=|a_1-a_2|+\max([\![a_3]\!],[\![a_{p+1}]\!])\,.\end{aligned}$$

Case 2.i Take K_2 as in (4.2.9) and determine the corresponding K_1. For $[\![b]\!]>K_1\max\left([\![a_3]\!],[\![a_{p+1}]\!]\right)$, we have

$$\begin{aligned}S(A_1,\dots,A_p,b)&=|a_1-\operatorname{argmax}([\![a_2]\!],[\![b]\!])|+\min([\![a_2]\!],[\![b]\!])\,,\\ S(A_{p+1},\dots,A_{p+q},b)&=|b-a_{p+1}|+[\![a_{p+2}]\!]\,,\end{aligned}$$

and moreover,

$$|a_{p+1}-b|\geq K_2[\![b]\!]\,.$$

Let us define

$$G_1 \coloneqq \{b\colon [\![b]\!] > K_1 \max\left([\![a_3]\!], [\![a_{p+1}]\!]\right), [\![b]\!] < [\![a_2]\!]\}\,.$$

For $b \in G_1$, we estimate

$$\begin{aligned}
&\sum_{b\in G_1} \frac{\mu(A_1,\dots,A_p,b)^{N+\nu_1}}{S(A_1,\dots,A_p,b)^N}\frac{\mu(A_{p+1},\dots,A_{p+q},b)^{N+\nu_2}}{S(A_{p+1},\dots,A_{p+q},b)^N}\\
&\quad= \sum_{b\in G_1} \frac{[\![b]\!]^{N+\nu_1}}{(|a_1-a_2| + [\![b]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{(|a_{p+1}-b| + [\![a_{p+2}]\!])^N}\\
&\quad\lesssim \sum_{b\in G_1} \frac{[\![b]\!]^{N+\nu_1}}{(|a_1-a_2| + [\![b]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{[\![b]\!]^N}\\
&\quad\lesssim \sum_{[\![b]\!]>K_1\max([\![a_3]\!],[\![a_{p+1}]\!])} \frac{[\![b]\!]^{\nu_1}}{(|a_1-a_2| + [\![b]\!])^N}[\![a_{p+2}]\!]^{N+\nu_2}\\
&\quad\lesssim \mu(A_1,\dots,A_{p+q})^{N+\nu_2}\frac{1}{(|a_1-a_2| + \max\left([\![a_3]\!],[\![a_{p+1}]\!]\right))^{N-\nu_1-d-1}}\\
&\quad\lesssim \frac{\mu(A_1,\dots,A_{p+q})^{N+\nu_2}}{S(A_1,\dots,A_{p+q})^{N-\nu_1-d-1}}\,,
\end{aligned} \tag{4.2.36}$$

where we are using the inequality

$$\sum_{|k|>A}\frac{|k|^l}{(|k|+B)^N} \le \sum_{|k|>A}\frac{(|k|+B)^l}{(|k|+B)^N} \le \frac{1}{(A+B)^{N-l-d-1}}\,.$$

This is the thesis with $N' = N-d-\nu_1-1$ and $\nu' = \nu_1+\nu_2+d$.
Let $G_2 \coloneqq \{b\colon [\![b]\!] > K_1\max\left([\![a_3]\!],[\![a_{p+1}]\!]\right), [\![b]\!] \ge [\![a_2]\!]\}$, then we have to estimate

$$\sum_{b\in G_2}\frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b|+[\![a_2]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{(|a_{p+1}-b|+[\![a_{p+2}]\!])^N}\,. \tag{4.2.37}$$

We observe that there exist two constants K_3, K_4 such that

$$[\![b]\!] \le K_4[\![a_1]\!] \quad \Rightarrow |a_1-b| \ge K_3|a_1-a_2|\,. \tag{4.2.38}$$

Then we estimate (4.2.37) by

$$\sum_{b\in G_2}\frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b|+[\![a_2]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{|a_{p+1}-b|^N}$$
$$\overset{(4.2.38)}{\lesssim}\sum_{b\in G_2,[\![b]\!]\le K_4[\![a_1]\!]}\frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b|+[\![a_2]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{[\![b]\!]^N} \tag{4.2.39}$$
$$+\sum_{b\in G_2,[\![b]\!]\ge K_4[\![a_1]\!]}\frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b|+[\![a_2]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{[\![b]\!]^N}.$$

The first term in (4.2.39) is controlled by

$$\frac{\max([\![a_{p+1}]\!],[\![a_3]\!])^{N+\nu_2}}{(|a_1-a_2|+\max([\![a_{p+1}]\!],[\![a_3]\!]))^{N-\nu_1-d-1}}\sum_{b\in G_2}\frac{[\![a_2]\!]^{N+\nu_1}[\![b]\!]^{-N}}{(|a_1-a_2|+[\![a_2]\!])^{\nu_1+d+1}}\lesssim$$
$$\lesssim\frac{\max([\![a_{p+1}]\!],[\![a_3]\!])^{N+\nu_2}}{(|a_1-a_2|+\max([\![a_{p+1}]\!],[\![a_3]\!]))^{N-\nu_1-d-1}}\sum_{b\in G_2}\frac{1}{[\![b]\!]^{d+1}}\lesssim$$
$$\lesssim\frac{\mu(A_1,\dots,A_{p+q})^{N'+\nu'}}{S(A_1,\dots,A_{p+q})^{N'}}.$$

For the second term in (4.2.39) we claim that $[\![b]\!]\ge K_4[\![a_1]\!]$ implies

$$\frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b|+[\![a_2]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{[\![b]\!]^N}\lesssim\frac{\max([\![a_{p+1}]\!],[\![a_3]\!])^{N'+\nu'}}{(|a_1-a_2|+\max([\![a_{p+1}]\!],[\![a_3]\!]))^{N'}}\frac{1}{[\![b]\!]^{d+1}},$$

with $N'=N-\nu_1-d-1$ and $\nu'=\nu_1+\nu_2+d+1$. Indeed, we have

$$\frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b|+[\![a_2]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{[\![b]\!]^N}\frac{(|a_1-a_2|+\max([\![a_{p+1}]\!],[\![a_3]\!]))^{N'}}{\max([\![a_{p+1}]\!],[\![a_3]\!])^{N'+\nu'}}[\![b]\!]^{d+1}$$
$$\lesssim[\![a_2]\!]^{\nu_1}\max([\![a_{p+1}]\!],[\![a_3]\!])^{N+\nu_2}\frac{1}{[\![b]\!]^{N-d-1}}\frac{[\![a_1]\!]^{N'}}{\max([\![a_{p+1}]\!],[\![a_3]\!])^{N'+\nu'}}$$
$$\overset{[\![b]\!]\ge K_4[\![a_2]\!]}{\lesssim}\frac{[\![a_1]\!]^{N'}}{[\![b]\!]^{N-d-1-\nu_1}}\lesssim\frac{[\![a_1]\!]^{N'}}{[\![a_1]\!]^{N-d-1-\nu_1}}\lesssim 1,$$

for any $N'\le N-d-1-\nu_1$. Then, we estimate

$$\sum_{b\in G_2,[\![b]\!]\ge K_4[\![a_1]\!]}\frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b|+[\![a_2]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{[\![b]\!]^N}$$
$$\lesssim\frac{\max([\![a_{p+1}]\!],[\![a_3]\!])^{N'+\nu'}}{(|a_1-a_2|+\max([\![a_{p+1}]\!],[\![a_3]\!]))^{N'}}\sum_{b\in G_2}\frac{1}{[\![b]\!]^{d+1}}$$
$$\lesssim\frac{\max([\![a_{p+1}]\!],[\![a_3]\!])^{N'+\nu'}}{(|a_1-a_2|+\max([\![a_{p+1}]\!],[\![a_3]\!]))^{N'}}.$$

Case 2.ii For $[\![b]\!] \leq K_1 \max([\![a_{p+1}]\!], [\![a_3]\!])$, we argue as in Case 1.ii, and it is sufficient to bound from above

$$\frac{\mu(A_1,\ldots,A_p,b)^{N+\nu_1}}{S(A_1,\ldots,A_p,b)^N}\frac{\mu(A_{p+1},\ldots,A_{p+q},b)^{N+\nu_2}}{S(A_{p+1},\ldots,A_{p+q},b)^N}\frac{S(A_1,\ldots,A_{p+q})^{N'}}{\mu(A_1,\ldots,A_{p+q})^{N'+\nu'}} \tag{4.2.40}$$

with a constant.

If $|a_1 - a_2| \leq 2\max([\![a_{p+1}]\!], [\![a_3]\!])$, we get

$$\begin{aligned}(4.2.40) \leq \max([\![b]\!], [\![a_3]\!])^{\nu_1} \mu(A_{p+1},\ldots,A_{p+q},b)^{\nu_2} \times \\ \times \frac{(|a_1-a_2| + \max([\![a_{p+1}]\!], [\![a_3]\!]))^{N'}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}} \\ \lesssim \max([\![a_{p+1}]\!], [\![a_3]\!])^{\nu_1+\nu_2}\frac{(\max([\![a_{p+1}]\!], [\![a_3]\!]))^{N'}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}} \lesssim 1\,,\end{aligned}$$

for $\nu' \geq \nu_1 + \nu_2$.

If $|a_1 - a_2| \geq 2\max([\![a_{p+1}]\!], [\![a_3]\!])$, we consider separately the case $[\![b]\!] \leq [\![a_2]\!]$ and the case $[\![b]\!] > [\![a_2]\!]$.

If $[\![b]\!] \leq [\![a_2]\!]$, we have

$$\begin{aligned}(4.2.40) \leq \frac{\max([\![b]\!], [\![a_3]\!])^{N+\nu_1}}{|a_1-a_2|^N}\mu(A_{p+1},\ldots,A_{p+q},b)^{\nu_2} \times \\ \times \frac{(|a_1-a_2| + \max([\![a_{p+1}]\!], [\![a_3]\!]))^{N'}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}} \\ \leq 2^{N+\nu_1+\nu_2}\frac{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N+\nu_1+\nu_2}}{|a_1-a_2|^N}\frac{\left(|a_1-a_2| + \frac{1}{2}|a_1-a_2|\right)^{N'}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}} \leq 3^{N'}\,,\end{aligned}$$

for $\nu' \geq \nu_1 + \nu_2$.

If $[\![b]\!] > [\![a_2]\!]$ and recalling that $\max([\![a_{p+1}]\!], [\![a_3]\!]) \leq \frac{1}{2}|a_1 - a_2|$, we need to bound

$$\begin{aligned}(4.2.40) \leq \frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b| + [\![a_2]\!])^N}\frac{[\![a_{p+2}]\!]^{N+\nu_2}}{(|a_{p+1}-b| + [\![a_{p+2}]\!])^N}\frac{\left(\frac{3}{2}|a_1-a_2|\right)^{N'}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}} \\ \lesssim \frac{[\![a_2]\!]^{N+\nu_1}}{(|a_1-b| + [\![a_2]\!])^N}\max([\![a_{p+1}]\!], [\![a_3]\!])^{\nu_2}\frac{\left(\frac{1}{C}|a_1-b| + [\![b]\!]\right)^{N'}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}}\,,\end{aligned} \tag{4.2.41}$$

where we are using the following triangular inequality,

$$|a_1 - a_2| \leq |a_1 - b| + |b - a_2| \leq |a_1 - b| + C[\![b]\!],$$

and $[\![a_2]\!] < [\![b]\!]$.

If $|a_1 - b| \leq [\![b]\!]$ we conclude, recalling that $[\![b]\!] \leq K_1 \max([\![a_{p+1}]\!], [\![a_3]\!])$, that

$$\begin{aligned}(4.2.41) &\lesssim \max([\![a_{p+1}]\!], [\![a_3]\!])^{\nu_1+\nu_2} \frac{[\![b]\!]^{N'}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}} \\ &\lesssim \frac{\max([\![a_{p+1}]\!], [\![a_3]\!])^{\nu_1+\nu_2+N}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}} \lesssim 1\,,\end{aligned}$$

for $N = N'$ and $\nu' = \nu_1 + \nu_2$. If instead $|a_1 - b| \geq [\![b]\!]$, we conclude that

$$(4.2.41) \lesssim \frac{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N+\nu_1+\nu_2}}{|a_1 - b|^N} \frac{|a_1 - b|^{N'}}{\max([\![a_{p+1}]\!], [\![a_3]\!])^{N'+\nu'}} \lesssim 1\,, \tag{4.2.42}$$

for $N \leq N'$ and $\nu' \geq \nu_1 + \nu_2$. $\square$

Corollary 4.2.10. *Let $F \in L^{\nu,N}_{r_1,r_2}$ and $G \in L^{\nu',N'}_{r'_1,r'_2}$, then $\{F;G\} \in L^{\nu'',N''}_{r_1+r'_1-2,r_2+r'_2-2}$, with*

$$\|\{F;G\}\|^{\nu'',N''}_R \lesssim \frac{1}{R^2} \|F\|^{\nu,N}_R \|G\|^{\nu',N}_R \quad , \tag{4.2.43}$$

with $N'' = N - d - 1 - \max\{\nu, \nu'\}$ and $\nu'' = \nu + \nu' + d + 1$.

4.2.3 High and low modes

In the definition of the normal form, we will distinguish between low and high modes. To this end, we fix some large K and we give the following definition.

Definition 4.2.11. *For $K > 1$, we define*

$$u^{\leq} = \Pi^{\leq} u := \sum_{\{A:\, [\![A]\!] \leq K\}} \Pi_A u\,, \tag{4.2.44}$$

$$u^{\perp} = \Pi^{\perp} u := \sum_{\{A:\, [\![A]\!] > K\}} \Pi_A u\,. \tag{4.2.45}$$

Definition 4.2.12. *A polynomial P of degree r is of order $k+1 \leq r$ in the high modes $u^{\perp}$ if, $\forall 1 \leq l \leq k$, $\forall u_1, \ldots, u_r \in \mathcal{H}^{\infty}_e$,*

$$\tilde{P}(\Pi^{\perp} u_1, \ldots, \Pi^{\perp} u_l, u^{\leq}_{l+1}, \ldots, u^{\leq}_r) = 0 \tag{4.2.46}$$

and there exist $u_1, \ldots, u_r$ s.t.

$$\tilde{P}(\Pi^{\perp} u_1, \ldots, \Pi^{\perp} u_{k+1}, u^{\leq}_{k+2}, \ldots, u^{\leq}_r) \neq 0\,. \tag{4.2.47}$$

From Theorem 4.2.5 and noticing that

$$\left\|\Pi^{\perp}u\right\|_{s_0} \le \frac{1}{K^{s-s_0}} \left\|\Pi^{\perp}u\right\|_{s} \ , \tag{4.2.48}$$

we immediately have the following Corollary.

Corollary 4.2.13. *Let $P \in L^{\nu,N}_{r+1}$.*
i) If P is of order at least three in $u^{\perp}$ then, for every $s_0 \in \left(\frac{3}{2}d+\nu, s\right)$, we have

$$\sup_{\|u\|_s \le R} \|X_P(u)\|_s \lesssim \frac{\|P\|^{\nu,N}_R}{K^{s-s_0}} \frac{1}{R} \tag{4.2.49}$$

ii) If P is at least of order two in $u^{\perp}$ then, for every $s_0 \in \left(\frac{3}{2}d+\nu, s\right)$, we have

$$\sup_{\|u\|_s \le R} \left\|\Pi^{\le} X_P(u)\right\|_s \lesssim \frac{\|P\|^{\nu,N}_R}{K^{s-s_0}} \frac{1}{R} \tag{4.2.50}$$

Proof of Corollary 4.2.13. Firstly recall the usual high modes estimate

$$\left\|u^{\perp}\right\|^2_{s_0} = \sum_{[\![A]\!]>K} (1+[\![a]\!])^{2s_0} \|\Pi_A u\|^2_0 \lesssim \frac{1}{K^{2(s-s_0)}} \|u\|^2_s \,. \tag{4.2.51}$$

i) If P is at least of order 3 in $u^{\perp}$, X_P is at least of order two in $u^{\perp}$. Then

$$\begin{aligned} X_P(u) &= X(u^{\le} + u^{\perp}) = \tilde{X}_P(u^{\le} + u^{\perp}, \dots, u^{\le} + u^{\perp}) \\ &= \sum_{l=2}^{r} \binom{r}{l} \tilde{X}_P(\underbrace{u^{\le}, \dots, u^{\le}}_{r-l-times}, \underbrace{u^{\perp}, \dots, u^{\perp}}_{l-times}) \,. \end{aligned}$$

Applying the tame estimate in Lemma 4.2.5, we get

$$\begin{aligned} &\|X_P(u)\|_s \\ &\quad \le \|P\|^{\nu,N} \sum_{l=2}^{r} \left(\|u^{\le}\|_s \|u^{\le}\|^{r-l-1}_{s_0} \|u^{\perp}\|^{l}_{s_0} + \|u^{\le}\|^{r-l}_{s_0} \|u^{\perp}\|^{l-1}_{s_0} \|u^{\perp}\|_s \right) \\ &\quad \overset{(4.2.51)}{\lesssim} \frac{\|P\|^{\nu,N}}{K^{s-s_0}} \|u\|^2_s \|u\|^{r-2}_{s_0} \,. \end{aligned}$$

ii) We reason as in the previous case, since again $\Pi^{\le} X_P$ is of order two in $u^{\perp}$. □

We also have the following easy, but important corollary.

Corollary 4.2.14. *Let $P \in L^{N,\nu}_{r+1}$ be of order 2 in $u^{\perp}$; assume that*

$$\tilde{P}(\Pi_{A_1} u_1, \dots, \Pi_{A_{r+1}} u_{r+1}) \ne 0 \implies |a_{\tau_{ord(1)}} - a_{\tau_{ord(2)}}| > K^{\delta} \tag{4.2.52}$$

with τ_{ord} the ordering permutation defined in Def. 3.1.11. Then, $\forall N' > N$ one has

$$\|P\|^{\nu,N} \leq \frac{\|P\|^{\nu,N'}}{K^{\delta(N'-N)}} \tag{4.2.53}$$

and therefore, for any s large enough,

$$\sup_{\|u\|_s \leq R} \|X_P(u)\|_s \leq (r+1) \frac{\|P\|_R^{\nu,N'}}{K^{\delta(N'-N)}} \frac{1}{R}\,. \tag{4.2.54}$$

Proof of Corollary 4.2.14. Writing $N' = N - M_1$, (4.2.53) amounts to show that

$$\begin{aligned}
\frac{\mu(A_1,\dots,A_r)^{N+\nu}}{S(A_1,\dots,A_r)^N} &= \frac{\mu(A_1,\dots,A_r)^{N'+\nu+M_1}}{S(A_1,\dots,A_r)^{N'+M_1}} \\
&\leq \frac{\mu(A_1,\dots,A_r)^{N'+\nu+M_1}}{S(A_1,\dots,A_r)^{N'}} \frac{1}{|a_{\tau_{ord(1)}} - a_{\tau_{ord(2)}}|^{M_1}} \\
&\leq \frac{\mu(A_1,\dots,A_r)^{N'+\nu+M_1}}{S(A_1,\dots,A_r)^{N'}} \frac{1}{K^{\delta M_1}}\,.
\end{aligned}$$

Then we apply directly Lemmas 4.2.5 and 4.2.5 and we get (4.2.54). □

Chapter 5

An abstract result of almost global existence

5.1 The main abstract result

We will study an equation of the form

$$\mathrm{i}\dot{u} = H_L u + \nabla_{\bar{u}} P(u, \bar{u}) \ , \tag{5.1.1}$$

where H_L is a linear selfadjoint operator on which we are going to make several assumptions and P is a nonlinear functional that we describe below. Here $\nabla_{\bar{u}}$ is the gradient of P with respect to the variable $\bar{u}$ and the L^2 metric.

We remark that the system (5.1.1) is Hamiltonian with Hamiltonian function $H = H_0 + P$ where

$$H_0 := \int_M \bar{u} H_L u \, dx \ . \tag{5.1.2}$$

We assume the following Hypotheses on H_L.

Hypothesis L.0. *[Integrability of H_L] H_L is a globally integrable quantum system, according to Def. 3.1.1.*

Hypothesis L.1 (Asymptotics)**.** *There exist constants C_1, C_2 and β, with $\beta > 0$, s.t.*

$$C_1 |a|^\beta \leq |\omega_a| \leq C_2 |a|^\beta \, , \tag{5.1.3}$$

where, for a vector $a \in \mathbb{R}^d$, we denote $|a| := \sqrt{\sum_{j=1}^d (a^j)^2}$.

As a corollary, one can partition Σ in separated pieces. Precisely we have the following lemma.

Lemma 5.1.1. *There exists a sequence of intervals*

$$\Sigma_n = [a_n, b_n] \ , \quad n \in \mathbb{N} \tag{5.1.4}$$

and a positive constant C, with the following properties:

$$a_n < b_n < a_{n+1} < 3n \ , \tag{5.1.5}$$

and

$$\Sigma \subset [0, a_1] \cup \left(\bigcup_n \Sigma_n \right) \ , \tag{5.1.6}$$

$$|b_n - a_n| \equiv |\Sigma_n| \leq 2 \tag{5.1.7}$$

$$d(\Sigma_n, \Sigma_{n+1}) \equiv a_{n+1} - b_n \geq \frac{2}{n^{d/\beta}} \ . \tag{5.1.8}$$

Proof. Given a set X, we denote with $\sharp X$ its cardinality. Fix $N > 0$, we have

$$\sharp \left\{ a \in \Lambda \colon |a|^\beta \leq N \right\} \leq \sharp \left\{ a \in \mathbb{Z}^d \colon |a| \leq N^{\frac{1}{\beta}} \right\} \leq (2\pi)^{-d} N^{\frac{d}{\beta}} \ .$$

Thus, from (5.1.3), there exists a constant $K = K(C_1, C_2, d)$ such that for any $N > 0$

$$\sharp \left\{ a \in \Lambda \colon \omega_a \leq N \right\} \leq K N^{\frac{d}{\beta}} \ .$$

Then, for any $n \in \mathbb{N}, n \geq 0$, we have

$$\sharp \left(\Sigma \cap [n, n+1] \right) \leq K(n+1)^{\frac{d}{\beta}}$$

and so there must be a gap in $\Sigma \cap [n, n+1]$ with length at least $K^{-1}(n+1)^{-\frac{d}{\beta}}$ centered in some $\gamma_n \in [n, n+1]$. Defining

$$\beta_n \coloneqq \gamma_n - \frac{1}{2K(n+1)^{\frac{d}{\beta}}}$$
$$\alpha_{n+1} \coloneqq \gamma_n + \frac{1}{2K(n+1)^{\frac{d}{\beta}}} \ , \qquad \forall n \geq 0$$

the intervals $\Sigma_n = [a_n, b_n]$ satisfy the theses by construction. □

We are now going to assume a non-resonance condition that allows to use of normal form theory to eliminate from the Hamiltonian the terms enforcing exchanges of energy among modes labeled by indexes belonging to different intervals Σ_n.

First, we keep into account that the system depends on both u and $\bar{u}$, extending the space of the indexes $a \in \Lambda$.

Definition 5.1.2. *We define* $\Lambda_e := \Lambda \times \{\pm 1\}$ *and we denote the index as* $A \equiv (a, \sigma) \in \Lambda_e$.

Definition 5.1.3 (Set of resonant indexes)**.** *A multi-index* $\boldsymbol{A} \equiv (A_1, ..., A_r) \in \Lambda_e^r$, $A_j \equiv (a_j, \sigma_j)$ *is said to be* resonant *if* r *is even and there exists a permutation* τ *of* $(1, ..., r)$ *and a sequence* $n_1, ..., n_{r/2}$ *s.t.*

$$\forall j = 1, ..., r/2, \quad \omega_{a_{\tau(j)}},\ \omega_{a_{\tau(j+r/2)}} \in \Sigma_{n_{\tau(j)}} \quad \textit{and} \ \ \sigma_{\tau(j)} = -\sigma_{\tau(j+r/2)} \,. \tag{5.1.9}$$

We denote with W *the set of resonant indexes.*

Hypothesis L.2. *[Non-resonance] For any* $r \geq 3$*, there are constants* γ, τ *such that for any multi-index* $\boldsymbol{A} = (A_1 \ldots, A_r) \in \Lambda_e^r \setminus W$ *one has*

$$\left| \sum_{j=1}^{r} \sigma_j \omega_{a_j} \right| \geq \frac{\gamma}{(\max_{j=1,\ldots,r} |a_j|)^{\tau}} \,. \tag{5.1.10}$$

Eventually, we assume a dyadic clusterization of the lattice Λ.

Hypothesis L.3 (Bourgain clusters)**.** *There exists a partition*

$$\Lambda = \bigcup_{\alpha \in \mathfrak{A}} \Omega_\alpha$$

with the following properties.

i. Each Ω_α *is dyadic, namely there exists a constant* C*, independent of* α*, such that*

$$\sup_{a \in \Omega_\alpha} |a| \leq C \inf_{a \in \Omega_\alpha} |a|.$$

ii. There exist $\delta > 0$ *such that, if* $a \in \Omega_\alpha$ *and* $b \in \Omega_\beta$ *with* $\alpha \neq \beta$*, then*

$$|a - b| + |\omega_a - \omega_b| \geq C_\delta (|a|^\delta + |b|^\delta).$$

Concerning the non-linearity P, we assume that it has the following form.

Hypothesis P. P *is a functional defined on a neighborhood of the origin of* $H^s(M) \times H^s(M)$ *for some positive* $s > d/2$*, that has a zero of third order at the origin and has* *the structure*

$$P(u,\bar u) = \left(\int_M F(N(u,\bar u), u(x), \bar u(x), x)dx\right) , \tag{5.1.11}$$

where

$$N(u,\bar u) := \int_M u(x)\bar u(x)dx , \tag{5.1.12}$$

and $F \in C^\infty(\mathcal{U}\times\mathcal{U}\times\mathcal{U}\times M;\mathbb{C})$ *is a smooth function and* $\mathcal{U}\subset\mathbb{C}$ *an open neighbourhood of the origin. We also assume that* $F(N,u,\bar u,x)\in\mathbb{R}$.

Our main result is the following.

Theorem 5.1.4. *Consider the Hamiltonian system* (5.1.1)*. Assume Hypotheses L.0, L.1, L.2, L.3, P, then for any integer* $r\ge 3$*, there exists* $s_r\in\mathbb{N}$ *such that, for any* $s\ge s_r$*, there are constants* $\epsilon_0>0$*,* $c>0$ *and* C *for which the following holds: if the initial datum* $u_0\in H^s(M,\mathbb{C})$ *fulfills*

$$\epsilon := \|u_0\|_s < \epsilon_0 ,$$

then the Cauchy problem has a unique solution $u\in\mathcal{C}^0((-T_\epsilon,T_\epsilon),H^s(M,\mathbb{C}))$ *with* $T_\epsilon > c\epsilon^{-r}$*. Moreover, one has*

$$\|u(t)\|_s \le C\epsilon, \quad \forall t\in(-T_\epsilon,T_\epsilon) .$$

5.2 Structure of the proof of Theorem 5.1.4

The proof of Theorem 5.1.4 is the combination of two main results. First, we prove an almost global existence result holding for nonlinearities P with localized coefficients, namely Theorem 5.2.1. This is the content of Sections 5.3 and 5.4. Then we prove that all nonlinearities P satisfying Assumption P have localized coefficients (see Theorem 5.2.2). This is the content of Section 5.5.
Consider a Hamiltonian function of the form

$$H(u) = H_0(u) + P(u) ,$$
$$H_0(u) = \int_M u_- H_L u_+ dx = \sum_{a\in\Lambda}\omega_a\int_M \Pi_{(a,+)}u\,\Pi_{(a,-)}u\,dx . \tag{5.2.1}$$

We will prove the following.

Theorem 5.2.1. *Consider the Hamiltonian* (5.2.1)*, assume that* ω_a *fulfill the Hypotheses L.1, L.2, L.3, and assume that* P *is a function with localized coefficients, which is real for real states and that has a zero of order at least three at the origin,* *then for any integer $r \geq 3$, there exists $s_r \in \mathbb{N}$ such that, for any $s \geq s_r$, there are constants $\epsilon_0 > 0$, $c > 0$ and C for which the following holds: if the initial datum $u_0 \in \mathcal{H}^s(M, \mathbb{C})$ fulfills*

$$\epsilon := \|u_0\|_s < \epsilon_0 ,$$

then the Cauchy problem has a unique solution $u \in \mathcal{C}^0((-T_\epsilon, T_\epsilon), H^s(M, \mathbb{C}))$ with $T_\epsilon > c\epsilon^{-r}$. Moreover, one has

$$\|u(t)\|_s \leq C\epsilon, \quad \forall t \in (-T_\epsilon, T_\epsilon) . \tag{5.2.2}$$

Theorem 5.1.4 is a consequence of Theorem 5.2.1, in view of the following.

Theorem 5.2.2. *A nonlinear functional of the form* (5.1.11) *fulfilling Hypothesis P is a function with localized coefficients.*

5.3 Birkhoff normal form

In this Section, we construct the normal form (see Theorem 5.3.19 and Corollary 5.3.5) in order to normalize the homogeneous term of low degree in P; this is the core of the proof. The normal form is the result of the application of a sequence of *near to the identity* Lie transforms. For a precise definition of a Lie transform see the Subsection 5.3.2. In Subsection 5.3.3 we show how to solve the homological equation which is needed in the iterative scheme of Subsection 5.3.4.

5.3.1 Normal form: definition and statement

We start defining the support of a polynomial.

Definition 5.3.1 (Support)**.** *Given a polynomial*

$$P(u) = \sum_{l=3}^{r} \sum_{\boldsymbol{A}=(A_1,\ldots,A_l)\in\Lambda_e^l} \tilde{P}(\Pi_{A_1}u, \ldots, \Pi_{A_l}u) \tag{5.3.1}$$

we define the support of P as

$$supp(P) = \left\{ \boldsymbol{A} = (A_1, \ldots, A_l) : \exists\, u_1, \ldots, u_l \; s.t.\; \tilde{P}(\Pi_{A_1}u_1, \ldots, \Pi_{A_l}u_l) \neq 0 \right\} . \tag{5.3.2}$$

We now define what we mean by normal form. Informally, it is a polynomial containing all the terms that we will not eliminate from the perturbation in the normal form procedure. For fixed $K \gg 1$ and $r \geq 3$ we have the following definitions.

Definition 5.3.2 (Block Resonant Normal Form). *A non homogeneous polynomial Z_{BR} of degree r will be said to be in* Block Resonant Normal Form *if*

$$Z_{BR} = Z_0 + Z_B \ , \tag{5.3.3}$$

with Z_0 and Z_B of degree, respectively, 0 and 2 in $\Pi^{\perp}u$ and

i. $\mathbf{A} \in Supp(Z_0)$ *implies* $\mathbf{A} \in W$ *(see Def. 5.1.3);*

ii. $\boldsymbol{A} \in Supp(Z_B)$ *implies that there exists a block Ω_α s.t.*

$$a_{\tau_{ord}(1)}, a_{\tau_{ord}(2)} \in \Omega_\alpha \ , \ \ and \quad \sigma_{\tau_{ord}(1)}\sigma_{\tau_{ord}(2)} = -1 \ .$$

where τ_{ord} is the ordering permutation defined in (3.1.11).

Definition 5.3.3 (Higher Order Normal Form). *A non homogeneous polynomial Z_{HO} of degree r will be said to be in* Higher Order Normal Form *if it has the structure*

$$Z_{HO} = Z_2 + Z_{\geq 3} \ ,$$

with Z_2 of degree 2 in $\Pi^{\perp}u$ and $Z_{\geq 3}$ of degree at least three in $\Pi^{\perp}u$ and the following holds:

$$\boldsymbol{A} \in supp(Z_2) \implies \left|a_{\tau_{ord}(1)} - a_{\tau_{ord}(2)}\right| > C_\delta K^\delta. \tag{5.3.4}$$

The terms in Z_0 are resonant in the standard sense of perturbation theory, namely, they do not enforce the exchange of energy between modes pertaining to different sets Σ_n; the terms in Z_B do not provoke the exchange of energy between modes pertaining to different blocks Ω_α and thus they conserve the total L^2 norm of the modes of a block Ω_α; finally, according to Corollaries 4.2.13 and 4.2.14, terms in Higher Order Normal Form will be shown to have a small vector field.

Definition 5.3.4. *A polynomial which is the sum of polynomials in normal form according to the above definitions will be said to be in normal form.*

The heart of the proof of Theorem 5.2.1 is the following proposition.

Proposition 5.3.5. *For any $r \geq 0$, $\exists s_r$ such that, $\forall s \geq s_r$, $\exists R_{s,r} > 0$, with the property that $\forall R < R_{s,r}$ $\exists K$ and a canonical transformation*

$$\mathcal{T}^{(r)} : \mathcal{B}^s_{R/2^{2r}} \to \mathcal{B}^s_R \quad \ with \quad [\mathcal{T}^{(r)}]^{-1} : \mathcal{B}^s_{R/4^{2r}} \to \mathcal{B}^s_{R/2^{2r}} \tag{5.3.5}$$

s.t.

$$H \circ \mathcal{T}^{(r)} = H_0 + Z_0 + Z_B + \mathcal{R}^{(\bar{r})}$$

with Z_0 *and* Z_B *as in K block-resonant normal form, see Def. 5.3.2, and*

$$\|X_{\mathcal{R}^{(\bar{r})}}(u)\|_s \lesssim R^{r+2}, \qquad \forall u \in \mathcal{B}^s_R .$$

Moreover, we have

$$\left\|\Pi^{\leq} X_{Z_B}(u)\right\|_s \lesssim R^{r+2}, \qquad \forall u \in \mathcal{B}^s_R . \tag{5.3.6}$$

5.3.2 Lie Transform

The transformation $\mathcal{T}^{(r)}$ will be constructed by the composition of Lie transforms, so we start by studying the properties of the Lie transform.

Given $G \in \mathcal{C}^\infty(\mathcal{H}^s_e, \mathbb{C})$, we denote by Φ^t_G the flow generated by the Hamilton equation $\dot{u} = X_G(u)$. From Lemma 4.2.5 one has the following result.

Lemma 5.3.6. *Fir* $\bar{r}$*, let* $3 \leq r \leq \bar{r}$*,* $\nu \in [0, +\infty)$*,* $N \geq 1$ *and* $G \in L^{\nu,N}_{r,\bar{r}}$*.* $\forall s > \frac{3}{2}d + \nu$ *there is a constant* $C_{\bar{r},N,s} > 0$ *such that* $\forall R > 0$ *satisfying*

$$\|G\|^{\nu,N}_R \leq \frac{R^2}{C_{\bar{r},N,s}} \tag{5.3.7}$$

the map $\Phi^t_G : \mathcal{B}^s_{R/2} \to \mathcal{B}^s_R$ *is well defined for* $|t| \leq 1$*, and moreover*

$$\sup_{\|u\|_s < R/2} \left\|\Phi^t_G(u) - u\right\|_s \lesssim_s |t| \frac{\|G\|^{\nu,N}_R}{R} .$$

Definition 5.3.7. *We call* $\Phi_G := \Phi^t_G\big|_{t=1}$ *Lie transform generated by* G*.*

Lemma 5.3.8. *Let* $G \in L^{\nu,N}_{r,\bar{r}}$*,* $3 \leq r \leq \bar{r}$ *and let* Φ_G *be the Lie transform it generates. For any* $s > \frac{3}{2}d + \nu$ *there exists* $R_s > 0$ *such that for any* $F \in C^\infty(B^s_{R_s})$ *satisfying*

$$\sup_{\|u\|_s \leq R_s} \|X_F(u)\|_s < \infty ,$$

one has

$$\sup_{\|u\|_s \leq R/2} \|X_{F\circ\Phi_G}(u)\|_s \leq 2 \sup_{\|u\|_s \leq R} \|X_F(u)\|_s , \quad \forall R < R_s .$$

Defining

$$Ad_G^0(P) := P \quad \text{and} \quad Ad_G^k(P) := \{Ad_G^{k-1}(P), G\} \text{ for } k \geq 1$$

we have the following standard lemma.

Lemma 5.3.9 (Lie transform)**.** *Let $3 \leq r \leq \bar{r}$ and $G \in L_{r,\bar{r}}^{\nu,N}$. Assume* (5.3.7)*, then for any $P \in \mathcal{C}^\infty(\mathcal{B}_{2R_s}^s, \mathbb{C})$, we have,*

$$P(\Phi_G(u)) = \sum_{k=0}^{n} \frac{1}{k!} \left(Ad_G^k P\right)(u) + \frac{1}{n!} \int_0^1 (1-\tau)^n \left(Ad_G^{n+1} P\right)(\Phi_G^\tau(u))\, d\tau \qquad (5.3.8)$$

for any $n \in \mathbb{N}$ and $\forall u \in \mathcal{B}_{R/2}^s$.

From the estimate of Poisson brackets (see Lemma 4.2.9), we deduce the following result.

Lemma 5.3.10. *Let $G \in L_{r_1+2,\bar{r}_1+2}^{\nu,N}$ and $F \in L_{r_2,\bar{r}_2}^{\nu,N}$, with $r_1 \leq \bar{r}_1$ and $r_2 \leq \bar{r}_2$. $\forall k \geq 0$ we have $Ad_G^k(F) \in L_{r_2+kr_1,\bar{r}_2+k\bar{r}_1}^{\nu_k,N_k}$ and*

$$\left\|Ad_G^k(F)\right\|_R^{\nu_k,N_k} \lesssim \left(\frac{\|G\|_R^{\nu,N}}{R^2}\right)^k \|F\|_R^{\nu,N}, \qquad (5.3.9)$$

with $N_k = N - k(d+\nu)$ and $\nu_k = k(d+2\nu)$.

Proof. We prove it by induction. For $k=1$, the thesis follows from (4.2.43). Suppose the thesis is true at step k. Exploiting again (4.2.43), we get

$$\left\|Ad_G^{k+1}(F)\right\|_R^{\nu_{k+1},N_{k+1}} = \left\|\{Ad_G^k(F), G\}\right\|_R^{\nu_{k+1},N_{k+1}} \lesssim$$
$$\frac{1}{R^2}\left(\frac{\|G\|_R^{\nu,N}}{R^2}\right)^k \|F\|_R^{\nu,N} \|G\|_R^{\nu,N} = \left(\frac{\|G\|_R^{\nu,N}}{R^2}\right)^{k+1} \|F\|_R^{\nu,N}.$$

□

It is useful for the perturbative iteration to summarize the last results in the following lemma.

Lemma 5.3.11. *Fix $\bar{r}$, let $P \in L_{r_1,\bar{r}_1}^{\nu,N}$ and $G \in L_{r_2+2,\bar{r}_2+2}^{\nu,N}$ with $r_1 < \bar{r}$, $r_2 + 2 < \bar{r}$. Let*

$$n = \frac{\bar{r} + 3 - r_1}{r_2}, \qquad (5.3.10)$$

if the r.h.s. is integer, otherwise we define n to be the r.h.s. of (5.3.10) *+1. Let $s > \frac{3}{2}d+2$ There exists $C_{\bar{r},s,N} > 0$ such that if R fulfills* (5.3.7)*, then the Lie transform*

$\Phi_G\colon \mathcal{B}^s_{R/2} \to \mathcal{B}^s_R$ is well defined. Moreover, one has

$$P \circ \Phi_G = P + P' + \mathcal{R}_{P,G}$$

and $\exists \nu', N'$ s.t., $P' \in L^{\nu',N'}_{r_1+r_2,\bar r_1+n\bar r_2}$. Furthermore $X_{\mathcal{R}_{P,G}} \in \mathcal{C}^\infty(\mathcal{B}^s_R, \mathcal{H}^s_e)$ and one has

$$\|P'\|^{\nu',N'}_R \lesssim \|P\|^{\nu,N}_R \frac{\|G\|^{\nu,N}_R}{R^2} \tag{5.3.11}$$

$$\sup_{\|u\|_s \le R/2} \left\| X_{\mathcal{R}_{P,G}}(u) \right\|_s \lesssim \|P\|^{\nu,N}_R \left(\frac{\|G\|^{\nu,N}_R}{R^2} \right)^{n+1} \frac{1}{R}\ , \tag{5.3.12}$$

so that $X_{\mathcal{R}_{P,G}}$ has a zero of order at least $\bar r + 2$ at the origin.

Proof. Define

$$P' := \sum_{k=1}^{n} \frac{1}{k!} Ad^k_G P$$

and $R_{P,G}$ be the integral term in (5.3.8). Then, by Lemma 5.3.10, one has

$$\|P'\|^{\nu',N'}_R \lesssim \sum_{k=1}^{n} \frac{1}{k!} \left(\frac{\|G\|^{\nu,N}}{R^2} \right)^k \|P\|^{\nu,N}_R \lesssim \frac{\|G\|^{\nu,N}}{R^2} \|P\|^{\nu,N}_R\ ,$$

provided $\left(\frac{\|G\|^{\nu,N}}{R^2}\right) \le 1/2$, which is ensured by (5.3.7). This gives (5.3.11). To get (5.3.12) just use the estimate (5.3.9), Lemma 4.2.2 and Lemma 5.3.8. □

5.3.3 Solution of the homological equation

In this subsection, we state and solve the homological equation.

Remark 5.3.12. *Recall that, following Lemma 5.1.1, the set of frequencies decomposes as the union of bounded and disjoint intervals Σ_n. In the following, we will choose the cut-off K lying between two intervals, namely such that for any $a, b \in \Lambda$ with $[\![a]\!] < K$ and $[\![b]\!] \ge K$ we have that, for any $n \ge 1$,*

$$\omega_a \in \Sigma_n \quad \Longrightarrow \quad \omega_b \notin \Sigma_n\,. \tag{5.3.13}$$

We start defining the set of non-resonant indexes.

Definition 5.3.13 (Block K non resonant indexes)**.** *We say that $\boldsymbol{A} = (A_1, ..., A_r) \in \Lambda^r_e$ is Block-K non resonant, and we write $\boldsymbol{A} \in \mathcal{I}^K(r)$, if $[\![A_{\tau_{ord}(3)}]\!] < K$ (namely there are at most two large indexes) and one of the following holds:*

- *$[\![A_{\tau_{ord}(1)}]\!] < K$ (there are no indexes larger than K) and $\mathbf{A} \notin W$;*

- $[\![A_{\tau_{ord}(1)}]\!] \geq K$ *and* $[\![A_{\tau_{ord}(2)}]\!] < K$ *(there is exactly one large index);*
- $[\![A_{\tau_{ord}(2)}]\!] \geq K$ *and* $[\![A_{\tau_{ord}(3)}]\!] < K$ *(there are exactly two large indexes) and one of the following holds, recalling Hyp. L.3:*
  - $\exists \alpha$ *s.t.* $a_{\tau_{ord}(1)}, a_{\tau_{ord}(2)} \in \Omega_\alpha$ *and* $\sigma_{\tau_{ord}(1)}\sigma_{\tau_{ord}(2)} = +1$,
  - $\exists \alpha \neq \beta$ *s.t.* $a_{\tau_{ord}(1)} \in \Omega_\alpha$, $a_{\tau_{ord}(2)} \in \Omega_\beta$ *and* $|a_{\tau_{ord}(1)} - a_{\tau_{ord}(2)}| \leq K^\delta$.

Remark 5.3.14. *By Definitions 5.3.2 and 5.3.3, a polynomial supported only on multi-indexes* $\boldsymbol{A} \notin \mathcal{I}^K(r)$ *is in normal form according to Definition 5.3.4*

In the following lemma, we show that there are no resonant multi-indexes with just one large index.

Lemma 5.3.15. *Fix* K *as in*(5.3.13)*. If* $\boldsymbol{A} \in \Lambda_e^r$ *is s.t.*

$$[\![A_{\tau_{ord}(1)}]\!] \geq K \quad and \quad [\![A_{\tau_{ord}(2)}]\!] < K\,, \tag{5.3.14}$$

then there exists a constant $\gamma'_r > 0$ *such that*

$$\Big|\sum_{l=1}^{r} \sigma_l \omega_{a_l}\Big| \geq \gamma'_r K^{-\tau}\,,$$

where $\tau > 0$ *is the constant appearing in Hyp. L.2.*

Proof. For simplicity, we can suppose that $A_{\tau_{ord}(1)} = A_1$. We distinguish two cases.
Case 1 If $[\![A_1]\!] \geq K_1 = 2(r-1)^{\frac{1}{\beta}} K$, recalling $\omega_{a_j} < K^\beta$ for any $j = 2, \ldots, r$, we have that

$$\left|\sum_{l=2}^{r} \sigma_l \omega_{a_l}\right| \leq (r-1)K^\beta\,. \tag{5.3.15}$$

From (5.3.15) and $[\![A_1]\!] \geq K_1$, we deduce

$$\left|\sum_{l=2}^{r} \sigma_l \omega_{a_l} + \sigma_1 \omega_{a_1}\right| \geq K_1^\beta - (r-1)K^\beta \gtrsim 1$$

that implies the thesis.
Case 2 If $[\![A_1]\!] < K_1$, we prove that $\mathbf{A} \notin W$. In fact, if by contradiction $\mathbf{A} \in W$, it should exist $A' \in \mathbf{A}$ with $A_1, A' \in \Sigma_n$ for some n. But then, from (5.3.13), it would follow that $[\![A_{\tau_{ord}(2)}]\!] \geq K$. This is in contradiction with (5.3.14).

Since $\mathbf{A} \notin W$, from Hypothesis (L.2) it follows that

$$\left| \sum_{j=1}^{r} \sigma_j \omega_{a_j} \right| \geq \frac{\gamma}{(\max_{j=1,\dots,r} |a_j|)^\tau} \gtrsim \frac{\gamma}{K_1^\tau} = \frac{\gamma_r'}{K^\tau} \,,$$

with $\gamma_r' = \frac{\gamma}{2(r-1)^{\frac{\tau}{\beta}}}$. □

In the following lemma, we take care of multi-indexes with exactly two indexes with modulus larger than K.

Lemma 5.3.16. *If $\boldsymbol{A} \in \mathcal{I}^K(r)$ is s.t.*

$$[\![A_{\tau_{ord}(2)}]\!] \geq K \quad \text{and} \quad [\![A_{\tau_{ord}(3)}]\!] < K \,, \tag{5.3.16}$$

then there exist constants γ_r' and τ such that

$$\left| \sum_{l=1}^{r} \sigma_l \omega_{a_l} \right| \geq \frac{\gamma_r'}{K^{\tau'}} \,.$$

Proof. For simplicity, suppose that

$$A_{\tau_{ord}(1)} = A_1 \quad \text{and} \quad A_{\tau_{ord}(2)} = A_2 \,.$$

Case 1 Consider first the case $\sigma_1 = \sigma_2$. Reasoning as in the proof of Lemma 5.3.15, we distinguish two cases.

Case 1.i If $[\![A_1]\!] \geq K_1 := 2(r-2)^{\frac{1}{\beta}} K$, the thesis follows trivially since

$$\left| \sum_{l=1}^{r} \sigma_l \omega_{a_l} \right| = \left| \sum_{l=3}^{r} \sigma_l \omega_{a_l} + \sigma_1 \omega_{a_1} + \sigma_2 \omega_{a_2} \right|$$

and we can reason as in case 1. in the Proof of Lemma 5.3.15.

Case 1.ii If $[\![A_1]\!] < K_1$, we observe that by definition $\mathbf{A} \notin W$, since $\sigma_1\sigma_2 = +1$ and recalling (5.3.13). Then, from Hyp. (L.2), it follows

$$\left| \sum_{j=1}^{r} \sigma_j \omega_{a_j} \right| \geq \frac{\gamma}{(\max_{j=1,\dots,r} |a_j|)^\tau} \gtrsim \frac{\gamma}{K_1^\tau} = \frac{\gamma'}{K^\tau}$$

for $\gamma' = \frac{\gamma}{2^\tau (r-1)^{\frac{\tau}{\beta}}}$.

Case 2 Consider now the case $\sigma_1\sigma_2 = -1$. It follows by definition of $\mathcal{I}^K(r)$ that $a_1 \in \Omega_\alpha$, $a_2 \in \Omega_\beta$ with $\alpha \neq \beta$ and moreover $|a_1 - a_2| \leq K^\delta$. From Hyp. L.3, it follows that

$$|\omega_{a_1} - \omega_{a_2}| \geq C_\delta |a_1|^\delta \,.$$

Case 2.i If $[\![a_1]\!] \geq K_2 := 2\,(r-2)^{\frac{1}{\delta}}\, K^{\frac{\beta}{\delta}}$ we observe that $|a_1| \gtrsim K_2$ and then

$$\left| \sum_{l=3}^{r} \sigma_l \omega_{a_l} + \omega_{a_1} - \omega_{a_2} \right| \geq C_\delta |a_1|^\delta - (r-2) K^\beta \gtrsim 1$$

and the thesis follows.

Case 2.ii If $[\![a_1]\!] < K_2$, one has that $\mathbf{A} \notin W$. To see this, remark that $|\omega_{a_1} - \omega_{a_2}| \geq C_\delta |a_1|^\delta > CK^\delta > 2$ but each Σ_n has a length smaller or equal to 2, so that a_1, a_2 belong to different set Σ_n and then the definition of W cannot be fulfilled. Then recalling Hyp. (L.2), we conclude that

$$\left| \sum_{j=1}^{r} \sigma_j \omega_{a_j} \right| \geq \frac{\gamma}{(\max_{j=1,\dots,r} |a_j|)^\tau} \gtrsim \frac{\gamma}{K_2^\tau} = \frac{\gamma'}{K^{\tau'}}$$

with $\gamma' = \frac{\gamma}{C(r-2)^{\frac{\tau}{\delta}}}$ and $\tau' = \frac{\beta}{\delta}\tau$. □

In the next lemma, we solve the homological equation.

Lemma 5.3.17 (Homological equation)**.** *Let $\bar{r}, l$ be given in such a way that $\bar{r} \geq l \geq 3$, then for any $F \in L^{\nu,N}_{l,\bar{r}}$ there exist $G, Z \in L^{\nu,N}_{l,\bar{r}}$ which solve the homological equation*

$$\{H_0, G\} + F = Z. \tag{5.3.17}$$

Furthermore there exist $\tau(\bar{r})$ and $\gamma(\bar{r})$ s.t.

$$\|G\|_R^{\nu,N} \leq \frac{K^\tau}{\gamma} \|F\|_R^{\nu,N} \,, \tag{5.3.18}$$

and $Z \in L^{\nu,N}_{l,\bar{r}}$ is in normal form according to Def. 5.3.4 and fulfills the estimate

$$\|Z\|_R^{\nu,N} \leq \|F\|_R^{\nu,N} \,.$$

Proof. Writing

$$F(u) = \sum_{l=3}^{\bar{r}} \sum_{\mathbf{A} \in \Lambda_e^l} \widetilde{F}_l(\Pi_{A_1} u, \dots, \Pi_{A_l} u) \,,$$

we define G, Z through their multilinear map. More precisely, recalling Definition

5.3.13, we set, for $3 \leq l \leq \bar{r}$,

$$\widetilde{Z}_l(\Pi_{A_1}u_1, \ldots, \Pi_{A_l}u_l) := \begin{cases} \widetilde{F}_l(\Pi_{A_1}u_1, \ldots, \Pi_{A_l}u_l) & \text{if } (A_1, \ldots, A_l) \notin \mathcal{I}^K(l) \\ 0 & \text{otherwise} \end{cases}$$

$$\widetilde{G}_l(\Pi_{A_1}u_1, \ldots, \Pi_{A_l}u_l) := \begin{cases} \dfrac{\widetilde{F}_l(\Pi_{A_1}u_1, \ldots, \Pi_{A_l}u_l)}{\sum_{j=1}^{l} \sigma_j \omega_{a_j}} & \text{if } (A_1, \ldots, A_l) \in \mathcal{I}^K(l) \\ 0 & \text{otherwise}\,. \end{cases}$$

These polynomials solve the homological equation (5.3.17) since

$$\{H_0, G\}(u) = -\sum_{l=3}^{\bar{r}} \sum_{\mathbf{A} \in \mathcal{I}^K(l)} \tilde{F}(\Pi_{A_1}u, \ldots, \Pi_{A_l}u)\,.$$

In particular $\widetilde{Z}$ is in normal form (recall Remark 5.3.14). The estimate (5.3.18) follows from Lemmas 5.3.15 and 5.3.16. □

5.3.4 Proof of the normal form Lemma

In this subsection, we complete the proof of Proposition 5.3.5.

Fix $\bar{r} \geq 2$ and Taylor expand P at order $\bar{r}+2$. Recalling (5.2.1) and Definition 4.1.12, we have

$$P = P^{(0)} + \mathcal{R}_{T,0}\,.$$

with $P^{(0)} \in L^{N,\nu}_{1,\bar{r}+2}$ for some positive N, ν. Furthermore $\mathcal{R}_{T,0}$ has a zero of order $\bar{r}+2$ at $u = 0$. Moreover, since $X_{R_{T,0}} \in \mathcal{C}^\infty(\mathcal{B}^s_R, \mathcal{H}^s_e)$, one has, for s large enough and $R > 0$ small enough

$$\|X_{\mathcal{R}_T}(u)\|_s \lesssim R^{\bar{r}+2}\,, \quad \forall u \in \mathcal{B}^s_R\,,$$

In the following lemma, we describe the iterative step that proves Theorem 5.3.19.

Lemma 5.3.18 (Iteration)**.** *Fix $\bar{r} \geq 2$; for any $0 \leq k \leq \bar{r}$, there is a small constant $\mu_k > 0$ s.t., denoting*

$$\mu = \mu(R) := \frac{\|P^{(0)}\|^{N,\nu}_R}{R^2} K^\tau\,,$$

if R_k, K are s.t.

$$\mu(R_k) \leq \mu_k \tag{5.3.19}$$

then there exist $N_k, \nu_k > 0$ such that, $\forall s_0 > d/2 + \nu_k$ the following holds: there exists *an invertible canonical transformation* $T^{(k)} : \mathcal{B}^{s_0}_{R_k/2^k} \to \mathcal{B}^{s_0}_{R_k}$ *such that*

$$H^{(k)} = H \circ T^{(k)} = H_0 + Z^{(k)} + P^{(k)} + \mathcal{R}^{(k)}_T$$

and, $\exists R_k > 0$ *s.t. one has*

- $Z^{(k)} \in L^{N_{k-1},\nu_{k-1}}_{3,\bar{r}_{k-1}+2}$ *is in normal form and*

$$\left\|Z^{(k)}\right\|^{N_{k-1},\nu_{k-1}}_R \lesssim_k \|P^{(0)}\|^{N,\nu}_R \ , \quad \forall R < R_k \ , \tag{5.3.20}$$

- $P^{(k)} \in L^{N_k,\nu_k}_{k+3,\bar{r}_k+2}$ *and*

$$\left\|P^{(k)}\right\|^{N_k,\nu_k}_R \lesssim_k \mu^k \|P^{(0)}\|^{N,\nu}_R \ , \quad \forall R < R_k \ , \tag{5.3.21}$$

- $\forall s \geq s_0 \ \exists R_{s,k}$ *s.t.* $T^{(k)} \in C^\infty\left(\mathcal{B}^s_{\frac{R_{s,k}}{2^k}}; \mathcal{H}^s_e\right)$ *and* $[T^{(k)}]^{-1} \in C^\infty\left(\mathcal{B}^s_{\frac{R_{s,k}}{4^k}}; \mathcal{H}^s_e\right)$

$$T^{(k)}(\mathcal{B}^s_{R/2^k}) \subset \mathcal{B}^s_R \ , \quad [T^{(k)}]^{-1}(\mathcal{B}^s_{R/4^k}) \subset \mathcal{B}^s_{R/2^k} \ , \quad \forall R < R_{s,k} \ , \tag{5.3.22}$$

- $X_{\mathcal{R}^{(k)}_T} \in \mathcal{C}^\infty(\mathcal{B}^s_{R_{s,k}/2^k}, \mathcal{H}^s_e)$ *and,* $\forall u \in \mathcal{B}^s_R$*, with* $R < R_{s,k}/2^k$ *one has*

$$\left\|X_{\mathcal{R}^{(k)}_T}(u)\right\|_s \lesssim_k R^2 (K^\tau R)^{\bar{r}} \tag{5.3.23}$$

Proof. The result is true for $k = 0$ with $T^{(0)} = Id$ and $N_0 = N$, $\nu_0 = \nu$. We prove the inductive step $k \rightsquigarrow k+1$.
We determine $G_{k+1}, Z_{k+1} \in L^{N_k,\nu_k}_{k+3,\bar{r}_{k+1}}$ solving the homological equation.

$$\{H_0, G_{k+1}\} + P^{(k)} = Z_{k+1} \, , \tag{5.3.24}$$

To this end, we remark that the maximal degree homogeneity of $P^{(k)}$ appears when $k = \bar{r} - 1$ and is smaller than $\bar{r}_{\bar{r}}$. So we take τ and γ in (5.3.18) to be $\tau(\bar{r}_{\bar{r}})$ and $\gamma(\bar{r}_{\bar{r}})$. Then we write

$$H^{(k+1)} := H^{(k)} \circ \Phi_{G_{k+1}} \, .$$

Exploiting (5.3.24), the Hamiltonian $H^{(k+1)}$ has the form,

$$\begin{aligned}H^{(k+1)} &= H_0 + Z^{(k)} + Z_{k+1} + \\ &+ H_0' - \{H_0, G_{k+1}\} + (P^{(k)})' + \left(Z^{(k)}\right)' + \\ &+ \mathcal{R}_{H_0,G_{k+1}} + \mathcal{R}_{P^{(k)},G_{k+1}} + \mathcal{R}_{Z^{(k)},G_{k+1}} + \mathcal{R}_{T,k} \circ \Phi_{G_{k+1}} ,\end{aligned}$$

with the primed quantities defined as in Lemma 5.3.11. Collecting the terms above, we define the following quantities:

$$\begin{aligned}Z^{(k+1)} &:= Z^{(k)} + Z_{k+1} , \\ P^{(k+1)} &:= H_0' - \{H_0, G_{k+1}\} + (P^{(k)})' + \left(Z^{(k)}\right)' , \\ \mathcal{R}_{T,k+1} &:= \mathcal{R}_{H_0,G_{k+1}} + \mathcal{R}_{P^{(k)},G_{k+1}} + \mathcal{R}_{Z^{(k)},G_{k+1}} + \mathcal{R}_{T,k} \circ \Phi_{G_{k+1}} .\end{aligned}$$

First, we check the order of the polynomials, we have $(P^{(k)})' \in L_{2k+2,\bar{r}_k+n_1\bar{r}_k}$, with a suitable n_1. Similarly one has $H_0' - \{H_0, G_{k+1}\} \in L_{2k+2,\bar{r}_k+n_2\bar{r}_k}$, and $(Z^k)' \in L_{3+k+1,\bar{r}_{k-1}+n_3\bar{r}_k}$ with suitable n_2, n_3. Defining

$$\bar{r}_{k+1} := \max\left\{\bar{r}_{k-1} + n_3\bar{r}_k, \bar{r}_k + n_2\bar{r}_k, \bar{r}_k + n_1\bar{r}_k\right\}$$

one gets the result on the order. We come to the estimates. First, we remark that

$$\mu \lesssim RK^\tau \ . \tag{5.3.25}$$

From Lemma 5.3.17 and the inductive hypothesis, we get the estimates

$$\begin{aligned}\|G_{k+1}\|_R^{N_k,\nu_k} &\le \frac{K^\tau}{\gamma}\|P^{(k)}\|_R^{N_k,\nu_k} \lesssim \frac{K^\tau}{\gamma}\mu^k\|P^{(0)}\|_R^{N,\nu} \lesssim \mu^{k+1}R^2 , \\ \|Z_{k+1}\|_R^{N_k,\nu_k} &\lesssim \|P^{(k)}\|_R^{N_k,\nu_k} \lesssim \mu^k\|P^{(0)}\|_R^{N,\nu} .\end{aligned} \tag{5.3.26}$$

The estimate (5.3.20) follows from (5.3.26).
For the estimate (5.3.21), we prove it for $\tilde{H}_0' - \{H_0, G_{k+1}\}$ applying Lemma 5.3.10. Denoting $N' = N_{k+1} = N_k - n(\nu_k + d)$ and $\nu' = \nu_{k+1} = n(2\nu_k + d)$, and taking profit

of (5.3.24), we have

$$
\begin{aligned}
\|H_0' - \{H_0, G_{k+1}\}\|_R^{N_k,\nu_k} &= \left\| \sum_{l=1}^{n} \frac{1}{l!} Ad^l_{G_{k+1}} \left(\{H_0, G_{k+1}\}\right) \right\|_R^{N_k,\nu_k} \\
&= \left\| \sum_{l=1}^{n} \frac{1}{l!} Ad^l_{G_{k+1}} \left(Z_{k+1} - P^{(k)}\right) \right\|_R^{N_k,\nu_k} \\
&\lesssim \sum_{l=1}^{n} \frac{1}{l!} \left(\frac{\|G_{k+1}\|_R^{N_k,\nu_k}}{R^2} \right)^l \|P^{(k)}\|_R^{N_k,\nu_k} \lesssim \mu^k \|P^{(0)}\|_k^{N,\nu} \sum_{l=1}^{n} \frac{1}{l!} \mu^{(k+1)l} \\
&\lesssim \mu^k \|P^{(0)}\|_R^{N,\nu} \mu^{k+1} \sum_{l=0}^{\infty} \frac{1}{l!} \mu^{(k+1)l} \lesssim \mu^{2k+1} \left\| \hat{P} \right\|_R^{N,\nu} ,
\end{aligned}
$$

that is (5.3.21) with k replaced by $k+1$ (also in the case $k=0$).

Concerning $\left(Z^{(k)}\right)'$ we just use (5.3.11) which gives

$$
\left\| \left(Z^{(k)}\right)' \right\|_R^{N_k,\nu_k} \lesssim \|P^{(0)}\|_R^{N,\nu} \mu^{k+1} .
$$

Similarly one gets the estimate of $\left(P^{(k)}\right)'$.

Finally, consider $\mathcal{R}_{Z^{(k)},G_{k+1}}$. One has $Z^{(k)} \in L_{3,\bar{r}_{k-1}}$, so that we apply (5.3.12) with $r_1 = k+1$, $r_2 = 3$, which gives

$$
n + 1 \geq \frac{\bar{r}}{k+1} ,
$$

so that the remainder is estimated by

$$
\left\| X_{\mathcal{R}_{Z^{(k)},G_{k+1}}} \right\| \lesssim \frac{\|Z^{(k)}\|_R^{N_k,\nu_k}}{R} (\mu^{k+1})^{n+1} \lesssim \frac{\|P^{(0)}\|_R^{N_k,\nu_k}}{R} \mu^{\bar{r}} \lesssim R^2 (RK^\tau)^{\bar{r}} .
$$

Proceeding in the same way for the other terms one gets the thesis. □

We thus have the following.

Corollary 5.3.19 (Normal form). *Assume Hypotheses L.1, L.2 and L.3 for the Hamiltonian $H = H_0 + P$ of the form* (5.2.1) *and that P is a function with localized coefficients according to Definition 4.1.12 and with a zero of order at least* 3 *at the origin. For any $\bar{r} > 3$ there exist τ and $s_{\bar{r}} \geq s_0 > d/2$ such that for any $s \geq s_{\bar{r}}$ the following holds. There exist constants $R_{\bar{r},s}$ and $C_{\bar{r},s}$ such that, if K and R fulfill*

$$
RK^\tau \leq R_{\bar{r},s} ,
$$

then there exists an invertible canonical transformation

$$T^{(\bar r)} : \mathcal{B}^s_{R/2^{\bar r}} \to \mathcal{B}^s_R \,, \quad \left[T^{(\bar r)}\right]^{-1} : \mathcal{B}^s_{R/4^{\bar r}} \to \mathcal{B}^s_{R/2^{\bar r}}$$

such that

$$H^{(\bar r)} = H \circ T^{(\bar r)} = H_0 + Z^{(\bar r)} + \mathcal{R}_T \tag{5.3.27}$$

where $Z^{(\bar r)} \in L^{\nu_{\bar r}, N_{\bar r}}_{3, \bar r_{\bar r}}$ *is in normal form and* $X_{\mathcal{R}_T} \in \mathcal{C}^\infty(\mathcal{B}^s_{R/2^{\bar r}}, \mathcal{H}^s_e)$*. Moreover, for any* $u \in \mathcal{B}^s_R$*, we have*

$$\left\| X_{\mathcal{R}_T}(u) \right\|_s \lesssim R^2 (K^\tau R)^{\bar r} \,.$$

Applying Corollary 5.3.19, we prove Proposition 5.3.5.

Proof of Proposition 5.3.5. We apply Corollary 5.3.19. Choosing K in such a way that

$$RK^\tau \sim R^{\frac{1}{2}} \tag{5.3.28}$$

we have

$$(RK^\tau)^{\bar r} R^2 \simeq R^{\frac{\bar r + 4}{2}} .$$

Choosing $\bar r := 2r$ we get

$$\left\| X_{\mathcal{R}_T}(u) \right\|_s \lesssim R^{r+2} .$$

From Theorem 5.3.19, Definitions 5.3.2 and 5.3.3, we can write

$$Z^{(\bar r)} = Z_0 + Z_B + Z_2 + Z_{\geq 3} .$$

We have to show that Z_2 and $Z_{\geq 3}$ can be considered remainder terms of order R^{r+1}. From Corollary 4.2.13, we have that, for any $u \in \mathcal{B}^s_{R/2^{\bar r}}$,

$$\left\| X_{Z_{\geq 3}}(u) \right\|_s \lesssim \frac{\| Z_{\geq 3} \|^{\nu, N}_R}{K^{s - s_0}} \frac{1}{R} \lesssim \frac{R^2}{K^{s - s_0}} .$$

Since (5.3.28) implies $K \sim R^{-\frac{1}{2\tau}}$, we have

$$\frac{R^2}{K^{s-s_0}} \sim R^{2 + \frac{s - s_0}{2\tau}} \lesssim R^{r+1}$$

if $s > s_r = s_0 + 2\tau(r-1)$ and so

$$\left\|X_{Z_{\geq 3}}(u)\right\|_s \lesssim R^{r+1}\,.$$

It remains to consider X_{Z_2}. From Lemma 4.2.14, it follows that, for any $u \in \mathcal{B}^s_R$,

$$\|X_{Z_2}(u)\|_s \lesssim \frac{\|Z_2\|_R^{\nu,N'}}{K^{\delta(N'-N)}}\frac{1}{R} \lesssim \frac{R^2}{K^{\delta(N'-N)}}\,,$$

for any $N' > N$. Denoting $N' = N + M_1$, we have

$$\frac{R^2}{K^{\delta M_1}} \sim R^{2+\frac{\delta M_1}{2\tau}} \sim R^{r+1}$$

for $M_1 = \frac{2\tau}{\delta}(r-2)$. We get the thesis denoting

$$\mathcal{R}^{(\bar r)} := Z_2 + Z_{\geq 3} + \mathcal{R}_T.$$

The estimates (5.3.6) follow from Lemma 4.2.13. □

5.4 Dynamics of the normal form

We conclude the proof of Theorem 5.2.1, exploiting Proposition 5.3.5 to study the dynamics corresponding to a smooth, small, real initial datum for the Hamilton equations of (5.2.1). Precisely, take some large s and assume that

$$\left\|u_0\right\|_s =: \epsilon < \frac{R}{2\cdot 4^{2r}}\,, \tag{5.4.1}$$

with a small $R < R_{s,r}$ and $R_{r,s}$ from Proposition 5.3.5. Denote

$$z_0 := \left(T^{(\bar r)}\right)^{-1}(u_0)\,,$$

and we consider the evolution in the z variables. Let $z = (z_+, z_-) \in \mathcal{H}^s \times \mathcal{H}^s \equiv \mathcal{H}^s_e$ and, for K defined in Proposition 5.3.5, recall that

$$\Pi^{\leq} z = z^{\leq} = \sum_{\{[\![A]\!]\leq K\}} \Pi_A z\,,$$
$$\Pi^{\perp} z = z^{\perp} = \sum_{\{[\![A]\!]> K\}} \Pi_A z\,.$$

Write the Hamilton equations of $H^{(\bar r)}$ as a system:

$$\begin{cases} \dot z^{\leq} = X_{H_0}(z^{\leq}) + X_{Z_0}(z^{\leq}) + \Pi^{\leq} X_{Z_B}(z) + \Pi^{\leq} X_{R^{(\bar r)}}(z)\,, \\ \dot z^{\perp} = X_{H_0}(z^{\perp}) + \Pi^{\perp} X_{Z_B}(z) + \Pi^{\perp} X_{R^{(\bar r)}}(z)\,. \end{cases} \tag{5.4.2}$$

We start considering the dynamics of $z^{\leq}$. We remark first that $\Pi^{\leq} X_{Z_B}$ and $\Pi^{\leq} X_{R^{(\bar r)}}$ are remainder terms of order R^{r+2} (see Proposition 5.3.5). Then, it remains to analyze the role of Z_0.
To this end, we define the set of indexes correspondent to each "band" of the spectrum Σ_n, the correspondent projector and the correspondent "superaction", namely

$$E_n := \{a \in \Lambda : \omega_a \in \Sigma_n\}\,, \qquad \Pi_n z := \sum_{\{A=(a,\sigma): a\in E_n\}} \Pi_A z\,,$$
$$J_n(z) := \sum_{a\in E_n} \int \Pi_{(a,+)} z\, \Pi_{(a,-)} z\, dx\,.$$

In particular, if z is real (as we assumed)

$$J_n(z) = \sum_{a\in E_n} \left\|\Pi_{(a,+)} z\right\|_0^2 = \frac{1}{2} \sum_{a\in E_n} \left\|\Pi_{(a,+)} z\right\|_0^2 + \left\|\Pi_{(a,-)} z\right\|_0^2.$$

In the next lemma, we prove that J_n is preserved under the dynamics associated to Z_0.

Lemma 5.4.1. *Let Z a polynomial supported on W, then*

$$\{Z, J_n\} = 0\,.$$

Proof. From Def. 5.3.1 follows that $\widetilde{Z}$ is the sum of terms of the form

$$\int_M \Pi_{A_1} u_1 \dots \Pi_{A_l} u_l$$

with $\mathbf{A} = (A_1, \dots, A_l) \in W$. First, note that we have

$$X_{J_n}(u) = i \sum_{a\in E_n} \left(\Pi_{(a,+)} u, -\Pi_{(a,-)} u\right) = \mathrm{i} \sum_{(a,\sigma)} \delta_{a\in\Sigma_n} (\delta_{\sigma,+} - \delta_{\sigma,-}) \Pi_{(a,\sigma)} u\,.$$

Moreover, for any homogeneous polynomial F, one has

$$\mathrm{d}F(u)X = \tilde F(X, u, \dots, u) + \tilde F(u, X, \dots, u) + \dots + \tilde F(u, \dots, u, X)\,.$$

Decompose $Z = \sum_l Z_l$ in homogeneous polynomials, we have

$$\{Z_l, J_n\}(u) = \mathrm{d}Z(u)X_{J_n}(u) =$$
$$= \mathrm{i} \sum_{A_1,...,A_l} \tilde{Z}(\Pi_{A_1}u, ..., \Pi_{A_l}u) \sum_{j=1}^{n} \delta_{a_j \in \Sigma_n}(\delta_{\sigma_j,+} - \delta_{\sigma_j,-}) \ .$$

We recall that Z is in normal form, which means that the sum can be restricted to multi-indexes belonging to W. So, fix one of the multi-indexs $A \in W$. The definition of W implies that there exists a permutation τ of $1, ..., l$, and indexes $n_1, ..., n_l$ s.t, $a_{\tau(j)}, a_{\tau(j+l/2)} \in \Sigma_{n_j}$ and $\sigma_{\tau(j)}\sigma_{\tau(j+l/2)} = -1$. Thus consider the sum

$$\sum_{j=1}^{n} \delta_{a_j \in \Sigma_n}(\delta_{\sigma_j,+} - \delta_{\sigma_j,-}) = \sum_{j=1}^{n} \delta_{a_{\tau(j)} \in \Sigma_n}(\delta_{\sigma_{\tau(j)},+} - \delta_{\sigma_{\tau(j)},-}) \ .$$

If $a_{\tau(j)} \in \Sigma_n$ with a sign, it means that also $a_{\tau(j+l/2)} \in \Sigma_n$ with the opposite sign. Thus the sum vanishes for any index in W. □

Corollary 5.4.2. *There exists a positive constant C_1 with the following property: assume that* (5.4.1) *holds and that there exists $T_e > 0$ s.t.*

$$\|z(t)\|_s \le \frac{R}{2 \cdot 2^{2r}} \ , \quad \forall |t| \le T_e \tag{5.4.3}$$

and some $R < R_{rs}$, then one has

$$\left\|z^{\le}(t)\right\|_s^2 \le C_1(\left\|z^{\le}(0)\right\|_s^2 + |t|R^{r+3}) \, . \tag{5.4.4}$$

Proof. Define

$$a_{n,s} := \inf_{a \in E_n} (1 + [\![a]\!])^{2s}$$

then there exist two constants C_3, C_4 such that, for any n, one has

$$C_3\|\Pi_n z\|_s^2 \le a_{n,s}J_n \le C_4\|\Pi_n z\|_s^2 \, ,$$

$$\|z^{\le}\|_s^2 \simeq \sum_{\substack{n \text{ s.t.} \\ \max E_n \le K}} a_{n,s}J_n \, . \tag{5.4.5}$$

Then, by Proposition 5.3.5 and Lemma 5.4.1, one has

$$\frac{d}{dt} \sum_{\substack{n \text{ s.t.} \\ \max E_n \le K}} a_{n,s} J_n \lesssim \sum_n a_{n,s} \left\{ J_n, Z_2 + R^{(\bar{r})} \right\} \lesssim R^{r+3} \,, \tag{5.4.6}$$

which is valid for $|t| \le T_e$. From (5.4.5) and (5.4.6) the thesis immediately follows. $\square$

Consider the dynamics of the high modes $z^{\perp}$ given by (5.4.2). From Proposition 5.3.5), $\Pi^{\perp} X_{R^{(r)}}$ is a remainder term of order R^{r+2} and Z_B is in Block Resonant Normal Form and of order 2 in $z^{\perp}$.

Recalling the dyadic partition $\Lambda = \bigcup_{\alpha \in \mathfrak{A}} \Omega_\alpha$, we define the correspondent projectors and the correspondent superactions. Since we are interested in the dynamics of the high modes, we consider only superactions defined on modes $|a| \ge K$. This amount to consider a cutoff of the Bourgain's blocks, which do not break the dyadicity of the partition.

$$\begin{aligned} \Pi_\alpha z &:= \sum_{\{A=(a,\sigma) :\, a \in \Omega_\alpha,\, [\![a]\!] \ge K\}} \Pi_A z \\ J_\alpha(z) &:= \sum_{\{a \in \Omega_\alpha,\, [\![a]\!] \ge K\}} \int \Pi_{(a,+)} z \, \Pi_{(a,-)} z \, dx = \|\Pi_\alpha z\|_0^2 \,. \end{aligned}$$

By definition of Block Resonant normal form (see Def. 5.3.2) $\Pi^{\perp} X_{Z_B}$ is linear in $z^{\perp}$. Then we exploit that Z_B is a real polynomial to show that the L^2 norm on each block is conserved along the dynamics induced by the normal form. Namely, the dynamics of the normal form Z_B enforces the exchange of energy only within high modes in the same block Ω_α. This is the content of the following lemma.

Lemma 5.4.3. *For any real $z \in \mathcal{H}^s_e$, we have that, for any $\alpha \in \mathfrak{A}$,*

$$\{J_\alpha, Z_B\}(z) = 0 \,.$$

Proof. In this proof, we denote again $u_A := \Pi_A u$. Since Z_B is real and recalling Def. 5.3.2, it is the sum of terms of the form

$$\begin{aligned} \widetilde{Z}_\beta(z) &:= \int z_{(a,+)} z_{(b,-)} z_{A_3} \dots z_{A_l} + \int z_{(a,-)} z_{(b,+)} z_{\overline{A}_3} \dots z_{\overline{A}_r} \\ &= 2\mathrm{Re} \left(\int z_{(a,+)} z_{(b,-)} z_{A_3} \dots z_{A_r} \right) , \quad \forall z \text{ real} \end{aligned} \tag{5.4.7}$$

with $\mathbf{A} = ((a,+), (b,-), A_3, \dots, A_r) \in \Lambda^r_e$ such that $a, b \in \Omega_\beta$, $[\![a]\!] > K$ and $[\![b]\!] > K$ for some β. Here, for the sake of simplicity, we are considering multi-indexes for which the two largest indexes are the first and the second.

If $\beta \neq \alpha$, then $\{J_\alpha, Z_\beta\} = 0$. Otherwise, we have

$$\left\{J_\alpha, \widetilde{Z}_\alpha\right\}(z) = i\int z_{(a,-)}z_{(b,+)}\dots z_{\bar{A}_r} - i\int z_{(a,+)}z_{(b,-)}\dots z_{A_r} + i\int z_{(b,-)}z_{(a,+)}\dots z_{A_r} - i\int z_{(b,+)}z_{(a,-)}\dots z_{\bar{A}_r} = 0\,,$$

with $a, b \in \Omega_\alpha$. □

The following Corollary is proved exactly in the same way as Corollary 5.4.2

Corollary 5.4.4. *There exists a positive constant C_1 with the following property: assume there exists $T_e > 0$ s.t.* (5.4.3) *holds for some $R < R_{rs}$, then one has*

$$\left\|z^\perp(t)\right\|_s^2 \leq C_1(\left\|z^\perp(0)\right\|_s^2 + |t|R^{r+3})\,. \tag{5.4.8}$$

Proof of Theorem 5.2.1. We prove by a bootstrap argument that, if ϵ is small enough, the escape time T_e fulfills $1/R^{r+1} \lesssim T_e$. First we make the canonical transformation $\mathcal{T}^{(r)}$ and apply the estimate (5.3.5) with $R = 2^{-1}\epsilon 4^{2r}$ where $\epsilon := \|u_0\|$,getting that

$$\|z_0\|_s \leq \frac{R}{2\cdot 2^{2r}}\ .$$

Then we can apply Lemmas 5.4.2 and Lemma 5.4.4, since the assumptions on the initial datum $z(0)$ are fulfilled. Assume now, by contradiction, that there exists $t_* < T_e := R^{-r-1}(2\cdot 2^{2r})^{-2}$ s.t. $z(t) \in \mathcal{B}^s_{R'_r}$ for all $|t| < t_*$ and $z(t_*) \in \partial\mathcal{B}^s_{R'_r}$, with $R'_r := \frac{\sqrt{2C_1}R}{2\cdot 2^{2r}}$ and C_1 as in Lemmas 5.4.4, 5.4.2. For $|t| \leq t_*$, Lemmas 5.4.2 and 5.4.4 give that

$$\left(\frac{\sqrt{2C_1}R}{2\cdot 2^{2r}}\right)^2 = \|z(t_*)\|_s^2 \leq C_1(\|z_0\|_s^2 + R^{r+3}|t|) \leq C_1\left(\frac{R^2}{2^2\cdot 2^{4r}} + \frac{R^2}{2^2\cdot 2^{4r}}\right)\ ,$$

which is absurd. Going back to the variables u, changing $r+1$ to r and adjusting the name of the constants one gets the result. □

5.5 Proof of Theorem 5.2.2

In this Section, we prove Theorem 5.2.2. It implies that the functional introduced in Hypothesis P is a function with localized coefficients. The proof follows immediately from the next lemma and Theorem 3.2.1.

Lemma 5.5.1. *Let P be a polynomial with localized coefficients, then also*

$$Q(u) := P(u) \int_M u\bar{u}dx \tag{5.5.1}$$

has localized coefficients.

Proof. Let r be the degree of P, then one has

$$|\widetilde{Q}(\Pi_{a_1}u_1, \Pi_{a_2}u_2, \Pi_{b_1}u_3, ..., \Pi_{b_r}u_{r+2})| = \delta_{a_1,a_2} \left| \widetilde{P}(\Pi_{b_1}u_3, ..., \Pi_{b_r}u_{r+2}) \right| \|\Pi_{a_1}u_1\| \, \|\Pi_{a_2}u_2\| \ . \tag{5.5.2}$$

Therefore, to get the thesis it is enough to show that

$$\delta_{a_1,a_2} \frac{\mu(\mathbf{b})^{N+\nu}}{S(\mathbf{b})^N} \lesssim \frac{\mu(a_1, a_2, \mathbf{b})^{N+\nu}}{S(a_1, a_2, \mathbf{b})^N} \ . \tag{5.5.3}$$

For simplicity, we will also denote $a \coloneqq a_1 = a_2$ and we consider, in full generality that $[\![b_1]\!] \geq [\![b_2]\!] \geq [\![b_3]\!]$ are the three largest indexes among $b_1, \ldots, b_r$.

Case 1. If $[\![a]\!] \geq [\![b_1]\!]$ we have the trivial estimate

$$\begin{aligned} \frac{\mu(\mathbf{b})^{N+\nu}}{S(\mathbf{b})^N} &= \frac{[\![b_3]\!]^{\nu+N}}{(|b_1 - b_2| + [\![b_3]\!])^N} \leq [\![b_3]\!]^\nu \\ &\leq [\![b_1]\!]^\nu = \frac{[\![b_1]\!]^{\nu+N}}{(|a_1 - a_2| + [\![b_1]\!])^N} = \frac{\mu(a_1, a_2, \mathbf{b})^{N+\nu}}{S(a_1, a_2, \mathbf{b})^N} \ , \end{aligned}$$

since $|a_1 - a_2| = 0$.

Case 2. If $[\![b_1]\!] > [\![a]\!] > [\![b_2]\!]$, we need to distinguish two cases.

Case 2.i. Consider first the case $[\![b_1]\!] > K_1[\![b_2]\!]$, with $K_1 > 0$ so large that

$$|b_1 - b_2| \geq K_2[\![b_1]\!] \tag{5.5.4}$$

for a constant $0 < K_2 < 1$; the existence of such constants is established in 4.2.4 . Then using $|b_1 - a| \lesssim [\![b_1]\!] + [\![a]\!] \lesssim [\![b_1]\!]$, we estimate

$$\frac{[\![b_3]\!]^{\nu+N}}{(|b_1 - b_2| + [\![b_3]\!])^N} \leq \frac{[\![b_3]\!]^{\nu+N}}{(K_2[\![b_1]\!] + [\![b_3]\!])^N} \lesssim \frac{[\![b_3]\!]^{\nu+N}}{(|b_1 - a| + [\![b_3]\!])^N} \ .$$

Since the function $f(x) = \frac{x^{N+\nu}}{(K+x)^N}$ is increasing for any $N, \nu > 0$, $K \geq 0$ and $x \geq 0$, the above quantity is bounded, up to a constant, by

$$\frac{[\![a]\!]^{\nu+N}}{(|b_1 - a| + [\![a]\!])^N} = \frac{\mu(a_1, a_2, \mathbf{b})^{N+\nu}}{S(a_1, a_2, \mathbf{b})^N} \ .$$

Case 2.ii. If $[\![b_1]\!] \leq K_1[\![b_2]\!]$ we observe that

$$K_1C[\![b_2]\!] \geq C[\![b_1]\!] \geq |b_1 - a|$$

and we estimate

$$\begin{aligned}\frac{[\![b_3]\!]^{\nu+N}}{(|b_1-b_2|+[\![b_3]\!])^N} &\leq [\![b_2]\!]^\nu = (K_1C+1)^N \frac{[\![b_2]\!]^{\nu+N}}{(K_1C[\![b_2]\!]+[\![b_2]\!])^N} \\ &\leq (K_1C+1)^N \frac{[\![b_2]\!]^{\nu+N}}{(|b_1-a|+[\![b_2]\!])^N} \,.\end{aligned} \tag{5.5.5}$$

Using again the monotonicity of the function $f(x) = \frac{x^{N+\nu}}{(K+x)^N}$, we get

$$\begin{aligned}(5.5.5) &\leq (K_1C+1)^N \frac{[\![a]\!]^{\nu+N}}{(|b_1-a|+[\![a]\!])^N} \\ &= (K_1C+1)^N \frac{\mu(a_1,a_2,\mathbf{b})^{N+\nu}}{S(a_1,a_2,\mathbf{b})^N} \,.\end{aligned}$$

Case 3. If $[\![b_2]\!] \geq [\![a]\!] > [\![b_3]\!]$ we get

$$\begin{aligned}\frac{\mu(\mathbf{b})^{N+\nu}}{S(\mathbf{b})^N} &= \frac{[\![b_3]\!]^{\nu+N}}{(|b_1-b_2|+[\![b_3]\!])^N} \\ &\leq \frac{[\![a]\!]^{\nu+N}}{(|b_1-b_2|+[\![a]\!])^N} = \frac{\mu(a_1,a_2,\mathbf{b})^{N+\nu}}{S(a_1,a_2,\mathbf{b})^N} \,.\end{aligned}$$

Case 4. If $[\![a]\!] \leq [\![b_3]\!]$ the estimates (4.2.27) is obvious since it does not involve a.

This concludes the proof. □

Proof of Theorem 5.2.2. The proof follows directly from Lemma 5.5.1 and the definition of a function with localized coefficients 4.1.12, since the Taylor expansion of a functional fulfilling Hypothesis P is the sum of terms P_m of the form (5.5.1), where any $P(u)$ is a polynomial with localized coefficients as a consequene of Theorem 3.2.1. □

Chapter 6

Applications of the abstract result

In this final Chapter, we prove some results of almost global existence and Sobolev stability by applying the abstract Theorem 5.1.4, proven in Chapter 5.

6.1 Manifolds

We present examples of manifolds on which the Laplacian-Beltrami operator $-\Delta$ is a globally integrable quantum system with steep and homogeneous Hamiltonian function. Thus, the associated set of frequencies satisfies Theorem 3.2.7 and so fulfills Hypothesis L.3 of the abstract Theorem 5.1.4. For some of these examples, the partition of the frequencies follows directly from the spectral structure of the Laplacian. For example, this is the case for tori [23]. Theorem 3.2.7 furnishes a more general theoretical framework and extends the result to Lie groups and surfaces of revolution.

1. **Flat tori**. Given a basis $e_1, \dots, e_d$ of $\mathbb{R}^d$ we define a maximal dimensional lattice $\Gamma \subset \mathbb{R}^d$ by

$$\Gamma := \left\{ x = \sum_{j=1}^{d} m_j e_j \ , \quad m_j \in \mathbb{Z} \right\}$$

   and the corresponding maximal dimensional torus $T^d_\Gamma := \mathbb{R}^d / \Gamma$. By using in $\mathbb{R}^d$ the basis e_j, one is reduced to the standard torus $\mathbb{T}^d$ endowed by a flat metric. Then the actions are given by the operators $-\mathrm{i}\partial_j$. Bourgain's clustering of the eigenvalues of the Laplacian was already proved in [29, 32].

2. **Rotation invariant surfaces**. Consider a real analytic function $f : \mathbb{R}^3 \to \mathbb{R}$, invariant by rotations around the z axis, and assume it is a submersion at $f(x, y, z) = 1$. Denote by M the level surface $f(x, y, z) = 1$ and endow it by the natural metric g induced by the Euclidean metric of $\mathbb{R}^3$. We introduce coordinates in M as follows: let N and S be the north and the south poles

(intersection of M with the z axis) and denote by $\theta \in [0, L]$ the curvilinear abscissa along the geodesic given by the intersection of M with the xz plane; we orient it as going from N to S and consider also the cylindrical coordinates (r, ϕ, z) of $\mathbb{R}^3$: on M we will use the coordinates

$$(\theta, \phi) \in (0, L) \times (0, 2\pi) \ .$$

Using such coordinates, one can write the equation of M by expressing the cylindrical coordinates of a point in $\mathbb{R}^3$ as a function of (θ, ϕ) getting

$$M = \left\{ (r(\theta), \phi, z(\theta)) \ , \quad (\theta, \phi) \in (0, L) \times \mathbb{T}^1 \right\} \ .$$

Since θ is a geodesic parameter, the metric takes the form

$$g = r^2(\theta) d\phi^2 + d\theta^2 \ .$$

We assume that the function $r(\theta)$ has only one critical point $\theta_0 \in (0, L)$. The fact that the Laplacian is a globally integrable quantum system that was proved by Colin de Verdier [94].

3. **Lie Groups and Homogeneous spaces**. M is either a compact, simply connected Lie group or a homogeneous space. The steepness of the Laplacian is proved in [12].

From now on, M will denote any of these manifolds.

6.2 Applications

We will we denote with $I_1, \ldots, I_d$ the quantum actions (see Def. 3.1.1) of the Laplacian and with h_{L0} the function (see Def. 3.1.3) such that

$$-\Delta = h_{L0}(I_1, \ldots, I_d) \, . \tag{6.2.1}$$

Moreover, we will denote with $\{\lambda_a\}_{a \in \Lambda}$ the eigenvalues of $-\Delta$, namely

$$\lambda_a = h_{L0}(a), \qquad \forall a \in \Lambda \, . \tag{6.2.2}$$

As emphasized in Section 6.1, they fulfill Hypothesis L.3.

In the applications, we apply Theorem 5.1.4 to a Hamiltonian with linear part H_L

that is a small correction of $-\Delta$; for each of them we will check that they still fulfill Hypotheses L.0, L.1 and L.3. We remark that considering systems that are small corrections of the Laplacian allows us to verify the non-resonance assumption L.2.

Finally, we remark that all the nonlinear perturbations that we will meet are of the form (5.1.11), so they fulfill Hypothesis P. In other words, following Theorem 5.2.2, they are functions with localized coefficients.

6.2.1 NLS equation with convolution potential

Our first result concerns a nonlinear Schrödinger equation with a spectral multiplier. To define it we recall the definition of the spectral projectors Π_a, with $a \in \Lambda$.

Definition 6.2.1. *For $a \equiv (a^1, \dots, a^d) \in \Lambda$,*

$$\Pi_a := \Pi_{a^1} \dots \Pi_{a^d} \tag{6.2.3}$$

where Π_{a^j} is the orthogonal projector on the eigenspace of I_j with eigenvalue a^j.

Given $u \in L^2(M, \mathbb{C})$, consider its spectral decomposition,

$$u = \sum_{a \in \Lambda} \Pi_a u \,,$$

and let $V = \{V_a\}_{a \in \Lambda}$ with $V_a \in \mathbb{R}$ be a sequence; correspondingly, we define a spectral multiplier by

$$V * u := \sum_a V_a \Pi_a u \,. \tag{6.2.4}$$

In the following, we will assume that V belongs to the space

$$\mathcal{V}_n := \left\{ V = \{V_a\}_{a \in \Lambda} \ : \ V_a \in \mathbb{R} \,, |V_a| \langle a \rangle^n \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\} \,, \tag{6.2.5}$$

that we endow with the product measure.

Consider the Cauchy problem

$$\begin{cases} i\partial_t \psi = -\Delta\psi + V * \psi + f(x, |\psi|^2)\psi, \quad x \in M \,, \\ \psi(0) = \psi_0 \end{cases} \tag{6.2.6}$$

where $V \in \mathcal{V}_n$ and the non-linearity f is of class $C^\infty(M \times \mathcal{U}, \mathbb{R})$, $\mathcal{U} \subset \mathbb{R}$ being a neighbourhood of the origin, and fulfills $f(x,0) = 0$, $\partial_y f(x,y)|_{y=0} = 0$.

Theorem 6.2.2. *There exists a set $\mathcal{V}^{(res)} \subset \mathcal{V}_n$ of zero measure, s.t., if $V \in \mathcal{V} \setminus \mathcal{V}^{(res)}$ the following holds. For any $r \in \mathbb{N}$, there exists $s_r > d/2$ such that for any $s > s_r$ there is $\epsilon_s > 0$ and $C > 0$ such that if the initial datum for (6.2.6) belongs to H^s and fulfills $\epsilon := |\psi_0|_s < \epsilon_s$ then*

$$\|\psi(t)\|_s \leq C\epsilon\,, \quad \text{for all} \quad |t| \leq C\epsilon^{-r}\,.$$

We begin the proof of Theorem 6.2.2 recalling that the nonlinear Schrödinger equation (6.2.6) is Hamiltonian with a Hamiltonian function

$$H = \int_M (\varphi(-\Delta\psi) + \varphi(V * \psi) + F(x, \psi\varphi))\, dx \tag{6.2.7}$$

where F is such that $f(x, u) = \partial_u F(x, u)$ and φ is a variable conjugated to ψ. To get equation (6.2.6) one has to restrict to the invariant manifold $\varphi = \overline{\psi}$.
The Hamiltonian (6.2.7) fits the abstract settings (5.1.2) with H_L the globally integrable quantum system

$$H_L = -\Delta + V* \tag{6.2.8}$$

The actions are $I_1, \ldots, I_d$, namely they are the actions of the Laplacian; the associated function h_L is

$$\mathbb{R}^d \ni \xi \mapsto h_{L0}(\xi) + v(\xi) \tag{6.2.9}$$

where $v(\xi)$ is any C^∞ interpolation of $V*$ on the lattice Λ, namely it is a function such that $v(\xi) = V_\xi, \forall \xi \in \Lambda$. Moreover, we remark that the frequencies are given by

$$\omega_a := \lambda_a + V_a\ . \tag{6.2.10}$$

In order to apply Theorem 5.1.4 and prove Theorem 6.2.2, it remains to verify that Hypotheses L.1,L.2 and L.3 hold.

Hypotheses L.1 and L.3 clearly hold, since they hold for $\{\lambda_a\}_{a\in\Lambda}$ and the coefficients V_a have strong decay, see (6.2.5). In fact, for any $a, b \in \Lambda$,

$$|a - b| + |\lambda_a - \lambda_b| \geq C_\delta(|a|^\delta + |b|^\delta) \tag{6.2.11}$$

implies

$$\begin{aligned} |a - b| + |\omega_a - \omega_b| &\geq |a - b| + |\lambda_a - \lambda_b| - |V_a - V_b| \geq \\ &\geq C_\delta(|a|^\delta + |b|^\delta) - 1/2 \geq C'_\delta(|a|^\delta + |b|^\delta)\,. \end{aligned} \tag{6.2.12}$$

We come to prove the non-resonance condition L.2. Actually, we will prove a stronger condition.

Lemma 6.2.3. *For any r there exists τ and a set $\mathcal{V}^{(res)} \subset \mathcal{V}$ of zero measure, s.t., if $V \in \mathcal{V} \setminus \mathcal{V}^{(res)}$ there exists $\gamma > 0$ s.t. for all $K \geq 1$ one has*

$$\begin{gathered} |\omega \cdot k| \geq \frac{\gamma}{K^\tau} \ , \\ \forall k = (k_{a_1}, ..., k_{a_r}) \ s.t. \ |a_j| \leq K \ \forall j = 1, ..., r \ , |k|_{\ell^1} \leq r \end{gathered} \tag{6.2.13}$$

Before proving this lemma we prove a preliminary result; to this end consider, for k fulfilling (6.2.13), consider

$$\mathcal{V}(k, \gamma) := \{V \in \mathcal{V}_n \ : \ |\omega \cdot k| < \gamma\} \ .$$

We have

Lemma 6.2.4. *One has*

$$|\mathcal{V}(k, \gamma)| \leq 2\gamma K^n \ . \tag{6.2.14}$$

Here we indicate with $|\cdot|$ the measure of a set.

Proof. We prove that the result is true for any arbitrary sequence λ_a, namely that the asymptotic behavior is not important.
First, if $\mathcal{V}(k, \gamma)$ is empty there is nothing to prove. Assume that $\tilde{V} \in \mathcal{V}(k, \gamma)$. Since $k \neq 0$, there exists $\bar{a}$ such that $k_{\bar{a}} \neq 0$ and thus $|k_{\bar{a}}| \geq 1$; so we have

$$\left| \frac{\partial \omega \cdot k}{\partial \hat{V}_{\bar{a}}} \right| \geq 1 \ .$$

It means that if $\mathcal{V}(k, \gamma)$ is not empty it is contained in the layer

$$\left| \widehat{\tilde{V}}_{\bar{a}}' - \hat{V}_{\bar{a}}' \right| \leq \gamma \ ,$$

whose measure is $\gamma \langle \bar{a} \rangle^n \leq 2\gamma N^n$. This implies (6.2.14). □

Proof of Lemma 6.2.3. From Lemma 6.2.4 it follows that the measure of the set

$$\mathcal{V}^{(res)}(\gamma) := \bigcup_{K \geq 1} \bigcup_{k} \mathcal{V}\left(k, \frac{\gamma}{K^{dr+2}}\right)$$

is estimated by a constant times γ. It follows that the set

$$\mathcal{V}^{(res)} := \cap_{\gamma > 0} \mathcal{V}^{(res)}(\gamma)$$

has zero measure and with this definition the lemma is proved. □

6.2.2 Sobolev stability of the ground state of NLS

Our second result is the long-time Sobolev stability of the ground state solution of the nonlinear Schrödinger equation

$$i\dot{\psi} = -\Delta\psi + f\left(|\psi|^2\right)\psi \ , \tag{6.2.15}$$

where $f \in C^\infty(\mathcal{U};\mathbb{R})$, $\mathcal{U} \subset \mathbb{R}$ is an open neighborhood of the origin, and f has a zero of order at least one at the origin. It is well known that for any $p_0 \in \mathcal{U} \cap \mathbb{R}^+$ equation (6.2.15) has a solution given by a plane wave of the form

$$\psi_*(t) = \sqrt{p_0} e^{-i\nu t} \ ,$$

provided $\nu = f(p_0)$. Denote by $\bar{\lambda}$ the lowest non vanishing eigenvalue of $-\Delta$, then we will prove the following result.

Theorem 6.2.5. *Assume there exists $\bar{p}_0 > 0$ such that $\bar{\lambda} + 2f(p_0) > 0$ for any $p_0 \in (0, \bar{p}_0]$. Then there exists a zero measure set $\mathcal{P}$ such that if $p_0 \in (0, \bar{p}_0] \setminus \mathcal{P}$ then for any $r \in \mathbb{N}$ there exists s_r for which the following holds. For any $s \geq s_r$, there exists constants ϵ_0 and C such that if the initial datum ψ_0 fulfills*

$$\|\psi_0\|_0^2 = p_0, \qquad \inf_{\alpha\in\mathbb{T}} \left\|\psi_0 - \sqrt{p_0}e^{-i\alpha}\right\|_s = \epsilon \leq \epsilon_0 \ ,$$

then the corresponding solution fulfills

$$\inf_{\alpha\in\mathbb{T}} \left\|\psi(t) - \sqrt{p_0}e^{-i\alpha}\right\|_s \leq C\epsilon \qquad \forall\, |t| \leq C\epsilon^{-r}$$

with $\psi(0) = \psi_0$.

Remark 6.2.6. *Note that if f is a positive function (the so-called* defocusing *case) there is no restriction in the L^2 norm of the initial datum, since $\bar{\lambda} + 2f(p_0) > 0$ for any p_0.*

The equation (6.2.15) is Hamiltonian with Hamiltonian

$$H(\psi,\bar{\psi}) = \int_M \left(\overline{\psi}(-\Delta\psi) + F(|\psi|^2)\right) dx \ , \tag{6.2.16}$$

where F is such that $F' = -f$. We introduce variables in which the ground state becomes an equilibrium point of a reduced system [54]. To this end, we consider the space

$$L^2_0(M;\mathbb{C}) := \left\{ \varphi \in L^2(M;\mathbb{C}) \ : \ \int_M \varphi(x) dx = 0 \right\} ,$$

of the functions with vanishing average and we denote

$$N(\varphi) = \int_M |\varphi|^2 \ dx \ .$$

Then we consider the map

$$\begin{aligned} &L^2_0(M,\mathbb{C}) \times \mathbb{R} \times \mathbb{T} \to L^2(M;\mathbb{C}) \\ &\qquad (\varphi, p, \theta) \mapsto \psi(\varphi, p, \theta) := e^{-i\theta} \left(\sqrt{p - \|\varphi\|^2} + \varphi(x) \right) \ . \end{aligned} \tag{6.2.17}$$

Lemma 6.2.7 (Faou,Lubich,Glouckler [54])**.** *The map* (6.2.17) *defines a local coordinate system close to* $\varphi = 0$*. Furthermore, such coordinates are symplectic, namely the Hamilton equations of a Hamiltonian function* H *have the form*

$$\dot{\theta} = \frac{\partial H}{\partial p} \ , \quad \dot{p} = -\frac{\partial H}{\partial \theta} \ , \quad \dot{z} = -i \nabla_{\bar{z}} H \ .$$

Then the Hamiltonian is just (6.2.16) with ψ given by (6.2.17). We now fix a value p_0 of p (which is an integral of motion) and expand in power series in φ getting (neglecting irrelevant terms independent of φ) a Hamiltonian of the form

$$H = H_0 + \hat{P}$$

with

$$H_0(\varphi) = \int_M \left[\bar{\varphi}(-\Delta\varphi) + 2f(p_0)\bar{\varphi}\varphi + \frac{f(p_0)}{2} \left(\varphi^2 + \bar{\varphi}^2 \right) \right] dx$$

and $\hat{P} = \hat{P}(\varphi, \bar{\varphi})$ of the form (5.1.11). Thus we have just to verify the assumptions on the linear part.

Introducing the spectral decomposition relative to the quantum actions of the Laplacian, we get

$$H_0 = \sum_{a \in \Lambda} \int_M \left((\lambda_a + 2f(p_0)) \, \Pi_a \bar{\varphi} \Pi_a \varphi + \frac{f(p_0)}{2} \left(\Pi_a \varphi \right)^2 + \frac{f(p_0)}{2} \left(\Pi_a \bar{\varphi} \right)^2 \right) dx \ ,$$

which can be diagonalized through a symplectic change of coordinates of the form

$$\begin{pmatrix} w_a \\ \bar{w}_a \end{pmatrix} = S_a \begin{pmatrix} \varphi_a \\ \bar{\varphi}_a \end{pmatrix}$$

where S_a are matrices uniformly bounded with respect to a. Such a change of coordinates of course does not change the nature of $\hat{P}$ of having localized coefficients. In the new coordinates, one gets

$$H_0(w,\overline{w}) = \sum_a \omega_a \int_M \Pi_a \bar{w} \Pi_a w dx$$

with

$$\omega_a = \sqrt{\lambda_a^2 + 2f(p_0)\lambda_a}\,. \tag{6.2.18}$$

Hypotheses L.1 and L.3 hold for $\{\omega_a\}_{a\in\Lambda}$, since they hold for $\{\lambda_a\}_{a\in\Lambda}$ and one has

$$\omega_a = \sqrt{\lambda_a^2 + 2f(p_0)\lambda_a} = \lambda_a \left(1 + \frac{f(p_0)}{\lambda_a} + o\left(\frac{f(p_0)}{\lambda_a}\right)\right)$$

so that

$$|a-b| + |\lambda_a - \lambda_b| \geq C_\delta(|a|^\delta + |b|^\delta)$$

implies

$$|a-b| + |\omega_a - \omega_b| \geq C'_\delta(|a|^\delta + |b|^\delta)\,.$$

with computations analogous to the ones in (6.2.12)

We prove now the non-resonance condition L.2, essentially identically as for the case of the Klein-Gordon equation on tori and Zoll Manifolds given by Delort and Szeftel [47] (see also [48]). In particular, the proof follows by applying the next theorem.

Theorem 6.2.8 ([47], Theorem 5.1)**.** *Let X be a closed ball $\mathcal{B}_{R_0}$ in $\mathbb{R}^p$ for some $R_0 > 0$ and by Y a compact interval in $\mathbb{R}$. Let $f\colon X \times Y \mapsto \mathbb{R}$ be a continuous subanalytic function, $\rho\colon X \mapsto \mathbb{R}$ a real analytic function, $\rho \not\equiv 0$. Assume*

- *f is real analytic on $\{x \in X; \rho(x) \neq 0\} \times Y$;*
- *for all $x_0 \in X$, the equation $f(x_0, y) = 0$ has only finitely many solutions $y \in Y$.*

Then there are $N_0 \in \mathbb{N}, \alpha_0 > 0, \delta > 0, C > 0$, such that for any $\alpha \in (0, \alpha_0)$, any $N \geq N_0$, any $x \in X$ with $\rho(x) \neq 0$,

$$meas\left\{y \in Y\colon |f(x,y)| \leq \alpha|\rho(x)|^N\right\} \leq C\alpha^\delta |\rho(x)|^{N\delta}\,.$$

The strategy to deduce the existence of large sets of values p_0 for which assumption L.2 holds is not trivial. In our case one can reproduce almost exactly the computations of [47, Sect. 5.2], substituting the function f of that paper with the function

$$\mathtt{f}(x,y) := \sum_{j=1}^{l} \sqrt{x_j^2 y + x_j} - \sum_{j=l+1}^{p} \sqrt{x_j^2 y + x_j} \ ,$$

which has the property that

$$m^2 \mathtt{f}\left(\lambda_{a_j}, \frac{1}{m^2}\right)$$

with $m^2 = 2f(p_0)$ are exactly the small divisors one has to control.

The only nontrivial point is to prove that for any fixed value of $x \in [0,1]^p$ the equation $\mathtt{f}(x,y) = 0$ has only isolated solutions in y. We give detailed proof just of this fact since its proof requires different computations.

First, we have to give a selection property for the sequences $(x_1, ..., x_p)$: we will say that a sequence $(x_1, ..., x_p)$, given an integer $l \le p$ satisfy condition Z, if one of the following holds:

- p is odd
- p is even and $l \neq p/2$
- p is even, $l = p/2$ and for any permutation τ of $1, ..., p/2$ exists j s.t. $x_{\tau(j)} \neq x_{p/2+j}$.

Lemma 6.2.9. *For any $(x_1, ..., x_p) \in [0,1]^p$ fulfilling condition Z, the equation $\mathtt{f}(x,y) = 0$ has a discrete set of solution.*

Proof. Following [48] we remark that since $\mathtt{f}$ is an analytic function of y, its roots can have accumulation points only if the function identically vanishes. We compute its Taylor expansion at $y = 0$ and show that it can be identically zero only if condition Z is violated. Denote

$$\nu_j = \sqrt{x_j^2 y + x_j} \ ,$$

by direct computation, we get

$$\frac{d^k \nu_j}{dy^k} = c_k \left(\frac{x_j}{\nu_j}\right)^{2k} \nu_j \ ,$$

with suitable constants c_k. Thus we have

$$\frac{\partial^k \mathtt{f}}{\partial y^k} = c_k \left[\sum_{j=1}^{j} \left(\frac{x_j}{\nu_j^2}\right)^{2k} \nu_j - \sum_{j=l+1}^{p} \left(\frac{x_j}{\nu_j^2}\right)^{2k} \nu_j\right] . \tag{6.2.19}$$

Let consider the equivalence relation

$$x_i \sim x_j \iff (x_i = x_j \text{ and } (i, j \leq l \text{ } or \text{ } i, j > l))$$

and denote by n_j the cardinality of the correspondent equivalence classes. So, we can write the condition (6.2.19) $= 0$ (renaming the indexes j), as

$$0 = \sum_{j=1}^{l_1} n_j \left(\frac{x_j}{\nu_j}\right)^{2k} \nu_j - \sum_{j=l_1+1}^{p_1} n_j \left(\frac{x_j}{\nu_j}\right)^{2k} \nu_j \ , \tag{6.2.20}$$

and now we have that $\forall i \neq j$ $x_i \neq x_j$. Remark that in our computation we have implicitly erased the terms with a plus sign with a corresponding term with a minus sign, and that. Moreover, there must be at least a couple of indexes i, j with $i \leq l$ and $j > l$ such that $x_i \neq x_j$, so that not all the n_j's vanish since condition Z is fulfilled.

Now, (6.2.20) is a linear equation in the unknown n_j and its determinant must vanish to have nontrivial solutions. However, it is a Vandermonde determinant, that can be explicitly computed, giving

$$\nu_1 \ldots \nu_{p_1} \prod_{1 \leq k < l \leq p_1} \left(\frac{x_l^2}{\nu_l^2} - \frac{x_k^2}{\nu_k^2}\right) \neq 0 \ .$$

This leads to a contradiction. □

As anticipated, the rest of the proof follows exactly as in [48] and thus is omitted.

6.2.3 Semilinear beam equation

A third result concerns the beam equation

$$\psi_{tt} + \Delta^2 \psi + m\psi = -\partial_\psi F(x, \psi) \ , \tag{6.2.21}$$

with $F \in C^\infty(M \times \mathcal{U}), \mathcal{U} \subset \mathbb{R}$ being a neighbourhood of the origin, and $m > 0$ a real positive parameter that we will call *mass*. We will assume F to have a zero of order at least 2 at $\psi = 0$. The precise statement of the main theorem is the following.

Theorem 6.2.10. *There exists a set of zero measure $\mathcal{M}^{(res)} \subset \mathbb{R}^+$ such that if $m \in \mathbb{R}^+ \setminus \mathcal{M}^{(res)}$ then for all $r \in \mathbb{N}$ there exist $s_r > d/2$ such that the following holds. For any $s > s_r$ there exist ϵ_{rs}, c, C such that if the initial datum for* (6.2.21) *fulfills*

$$\epsilon := \left\|(\psi_0, \dot{\psi}_0)\right\|_s := \|\psi_0\|_{H^{s+2}} + \left\|\dot{\psi}_0\right\|_{H^s} < \epsilon_{sr} \ ,$$

then the corresponding solution satisfies

$$\left\| \left(\psi(t), \dot{\psi}(t) \right) \right\|_s \le C\epsilon \,, \quad \textit{for} \quad |t| \le c\epsilon^{-r} \,.$$

Introducing the variable $\varphi = \dot{\psi} \equiv \psi_t$, it is well known that (6.2.21) is an Hamiltonian system in the variables (ψ, φ), with Hamiltonian function

$$H(\psi, \varphi) := \int_M \left(\frac{\varphi^2}{2} + \frac{\psi(\Delta_g^2 + m)\psi}{2} + P(x, \psi) \right) dx \,.$$

To prove Theorem 6.2.10 we first show how to put the system in the form (5.1.2) and then we prove that the Hypotheses of Theorem 5.1.4 are verified.
We introduce new variables [16]

$$u(x) := \frac{1}{\sqrt{2}} \left(\left(\Delta_g^2 + m \right)^{1/4} \psi + i \left(\Delta_g^2 + m \right)^{-1/4} \varphi \right) \,,$$
$$\overline{u}(x) := \frac{1}{\sqrt{2}} \left(\left(\Delta_g^2 + m \right)^{1/4} \psi - i \left(\Delta_g^2 + m \right)^{-1/4} \varphi \right) \,,$$

such that the Hamiltonian takes the form

$$H(u, \bar{u}) := \int_M \bar{u} \left(H_L u \right) + P(x, u, \bar{u}) \, dx$$

with

$$H_L = \sqrt{-\Delta_g^2 + m} \,.$$

We point out that H_L is a globally integrable quantum system; in particular, its actions are given by the actions of the Laplacian and the associated function is

$$\mathbb{R}^d \ni \xi \mapsto h_L(\xi) = \sqrt{h_{L0}^2(\xi) + m} \,.$$

Moreover, the frequencies $\{\omega_a\}_{a \in \Lambda}$ are given by

$$\omega_a := \sqrt{\lambda_a^2 + m} \,.$$

As for the other applications, Hypotheses L.1 and L.3 hold since they hold for $\{\lambda_a\}_{a \in \Lambda}$ and

$$\omega_a = \lambda_a \left(1 + \frac{m}{2\lambda_a^2} + o\left(\frac{m}{\lambda_a^2} \right) \right) \,.$$

The verification of non-resonance condition L.2 is again a straightforward application of Theorem 6.2.8, thus we omit the details.

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