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Hilbert curves of quadric fibrations over smooth surfaces

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Dedicated to Giorgio Ottaviani on the occasion of his 60th birthday

Abstract. *Let (X, L) be a complex polarized n-fold with the structure of a geometric quadric fibration over a smooth projective surface. The Hilbert curve of (X, L) is a complex affine plane curve of degree n, containing $n - 3$ evenly spaced parallel lines. This paper is devoted to a detailed study of the cubic representing the residual component. Reducibility, existence of triple points, and properties of the irreducible components are analyzed in connection with the structure of (X, L).*

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1. Introduction

The Hilbert curve of a polarized manifold (X, L) with $\dim(X) = n \geq 2$ is the complex affine plane curve $\Gamma = \Gamma_{(X,L)}$, of degree n, defined by the Hilbert-like polynomial $\chi(xK_X + yL)$, where K_X is the canonical bundle of X and x and y are regarded as complex variables. This notion was introduced in [3] and extensively studied in [10, 11, 13, 6] for varieties which are special from the adjunction theoretic point of view. The natural expectation is that several properties of the polarized manifold that one considers are encoded by its Hilbert curve, as suggested by [3, Theorem 6.1]. In particular, if X is endowed with a fibration $\varphi : X \to Y$ over a normal variety Y of dimension $m < n - 1$ and $K_X + (n - m)L = \varphi^* A$, for some ample $\mathbb{Q}$-line bundle A on Y, then Γ contains $n - m - 1$ parallel lines of prescribed equations as components, and therefore it becomes important to understand the properties of the residual curve of the union of such lines in Γ.

In this paper, relying on our previous study of the Hilbert curve of threefolds which are conic fibrations over a smooth surface [6], we investigate n-dimensional pairs (X, L) with $n \geq 4$, where X is a quadric fibration in the classical sense over a smooth surface and L makes it an adjunction theoretic quadric fibration at the same time. We refer to pairs (X, L) of this type as geometric quadric fibrations. In this setting, $\Gamma = \ell_1 + \cdots + \ell_{n-3} + C$, where the ℓ_i's are certain $n-3$ parallel lines and C is the residual cubic; moreover, both Γ and C are Serre-invariant, i.e. invariant under the involution induced on the affine plane by Serre duality on X.

In order to make the equation of C explicit in terms of the numerical invariants associated with (X, L) (Proposition 3.1) we describe X as a divisor of relative degree 2 inside the projective bundle defined by $\mathcal{E} := \varphi_* L$, where $\varphi : X \to S$ is the fibration morphism. The whole Section 3 is devoted to computations involving Chern classes which lead to the equation of C. Various consequences of these computations are discussed in Section 4 and Section 5. A first crucial implication is that the projective closure of our cubic C intersects the line at infinity transversely at a special point, say P_∞, whose homogeneous coordinates depend on n (Proposition 4.2). The cubic C is irreducible in general. The above property allows us to prove that C contains a special line of the affine plane whose direction is given by P_∞ if and only if the quadric fibration has no singular fibers, and also to characterize the existence of a triple point for C in terms of the structure and the numerical invariants of X (Theorem 4.5). This provides a complete generalization of [6, Theorem 5.2]. Moreover, this in turn leads to investigate other significant circumstances, for instance, under what conditions: a) Γ is nonreduced (Proposition 4.7), b) C is reducible (Corollary 5.3), c) C contains a general line, at least in the case when (X, L) is a 4-dimensional geometric quadric fibration over $\mathbb{P}^2$ (Proposition 5.4).

Next we consider a special class of geometric quadric fibrations that we call "deriving from cones", in view of their construction (Section 6). They generalize the geometric conic fibrations studied in [6, Section 6]. When the base surface S of such a pair (X, L) is a minimal surface of Kodaira dimension zero, we prove that the residual cubic of the Hilbert curve is always irreducible unless $n \geq 4$, S is abelian or bielliptic and the Chern classes of the vector bundle $\mathcal{E}$ satisfy a precise numerical condition depending on n (Theorem 6.2). In particular, this result amends the sentence given for $n = 3$ in [6, Proposition 6.3 (ii)] and at the same time provides a generalization to higher dimensions.

Clearly, $\Gamma = C$ for $n = 3$, and several results established here for C specialize to those proven for Γ in [6]. As it is natural to expect, passing from threefolds to varieties of higher dimensions, new situations arise, for instance this happens when we investigate the nonreducedness of Γ (Proposition 4.7). This fact makes case $n = 4$ particularly relevant in our study. For this reason, in Section 7 we discuss several examples in the setting of fourfolds, taking also advantage of the fact that the Riemann–Roch formula, which is crucial to determine the equation of Γ, is still handleable for $n = 4$. In particular, we discuss three types of geometric quadric fibrations (X, L) whose underlying varieties X arise in the classification of Fano fourfolds of index 2 with Picard number ≥ 2 [14]. For all of them the residual cubic C is reducible, containing a line that depends on the polarization L.

In Section 8, in the framework of plane cubic curves we provide a unifying perspective for residual cubics of our Γ's and for Serre-invariant cubics, which constitute a dense Zariski open subset of $\mathbb{P}^5$. In particular we describe the varieties whose points represent the cubics satisfying the various properties discussed in the previous sections, like reducibility, existence of triple points, etc. . This offers a global view of the families in which the residual cubic of the Hilbert curve of a geometric quadric fibration (X, L) can fit into. It is worth noting that while the families we describe are "continua", only points with rational coordinates on them can represent a residual cubic, because, as for Γ, its equation has rational coefficients.

2. The leading idea

Let (X, L) be a quadric fibration with $\dim(X) = n$ over a smooth projective surface S, via a morphism $\varphi : X \to S$. In view of [6], we will assume that $n \geq 4$. We say that (X, L) is a *geometric quadric fibration*, to mean that the following two facts hold. 1) The morphism φ is equidimensional with connected fibers, and all of them are irreducible quadric hypersurfaces of $\mathbb{P}^{n-1}$ with L inducing the hyperplane bundle. In particular, φ is flat, and for the general fiber F of φ we have $(F, L_F) = \big(\mathbb{Q}^{n-2}, \mathcal{O}_{\mathbb{Q}}(1)\big)$, where $\mathbb{Q}^{n-2}$ stands for a smooth quadric hypersurface in $\mathbb{P}^{n-1}$. 2) $K_X + (n-2)L = \varphi^* H$ for some ample line bundle H on S. Condition 1) means that $\varphi : X \to S$ is a *fibré en quadriques* in the sense of [1] and, to emphasize the role of the polarization L we can say that (X, L) is a *classical quadric fibration*, while condition 2) says that (X, L) is also an *adjunction theoretic quadric fibration* over S (in the sense of [4, p. 81]). Thanks to Grauert's theorem, conditions 1) and 2) are enough to guarantee that $\mathcal{E} := \varphi_* L$ is a locally free sheaf, i.e. a vector bundle, of rank n on S, [9, Corollary 19.2]. If we consider its projectivization $P := \mathbb{P}(\mathcal{E})$ and we denote by ξ the tautological line bundle on it, then X is fiberwise embedded in the $\mathbb{P}^{n-1}$-bundle P as a divisor of relative degree 2; more precisely, letting $\pi : P \to S$ denote the bundle projection of P, we have that $X \in |2\xi + \pi^*\mathcal{B}|$ for some line bundle $\mathcal{B}$ on S, $\varphi = \pi|_X$, and $L = \xi_X$.

The discriminant curve of (X, L) is the possible empty curve $\mathcal{D} \subset S$ parameterizing the singular fibers of π. By [7, (3.3)] we know that $\mathcal{D} \in |2c_1(\mathcal{E}) + n\mathcal{B}|$ (for $n = 3$ see also [6, (5)]).

Let $p(x, y) = 0$ be the equation of the Hilbert curve of (X, L). Recall that $p(x, y) = \chi(xK_X + yL)$, the polynomial expressing the Euler–Poincaré characteristic of $xK_X + yL$, when x and y are regarded as complex variables.

According to [3, Theorem 6.1], we have that

$$p(x,y) = \prod_{i=1}^{n-3} \big((n-2)x - y - i\big) R(x,y), \tag{1}$$

where $R(x,y)$ is a polynomial of degree 3. From the qualitative point of view, this means that the Hilbert curve Γ of (X,L) can be written as

$$\Gamma = \ell_1 + \cdots + \ell_{n-3} + C, \tag{2}$$

i.e., it consists of $n-3$ evenly spaced parallel lines with slope $(n-2)$ (the nef value of (X,L)) plus a cubic C, which we call the *residual cubic*.

We call Serre involution the map $s : \mathbb{A}^2 \to \mathbb{A}^2$ sending (x,y) to $(1-x,-y)$, induced by Serre duality. Note that Γ is *Serre-invariant*, i.e., invariant under s. Moreover, s exchanges the line ℓ_i of equation $(n-2)x - y - i = 0$ with ℓ_{n-2-i} $(i = 1, \ldots, n-3)$, hence the set consisting of the $n-3$ lines $\ell_1, \ldots, \ell_{n-3}$ is globally Serre-invariant. It thus follows that the cubic C itself is also Serre-invariant. We use coordinates (u,v) in place of $(x = \frac{1}{2} + u, y = v)$ in order to make this invariance more evident. Since the degree of C is odd, then $R(\frac{1}{2} + u, v)$ is the sum of two homogeneous polynomials in u and v of degrees 3 and 1 respectively [3, Lemma 7.1]. Thus we can write

$$R\left(\frac{1}{2} + u, v\right) = R_3(u,v) + R_1(u,v). \tag{3}$$

where

$$R_3(u,v) = \alpha u^3 + \beta u^2 v + \gamma u v^2 + \delta v^3 \tag{4}$$

with $(\alpha, \beta, \gamma, \delta) \neq (0,0,0,0)$, because $\deg C = 3$, and

$$R_1(u,v) = \sigma u + \tau v. \tag{5}$$

Note that the property of having an equation of this type characterizes any Serre-invariant plane cubic, which is not necessarily the residual cubic of Γ.

Our aim is to obtain the explicit expression of $R\left(\frac{1}{2} + u, v\right)$ in our specific case, which in particular describes our cubic C. To do that, first recall that for any divisor D on X,

$$\chi(D) = \frac{1}{n!} D^n + \ldots,$$

where the dots stand for lower degree terms. So, by using homogeneous coordinates $(x : y : z)$, where z is the homogenizing coordinate, and letting $p_0(x,y,z)$

denote the homogeneous polynomial associated to p, we have:

$$\begin{aligned} p_0(x,1,0) &= \frac{1}{n!}(xK_X+L)^n \\ &= \frac{1}{n!}\Big[d_n x^n + \binom{n}{1} d_{n-1}x^{n-1} + \binom{n}{2} d_{n-2}x^{n-2} + \dots \\ &\quad \dots + \binom{n}{n-3} d_3 x^3 + \binom{n}{n-2} d_2 x^2 + \binom{n}{n-1} d_1 x + d\Big], \end{aligned} \tag{6}$$

where $d_i := K_X^i \cdot L^{n-i}$ for $i = 0, 1, \dots, n$ ($d_0 = d$ being the degree of (X, L)). On the other hand, from (1) and (3) we see that $p_0(x,y,0) = R_3(x,y)\big((n-2)x-y\big)^{n-3}$. Hence (4) gives

$$p_0(x,1,0) = (\alpha x^3 + \beta x^2 + \gamma x + \delta)\big((n-2)x-1\big)^{n-3}. \tag{7}$$

By comparing (6) with (7), we can get the explicit expressions of α, β, γ and δ in terms of the natural invariants of (X, L). Next, recalling that $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S)$ and Serre duality, we have

$$p(1,0) = \chi(K_X) = (-1)^n \chi(\mathcal{O}_X) = (-1)^n \chi(\mathcal{O}_S). \tag{8}$$

On the other hand, from (1) and (3) we get

$$p(1,0) = \prod_{i=1}^{n-3}(n-2-i)\Big(\frac{\alpha}{8} + \frac{\sigma}{2}\Big) = \frac{(n-3)!}{8}\,(\alpha + 4\sigma). \tag{9}$$

So, taking into account the previous discussion, we obtain the expression of σ. It remains to determine τ. To do it, recall that $K_X + (n-2)L = \varphi^* H$. We have, for every $i \geq 0$,

$$H^i\big(K_X + (n-2)L\big) = H^i(\varphi^* H) \cong H^i\big(\varphi_*(\varphi^* H)\big) = H^i(H). \tag{10}$$

The last equality will follow once we prove that $R^i\varphi_*(\varphi^* H)) = 0$ for $i > 0$, see [9, p. 252, Ex. 8.1].

Because by projection formula $R^i\varphi_*(\varphi^* H) \cong R^i\varphi_*\mathcal{O}_X \otimes H$, it is enough to show that $R^i\varphi_*\mathcal{O}_X = 0$ for $i > 0$. As $X \subset P$ and $X \in |2\xi + \pi^*\mathcal{B}|$, we have the following exact sequence

$$0 \to \mathcal{O}_P(-2\xi - \pi^*\mathcal{B}) \to \mathcal{O}_P \to \mathcal{O}_X \to 0, \tag{11}$$

and applying to it the direct image functor and using [9, p. 281, Ex. 11.8] we obtain the following long exact sequence

$$\begin{aligned}0 \to R^0\pi_*\mathcal{O}_P(-2\xi - \pi^*\mathcal{B}) \to R^0\pi_*\mathcal{O}_P \to R^0\varphi_*\mathcal{O}_X \to \\ R^1\pi_*\mathcal{O}_P(-2\xi - \pi^*\mathcal{B}) \to \cdots\cdots\cdots \to R^{n-2}\varphi_*\mathcal{O}_X \to \\ R^{n-1}\pi_*\mathcal{O}_P(-2\xi - \pi^*\mathcal{B}) \to R^{n-1}\pi_*\mathcal{O}_P \to R^{n-1}\varphi_*\mathcal{O}_X = 0,\end{aligned} \tag{12}$$

the last equality coming from the fact that the fibers of φ have dimension $n-2$. By [9, p. 253, Ex. 8.4 (a)] we thus conclude that $\varphi_*\mathcal{O}_X = R^0\varphi_*\mathcal{O}_X = R^0\pi_*\mathcal{O}_P = \mathcal{O}_S$ and $R^i\varphi_*\mathcal{O}_X = R^i\pi_*\mathcal{O}_P = 0$ for $i > 0$. Therefore,

$$p(1, n-2) = \chi(K_X + (n-2)L) = \chi(\varphi^* H) = \chi(H), \tag{13}$$

in view of (10). Now, recalling the canonical bundle formula for $\mathbb{P}$-bundles, by adjunction we have

$$\begin{aligned} K_X = (K_P + X)_X &= \big(- n\xi + \pi^*(K_S + \det\mathcal{E}) + 2\xi + \pi^*\mathcal{B}\big)_X \\ &= \big(- (n-2)\xi + \pi^*(K_S + \det\mathcal{E} + \mathcal{B})\big)_X \\ &= -(n-2)L + \varphi^*(K_S + \det\mathcal{E} + \mathcal{B}). \end{aligned} \tag{14}$$

Hence, due to the injectivity of the homomorphism induced by φ between the Picard groups of S and X, we get

$$H = K_S + c_1(\mathcal{E}) + \mathcal{B}. \tag{15}$$

Thus (13) allows us to express $p(1, n-2)$ in terms of K_S, $c_1(\mathcal{E})$ and $\mathcal{B}$ via the Riemann–Roch theorem. On the other hand, from (1)–(5) we get

$$\begin{aligned} p(1, n-2) &= \prod_{i=1}^{n-3} \big(n - 2 - (n-2) - i\big) R(1, n-2) \\ &= (-1)^{n-3}(n-3)! \, \Big(\frac{\alpha}{8} + \frac{\beta}{4}(n-2) + \frac{\gamma}{2}(n-2)^2 \\ &\qquad + \delta(n-2)^3 + \frac{\sigma}{2} + \tau(n-2)\Big). \end{aligned} \tag{16}$$

So (13) and (16) give another equation, which, added to the previous ones, allows us to determine τ. For the explicit computations see Section 3, which leads to Proposition 3.1.

3. Some computations

First of all we make explicit the coefficients of some of the powers of x from (6), and precisely

$\operatorname{coeff}(x^n) = \frac{1}{n!} d_n,$
$\operatorname{coeff}(x^2) = \frac{1}{2(n-2)!} d_2,$
$\operatorname{coeff}(x) = \frac{1}{(n-1)!} d_1,$
$\operatorname{coeff}(1) = \frac{1}{n!} d.$

On the other hand, doing the same with (7), we get:

$\operatorname{coeff}(x^n) = (n-2)^{n-3}\alpha,$
$\operatorname{coeff}(x^2) = (-1)^{n-5}\binom{n-3}{2}(n-2)^2\delta + (-1)^{n-4}(n-3)(n-2)\gamma + (-1)^{n-3}\beta,$
$\operatorname{coeff}(x) = (-1)^{n-4}(n-3)(n-2)\delta + (-1)^{n-3}\gamma,$
$\operatorname{coeff}(1) = (-1)^{n-3}\delta.$

So, by equating the corresponding expressions of the coefficients of x^n, x^2, $x, 1$, we obtain:

$$\alpha = \frac{1}{n!(n-2)^{n-3}}\ d_n, \tag{17}$$

$$\delta = (-1)^{n-1}\frac{1}{n!}\ d, \tag{18}$$

$$\gamma = (-1)^{n-1}\frac{1}{(n-1)!}\left(d_1 + \frac{(n-2)(n-3)}{n}\ d\right), \tag{19}$$

$$\beta = (-1)^{n-1}\left(\frac{1}{2(n-2)!}\ d_2 + \frac{(n-2)(n-3)}{(n-1)!}\ d_1 + \frac{(n-2)^3(n-3)}{2n!}\ d\right). \tag{20}$$

This shows that the values of d, d_1, d_2 and d_n are enough to compute the coefficients of $R_3(u,v)$. As to $R_1(u,v)$, combining (8), (9) and (17) it follows that

$$\sigma = \frac{(-1)^n 8}{4(n-3)!}\ \chi(\mathcal{O}_S) - \frac{\alpha}{4} = (-1)^n\frac{2}{(n-3)!}\chi(\mathcal{O}_S) - \frac{1}{4n!(n-2)^{n-3}}d_n. \tag{21}$$

Furthermore, using (13), (16) it follows that

$$\tau = \frac{(-1)^{n-3}\chi(H)}{(n-2)!} - \frac{1}{(n-2)}\left(\frac{\alpha}{8} + \frac{\beta(n-2)}{4} + \frac{\gamma(n-2)^2}{2} + \delta(n-2)^3 + \frac{\sigma}{2}\right)$$

and plugging in such expression the values from (17), (20), (19), (18), (21) we get

$$\tau = \frac{(-1)^{n-1}}{(n-2)!}\left(\chi(H) + \chi(\mathcal{O}_S) - \frac{1}{8}d_2 - \frac{n-2}{4}d_1 - \frac{(n-2)^2(n^2-n+2)}{8n(n-1)}d\right). \tag{22}$$

Hence, for the time being, we obtain the following expression for the polynomial defining the residual cubic C:

$$
\begin{aligned}
R\left(\frac{1}{2}+u,v\right) =& \frac{1}{n!(n-2)^{n-3}}\, d_n u^3 \\
&+(-1)^{n-1}\Big(\frac{1}{2(n-2)!}d_2+\frac{(n-2)(n-3)}{(n-1)!}d_1+\frac{(n-2)^3(n-3)}{2n!}d\Big)u^2v \\
&+(-1)^{n-1}\frac{1}{(n-1)!}\left(d_1+\frac{(n-2)(n-3)}{n}d\right)\, uv^2+(-1)^{n-1}\frac{1}{n!}dv^3 \\
&+\Big((-1)^n\frac{2}{(n-3)!}\chi(\mathcal{O}_S)-\frac{1}{4n!(n-2)^{n-3}}d_n\Big)u \\
&+\frac{(-1)^{n-1}}{(n-2)!}\Big(\chi(H)+\chi(\mathcal{O}_S)-\frac{1}{8}d_2-\frac{n-2}{4}d_1-\frac{(n-2)^2(n^2-n+2)}{8n(n-1)}d\Big)v.
\end{aligned}
$$

To determine the value of d_i we need several computations involving Chern classes. From now on, for simplicity we set $c_i = c_i(\mathcal{E})$, $i = 1, 2$. First of all, we recall the following facts. In the projective bundle $P := \mathbb{P}(\mathcal{E})$, since $\dim(P) = n+1$, for any divisors $\mathcal{D}_1, \mathcal{D}_2$ on S, we have

$$\xi^n\pi^*\mathcal{D}_1 = c_1\mathcal{D}_1 \qquad \text{and} \qquad \xi^{n-1}\pi^*\mathcal{D}_1\pi^*\mathcal{D}_2 = \mathcal{D}_1\mathcal{D}_2.$$

Moreover, according to the Chern–Wu relation

$$\xi^n - \xi^{n-1}\pi^*c_1 + \xi^{n-2}\pi^*c_2 = 0,$$

we get

$$\xi^n = \xi^{n-1}\pi^*c_1 - \xi^{n-2}\pi^*c_2 \qquad \text{and} \qquad \xi^{n+1} = c_1^2 - c_2. \tag{23}$$

Then standard computations relying on the above relations lead to the following expression:

$$d = L^n = 2\big(c_1^2 - c_2\big) + c_1\mathcal{B}. \tag{24}$$

Next, recalling that X is contained in P as an element of $|2\xi + \pi^*\mathcal{B}|$ and the expression of K_X given by (14), we get

$$
\begin{aligned}
d_1 = K_XL^{n-1} =& 2(n-2)c_2 - 2(n-3)c_1^2 + 2K_Sc_1 \\
&- (n-5)c_1\mathcal{B} + K_S\mathcal{B} + \mathcal{B}^2,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
d_2 = K_X^2L^{n-2} =& 2(n-3)^2c_1^2 - 2(n-2)^2c_2 - 4(n-3)K_Sc_1 \\
&+ 2K_S^2 - 2(n-4)K_S\mathcal{B} + (n^2-10n+20)c_1\mathcal{B} - 2(n-3)\mathcal{B}^2,
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
d_n = K_X^n = (-1)^n(n-2)^{n-2}\Big[&(n^2-5n+8)c_1^2 - 2(n-2)^2c_2 \\
&+ n(n-1)K_S^2 + 2nK_Sc_1 + n^2K_S\mathcal{B} + 4c_1\mathcal{B} + n\mathcal{B}^2\Big]\,.
\end{aligned} \tag{27}
$$

Plugging (24), (25), (26), (27), in (21) and (22), respectively, we see that

$$\sigma = \tfrac{(-1)^n}{(n-3)!}\, 2\chi(\mathcal{O}_S) - \tfrac{(-1)^n(n-2)}{4n!}\Big[(n^2-5n+8)c_1^2 - 2(n-2)^2c_2 \\ + n(n-1)K_S^2 + 2nK_Sc_1 + n^2K_S\mathcal{B} + 4c_1\mathcal{B} + n\mathcal{B}^2\Big] \tag{28}$$

and

$$\begin{aligned}\tau &= \tfrac{(-1)^n}{4}\Big(\tfrac{(3n^2-9n+8)}{n!}c_1^2 - \tfrac{2(n-2)^2}{n!}c_2 + \tfrac{(3n^2-6n+4)}{n!}c_1\mathcal{B} \\ &\quad + \tfrac{1}{(n-2)!}(\mathcal{B}^2 + 2K_S\mathcal{B}) + \tfrac{1}{(n-2)!}\big(K_S^2 - 4\chi(\mathcal{O}_S) - 4\chi(H)\big)\Big) \\ &= \tfrac{(-1)^n}{4}\Big[\tfrac{(n^2-7n+8)}{n!}c_1^2 - \tfrac{2(n-2)^2}{n!}c_2 - \tfrac{(n^2+2n-4)}{n!}c_1\mathcal{B} \\ &\quad + \tfrac{1}{(n-2)!}\big(K_S^2 - \mathcal{B}^2 - 8\chi(\mathcal{O}_S)\big)\Big],\end{aligned} \tag{29}$$

after replacing $\chi(H)$ with its expression provided by the Riemann–Roch theorem.

Similarly, plugging (24), (25), (26), (27), in (17), (20), (19), (18), respectively, we see that

$$\alpha = \frac{(-1)^n(n-2)}{n!}\Big[(n^2-5n+8)c_1^2 - 2(n-2)^2c_2 \\ + n(n-1)K_S^2 + 2nK_Sc_1 + n^2K_S\mathcal{B} + 4c_1\mathcal{B} + n\mathcal{B}^2\Big], \tag{30}$$

$$\delta = (-1)^{n-3}\frac{1}{n!}\Big(2c_1^2 - 2c_2 + c_1\mathcal{B}\Big), \tag{31}$$

$$\gamma = \frac{(-1)^{n-1}}{n!}\Big[4(3-n)c_1^2 + 6(n-2)c_2 + 6c_1\mathcal{B} \\ + n(2K_Sc_1 + K_S\mathcal{B} + \mathcal{B}^2)\Big], \tag{32}$$

$$\beta = (-1)^{n-1}\Big[\frac{1}{n!}(3n^2-17n+24)c_1^2 - \frac{6}{n!}(n-2)^2c_2 + \frac{1}{(n-2)!}K_S^2 \\ - \frac{(n-3)}{(n-1)!}(2K_Sc_1 + \mathcal{B}^2) + \frac{2}{(n-1)!}K_S\mathcal{B} - \frac{(n^2+2n-12)}{n!}c_1\mathcal{B}\Big]. \tag{33}$$

The above discussion proves the following result.

Proposition 3.1. *Let (X, L) be a geometric quadric fibration over a smooth surface S, as in Section 2. Then the residual cubic of its Hilbert curve is defined by* (3)*, where the homogeneous part of degree* 3 *is*

$$\begin{aligned}R_3(u,v) = \frac{(-1)^{n-1}}{n!}\Big\{ & -(n-2)\Big[(n^2-5n+8)c_1^2 - 2(n-2)^2c_2 \\ & + n^2K_S\mathcal{B} + n(n-1)K_S^2 + 2nK_Sc_1 + 4c_1\mathcal{B} + n\mathcal{B}^2\Big]u^3 \\ & + \Big[(3n^2-17n+24)c_1^2 - 6(n-2)^2c_2 + n(n-1)K_S^2 \\ & - n(n-3)(2K_Sc_1+\mathcal{B}^2) + 2nK_S\mathcal{B} - (n^2+2n-12)c_1\mathcal{B}\Big]u^2v \\ & + \Big[4(3-n)c_1^2 + 6(n-2)c_2 + 6c_1\mathcal{B} + n(2K_Sc_1 + K_S\mathcal{B} + \mathcal{B}^2)\Big]uv^2 \\ & + \big(2c_1^2 - 2c_2 + c_1\mathcal{B}\big)v^3\Big\},\end{aligned} \tag{34}$$

while the homogenous part of degree 1 *is*

$$\begin{aligned}R_1(u,v) = \frac{(-1)^n}{4n!}\Big\{ & \Big(8n(n-1)(n-2)\chi(\mathcal{O}_S) - (n-2)\Big[(n^2-5n+8)c_1^2 \\ & - 2(n-2)^2c_2 + n(n-1)K_S^2 + 2nK_Sc_1 + n^2K_S\mathcal{B} + 4c_1\mathcal{B} \\ & + n\mathcal{B}^2\Big]\Big)u + \Big[(n^2-7n+8)c_1^2 - 2(n-2)^2c_2 - (n^2+2n-4)c_1\mathcal{B} \\ & + 4n(n-1)\big(K_S^2 - \mathcal{B}^2 - 8\chi(\mathcal{O}_S)\Big]v\Big\}.\end{aligned} \tag{35}$$

We like to point out that if we plug $n = 3$ in (34) and (35) then their sum gives the equation of Γ in [6, Proposition 4.1].

4. First properties of the Hilbert curve

Let ℓ_∞ be the line at infinity of the (u, v) plane. We denote by ℓ_0 the line of equation $(n-2)u - v = 0$, whose point at infinity is $P_\infty := (1 : n-2 : 0)$.

LEMMA 4.1. *Let C be any Serre-invariant plane cubic and let* (3) *be its equation, with R_3 and R_1 given by* (4) *and* (5) *respectively.*
a) *The projective closure $\overline{C}$ of C contains the point P_∞ if and only if*

$$\alpha + (n-2)\beta + (n-2)^2\gamma + (n-2)^3\delta = 0. \tag{36}$$

b) *C contains the line ℓ_0 if and only if, in addition to* (36)*, we have*

$$\sigma + (n-2)\tau = 0. \tag{37}$$

Proof. If we put $v = (n-2)u$ in (3) then (36) and (37) express the vanishing of the homogeneous parts of degree 3 and 1 of the polynomial in (3), respectively. This proves a) and b). □

The computations done for Proposition 3.1 have the following consequence.

PROPOSITION 4.2. *Let (X, L) be a geometric quadric fibration over S as in Section 2 and let C be the residual cubic of its Hilbert curve. Then condition* (36) *is always satisfied for C. Moreover, $\overline{C}$ intersects ℓ_∞ at P_∞, transversely.*

Proof. If we plug the values (30), (33), (32), (31) in (36) we get an expression involving c_1^2, c_2, K_S^2, $K_S c_1$, $K_S\mathcal{B}$, $\mathcal{B}^2$, $c_1\mathcal{B}$, with appropriate coefficients. At a close look such coefficients are all zeroes, hence our former claim follows. To prove the latter, suppose, by contradiction, that $\overline{C}$ intersects ℓ_∞ at P_∞ with multiplicity > 1. Then, dividing $R_3(u, v)$ by u^3 and letting $t := v/u$, the value $n-2$ has to be a common root of the polynomial

$$\delta t^3 + \gamma t^2 + \beta t + \alpha$$

and of its derivative. Hence

$$3(n-2)^2\delta + 2(n-2)\gamma + \beta = 0. \tag{38}$$

However, taking into account (31), (32) and (33), the term on the left hand side of (38) becomes

$$3(n-2)^2\delta + 2(n-2)\gamma + \beta = \frac{(-1)^n}{(n-2)!}(K_S + c_1 + \mathcal{B})^2,$$

and since $H = K_S + c_1 + \mathcal{B}$ is the ample divisor in (15), this cannot be zero, a contradiction. □

As to the residual intersections of $\overline{C}$ with ℓ_∞ we have the following result.

PROPOSITION 4.3. *Let C be a Serre-invariant plane cubic as in Lemma 4.1 such that $P_\infty \in \overline{C}$, and let Q_∞ be a point at infinity distinct from P_∞. The cubic $\overline{C}$ intersects ℓ_∞ at Q_∞ with multiplicity* 2 *if and only if the following condition*

$$4\alpha\delta + (n-2)\big(\gamma + (n-2)\delta\big)^2 = 0 \tag{39}$$

is satisfied, in addition to (36). *Moreover, in this case, if $\overline{C}$ is irreducible, then $\overline{C}$ is singular at Q_∞.*

Proof. Let $Q_\infty = (-a : b : 0)$. Dividing $R_3(u, v)$ by u^3 and letting $t = v/u$, as before, we see that

$$\overline{C} \cap \ell_\infty = P_\infty + 2Q_\infty \tag{40}$$

if and only if

$$\delta t^3 + \gamma t^2 + \beta t + \alpha = (t - n + 2)(at + b)^2, \tag{41}$$

identically with respect to t. This is equivalent to

$$\delta = a^2, \quad \gamma = 2ab - (n-2)a^2, \quad \beta = b^2 - 2ab(n-2), \quad \alpha = -(n-2)b^2, \tag{42}$$

and eliminating a, b from these equations gives (36) and (39). Now suppose that $\overline{C}$ is irreducible and smooth at Q_∞. The Serre involution $(u, v) \mapsto (-u, -v)$ induces an involution $\iota : \overline{C} \to \overline{C}$ such that $\iota(Q_\infty) = Q_\infty$. Then, as in [3, Lemma 3.3], we see that either ι is the identity map, or its differential, acting on the tangent space $T_{Q_\infty}(\overline{C})$ to $\overline{C}$ at Q_∞, is the multiplication by -1. But the projective closure of $T_{Q_\infty}(\overline{C})$ is ℓ_∞ itself because the intersection index of $\overline{C}$ and ℓ_∞ at Q_∞ is 2. We thus get a contradiction since the Serre involution induces the identity on ℓ_∞ but not on $\overline{C}$. It thus follows that Q_∞ is a singular point of $\overline{C}$. □

Actually, more can be said about the singular point Q_∞.

Proposition 4.4. *Let C be a Serre-invariant plane cubic such that $P_\infty \in \overline{C}$. If C is irreducible and Q_∞ is a double point of $\overline{C}$, then Q_∞ is necessarily a node. Moreover the Serre involution exchanges the principal tangents to $\overline{C}$ at Q_∞.*

Proof. Let $Q_\infty = (-a : b : 0)$ be a double point and suppose that $a \neq b$. Up to the change of homogenous coordinates $u = U - aW, v = V + bW, w = U + V$, with $a \neq b$ we can assume that Q_∞ is the origin. In the new affine coordinates U, V (if we set $W = 1$) the equation of $\overline{C}$, after using (42), is:

$$(bU + aV)^2[b + (n-2)a] - (U + V)^2(a\sigma - b\tau) + [\sigma - b^2(n-2)]U^3 + [b^2 - 2ab(n-2) + 2\sigma + \tau]U^2V + [2ab - a^2(n-2) + \sigma + 2\tau]UV^2 + (a^2 + \tau)V^3 = 0.$$

The coefficient of the first term is not zero because $Q_\infty \neq P_\infty$. We thus see that Q_∞ is a cusp, the line of equation $bU + aV = 0$ being the unique principal tangent to $\overline{C}$ at Q_∞, if and only if $\sigma a - \tau b = 0$. Next, note that the point O, the origin of the affine coordinates (u, v), is a smooth point of C, due to the assumptions. Thus the condition $\sigma a - \tau b = 0$ is equivalent to saying that the line tangent to $\overline{C}$ at O (whose equation is $\sigma u + \tau v = 0$) contains the point Q_∞. But then, the intersection index of this line and $\overline{C}$ would be greater than 3 (2 intersections at O, due to the tangency plus 2 intersections at least at the singular point Q_∞), a contradiction. If $a = b$, the same argument as above works by using the following change of homogeneous coordinates: $u = U + V - W$, $v = U + W$, $w = U + V$.

To see that the Serre involution exchanges the principal tangents at the node Q_∞, let $\overline{s}$ denote the extension of the Serre involution to $\mathbb{P}^2$. If ℓ is a principal tangent at Q_∞, then $\overline{s}(\ell)$ is also a principal tangent. But if $\overline{s}(\ell) = \ell$, then necessarily ℓ must contain O. This comes from the fact that the only lines fixed by $\overline{s}$ are those in the pencil through O plus ℓ_∞. The latter, however, cannot be a principal tangent to $\overline{C}$ at Q_∞, since the multiplicity of intersection is just 2. But then the intersection index of ℓ and $\overline{C}$ would be greater than 3 (1 intersection at O and 3 at Q_∞, since it is a principal tangent), a contradiction again. $\square$

The following result is a generalization of [6, Theorem 5.2].

THEOREM 4.5. *Let* (X, L) *be a geometric quadric fibration of dimension* n *over* S *and let* C *be the residual cubic of the Hilbert curve* Γ *of* (X, L). *Then*

(i) ℓ_0 *is contained in* C *if and only if* X *is a bundle.*

(ii) C *has a triple point if and only if* X *is a bundle and*

$$K_X^n + (-1)^{n-1} 8n(n-1)(n-2)^{n-2}\chi(\mathcal{O}_X) = 0.$$

(iii) If ℓ_0 *is contained in* C, *then it is an irreducible component of multiplicity* 1 *of* C.

Proof. A tedious check shows that

$$\sigma + (n-2)\tau = \frac{(-1)^{n+1}}{4n!}\, n(n-2)(2c_1 + n\mathcal{B})(K_S + c_1 + \mathcal{B}). \tag{43}$$

Recalling that the discriminant curve $\mathcal{D} \in |2c_1 + n\mathcal{B}|$ and that $H = K_S + c_1 + \mathcal{B}$ is the ample divisor in (15) this shows that

$$\sigma + (n-2)\tau = \frac{(-1)^{n+1}}{4n!}\, n(n-2)\mathcal{D}H. \tag{44}$$

Therefore $\sigma + (n-2)\tau = 0$ if and only if $\mathcal{D} = 0$, i.e. X has no singular fibers. Then *(i)* follows from Lemma 4.1, b) taking into account Proposition 4.2. To prove *(ii)* note that if C has a triple point, then the origin must be a triple point, and this happens if and only if $\sigma = \tau = 0$. This is equivalent to $\sigma = \sigma + (n-2)\tau = 0$ and we know from *(i)* that the latter of these two conditions is equivalent to X being a bundle, hence to the fact that $\mathcal{B} = -\frac{2}{n}c_1$. Replace $\mathcal{B}$ with this value in the expression of σ provided by (21). Recalling that $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S)$ since X is a bundle, we thus get *(ii)*. Finally, *(iii)* follows from the latter assertion in Proposition 4.2. $\square$

The next question we want to address is about the nonreducedness of Γ, where Γ is the Hilbert curve of a geometric quadric fibration (X, L) as in Section 2. First of all consider the residual cubic C. As a consequence of Theorem 4.5 we have the following result.

COROLLARY 4.6. *Let* (X, L) *be a geometric quadric fibration of dimension* n *over* S *and let* C *be the residual cubic of the Hilbert curve* Γ *of* (X, L). *Then* C *is nonreduced if and only if* $C = \ell_0 + 2\ell'$, *with* ℓ' *a line through the origin transverse to* ℓ_0. *This happens if and only if* (40) *holds, where* $Q_\infty \neq P_\infty$, *i.e. if and only if, letting* $Q_\infty = (-a : b : 0)$, *the coefficients of* $R_3(u, v)$ *satisfy conditions* (36) *and* (39).

Proof. Suppose that C is nonreduced. Clearly for no line ℓ it can happen that $C = 3\ell$, in view of Proposition 4.2. Therefore $C = \ell + 2\ell'$ where ℓ and ℓ' are two distinct lines, which cannot be parallel, by Proposition 4.2. Thus C has a single triple point at $\ell \cap \ell'$, which necessarily has to be the origin, and then Theorem 4.5 and Proposition 4.2 again imply that X is a bundle and $\ell = \ell_0$. Moreover, (40) holds, where Q_∞ is the point at infinity of ℓ'. Then the last assertion follows from Proposition 4.3. The converse is obvious. □

For an example of the situation described in Corollary 4.6, see [6, Example 5.3, case (ii) on p. 556 and Remark 5.4].

Next look at Γ. In view of Corollary 4.6 we can suppose that C is reduced. Assume that $\Gamma = \ell_1 + \cdots + \ell_{n-3} + C$ is nonreduced; then $C = \ell + \gamma$ is necessarily reducible into a line ℓ and a conic γ which could possibly be reducible. Recall that $\ell_1, \ldots, \ell_{n-3}$ have the same point at infinity, which is P_∞. Due to Proposition 4.2 there are two possibilities: either

i) P_∞ is the point at infinity of ℓ but it does not belong to $\overline{\gamma}$, or

ii) P_∞ is a point at infinity of γ but not of ℓ.

In case i), even if γ is reducible no line constituting γ can overlap one of the ℓ_i's, having a point at infinity distinct from P_∞. On the other hand, ℓ has to contain the origin O regardless of the rank of γ, in view of the symmetry of C, hence $\ell =< O, P_\infty >= \ell_0$. Therefore ℓ_0 must coincide with one of the ℓ_i's ($i = 1, \ldots, n-3$). Since ℓ_i is described, in coordinates u, v by $(n-2)u - v - (i + 1 - \frac{n}{2}) = 0$, we have that $\ell_0 = \ell_i$ if and only if $n = 2m \geq 4$ and $i = m - 1$.

In case ii), since C is reduced, the nonreducedness of Γ implies that γ is reducible in two lines, one of which, say ℓ', has P_∞ as point at infinity. Since C also contains the line ℓ, we conclude that C has a triple point, which is the origin O, due to the symmetry, hence $\ell' =< O, P_\infty >= \ell_0$. Then up to exchanging ℓ with ℓ' we fall in case i) again and we get the same conclusion.

As we have seen, if Γ is nonreduced, then $\ell_0 \subset C$ regardless of the fact that C is reduced or not; hence X is a bundle; moreover, if C is reduced, ℓ_0 is the unique irreducible multiple component of Γ.

We want to stress the following fact. Suppose that C has no triple point (or, equivalently, that γ has not a double point at the origin). This is equivalent to requiring that the polynomial $R_1(u, v)$ is not identically zero. In this case, it represents ℓ_0, hence it divides $R_3(u, v)$, since $C = \ell_0 + \gamma$. So $R_3(u, v) = Q(u, v)R_1(u, v)$, where Q is a homogeneous polynomial in u, v of degree 2 and then, recalling (3), C has equation

$$R\left(\frac{1}{2} + u, v\right) = \big(Q(u, v) + 1\big)R_1(u, v) = 0. \tag{45}$$

Therefore the conic γ is described by $Q(u,v)+1=0$; this clearly shows that its rank is ≥ 2. In particular, if equality holds, our assumptions imply that γ consist of two parallel lines, symmetric with respect to the origin. This situation does really occur, as [6, Example 7.1, equation (41) at p. 563] shows.

In conclusion, we have

PROPOSITION 4.7. *Let (X,L) be a geometric quadric fibration of dimension n over a smooth surface S as in Section 2. Its Hilbert curve Γ is nonreduced if and only if either*

1. *$C=\gamma+\ell_0$, where γ is a conic of rank ≥ 2 and $n=2m\geq 4$, or*
2. *C is non reduced.*

In both cases X is a bundle. In the former case ℓ_0 is the only multiple component of Γ and its multiplicity is 2; in the latter, $\gamma=2\ell'$, where $\ell'\neq\ell_0$ is a line; ℓ' is the only component of multiplicity 2 of Γ, unless $n=2m\geq 4$, in which case ℓ_0 is a further component of multiplicity 2.

Case *1.* in Proposition 4.7 is clearly a novelty with respect to what is known for $n=3$.

5. More on the residual cubic C

In this Section we analyze further the reducibility of the residual cubic C. More generally, we first look at reducible Serre-invariant plane cubics.

PROPOSITION 5.1. *Let $C\subset\mathbb{A}^2$ be a Serre-invariant plane cubic such that $\overline{C}$ meets ℓ_∞ transversely at P_∞, and let O be the origin of coordinates (u,v). If C is reducible then $C=\ell+\gamma$, where ℓ is a line passing through O and γ is a conic, possibly reducible. Moreover, either*

a) γ is of hyperbolic type, with center at O (in particular it has two distinct points at infinity), or

b) γ consist of two parallel lines.

Proof. Clearly, if C is reducible, then it contains a line, say ℓ. There are two possibilities: either

i) the line ℓ contains the origin O, or

ii) the line ℓ does not contain O.

We claim that in case ii) ℓ is an irreducible component of a conic residual with respect to another line, which is also contained in C, and passes through O. So, up to renaming, case ii) reduces to i), which gives the first assertion in the statement. To prove the claim, note that the map $\iota : C \to C$ induced by the Serre involution maps ℓ to another line $\iota(\ell)$, which also does not contain O. Thus C consists of three lines, two of which do not contain O, hence O belongs to the third line, say λ. Note that ℓ and its conjugate $\iota(\ell)$ are parallel, due to the symmetry properties of C. Thus their projective closures cannot contain the point $P_\infty = (1 : n-2 : 0)$, in view of the assumption on $\overline{C}$. It thus follows from Lemma 4.1 a) that P_∞ is the point at infinity of λ and therefore $\lambda = \ell_0$, since it contains both the origin and P_∞. In conclusion, in case ii) we have that $C = \ell_0 + \ell' + \ell''$, where ℓ' and ℓ'' are two parallel lines, and letting $\gamma = \ell' + \ell''$ this gives *b)* in the statement. Next come to case i). Clearly $\gamma := C - \ell$ is symmetric with respect to O. Hence γ is as in *a)* (regardless the fact that it is irreducible or not) if O is its unique center. Otherwise it is as in *b)*, since it cannot be a parabola, because it is Serre-invariant itself. □

Now let C be any Serre-invariant plane cubic. If C has a triple point then necessarily it has a triple point at the origin, hence assuming that C has not a triple point is equivalent to requiring that $(\sigma, \tau) \neq (0, 0)$. So, let C be a Serre-invariant reducible plane cubic again. Suppose that C has not a triple point. Then $R_1(u, v) = 0$ represents a line ℓ through O. Moreover, since C is reducible, $R_1(u, v)$ divides $R_3(u, v)$, hence $R_3(u, v) = Q(u, v)R_1(u, v)$, Q being a homogeneous (nontrivial) polynomial of degree two in u and v. Thus, C is described by (45). This shows that ℓ is a component of C and the conic residual of ℓ in C has rank ≥ 2, in accordance with the assumption that C has not a triple point. Now, by applying the same argument as in [6, p. 551] we see that the existence of a polynomial Q as above is equivalent to the condition

$$\sigma^2(\sigma\delta - \tau\gamma) + \tau^2(\sigma\beta - \tau\alpha) = 0. \tag{46}$$

Note that (46) is trivially satisfied also when C has a triple point. On the other hand (45) obviously implies reducibility. Therefore, we have

PROPOSITION 5.2. *Let C be a Serre-invariant plane cubic and let* (3) *be its equation, with R_3 and R_1 given by* (4) *and* (5) *respectively. Then C is reducible if and only if* (46) *holds.*

In particular, we get the following consequence.

COROLLARY 5.3. *Let (X, L) be a geometric quadric fibration over a smooth surface, as in Section 2, and let C be the residual cubic of its Hilbert curve with respect to the lines $\ell_1, \dots, \ell_{n-3}$. Then C is reducible if and only if* (46) *holds.*

For $n = 4$, assuming that $S = \mathbb{P}^2$, we can characterize the fact that C contains a given line ℓ through the origin even more explicitly. In view of Theorem 4.5*(i)*, we can suppose that $\ell \neq \ell_0$.

PROPOSITION 5.4. *Let (X,L) be a 4-dimensional geometric quadric fibration over $\mathbb{P}^2$ and let Γ be its Hilbert curve. Then Γ contains the line $\ell : pu - qv = 0$ ($(p,q) \neq (0,0)$), with $p \neq 2q$, if and only if $[p(c_1+b-1)-4q][p(c_1+b+1)-8q] = 0$, where c_1 and b are such that $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(c_1)$ and $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(b)$.*

Proof. Because $\dim(X) = 4$ then $\Gamma = \ell_1 + C$, where $\ell_1 : 2u - v = 0$ and C is the residual cubic. Thus if the line $\ell : pu - qv = 0$, with $p \neq 2q$ is contained in Γ it follows that it is a component of the residual cubic C. Because the base of the geometric quadric fibration (X,L) is $\mathbb{P}^2$, $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(c_1)$ and $c_2(\mathcal{E}) = c_2$, for some $c_1, c_2 \in \mathbb{Z}$. Thus the coefficients of the terms in $R_3(u,v)$ become, up to the factor $\frac{1}{24}$:

$$\alpha = 8c_1^2 - 16c_2 - 48c_1 - 96b + 8c_1b + 8b^2 + 216,$$

$$\beta = -\left(4c_1^2 - 24c_2 + 24c_1 - 12c_1b - 4b^2 - 24b + 108\right),$$

$$\gamma = 4c_1^2 - 12c_2 + 24c_1 - 6c_1b + 12b - 4b^2,$$

$$\delta = -2c_1^2 + 2c_2 - c_1b.$$

Likewise the coefficients of u and v in $R_1(u,v)$ are, up to the factor $\frac{1}{24}$, respectively

$$\sigma = -\left(2c_1^2 - 4c_2 - 12c_1 - 24b + 2c_1b + 2b^2 + 6\right),$$

$$\tau = -\left(c_1^2 + 2c_2 + 5c_1b + 3b^2 - 3\right).$$

In view of Proposition 5.3, the line $\ell : pu - qv = 0$ is a component of C if and only if (46) holds with $\sigma = pk$ and $\tau = -qk$ for some non zero $k \in \mathbb{Z}$ (since σ and τ, expressed by the above equalities, are integers). Recalling that $p \neq 2q$, the last two conditions, combined with the above expressions of σ and τ, give

$$c_2 = -\frac{1}{2}\,\frac{1}{p-2q}\,(2qc_1^2 + 2qb^2 - 12qc_1 - 24qb + 2qc_1b + pc_1^2 + 5pc_1b + 3pb^2 - 3p + 6q).$$

The relation (46), after replacing $\sigma = pk$, $\tau = -qk$ and α, β, γ, δ with the above expressions, becomes

$$k^3(2q-p)(4q^2c_1^2 - 8q^2c_2 + 108q^2 - 24q^2c_1 - 48q^2b + 4q^2bc_1 + 4q^2b^2 + 8qpc_2 - 24qpc_1 + 8qpc_1b + 4qpb^2 - 12qpb - 2p^2c_2 + 2p^2c_1^2 + p^2c_1b) = 0. \quad (47)$$

Because $k \neq 0$, after dividing out (47) with k^3 and replacing the value of c_2, we see that (47) can be rewritten as

$$3(2q-p)(-4q - p + c_1p + bp)(-8q + p + c_1p + bp) = 0,$$

and this proves the assertion, since, as we said, $p - 2q \neq 0$. □

As a further comment, we note the following. If $p(c_1+b-1)-4q=0$ then $b=-c_1+\frac{4q}{p}+1$ and in this case $C=\ell+\gamma_1$, where the equation of γ_1 is

$$(16p+4pc_1-32q)u^2+(-4pc_1-10p+20q)uv+(pc_1+p-2q)v^2 \\ +2p-pc_1+8q=0.$$

On the other hand, if $p(c_1+b+1)-8q=0$ then $b=-c_1+\frac{8q}{p}-1$ and in this case $C=\ell+\gamma_2$, where the equation of γ_2 is

$$(20p+4pc_1-64q)u^2+(-4pc_1-8p+40q)uv+(pc_1-p-4q)v^2 \\ -2p-pc_1+16q=0.$$

For instance, as to γ_2, the determinant of its matrix is $36(p-2q)^2\big(p(c_1+2)-16q\big)$. Hence γ_2 is reducible if and only if $p=16q/(c_1+2)$.

6. A special class of geometric quadric fibrations

Here we introduce a special class of quadric fibrations (X,L) which generalize conic fibrations considered in [6, Section 6]. In line with [6], we call them quadric fibrations deriving from cones since they are defined by generic quadric sections of a cone with vertex a point over a scroll on a surface. As we will see, they can never be quadric bundles, however the equation of the corresponding Hilbert curve simplifies considerably with respect to that of a general quadric fibration. The construction goes as follows.

6.1. Construction.

Let S be a smooth surface and let $\mathcal{V}$ be a very ample vector bundle of rank $n-1\geq 2$ on S. Set $T:=\mathbb{P}(\mathcal{V})$ and denote by h the tautological line bundle. Then h embeds T as an n-dimensional scroll over S in some projective space, say $\mathbb{P}^m$. Now set $\mathcal{E}:=\mathcal{V}\oplus\mathcal{O}_S$, and $R:=\mathbb{P}(\mathcal{E})$. Then R is a $\mathbb{P}^{n-1}$-bundle over S, with projection $\pi: R\to S$. Let ξ be the tautological line bundle of $\mathcal{E}$ on R and denote by $\phi: R\to\mathbb{P}^N$ the map defined by ξ. Clearly ϕ is a morphism, since $\mathcal{E}$ is spanned. We have

$$c_i(\mathcal{E})=c_i(\mathcal{V}) \qquad \text{for } i=1,2, \tag{48}$$

by construction. Furthermore, from the additivity of H^0, we get $h^0(\xi)=h^0(\mathcal{E})=h^0(\mathcal{V})+1=h^0(h)+1$, hence $N=m+1$. Note that ξ restricts trivially to the section, say σ, of R corresponding to the obvious surjection $\mathcal{E}\to\mathcal{O}_S$. Hence ϕ contracts σ to a point, say v, of the image $Y:=\phi(R)$. Due to the properties of ϕ, $Y\subset\mathbb{P}^{m+1}$ is the cone over T with vertex v, $\phi: R\to Y$ being the desingularization morphism; in fact, any fiber $F_s = \pi^{-1}(s)$ of $\pi : R \to S$ is a $\mathbb{P}^{n-1}$, and ϕ maps it isomorphically to the linear subspace of $\mathbb{P}^N$ spanned by v and the $\mathbb{P}^{n-2}$ which is the fiber of the scroll T over s. Now consider a general quadric hypersurface $Q \subset \mathbb{P}^N$ (i.e., not containing v) and let $X \subset R$ be its inverse image via ϕ. Then $X \in |2\xi|$, because $\xi = \phi^*\mathcal{O}_{\mathbb{P}^{m+1}}(1)$. Note that X is smooth and $L := \xi_X$ is ample since $X \subset R \setminus \sigma$. Moreover, X intersects every fibre F_s of π along a quadric. Therefore, by restricting π to X we get a fibration $\varphi := \pi|_X : X \to S$ in quadrics over S.

Because $L = \xi_X$ we have that $\mathcal{E} = \pi_*\xi = \varphi_*L$, thus R is exactly the $\mathbb{P}^{n-1}$-bundle P introduced in Section 2.

By the canonical bundle formula, recalling (48), we know that $K_R = -n\xi + \pi^*(K_S + \det\mathcal{V})$, thus, since $X \in |2\xi|$, we get by adjunction

$$\begin{aligned} K_X = (K_R + X)_X &= \big(-(n-2)\xi + \pi^*(K_S + \det\mathcal{V})\big)_X \\ &= -(n-2)L + \varphi^*(K_S + \det\mathcal{V}). \end{aligned}$$

The fact that $\mathcal{B}$ is trivial implies that $H := K_S + \det\mathcal{E} = K_S + \det\mathcal{V}$, hence H is ample unless $(S, \mathcal{V})$ is in a precise list of exceptions [8, Main Theorem]. Therefore,

Proposition 6.1. *Let (X, L) be a quadric fibration over S deriving from cones. Then (X, L) is a geometric quadric fibration if and only if $(S, \mathcal{V})$ is not one of the following pairs:*

(i) $(S, \mathcal{V}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus r}))$, *with* $r = 2, 3$,

(ii) $(S, \mathcal{V}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)))$,

(iii) $(S, \mathcal{V}) = (\mathbb{P}^2, T_{\mathbb{P}^2})$,

(iv) S is a $\mathbb{P}^1$-bundle over a smooth curve and $\mathcal{V}$ restricts as $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ to every fiber.

Note that the exception $(S, \mathcal{V}) = (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2}))$ (case 6 in [8, Main Theorem]) is included in case *(iv)*.

According to what we said before, for quadric fibrations (X, L) as in the above construction, the line bundle $\mathcal{B}$ is trivial. This has strong implications. As a first thing, we observe the following fact.

Remark 6.1. If (X, L) is quadric fibration deriving from cones then X cannot be a bundle. Actually, were X a bundle, the fact that $\mathcal{D} \in |2c_1(\mathcal{E}) + n\mathcal{B}|$ would imply $c_1(\mathcal{E}) = 0$, hence $c_1(\mathcal{V}) = 0$ by (48), but this would prevent $\mathcal{V}$ from being ample.

By Proposition 3.1 the equation of the residual cubic of its Hilbert curve is such that the homogeneous part of degree 3 is

$$\begin{aligned} R_3(u,v) = \frac{(-1)^n}{n!}\Big\{(n-2)\Big[(n^2-5n+8)c_1^2 - 2(n-2)^2c_2 + n(n-1)K_S^2 + 2nK_Sc_1\Big]u^3 \\ +\Big[(3n^2-17n+24)c_1^2 - 6(n-2)^2c_2 + n(n-1)K_S^2 - 2n(n-3)K_Sc_1\Big]u^2v \\ +\Big[4(3-n)c_1^2 + 6(n-2)c_2 + 2nK_Sc_1\Big]uv^2 + \big(2c_1^2-2c_2\big)v^3\Big\}, \end{aligned} \tag{49}$$

while the homogenous part of degree 1 is

$$\begin{aligned} R_1(u,v) = \frac{(-1)^n}{4n!}\Big\{\Big(8n(n-1)(n-2)\chi(\mathcal{O}_S) - (n-2)\Big[(n^2-5n+8)c_1^2 \\ - 2(n-2)^2c_2 + n(n-1)K_S^2 + 2nK_Sc_1\Big]\Big)u \\ + \Big[(n^2-7n+8)c_1^2 - 2(n-2)^2c_2 + 4n(n-1)\big(K_S^2 - 8\chi(\mathcal{O}_S)\big)\Big]v\Big\}. \end{aligned} \tag{50}$$

Moreover, if the base surface S is a minimal surface of Kodaira dimension zero, then the fact that K_S is numerically trivial produces a further simplification, which leads to the following result.

THEOREM 6.2. *Let (X,L) be a quadric fibration deriving from cones over a minimal surface S with $\kappa(S)=0$ and $\dim(X)=n\geq 3$.*

(1) If $n=3$ then $C=\Gamma$ is always irreducible.

(2) If $n\geq 4$ then C is irreducible in the following cases:

(i) if S is a $K3$ surface or an Enriques surface, and

(ii) if S is an abelian or a bielliptic surface and $d\neq \frac{2n(n-3)}{n^2-5n+8}c_2$.

Proof. Due to Proposition 5.2, and Proposition 4.2 we know that (46) can be rewritten as

$$[\sigma+(n-2)\tau]\ U=0,$$

where

$$U := n^2\delta\tau^2 - n\delta\sigma\tau - 4n\delta\tau^2 + n\gamma\tau^2 + \beta\tau^2 + \delta\sigma^2 + 2\delta\sigma\tau + 4\delta\tau^2 - \gamma\sigma\tau - 2\gamma\tau^2.$$

Now, taking into account that $\mathcal{B}$ is trivial, we can compute the coefficients $\sigma,\tau,\beta,\gamma,\delta$ from (50) and (49). Moreover, adding the information that K_S is numerically trivial and recalling (15), we get

$$\sigma+(n-2)\tau = \frac{(-1)^n n(n-2)}{2n!}\ c_1^2 = \frac{(-1)^n n(n-2)}{2n!}\ H^2\neq 0,$$

Therefore C is reducible if and only if $U=0$. Plugging in the expression of U the values of the coefficients $\sigma,\tau,\beta,\gamma,\delta$ we see that, up to the scalar factor

$-\frac{(-1)^{3n} n(n-1)}{16n!^3}\ c_1^2$

$$U = \begin{cases} \big[-(n^2-5n+8)c_1^2+2(n-2)^2c_2+16n(n-1)\big]^2+64n^2c_1^2 & \text{if } S \text{ is } K3 \\ \big[-(n^2-5n+8)c_1^2+2(n-2)^2c_2+8n(n-1)\big]^2+32n^2c_1^2 & \text{if } S \text{ is Enriques} \\ \big[(n^2-5n+8)c_1^2-(2n^2-8n+8)c_2\big]^2 & \text{if } S \text{ is abelian or bielliptic.} \end{cases}$$

Thus for every $n \geq 3$, U is different from zero if S is a $K3$ surface or an Enriques surface, being U the sum of two quantities in which the first one is greater than or equal to zero and the second one is strictly greater than zero because $c_1^2 = H^2 > 0$. If S is an abelian or a bielliptic surface we see that if $n = 3$ then $U = \big[(n^2-5n+8)c_1^2-(2n^2-8n+8)c_2\big]^2 = 2c_1^2 - 2c_2$ and recalling that $d = L^3 = 2(c_1^2 - c_2)$ our claim follows. This proves *(1)* and *(i)* of *(2)*. On the other hand, if $n \geq 4$ then $U = \big[(n^2-5n+8)c_1^2-(2n^2-8n+8)c_2\big]^2 = 0$ if and only if $c_1^2 = \frac{2n^2-8n+8}{n^2-5n+8}\ c_2$, in which case $d = 2(c_1^2 - c_2) = [2 + \frac{4(n-4)}{n^2-5n+8}]c_2$. This proves *(ii)* of *(2)*. □

We have to point out that in [6, Proposition 6.3, (ii)] the statement is not correct, in fact no condition on L^3 is needed in order to have the irreducibility of Γ. As to case $n \geq 4$ with S abelian or bielliptic we observe that C is certainly irreducible if $(n-1)c_1^2 > 2nc_2$ (that is if $\mathcal{E}$ is not Bogomolov stable, being $\mathrm{rk}(\mathcal{E}) = n$), because this prevents the term U from being zero.

Example 6.2. If in the construction 6.1, as $T = \mathbb{P}(\mathcal{V})$ we take the 5-dimensional scroll in $\mathbb{P}^{11}$, over $\mathbb{P}^2$, of degree 10 and sectional genus 3, that is $T = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4})$ $= \mathbb{P}^2 \times \mathbb{P}^3$, then (X, L) is a geometric quadric fibration, with $c_1(\mathcal{E}) = c_1(\mathcal{V}) =$ $\mathcal{O}_{\mathbb{P}^2}(4)$ and $c_2(\mathcal{E}) = c_2(\mathcal{V}) = 6$ and plugging such values in (49) and (50) we get

$$p_{(X,L)}\left(\frac{1}{2}+u, v\right) = -\frac{1}{12}(2u-v)(12u^2-10uv+2v^2+3).$$

Note that the linear factor $2u - v$ is not the one defining the line ℓ_0 whose equation is $(n-2)u - v = 0$, that is $3u - v = 0$.

Example 6.3. If in the construction 6.1, as $T = \mathbb{P}(\mathcal{V})$ we take the 4-dimensional scroll in $\mathbb{P}^{10}$, over $\mathbb{P}^2$, of degree 10 and sectional genus 3, that is $T = \mathbb{P}(T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$, then (X, L) is a geometric quadric fibration, with $c_1(\mathcal{E}) = c_1(\mathcal{V}) =$ $\mathcal{O}_{\mathbb{P}^2}(4)$ and $c_2(\mathcal{E}) = c_2(\mathcal{V}) = 6$ and plugging such values in (49) and (50) we get

$$p_{(X,L)}\left(\frac{1}{2}+u, v\right) = \frac{7}{3}u^3 - \frac{31}{6}u^2v + \frac{11}{3}uv^2 - \frac{5}{6}v^3 + \frac{17}{12}u - \frac{25}{24}v.$$

We like to stress the following fact. Let T_1 and T_2 be the scrolls in Example 6.2 and Example 6.3 respectively. Adding $\mathcal{O}_{\mathbb{P}^2}(1)$ to the three terms of the Euler sequence on $\mathbb{P}^2$, we get the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4} \to T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to 0,$$

where $T_{\mathbb{P}^2}$ is the tangent bundle to $\mathbb{P}^2$. Then by using, for instance, [12, Lemma 0.7], we see that $T_2 \subset \mathbb{P}^{10}$ is the general hyperplane section of T_1, Segre embedded in $\mathbb{P}^{11}$.

7. Examples

Let (X, L) be a geometric quadric fibration over a smooth surface as in Section 2. The key point in passing from the case of threefolds studied in [6] to higher dimensions is clearly $n = 4$, as already (1), and what we proved in the previous sections, show. For this reason the examples we discuss in this Section are concerned with $n = 4$. First of all, note that if (X, L) is general, then the residual cubic C of its Hilbert curve is irreducible according to Proposition 5.3. Here is an example.

Example 7.1. Let $\mathbb{P}(\mathcal{E})$ be the $\mathbb{P}^3$-bundle over the smooth quadric $\mathbb{Q}^2$, defined by the rank four vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^2}(1,2)^{\oplus 4}$. Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{Q}^2$ be the projection morphism, let X be a general element in $|2\xi + \pi^*\mathcal{O}_{\mathbb{Q}^2}(1,2)|$, where ξ denotes the tautological line bundle of $\mathcal{E}$ on $\mathbb{P}(\mathcal{E})$, and call $\varphi : X \to \mathbb{Q}^2$ the restriction of π to X. On X we consider the polarization given by $L = (\xi)_X$ Note that $K_X = \big(-4\xi + \pi^*\mathcal{O}_{\mathbb{Q}^2}(4,8) + \pi^*\mathcal{O}_{\mathbb{Q}^2}(-2,-2) + 2\xi + \pi^*\mathcal{O}_{\mathbb{Q}^2}(1,2)\big)_X = \big(-2\xi + \pi^*\mathcal{O}_{\mathbb{Q}^2}(3,8)\big)_X$. The polarized pair (X, L) is a geometric quadric fibration over $\mathbb{Q}^2$. Using (28) through (33) and the fact that $c_1 = \mathcal{O}_{\mathbb{Q}^2}(4,8)$, $c_2 = 24$ and $\mathcal{B} = \mathcal{O}_{\mathbb{Q}^2}(1,2)$, we see that the residual cubic C of the Hilbert curve Γ of (X, L) has equation

$$4u^3 - 12u^2v - 3uv^2 + 4v^3 - 3u + \frac{17}{2}v = 0. \tag{51}$$

In this case C is irreducible, the term on the left hand side of (46) taking the value 266.

The remainder of this Section is devoted to examples for which C is reducible.

Example 7.2. Let $Y := \mathbb{P}^2 \times \mathbb{P}^3$ and let π and ρ be the projections onto the first and the second factors respectively. Set $A := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, $B := \rho^*\mathcal{O}_{\mathbb{P}^3}(1)$, and write $\mathcal{O}(r, s)$ for $rA + sB$. Let $X \subset Y$ is a smooth element in the linear system $|\mathcal{O}(1,2)|$, hence

$$X \sim A + 2B. \tag{52}$$

By adjunction, $K_X = [\mathcal{O}(-3,-4) + X]_X = \mathcal{O}(-2,-2)_X = -2\,\mathcal{O}(1,1)_X$, so that X is a Fano 4-fold of index 2; moreover, taking into account that $A^3 = B^4 = 0$ and $A^2B^3 = 1$, we have

$$\big(\mathcal{O}(1,1)_X\big)^4 = \mathcal{O}(1,1)^4 X = (A+B)^4(A+2B) = (4+12)A^2B^3 = 16$$

(i. e. $(X, \mathcal{O}(1,1)_X)$ has degree 16).

Up to now $\varphi := \pi_{|X} : X \to \mathbb{P}^2$ is only a classical quadric fibration. To make it a geometric quadric fibration consider on X the ample line bundle $L := \mathcal{O}(a,1)_X$ for some positive integer a. Clearly L induces the hyperplane bundle on every quadric surface, fiber of φ. Moreover,

$$K_X + 2L = \mathcal{O}\big(2(a-1), 0\big)_X = \varphi^*\mathcal{O}_{\mathbb{P}^2}\big(2(a-1)\big).$$

Therefore, for (X, L) to be a geometric quadric fibration we need $a \geq 2$.

Note that $Y = \mathbb{P}(\mathcal{V})$, where $\mathcal{V} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$, the corresponding tautological line bundle being $\zeta := \mathcal{O}(1,1)$. This is clear once we compare the two expression of the canonical bundle of Y, viewed both as a product and as $\mathbb{P}(\mathcal{V})$ respectively. Then $\mathcal{V} = \pi_*\zeta$ and, recalling that $\varphi = \pi_{|X}$, we also have $\mathcal{V} = \varphi_*(\zeta_X)$. Next let us determine the vector bundle $\mathcal{E} := \varphi_* L$. Since $L = \big(\zeta + (a-1)\pi^*\mathcal{O}_{\mathbb{P}^2}(1)\big)_X$, we get

$$\mathcal{E} = \varphi_*[\big(\zeta + \pi^*\mathcal{O}_{\mathbb{P}^2}(a-1)\big)_X] = \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^2}(a-1) = \mathcal{O}_{\mathbb{P}^2}(a)^{\oplus 4}.$$

In particular, this gives

$$c_1 = \mathcal{O}_{\mathbb{P}^2}(4a). \tag{53}$$

Since $\mathcal{E}$ is $\mathcal{V}$ twisted by a line bundle, we see that $P := \mathbb{P}(\mathcal{E}) \cong Y$ itself; note however that the tautological line bundle corresponding to $\mathcal{E}$ is $\xi = \zeta + (a-1)A$ (in accordance with the fact that $L = \xi_X$). Now recall that, in our setting, $X \in |2\xi + \pi^*\mathcal{B}|$. So, letting $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(b)$, we get

$$\begin{aligned} 2\xi + \pi^*\mathcal{B} &= \big(2\zeta + 2(a-1)A + bA\big) = [2(A+B) + (2a+b-2)A] \\ &= [(2a+b)A + 2B], \end{aligned}$$

and from a comparison with (52) we deduce that $b = -2a+1$, i.e. $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(-2a+1)$. Finally, look at $\mathcal{D}$, the discriminant curve of our quadric fibration (X, L). From a general result already mentioned in Section 2 combined with (53), since $n = 4$ we get

$$\mathcal{D} \in |2c_1 + n\mathcal{B}| = |\mathcal{O}_{\mathbb{P}^2}(4)|.$$

Therefore certainly X is not a $\mathbb{Q}^2$-bundle over $\mathbb{P}^2$, $\mathcal{D}$ being non-trivial. It remains to determine the canonical equation of the Hilbert curve Γ of (X, L). By using the Riemann–Roch–Hirzebruch formula in the following form (see [2, (8)])

$$\chi(D) = \frac{1}{24}E^4 + \frac{1}{48}\big(2c_2(X) - K_X^2\big)E^2 + \frac{1}{384}\big(K_X^2 - 4c_2(X)\big)K_X^2 + \chi(\mathcal{O}_X), \tag{54}$$

where $D = \frac{1}{2}K_X + E$ and $E = uK_X + vL$, after standard Chern class computations we get the canonical equation of Γ, which is

$$p\left(\frac{1}{2} + u, v\right) = \frac{1}{6}(v-2u)(av-2u)\big(16u^2 - 2(3a+5)uv + (3a+1)v^2 + 2\big) = 0.$$

We note that Γ is reducible, but, in addition to the line $2u-v=0$, which is a prescribed component of Γ according to [3, Theorem 6.1], there is another linear component for any $a \geq 2$, namely the line $av-2u=0$.

As to the conic component, say γ, note that it is irreducible since $a \geq 2$. On the other hand, if $a=1$, then (X, L) is not a geometric quadric fibration, as already observed; moreover the equation of Γ becomes $(2u-v)^2\big(4(2u-v)^2+2\big)=0$. In particular we see that the projective closure $\overline{\Gamma}$ of Γ has a singular point of multiplicity 4 at $(2:1:0)$. Note that this is in accordance with [3, Lemma 3.2], since for $a=1$, we have $K_X+2L=0$, hence (X, L) fits into the degenerate case.

Example 7.3. Consider $\mathbb{P}^2 \times \mathbb{Q}^3$ and let p_1 and p_2 be the projections onto the first and the second factor respectively. Set $H_1 := p_1^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)$ and $H_2 := p_2^*\big(\mathcal{O}_{\mathbb{Q}^3}(1)\big)$, and write $\mathcal{O}(r,s)$ for rH_1+sH_2. Let $X \subset \mathbb{P}^2 \times \mathbb{Q}^3$ be a smooth element in the linear system $|\mathcal{O}(1,1)|$. By adjunction, $K_X = [\mathcal{O}(-3,-3)+X]_X = -2\,\mathcal{O}(1,1)_X$, so that X is a Fano 4-fold of index 2. Taking into account that $H_1^3 = H_2^4 = 0$ and $H_1^2H_2^3 = 2$, we see that $\big(X, \mathcal{O}(1,1)_X\big)$ has degree 20. Let $\varphi : X \to \mathbb{P}^2$ be the restriction of p_1 to X, and take on X the polarization given by $L := \mathcal{O}(a,1)_X$ for some positive integer a. Clearly L induces the hyperplane bundle on every quadric surface, fiber of φ. Moreover,

$$K_X + 2L = \mathcal{O}\big(2(a-1),0\big)_X = \varphi^*\mathcal{O}_{\mathbb{P}^2}\big(2(a-1)\big).$$

Therefore, (X, L) will be a geometric quadric fibration as soon as $a \geq 2$.

In order to compute the canonical equation of Γ, we tensor the structure sequence

$$0 \to \mathcal{O}(-1,-1) \to \mathcal{O}(0,0) \to \mathcal{O}_X \to 0$$

with $\mathcal{O}(ay-2x, y-2x)$ and we get

$$0 \to \mathcal{O}(ay-2x-1, y-2x-1) \to \mathcal{O}(ay-2x, y-2x) \to xK_X + yL \to 0. \quad (55)$$

Using the fact that $\chi(\mathbb{P}^2 \times \mathbb{Q}^3, \mathcal{O}(r,s)) = \chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r)) \cdot \chi(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(s))$ and standard computations, after replacing $x = u + \frac{1}{2}$, $y = v$ we see that the canonical equation of Γ is

$$p\left(\frac{1}{2}+u, v\right) = \frac{1}{6}(v-2u)(av-2u)\big(20u^2 - 2(3a+7)uv + (3a+2)v^2 + 1\big) = 0.$$

Thus the residual cubic C has equation

$$\frac{1}{6}(av-2u)\big(20u^2 - 2(3a+7)uv + (3a+2)v^2 + 1\big) = 0.$$

Even in this case the conic γ is irreducible, since $a \geq 2$. Because X sits in $P := \mathbb{P}(\mathcal{E})$ as a divisor, $X \in |2\xi + \pi^*\mathcal{B}|$ where ξ is the tautological line bundle, we compute the values of c_i. Let $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(b)$ for some integer b. From (15) we have

$$2a - 2 = c_1 - 3 + b$$

and thus $c_1 = 2a + 1 - b$. Easy computations show that

$$d = 12a^2 + 8a. \tag{56}$$

$$d_1 = -24a - 4 - 12a^2. \tag{57}$$

On the other hand from (24) and (25), since $K_S = \mathcal{O}_{\mathbb{P}^2}(-3)$, $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(b)$, $n = 4$, and $c_1 = 2a + 1 - b$ we get that

$$d = 8a^2 + 8a - 6ab + 2 - 3b + b^2 - 2c_2. \tag{58}$$

and

$$d_1 = -8a^2 - 20a + 10ab - 8 + 8b - 2b^2 + 4c_2. \tag{59}$$

Using (56),(57),(58),(59) we get that $b = -2a$, $c_1 = 4a + 1$, $c_2 = 6a^2 + 3a + 1$.

As to the discriminant curve $\mathcal{D}$ of our quadric fibration (X, L), we have

$$\mathcal{D} \in |2c_1 + 4\mathcal{B}| = |\mathcal{O}_{\mathbb{P}^2}(2)|.$$

EXAMPLE 7.4. Let $\pi : X \to \mathbb{P}^2 \times \mathbb{P}^2$ be a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$, branched along a smooth divisor of type $(2, 2)$ and let $\mathscr{R} \subset X$ be the ramification divisor. Then $\mathscr{R}$ is a smooth hypersurface and $\pi(\mathscr{R}) \in |2H|$, with $H = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1, 1)$. We have a short exact sequence

$$0 \to \pi^* T^*_{\mathbb{P}^2\times\mathbb{P}^2} \to T^*_X \to N^*_{\mathscr{R}/X} \to 0, \tag{60}$$

where $N^*_{\mathscr{R}/X}$ is the conormal bundle of $\mathscr{R} \subset X$. It comes from a local computation combined with the fact that $N^*_{\mathscr{R}/X} = \mathcal{J}/\mathcal{J}^2$, where $\mathcal{J}$ is the ideal sheaf of $\mathscr{R}$ in X. We will use (60) and the short exact sequence

$$0 \to -2\mathscr{R} \to -\mathscr{R} \to N^*_{\mathscr{R}/X} \to 0 \tag{61}$$

to determine $c_i(X)$. In fact arguing as in [5, Lemma 2.6] (which holds in any dimension) we see that

$$c_1(X) = \pi^*\big(c_1(\mathbb{P}^2 \times \mathbb{P}^2) - H\big), \tag{62}$$

$$c_2(X) = \pi^*\big(c_2(\mathbb{P}^2 \times \mathbb{P}^2) - c_1(\mathbb{P}^2 \times \mathbb{P}^2)H + 2H^2\big). \tag{63}$$

Let $p_i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the projection onto the i-th factor. Let $p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) = H_1$ and $p_2^*(\mathcal{O}_{\mathbb{P}^2}(1)) = H_2$, where H_1 and H_2 satisfy $H_1^3 = H_2^3 = 0$ and $H_1^2 H_2^2 = 1$.

In order to compute $c_i(\mathbb{P}^2 \times \mathbb{P}^2)$ we use the following exact sequence, deriving from the Euler sequence,

$$0 \to p_1^*(\mathcal{O}_{\mathbb{P}^2}) \oplus p_2^*(\mathcal{O}_{\mathbb{P}^2}) \to p_1^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)^{\oplus 3} \oplus p_2^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)^{\oplus 3} \to \\ p_1^*(T_{\mathbb{P}^2}) \oplus p_2^*(T_{\mathbb{P}^2}) = T_{\mathbb{P}^2 \times \mathbb{P}^2} \to 0$$

and we see that

$$c_1(\mathbb{P}^2 \times \mathbb{P}^2) = 3(H_1 + H_2),\ c_2(\mathbb{P}^2 \times \mathbb{P}^2) = 3(H_1^2 + H_2^2 + 3H_1H_2).$$

Thus

$$c_1(X) = \pi^*\big(c_1(\mathbb{P}^2 \times \mathbb{P}^2) - H\big) = \pi^*(2H_1 + 2H_2),$$

$$c_2(X) = \pi^*\big(c_2(\mathbb{P}^2 \times \mathbb{P}^2) - c_1(\mathbb{P}^2 \times \mathbb{P}^2)H + 2H^2\big) = \pi^*(2H_1^2 + 2H_2^2 + 7H_1H_2).$$

Let $\varphi = p_1 \circ \pi : X \to \mathbb{P}^2$, which is a classical quadric fibration. To make it a geometric quadric fibration we consider on X the ample line bundle $L := \pi^*\big(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(a, 1)\big)$ for some positive integer a. Because

$$K_X + 2L = \pi^*\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}\big(2(a-1), 0\big) = \varphi^*\mathcal{O}_{\mathbb{P}^2}\big(2(a-1)\big)$$

it follows that (X, L) will be a geometric quadric fibration if $a \geq 2$.

By (54), after standard computations, we get the canonical equation of Γ, which is

$$p\left(\frac{1}{2} + u, v\right) = \frac{1}{2}(v - 2u)(av - 2u)\big(4u^2 - 2(a+1)uv + av^2 + 1\big) = 0.$$

Thus the residual cubic C has equation

$$\frac{1}{2}(2u - av)\big(4u^2 - 2(a+1)uv + av^2 + 1\big) = 0.$$

Even in this case the conic γ is irreducioble, provided that $a \geq 2$. For such (X, L) we see that

$$d = L^4 = \Big(\pi^*\big(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(a, 1)\big)\Big)^4 = 2 \cdot 6(aH_1)^2H_2^2 = 12a^2. \tag{64}$$

Similarly

$$d_1 = 2(-2H_1 - 2H_2)(aH_1 + H_2)^3 = -12a^2 - 12a. \tag{65}$$

We now compute the values of c_i. Let $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(b)$ for some integer b. From (15) we have

$$2a - 2 = c_1 - 3 + b$$

and thus $c_1 = 2a + 1 - b$.

On the other hand from (24) and (25), since $K_S = \mathcal{O}_{\mathbb{P}^2}(-3)$, $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(b)$, $n = 4$, and $c_1 = 2a + 1 - b$ we get that

$$d = 8a^2 + 8a - 6ab + 2 - 3b + b^2 - 2c_2. \tag{66}$$

and

$$d_1 = -8a^2 - 20a + 10ab - 8 + 8b - 2b^2 + 4c_2. \tag{67}$$

Using (64), (66), (65), (67) we obtain $b = -2a + 2$, $c_1 = 4a - 1$, $c_2 = 6a^2 - 3a$ and thus for the discriminant curve $\mathcal{D}$, of our quadric fibration (X, L), we have

$$\mathcal{D} \in |2c_1 + 4\mathcal{B}| = |\mathcal{O}_{\mathbb{P}^2}(6)|.$$

We like to point out that in Examples 7.2 - 7.4 X is a Fano 4-fold as in [14, Table 0.3, No. 5, 8, 4, respectively]. For $a \geq 2$ (X, L) is a 4-dimensional geometric quadric fibration over $\mathbb{P}^2$ and fits into the situation described by Proposition 5.4, satisfying the condition $p(c_1 + b - 1) - 4q = 0$ in all three cases. For instance, in Example 7.2, $p = 2, b = -\frac{c_1}{2} + 1, q = \frac{c_1}{4}$. On the other hand, if $a = 1$ then $K_X = -2L$, hence Γ itself is reducible into 4 parallel lines, in accordance with [13, Lemma 3.1].

Example 7.5. Let $\mathbb{P}(\mathcal{E})$ be the $\mathbb{P}^3$-bundle over the Segre–Hirzebruch surface $\mathbb{F}_e$ of invariant e (≥ 0), defined by the rank four vector bundle $\mathcal{E} = [C_0 + (e+1)f]^{\oplus 4}$, where C_0 is a section of self-intersection $C_0^2 = -e$ and f a fiber of the bundle projection $\mathbb{F}_e \to \mathbb{P}^1$. Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{F}_e$ be the projection morphism, let X be a general element in $|2\xi + \pi^*\mathcal{B}|$, where ξ denotes the tautological line bundle of $\mathcal{E}$ on $\mathbb{P}(\mathcal{E})$, $\mathcal{B} = aC_0 + bf$ and call $\varphi : X \to \mathbb{F}_e$ the restriction of π to X. Note that $K_X = \big(-4\xi + \pi^*(4C_0 + 4(e+1)f - 2C_0 - (e+2)f) + 2\xi + \pi^*\mathcal{B}\big)_X = \big(-2\xi + \pi^*((2+a)C_0 + (3e+2+b)f)\big)_X$. The polarized pair (X, L) where $L = (\xi)_X$ is a geometric quadric fibration over $\mathbb{F}_e$. Using (28) through (33) and the fact that $c_1 = 4C_0 + 4(e+1)f$, $c_2 = 6(e+2)$, we see that the residual cubic C of the Hilbert curve Γ of (X, L) has equation

$$\frac{2}{3}(2u - 3v)(v^2 - uv - 2u^2 + 2) = 0, \tag{68}$$

if the base surface is $\mathbb{F}_0$ and $\mathcal{B} = 2C_0$, and

$$\frac{1}{6}(2u - 3v)(5v^2 - 8uv - 4u^2 + 7) = 0. \tag{69}$$

if the base surface is $\mathbb{F}_1$ and $\mathcal{B} = \mathcal{O}_{\mathbb{F}_1}$.

8. A unifying perspective

Here we discuss a natural framework in which the residual cubics of Hilbert curves of geometric quadric fibrations over a smooth surface fit into, offering a unifying perspective to many results proved in the previous Sections. To start with let $\mathscr{V}$ be the family of Serre-invariant cubics, see for instance [3, Section 7]. As observed in Section 2, a cubic C in the complex affine plane of coordinates u and v belongs to $\mathscr{V}$ if and only if it is described by an equation of type (3), with $R_3(u,v)$ and $R_1(u,v)$ as in (4) and (5), respectively, for some complex numbers $\alpha,\dots,\tau$, with $(\alpha,\beta,\gamma,\delta)\neq(0,0,0,0)$. So we can look at C as the point $(\alpha:\beta:\gamma:\delta:\sigma:\tau)$ of $\mathbb{P}^5$ lying outside the line, say Λ, defined by $\alpha=\beta=\gamma=\delta=0$. Thus we have a natural identification

$$\mathscr{V}=\mathbb{P}^5\setminus\Lambda. \tag{70}$$

According to Proposition 5.2, we can identify the reducible $C\in\mathscr{V}$ with the points of the quartic hypersurface $V\subset\mathbb{P}^5$ of equation (46), lying outside Λ. Let us rewrite the equation of V in the form

$$f(\alpha,\beta,\gamma,\delta,\sigma,\tau)=\sigma^2(\sigma\delta-\tau\gamma)+\tau^2(\sigma\beta-\tau\alpha)=0. \tag{71}$$

Clearly, V contains Λ and also the 3-plane $\sigma=\tau=0$. In fact we have

PROPOSITION 8.1. *The singular locus* $\mathrm{Sing}(V)$ *is exactly the* 3*-plane of equations* $\sigma=\tau=0$.

Proof. The assertion follows immediately from the Jacobian criterion. Actually, from (71) we get

$$\mathrm{grad}(f)=(-\tau^3,\tau^2\sigma,-\sigma^2\tau,\sigma^3,3\delta\sigma^2-2\gamma\sigma\tau+\beta\tau^2,-\gamma\sigma^2+2\beta\sigma\tau-3\alpha\tau^2).$$

This shows that condition $\mathrm{grad}(f)=0$ is equivalent to the vanishing of the first and the fourth components only. □

As a consequence of Proposition 8.1 we have that also the singular locus of $V\setminus\Lambda$ is the 3-plane $\sigma=\tau=0$, since the latter and the line Λ are skew.

Now we consider another relevant locus in $\mathbb{P}^5$. We denote by $\mathscr{S}$ the family of cubics $C\in\mathscr{V}$ such that the projective closure $\overline{C}$ contains the point at infinity $P_\infty=(1:n-2:0)$ and intersects the line at infinity ℓ_∞ at that point transversely. The first condition says that the point corresponding to C lies on the hyperplane $\mathscr{H}$ of equation (36), while the latter means that it does not belong to the hyperplane h of equation (38), according to the proof of Proposition 4.2. So $\mathscr{S}$ seems the most appropriate locus of $\mathscr{V}$ to include the residual cubics of the Hilbert curves of geometric quadric fibrations as in Section 2.

Because the coefficients in (1) are rationals, all residual cubics of the Hilbert curves of our quadric fibrations correspond to rational points of the locus $\mathscr{S}$.

By definition $\mathscr{S}$ is the complement of h in $\mathscr{V} \cap \mathscr{H}$, hence it is a quasi-projective variety of dimension 4. In fact, noting that both $\mathscr{H}$ and h contain Λ, we have that

$$\mathscr{S} := \mathscr{V} \cap (\mathscr{H} \setminus h) = (\mathbb{P}^5 \setminus \Lambda) \cap (\mathscr{H} \setminus h) = \mathscr{H} \setminus h. \tag{72}$$

In particular, $\overline{\mathscr{S}} = \mathscr{H}$. Next, consider

$$\mathscr{T} := \{C \in \mathscr{S} \mid C \text{ has a triple point}\}. \tag{73}$$

According to the discussion in Section 4, any $C \in \mathscr{T}$ has a triple point at the origin, hence it is reducible into three lines through the origin. Moreover, since $\overline{C}$ contains P_∞, any such C consists of the line ℓ_0 and two other lines distinct from ℓ_0 belonging to the same pencil. Thus $\mathscr{T} = \mathrm{S}^{(2)}(\mathbb{P}^1 \setminus \{\mathrm{o}\})$, the second symmetric product of $\mathbb{P}^1 \setminus \{o\}$ with itself, o representing the line ℓ_0, which is removed from the pencil. Recall that for a $C \in \mathscr{V}$, having a triple point at the origin is equivalent to satisfying the equations $\sigma = \tau = 0$ with the further condition that $C \in \mathscr{H}$. Removing the intersection with the hyperplane h, this shows that $\mathscr{T}$ is a $\mathbb{P}^2$ minus a line, which agrees with the previous description, since $\mathrm{S}^{(2)}(\mathbb{P}^1) = \mathbb{P}^2$. Proposition 8.1 has the following consequence.

COROLLARY 8.2. *We have* $\mathrm{Sing}(V) \cap \mathscr{S} = \mathscr{T}$.

REMARK 8.1. Since all reducible cubics of $\mathscr{S}$ lie in $V \cap \mathscr{H}$, one could be tempted to think that the singular locus $\mathrm{Sing}(V \cap \mathscr{H})$ is more related to our analysis than $\mathrm{Sing}(V)$. However, this is not the case, as we will see in a moment. Of course $\mathrm{Sing}(V \cap \mathscr{H})$ is larger than $\mathrm{Sing}(V) \cap \mathscr{H}$ and using the Jacobian criterion one can see that it consists of two components. Precisely, $\mathrm{Sing}(V \cap \mathscr{H}) = \mathscr{T} \cup \mathscr{Z}$, where $\mathscr{Z}$ is defined by the following three equations: $\sigma = -(n-2)\tau$, $\beta = -(n-2)\big(2\gamma + 3(n-2)\delta\big)$ and $\alpha = (n-2)^2\big(\gamma + 2(n-2)\delta\big)$. However

$$(\mathscr{Z} \setminus \mathscr{T}) \cap \mathscr{S} = \emptyset, \tag{74}$$

which says that the component $\mathscr{Z}$ is irrelevant for $\mathscr{S}$. To see this, suppose that $C \in \mathscr{Z}$. Then its equation is

$$\begin{aligned}(n-2)^2\big(\gamma + 2(n-2)\delta\big)u^3 - (n-2)\big(2\gamma + 3(n-2)\delta\big)u^2 v \\ + \gamma u v^2 + \delta v^3 - (n-2)\tau u + \tau v = 0,\end{aligned}$$

where $\tau \neq 0$ if, in addition, $C \notin \mathscr{T}$. Note that the polynomial at the left hand side is divisible by $(n-2)u - v$, hence the above equation can be rewritten as

$$\big((n-2)u - v\big)\big[\gamma u\big((n-2)u - v\big) + \delta\big(2(n-2)^2u^2 - (n-2)uv - v^2\big) - \tau\big] = 0. \tag{75}$$

This shows that $C = \ell_0 + G$, where G is the conic described by the factor in brackets. Looking for the points at infinity of $\overline{G}$ we immediately see that they are $Q'_\infty = (1 : n-2 : 0)$ and $Q''_\infty = \big(-\delta : \gamma + 2(n-2)\delta : 0\big)$, up to renaming. But $Q'_\infty = P_\infty$, the point at infinity of ℓ_0, hence $\overline{C}$ intersects ℓ_∞ at P_∞ with multiplicity ≥ 2. Therefore $C \notin \mathscr{S}$.

We can also revisit Proposition 4.3 in the current setting. Let $\mathscr{Q} \subset \mathbb{P}^5$ be the quadric hypersurface of equation (39). According to Proposition 4.3, the section of $\mathscr{Q}$ with the hyperplane $\mathscr{H}$ of equation (36), outside of Λ, that is

$$\mathscr{Q}' := \mathscr{Q} \cap \mathscr{H} \setminus \Lambda,$$

describes the Serre-invariant cubics C such that $\overline{C} \cap \ell_\infty = P_\infty + 2Q_\infty$, where Q_∞ is a point at infinity, distinct from P_∞. An immediate check shows that $\mathrm{Sing}(\mathscr{Q})$ is the plane of equations $\delta = \gamma = \alpha = 0$, and combining them with (36), we see that $\mathrm{Sing}(\mathscr{Q}) \cap \mathscr{H} = \Lambda$. But Λ is not included in $\mathscr{S}$, hence

$$\mathrm{Sing}(\mathscr{Q}) \cap \mathscr{S} = \emptyset.$$

On the other hand, replacing α in (39) with its expression provided by (36) we get an equation in β, γ, δ, mute in σ and τ, representing a quadric hypersurface of $\mathbb{P}^4$, say $\mathscr{Q}''$, which is the image of $\mathscr{Q}'$ in the hyperplane Π of equation $\alpha = 0$ via the projection $\rho : \mathbb{P}^5 \setminus \{A\} \to \Pi$ from the point $A = (1 : 0 : 0 : \cdots : 0) \in \mathbb{P}^5$. A straightforward computation shows that the singular locus of $\mathscr{Q}''$ in the hyperplane Π is described by the equations $\beta = \gamma = \delta = 0$, i.e. $\mathrm{Sing}(\mathscr{Q}'') = \Lambda$. So, coming back to $\mathscr{Q}'$ through the projection ρ to describe $\mathrm{Sing}(\mathscr{Q}')$, we see that

$$\mathrm{Sing}(\mathscr{Q}') \cap \mathscr{S} = \emptyset.$$

To complete the picture it remains to understand which loci of $\mathscr{Q}'$ represent the different kinds of cubics C fitting in Proposition 4.3. They are:

1) Serre-invariant cubics (the general being irreducible) passing through P_∞ and with a node on ℓ_∞, not in P_∞;

2) reducible Serre-invariant cubics passing through P_∞ and having a double intersection with ℓ_∞, not in P_∞.

Now, come back to reducible Serre-invariant cubics. According to Proposition 5.1, in case 2) we have $C = \ell + \gamma$; moreover, if γ is irreducible, then ℓ contains O and γ has center in O, due to the Serre invariance, hence it cannot be a parabola. As a consequence, $\overline{\gamma} \cap \ell_\infty$ consists of two distinct points. Since we are dealing with cubics C such that $\overline{C} \cap \ell_\infty = P_\infty + 2Q_\infty$, we have that $\ell \neq \ell_0$ and γ is a hyperbola whose asymptotes are ℓ_0 and the line $< O, Q_\infty >$. In this case $\overline{C}$ intersects ℓ_∞ with multiplicity one in P_∞ and two in Q_∞. Note that for these cubics C, the admissible conics γ constitute a pencil; however, since Q_∞ can vary on ℓ_∞, with the only restriction of being different from P_∞, the family of such conics depends on two parameters. In addition, the line ℓ moves in a pencil, since it has to contain O. Therefore the family of cubics C of this type is 3-dimensional. Let us call this case 2a). Clearly, case 2) also includes the following two possibilities.

2b) $\ell = \ell_0$ and γ consists of two lines through O. This is the situation in which C has a triple point, i.e., $C \in \mathscr{T}$. We already know that this is a 2-dimensional family.

2c) $\ell = \ell_0$ again, but C has no triple points. In this case, according to Proposition 5.1 *(b)*, γ consists of two parallel lines ℓ' and ℓ'' with direction corresponding to Q_∞ (and obviously symmetric with respect to O). This family too has dimension 2, as one can see from the following computation.

Concerning the dimensions of the various families, we note the following. We know that case 1) is effective, as [6, Example 6.3] shows. Suppose that the node is $Q_\infty = (-b : a : 0)$ and for simplicity, to treat equations in affine coordinates, call $m = -a/b$ the slope; then $Q_\infty = (1 : m : 0)$. The equation of $\overline{C}$, in homogeneous coordinates u, v, w is

$$f_0 := f_0(u, v, w) = \alpha u^3 + \beta u^2 v + \gamma u v^2 + \delta v^3 + \sigma u w^2 + \tau v w^2 = 0.$$

Imposing the vanishing of the three partial derivatives of f_0 evaluated at Q_∞, we get the following system:

$$\begin{cases} f_{0,u} = 3\alpha + 2\beta m + \gamma m^2 = 0 \\ f_{0,v} = \beta + 2\gamma m + 3\delta m^2 = 0. \end{cases} \tag{76}$$

Note that we get only two nontrivial equations, since the derivative of f_0 with respect to w evaluated at any point of ℓ_∞ is zero, because only the last two terms of f_0 contain w, and in fact only w^2. Equivalently, the homogeneous equation of the tangent line to $\overline{C}$ at any of its points at infinity does not contain w, hence it passes through O (cf. [3, Theorem 3.4]). Now, the occurrence of a singular point of $\overline{C}$ at Q_∞ is equivalent to the fact that m is a common root of both equations in (76), This happens if and only if the resultant $\mathrm{Res}(f_{0,u}, f_{0,v})$ of the two polynomials in m at the left hand side of the two equations vanishes. On the other hand,

$$\mathrm{Res}(f_{0,u}, f_{0,v}) = \begin{vmatrix} \gamma & 2\beta & 3\alpha & 0 \\ 0 & \gamma & 2\beta & 3\alpha \\ 3\delta & 2\gamma & \beta & 0 \\ 0 & 3\delta & 2\gamma & \beta \end{vmatrix},$$

which, up to the multiplicative factor 3, is given by

$$F := 4\alpha\gamma^3 + 4\beta^3\delta - \beta^2\gamma^2 + 27\alpha^2\delta^2 - 18\alpha\beta\gamma\delta.$$

An easy check shows that this quartic polynomial is irreducible. It defines a quartic hypersurface in $\mathbb{P}^5$ and then the locus of the cubics C as in 1) corresponds to the section of the quadric $\mathscr{Q}'$ with the quartic of equation $F = 0$. In particular this says that the family corresponding to case 1) depends on three parameters. This is in accordance with the following fact: Serre-invariant cubics passing through P_∞ depend on 4 parameters (general point of the hyperplane $\mathscr{H}$) and imposing a singularity at a point that can vary on ℓ_∞ requires only one condition; hence the dimension is $4 - 1 = 3$. We already said about the dimensions of the families corresponding to cases $2a$) and $2b$), hence we come to case $2c$). Since ℓ_0 is fixed, having equation $(n-2)u - v = 0$, C is determined by the slope of ℓ' (the same as that of ℓ'') and e. g. the distance between ℓ' and ℓ''; hence the family depends on two parameters. In fact, letting $Q_\infty = (-a : b : 0)$, ℓ' has equation $bu + av - c = 0$ for some $c \in \mathbb{C}$, and then the equation of C has the form:

$$[(n-2)u - v][(bu + av)^2 - c^2] = 0.$$

Therefore C depends on the two parameters $m := -b/a$ and c/a. In conclusion, letting $F_{i)}$ denote the family of the cubics C as in case i), where $i = 1, 2a$, etc., we have

$$\dim[F_{1)}] = \dim[F_{2a)}] = 3, \text{ while } \dim[F_{2b)}] = \dim[F_{2c)}] = 2.$$

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