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A STUDY OF DE BRANGES–ROVNYAK AND DIRICHLET SPACES

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Introduction

This doctoral thesis is a presentation of the work done during the Ph.D. program, carried out by the author in Università degli Studi di Milano - Statale, under the supervision of professor Marco M. Peloso, from October 2021 to September 2024. This work is devoted to the study of certain Hilbert spaces of analytic functions on the unit disk $\mathbb{D}$ of the complex plain, $\mathbb{D} := \{w \in \mathbb{C} \colon |w| < 1\}$. The classical Dirichlet space $\mathcal{D}$ [3, 23], the Hardy spaces H^p, $0 < p \leq \infty$, [21, 36, 40], and the Bergman space A^2 [11, 20, 35] are the most well-known function spaces on the open unit disk $\mathbb{D}$. These spaces have been extended in several diverse directions. In this thesis, we are interested in a generalized version of the Dirichlet space $\mathcal{D}$, and a class of Hilbert spaces that arise from the Hardy space H^2.

This dissertation is divided into four parts. The first chapter contains the relevant background definitions and results. The remaining three chapters contain the original results, and are organized as three different papers or preprints. In particular, Chapters 2 and 3 stem from a broad investigation into the relations between the two different classes of spaces of interest for this work, whereas Chapter 4 focuses on a specific problem in Dirichlet spaces.

In this introduction, we give an overview of the content of this thesis and we provide a selection of the original results that were obtained. The third chapter contains a joint paper with Carlo Bellavita, that has already been published, which corresponds to Subsections 3.1 and 3.2. The second and fourth chapters are based on two separate projects that gave rise to two preprints, that are currently under review: the first one in collaboration with Carlo Bellavita and Javad Mashreghi, and the second one with Javad Mashreghi, Mostafa Nasri and William Verreault.

In Chapter 1, we introduce the setting, the main stage where this work takes place, that is, the Hardy space on the unit disk H^2. This mathematical object lies at the intersection between harmonic analysis, complex analysis and operator theory. In this dissertation we recall some of the aspects that have made the space H^2 one of the most studied topics in mathematical analysis in the last century. In Chapter 1, we also define the protagonists of this study, that are two different classes of subsets of H^2, the *de Branges–Rovnyak spaces* $H(b)$ and the *harmonically weighted Dirichlet spaces* $\mathcal{D}_\mu$.

The first class of spaces have been introduced in 1966 by Louis de Branges and James Rovnyak [18], as a generalization of the range space for the multiplication operator T_b. Here, b is an analytic function on $\mathbb{D}$ that is uniformly bounded by 1. In 1994, Donald Sarason [54] started a new approach to this theory and suggested a new equivalent definition for the $H(b)$ spaces. In this formulation, that is the one that we follow in this thesis, the de Branges–Rovnyak spaces are realized as the range of the square root of the positive operator $I - T_b T_b^*$. They are subsets of H^2 that are not necessarily closed in the topology induced by the Hardy norm. In general, the de Branges–Rovnyak space $H(b) = \mathrm{Ran}(I - T_b T_b^*)^{\frac{1}{2}}$ is a Hilbert space with respect to the so-called *range norm*, that in general can be somewhat mysterious. As we will see, the construction is rather abstract, there is no explicit formula that one can use to establish whether a function $f \in H^2$ belongs to a certain $H(b)$ space. This aspect is one of the features that makes this whole theory complicated, yet intriguing.

A special class of de Branges–Rovnyak spaces that we will discuss in great detail are the *model spaces*. They correspond to the $H(b)$ spaces that are, in fact, closed in the Hardy norm and satisfy $\|\cdot\|_{H(b)} = \|\cdot\|_{H^2}$. They are characterized by the fact that the associated function b is *inner*. Their name, and a part of their fortune in literature, is due to certain results by Sz. Nagy and Foias, who understood that the model spaces act as *models* for a certain type of isometries: they appear in representation theorems for completely non-unitary isometries, a class of operators that the very powerful spectral theorem cannot reach. Proper references will be given throughout this work.

The other class of spaces that we treat in this thesis is a weighted version of the Dirichlet space $\mathcal{D}$. The *harmonically weighted* Dirichlet spaces $\mathcal{D}_\mu$ have been introduced by Stefan Richter in 1991 [48], in the context of the representation of cyclic analytic 2-isometries. They also play an important role in the study of the forward shift S on the classic Dirichlet space $\mathcal{D}$: they appear in a very beautiful Beurling-type classification theorem for closed S-invariant subspaces of $\mathcal{D}$ [50]. An important example in this class of spaces is the so-called *local* Dirichlet space. Given a point ζ in the unit circle $\mathbb{T} := \partial\mathbb{D}$ and a function f that is analytic on $\mathbb{D}$, we define the *local* Dirichlet integral of f at ζ as

$$\mathcal{D}_\zeta(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|z-\zeta|^2}\, dA(z).$$

We define the local Dirichlet space as

$$\mathcal{D}_\zeta := \{f \in \mathrm{Hol}(\mathbb{D})\colon \mathcal{D}_\zeta(f) < \infty\},$$

where dA is the two-dimensional Lebesgue measure, i.e., the area measure, and $\mathrm{Hol}(\mathbb{D})$ denotes the space of analytic functions on $\mathbb{D}$. This space is of great importance in this theory because of a disintegration formula that allows to express more general Dirichlet integrals in terms of the local one. Because of this formula, that we will discuss later, in some contexts the study of the quantities $\mathcal{D}_\zeta(f)$ allows one to recover information on the membership of the function f in any harmonically weighted Dirichlet space. One thing that we mention in passing is that every harmonically weighted Dirichlet space contains all polynomials, hence it is dense in H^2. In 1991, Stefan Richter and Carl Sundberg [49] characterized the functions in H^2 that belong to $\mathcal{D}_\zeta$: a function $f \in H^2$ belongs to $\mathcal{D}_\zeta$ if and only if the difference quotient function

$$Q_\zeta f(z) := \frac{f(z) - f(\zeta)}{z - \zeta}, \qquad z \in \mathbb{D}, \tag{1}$$

belongs to H^2. Moreover, in this case, $\mathcal{D}_\zeta(f) = \|Q_\zeta f\|_{H^2}^2$. This result is very important to us, and we show it in this introduction to motivate some of the work that appears in this thesis.

The two classes of spaces that we introduced, the $H(b)$ spaces and the $\mathcal{D}_\mu$ spaces, have both a beautiful structure and interesting properties that will be dealt with in more detail. In 1997, Donald Sarason [55] showed that they are not two completely separate species. From the realization that a special example of a harmonically weighted Dirichlet space, the local Dirichlet space $\mathcal{D}_\zeta$, is in fact also a de Branges–Rovnyak space, many authors have started to study the possible connections between these two classes of spaces. In Chapter 3 we say more about the state of the art of this fascinating line of research, and we provide some original contributions.

In Chapter 2, we study a special class of operators on the spaces $H(b)$, the *difference quotient operators* Q_ζ^b. The idea is to study from an operator-theoretic point of view the fundamental quantities

$$\frac{f(z)-f(\zeta)}{z-\zeta},$$

for functions $f \in H(b)$ and boundary points $\zeta \in \mathbb{T}$. As we briefly explained with Equation (1), such difference quotients characterize the membership in the local Dirichlet space $\mathcal{D}_\zeta$, and can give much information about the membership in more general Dirichlet spaces. In this chapter, there are original results that are interesting on their own, and the analysis that is carried out is instrumental for Chapter 3. In these results that we report, a special role is played by the *spectrum*: given a bounded analytic function b with $\|b\|_{H^\infty} = 1$, we define its spectrum as the set

$$\sigma(b) := \{w \in \mathbb{D} : b(w) = 0\} \cup \{\lambda \in \mathbb{T} : \liminf_{z\to\lambda} |b(z)| < 1\}.$$

We will discuss the set $\sigma(b)$ in much more detail in the preliminaries chapter. In particular, we will show that, if Δ is an open arc that is contained in the complement of the closure of the spectrum $\overline{\sigma(b)}^{\mathrm{cl}}$, i.e., $\Delta \subset \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$, then the function b extends analytically on Δ with $|b| = 1$ on Δ. For a boundary point $\zeta \in \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$, the difference quotient operator

$$Q_\zeta^b f(z) := \frac{f(z)-f(\zeta)}{z-\zeta}, \qquad z \in \mathbb{D},$$

is well-defined and bounded on $H(b)$. In the following original result, we describe the spectrum $\sigma(Q_\zeta^b)$ and we give a lower estimate for the norm of Q_ζ^b. We consider the usual operator norm in the space $\mathcal{B}(H(b))$ of bounded operators on $H(b)$.

Theorem 1. *Let b be a bounded analytic function on $\mathbb{D}$ with $\|b\|_{H^\infty} = 1$, and $\zeta \in \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$. Then, the spectrum of the operator Q_ζ^b is the set*

$$\sigma(Q_\zeta^b) = \left\{ \frac{\overline{\eta}}{1-\zeta\overline{\eta}} : \eta \in \overline{\sigma(b)}^{\mathrm{cl}} \right\}.$$

Also, the following lower estimate holds:

$$\|Q^b_\zeta\|_{\mathcal{B}(H(b))} \geq \frac{1}{\operatorname{dist}(\zeta, \overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T})}. \tag{2}$$

Notice that the inequality in (2) shows that it is actually necessary to consider points outside of the closure of the spectrum of b, to have a bounded operator Q^b_ζ.

Specializing to the case of inner functions, we provide another lower estimate for the norm $\|Q^b_\zeta\|_{\mathcal{B}(H(b))}$. As we will see, the case of inner functions is easier, because the norm of the associated model space coincides with the Hardy norm. Moreover, the spectrum of an inner function is always closed.

Theorem 2. *Let u be an inner function and $\zeta \in \mathbb{T} \setminus \sigma(u)$. Then,*

$$\|Q^u_\zeta\|_{\mathcal{B}(K_u)} \geq \frac{|u''(\zeta)|}{2|u'(\zeta)|}.$$

The next results concern a special class of inner functions, the *one-component* inner functions. As we will see, these special functions have certain properties that allow for easier computations, in the context of difference quotients. The following upper estimate for the norm holds.

Theorem 3. *Let u be a one-component inner function, and let $\zeta \in \mathbb{T}\setminus\sigma(u)$. Then,*

$$\|Q^u_\zeta\|_{\mathcal{B}(K_u)} \leq C_u|u'(\zeta)|, \tag{3}$$

where C_u is a positive quantity not depending on ζ.

This beautiful match between one-component inner functions and the difference quotient operator reaches its climax in the next theorem, where a characterization of the property of being one-component is given in terms of the difference quotient operator.

Theorem 4. *Let u be an inner function. Then, u is a one-component inner function if and only if there exists a positive constant C_u such that for every $\zeta \in \mathbb{T}\setminus\sigma(u)$ the inequality (3) holds and $m(\sigma(u)) = 0$, where m denotes the normalized Lebesgue measure on $\mathbb{T}$.*

In Chapter 3, as we mentioned before, we treat the topic of the different relations between $H(b)$ spaces and $\mathcal{D}_\mu$ spaces. We report some important results and recent contributions to this study, concerning when a de Branges–Rovnyak space is also a Dirichlet space, and vice versa, with equality or equivalence of norms. As we will further motivate, we are also interested in the possibility that a $H(b)$ space simply *embeds* into $\mathcal{D}_\mu$, without necessarily a total identity of the sets.

Before we give more details about the content of this chapter, we have to tell something more about the structure of the $H(b)$ spaces. Basically, the theory splits into two main categories, depending on whether the function b associated to the $H(b)$ space is an extreme point of the closed unit ball of H^∞. For functions b that are extreme, the corresponding $H(b)$ space cannot contain all polynomials, and in this case an embedding result $H(b) \hookrightarrow \mathcal{D}_\mu$ is the best that we can hope for. In general, $H(b)$ spaces associated to extreme functions b may be very "small", even finite-dimensional. We mention that inner functions are always extreme, hence this is the case of the model spaces. On the other hand, when the function b is non-extreme, the space $H(b)$ contains all polynomials. In particular, the corresponding $H(b)$ space is infinite-dimensional, it is dense in H^2, and we may have an equality of sets $H(b) = \mathcal{D}_\mu$. When the function b is non-extreme, one can construct an auxiliary function a that plays an important role in the study of the associated $H(b)$ space. We will say that a is the *Pythagorean mate* of b.

In Chapter 3, we study the embedding phenomenon in some special cases. We give a sufficient condition for the embedding into local Dirichlet spaces.

Theorem 5. *Let b be a bounded analytic function with $\|b\|_{H^\infty} = 1$, and let $\zeta \in \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$. Then, we have the embedding $H(b) \hookrightarrow \mathcal{D}_\zeta$.*

We also give a necessary condition for the embedding $H(b) \hookrightarrow \mathcal{D}_\zeta$.

Theorem 6. *Let b be a bounded analytic function with $\|b\|_{H^\infty} = 1$, and let $\zeta \in \mathbb{T} \cap \sigma(b)$. Then, the de Branges–Rovnyak space $H(b)$ does not embed into the local Dirichlet space $\mathcal{D}_\zeta$.*

In some special cases, these two conditions combined provide a complete characterization of the embedding. For example, this is the case of the embedding of model spaces into the local Dirichlet spaces, or any $H(b)$ space that is associated to a function b whose spectrum $\sigma(b)$ is closed. We also completely characterize the embedding of a $H(b)$ space associated to a non-extreme function into a $\mathcal{D}_\mu$ space, where the associated measure μ is a finite sum of Dirac deltas,

$$\mu = \sum_{j=1}^{N} \alpha_j \delta_{\zeta_j}, \tag{4}$$

with positive weights $\alpha_j > 0$.

Theorem 7. *Let b be a non-extreme function in H^∞, a its Pythagorean mate, and μ an atomic measure as in (4). Then, we have the embedding $H(b) \hookrightarrow \mathcal{D}_\mu$ if and only if the following conditions hold:*

(i) There exists $g \in H^\infty$ such that a has the form

$$a = \left(\prod_{j=1}^{N} (z - \zeta_j) \right) g;$$

(ii) $\{\zeta_1, \dots, \zeta_N\} \cap \sigma(b) = \emptyset$.

We conclude the discussion of this embedding with a characterization of the identity $H(b) = \mathcal{D}_\mu$, completing a previous result in literature.

Theorem 8. *Let b be a non-extreme function in H_1^∞, a its Pythagorean mate, and μ an atomic measure as in (4). Then, we have the equality $H(b) = \mathcal{D}_\mu$ if and only if the following conditions hold:*

(i) There exists $g \in H^\infty$ with $\inf_{\mathbb{D}} |g| > 0$ such that a has the form

$$a = \left(\prod_{j=1}^{n}(z - \zeta_j)\right) g;$$

(ii) $\{\zeta_1, \ldots, \zeta_N\} \cap \sigma(b) = \emptyset$.

As previously indicated in the introduction, the last chapter diverges from the main topic of the thesis. This final piece of work was carried out during a visiting period at Université Laval, in Québec City (CA), under the supervision of professor Javad Mashreghi. In Chapter 4, we deal with a specific question about polynomial approximation schemes in Dirichlet spaces. Given the analytic function $f(z) = \sum_{k=0}^{n} a_k z^k$ in an appropriate weighted Dirichlet space, we study the generalized Cesàro means

$$(\sigma_n^\alpha f)(z) = \binom{n+\alpha}{\alpha}^{-1} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} a_k z^k,$$

where α is a parameter in the interval $[0,1]$. We point out that, for $\alpha = 0$, we recover the n-th partial Taylor sum

$$(\sigma_n^0 f)(z) = \sum_{k=0}^{n} a_k z^k,$$

while for $\alpha = 1$, we obtain the standard Cesàro mean

$$(\sigma_n^1 f)(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k z^k.$$

We investigate the convergence $\sigma_n^\alpha f \to f$ as $n \to \infty$. To this end, it is enough to study such convergence in the local Dirichlet space $\mathcal{D}_1$, to recover analogous results in more general weighted Dirichet spaces. In particular, we are interested in the asymptotical behavior of the norm $\|\sigma_n^\alpha\|$, as $n \to \infty$, and its dependence on the parameter $\alpha \in [0,1]$. In [41] it was proved that the approximation $\|\sigma_n^\alpha f - f\|_{\mathcal{D}_1} \to 0$ is valid if and only if $\alpha > \frac{1}{2}$. Hence, the value $\alpha = 1/2$ is a threshold point, and that is why in the following we have different theorems about the behavior of $\|\sigma_n^\alpha\|$, with different flavors, corresponding to whether $\alpha > 1/2$, $\alpha = 1/2$, or $\alpha < 1/2$. The notation $f(n) \sim g(n)$ means that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Theorem 9. *Let* $\alpha < \frac{1}{2}$. *Then*

$$\|\sigma_n^\alpha\| \sim C_\alpha n^{\frac{1}{2}-\alpha},$$

where

$$C_\alpha := \Gamma(\alpha+1)\frac{\Gamma(1-2\alpha)^{1/2}}{\Gamma(1-\alpha)}$$

is a finite positive constant.

Theorem 10. *Let* $\alpha = \frac{1}{2}$. *Then*

$$\|\sigma_n^{\frac{1}{2}}\| \sim \frac{1}{2}\log^{1/2} n.$$

The result concerning the case $\alpha > \frac{1}{2}$ is more complicated, and we will not discuss it in this introduction.

Chapter 1

Preliminaries

1.1 Measures and Poisson kernels

We will be needing some basic yet fundamental results in measure theory. For this part, we follow [30]. Standard references are [24, 52]. We will consider mainly measures on the unit circle $\mathbb{T}$, defined on the Borel σ-algebra, that is, the smallest σ-algebra that contains all open arcs of $\mathbb{T}$. A Borel measure on $\mathbb{T}$ is a countably additive function that assigns to each Borel subset of $\mathbb{T}$ a complex number. We denote $M(\mathbb{T})$ the set of all such measures. An example of an element of $M(\mathbb{T})$, and perhaps the most important, is the Lebesgue measure m, normalized so that $m(\mathbb{T}) = 1$. We are mainly interested in the set of positive measures, denoted $M_+(\mathbb{T})$, i.e. measures μ such that $\mu(E) \geq 0$ for every Borel set E.

We distinguish between the notions of *support* and *carrier* for a positive measure μ. Consider the union $\mathcal{U}$ of all the open sets O of $\mathbb{T}$ having measure $\mu(O) = 0$. Then, we define the support of μ as the complement

$$\operatorname{supp}(\mu) := \mathbb{T} \setminus \mathcal{U}.$$

Equivalently, we can define the support of μ as the set of points $\zeta \in \mathbb{T}$ such that every arc Δ containing ζ has positive measure $\mu(\Delta) > 0$. Notice that the support is a closed subset of $\mathbb{T}$. On the other hand, a carrier for μ is any set E such that, for every Borel set $A \subset \mathbb{T}$,

$$\mu(A) = \mu(A \cap E).$$

We also say that μ is *carried* by the set E.

In general, the support of a measure is unique while a carrier is not. Also, the support is always a carrier, however a carrier need not be the support. For example, taking the normalized Lebesgue measure m on $\mathbb{T}$, its support is the whole circle $\mathbb{T}$, but if we remove any collection of finitely many points from $\mathbb{T}$ we would get a carrier for m. Note also that, in fact, a carrier might not even be closed.

We have the following notion of derivative for elements of $M(\mathbb{T})$.

Definition 1.1. For a measure $\mu \in M(\mathbb{T})$ we define the *symmetric derivative*

$D\mu(\zeta)$ at $\zeta \in \mathbb{T}$ as the limit

$$D\mu(\zeta) := \lim_{t \to 0^+} \frac{\mu\big((e^{-it}\zeta, e^{it}\zeta)\big)}{m\big((e^{-it}\zeta, e^{it}\zeta)\big)},$$

when this limit exists, where $(e^{-it}\zeta, e^{it}\zeta)$ is the arc having extremes $e^{-it}\zeta$ and $e^{it}\zeta$, for $t > 0$.

This notion is strongly related to the Radon-Nikodym decomposition of a measure. The notation $\mu \perp m$ denotes that μ is singular with respect to m, that is, that the measures μ and m are carried by two disjoint sets. The following result holds.

Theorem 1.2. *Let $\mu \in M(\mathbb{T})$. The following properties hold:*

(i) $D\mu(\zeta)$ exists for m-almost every $\zeta \in \mathbb{T}$ and

$$D\mu = \frac{d\mu}{dm} \qquad m - a.e.,$$

where $d\mu/dm$ is the Radon-Nikodym derivative of μ with respect to m;

(ii) $\mu \perp m$ if and only if $D\mu = 0$ m-almost everywhere;

(iii) If μ is positive and $\mu \perp m$, then $D\mu = \infty$ μ-almost everywhere. In particular, μ is carried by the set $\{\zeta \in \mathbb{T} \colon D\mu(\zeta) = \infty\}$.

Proof. See Theorems 7.14 and 7.15 in [52]. □

We introduce the Poisson kernel. The function

$$P_z(\zeta) := \frac{1 - |z|^2}{|z - \zeta|^2}, \qquad z \in \mathbb{D}, \quad \zeta \in \mathbb{T},$$

is called the *Poisson kernel*. This function plays a crucial role in many areas of mathematics, in particular in harmonic analysis on the unit circle $\mathbb{T}$ and in complex analysis, due to his many properties. We report some of them.

A simple computation shows that

$$P_z(\zeta) = \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right), \qquad z \in \mathbb{D}, \quad \zeta \in \mathbb{T}.$$

Hence, for fixed $\zeta \in \mathbb{T}$, the mapping $z \in \mathbb{D} \mapsto P_z(\zeta) \in (0, +\infty)$ is a positive harmonic function on $\mathbb{D}$. A computation using the geometric series expression

$$P_{r\lambda}(\zeta) = \sum_{n \in \mathbb{Z}} r^{|n|} \lambda^n \overline{\zeta}^n, \qquad r \in (0,1), \quad \lambda \in \mathbb{T},$$

shows that

$$\int_{\mathbb{T}} P_z(\zeta)\, dm(\zeta) = 1, \qquad z \in \mathbb{D}.$$

Given a measure $\mu \in M(\mathbb{T})$, its Poisson integral is

$$P\mu(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2}\, d\mu(\zeta),$$

which is well-defined since the mapping $\mathbb{T} \ni \zeta \mapsto P_z(\zeta)$ is continous for every z in $\mathbb{D}$. A crucial result that we will need is Fatou's Theorem. The following is Theorem 1.10 in [30].

Theorem 1.3 (Fatou)**.** *Let $\mu \in M(\mathbb{T})$ and $\lambda \in \mathbb{T}$. If the symmetric derivative $D\mu(\lambda)$ exists, then*

$$\lim_{r \to 1^-} P\mu(r\lambda) = D\mu(\lambda).$$

Thus, $P\mu$ has a finite radial limit m-almost everywhere on $\mathbb{T}$.

Poisson kernels are instrumental in representing harmonic functions. The following fundamental result by Herglotz is Theorem 1.18 in [30].

Theorem 1.4 (Herglotz)**.** *A function u on $\mathbb{D}$ is positive and harmonic if and only if there exists a positive measure $\mu \in M_+(\mathbb{T})$ such that $u = P\mu$. Also, such measure μ is unique.*

In the last chapter, we will also see an analogous for a more general class of functions, the *super-harmonic* functions.

Given a finite positive Borel measure μ on the unit circle $\mathbb{T}$, another function associated to μ that is important for this work is the *potential*

$$V_\mu \colon \mathbb{C} \ni z \mapsto \int_{\mathbb{T}} \frac{1}{|\lambda - z|^2}\, d\mu(\lambda) \in [0, +\infty].$$

Following [17], we list some properties of V_μ that we will use in this work. We also give a short proof.

Proposition 1.5. *Let μ be a finite positive Borel measure on the unit circle $\mathbb{T}$. The following properties hold:*

1. *V_μ is lower-semicontinuous on $\mathbb{C}$ and continuous on $\mathbb{C} \setminus supp(\mu)$;*
2. *For $z \in \mathbb{C}$, it holds*

$$\frac{\mu(\mathbb{T})}{(1+|z|)^2} \leq V_\mu(z) \leq \frac{\mu(\mathbb{T})}{\operatorname{dist}(z, supp(\mu))^2};$$

3. *$V_\mu = \infty$ μ-a.e. on $\mathbb{T}$.*

Proof. The first two properties are trivial. The last one follows from the following claim: given $e^{i\alpha} \in \mathbb{T}$, if the symmetric derivative at $e^{i\alpha}$ exists and $D\mu(e^{i\alpha}) > 0$, then $V_\mu(e^{i\alpha}) = +\infty$. Indeed, for $t \in (0, 2\pi)$, let I_t be the arc

$$I_t := \{e^{i\theta} \colon \alpha - t < \theta < \alpha + t\}.$$

Since $e^{i\alpha} \in I_t$ for every $t > 0$, it holds

$$V_\mu(e^{i\alpha}) \geq \int_{I_t} \frac{1}{|e^{i\alpha} - e^{i\theta}|^2} \, d\mu(e^{i\theta}) \geq \frac{1}{|I_t|} \frac{\mu(I_t)}{|I_t|}.$$

Since the arc length $|I_t|$ is comparable with $m(I_t)$, and $D\mu(e^{i\alpha}) > 0$, it follows that $V_\mu(e^{i\alpha}) = +\infty$, as we claimed. Now, the proof follows from taking the Radon-Nikodym decomposition of μ, namely $d\mu = f dm + d\mu_s$, with $f \in L^1(m)$ and $\mu_s \perp m$, and the properties of the symmetric derivatives of measures in Theorem 1.2. □

1.2 Hardy spaces

In this section we introduce the Hardy spaces H^p. These are arguably the most important and most studied spaces of holomorphic functions in the unit disk. As we will briefly explain, H^p spaces correspond to an analytic version of the L^p spaces on the unit circle $\mathbb{T}$, with respect to the Lebesgue measure. There exist versions of Hardy spaces in other domains, for example in the upper half plane $\{z \in \mathbb{C} \colon \Im(z) > 0\}$, and also in several complex variables. For the presentation of this very classic topic, we follow [28] and [30]. See also [31].

For an analytic function f on the unit disk $\mathbb{D}$ and $r \in (0, 1)$, we denote by f_r the function

$$f_r(z) := f(rz), \qquad z \in \mathbb{D}.$$

The function f_r is holomorphic on the closed disk $\overline{\mathbb{D}}^{\mathrm{cl}}$, and this allows us to give the following definition.

Definition 1.6. For $p \in (0, +\infty]$, the Hardy space H^p is the space of holomorphic functions on $\mathbb{D}$ such that

$$\|f\|_{H^p} := \sup_{0<r<1} \|f_r\|_{L^p(\mathbb{T})} < \infty.$$

For $p \geq 1$, the space H^p paired with the norm $\|\cdot\|_{H^p}$ is a Banach space and, as per usual, the special case $p = 2$ provides a Hilbert space. For $p \in (0, 1)$, the function $\|\cdot\|_{H^p}$ provides a seminorm that is not a norm, and the corresponding H^p spaces are quasi-Banach spaces. In this thesis, we will only deal with the case $p \geq 1$, and we will be mainly interested in H^2 and H^∞. However, an application of Hölder's inequality gives the containment property

$$H^p \subseteq H^q \iff p \geq q,$$

and thus we will present some properties of the space H^1, which contains all H^p spaces with $p \geq 1$. We point out that some of these properties that we discuss for H^1 also hold for the quasi-Banach case $p \in (0, 1)$, but we will not mention it again.

The following result clarifies the connection with the L^p spaces that was mentioned at the beginning of this chapter.

Theorem 1.7. *Let $f \in H^p, 1 \leq p \leq \infty$. Then the radial limit*

$$f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta) \tag{1.1}$$

exists for almost every $\zeta \in \mathbb{T}$, the boundary value function f^* belongs to L^p, and it holds $\|f\|_{H^p} = \|f^*\|_{L^p}$. Also, if $f \in H^1$, then the Fourier coefficients of f^* are described by

$$\widehat{f^*}(n) = \begin{cases} \frac{f^{(n)}(0)}{n!}, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Finally, we have the Poisson and Cauchy integral representations,

$$f(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} f^*(\zeta)\, dm(\zeta) = Pf^*(z), \qquad z \in \mathbb{D}; \tag{1.2}$$

$$f(z) = \int_{\mathbb{T}} \frac{f^*(\zeta)}{1 - \overline{\zeta} z}\, dm(\zeta), \qquad z \in \mathbb{D}. \tag{1.3}$$

The last theorem gathers many important facts about Hardy spaces. First of all, that H^p is isometrically isomorphic to the (closed) subspace of L^p consisting of functions whose Fourier coefficients vanish on the negative integers. In particular, the Taylor coefficients of f coincide with the Fourier coefficients of the boundary function f^*. In the Hilbert case $p = 2$, we can consider the orthogonal projection $P_+ \colon L^2 \to H^2$. This map is called the Riesz projection, and it acts as

$$P_+ \left(\sum_{n \in \mathbb{Z}} \widehat{f}(n) \zeta^n \right) = \sum_{n \geq 0} \widehat{f}(n) z^n, \qquad f \in L^2.$$

The Riesz projection returns the "analytic part" of a Fourier series in L^2. As we will see later, the Riesz projection can also be expressed in terms of the Cauchy integral.

We have an explicit formula to compute the Hardy norm, by means of the L^p norm. In particular, for $f, g \in H^2$,

$$\langle f, g \rangle_{H^2} = \langle f, g \rangle_{L^2} = \int_{\mathbb{T}} f \overline{g}\, dm.$$

Also, Equations (1.2) and (1.3) tell us that the Poisson integral and the Cauchy transform act as bridges between these two different realms. Finally, we remark that the existence of the radial limit $f(\zeta)$ in (1.1) can actually be improved to the existence of a limit in a bigger, non-tangential, region. Given a point $\zeta \in \mathbb{T}$ and $\alpha > 1$, we define the *Stolz* region as

$$\Gamma_\alpha(\zeta) := \{z \in \mathbb{D} \colon |z - \zeta| \leq \alpha(1 - |z|)\}.$$

We say that f has non-tangential limit L at ζ provided that, for every $\alpha > 1$, it holds

$$\lim_{z \to \zeta, z \in \Gamma_\alpha(\zeta)} f(z) = L.$$

We denote this special non-tangential limit with

$$L = \angle \lim_{z \to \zeta} f(z).$$

It can be proved that a function $f \in H^1$ has non-tangential limit at almost every point in $\mathbb{T}$. In what follows, with a standard abuse of notation, we will use the same symbol f to denote both the function on $\mathbb{D}$ and its boundary values on $\mathbb{T}$.

We recall the inner-outer factorisation for H^p functions. To this end, we follow Chapter 17 of [52] and Chapter 2 of [30]. We say that

1. an *inner* function is a function $u \in H^\infty$ for which $|u| = 1$ m-a.e. on $\mathbb{T}$,

2. an *outer* function is an analytic function O on $\mathbb{D}$ of the form

$$O(z) = \exp\left\{ \int_{\mathbb{T}} \frac{\lambda + z}{\lambda - z} \varphi(\lambda)\, dm(\lambda) \right\}, \qquad z \in \mathbb{D},$$

   where φ is a real-valued L^1 function on $\mathbb{T}$.

We are able to explicitly describe the structure of an inner function. They are made up by two main factors: a Blaschke product and a singular inner function. We recall that a sequence $(a_n)_n \subseteq \mathbb{D}$ satisfies the *Blaschke condition* if

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty. \tag{1.4}$$

A sequence that satisfies (1.4) is called a *Blaschke sequence.*

Theorem 1.8 (Blaschke, F. Riesz)**.** *If $(a_n)_n \subseteq \mathbb{D} \setminus \{0\}$ is a Blaschke sequence, then the infinite product*

$$B(z) := \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}, \qquad z \in \mathbb{D},$$

converges uniformly on compact subsets of $\mathbb{D}$. Moreover, B is an inner function.

We call *Blaschke product* any function of the form

$$B(z) = e^{i\gamma} z^N \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}, \qquad z \in \mathbb{D},$$

where $(a_n)_n \subseteq \mathbb{D} \setminus \{0\}$ is a Blaschke sequence, $N \in \mathbb{N}$ and $\gamma \in \mathbb{R}$. Note that, by construction, every Blaschke product has some zeros inside the disk: the zero set is precisely $\{a_n\}_n$. The second class of inner functions that we mentioned, however, are distinguished by the property of being nowhere zero on $\mathbb{D}$.

Theorem 1.9. *Let τ be a positive measure on $\mathbb{T}$ that is singular with respect to the Lebesgue measure m. The function*

$$S_\tau(z) := \exp\left\{ - \int_{\mathbb{T}} \frac{\lambda + z}{\lambda - z}\, d\tau(\lambda) \right\}, \qquad z \in \mathbb{D},$$

is inner.

A function of the form $e^{i\gamma}S_\tau$, with $\gamma \in \mathbb{R}$, is called a singular inner function. Blaschke products and singular inner functions form the building blocks of all non-constant inner functions.

Theorem 1.10 (Nevanlinna, F. Riesz)**.** *Let u be an inner function. Let*

$$\Lambda := \{z \in \mathbb{D} \setminus \{0\} : u(z) = 0\} = (a_n)_n,$$

counting multiplicity. Then, Λ is a Blaschke sequence and u can be written as the product

$$u = z^N B_\Lambda S_\tau,$$

where N is the multiplicity of the zero of u at $z = 0$, B_Λ is the Blaschke product associated to zero set $(a_n)_n \subseteq \mathbb{D} \setminus \{0\}$ and S_τ is a singular inner function. This factorisation is unique up to a unimodular constant.

Now, we recall the inner-outer factorisation for H^p functions.

Theorem 1.11. *Let $f \in H^p \setminus \{0\}$, for $1 \leq p \leq \infty$. Then, $\log|f| \in L^1(\mathbb{T})$, the outer function*

$$O_f(z) := \exp\left\{ \int_{\mathbb{T}} \frac{\lambda + z}{\lambda - z} \log|f(\lambda)|\, dm(\lambda) \right\}, \qquad z \in \mathbb{D},$$

belongs to H^p, and there exists an inner function u_f such that

$$f = u_f O_f.$$

Notice that, in particular, if $f \in H^p \setminus \{0\}$ is an outer function, then

$$f(z) = \exp\left\{ \int_{\mathbb{T}} \frac{\lambda + z}{\lambda - z} \log|f(\lambda)|\, dm(\lambda) \right\}, \qquad z \in \mathbb{D}. \tag{1.5}$$

In other words, outer functions are entirely determined by the modulus of their boundary values. This of course does not hold for inner functions, since by definition every inner function u satisfies $|u| = 1$ a.e. on $\mathbb{T}$.

We conclude this part on the inner-outer factorisation with some concrete examples of outer functions. This simple class of functions will come up in Section 3.3. Using Equation (1.5), one can deduce the following characterization. For a proof, see [30, Proposition 3.22].

Proposition 1.12. *Suppose $f \in H^2$. Then f is outer if and only if*

$$\log|f(0)| = \int_{\mathbb{T}} \log|f(\lambda)|\, dm(\lambda).$$

It also holds the following sufficient condition. For a proof, see [30, Corollary 3.23].

Proposition 1.13. *If $f \in H^2$ satisfies $\mathrm{Re}(f) > 0$ on $\mathbb{D}$, then f is outer. In particular, given any self-map of the unit disk $\varphi\colon \mathbb{D} \to \mathbb{D}$, the function $\varphi + 1$ is outer.*

It follows that the binomial $z-\gamma$ is outer, for any $\gamma \in \mathbb{C} \setminus \mathbb{D}$. It is not relevant for the thesis, but for $\gamma \in \mathbb{D}$, the binomial $z-\gamma$ cannot be outer, for it vanishes in γ. In this case, its inner-outer factorization is given by

$$z - \gamma = \frac{\gamma - z}{1 - \overline{\gamma} z} \, \overline{\gamma} \left(z - \frac{1}{\overline{\gamma}} \right).$$

Now that we have described the inner-outer factorisation, we can introduce two special classes of functions.

Definition 1.14. The Nevanlinna class $\mathcal{N}$ is the set of all analytic functions on $\mathbb{D}$ that can be written as the quotient of two bounded analytic functions,

$$\mathcal{N} := \left\{ \frac{h_1}{h_2} : h_1 \in H^\infty, h_2 \in H^\infty \setminus \{0\} \right\}.$$

The Smirnov class $\mathcal{N}_+$ is a special subset of $\mathcal{N}$, where we ask that the denominator is outer:

$$\mathcal{N}_+ := \left\{ \frac{h_1}{h_2} : h_1 \in H^\infty, h_2 \in H^\infty \setminus \{0\} \text{ outer} \right\}.$$

The Smirnov class has the following important property.

Theorem 1.15 (Smirnov maximum principle)**.** *If $f \in \mathcal{N}_+$ and*

$$\int_{\mathbb{T}} |f|^2 dm < \infty,$$

then $f \in H^2$. If the boundary function f belongs to L^∞, then $f \in H^\infty$.

1.2.1 H^2: kernels and shifts

A central fact in the theory of H^p spaces is that the polynomials are dense in H^p. In particular, for $p = 2$, starting from the Taylor series $f(z) = \sum_k a_k z^k$, one can deduce the following formula:

$$\begin{aligned}
\|f\|_{H^2}^2 &= \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \\
&= \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \right) \left(\sum_{j=0}^{\infty} \overline{a_j} r^j e^{-ij\theta} \right) d\theta \\
&= \lim_{r \to 1} \sum_k |a_k|^2 r^2 \\
&= \sum_{k=0}^{\infty} |a_k|^2.
\end{aligned}$$

Not only is the formula $\|f\|_{H^2}^2 = \sum_{k \geq 0} |a_k|^2$ very beautiful and useful on its own, but we can also apply it to show that the partial Taylor sums

$$S_n f = \sum_{k=0}^{n} a_k z^k$$

converge to f in H^2, as $n \to \infty$. One has

$$\|f - S_n f\|_{H^2}^2 = \sum_{k=n+1}^{\infty} |a_k|^2 \xrightarrow[n\to\infty]{} 0.$$

The same result does not hold for example in H^1. One can use the Cesàro means to produce an explicit example of approximation with polynomials in this space, but the Taylor approximation scheme fails. We will say more about these polynomial approximations in the last chapter. In the case $p = 2$, in analogy with the Fourier basis in L^2, the monomials $\{z^n\}_{n\geq 0}$ form an orthonormal basis for H^2.

For $w \in \mathbb{D}$, we introduce the Cauchy-Szegö kernel

$$c_w(z) = \frac{1}{1 - \overline{w}z}, \qquad z \in \mathbb{D}.$$

Due to the Cauchy integral formula (1.3), we have that

$$f(w) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta}w}\, dm(\zeta) = \langle f, c_w \rangle_{H^2}, \qquad f \in H^2.$$

This tells us that H^2 is a *reproducing kernel Hilbert space.*

Definition 1.16. Let H be a Hilbert space of functions on $\mathbb{D}$. We say that H is a reproducing kernel Hilbert space (RKHS) if there exists a function $k\colon \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that for every $w \in \mathbb{D}$ the function $k(\cdot, w) \in H$ and it satisfies the reproducing kernel formula

$$f(w) = \langle f, k(\cdot, w) \rangle_H, \qquad f \in H.$$

Equivalently, H is a RKHS if for every $w \in H$ the evaluation functional $H \ni f \mapsto f(w) \in \mathbb{C}$ is bounded.

Using the reproducing kernels c_w, we have the following expression for the Riesz projection $P_+\colon L^2 \to H^2$. For $f \in L^2$, we have that

$$P_+ f(w) = \langle f, c_w \rangle_{L^2} = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta}w}\, dm(\zeta), \qquad w \in \mathbb{D}.$$

This is because, since $P_+ f \in H^2$ and P_+ is an orthogonal projection,

$$P_+ f(w) = \langle P_+ f, c_w \rangle_{H^2} = \langle f, P_+ c_w \rangle_{L^2} = \langle f, c_w \rangle_{L^2}.$$

We introduce the (forward) shift operator S on the Hardy space H^2. We follow [30]. For $f \in H^2$, we define

$$Sf(z) := zf(z), \qquad z \in \mathbb{D}.$$

One can easily check that S is bounded on H^2. More precisely, it is an isometry, i.e. $\|Sf\|_{H^2} = \|f\|_{H^2}$. The forward shift S plays a crucial role in operator theory, for it represents a prototype for non-unitary isometries. We would need an entire section to explain this last statement, and this topic is not really in the goals of this thesis, so we will not add anything to this.

If we study the action of S on the Taylor coefficients, with the standard identification

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^2 \longleftrightarrow (a_0, a_1, a_2, \dots,) \in \ell^2,$$

then it is easy to see that S acts on ℓ^2

$$S \colon \ell^2 \ni (a_0, a_1, a_2, \dots,) \mapsto (0, a_0, a_1, \dots,) \in \ell^2.$$

With this in mind, it is clear that

$$\operatorname{Ker}(S) = \{0\}, \qquad \operatorname{Ran}(S) = \{f \in H^2 \colon f(0) = 0\} = zH^2.$$

Given an Hilbert space H, we denote $\mathcal{B}(H)$ the algebra of bounded linear operators on H and $\|\cdot\|_{\mathcal{B}(H)}$ the operator norm, i.e.

$$\|A\|_{\mathcal{B}(H)} := \sup_{x\in H, \|x\|_H = 1} \|Ax\|_H.$$

We recall that, given an operator $A \in \mathcal{B}(H)$, its *resolvent* $\rho(A)$ is the set

$$\rho(A) := \{\lambda \in \mathbb{C} \colon \lambda I - A \text{ is invertible in } \mathcal{B}(H)\},$$

and its *spectrum* is the complement

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

Also, the *point spectrum* is the set

$$\sigma_p(A) := \{\lambda \in \mathbb{C} \colon \lambda I - A \text{ is not injective}\}.$$

In other words, $\sigma_p(A)$ is the set of the eigenvalues of A, and clearly $\sigma_p(A) \subseteq \sigma(A)$. By Gelfand Theorem, the spectrum $\sigma(A)$ is a non-empty compact set. We recall the spectral properties of the shift S. We recall that, to avoid ambiguity with the notation for the conjugation on $\mathbb{C}$, we denote the closure of a set E with $\overline{E}^{\text{cl}}$.

Proposition 1.17. *Let S be the (forward) shift operator on H^2. Then:*

- *The spectrum $\sigma(S)$ is the closed disk $\overline{\mathbb{D}}^{\text{cl}}$;*
- *The point spectrum $\sigma_p(S)$ is the empty set.*

We discuss the adjoint S^*. Let $f \in H^2$. First, notice that the function

$$\mathbb{D} \ni z \mapsto \frac{f(z) - f(0)}{z} \in \mathbb{C}$$

is analytic and in H^2. We can write

$$S^* f(z) = \langle S^* f, c_z \rangle_{H^2} = \langle f, z c_z \rangle_{H^2} = \langle f - f(0), z c_z \rangle_{H^2},$$

since in H^2 the set $\mathbb{C}$ is orthogonal to zH^2. Thus,

$$\begin{aligned} S^*f(z) &= \langle f - f(0), zc_z\rangle_{H^2} \\ &= \int_{\mathbb{T}} \big(f(\zeta) - f(0)\big)\,\overline{\zeta c_z(\zeta)}\,dm(\zeta) \\ &= \int_{\mathbb{T}} \frac{f(\zeta) - f(0)}{\zeta}\,\overline{c_z(\zeta)}\,dm(\zeta) \\ &= \frac{f(z) - f(0)}{z}. \end{aligned}$$

The operator S^* is called the backward shift operator. Notice that S^* acts on the Taylor coefficient as

$$S^*\colon \ell^2 \ni (a_0, a_1, a_2, \dots,) \mapsto (a_1, a_2, a_3 \dots,) \in \ell^2.$$

In particular, it is clear that $\operatorname{Ker}(S^*) = \mathbb{C}$ and $\operatorname{Ran}(S^*) = H^2$. We introduce the following notation for a special class of operators. Given two points in a Hilbert space, $x, y \in H \setminus \{0\}$, we define $x \otimes y$ as the rank-one operator

$$\begin{array}{rccc} x \otimes y: & H & \longrightarrow & H \\ & h & \longmapsto & \langle h, y\rangle_H x \end{array}. \tag{1.6}$$

Notice that $\operatorname{Ran}(x \otimes y) = \mathbb{C}x$.

The operators S and S^* satisfy the relations $S^*S = I$ and

$$SS^* = I - c_0 \otimes c_0,$$

where $c_0 \otimes c_0$ is the rank-one operator given by

$$(c_0 \otimes c_0)f = \langle f, c_0\rangle_{H^2}\, c_0 = f(0), \qquad f \in H^2,$$

for the Cauchy-Szegö kernel c_0 is identically 1. Concerning the spectrum of S^*, it follows from Proposition 1.17 that $\sigma(S^*)$ is the closed disk $\overline{\mathbb{D}}^{\mathrm{cl}}$. However, unlike the forward shift S, in this case we have that the point spectrum $\sigma_p(S^*)$ is the open disk $\mathbb{D}$. In particular, it is not hard to show that

$$S^*c_w = \overline{w}c_w, \qquad w \in \mathbb{D}.$$

We conclude this section discussing the shift-invariant subspaces of H^2. Notice that the multiplication operator with an inner function u

$$T_u\colon H^2 \ni f \mapsto uf \in H^2$$

is an isometry on H^2, due to the fact that $|u| = 1$ a.e. on $\mathbb{T}$. The shift S is a special case of this class of operators, for $S = T_z$. In particular, the subspace uH^2 is always closed in H^2, whenever u is inner. It is easy to check that uH^2 is a shift-invariant subspace of H^2, i.e. $S(uH^2) \subseteq uH^2$. Surprisingly, these simple spaces uH^2 turn out to be exactly all the closed S-invariant subspaces of H^2.

Theorem 1.18 (Beurling)**.** *Let M be a non-zero closed shift-invariant subspace of H^2. Then, there exists an inner function u such that $M = uH^2$. Also, the function u is unique up to unimodular constant.*

We will devote an entire section to the closed S^*-invariant subspaces of H^2, the so-called *model spaces*.

1.2.2 H^∞: spectrum and Corona theorem

Given a function $b \in H^\infty$ with $\|b\|_{H^\infty} \leq 1$, we are interested in characterising the open arcs $\Delta \subseteq \mathbb{T}$ for which b has an analytic extension across Δ with the property that $|b| = 1$ on Δ. We introduce the notation

$$H_1^\infty := \{b \in H^\infty \colon \|b\|_{H^\infty} \leq 1\}.$$

To extend analytically our functions, we appeal to the Schwarz reflection principle, see for example [52, Theorem 11.14]. We recall the result for the unit disk.

Theorem 1.19. *[28, Theorem 4.21] Let Δ be an open subarc of $\mathbb{T}$ and f an analytic function on $\mathbb{D}$. Suppose that*

$$\lim_n Im(f(z_n)) = 0$$

for every sequence $(z_n)_n$ in $\mathbb{D}$ converging to a point in Δ. Then, f can be analytically extended through Δ.

We will use the following corollary, for which we could not provide any specific reference. We also give a short proof.

Corollary 1.20. *Let Δ be an open subarc of $\mathbb{T}$ and f an analytic function on $\mathbb{D}$. Suppose that*

$$\lim_n |f(z_n)| = 1$$

for every sequence $(z_n)_n$ in $\mathbb{D}$ converging to a point in Δ. Then, f can be analytically extended through Δ.

Proof. Let $\zeta \in \Delta$. By assumption,

$$\lim_{z \in \mathbb{D}, z \to \zeta} |f(z)| = 1,$$

and we can select an open disk $D_\zeta \ni \zeta$ such that f never vanishes on $D_\zeta \cap \mathbb{D}$. The function $g := i \log f$ is well defined and analytic on $D_\zeta \cap \mathbb{D}$. Also,

$$\operatorname{Im}(g(z)) = \operatorname{Re}\big(\log f(z)\big) = \log |f(z)|, \qquad z \in D_\zeta \cap \mathbb{D},$$

thus by Theorem 1.19 the function g can be analytically extended across the arc $D_\zeta \cap \Delta$. Let G be this analytic extension. The function

$$F := \exp(-iG)$$

is an analytic extension for f across $D_\zeta \cap \Delta$, and we conclude the proof. □

We introduce the notion of *spectrum* of a bounded analytic function. This is one of the central definitions for this work.

Definition 1.21. For a bounded analytic function b on $\mathbb{D}$ with $\|b\|_{H^\infty} = 1$, we define its *spectrum* as the set

$$\sigma(b) := \{w \in \mathbb{D} \colon b(w) = 0\} \cup \{\lambda \in \mathbb{T} \colon \liminf_{z \to \lambda} |b(z)| < 1\}.$$

We point out that in literature there exist different definitions for the spectrum $\sigma(b)$. For example, in [28], the spectrum of b introduced in section 5.2 is always a closed set, and it corresponds to the closure $\overline{\sigma(b)}^{\mathrm{cl}}$ according to our definition. In this work we follow [8], where the boundary spectrum introduced in Section 5 is not necessarily closed.

Later in this work it will be more apparent why this set is called *spectrum*, and why it contains some points in the inside of the unit disk, even though we are introducing it in the context of boundary regularity. The following proposition holds.

Proposition 1.22. *Let $b \in H^\infty$ with $\|b\|_{H^\infty} = 1$ and Δ be an open arc in $\mathbb{T}$. Then, b has an analytic extension across Δ with $|b| = 1$ on Δ if and only if*

$$\Delta \subseteq \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}.$$

In particular, the boundary part $\overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T}$ is the smallest closed subset of $\mathbb{T}$ having the property that b admits an analytic extension on every open arc Δ contained in $\mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$ with $|b| = 1$ on Δ.

Proof. On the one hand, if $\Delta \subseteq \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$, then

$$\liminf_{z \in \mathbb{D}, z \to \zeta} |b(z)| \geq 1, \qquad \zeta \in \Delta.$$

Since $\|b\|_{H^\infty} = 1$, it follows that

$$\limsup_{z \in \mathbb{D}, z \to \zeta} |b(z)| \leq \liminf_{z \in \mathbb{D}, z \to \zeta} |b(z)|, \qquad \zeta \in \Delta.$$

This shows that $|b| = 1$ on Δ, and therefore by Corollary 1.20 the function b admits an analytic extension through Δ. On the other hand, assume that b has an analytic extension across Δ with $|b| = 1$ on Δ. By contradiction, if $\zeta \in \Delta \cap \overline{\sigma(b)}^{\mathrm{cl}}$, then there would exist a sequence $(\zeta_n)_n$ in $\mathbb{T}$ converging to ζ and such that

$$\liminf_{z \to \zeta_n} |b(z)| < 1, \qquad n \in \mathbb{N}.$$

Since $\zeta \in \Delta$, there exists $N \in \mathbb{N}$ such that for any $n > N$ we have $\zeta_n \in \Delta$. But then b is analytic on each ζ_n with $|b(\zeta_n)| = 1$, which is a contradiction. This shows that necessarily $\Delta \subseteq T \setminus \overline{\sigma(b)}^{\mathrm{cl}}$. The last part of the statement is a trivial consequence: let E be a closed subset ot $\mathbb{T}$ such that b admits an analytic extension on every open arc Δ contained in $\mathbb{T} \setminus E$ with $|b| = 1$ on Δ. Being E closed, we can write its complement as a union of open arcs,

$$\mathbb{T} \setminus E = \bigcup_{n \in \mathbb{N}} \Delta_n.$$

Since every Δ_n is contained in $\mathbb{T} \setminus E$, b admits an analytic extension across Δ_n with $|b| = 1$ on Δ_n. This shows that $\Delta_n \subseteq \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$, whence

$$\mathbb{T} \setminus E = \bigcup_{n \in \mathbb{N}} \Delta_n \subseteq \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}},$$

proving that $\overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T} \subseteq E$. □

The last proposition shows that the closure $\overline{\sigma(b)}^{\text{cl}}$ is the right set to consider if we want to study when b admits analytic extensions across an arc Δ of the unit circle, with $|b| = 1$ on Δ. For some finer results which will be discussed later, it is important for us to distinguish between $\sigma(b)$ and its closure $\overline{\sigma(b)}^{\text{cl}}$.

Now, we move on to describe more explicitly the spectrum. In order to do so, we appeal to the inner-outer factorisation. In the case of inner functions, we have the following explicit description of the spectrum, in terms of the Blaschke factor and the singular inner part. We use the convention that, if $z = 0$ is a zero for the function u, then the corresponding Blaschke factor is z^k, where $k \in \mathbb{N}$ is the multiplicity of the zero of u at $z = 0$.

Theorem 1.23. *Let $u = B_\Lambda S_\tau$ be an inner function, where B_Λ is the Blaschke product associated to the zero set $\Lambda = \{a_n\}_n$ of u and S_τ is the singular inner function associated to the singular measure τ. Then,*

$$\sigma(u) = \{w \in \overline{\mathbb{D}}^{\text{cl}} \colon \liminf_{z \to w} |u(z)| = 0\} = \overline{\{a_n\}_n}^{\text{cl}} \cup supp(\tau).$$

In particular, $\sigma(u)$ is closed.

Parts of the following proof are taken from Chapter 7 of [30]. We point out that the definition of spectrum of an inner function in [30], which is Definition 7.17, at first glance does not coincide with ours. The goal of Theorem 1.23 is precisely to show that there is no ambiguity, for the two definitions are equivalent.

Proof of Theorem 1.23. The set equality

$$\{w \in \overline{\mathbb{D}}^{\text{cl}} \colon \liminf_{z \to w} |u(z)| = 0\} = \overline{\{a_n\}_n}^{\text{cl}} \cup \text{supp}(\tau)$$

is proved in Proposition 7.19 in [30]. That

$$\{w \in \overline{\mathbb{D}}^{\text{cl}} \colon \liminf_{z \to w} |u(z)| = 0\} \subseteq \sigma(u)$$

follows trivially from Definition 1.21 of the spectrum. To conclude, we show that

$$\sigma(u) \subseteq \overline{\{a_n\}_n}^{\text{cl}} \cup \text{supp}(\tau).$$

In Theorem 7.18 of [30], the authors show that u admits an analytic extension with $|u| = 1$ across any open arc contained in the set

$$\mathbb{T} \setminus \left(\overline{\{a_n\}_n}^{\text{cl}} \cup \text{supp}(\tau)\right).$$

Therefore, by Proposition 1.22, we have that

$$\overline{\sigma(u)} \cap \mathbb{T} \subseteq \left(\overline{\{a_n\}_n}^{\text{cl}} \cup \text{supp}(\tau)\right) \cap \mathbb{T}.$$

Again by definition, the part of $\sigma(u)$ that is inside of $\mathbb{D}$ is precisely the zero set of u, in symbols $\sigma(u) \cap \mathbb{D} = \{a_n\}_n$. This shows that

$$\sigma(u) \subseteq \overline{\{a_n\}_n}^{\text{cl}} \cup \text{supp}(\tau),$$

concluding the proof. □

For a bounded outer function F, we have the following interpretation of the spectrum $\sigma(F)$, in terms of the measure $\log|F|\,dm$. Although this result is well-known, we are not able to provide any reference for this specific formulation.

Proposition 1.24. *Let F be an outer function with $\|F\|_{H^\infty} = 1$. Then,*

$$\overline{\sigma(F)}^{\mathrm{cl}} = \operatorname{supp}(-\log|F|\,dm). \tag{1.7}$$

Proof. First of all, notice that

$$\overline{\sigma(F)}^{\mathrm{cl}} = \overline{\{\zeta \in \mathbb{T} \colon \liminf_{z\in\mathbb{D}, z\to\zeta} |F(z)| < 1\}}^{\mathrm{cl}},$$

since F does not vanish on $\mathbb{D}$. Let $\zeta \in \operatorname{supp}(-\log|F|\,dm)$. By the definition of support of a positive measure, for every open neighbourhood $U \subset \mathbb{T}$ containing ζ, we have that

$$-\int_U \log|F|\,dm > 0.$$

Let us consider

$$\Delta(\zeta,\varepsilon) := \{z \in \mathbb{T} \colon |z-\zeta| < \varepsilon\}.$$

Notice that for every $n \in \mathbb{N}$ there exists $\zeta_n \in \Delta(\zeta, \frac{1}{n})$ such that

$$\liminf_{z\in\mathbb{D}, z\to\zeta_n} |F(z)| < 1.$$

Otherwise, we would have $|F| = 1$ on some arc $\Delta(\zeta, \frac{1}{N})$, which gives the contradiction

$$-\int_{\Delta(\zeta,\frac{1}{N})} \log|F(\lambda)|\,dm(\lambda) = 0.$$

We have constructed the sequence $(\zeta_n)_n$ in $\sigma(F)$ converging to ζ, proving that $\zeta \in \overline{\sigma(F)}^{\mathrm{cl}}$. On the other hand, we consider $\zeta \in \mathbb{T} \setminus \operatorname{supp}(-\log|F|\,dm)$. Let $\Delta \subset \mathbb{T}$ be an open arc containing ζ such that

$$\int_\Delta \log|F|\,dm = 0.$$

Since F is an outer function, for every $z \in \mathbb{D}$ we have that

$$\begin{aligned} |F(z)| &= \exp\left(\int_{\mathbb{T}} \frac{1-|z|^2}{|z-\lambda|^2} \log|F(\lambda)|\,dm(\lambda)\right) \\ &= \exp\left(\int_{\mathbb{T}\setminus\Delta} \frac{1-|z|^2}{|z-\lambda|^2} \log|F(\lambda)|\,dm(\lambda)\right). \end{aligned}$$

This proves that F can be analytically extended across Δ, and that $|F| = 1$ on Δ. By Proposition 1.22, this means that

$$\Delta \subseteq \mathbb{T} \setminus \overline{\sigma(F)}^{\mathrm{cl}},$$

and in particular that $\zeta \notin \overline{\sigma(F)}^{\mathrm{cl}}$, concluding the proof. □

For an analytic function b with $\|b\|_{H^\infty} = 1$, we have the following description of $\overline{\sigma(b)}^{\mathrm{cl}}$ in terms of its inner-outer factorisation.

Theorem 1.25. *Given b in H^∞ with $\|b\|_{H^\infty} = 1$, let $b = b_i b_o$ be its inner-outer factorisation. Then,*

$$\overline{\sigma(b)}^{\mathrm{cl}} = \sigma(b_i) \cup \mathit{supp}\,(-\log|b_o|\,dm)\,.$$

Proof. By Theorem 1.23,

$$\sigma(b_i) = \{w \in \overline{\mathbb{D}}^{\mathrm{cl}} \colon \liminf_{z\to w} |b_i(z)| = 0\} \subseteq \sigma(b).$$

Also, by Proposition 1.24, since $b_o \in H^\infty$ with $\|b_o\|_\infty = 1$,

$$\mathrm{supp}(-\log|b_o|\,dm) = \overline{\{\zeta \in \mathbb{T} \colon \liminf_{z\to\zeta} |b_o(z)| < 1\}}^{\mathrm{cl}} \subseteq \overline{\sigma(b)}^{\mathrm{cl}},$$

for

$$\liminf_{z\to\zeta} |b(z)| \le \liminf_{z\to\zeta} |b_o(z)|, \qquad \zeta \in \mathbb{T}.$$

To show the reverse inequality, we note that if

$$\zeta \in \mathbb{T} \setminus \Big(\sigma(b_i) \cup \mathrm{supp}\,(-\log|b_o|\,dm)\Big),$$

then both b_i and b_o have an analytic extension to an open arc Δ containing ζ, with $|b_i| = |b_o| = 1$ on Δ. In particular, the same holds for the product $b = b_i b_o$, and then by Proposition 1.22

$$\Delta \subseteq \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}.$$

This shows that $\zeta \notin \overline{\sigma(b)}^{\mathrm{cl}}$, concluding the proof. □

We conclude this section with a discussion on the Corona theorem. This was proved in 1962 by Carleson [13]. This is a very profound result, concerning the structure of the maximal ideals in the Banach algebra H^∞. However, for the goals of this thesis, we are only interested in the following statement, that is also very important on its own.

Theorem 1.26 (Corona Theorem)**.** *Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in H^\infty$. Then, the two following conditions are equivalent.*

(i) There exists $\delta > 0$ such that

$$\sum_{k=1}^{n} |f_k(z)| \ge \delta, \qquad z \in \mathbb{D}.$$

(ii) There exist functions $g_1, \ldots, g_n \in H^\infty$ such that

$$\sum_{k=1}^{n} f_k(z) g_k(z) = 1, \qquad z \in \mathbb{D}.$$

1.3 Spaces of analytic functions

In the following chapters of this thesis, we will discuss several spaces of analytic functions on the unit disk. In this section, we gather the preliminary and central facts that are important to us for our goals. We split this discussion in three different parts, corresponding to three different subsections. First, we introduce the de Branges–Rovnyak spaces $H(b)$, then we focus on a special class, the so-called model spaces K_u, and finally we define the harmonically weighted Dirichlet spaces $\mathcal{D}_\mu$.

1.3.1 $H(b)$ spaces

We define the de Branges–Rovnyak spaces $H(b)$. These spaces were originally introduced by Louis de Branges and James Rovnyak in 1966 as a generalization of the orthogonal complement of the range of multiplication by b on H^2, see [18]. Ever since, this class of spaces has been of great interest because of the role that they play in operator theory. We will see some examples of their importance, and we will briefly present some aspects of their beautiful structure.

There are several equivalent definitions for the de Branges–Rovnyak spaces $H(b)$. In this work, we follow the approach of [28] and [29]. See also [54].

We start in a rather abstract setting. Given a Hilbert space H, let $A \in \mathcal{B}(H)$ be a bounded linear operator on H. We can give a Hilbert space structure to the range of A, $\mathrm{Ran}(A)$, regardless if it is closed in the topology of H. We call $\mathcal{M}(A)$ the pair $(\mathrm{Ran}(A), \langle \cdot, \cdot \rangle_{\mathcal{M}(A)})$, where

$$\langle Ax, Ay \rangle_{\mathcal{M}(A)} := \langle P_{(\mathrm{Ker}A)^\perp} x, P_{(\mathrm{Ker}A)^\perp} y \rangle_H, \qquad x, y \in H.$$

Here, $P_{(\mathrm{Ker}A)^\perp}$ denotes the orthogonal projection on $(\mathrm{Ker}A)^\perp$, and the symbol $\perp$ denotes the orthogonal complement. Notice that the definition of the inner product is well-given, because if $x_1, x_2 \in H$ satisfy $Ax_1 = Ax_2$, then $x_1 - x_2 \in \mathrm{Ker}A$ and therefore $P_{(\mathrm{Ker}A)^\perp} x_1 = P_{(\mathrm{Ker}A)^\perp} x_2$. Also, if either x or y belongs to $(\mathrm{Ker}A)^\perp$, then

$$\langle Ax, Ay \rangle_{\mathcal{M}(A)} = \langle x, y \rangle_H. \tag{1.8}$$

Finally, it is not hard to check that $\mathcal{M}(A)$ is a Hilbert space. We move on to the so-called *complementary* spaces $H(A)$, that will be our de Branges–Rovnyak spaces. We consider a contraction $A \in \mathcal{B}(H)$, i.e.

$$\|Ax\|_H \leq \|x\|_H, \qquad x \in H.$$

Notice that it holds $AA^* \leq I$, meaning that the operator $I - AA^*$ is positive. Thus, its square root $(I - AA^*)^{\frac{1}{2}}$ is a well-defined positive operator. We define $H(A)$ as the range space

$$H(A) := \mathcal{M}\big((I - AA^*)^{\frac{1}{2}}\big).$$

Notice that both $I - AA^*$ and $(I - AA^*)^{\frac{1}{2}}$ are self-adjoint. For more information on the construction of the square root of a positive operator, see [28, Ch. 7]. The condition that A is a contraction is equivalent to $\|A\|_{\mathcal{B}(H)} \leq 1$. If $\|A\|_{\mathcal{B}(H)} < 1$, then we have the set equality $H(A) = H$. This is because

$$H(A) = \operatorname{Ran}(I - AA^*)^{\frac{1}{2}} = \left(\operatorname{Ker}(I - AA^*)^{\frac{1}{2}}\right)^{\perp},$$

and

$$\operatorname{Ker}(I - AA^*)^{\frac{1}{2}} \subseteq \operatorname{Ker}(I - AA^*) = \{0\}.$$

That $\operatorname{Ker}(I - AA^*) = \{0\}$ follows from the following fact. If $x \in \operatorname{Ker}(I - AA^*)$, then

$$\|x\|_H = \|AA^*x\|_H \leq \|A\|^2_{\mathcal{B}(H)}\|x\|_H.$$

If $x \neq 0$, we would have the contradiction $\|x\|_H < \|x\|_H$. It can also be shown that, in the case with $\|A\|_{\mathcal{B}(H)} < 1$, the $H(A)$-norm is equivalent to the H-norm, being that $(I - AA^*)^{\frac{1}{2}}$ is bounded, invertible and positive, thus also bounded from below.

Another equivalent definition for $H(A)$, the complementary space for the range $\mathcal{M}(A)$, follows from the next theorem (that is Corollary 16.27 in [29]). This was actually the original construction of $H(A)$, done by de Branges and Rovnyak in 1966 [18].

Theorem 1.27. *Let $A \in \mathcal{B}(H)$ be a contraction and let $x \in H$. Then, the following are equivalent:*

- *$x \in H(A)$;*
- $\sup_{y \in \mathcal{M}(A)} \left(\|x + y\|^2_H - \|y\|^2_{\mathcal{M}(A)}\right) < \infty.$

Moreover, for such an element, we have

$$\|x\|^2_{H(A)} = \sup_{y \in \mathcal{M}(A)} \left(\|x + y\|^2_H - \|y\|^2_{\mathcal{M}(A)}\right).$$

This theorem is due to Sarason, who originally showed that the spaces $H(A)$ defined by de Branges and Rovnyak coincide with the range of the operator $(I - AA^*)^{\frac{1}{2}}$. See [54].

Notice that, if A is a contraction, then the adjoint A^* is also a contraction. There is a relation between the space $H(A)$ and the space $H(A^*)$, that gives us an important inequality for the norm.

Proposition 1.28. *Let A be a contractive operator on a Hilbert space H, and let $x \in H$. Then, $x \in H(A)$ if and only if $A^*x \in H(A^*)$. Moreover, in this case,*

$$\|x\|^2_{H(A)} = \|A^*x\|^2_{H(A^*)} + \|x\|^2_H.$$

In particular,

$$\|x\|_H \leq \|x\|_{H(A)}, \qquad x \in H(A). \tag{1.9}$$

We are ready to define the de Branges–Rovnyak spaces. We recall the notation H_1^∞ for the closed unit ball in H^∞. Given $b \in H_1^\infty$, we introduce the multiplication operator

$$T_b \colon H^2 \ni f \mapsto bf \in H^2.$$

Since $\|b\|_{H^\infty} \leq 1$, the operator T_b is a contraction, for

$$\|T_b f\|_{H^2}^2 = \int_{\mathbb{T}} |bf|^2 \, dm \leq \|b\|_{H^\infty}^2 \|f\|_{H^2}^2 \leq \|f\|_{H^2}^2.$$

We define the de Branges–Rovnyak space $H(b)$ as the complementary space

$$H(b) := H(T_b) = \mathcal{M}\big((I - T_b T_b^*)^{\frac{1}{2}}\big).$$

If $\|b\|_{H^\infty} < 1$, then the space $H(b)$ is just a renormed version of H^2. In some parts of this thesis, we will make the non-restricting assumption that $\|b\|_{H^\infty} = 1$.

We have introduced a class of Hilbert spaces that are contained in H^2, but are not necessarily closed in H^2. We will discuss their closure in the H^2 norm in Section 1.3.2. This definition is rather mysterious, and we have no concrete examples of elements in an $H(b)$ space, nor much information on the norm $\| \cdot \|_{H(b)}$. Equation (1.9) gives us

$$\|f\|_{H^2} \leq \|f\|_{H(b)}, \qquad f \in H(b). \tag{1.10}$$

Using the characterization in Theorem 1.27, we can show that $H(b)$ spaces are all invariant under the action of the backward shift S^*. In fact, this is one of the central properties of these spaces.

Theorem 1.29. *Let $b \in H_1^\infty$. If $f \in H(b)$, then $S^* f \in H(b)$ and*

$$\|S^* f\|_{H(b)} \leq \|f\|_{H(b)}.$$

In particular, the operator

$$\begin{array}{rccc} X_b : & H(b) & \longrightarrow & H(b) \\ & f & \longmapsto & S^* f \end{array} \tag{1.11}$$

is bounded.

Proof. We fix $f \in H(b)$ and we show that $S^* f \in H(b)$. In light of Theorem 1.27, it suffices to show that

$$\sup_{g \in \mathcal{M}(T_b)} \big(\|S^* f + g\|_{H^2}^2 - \|g\|_{\mathcal{M}(T_b)}^2 \big) < \infty.$$

Notice that the operator T_b is injective, therefore for every $g \in \mathcal{M}(T_b) = bH^2$ there exists a unique $h \in H^2$ such that $g = bh$, and it holds $\|g\|_{\mathcal{M}(T_b)} = \|h\|_{H^2}$. Since the (forward) shift operator S is an isometry on H^2, one can check that it is well-defined and isometric also on $\mathcal{M}(T_b)$. Also, recall that the range $\operatorname{Ran}(S) = zH^2$ is orthogonal in H^2 to $\mathbb{C}$. It follows that, for $g \in \mathcal{M}(T_b)$,

$$\|S^* f + g\|_{H^2}^2 = \|SS^* f + Sg\|_{H^2}^2 = \|SS^* f + Sg + f(0)\|_{H^2}^2 - |f(0)|^2.$$

From the identity $SS^* = I - c_0 \otimes c_0$, we have that

$$\begin{aligned}\|S^*f+g\|_{H^2}^2 - \|g\|_{\mathcal{M}(T_b)}^2 &= \|f+Sg\|_{H^2}^2 - |f(0)|^2 - \|g\|_{\mathcal{M}(T_b)}^2 \\ &= \|f+Sg\|_{H^2}^2 - \|Sg\|_{\mathcal{M}(T_b)}^2 - |f(0)|^2.\end{aligned}$$

In particular, by Theorem 1.27, for every $g \in \mathcal{M}(T_b)$

$$\begin{aligned}\|S^*f+g\|_{H^2}^2 - \|g\|_{\mathcal{M}(T_b)}^2 &\leq \sup_{G\in\mathcal{M}(T_b)} \left(\|f+G\|_{H^2}^2 - \|G\|_{\mathcal{M}(T_b)}^2\right) - |f(0)|^2 \\ &= \|f\|_{H(b)}^2 - |f(0)|^2 < \infty.\end{aligned}$$

In particular, $S^*f \in H(b)$ and

$$\|S^*f\|_{H(b)}^2 = \sup_{g\in\mathcal{M}(T_b)} \left(\|S^*f+g\|_{H^2}^2 - \|g\|_{\mathcal{M}(T_b)}^2\right) \leq \|f\|_{H(b)}^2 - |f(0)|^2 \leq \|f\|_{H(b)}^2.$$

□

For concrete examples of functions in $H(b)$, we discuss its reproducing kernel Hilbert space structure. We recall that

$$c_w(z) = \frac{1}{1-\overline{w}z}, \qquad z, w \in \mathbb{D},$$

is the reproducing kernel of H^2. Notice that, for $b \in H_1^\infty$,

$$(T_b)^*c_w(z) = \langle c_w, bc_z\rangle_{H^2} = \overline{b(w)c_z(w)} = \overline{b(w)}c_w(z), \qquad z \in \mathbb{D}. \tag{1.12}$$

Theorem 1.30. *Let $b \in H_1^\infty$. Then, $H(b)$ is a RKHS with kernel given by the function*

$$k_w^b(z) := \frac{1-\overline{b(w)}b(z)}{1-\overline{w}z}, \qquad z, w \in \mathbb{D}.$$

Also, it holds

$$\|k_w^b\|_{H(b)}^2 = \frac{1-|b(w)|^2}{1-|w|^2}, \qquad w \in \mathbb{D}.$$

Proof. First of all, by (1.12) it holds

$$(I - T_bT_b^*)c_w(z) = (1 - b(z)\overline{b(w)})c_w(z), \qquad z \in \mathbb{D},$$

so that

$$k_w^b = (I - T_bT_b^*)^{\frac{1}{2}}(I - T_bT_b^*)^{\frac{1}{2}}c_w \in H(b), \qquad w \in \mathbb{D}.$$

Now, we show that the reproducing formula holds. By definition, given $f \in H(b)$, there exists $g \in H^2$ such that

$$f = (I - T_bT_b^*)^{\frac{1}{2}}g.$$

Also, since $(I - T_bT_b^*)^{\frac{1}{2}}$ is self-adjoint,

$$\left(\mathrm{Ker}(I - T_bT_b^*)^{\frac{1}{2}}\right)^\perp = \mathrm{Ran}(I - T_bT_b^*)^{\frac{1}{2}}.$$

By the reproducing formula for H^2 and (1.8), for $f \in H(b)$ it holds

$$\begin{aligned} f(w) &= \langle f, c_w \rangle_{H^2} \\ &= \langle g, (I - T_b T_b^*)^{\frac{1}{2}} c_w \rangle_{H^2} \\ &= \langle (I - T_b T_b^*)^{\frac{1}{2}} g, (I - T_b T_b^*) c_w \rangle_{H(b)} \\ &= \langle f, k_w^b \rangle_{H(b)}. \end{aligned}$$

Finally, for $w \in \mathbb{D}$,

$$\|k_w^b\|_{H(b)}^2 = \langle k_w^b, k_w^b \rangle_{H(b)} = k_w^b(w) = \frac{1 - |b(w)|^2}{1 - |w|^2},$$

concluding the proof. □

It is easy to check that the kernels k_w^b span a dense subset of $H(b)$: if $f \in H(b)$ is orthogonal to every kernel k_w^b, then

$$f(w) = \langle f, k_w^b \rangle_{H(b)} = 0, \qquad w \in \mathbb{D},$$

so that $f \equiv 0$. Following the abstract construction of a reproducing kernel Hilbert space of Aronszjan [4], one could equivalently define the $H(b)$ space as the RKHS having for kernel the function

$$k^b(z, w) := \frac{1 - \overline{b(w)} b(z)}{1 - \overline{w} z}, \qquad z, w \in \mathbb{D}.$$

In this case, the space $H(b)$ is defined as the closure of the linear span of the reproducing kernels,

$$\overline{\operatorname{span}(\{k^b(\cdot, w) \colon w \in \mathbb{D}\})}^{\text{cl}},$$

with respect to a suitable norm so that the reproducing kernel property is satisfied.

An interesting feature of the $H(b)$ spaces is that, not only do we have reproducing kernels for points $w \in \mathbb{D}$, but we also have kernels for some special points $\zeta \in \mathbb{T}$. We recall the following definition. Given a function $b \in H_1^\infty$ and a point $\zeta \in \mathbb{T}$, we say that b has angular derivative in the sense of Carathéodory (ADC) at ζ if the boundary value $b(\zeta)$ exists and has modulus 1, and the derivative b' admits non-tangential limit $b'(\zeta)$ at ζ. For more details on this definition, see for example [30, Section 2.5]. The following is Theorem 21.1 in [29].

Theorem 1.31. *Let b be an analytic function on $\mathbb{D}$ with $\|b\|_{H^\infty} \leq 1$ and let $\zeta \in \mathbb{T}$. Put*

$$c := \liminf_{z \to \zeta} \frac{1 - |b(z)|}{1 - |z|}.$$

The following are equivalent:

(i) $c < \infty$;

(ii) There exists $\lambda \in \mathbb{T}$ *such that the function*

$$\mathbb{D} \ni z \mapsto \frac{b(z) - \lambda}{z - \zeta}$$

belongs to $H(b)$*;*

(iii) Every function f *in* $H(b)$ *admits non-tangential limit at* ζ*;*

(iv) The function b *has ADC at* ζ*.*

Moreover, under the preceding equivalent conditions, the following results hold:

(a) $\lambda = b(\zeta)$ *and for every* $f \in H(b)$ *one has* $f(\zeta) = \langle f, k_\zeta^b \rangle_{H(b)}$*, where*

$$k_\zeta^b(z) = \frac{1 - \overline{b(\zeta)} b(z)}{1 - \overline{\zeta} z} \in H(b);$$

(b) We have

$$c = \angle \lim_{z \to \zeta} \frac{1 - |b(z)|}{1 - |z|} = \|k_\zeta^b\|_{H(b)}^2 = |b'(\zeta)| = \overline{b(\zeta)} \zeta b'(\zeta) > 0;$$

(c) We have the convergence

$$\angle \lim_{w \to \zeta} \|k_w^b - k_\zeta^b\|_{H(b)}^2 = 0.$$

The function k_ζ^b is called *boundary kernel* at the point $\zeta \in \mathbb{T}$. The previous theorem says that, roughly speaking, boundary regularity of the function b translates to boundary properties for the elements of $H(b)$. We see another result in this direction, that concerns analyticity. The spectrum $\sigma(b)$, introduced in Section 1.2.2., and the operator X_b, introduced in Section 1.3.1, come into the picture.

Theorem 1.32. *Let* Δ *be an open arc of* $\mathbb{T}$*. The following are equivalent:*

(i) Each function $f \in H(b)$ *has an analytic continuation across* Δ*;*

(ii) b *has an analytic continuation across* Δ *and* $|b| = 1$ *on* Δ*;*

(iii) Δ *is contained in the resolvent of* X_b^**, i.e.* $\Delta \subseteq \mathbb{T} \setminus \sigma(X_b^*)$*;*

(iv) Δ *is disjoint from the closure of the spectrum of* b*, i.e.* $\Delta \subseteq \mathbb{T} \setminus \overline{\sigma(b)}^{\text{cl}}$*.*

Proof. The equivalence of the first three statements follows from Theorem 20.13 of [29]. The equivalence of (*ii*) and (*iv*) follows from Proposition 1.22. □

Notice that, as a Corollary of Theorem 1.32, we have that

$$\mathbb{T} \cap \overline{\sigma(b)}^{\mathrm{cl}} = \mathbb{T} \cap \sigma(X_b^*).$$

This relation between the spectrum $\sigma(b)$ and the spectrum of the operator X_b^* will be discussed in greater detail.

In Theorem 1.29, we showed that all $H(b)$ spaces are invariant under the backward shift operator S^*, and its restriction X_b is a bounded operator on $H(b)$. However, the same does not hold for the forward shift S: not all $H(b)$ spaces are S-invariant. To discuss this matter, we introduce an important notion that allows to categorize $H(b)$ spaces into two main groups: this is the notion of *extremality*. We follow [28, Chapter 6]. Given a normed vector space $\mathcal{X}$ and two distinct points $x, y \in \mathcal{X}$, we define the interval $[x, y]$ as the set of all convex combinations

$$[x, y] := \{tx + (1-t)y \colon t \in [0,1]\}.$$

We recall that a set $B \subseteq \mathcal{X}$ is convex if, for each pair of distinct points $x, y \in B$, the interval $[x, y]$ is entirely contained in B. Given a convex set $B \subseteq \mathcal{X}$ and $p \in B$, we say that p is an *extreme point* for B if it is not in the interior of any interval contained in B. Equivalently, a point $p \in B$ is an extreme point for B if and only if the set $B \setminus \{p\}$ is convex.

There is a very practical characterization for the property of extremality in the context that interests us: $b \in H_1^\infty$ is an extreme point of H_1^∞ if and only if

$$\int_{\mathbb{T}} \log(1 - |b|)\, dm = -\infty.$$

For a proof, see [28, Theorem 6.7]. Notice that, in particular, every inner function u is an extreme point of H_1^∞, for $|u| = 1$ a.e. on $\mathbb{T}$. Also, notice that the property of being non-extreme forces a condition on the size of the spectrum $\sigma(b)$. This is because, if $\zeta \notin \sigma(b)$, then $\liminf_{z\to\zeta} |b(z)| \geq 1$. Since $\|b\|_{H^\infty} \leq 1$, this implies that the limit $\lim_{z\to\zeta} |b(z)|$ exists and it is equal to 1. In particular, the complement $\mathbb{T}\setminus\sigma(b)$ has to have Lebesgue measure $m(\mathbb{T}\setminus\sigma(b)) = 0$, otherwise, by monotonicity, since $\log(1 - |b|) \leq 0$,

$$\int_{\mathbb{T}} \log(1 - |b|)\, dm \leq \int_{\mathbb{T}\setminus\sigma(b)} \log(1 - |b|)\, dm = -\infty.$$

This means that the spectrum $\sigma(b)$ has full measure, and in particular the closure is $\overline{\sigma(b)}^{\mathrm{cl}} = \mathbb{T}$. Theorem 1.32 tells us that, given any open arc $\Delta \subseteq \mathbb{T}$, then for every non-extreme symbol b there exists a function in $H(b)$ that does not have analytic continuation across Δ.

If the function b is non-extreme, i.e. $\log(1 - |b|) \in L^1$, then there exists an unique outer function a whose modulus on $\mathbb{T}$ is a.e. $(1 - |b|^2)^{\frac{1}{2}}$ and $a(0) > 0$. This function is defined by

$$a(z) := \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(1 - |b(\zeta)|^2)^{\frac{1}{2}}\, dm(\zeta)\right), \qquad z \in \mathbb{D},$$

and it is called the *Pythagorean mate* of b. The name comes from the following relation that the pair (b, a) satisfies:

$$|b|^2 + |a|^2 = 1 \qquad \text{a.e. on } \mathbb{T}.$$

This function a plays a pivotal role for the structure of $H(b)$. We refer to [29, Ch. 23-24] for more details.

The structure of $H(b)$ heavily depends on whether the function b is an extreme point for H_1^∞ or not. Informally, the $H(b)$ spaces associated to a non-extreme function b are "bigger" than the ones associated to an extreme function, and they are more similar to the Hardy space H^2. We list some important features of the structure of $H(b)$ spaces with a non-extreme symbol b. We always have that $b \in H(b)$. The space $H(b)$ contains all (analytic) polynomials, and the set of polynomials is dense in $H(b)$. For every $f \in H(b)$, we have that $Sf \in H(b)$, where S denotes the (forward) shift operator. Also, S defines a bounded operator on $H(b)$. We denote Y_b the restriction of S to $H(b)$, and we recall that X_b denotes the restriction of S^*. We have that Y_b and X_b are not one the adjoint of the other. There is a rank-one operator that ruins this analogy with the H^2 case. The following relations hold:

$$Y_b = X_b^* + b \otimes S^*b, \qquad Y_b^* = X_b + S^*b \otimes b.$$

In this work, it is more important to us the adjoint operator X_b^*, rather than the restricted shift Y_b. In Chapter 2, X_b^* will be used to define the difference quotient operator, that will be crucial in Chapter 3. The following spectral properties hold, in the non-extreme case: $\sigma_p(X_b) = \mathbb{D}$, $\sigma_p(X_b^*) = \emptyset$, and $\sigma(X_b) = \sigma(X_b^*) = \overline{\mathbb{D}}^{\mathrm{cl}}$. For proofs, we refer to [29, Section 24.3]. Notice the analogy between the spectra of the operator X_b, and the spectra of the operator S^* on H^2, that was previously discussed. We conclude this part with an important result that characterizes the property of being non-extreme in terms of the associated $H(b)$ space.

Theorem 1.33. *Let $b \in H_1^\infty$. Then, the following are equivalent.*

(i) b is a non-extreme point of H_1^∞;

(ii) $b \in H(b)$;

(iii) $H(b)$ is invariant under the forward shift operator S;

(iv) $H(b)$ contains all the (analytic) polynomials.

Proof. See Corollary 25.5 and Corollary 25.10 in [29]. □

Now, we treat the $H(b)$ spaces with symbol b extreme. In this case, it is no longer necessary that $\sigma(b)$ is dense in $\mathbb{T}$, although it still might occur. In light of Theorem 1.32, we can expect boundary regularity on larger subsets of $\mathbb{T}$ from both b and the elements of $H(b)$. This has also consequences on the spectrum of X_b. A very important example of extreme functions of H_1^∞ are inner functions, and the corresponding class of $H(b)$ spaces will be discussed in detail in the next section.

By Theorem 1.33, we know that $b \notin H(b)$, $H(b)$ does not contain all polynomials and it is not invariant under the shift S. We have the following description of the spectrum of X_b^*, when the function b is extreme.

Theorem 1.34. *Let b be an extreme function in H_1^∞. Then the spectrum of the operator X_b^* coincides with the closure of the spectrum of b,*

$$\sigma(X_b^*) = \overline{\sigma(b)}^{\mathrm{cl}}.$$

In particular,

$$\sigma_p(X_b^*) = \sigma(X_b^*) \cap \mathbb{D} = \{w \in \mathbb{D} \colon b(w) = 0\}.$$

Proof. This follows from combining Corollary 26.3 of [29], for the part of $\sigma(X_b^*)$ that is inside $\mathbb{D}$, and Theorem 1.32 for the part of $\sigma(X_b^*)$ that is on the boundary $\mathbb{T}$. □

In the case of b inner, this celebrated result is due to Livšic–Möller [37, 43]. Note that a similar result holds for X_b: we have that

$$\sigma(X_b) = \{w \in \overline{\mathbb{D}}^{\mathrm{cl}} : \overline{w} \in \sigma(b)\},$$

and

$$\sigma_p(X_b) = \sigma(X_b) \cap \mathbb{D} = \{w \in \mathbb{D} \colon b(\overline{w}) = 0\}.$$

We conclude this section showing a class of functions b that will appear in many examples in Chapter 3. For these special functions b, much can be said about the structure of the associated $H(b)$ space. For reference, see [25].

Proposition 1.35. *Let u be an inner function. Then, the function*

$$b(z) := \frac{1 + u(z)}{2}, \qquad z \in \mathbb{D},$$

is a non-extreme element of H_1^∞. Its Pythagorean mate a satisfies

$$a(z) = e^{i\gamma} \frac{1 - u(z)}{2}, \qquad z \in \mathbb{D},$$

where $\gamma \in \mathbb{R}$.

Proof. Since u is inner, we have that $\|b\|_{H^\infty} \leq 1$. Now we show that b is non-extreme. For every $z \in \mathbb{D}$ it holds that

$$|1 + u(z)|^2 + |1 - u(z)|^2 = 2(1 + |u(z)|^2).$$

In particular, the following equality holds $m-$a.e. on $\mathbb{T}$:

$$|b|^2 + \left|\frac{1 - u}{2}\right|^2 = \frac{1 + |u|^2}{2} = 1.$$

By proposition 1.13, the function $1-u$ is outer, and therefore by 1.12 we have that

$$\begin{aligned}\int_{\mathbb{T}} \log(1-|b|^2)\,dm &= \int_{\mathbb{T}} \log\left|\frac{1-u}{2}\right|^2 dm \\ &= 2\int_{\mathbb{T}} \log|1-u|\,dm - 2\log 2 \\ &= 2\log|1-u(0)| - 2\log 2.\end{aligned}$$

This shows that b is non-extreme. Also, the function $\tilde{a} := (1-u)/2$ is outer and it satisfies $|b|^2 + |\tilde{a}|^2 = 1$ a.e. on $\mathbb{T}$. Therefore, the Pythagorean mate a of b is obtained through a suitable rotation

$$a(z) = e^{i\gamma}\frac{1-u(z)}{2}, \qquad z \in \mathbb{D},$$

with $\gamma \in \mathbb{R}$ chosen in a way that $a(0) > 0$. □

1.3.2 Model spaces

In this subsection, we are interested in the de Branges–Rovnyak spaces that are actually closed in H^2, and that inherit the Hilbert space structure from H^2, with $\|\cdot\|_{H(b)} = \|\cdot\|_{H^2}$. The following result, that is Lemma 16.14 in [29], gives us a characterization for the complementary spaces $H(A)$ that are closed in H with $\|\cdot\|_{H(A)} = \|\cdot\|_H$. We say that an operator A is a *partial isometry* if

$$\|Ax\|_H = \|x\|_H, \qquad x \in (\mathrm{Ker}A)^{\perp}.$$

Proposition 1.36. *Let $A \in \mathcal{B}(H)$ be a contraction. Then, $H(A)$ is a closed subspace of H and $\|x\|_{H(A)} = \|x\|_H$, for each $x \in H(A)$, if and only if A is a partial isometry. In this case, the set identity $H(A) = Ran(I - AA^*)$ holds.*

In our context, Proposition 1.36 says that the space $H(b)$ is closed in H^2, with $\|\cdot\|_{H(b)} = \|\cdot\|_{H^2}$, if and only if the operator $T_{\bar{b}}$ is a partial isometry. In light of Theorem 12.18 in [28], this happens if and only if the function b is inner. The special class of $H(b)$ spaces with b inner are called *model spaces*. It is common to denote the inner function with the letter u, and the corresponding model space with K_u.

Being K_u closed in H^2, there exists $P_u\colon H^2 \to K_u$, the orthogonal projection onto K_u. We can express P_u in terms of the reproducing kernels k_w^u: for $f \in H^2$, it holds

$$P_u f(w) = \langle P_u f, k_w^u\rangle_{H^2} = \langle f, P_u k_w^u\rangle_{H^2} = \langle f, k_w^u\rangle_{H^2}.$$

We can also express P_u in terms of the Riesz projection $P_+\colon L^2 \to H^2$, and this will have important consequences for us. This is taken from Proposition 5.14 in [30]. We write the (short) proof for the sake of completeness.

Proposition 1.37. *Let $f \in H^2$. Then,*

$$P_u f = f - uP_+(\overline{u}f). \tag{1.13}$$

Proof. We recall that $k_w^u = (1 - \overline{u(w)}u)c_w$, where

$$c_w(z) = \frac{1}{1 - \overline{w}z}, \qquad z \in \mathbb{D},$$

is the reproducing kernel of H^2. We have that

$$\begin{aligned} P_u f(w) &= \langle f, k_w^u \rangle_{H^2} \\ &= \langle f, (1 - \overline{u(w)}u)c_w \rangle_{H^2} \\ &= \langle f, c_w \rangle_{H^2} - u(w)\langle f, uc_w \rangle_{H^2} \\ &= f(w) - u(w)\langle \overline{u}f, c_w \rangle_{L^2} \\ &= f(w) - u(w)P_+(\overline{u}f)(w). \quad \square \end{aligned}$$

We remark that it follows directly from (1.13) that $P_u = I - T_u T_u^*$. This is coherent with the general result contained in Proposition 1.36: when the operator A is a partial isometry, then $H(A) = \mathrm{Ran}(I - AA^*)$.

Thanks to this expression for the orthogonal projection P_u, we obtain the following description for the model space K_u.

Theorem 1.38. *Let u be an inner function. Then, K_u is the orthogonal complement of the space uH^2,*

$$K_u = H^2 \ominus uH^2.$$

Proof. On the one hand, we show that every function f in K_u is orthogonal to uH^2. We start with the reproducing kernel $f = k_w^u$. For $g \in H^2$, it holds

$$\begin{aligned} \langle ug, k_w^u \rangle_{H^2} &= \langle ug, (1 - \overline{u(w)}u)c_w \rangle_{H^2} \\ &= \langle ug, c_w \rangle_{H^2} - u(w)\langle ug, uc_w \rangle_{H^2}. \end{aligned}$$

Being that $|u| = 1$ a.e. on $\mathbb{T}$,

$$\langle ug, k_w^u \rangle_{H^2} = u(w)g(w) - u(w)\langle g, c_w \rangle_{H^2} = 0.$$

In particular, the orthogonal complement $H^2 \ominus uH^2$ contains every finite linear combination of kernels k_w^u, and therefore its closure as well. This means that

$$H^2 \ominus uH^2 \supseteq \overline{\mathrm{span}(\{k_w^u \colon w \in \mathbb{D}\})} = K_u.$$

On the other hand, we show that every function $f \in H^2 \ominus uH^2$ belongs to K_u. Notice that

$$P_+(\overline{u}f)(w) = \langle \overline{u}f, c_w \rangle_{L^2} = \langle f, uc_w \rangle_{H^2} = 0.$$

It follows from Proposition 1.37 that

$$f = P_u f \in K_u,$$

concluding the proof. $\square$

Thanks to this expression for K_u, it follows from Beurling's theorem 1.18 that the model spaces are exactly all the closed S^*-invariant subspaces of H^2. In literature, it is more common to find the model spaces introduced in this way, as the orthogonal complement $H^2 \ominus uH^2$, with u inner. For a treatise on model spaces, see for example [30, 46].

We can characterize the membership in model spaces in terms of the boundary values.

Proposition 1.39. *For an inner function u, the model space K_u is the set of all $f \in H^2$ such that $f = u\overline{zg}$ a.e. on $\mathbb{T}$ for some $g \in H^2$. In other words,*

$$K_u = H^2 \cap u\overline{zH^2}.$$

Proof. We have that $f \in K_u$ if and only if $\langle f, uh \rangle_{L^2} = 0$ for every $h \in H^2$. This is equivalent to saying that $\overline{u}f$ belongs to the orthogonal complement of H^2 in L^2, which is exactly $\overline{zH^2}$. Thus, $f \in K_u$ if and only if $\overline{u}f \in \overline{zH^2}$. This concludes the proof, for $|u| = 1$ a.e. on $\mathbb{T}$. □

This characterization allows to define a special *conjugation* operator on K_u, defined in terms of the boundary function:

$$C: \begin{array}{ccc} K_u & \longrightarrow & K_u \\ f & \longmapsto & \overline{fz}u \end{array}. \tag{1.14}$$

C is a conjugation, i.e. conjugate linear, involutive ($C^2 = I$), and isometric.

As we saw in the previous subsection, being that inner functions are a special class of extreme functions of H^∞_1, the model spaces K_u are not S-invariant. Thanks to the orthogonal projection P_u, we have a substitute for this important operator. We define the *compressed (forward) shift* on K_u as $S_u := P_u S$. It is easy to check that S_u is well-defined and bounded on K_u. In fact,

$$\|S_u f\|_{H^2} = \|P_u S f\|_{H^2} \leq \|f\|_{H^2}, \qquad f \in K_u.$$

This compressed shift plays a very important role in our analysis.

Proposition 1.40. *Let u be an inner function. Then, $X_u^* = S_u$.*

Proof. Let $f, g \in K_u$. Then,

$$\begin{aligned} \langle X_u^* f, g \rangle_{H^2} &= \langle f, S^* g \rangle_{H^2} \\ &= \langle Sf, g \rangle_{H^2} \\ &= \langle Sf, P_u g \rangle_{H^2} \\ &= \langle P_u S f, g \rangle_{H^2} \\ &= \langle S_u f, g \rangle_{H^2}. \end{aligned}$$

□

The compressed shift S_u is a natural substitute for the forward shift S, in the context of model spaces. By the Livšic–Möller theorem, the spectrum of the operator S_u is exactly the spectrum of u, i.e. $\sigma(S_u) = \sigma(u)$. Just like in the H^2 space, the shift S_u is a rank-one operator away from being unitary. It holds that [30, Lemma 9.9]

$$S_u X_u = I - k_0^u \otimes k_0^u, \qquad X_u S_u = I - Ck_0^u \otimes Ck_0^u,$$

where C is the conjugation defined in (1.14). We will discuss a special class of rank-one perturbations of S_u, that will provide a family of unitary operators, the Clark operators U_α. These operators are very important to us, for they allow us to interpret the model space K_u as the Lebesgue space $L^2(\sigma_\alpha)$, for a suitable measure σ_α, called Clark measure. This identification will give us crucial information about the model spaces, that will be used in the following chapters. We introduce the Clark theory. For every $\alpha \in \mathbb{T}$, the function

$$z \in \mathbb{D} \mapsto \frac{1 + \overline{\alpha} u(z)}{1 - \overline{\alpha} u(z)} \in \mathbb{C}$$

is analytic on $\mathbb{D}$, and the function

$$\mathbb{D} \ni z \mapsto \operatorname{Re}\left(\frac{1 + \overline{\alpha} u(z)}{1 - \overline{\alpha} u(z)}\right) = \frac{1 - |u(z)|^2}{|\alpha - u(z)|^2}$$

is positive and harmonic on $\mathbb{D}$. By Herglotz's Theorem 1.4, there exists a positive measure σ_α on $\mathbb{T}$ such that

$$\frac{1 - |u(z)|^2}{|\alpha - u(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2}\, d\sigma_\alpha(\zeta), \qquad z \in \mathbb{D}.$$

The set of measures $\{\sigma_\alpha\}_{\alpha \in \mathbb{T}}$ is the set of the *Clark measures* for the inner function u. We cite [51] and [53] as standard references for the Clark measures, and we follow Chapter 11 of [30].

Proposition 1.41. *For an inner function u, the corresponding family of Clark measures $\{\sigma_\alpha\}_{\alpha \in \mathbb{T}}$ satisfies the following properties:*

(i) $\sigma_\alpha \perp m$ *for every* $\alpha \in \mathbb{T}$;

(ii) $\sigma_\alpha \perp \sigma_\beta$ *for every* $\alpha \neq \beta$;

(iii) σ_α *has a point mass at* $\zeta \in \mathbb{T}$ *if and only if*

$$u(\zeta) = \lim_{r \to 1} u(r\zeta) = \alpha$$

and u has finite angular derivative in the sense of Carathéodory at ζ. Furthermore,

$$\sigma_\alpha(\{\zeta\}) = \frac{1}{|u'(\zeta)|} \quad \textit{and} \quad u'(\zeta) = \frac{\alpha\overline{\zeta}}{\sigma_\alpha(\{\zeta\})};$$

(iv) A carrier for σ_α is the set

$$\{\zeta \in \mathbb{T} \colon \lim_{r\to 1} u(r\zeta) = \alpha\}.$$

We call *Clark points* the *atoms*, i.e. the point masses, of a Clark measure, that are described in *(iii)*.

Proof. See Proposition 11.2 of [30]. Notice that the assumption that $u(0) = 0$ is not really necessary for the proof. □

We introduce the Clark unitary operator, the Clark transform and we discuss a special orthonormal basis for K_u. For simplicity, it is customary to assume that $u(0) = 0$. At the end of this section, we will quickly discuss how to move to the general case. We fix an inner function u such that $u(0) = 0$. For each $\alpha \in \mathbb{T}$, we define the Clark operator on K_u as the rank-one perturbation

$$U_\alpha := S_u + \alpha(k_0^u \otimes Ck_0^u),$$

where C is the conjugation defined in (1.14). It holds the following, that is Theorem 11.4 in [30].

Theorem 1.42 (Clark)**.** *For each $\alpha \in \mathbb{T}$, the operator U_α is a unitary operator. The eigenvalues of U_α are the points $\zeta \in \mathbb{T}$ such that $u(\zeta) = \alpha$ and u has an angular derivative in the sense of Carathéodory at ζ. The corresponding eigenvectors are the boundary kernels k_ζ^u.*

We have a concrete spectral representation for the operator U_α, that uses the Clark measure σ_α [30, Theorem 11.6].

Theorem 1.43 (Clark)**.** *For an inner function with $u(0) = 0$ and $\alpha \in \mathbb{T}$, the Clark transform defined by*

$$(V_\alpha f)(z) = (1 - \overline{\alpha}u(z)) \int_{\mathbb{T}} \frac{f(\lambda)}{1 - \overline{\lambda}z}\, d\sigma_\alpha(\lambda), \qquad z \in \mathbb{D},$$

is a unitary operator from $L^2(\sigma_\alpha)$ onto K_u. Furthermore, considering the bilateral shift

$$Z_\alpha \colon L^2(\sigma_\alpha) \to L^2(\sigma_\alpha) \qquad (Z_\alpha f)(\lambda) = \lambda f(\lambda),$$

then $V_\alpha Z_\alpha = U_\alpha V_\alpha$. In particular, U_α is unitarily equivalent to Z_α.

Some collections of boundary kernels provide orthonormal basis for the model space K_u. In what follows, we take u inner with $u(0) = 0$ and $\alpha \in \mathbb{T}$ such that the Clark measure σ_α has countable support. If $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$ are the Clark points for the measure σ_α, we denote

$$\widetilde{k}_n(z) := \frac{1}{\|k_{\zeta_n}^u\|_{H^2}} \frac{1 - \overline{u(\zeta_n)}u(z)}{1 - \overline{\zeta_n}z}, \qquad z \in \mathbb{D} \tag{1.15}$$

the associated normalized boundary kernels.

Lemma 1.44. *Let σ_α be a Clark measure with countable support. Then, the collection $\{\widetilde{k_n}\}_n$ defined as in* (1.15) *forms a complete orthonormal basis for the model space K_u.*

Proof. The Clark transform $V_\alpha \colon L^2(\sigma_\alpha) \to K_u$ is a unitary operator. We note that the set $\{\varphi_n\}_n$ of the normalized characteristic functions

$$\varphi_n(\lambda) := \frac{1}{\sqrt{\sigma_\alpha(\{\zeta_n\})}}\, \chi_{\{\zeta_n\}}(\lambda), \qquad \lambda \in \mathbb{T},$$

is a complete orthonormal basis for $L^2(\sigma_\alpha)$. Finally, since by Theorem 1.31 and Proposition 1.41,

$$\sigma_\alpha(\{\zeta_n\}) = \frac{1}{|u'(\zeta_n)|} = \frac{1}{\|k^u_{\zeta_n}\|},$$

then for $z \in \mathbb{D}$ it holds

$$\begin{aligned} (V_\alpha \varphi_n)(z) &= \frac{1-\overline{\alpha}u(z)}{\sqrt{\sigma_\alpha(\{\zeta_n\})}} \int_{\mathbb{T}} \frac{\chi_{\zeta_n}(\lambda)}{1-\overline{\lambda}z}\, d\sigma_\alpha(\lambda) \\ &= \frac{1}{\sqrt{|u'(\zeta_n)|}} \frac{1-\overline{\alpha}u(z)}{1-\overline{\zeta_n}z} \\ &= \frac{1}{\|k^u_{\zeta_n}\|_{H^2}} \frac{1-\overline{u(\zeta_n)}u(z)}{1-\overline{\zeta_n}z} \\ &= \widetilde{k_n}(z). \end{aligned}$$

Therefore, $(\widetilde{k_n})_n$ is a complete orthonormal basis in K_u. □

So far, we have assumed that $u(0) = 0$. To pass to the general case, one can consider the *Crofoot transform.* First of all, we recall that every (analytic) disk automorphism takes the form

$$\varphi_{\eta,a}(z) = \eta \frac{a-z}{1-\overline{a}z}, \qquad z \in \mathbb{D},$$

with $\eta \in \mathbb{T}$ and $a \in \mathbb{D}$. Notice that, given an inner function u and $\varphi_{\eta,a}$ an automorphism of $\mathbb{D}$, the composition $\varphi_{\eta,a} \circ u$ is inner, since $\varphi_{\eta,a}$ maps the cirle $\mathbb{T}$ in itself. It is less evident, but the composition of two inner functions always produces an inner function [30, Chapter 6].

Theorem 1.45 (Theorem 6.7 in [30])**.** *For an inner function u and $w \in \mathbb{D}$, the Crofoot transform*

$$J_w f = \frac{\sqrt{1-|w|^2}}{1-\overline{w}u} f$$

defines a unitary operator from K_u onto K_{u_w}, where

$$u_w = \frac{u-w}{1-\overline{w}u} = \varphi_{-1,w} \circ u.$$

Considering the Crofoot transform J_w with $w = u(0)$, notice that $u_w(0) = 0$. This shows that any model space K_u admits an unitary operator onto another model space K_{u_w}, with $u_w(0) = 0$. This is the reason why, in our context, in the study of the Clark theory it was not restrictive to assume that $u(0) = 0$.

1.3.3 Dirichlet spaces

We move on to the third class of spaces that are of interest for this work. The Dirichlet space

$$\mathcal{D} := \{f \in \mathrm{Hol}(\mathbb{D}) \mid \mathcal{D}(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty\},$$

endowed with the norm $\|f\|_{\mathcal{D}}^2 := \|f\|_{H^2}^2 + \mathcal{D}(f)$, is one of the most interesting and studied spaces of analytic functions on the unit disk. Here, dA is the two-dimensional Lebesgue measure, i.e. the area measure. Many questions about Dirichlet functions are still unanswered today. For example, it is not known if there exists a characterization for the zero sets of Dirichlet functions, or a characterization for the *cyclicity* condition in $\mathcal{D}$. We mention that there is a formula for $\mathcal{D}(f)$ expressed purely in terms of the boundary values of f, due to Douglas [19]:

$$\mathcal{D}(f) = \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{f(\lambda)-f(\zeta)}{\lambda-\zeta}\right|^2 dm(\lambda)\, dm(\zeta), \qquad f \in \mathrm{Hol}(\mathbb{D}). \tag{1.16}$$

In this work, we will be interested in a weighted version of $\mathcal{D}$. Given a finite positive Borel measure μ on the unit circle $\mathbb{T}$, the associated $\mathcal{D}_\mu$ space is the space of holomorphic functions on $\mathbb{D}$ having finite *harmonically weighted* Dirichlet integral, that is

$$\mathcal{D}_\mu(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2\, P\mu(z)\, dA(z) < \infty, \tag{1.17}$$

where $P\mu$ is the Poisson integral of μ,

$$P\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\lambda - z|^2}\, d\mu(\lambda), \qquad z \in \mathbb{D}.$$

These spaces were introduced by Stefan Richter in 1991 for the representation of cyclic analytic two-isometries, see [48]. Also, they play a key role in the description of the closed shift-invariant subspaces of the classical Dirichlet space $\mathcal{D}$, see [50]. In the last chapter, we will consider a more general weighted version, the *super*-harmonically weighted Dirichlet spaces.

We recall a few basic properties; for a treatise of Dirichlet spaces we refer to [23]. If μ is a finite positive Borel measure on $\mathbb{T}$, then $\mathcal{D}_\mu$ is a subset of the Hardy space H^2 which contains all polynomials. Notice that the weighted Dirichlet integral annihilates the constants, thus, on its own, it does not produce a norm. However, $\mathcal{D}_\mu$ is a Hilbert space with respect to the norm

$$\|f\|_{\mathcal{D}_\mu}^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

Considering $\mu = m$, the normalized Lebesgue measure on $\mathbb{T}$, one has $Pm \equiv 1$, so that $\mathcal{D}_m(f) = \mathcal{D}(f)$ and the space $\mathcal{D}_m$ is exactly the classical Dirichlet space $\mathcal{D}$. For $\zeta \in \mathbb{T}$, considering the Dirac delta δ_ζ we obtain the so-called *local Dirichlet space*, which we simply denote by $\mathcal{D}_\zeta$. Also, we write $\mathcal{D}_\zeta(f)$ instead of $\mathcal{D}_{\delta_\zeta}(f)$, to denote the *local* Dirichlet integral. For $f \in H^2$, by Fubini's theorem, $\mathcal{D}_\mu(f)$ given in (1.17) can be expressed as

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}} \mathcal{D}_\zeta(f)\, d\mu(\zeta). \tag{1.18}$$

In [49], Richter and Sundberg showed the following characterization of $\mathcal{D}_\zeta$.

Theorem 1.46. *Let $f \in H^2$. Then, $f \in \mathcal{D}_\zeta$ if and only if there exist $c \in \mathbb{C}$ and $g \in H^2$ such that*

$$f(z) = c + (z - \zeta)g(z), \qquad z \in \mathbb{D}.$$

In this case, $c = f(\zeta)$ and $\mathcal{D}_\zeta(f) = \|g\|^2_{H^2}$.

We can also write

$$\mathcal{D}_\zeta = \mathbb{C} + (S - \zeta I)H^2.$$

The proof of Theorem 1.46 is technical, and we do not show it here. See [49] or [30]. In particular, Theorem 1.46 implies that all functions in $\mathcal{D}_\zeta$ admit boundary value at ζ. We remark that, in fact, stronger results hold: the limit $f(\zeta)$ exists in any oricyclic approach region of the form

$$\{z \in \mathbb{D} \colon |z - \zeta| \le \kappa(1 - |z|^2)^{\frac{1}{2}}\},$$

for every $\kappa > 0$, the Fourier series of f converges at ζ and

$$f(\zeta) = \sum_{n \ge 0} \widehat{f}(n)\zeta^n \in \mathbb{C}.$$

As a corollary, we get the following useful formula for $\mathcal{D}_\zeta(f)$, which includes the boundary value $f(\zeta) = \lim_{r \to 1^-} f(r\zeta)$, whenever it exists.

Theorem 1.47 (Local Douglas formula)**.** *Let $f \in H^2$ and $\zeta \in \mathbb{T}$. If the boundary value $f(\zeta)$ exists, then*

$$\mathcal{D}_\zeta(f) = \int_{\mathbb{T}} \left| \frac{f(\lambda) - f(\zeta)}{\lambda - \zeta} \right|^2 dm(\lambda). \tag{1.19}$$

Otherwise, $\mathcal{D}_\zeta(f) = \infty$.

Notice that, integrating both sides of (1.19) with respect to $dm(\zeta)$, we recover the Douglas formula (1.16).

Proof of Theorem 1.47. We follow [30]. Suppose first that $\mathcal{D}_\zeta(f) < \infty$. Then, by Theorem 1.46, $f(\zeta)$ exists and $\mathcal{D}_\zeta(f) = \|g\|^2_{H^2}$, where

$$g(z) := \frac{f(z) - f(\zeta)}{z - \zeta}, \qquad z \in \mathbb{D}. \tag{1.20}$$

In particular, (1.19) holds. To conclude the proof, we have to show that if $f(\zeta)$ exists and $\mathcal{D}_\zeta(f) = \infty$, then the right-hand side of (1.19) is also ∞. Consider again g as in (1.20). The numerator of g is in L^2, and the denominator is an outer function. If we had $g \in L^2$, then by Theorem 1.15 we would have that $g \in H^2$. We would have that $f \in \mathcal{D}_\zeta$, by Theorem 1.46, which is a contradiction. In particular, $\|g\|_{L^2} = \infty$, concluding the proof. ∎

This formula shows that the difference quotient at ζ plays an important role for membership in the local Dirichlet space. We will study these quotients in great detail in Chapter 2.

Inner functions behave nicely with respect to the local Dirichlet integral.

Theorem 1.48. *Let u be an inner function and $\zeta \in \mathbb{T}$. Then, $u \in \mathcal{D}_\zeta$ if and only if u has finite ADC at ζ. Moreover,*

$$\mathcal{D}_\zeta(u) = |u'(\zeta)|.$$

Proof. By Theorem 7.6.5 in [23], we have that

$$\mathcal{D}_\zeta(u) = \lim_{r\to 1} \frac{1-|u(r\zeta)|^2}{1-r^2}. \tag{1.21}$$

Using the notation of Theorem 1.31, if $u \in \mathcal{D}_\zeta$, then the quantity

$$c := \liminf_{z\to\zeta} \frac{1-|u(z)|}{1-|z|}$$

is finite. Thus, u admits ADC at ζ, and it follows that

$$c = \angle\lim_{z\to\zeta} \frac{1-|u(z)|}{1-|z|} = |u'(\zeta)|.$$

In particular, by (1.21), we have that $\mathcal{D}_\zeta(u) = |u'(\zeta)|$. On the other hand, if u has finite ADC at ζ, then by (b) in Theorem 1.31 and (1.21) we conclude that $\mathcal{D}_\zeta(u) = |u'(\zeta)| < \infty$. □

The previous result is taken from Proposition 3.5 of [49]. In the same work, the authors also showed that for any analytic self-map of the disk $b\colon \mathbb{D} \to \mathbb{D}$, if b has finite ADC at ζ, then $b \in \mathcal{D}_\zeta$ with $\mathcal{D}_\zeta(b) \le |b'(\zeta)|$. We can prove a simple generalization for an arbitrary measure μ.

Corollary 1.49. *Let μ be a finite positive Borel measure on $\mathbb{T}$. Then, $u \in \mathcal{D}_\mu$ is and only if $u' \in L^1(\mu)$. Moreover,*

$$\mathcal{D}_\mu(u) = \|u'\|_{L^1(\mu)}.$$

Proof. If $u \in \mathcal{D}_\mu$, then by Fubini's Theorem

$$\mathcal{D}_\mu(u) = \int_{\mathbb{T}} \mathcal{D}_\zeta(u)\, d\mu(\zeta) < \infty.$$

In particular, for μ-a.e. $\zeta \in \mathbb{T}$ we have that $\mathcal{D}_\zeta(u) < \infty$. By Theorem 1.48, we have that u has finite ADC at μ-a.e. $\zeta \in \mathbb{T}$, and

$$\|u'\|_{L^1(\mu)} = \int_{\mathbb{T}} |u'(\zeta)|\, d\mu(\zeta) = \int_{\mathbb{T}} \mathcal{D}_\zeta(u)\, d\mu(\zeta) = \mathcal{D}_\mu(u) < \infty.$$

Similarly, if $u' \in L^1(\mu)$, then for μ-a.e. $\zeta \in \mathbb{T}$ we have that $\mathcal{D}_\zeta(u) < \infty$ and

$$\mathcal{D}_\mu(u) = \int_{\mathbb{T}} \mathcal{D}_\zeta(u)\, d\mu(\zeta) = \int_{\mathbb{T}} |u'(\zeta)|\, d\mu(\zeta) = \|u'\|_{L^1(\mu)} < \infty.$$

□

There is a nice formula for the angular derivative $|u'(\zeta)|$, that uses the Nevanlinna factorization of the inner function u.

Proposition 1.50. *Let $\zeta \in \mathbb{T}$ and u be an inner function with Nevanlinna decomposition*

$$u(z) = \left(\prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z} \right) \exp\left\{ - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\tau(\zeta) \right\},$$

where $\{a_n\}_n$ are the zeros of u and τ a singular measure on $\mathbb{T}$. Then, u has finite ADC at ζ if and only if

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|a_n - \zeta|^2} + \int_{\mathbb{T}} \frac{2}{|\lambda - \zeta|^2}\, d\tau(\lambda) < \infty.$$

Moreover, in this case,

$$|u'(\zeta)| = \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|a_n - \zeta|^2} + \int_{\mathbb{T}} \frac{2}{|\lambda - \zeta|^2}\, d\tau(\lambda).$$

Proof. The proof is taken from Theorem 2.21 of [30] and Theorem 21.11 of [29]. □

It follows a characterization of the membership of inner functions in the classical Dirichlet space $\mathcal{D}$. For the details, see Corollary 7.6.10 of [23].

Corollary 1.51. *The only inner functions in $\mathcal{D}$ are finite Blaschke products.*

We conclude this part showing another property of inner functions, in relation to the local Dirichlet integral. For a proof, see [23, Theorem 7.6.1].

Proposition 1.52. *Let $f \in H^2$, u an inner function and $\zeta \in \mathbb{T}$. Then,*

$$\mathcal{D}_\zeta(uf) = \mathcal{D}_\zeta(f) + |f(\zeta)|^2 \mathcal{D}_\zeta(u). \tag{1.22}$$

If the radial limit $f(\zeta)$ does not exist, then the right-hand side of (1.22) should be interpreted as ∞. If the radial limit $f(\zeta)$ is equal to 0, then the product $|f(\zeta)|^2 \mathcal{D}_\zeta(u)$ should always be interpreted as 0.

As a corollary of the previous proposition, we deduce that the forward shift S defines a bounded operator on $\mathcal{D}_\mu$, for every measure μ. This is taken from Theorem 8.1.2 in [23].

Corollary 1.53. *Let μ be a finite, positive, Borel measure on $\mathbb{T}$. Then, the following hold.*

(i) For every $f \in \mathcal{D}_\mu$, we have that $f \in L^2(\mu)$ with

$$\|f\|_{L^2(\mu)} \le (1 + \mu(\mathbb{T})^{\frac{1}{2}}) \|f\|_{\mathcal{D}_\mu}$$

(ii) The forward shift

$$\begin{array}{rccc} S: & \mathcal{D}_\mu & \longrightarrow & \mathcal{D}_\mu \\ & f & \longmapsto & zf \end{array}$$

is well-defined and bounded.

Proof. We prove (i). For $f \in \mathcal{D}_\mu \subseteq H^2$, consider $g := S^* f \in H^2$. It holds that

$$zg(z) = f(z) - f(0), \qquad z \in \mathbb{D},$$

and that, by (1.22),

$$\mathcal{D}_\zeta(zg) = \mathcal{D}_\zeta(g) + |g(\zeta)|^2 \mathcal{D}_\zeta(z).$$

As $\mathcal{D}_\zeta(z) = 1$ and $f = zg + f(0)$, it holds

$$\mathcal{D}_\zeta(f) = \mathcal{D}_\zeta(zg) = \mathcal{D}_\zeta(g) + |f(\zeta) - f(0)|^2.$$

In particular,

$$\int_{\mathbb{T}} |f(\zeta) - f(0)|^2 \, d\mu(\zeta) \leq \int_{\mathbb{T}} \mathcal{D}_\zeta(f) \, d\mu(\zeta) = \mathcal{D}_\mu(f) < \infty.$$

It follows that $f \in L^2(\mu)$ and

$$\begin{aligned}
\|f\|_{L^2(\mu)} &\leq \|f - f(0)\|_{L^2(\mu)} + |f(0)| \mu(\mathbb{T})^{\frac{1}{2}} \\
&\leq \mathcal{D}_\mu(f)^{\frac{1}{2}} + \|f\|_{H^2} \mu(\mathbb{T})^{\frac{1}{2}} \\
&\leq (1 + \mu(\mathbb{T})^{\frac{1}{2}}) \|f\|_{\mathcal{D}_\mu}.
\end{aligned}$$

Now, we prove (ii). For $f \in \mathcal{D}_\mu$, we have that

$$\begin{aligned}
\mathcal{D}_\mu(zf) &= \int_{\mathbb{T}} \mathcal{D}_\zeta(zf) \, d\mu(\zeta) \\
&= \int_{\mathbb{T}} \mathcal{D}_\zeta(f) \, d\mu(\zeta) + \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{D}_\zeta(z) \, d\mu(\zeta) \\
&= \mathcal{D}_\mu(f) + \|f\|^2_{L^2(\mu)} < \infty,
\end{aligned}$$

so that $zf \in \mathcal{D}_\mu$. The boundedness of S follows from (i). □

Chapter 2

Difference quotient operator

In this chapter, we discuss the *difference quotient operator* on de Branges–Rovnyak spaces, with a special attention to the model spaces. We use the tools of operator theory to study the quantities

$$\frac{f(z) - f(w)}{z - w},$$

given a function f in a suitable function space and appropriate points z, w. Obviously, difference quotients are a key concept of mathematical analysis, they are instrumental for the definition of the derivative, that is an idea that revolutionised the history of mathematics and, thus, the history of mankind. Here, with a much more humble and modest approach, we discuss the operator on $H(b)$ that naturally arises from different quotients at boundary points $\zeta \in \mathbb{T}$, and we study its properties.

The analysis that we do in this chapter is very important for Chapter 3: as we already mentioned in Subsection 1.3.3, difference quotients play an important role for membership in the Dirichlet spaces. However, there are also original results that are very significant on their own: for example, Theorem 2.12 provides a new equivalent characterization for the property of being a *one-component inner function*. We conclude this introduction saying that the content of this chapter comes from a joint work with Carlo Bellavita and Javad Mashreghi. Now, we give a proper introduction to the difference quotient operators.

Let $\mathcal{X}$ be a family of functions which are defined on the set $\Omega \subset \mathbb{C}$. Fix a point $w \in \Omega$. The mapping

$$(Q_w f)(z) := \frac{f(z) - f(w)}{z - w}, \qquad z \in \Omega \setminus \{w\}, \quad f \in \mathcal{X}, \tag{2.1}$$

appears in many discussions and it is called the *difference quotient operator*. In particular, for the Hardy space on the unit disk H^2, the very special but important case $w = 0$ leads to the *backward shift operator*

$$S^* f(z) = Q_0 f(z) = \frac{f(z) - f(0)}{z}, \qquad z \in \mathbb{D} \setminus \{0\}. \tag{2.2}$$

In this introductory part, we will report some results on the operator Q_w on the Hardy space H^2, for $w \in \mathbb{D}$. We follow [30, Chapter 4.6]. First of all, we recall that the reproducing kernel of H^2 is

$$c_w(z) = \frac{1}{1 - \overline{w}z}, \qquad z, w \in \mathbb{D}.$$

For $z, w \in \mathbb{D}$, we have that

$$(\mathrm{I} - \overline{w}S)c_w(z) = 1 = c_0(z).$$

We recall that $\sigma(S) = \overline{\mathbb{D}}$, so that the operator $\mathrm{I} - \overline{w}S$ is invertible for every $w \in \mathbb{D}$. In particular, $c_w = (\mathrm{I} - \overline{w}S)^{-1}c_0$, and it follows that

$$f(w) = \langle f, c_w \rangle_{H^2} = \langle f, (\mathrm{I} - \overline{w}S)^{-1}c_0 \rangle_{H^2} = \langle (\mathrm{I} - wS^*)^{-1}f, c_0 \rangle_{H^2}.$$

We define

$$Q_w := (\mathrm{I} - wS^*)^{-1}S^*, \qquad w \in \mathbb{D}, \tag{2.3}$$

and we show that the operator Q_w satisfies the difference quotient formula (2.1). Note that Q_w is bounded on H^2 and, in particular, $Q_0 = S^*$.

Proposition 2.1. *For $f \in H^2$ and $w \in \mathbb{D}$, it holds*

$$(Q_w f)(z) = \frac{f(z) - f(w)}{z - w}, \qquad z \in \mathbb{D} \setminus \{w\}.$$

Proof. We start with the following resolvent computation:

$$\begin{aligned}(\mathrm{I} - zS^*)^{-1} - (\mathrm{I} - wS^*)^{-1} =& (\mathrm{I} - zS^*)^{-1}(\mathrm{I} - wS^*)(\mathrm{I} - wS^*)^{-1} \\ &- (\mathrm{I} - zS^*)^{-1}(\mathrm{I} - zS^*)(\mathrm{I} - wS^*)^{-1} \\ =& (\mathrm{I} - zS^*)^{-1}(wS^* + zS^*)(\mathrm{I} - wS^*)^{-1} \\ =& (z - w)(\mathrm{I} - zS^*)^{-1}(\mathrm{I} - wS^*)^{-1}S^*.\end{aligned}$$

We write this as

$$(\mathrm{I} - zS^*)^{-1}(\mathrm{I} - wS^*)^{-1}S^* = \frac{(\mathrm{I} - zS^*)^{-1} - (\mathrm{I} - wS^*)^{-1}}{z - w}.$$

We now have

$$\begin{aligned}Q_w f(z) &= \langle Q_w f, c_z \rangle_{H^2} \\ &= \langle Q_w f, (\mathrm{I} - \overline{z}S)^{-1}c_0 \rangle_{H^2} \\ &= \langle (\mathrm{I} - zS^*)^{-1}(\mathrm{I} - wS^*)^{-1}S^* f, c_0 \rangle_{H^2} \\ &= \frac{1}{z - w}\left(\langle (\mathrm{I} - zS^*)^{-1}f, c_0 \rangle_{H^2} + \langle (\mathrm{I} - wS^*)^{-1}f, c_0 \rangle_{H^2} \right) \\ &= \frac{f(z) - f(w)}{z - w}.\end{aligned}$$

□

2.1 Difference quotient operator on $H(b)$

So far, we have formally introduced the difference quotient operator on H^2 at a point $w \in \mathbb{D}$, as

$$Q_w f(z) = (\mathrm{I} - wS^*)^{-1} S^* f(z) = \frac{f(z) - f(w)}{z - w}, \qquad z \in \mathbb{D},\ f \in H^2.$$

On the space H^2, it makes no sense to consider difference quotient operators Q_ζ at points in the boundary $\zeta \in \mathbb{T}$. One the one hand, from a functional point of view, not every function f in H^2 admits the boundary value $f(\zeta)$, for the same fixed ζ. On the other hand, from an operator-theoretic point of view, the spectrum of S^* is the whole closed disk $\overline{\mathbb{D}}^{\mathrm{cl}}$, and there is no natural extension for the resolvent operator in formula (2.3) for points on the boundary. The situation changes if we move to the special class of subsets of H^2 that are $H(b)$ spaces. In this context, instead of S^* we consider the operator X_b. We recall that, if b is non-extreme, then $\sigma(X_b) = \sigma(X_b^*) = \overline{\mathbb{D}}^{\mathrm{cl}}$, and if b is extreme then $\sigma(X_b^*) = \overline{\sigma(b)}^{\mathrm{cl}}$. We focus on the latter case: the set $\overline{\sigma(b)}^{\mathrm{cl}}$ in general can be smaller than $\overline{\mathbb{D}}^{\mathrm{cl}}$. For $\zeta \notin \overline{\sigma(b)}^{\mathrm{cl}}$, the operator $\mathrm{I} - \zeta X_b$ is invertible, for

$$\mathrm{I} - \zeta X_b = \zeta(\bar{\zeta}\mathrm{I} - X_b) = \zeta(\zeta \mathrm{I} - X_b^*)^*.$$

In light of the formula (2.3), we give the following definition.

Definition 2.2. Let $b \in H_1^\infty$ be such that $\overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T} \neq \mathbb{T}$. Then, for $\zeta \in \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$, we define the difference quotient operator Q_ζ^b on $H(b)$ as

$$Q_\zeta^b := (\mathrm{I} - \zeta X_b)^{-1} X_b. \tag{2.4}$$

Notice that the assumption $\overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T} \neq \mathbb{T}$ is possible only for functions b that are extreme. For the rest of this chapter, we will only work with extreme functions b with $\overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T} \neq \mathbb{T}$. Choosing $\zeta \in \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$, then by Theorem 1.32 every function f in the de Branges-Rovnyak space $H(b)$ admits an analytic extension in a neighbourhood of the point ζ and, in particular, the boundary value $f(\zeta) = \lim_{r \to 1} f(r\zeta)$ is always well-defined.

Following the same steps of the proof of Proposition 2.1, using the boundary kernel k_ζ^b instead of c_w, one can check that for $f \in H(b)$ it holds the difference quotient formula

$$(Q_\zeta^b f)(z) = \frac{f(z) - f(\zeta)}{z - \zeta}, \qquad z \in \mathbb{D}.$$

This operator will be the object of study for this chapter. We start with the study of the spectrum of the operator Q_ζ^b, and we provide a lower bound for the norm. We will simply write $\|Q_\zeta^b\|$ to denote the operator norm $\|Q_\zeta^b\|_{\mathcal{B}(H(b))}$.

Theorem 2.3. *Let $b \in H_1^\infty$ extreme and $\zeta \in \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$. Then, the spectrum of the operator Q_ζ^b is the set*

$$\sigma(Q_\zeta^b) = \left\{ \frac{\overline{\eta}}{1 - \zeta\overline{\eta}} : \eta \in \overline{\sigma(b)}^{\mathrm{cl}} \right\}. \tag{2.5}$$

Also, it holds the lower estimate

$$\|Q_\zeta^b\| \geq \frac{1}{\operatorname{dist}(\zeta, \overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T})}. \tag{2.6}$$

Proof. We recall the definition (2.4), $Q_\zeta^b := (\mathrm{I} - \zeta X_b)^{-1} X_b$. For $\lambda \in \mathbb{C}$, an easy computation leads to the identity

$$\lambda \mathrm{I} - Q_\zeta^b = (\mathrm{I} - \zeta X_b)^{-1} [\lambda \mathrm{I} - (\lambda\zeta + 1) X_b].$$

This shows that, for $\lambda = -\overline{\zeta}$, the operator $\lambda \mathrm{I} - Q_\zeta^b$ is invertible. Moreover, for $\lambda \neq -\overline{\zeta}$ it holds

$$\lambda \mathrm{I} - Q_\zeta^b = (\lambda\zeta + 1)(\mathrm{I} - \zeta X_b)^{-1} \left(\frac{\lambda}{1 + \zeta\lambda} \mathrm{I} - X_b \right). \tag{2.7}$$

It would be more natural to work with the operator X_b. However, since we want the set $\sigma(b)$ to appear, which is directly related to its adjoint X_b^*, we write

$$\lambda \mathrm{I} - Q_\zeta^b = (\lambda\zeta + 1)(\mathrm{I} - \zeta X_b)^{-1} \left(\frac{\overline{\lambda}}{1 + \overline{\zeta\lambda}} \mathrm{I} - X_b^* \right)^*.$$

This shows that

$$\sigma(Q_\zeta^b) = \left\{ \lambda \in \mathbb{C} \setminus \{-\overline{\zeta}\} \colon \frac{\overline{\lambda}}{1 + \overline{\zeta\lambda}} \in \sigma(X_b^*) \right\}.$$

For $\eta \in \sigma(X_b^*)$, one can check that

$$\frac{\overline{\lambda}}{1 + \overline{\zeta\lambda}} = \eta \iff \overline{\lambda} = \frac{\eta}{1 - \overline{\zeta}\eta},$$

proving the set identity in Equation (2.5), since by Theorem 1.34 we have that $\sigma(X_b^*) = \overline{\sigma(b)}^{\mathrm{cl}}$. Now, the lower estimate (2.6) follows from the basic fact that each point in spectrum of an operator has modulus less or equal than the operator norm. In particular, for every $\eta \in \overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T}$ we have that

$$\|Q_\zeta^b\| \geq \frac{1}{|\zeta - \eta|},$$

and we conclude by taking the supremum. Notice that, if the intersection $\overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T}$ is empty, then $\operatorname{dist}(\zeta, \overline{\sigma(b)}^{\mathrm{cl}} \cap \mathbb{T}) = \inf \emptyset = +\infty$, and the statement is trivial. $\square$

Theorem 2.3 shows in a quantitative way that, when a point $\zeta \in \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$ approaches the closure of the spectrum of b moving on the boundary $\mathbb{T}$, then the operator norm $\|Q_\zeta^b\|$ explodes. This behaviour suggests that the choice of considering points $\zeta \in \mathbb{T} \setminus \overline{\sigma(b)}^{\mathrm{cl}}$ is optimal, in order to have a bounded operator associated to the difference quotient at ζ.

For what follows, we restrict to the case where $b = u$ is an inner function. We recall that the spectrum $\sigma(u)$ is always a closed set. We prove another lower estimate for $\|Q_\zeta^u\|$, which turns out to be somewhat sharp at least in some special cases that we will discuss. First, a lemma concerning the pointwise derivative $f'(\zeta)$ for functions f in K_u.

Lemma 2.4. *Let u be an inner function, and let $\zeta \in \mathbb{T} \setminus \sigma(u)$. Then, for every $f \in K_u$ the derivative $f'(\zeta)$ exists and the linear mapping*

$$\begin{array}{cccc} \Lambda_\zeta : & K_u & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & f'(\zeta) \end{array}$$

is an element of the dual space $(K_u)^*$, with norm

$$\|\Lambda_\zeta\|_{(K_u)^*} = \|Q_\zeta^u(k_\zeta^u)\|_{H^2}.$$

In particular, we have that

$$|f'(\zeta)| \leq \|Q_\zeta^u(k_\zeta^u)\|_{H^2}\|f\|_{H^2}, \qquad f \in K_u. \tag{2.8}$$

Proof. The first part of the proof is taken from Section 21.6 of [29]. Let $f \in K_u$. Since $\zeta \in \mathbb{T} \setminus \sigma(u)$, there exists $\delta > 0$ such that f has an analytic extension in the open ball $B_{\mathbb{C}}(\zeta, \delta) := \{w \in \mathbb{C} \colon |w - \zeta| < \delta\}$. For $w \in \mathbb{D} \cap B_{\mathbb{C}}(\zeta, \delta)$, we have that

$$\begin{aligned} \left| \left\langle f, \frac{k_w^u - k_\zeta^u}{\overline{w} - \overline{\zeta}} \right\rangle_{H^2} \right| &= \left| \frac{f(w) - f(\zeta)}{w - \zeta} \right| \\ &= \left| \frac{1}{w - \zeta} \int_\zeta^w f'(z)\, dz \right| \leq C_f < \infty. \end{aligned}$$

Hence, by the uniform boundedness principle, there exists a constant $C > 0$ such that

$$\left\| \frac{k_w^u - k_\zeta^u}{\overline{w} - \overline{\zeta}} \right\|_{H^2} \leq C, \qquad w \in \mathbb{D} \cap B_{\mathbb{C}}(\zeta, \delta).$$

We can extract a sequence $(w_n)_n$ in $\mathbb{D}$ that converges to ζ such that the difference quotients $(\overline{w_n} - \overline{\zeta})^{-1}(k_{w_n}^u - k_\zeta^u)$ weakly converge to a function $\psi_\zeta \in K_u$. Now, for every $f \in K_u$ one has that

$$\langle f, \psi_\zeta \rangle_{H^2} = \lim_n \left\langle f, \frac{k_{w_n}^u - k_\zeta^u}{\overline{w_n} - \overline{\zeta}} \right\rangle_{H^2} = \lim_n \frac{f(w_n) - f(\zeta)}{w_n - \zeta} = f'(\zeta).$$

This proves that the operator Λ_ζ is an element of the dual $(K_u)^*$, and its norm $\|\Lambda_\zeta\|_{(K_u)^*}$ is equal to $\|\psi_\zeta\|_{H^2}$. To conclude the proof, it suffices to show that $\|\psi_\zeta\|_{H^2} = \|\mathrm{Q}_\zeta^u(k_\zeta^u)\|_{H^2}$. We compute the function ψ_ζ.

For $z \in \mathbb{D}$, we have that

$$\begin{aligned}\psi_\zeta(z) = \langle \psi_\zeta, k_z^u \rangle_{H^2} &= \lim_n \left\langle \frac{k_{w_n}^u - k_\zeta^u}{\overline{w_n} - \overline{\zeta}}, k_z^u \right\rangle_{H^2} \\ &= \lim_n \frac{\overline{k_z^u(w_n)} - \overline{k_z^u(\zeta)}}{\overline{w_n} - \overline{\zeta}} = \overline{(k_z^u)'(\zeta)}.\end{aligned}$$

Differentiating the kernel k_z^u, we obtain

$$\psi_\zeta(z) = -\frac{\overline{u'(\zeta)}u(z)}{1 - \overline{\zeta}z} + \frac{z\left(1 - \overline{u(\zeta)}u(z)\right)}{(1 - \overline{\zeta}z)^2}, \qquad z \in \mathbb{D}.$$

Hence,

$$\begin{aligned}\|\psi_\zeta\|_{H^2}^2 &= \int_{\mathbb{T}} \left| \frac{-\overline{u'(\zeta)}u(\lambda)}{\lambda(\overline{\lambda} - \overline{\zeta})} + \frac{\lambda u(\lambda)\left(\overline{u(\lambda)} - \overline{u(\zeta)}\right)}{\lambda^2(\overline{\lambda} - \overline{\zeta})^2} \right|^2 dm(\lambda) \\ &= \int_{\mathbb{T}} \left| \frac{u(\lambda)}{\lambda(\overline{\lambda} - \overline{\zeta})} \left(-\overline{u'(\zeta)} + \frac{\overline{u(\lambda)} - \overline{u(\zeta)}}{\overline{\lambda} - \overline{\zeta}} \right) \right|^2 dm(\lambda) \\ &= \int_{\mathbb{T}} \left| \frac{1}{\lambda - \zeta} \left(-\overline{u'(\zeta)} + \frac{\overline{u(\zeta)}}{\overline{\zeta}} \frac{u(\zeta)\overline{u(\lambda)} - 1}{\zeta\overline{\lambda} - 1} \right) \right|^2 dm(\lambda).\end{aligned}$$

Factoring out $\zeta\overline{u(\zeta)}$, which has modulus 1, it follows that

$$\|\psi_\zeta\|_{H^2}^2 = \int_{\mathbb{T}} \left| \frac{1}{\lambda - \zeta} \left(-\overline{u'(\zeta)}u(\zeta)\overline{\zeta} + \overline{k_\zeta^u(\lambda)} \right) \right|^2 dm(\lambda).$$

Since by Theorem 1.31 it holds $k_\zeta^u(\zeta) = |u'(\zeta)| = u'(\zeta)\zeta\overline{u(\zeta)}$, we have that

$$\|\psi_\zeta\|_{H^2}^2 = \int_{\mathbb{T}} \left| \frac{k_\zeta^u(\lambda) - k_\zeta^u(\zeta)}{\lambda - \zeta} \right|^2 dm(\lambda) = \|\mathrm{Q}_\zeta^u(k_\zeta^u)\|_{H^2}^2. \quad \square$$

Notice that the same operator Λ_ζ is bounded also in the context of de Branges–Rovnyak spaces, *mutatis mutandis*, but the computation for the norm does not hold: the assumption that $b = u$ is inner allows us to compute the norm in $H(b)$ of the element ψ_ζ simply as a norm in H^2.

Using this lemma, we prove the following lower estimate for the norm of the difference quotient operator.

Theorem 2.5. *Let u be an inner function and $\zeta \in \mathbb{T} \setminus \sigma(u)$. Then,*

$$\|Q_\zeta^u\| \geq \frac{|u''(\zeta)|}{2|u'(\zeta)|}. \tag{2.9}$$

Proof. From Lemma 2.4, we have that

$$\|\mathrm{Q}^u_\zeta(k^u_\zeta)\|^2_{H^2} = \sup_{\|f\|_{K_u}=1} |f'(\zeta)|^2. \tag{2.10}$$

We test (2.10) on the (normalized) boundary kernel itself. By direct computation, it follows that

$$(k^u_\zeta)'(z) = \frac{\overline{u(\zeta)}}{\overline{\zeta}(\zeta - z)}\left[\frac{u(z)-u(\zeta)}{z-\zeta} - u'(z)\right], \qquad z \in \mathbb{D}.$$

Expanding $u(z)$ and $u'(z)$ in Taylor series in a neighborhood of ζ, one can check that

$$|(k^u_\zeta)'(\zeta)| = \frac{|u''(\zeta)|}{2}.$$

In particular, (2.10) yields

$$\|\mathrm{Q}^u_\zeta(k^u_\zeta)\|^2_{H^2} \geq \frac{|(k^u_\zeta)'(\zeta)|^2}{\|k^u_\zeta\|^2_{H^2}} = \frac{|u''(\zeta)|^2}{4|u'(\zeta)|}.$$

We conclude that

$$\|\mathrm{Q}^u_\zeta\|^2 \geq \frac{\|\mathrm{Q}^u_\zeta(k^u_\zeta)\|^2_{H^2}}{\|k^u_\zeta\|^2_{H^2}} \geq \frac{1}{\|k^u_\zeta\|^2_{H^2}}\frac{|u''(\zeta)|^2}{4|u'(\zeta)|} = \frac{|u''(\zeta)|^2}{4|u'(\zeta)|^2}.$$

Notice that we used the identity $|u'(\zeta)| = \|k^u_\zeta\|^2_{H^2}$ from Theorem 1.31. □

We give more information on the quantity appearing in Theorem 2.5.

Proposition 2.6. *Let u be an inner function, and consider an open arc $\Delta \subseteq \mathbb{T} \setminus \sigma(u)$ such that $\overline{\Delta}^{\mathrm{cl}} \cap \sigma(u) \neq \emptyset$. If $|u'|$ is unbounded on Δ, then also $\frac{|u''|}{|u'|}$ is unbounded on Δ.*

Proof. Since $\overline{\Delta}^{\mathrm{cl}} \cap \sigma(u) \neq \emptyset$, the function u admits an analytic extension across Δ. Also, by Theorem 1.31, $|u'| > 0$ on Δ. This means that there exists an open set $\Omega \subset \mathbb{C}$ containing Δ on which u' is holomorphic and does not vanish. Without loss of generality, we can assume that Ω is simply connected, so that the function $\log(u')$ is well defined and holomorphic on Ω, a neighbourhood of Δ. Notice that it holds

$$\frac{u''(z)}{u'(z)} = \big(\log(u')\big)'(z), \qquad z \in \Omega.$$

Since $|u'|$ is unbounded on Δ, so is $|\log(u')|$. A standard argument using the fundamental theorem of calculus shows that also $|\big(\log(u')\big)'|$ must be unbounded on Δ, concluding the proof. □

2.2 One-component inner functions

In this section, we restrict to a special class of inner functions, the *one-component* inner functions, for which we establish an upper estimate for the norm $\|Q^u_\zeta\|$. We will also show that, actually, the upper estimate that we provide is in fact characterizing for such special class of inner functions. First, we introduce the one-component inner functions.

Definition 2.7. We say that an inner function u is one-component if there exists $\varepsilon \in (0,1)$ such that the sublevel set

$$\Omega_\varepsilon := \{z \in \mathbb{D} \colon |u(z)| < \varepsilon\}$$

is connected.

One-component inner functions were introduced in 1982 by Cohn [16], to study the embeddings of model spaces in $L^p(\mu)$ spaces. Other equivalent descriptions of the one-component inner functions have been given in [1], [12] and [45]. The model spaces associated to one-component inner functions have been widely studied, see for references [9] and [59]. In particular, the following characterization holds.

Theorem 2.8 (Aleksandrov, [1]). *An inner function u is one-component if and only if the following two conditions hold:*

(i) The Lebesgue measure of its boundary spectrum $\sigma(u) \cap \mathbb{T}$ is zero, and $|u'|$ is unbounded on every open arc $\Delta \subset \mathbb{T} \setminus \sigma(u)$ such that $\overline{\Delta}^{\mathrm{cl}} \cap \sigma(u) \neq \emptyset$;

(ii) There exists a positive constant C_u such that for every $\zeta \in \mathbb{T} \setminus \sigma(u)$ it holds $|u''(\zeta)| \leq C_u |u'(\zeta)|^2$.

Using Theorem 2.8, Bessonov characterized the Clark measures of the one-component inner functions, see [12, Theorem 1]. We recall that, given an inner function u and $\alpha \in \mathbb{T}$, the Clark measure σ_α is the unique finite, positive, Borel measure on $\mathbb{T}$ such that

$$\frac{1-|u(z)|^2}{|\alpha - u(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2}\, d\sigma_\alpha(\zeta)\ , \qquad z \in \mathbb{D}.$$

Following Bessonov's notation, for every Borel measure μ on the unit circle $\mathbb{T}$, we denote by $a(\mu)$ the set of the isolated atoms of μ, and we define $\rho(\mu) := \mathrm{supp}(\mu) \setminus a(\mu)$. We say that an atom $\zeta \in a(\mu)$ has two neighbors if there is an open arc (ζ_-, ζ_+) of $\mathbb{T}$ with endpoints $\zeta_\pm \in a(\mu)$ such that ζ is the only point in $(\zeta_-, \zeta_+) \cap \mathrm{supp}(\mu)$.

Theorem 2.9 (Bessonov). *Let u be a one-component inner function, and let $\alpha \in \mathbb{T}$. Then the associated Clark measure σ_α is a discrete measure on $\mathbb{T}$ with isolated atoms such that $m(supp(\sigma_\alpha)) = 0$. Moreover, every arc $\Delta \subset \mathbb{T} \setminus \sigma(u)$ contains infinitely many atoms of $a(\sigma_\alpha)$ and every atom $\zeta \in a(\sigma_\alpha)$ has two neighbors $\zeta_\pm \in a(\sigma_\alpha)$ such that*

$$A_u|\zeta - \zeta_\pm| \leq \sigma_\alpha(\{\zeta\}) \leq B_u|\zeta - \zeta_\pm| \tag{2.11}$$

for some positive constants A_u, B_u depending only on u.

We will assume that the atoms are indexed on the set of positive natural numbers. Using the information on the atoms of σ_α in Theorem 2.9 and Proposition 1.41, one can explicitly describe the Clark measures σ_α associated to the one-component inner functions. In fact, σ_α is the purely atomic measure given by

$$\sigma_\alpha = \sum_{n\in\mathbb{N}} \sigma_\alpha(\{\zeta_n\})\,\delta_{\zeta_n}, \tag{2.12}$$

where the boundary points ζ_n are the Clark points, the points where the atoms are anchored, and they satisfy

$$\angle \lim_{z\to\zeta_n} u(z) = \alpha$$

and

$$u'(\zeta_n) = \angle \lim_{z\to\zeta_n} \frac{u(z)-u(\zeta_n)}{z-\zeta_n} = \frac{\alpha\overline{\zeta_n}}{\sigma_\alpha(\{\zeta_n\})} \neq 0. \tag{2.13}$$

In particular,

$$\sigma_\alpha(\{\zeta_n\}) = \frac{1}{|u'(\zeta_n)|}.$$

For each of these points ζ_n, we consider the associated (normalized) boundary kernel, as we did in (1.15):

$$\widetilde{k_n}(z) := \frac{k^u_{\zeta_n}(z)}{\|k^u_{\zeta_n}\|_{H^2}} = \frac{1}{\|k^u_{\zeta_n}\|_{H^2}} \frac{1-\overline{u(\zeta_n)}u(z)}{1-\overline{\zeta_n}z}, \qquad z\in\mathbb{D}.$$

With all of this in mind, we provide an upper bound for $\|Q^u_\zeta\|$, when u is one-component.

Theorem 2.10. *Let u be a one-component inner function, and let $\zeta\in\mathbb{T}\setminus\sigma(u)$. Then, it holds*

$$\|Q^u_\zeta\| \leq C_u|u'(\zeta)|, \tag{2.14}$$

where C_u is a positive quantity not depending on ζ.

Before the proof, we need a lemma. Since ζ does not belong to the spectrum of u, we consider $\alpha := u(\zeta)\in\mathbb{T}$ and the associated Clark measure σ_α, that is described in (2.12). Denoting $(\zeta_n)_{n\in\mathbb{N}}$ the atoms of the measure σ_α, thanks to Lemma 1.44 we know that the collection of normalized boundary kernels $\{\widetilde{k_n}\}_n$ defined as in (1.15) forms a complete orthonormal basis for the model space K_u. We note that our original point $\zeta\in\mathbb{T}\setminus\sigma(u)$ is an atom for σ_α, since by definition $\alpha = u(\zeta)$, and u is analytic in a neighbourhood of ζ. Let $\ell\in\mathbb{N}$ be the index such that $\zeta_\ell = \zeta$. This index ℓ will play a special role. We write $\widetilde{k_\ell}$ to denote the normalized kernel $k^u_\zeta/\|k^u_\zeta\|$. In the following lemma we compute the entries of the infinite matrix associated to Q^u_ζ, with respect to the basis $\{\widetilde{k_n}\}_n$.

Lemma 2.11. *Let u be a one-component inner function and $\zeta \in \mathbb{T} \setminus \sigma(u)$, as above. Let $(a_{ij})_{i,j}$ be the infinite matrix representing the operator Q_ζ^u with respect to the orthonormal basis $\{\widetilde{k}_n\}_n$, i.e.*

$$a_{ij} := \langle Q_\zeta^u \widetilde{k}_j, \widetilde{k}_i \rangle_{H^2}, \qquad i, j \geq 1.$$

Then, the matrix $(a_{ij})_{i,j}$ has the form

$$\begin{pmatrix} \frac{1}{\zeta_1 - \zeta} & 0 & \cdots & 0 & \frac{\|k_\zeta^u\|}{\|k_{\zeta_1}^u\|} \frac{1}{\zeta - \zeta_1} & 0 & \cdots \\ 0 & \frac{1}{\zeta_2 - \zeta} & \cdots & 0 & \frac{\|k_\zeta^u\|}{\|k_{\zeta_2}^u\|} \frac{1}{\zeta - \zeta_2} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\zeta_{\ell-1} - \zeta} & 0 & 0 & \cdots \\ \frac{(k_{\zeta_1}^u)'(\zeta)}{\|k_{\zeta_1}^u\| \|k_\zeta^u\|} & \frac{(k_{\zeta_2}^u)'(\zeta)}{\|k_{\zeta_2}^u\| \|k_\zeta^u\|} & \cdots & \frac{(k_{\zeta_{\ell-1}}^u)'(\zeta)}{\|k_{\zeta_{\ell-1}}^u\| \|k_\zeta^u\|} & \frac{(k_\zeta^u)'(\zeta)}{\|k_\zeta^u\|^2} & \frac{(k_{\zeta_{\ell+1}}^u)'(\zeta)}{\|k_{\zeta_{\ell+1}}^u\| \|k_\zeta^u\|} & \cdots \\ 0 & 0 & \cdots & 0 & \frac{\|k_\zeta^u\|}{\|k_{\zeta_{\ell+1}}^u\|} \frac{1}{\zeta - \zeta_{\ell+1}} & \frac{1}{\zeta_{\ell+1} - \zeta} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, all the non-zero coefficients are either on the diagonal, on the ℓ-th row or on the ℓ-column, where ℓ is the index such that $\zeta_\ell = \zeta$.

We point out a necessary abuse of notation: in the matrix above, $\|\cdot\|$ represents the H^2-norm.

Proof of Lemma 2.11. The proof is technical and it splits in different cases. Keep in mind that $\zeta_\ell = \zeta$. We start away from the ℓ-th row, taking $i \neq \ell$. For all $j's$, by the reproducing kernel property and writing $Q_\zeta^u \widetilde{k}_j$ as the difference quotient, it holds

$$\begin{aligned} a_{ij} &= \langle Q_\zeta^u \widetilde{k}_j, \widetilde{k}_i \rangle_{H^2} \\ &= \frac{\big(Q_\zeta^u \widetilde{k}_j\big)(\zeta_i)}{\|k_{\zeta_i}^u\|} \\ &= \frac{1}{\|k_{\zeta_i}^u\|} \frac{\widetilde{k}_j(\zeta_i) - \widetilde{k}_j(\zeta)}{\zeta_i - \zeta} \\ &= \frac{1}{\|k_{\zeta_i}^u\|} \frac{1}{\|k_{\zeta_j}^u\|} \frac{\langle k_{\zeta_j}^u, k_{\zeta_i}^u \rangle_{H^2} - \langle k_{\zeta_j}^u, k_{\zeta_\ell}^u \rangle_{H^2}}{\zeta_i - \zeta}. \end{aligned}$$

We distinguish three cases and we use the orthogonality relations. If $j = i$, then $\langle k_{\zeta_j}^u, k_{\zeta_i}^u \rangle_{H^2} = \|k_{\zeta_i}^u\|^2$ and $\langle k_{\zeta_j}^u, k_{\zeta_\ell}^u \rangle_{H^2} = 0$, so that

$$i \neq \ell, \quad j = i \implies a_{ii} = \frac{1}{\zeta_i - \zeta}.$$

If $j = \ell$, then $\langle k_{\zeta_j}^u, k_{\zeta_i}^u \rangle_{H^2} = 0$ and $\langle k_{\zeta_j}^u, k_{\zeta_\ell}^u \rangle_{H^2} = \|k_{\zeta_\ell}^u\|^2$, so that

$$i \neq \ell, \quad j = \ell \implies a_{i\ell} = -\frac{\|k_\zeta^u\|}{\|k_{\zeta_i}^u\|} \frac{1}{\zeta_i - \zeta}.$$

If $j \neq \ell, j \neq i$, then all the inner products involved equal 0, and $a_{ij} = 0$. Now, we move to the ℓ-th row: we take $i = \ell$. This is more delicate, because we are evaluating $Q^u_\zeta \widetilde{k}_j$, the difference quotient at ζ, precisely in the point ζ. Since $\zeta \in \mathbb{T} \setminus \sigma(u)$, by Theorem 1.32 every function in the model space K_u admits an analytic extension in a neighborhood of ζ. In particular, for all $j's$, we can write the Taylor series

$$k^u_{\zeta_j}(z) = \sum_{m=0}^{\infty} \frac{\left(k^u_{\zeta_j}\right)^{(m)}(\zeta)}{m!}(z-\zeta)^m$$

where $\left(k^u_{\zeta_j}\right)^{(m)}$ is the m^{th} derivative of $k^u_{\zeta_j}$. We compute

$$\begin{aligned}\left(Q^u_\zeta\, k^u_{\zeta_j}\right)(z) &= \frac{k^u_{\zeta_j}(z) - k^u_{\zeta_j}(\zeta)}{z-\zeta} \\ &= \frac{1}{z-\zeta}\left[\sum_{m=0}^{\infty} \frac{\left(k^u_{\zeta_j}\right)^{(m)}(\zeta)}{m!}(z-\zeta)^m - k^u_{\zeta_j}(\zeta)\right] \\ &= \sum_{m=1}^{\infty} \frac{\left(k^u_{\zeta_j}\right)^{(m)}(\zeta)}{m!}(z-\zeta)^{m-1}.\end{aligned}$$

One has $\left(Q^u_\zeta\, k^u_{\zeta_j}\right)(\zeta) = \left(k^u_{\zeta_j}\right)'(\zeta)$, so that

$$a_{\ell j} = \frac{\left(Q^u_\zeta \widetilde{k}_j\right)(\zeta_\ell)}{\|k^u_{\zeta_\ell}\|} = \frac{\left(Q^u_\zeta\, k^u_{\zeta_j}\right)(\zeta)}{\|k^u_{\zeta_j}\|\|k^u_{\zeta_\ell}\|} = \frac{\left(k^u_{\zeta_j}\right)'(\zeta)}{\|k^u_{\zeta_j}\|\|k^u_{\zeta}\|}.$$

In particular, for $j = \ell$,

$$a_{\ell\ell} = \frac{\left(k^u_\zeta\right)'(\zeta)}{\|k^u_\zeta\|^2},$$

and the proof is complete. □

Using the coefficients $a_{ij} = \langle Q^u_\zeta \widetilde{k}_j, \widetilde{k}_i \rangle_{H^2}$, we prove Theorem 2.10.

Proof of Theorem 2.10. We can decompose any $f \in K_u$ using the complete orthonormal basis of normalized boundary kernels $\{\widetilde{k}_n\}$ as

$$f = \sum_i \gamma_i \widetilde{k}_i.$$

The coefficients $\gamma_i := \langle f, \widetilde{k}_i \rangle_{H^2}$ satisfy the Parseval identity

$$\sum_i |\gamma_i|^2 = \|f\|^2_{H^2}.$$

By linearity, it holds

$$\|Q^u_\zeta(f)\|_{H^2} \leq |\gamma_\ell| \|Q^u_\zeta(\widetilde{k}_\ell)\|_{H^2} + \left\| Q^u_\zeta\Big(\sum_{i\neq\ell} \gamma_i \widetilde{k}_i\Big)\right\|_{H^2}. \tag{2.15}$$

Now, we work separately on each summand of (2.15). We start with the first one. Using the orthonormal basis $\{\widetilde{k}_n\}$ and coefficients a_{ij}, the Parseval identity yields

$$\|Q^u_\zeta(\widetilde{k}_\ell)\|^2_{H^2} = \sum_i \left|\langle Q^u_\zeta(\widetilde{k}_\ell), \widetilde{k}_i\rangle_{H^2}\right|^2 = \sum_i |a_{i\ell}|^2.$$

Now, by Lemma 2.11, separating the index ℓ it holds

$$\|Q^u_\zeta(\widetilde{k}_\ell)\|^2_{H^2} = \sum_i |a_{i\ell}|^2 = \frac{|(k^u_\zeta)'(\zeta)|^2}{\|k^u_\zeta\|^4_{H^2}} + \sum_{i\neq\ell} \frac{\|k^u_\zeta\|^2_{H^2}}{\|k^u_{\zeta_i}\|^2_{H^2}} \frac{1}{|\zeta_i - \zeta|^2}. \tag{2.16}$$

By direct computation, it follows that

$$(k^u_\zeta)'(z) = \frac{\overline{u(\zeta)}}{\bar{\zeta}(\zeta - z)}\left[\frac{u(z)-u(\zeta)}{z-\zeta} - u'(z)\right], \qquad z \in \mathbb{D}.$$

Expanding $u(z)$ and $u'(z)$ in Taylor series in a neighborhood of ζ, one can check that

$$|(k^u_\zeta)'(\zeta)| = \frac{|u''(\zeta)|}{2}.$$

Using property (ii) of Theorem 2.8, it holds that

$$\frac{|(k^u_\zeta)'(\zeta)|^2}{\|k^u_\zeta\|^4_{H^2}} = \frac{|u''(\zeta)|^2}{4|u'(\zeta)|^2} \leq C_u|u'(\zeta)|^2, \tag{2.17}$$

where C_u is a positive constant that depends only on u. We can rewrite the second summand in (2.16) as

$$\begin{aligned}\sum_{i\neq\ell} \frac{\|k^u_\zeta\|^2}{\|k^u_{\zeta_i}\|^2} \frac{1}{|\zeta_i - \zeta|^2} &= |u'(\zeta)| \sum_{i\neq\ell} \frac{1}{|\zeta_i-\zeta|^2}\sigma_\alpha(\{\zeta_i\}) \\ &= |u'(\zeta)| \int_{\mathbb{T}\setminus\{\zeta\}} \frac{1}{|\lambda-\zeta|^2} d\sigma_\alpha(\lambda).\end{aligned}$$

Now, by [12, Equation (16)],

$$\int_{\mathbb{T}\setminus\{\zeta\}} \frac{1}{|\lambda-\zeta|^2} d\sigma_\alpha(\lambda) \leq \frac{C_u}{\sigma_\alpha(\{\zeta\})},$$

and we conclude that

$$\|Q^u_\zeta(\widetilde{k}_\ell)\|^2_{H^2} \leq C_u|u'(\zeta)|^2. \tag{2.18}$$

Note that we use the same symbol C_u for a positive constant that depends only on u, although the various constants involved may be different from term to term. We have obtained the desired estimate for the first of the two summands of (2.15), since

$$|\gamma_\ell|\|Q^u_\zeta(\widetilde{k}_\ell)\|_{H^2} \leq \left(\sum_i |\gamma_i|^2\right)^{\frac{1}{2}} \|Q^u_\zeta(\widetilde{k}_\ell)\|_{H^2} \leq C_u|u'(\zeta)|\|f\|_{H^2}.$$

To conclude the proof, we exhibit a similar estimate for the second summand in (2.15). Notice that, now, the indexes i involved are different from ℓ. For $i \neq \ell$, by orthogonal decomposition it holds

$$Q^u_\zeta(\widetilde{k_i}) = \sum_j \langle Q^u_\zeta(\widetilde{k_i}), \widetilde{k_j} \rangle_{H^2} \, \widetilde{k_j} = \sum_j a_{ji} \widetilde{k_j} = a_{ii} \widetilde{k_i} + a_{\ell i} \widetilde{k_\ell},$$

for all the other coefficients a_{ji} equal 0, by Lemma 2.11. Therefore, by linearity and again Lemma 2.11,

$$\begin{aligned}
\left\| Q^u_\zeta\Big(\sum_{i\neq\ell} \gamma_i \widetilde{k_i}\Big) \right\|^2_{H^2} &= \left\| \sum_{i\neq\ell} \gamma_i Q^u_\zeta(\widetilde{k_i}) \right\|^2_{H^2} \\
&= \left\| \sum_{i\neq\ell} \gamma_i \left(a_{ii}\widetilde{k_i} + a_{\ell i}\widetilde{k_\ell} \right) \right\|^2_{H^2} \\
&= \left\| \sum_{i\neq\ell} \gamma_i \left(\frac{1}{\zeta_i - \zeta}\, \widetilde{k_i} + \frac{\big(k^u_{\zeta_i}\big)'(\zeta)}{\|k^u_{\zeta_i}\|_{H^2} \|k^u_\zeta\|_{H^2}} \widetilde{k_\ell} \right) \right\|^2_{H^2} \\
&= \left\| \sum_{i\neq\ell} \frac{\gamma_i}{\zeta_i - \zeta}\, \widetilde{k_i} + \frac{1}{\|k^u_\zeta\|_{H^2}} \left(\sum_{i\neq\ell} \gamma_i \frac{\big(k^u_{\zeta_i}\big)'(\zeta)}{\|k^u_{\zeta_i}\|_{H^2}} \right) \widetilde{k_\ell} \right\|^2_{H^2}.
\end{aligned}$$

Since $\{\widetilde{k_n}\}$ is an orthonormal basis, the previous squared norm can be rewritten as the sum of squares

$$\left\| Q^u_\zeta\Big(\sum_{i\neq\ell} \gamma_i \widetilde{k_i}\Big) \right\|^2_{H^2} = \sum_{i\neq\ell} \frac{|\gamma_i|^2}{|\zeta_i - \zeta|^2} + \frac{1}{\|k^u_\zeta\|^2_{H^2}} \left| \sum_{i\neq\ell} \gamma_i \frac{\big(k^u_{\zeta_i}\big)'(\zeta)}{\|k^u_{\zeta_i}\|_{H^2}} \right|^2. \tag{2.19}$$

In this last equation, we have rewritten the second and last summand of the original (2.15) in a more suitable form. Once again, we work separately on the two summands that appear in (2.19). We start with the first one. Notice that

$$\sum_{i\neq\ell} \frac{|\gamma_i|^2}{|\zeta_i - \zeta|^2} \leq \sup_{i\neq\ell} \frac{1}{|\zeta_i - \zeta|^2} \sum_i |\gamma_i|^2 = \frac{\|f\|^2_{H^2}}{(\inf_{i\neq\ell} |\zeta_i - \zeta|)^2}.$$

By assumption, u is a one-component inner function, and this gives us information about the atoms. In particular, the atom ζ_i that is closest to ζ is one of the neighbors $\zeta_\pm$, so that

$$\inf_{i\neq\ell} |\zeta_i - \zeta| = \min\{|\zeta - \zeta_+|, |\zeta - \zeta_-|\}.$$

By Theorem 2.9, we conclude that

$$\sum_{i\neq\ell} \frac{|\gamma_i|^2}{|\zeta_i - \zeta|^2} \leq \frac{C_u}{\sigma_\alpha(\{\zeta\})^2} \|f\|^2_{H^2} = C_u |u'(\zeta)|^2 \|f\|^2_{H^2}.$$

Now, we move on to estimate the second (and last) summand of (2.19). Simply by adding and subtracting the same term and using Young's inequality, it holds that

$$\left|\sum_{i\neq\ell}\gamma_i\frac{\left(k^u_{\zeta_i}\right)'(\zeta)}{\|k^u_{\zeta_i}\|_{H^2}}\right|^2 \leq 2\left|\sum_{i}\gamma_i\frac{\left(k^u_{\zeta_i}\right)'(\zeta)}{\|k^u_{\zeta_i}\|_{H^2}}\right|^2 + 2|\gamma_\ell|^2\frac{\left|\left(k^u_{\zeta_\ell}\right)'(\zeta)\right|^2}{\|k^u_{\zeta_\ell}\|^2_{H^2}}.$$

By Lemma 2.4, since the operator $\Lambda_\zeta\colon f\mapsto f'(\zeta)$ is linear and bounded on K_u, we have

$$\begin{aligned}\sum_{i}\gamma_i\frac{\left(k^u_{\zeta_i}\right)'(\zeta)}{\|k^u_{\zeta_i}\|_{H^2}} &= \sum_i\gamma_i\frac{\Lambda_\zeta(k^u_{\zeta_i})}{\|k^u_{\zeta_i}\|_{H^2}}\\ &= \Lambda_\zeta\left(\sum_i\gamma_i\widetilde{k}_i\right)\\ &= \Lambda_\zeta(f)\\ &= f'(\zeta).\end{aligned}$$

Notice that, by definition of the index ℓ, $\widetilde{k}_\ell$ is the normalization of k^u_ζ. By (2.8) and (2.18), we have that

$$\begin{aligned}|f'(\zeta)|^2 &\leq \|f\|^2_{H^2}\|Q^u_\zeta(k^u_\zeta)\|^2_{H^2}\\ &= \|f\|^2_{H^2}\|k^u_\zeta\|^2_{H^2}\|Q^u_\zeta(\widetilde{k}_\ell)\|^2_{H^2}\\ &\leq C_u\|f\|^2_{H^2}|u'(\zeta)|^3.\end{aligned}$$

This shows that

$$\left|\sum_{i\neq\ell}\gamma_i\frac{\left(k^u_{\zeta_i}\right)'(\zeta)}{\|k^u_{\zeta_i}\|_{H^2}}\right|^2 \leq 2C_u\|f\|^2_{H^2}\,|u'(\zeta)|^3 + 2|\gamma_\ell|^2\frac{\left|\left(k^u_{\zeta}\right)'(\zeta)\right|^2}{\|k^u_{\zeta}\|^2_{H^2}}.$$

For the last term in the previous sum, as in (2.17) it holds

$$|\gamma_\ell|^2\frac{\left|\left(k^u_{\zeta}\right)'(\zeta)\right|^2}{\|k^u_{\zeta}\|^2_{H^2}} \leq C_u|\gamma_\ell|^2|u'(\zeta)|^3 \leq C_u\|f\|^2_{H^2}|u'(\zeta)|^3.$$

Therefore, we can conclude that for (2.19) it holds

$$\begin{aligned}\left\|Q^u_\zeta\Big(\sum_{i\neq\ell}\gamma_i\widetilde{k}_i\Big)\right\|^2_{H^2} &= \sum_{i\neq\ell}\frac{|\gamma_i|^2}{|\zeta_i-\zeta|^2} + \frac{1}{\|k^u_\zeta\|^2_{H^2}}\left|\sum_{i\neq\ell}\gamma_i\frac{\left(k^u_{\zeta_i}\right)'(\zeta)}{\|k^u_{\zeta_i}\|_{H^2}}\right|^2\\ &\leq C_u|u'(\zeta)|^2\|f\|^2_{H^2} + \frac{2}{|u'(\zeta)|}\left(C_u\|f\|^2_{H^2}|u'(\zeta)|^3\right)\\ &\leq C_u|u'(\zeta)|^2\|f\|^2_{H^2},\end{aligned}$$

concluding the proof. □

Theorem 2.10 has a strong consequence: it provides a characterization of the property of being one-component.

Theorem 2.12. *Let u be an inner function. Then, u is a one-component inner function if and only if $m(\sigma(u)) = 0$ and for every $\zeta \in \mathbb{T} \setminus \sigma(u)$ the inequality* (2.14) *holds. More explicitly, u is one-component if and only if $m(\sigma(u)) = 0$ and there exists a positive constant C_u depending only on u such that, for every $\zeta \in \mathbb{T} \setminus \sigma(u)$ and every $f \in K_u$, it holds that*

$$\int_{\mathbb{T}} \left| \frac{f(\lambda) - f(\zeta)}{\lambda - \zeta} \right|^2 dm(\lambda) \leq C_u |u'(\zeta)|^2 \|f\|_{H^2}^2.$$

Proof. If u is one-component, then by Theorem 2.8 the Lebesgue measure of $\sigma(u)$ is 0, and by Theorem 2.10 it holds (2.14). Viceversa, confronting (2.14) with (2.6) we obtain

$$|u'(\zeta)| \geq \frac{C_u}{\operatorname{dist}(\zeta, \sigma(u) \cap \mathbb{T})}, \qquad \zeta \in \mathbb{T} \setminus \sigma(u),$$

showing that $|u'|$ is unbounded on every open arc $\Delta \subseteq \mathbb{T} \setminus \sigma(u)$ such that $\overline{\Delta}^{\mathrm{cl}} \cap \sigma(u) \neq \emptyset$. Also, confronting (2.14) with (2.9), we obtain

$$|u''(\zeta)| \leq C_u |u'(\zeta)|^2, \qquad \zeta \in \mathbb{T} \setminus \sigma(u),$$

for some positive constant C_u depending only on u. By the characterization in Theorem 2.8, we conclude that u is one-component. □

Now, we show an example for which the lower estimate in Theorem 2.5 and the upper estimate in Theorem 2.10 that we have produced have the same order of infinity. Note that this example also shows that the exponents of the derivatives of u involved in Theorems 2.5 and 2.10 are sharp.

Proposition 2.13. *Let u be the singular inner function associated to the measure δ_1,*

$$u(z) = \exp\left(\frac{z+1}{z-1}\right), \qquad z \in \mathbb{D}.$$

Then, there exists a positive constant C_u such that for every $\zeta \in \mathbb{T} \setminus \{1\}$ the couple of inequalities

$$\frac{1}{2}|u'(\zeta)| = \frac{1}{|\zeta - 1|^2} \leq \|Q_\zeta^u\| \leq \frac{2C_u}{|\zeta - 1|^2} = C_u |u'(\zeta)|$$

hold.

Proof. The spectrum of the inner function u is the singleton $\{1\}$. A simple computation shows that

$$|u'(\zeta)| = \frac{2}{|\zeta - 1|^2}, \qquad \zeta \neq 1. \tag{2.20}$$

Also, by directly computing the second derivative of u, we see that

$$u''(z) = \frac{4z\, u(z)}{(z-1)^4}, \qquad z \in \mathbb{D}. \tag{2.21}$$

In light of Aleksandrov's Theorem 2.8, we have that u is a one-component inner function. We note that this fact could also be easily proved directly, computing the sublevel sets Ω_ε. According to Theorem 2.10, for every $\zeta \neq 1$,

$$\|Q_\zeta^u\| \leq C_u|u'(\zeta)| = \frac{2C_u}{|\zeta - 1|^2}.$$

On the other hand, by Theorem 2.5 and equations (2.20), (2.21)

$$\|Q_\zeta^u\| \geq \frac{1}{2}\frac{4}{|\zeta - 1|^4}\frac{|\zeta - 1|^2}{2} = \frac{1}{|\zeta - 1|^2}, \qquad \zeta \neq 1.$$

We have obtained the desidered inequalities

$$\frac{1}{2}|u'(\zeta)| = \frac{1}{|\zeta - 1|^2} \leq \|Q_\zeta^u\| \leq \frac{2C_u}{|\zeta - 1|^2} = C_u|u'(\zeta)|,$$

and the proof is complete. □

As a concluding remark, we note that the operator Q_ζ^u appears in an explicit expression of the norm of $R(\zeta, S_u) = (\zeta I - S_u)^{-1}$, the resolvent operator of the compressed shift.

Proposition 2.14. *Let u be an inner function and let $\zeta \in \mathbb{T} \setminus \sigma(u)$. Then,*

$$\|R(\zeta, S_u)\|^2 = \|Q_\zeta^u\|^2 + |u'(\zeta)|. \tag{2.22}$$

Proof. First, since $X_u^* = S_u$, notice that

$$\|R(\zeta, S_u)\| = \|R(\overline{\zeta}, X_u)\| = \|(\overline{\zeta}I - X_u)^{-1}\| = \|(I - \zeta X_u)^{-1}\|.$$

From the definition (2.4) of the operator Q_ζ^u, it follows that

$$(I - \zeta X_u)^{-1} = I + \zeta Q_\zeta^u.$$

In fact,

$$\begin{aligned} I + \zeta Q_\zeta^u &= I + \zeta(I - \zeta X_u)^{-1}X_u \\ &= (I - \zeta X_u)(I - \zeta X_u)^{-1} + \zeta(I - \zeta X_u)^{-1}X_u \\ &= (I - \zeta X_u)^{-1}(I - \zeta X_u + \zeta X_u) \\ &= (I - \zeta X_u)^{-1}. \end{aligned}$$

Thus, we have that

$$(I - \zeta X_u)^{-1}f(z) = f(z) + \zeta\frac{f(z) - f(\zeta)}{z - \zeta} = \frac{zf(z) - \zeta f(\zeta)}{z - \zeta}.$$

Computing the norm, we obtain that

$$\|(I - \zeta X_u)^{-1}f\|_{H^2}^2 = \int_{\mathbb{T}}\left|\frac{\lambda f(\lambda) - \zeta f(\zeta)}{\lambda - \zeta}\right|^2 dm(\lambda) = \mathcal{D}_\zeta(Sf).$$

We appeal to the local Dirichlet integral for convenience, since by (1.22) it holds

$$\mathcal{D}_\zeta(Sf) = \mathcal{D}_\zeta(zf) = \mathcal{D}_\zeta(f) + |f(\zeta)|^2 = \|Q_\zeta^u f\|^2 + |f(\zeta)|^2.$$

Finally, by Theorem 1.31, we have that

$$\|R(\zeta, S_u)\|^2 = \|Q_\zeta^u\|^2 + \|k_\zeta^u\|_{H^2}^2 = \|Q_\zeta^u\|^2 + |u'(\zeta)|,$$

concluding the proof. □

Chapter 3

$H(b)$ spaces and $\mathcal{D}_\mu$ spaces

In this chapter, we discuss the embeddings and, in a special case, the equality, between de Branges–Rovnyak spaces and harmonically weighted Dirichlet spaces. For the sake of clarity, we say that $H(b)$ *embeds* into $\mathcal{D}_\mu$, and we write $H(b) \hookrightarrow \mathcal{D}_\mu$, if $H(b) \subset \mathcal{D}_\mu$ and

$$\|f\|_{\mathcal{D}_\mu} \leq C\|f\|_{H(b)}, \qquad f \in H(b),$$

where C is a positive constant. Notice that the simple set inequality would suffice, for the boundedness follows automatically from the closed graph theorem. In Section 3.1 we show a sufficient condition and a necessary condition for the embedding $H(b) \hookrightarrow \mathcal{D}_\zeta$, with no particular assumptions on the function b. In Sections 3.2 and 3.3, respectively, we show a complete characterization for the embedding $H(b) \hookrightarrow \mathcal{D}_\mu$ when b is inner and when b is non-extreme. In the latter case, we also discuss the equality of spaces $H(b) = \mathcal{D}_\mu$. The content of Section 3.1 and Section 3.2 is taken from a recent paper, that is joint work of the author of this thesis and Carlo Bellavita, [10].

Before entering into the technical details, we give a brief overview on the state of the art. The first work in this direction dates back to 1997. In [55], Sarason showed that the local Dirichlet space $\mathcal{D}_\zeta$ coincides with equality of norms with the de Branges–Rovnyak space $H(b_\zeta)$ associated to the rational function

$$b_\zeta(z) := \frac{(1 - s_0)\overline{\zeta}z}{1 - s_0\overline{\zeta}z}, \qquad z \in \mathbb{D},$$

where $s_0 = (3 - \sqrt{5})/2$ is the smaller of the two solutions of the equation $s = (1 - s)^2$. This means that $H(b_\zeta) = \mathcal{D}_\zeta$ as sets and

$$\|f\|_{H(b_\zeta)} = \|f\|_{\mathcal{D}_\zeta}, \qquad f \in H(b_\zeta). \tag{3.1}$$

The author used this identification to give a new proof that the local Dirichlet spaces are star-shaped, i.e. for every function f in $\mathcal{D}_\zeta$ the radial approximations $f_r(z) := f(rz)$, for $0 < r < 1$, converge to f in the norm of the space.

Later, in 2010 [15], Chevrot, Guillot and Ransford showed that the case studied by Sarason is basically the only case in which it holds $H(b) = \mathcal{D}_\mu$ with equality of norms as in (3.1). More precisely, they showed that $H(b) = \mathcal{D}_\mu$ with equality of norms if and only if $\mu = c\delta_\zeta$ with $c > 0$, i.e., μ is an atomic measure with a single atom, and the function b is given by

$$b(z) = \alpha \frac{\sqrt{\tau}\bar{\zeta} z}{1 - \tau\bar{\zeta} z},$$

where $\alpha \in \mathbb{C}$ and $\tau \in (0, 1]$ satisfy $|\alpha|^2 = c$ and $\tau + 1/\tau = 2 + c$.

In [22], the authors studied the equality of norms in $H(b) = \mathcal{D}_\omega$, with ω a super-harmonic weight. In 2013 [17], Costara and Ransford showed that there can be an equality of sets $H(b) = \mathcal{D}_\mu$ with just an equivalence of norms. In particular, they provided sufficient and necessary conditions for the equality $H(b) = \mathcal{D}_\mu$, where b is a rational function and μ is a finitely supported measure, i.e. a finite sum of Dirac deltas centered in boundary points.

There exist also results on the embedding between different de Branges–Rovnyak spaces. We will use the following result, due to Ball and Kriete [6]. This is Corollary 27.18 in [29].

Proposition 3.1. *Suppose b_1 is a non-extreme point of the closed unit ball of H^∞, and assume b_1 is continuous on the closed unit disk. Let b_2 be a function in H^∞ and θ_2 its inner factor. Then the following are equivalent:*

(i) It holds the inclusion of de Branges–Rovnyak spaces $H(b_2) \subset H(b_1)$.

(ii) The following conditions hold:

- $\{\lambda \in \mathbb{T}\colon |b_1(\lambda)| = 1\} \cap \sigma(\theta_2) = \emptyset$.
- *There exists $\gamma > 0$ such that $1 - |b_2|^2 \leq \gamma(1 - |b_1|^2)$ a.e. on $\mathbb{T}$.*

3.1 Embedding $H(b) \hookrightarrow \mathcal{D}_\zeta$

This section comes from a joint work [10] by E. Dellepiane and C. Bellavita. We provide a sufficient condition and a necessary one in order to have an embedding $H(b) \hookrightarrow \mathcal{D}_\zeta$ between de Branges–Rovnyak spaces and local Dirichlet spaces. We make no special assumption on the function b, we only ask that $\|b\|_{H^\infty} = 1$. The conditions that we will discuss involve the boundary spectrum

$$\sigma(b) = \{\lambda \in \mathbb{T}\colon \liminf_{z \to \lambda} |b(z)| < 1\},$$

introduced in the preliminaries section. We recall that every analytic function b with $\|b\|_{H^\infty} = 1$ can be factorized as $b = Fu$, where F is the outer function

$$F(z) := \exp\left\{ \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|b(\zeta)|\, dm(\zeta) \right\} \tag{3.2}$$

that satisfies $\|F\|_{H^\infty} = 1$ and u is an inner function. In particular, according to the Nevanlinna factorization, we can write

$$u(z) = \left(\prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z} \right) \exp\left\{ -\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\tau(\zeta) \right\}, \tag{3.3}$$

where $\{a_n\}_{n\geq 1}$ are the zeros of u and τ a positive singular measure.

The closure of the spectrum $\overline{\sigma(b)}^{\mathrm{cl}}$ is the smallest closed subset of $\mathbb{T}$ containing the closure of the zero set $\{a_n\}_n$ and the supports of the (positive, finite) measures τ and $-\log|b|\,dm$. Also, the function b and every element of the space $H(b)$ admit an analytic extension through any open arc contained in $\mathbb{T}\setminus\overline{\sigma(b)}^{\mathrm{cl}}$. In particular, $|b|=1$ on such arcs. If $b=u$ is an inner function, then it holds that

$$\sigma(u)=\{\lambda\in\mathbb{T}\colon \liminf_{z\to\lambda}|u(z)|=0\}.$$

In particular, the spectrum of inner functions is a closed set. We have the following sufficient condition for the embedding $H(b)\hookrightarrow\mathcal{D}_\zeta$.

Theorem 3.2. *Let b be a bounded analytic function with $\|b\|_{H^\infty}=1$, and let $\zeta\in\mathbb{T}$ be such that $\zeta\notin\overline{\sigma(b)}^{\mathrm{cl}}$. Then, we have the embedding $H(b)\hookrightarrow\mathcal{D}_\zeta$.*

Proof. The proof is short, it follows from the boundedness of the operator Q^b_ζ discussed in Section 2.1. By the local Douglas formula (1.19), and (1.10), for every $f\in H(b)$ we have

$$\mathcal{D}_\zeta(f)=\int_{\mathbb{T}}\left|\frac{f(\lambda)-f(\zeta)}{\lambda-\zeta}\right|^2 dm(\lambda)=\|Q^b_\zeta f\|^2_{H^2}\leq\|Q^b_\zeta f\|^2_{H(b)}<\infty.$$

This proves the embedding. We also have the following estimate for the norm:

$$\|f\|^2_{\mathcal{D}_\zeta}=\|f\|^2_{H^2}+\mathcal{D}_\zeta(f)\leq\big(1+\|Q^b_\zeta\|^2\big)\|f\|^2_{H(b)}.\quad\square$$

Notice that the sufficient condition of Theorem 3.2 can only hold for extreme functions b. We have the following necessary condition, that is also a partial converse result. For this, we do not require that b is extreme.

Theorem 3.3. *Let b be a bounded analytic function with $\|b\|_{H^\infty}=1$, and let $\zeta\in\mathbb{T}$ be such that $\zeta\in\sigma(b)$. Then, the de Branges–Rovnyak space $H(b)$ does not embed into the local Dirichlet space $\mathcal{D}_\zeta$.*

Proof. By contradiction, let us suppose that the embedding $H(b)\hookrightarrow\mathcal{D}_\zeta$ holds. Let $C>0$ be such that

$$\mathcal{D}_\zeta(f)\leq C\|f\|^2_{H(b)},\qquad f\in H(b).\tag{3.4}$$

By assumption, $\zeta\in\sigma(b)$, hence there exists a sequence $(\omega_n)_n$ in $\mathbb{D}$ converging to ζ such that

$$\beta:=\lim_n|b(\omega_n)|<1.$$

Let us consider the family of kernels

$$k_n(z):=k^b_{\omega_n}(z)=\frac{1-\overline{b(\omega_n)}b(z)}{1-\overline{\omega_n}z}.$$

Since $H(b) \subseteq \mathcal{D}_\zeta$, by Theorem 1.47 every function of $H(b)$ admits boundary value at ζ. By Theorem 1.31, $b(\zeta)$ is well defined and unimodular. Therefore, one can compute

$$\begin{aligned} k_n(z) - k_n(\zeta) &= \frac{1 - \overline{b(\omega_n)}b(z)}{1 - \overline{\omega_n}z} - \frac{1 - \overline{b(\omega_n)}b(\zeta)}{1 - \overline{\omega_n}\zeta} \\ &= \frac{\overline{\omega_n}(z-\zeta) - \overline{b(\omega_n)}\big(b(z) - b(\zeta)\big) + \overline{\omega_n b(\omega_n)}\big(\zeta b(z) - z b(\zeta)\big)}{(1 - \overline{\omega_n}z)(1 - \overline{\omega_n}\zeta)} \\ &= \frac{\overline{\omega_n}(z-\zeta)(1 - \overline{b(\omega_n)}b(\zeta)) - \overline{b(\omega_n)}\big(b(z) - b(\zeta)\big)(1 - \overline{\omega_n}\zeta)}{(1 - \overline{\omega_n}z)(1 - \overline{\omega_n}\zeta)}. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{k_n(z) - k_n(\zeta)}{z - \zeta} &= \frac{\overline{\omega_n}}{1 - \overline{\omega_n}z}\frac{1 - \overline{b(\omega_n)}b(\zeta)}{1 - \overline{\omega_n}\zeta} - \frac{\overline{b(\omega_n)}}{1 - \overline{\omega_n}z}\frac{b(z) - b(\zeta)}{z - \zeta} \\ &= \overline{\omega_n}c_{\omega_n}(z)k_n(\zeta) - \overline{b(\omega_n)}c_{\omega_n}(z)b(\zeta)\overline{\zeta}k^b_\zeta(z), \end{aligned} \tag{3.5}$$

where

$$c_{\omega_n}(z) = \frac{1}{1 - \overline{\omega_n}z}$$

is the usual Cauchy-Szegö kernel, the reproducing kernel of the Hardy space H^2. The local Dirichlet integral can be computed as in (1.19), yielding

$$\begin{aligned} \mathcal{D}_\zeta(k_n) =& \left\| \frac{k_n - k_n(\zeta)}{\cdot - \zeta} \right\|^2_{H^2} \\ =& \langle \overline{\omega_n}\, k_n(\zeta)c_{\omega_n} - \overline{b(\omega_n)}b(\zeta)\overline{\zeta}c_{\omega_n}k^b_\zeta, \overline{\omega_n}\, k_n(\zeta)c_{\omega_n} - \overline{b(\omega_n)}b(\zeta)\overline{\zeta}c_{\omega_n}k^b_\zeta \rangle_{H^2} \\ =& |\omega_n|^2|k_n(\zeta)|^2\|c_{\omega_n}\|^2_{H^2} - 2\Re\Big(\overline{\omega_n}k_n(\zeta)b(\omega_n)\overline{b(\zeta)}\zeta\langle c_{\omega_n}, c_{\omega_n}k^b_\zeta\rangle_{H^2}\Big) \\ &+ |b(\omega_n)|^2\|c_{\omega_n}k^b_\zeta\|^2_{H^2}. \end{aligned}$$

We have written the local Dirichlet integral $\mathcal{D}_\zeta(k_n)$ as a sum of three terms. We leave the first one as it is, and work on the other two. We use the reproducing property of the Cauchy-Szegö kernel, the fact that $c_{\omega_n}k^b_\zeta$ is an H^2 function and we estimate the real part with the modulus, obtaining

$$\begin{aligned} \Re\Big(\overline{\omega_n}k_n(\zeta)b(\omega_n)\overline{b(\zeta)}\zeta\langle c_{\omega_n}, c_{\omega_n}k^b_\zeta\rangle_{H^2}\Big) &= \Re\Big(\overline{\omega_n}k_n(\zeta)b(\omega_n)\overline{b(\zeta)}\zeta\overline{c_{\omega_n}(\omega_n)k^b_\zeta(\omega_n)}\Big) \\ &= \|c_{\omega_n}\|^2_{H^2}\Re\big(\overline{\omega_n}b(\omega_n)\overline{b(\zeta)}\zeta k_n(\zeta)^2\big) \\ &\le \|c_{\omega_n}\|^2_{H^2}|k_n(\zeta)|^2\,|\omega_n b(\omega_n)|. \end{aligned}$$

For the third summand, using the triangular inequality, we have

$$\begin{aligned} \|c_{\omega_n}k^b_\zeta\|^2_{H^2} &= \int_{\mathbb{T}} \left| \frac{1}{1 - \overline{\omega_n}\lambda}\frac{1 - \overline{b(\zeta)}b(\lambda)}{1 - \overline{\zeta}\lambda} \right|^2 dm(\lambda) \\ &= \int_{\mathbb{T}} \left| \frac{1}{1 - \omega_n\overline{\lambda}}\frac{1}{1 - \overline{\omega_n}\lambda}\left(\frac{1 - \overline{b(\zeta)}b(\lambda)}{1 - \overline{\zeta}\lambda}\right)^2 \right| dm(\lambda) \\ &\ge \left| \int_{\mathbb{T}} \frac{1}{1 - \omega_n\overline{\lambda}}\frac{1}{1 - \overline{\omega_n}\lambda}\left(\frac{1 - \overline{b(\zeta)}b(\lambda)}{1 - \overline{\zeta}\lambda}\right)^2 dm(\lambda) \right|. \end{aligned}$$

The function $c_{\omega_n}\big(k_\zeta^b\big)^2$ belongs to H^1 and, in particular, by the Cauchy integral formula (1.3) it holds

$$\int_{\mathbb{T}} \frac{1}{1-\omega_n\overline{\lambda}}\frac{1}{1-\overline{\omega_n}\lambda}\left(\frac{1-\overline{b(\zeta)}b(\lambda)}{1-\overline{\zeta}\lambda}\right)^2 dm(\lambda) = \int_{\mathbb{T}} \frac{c_{\omega_n}(\lambda)\big(k_\zeta^b\big)^2(\lambda)}{1-\omega_n\overline{\lambda}}\, dm(\lambda)$$
$$= c_{\omega_n}(\omega_n)\big(k_\zeta^b\big)^2(\omega_n).$$

Using this, we obtain

$$|b(\omega_n)|^2\|c_{\omega_n}k_\zeta^b\|^2_{H^2} \geq |b(\omega_n)|^2\|c_{\omega_n}\|^2_{H^2}|k_n(\zeta)|^2.$$

Now, computing the norms of the kernels

$$\|c_{\omega_n}\|^2_{H^2} = \frac{1}{1-|\omega_n|^2}, \qquad \|k_n\|^2_{H(b)} = \frac{1-|b(\omega_n)|^2}{1-|\omega_n|^2},$$

we obtain the lower bound

$$\begin{aligned}\frac{\mathcal{D}_\zeta(k_n)}{\|k_n\|^2_{H(b)}} &\geq \frac{|\omega_n|^2|k_n(\zeta)|^2}{1-|b(\omega_n)|^2} - \frac{2|k_n(\zeta)|^2|\omega_n b(\omega_n)|}{1-|b(\omega_n)|^2} + \frac{|b(\omega_n)|^2|k_n(\zeta)|^2}{1-|b(\omega_n)|^2}\\ &= |k_n(\zeta)|^2\frac{|\omega_n|^2 - 2|\omega_n b(\omega_n)| + |b(\omega_n)|^2}{1-|b(\omega_n)|^2}\\ &= \left|\frac{1-\overline{b(\omega_n)}b(\zeta)}{1-\overline{\omega_n}\zeta}\right|^2\frac{\big(|\omega_n|-|b(\omega_n)|\big)^2}{1-|b(\omega_n)|^2}\\ &\geq \frac{\big(1-|b(\omega_n)|\big)^2}{|1-\overline{\omega_n}\zeta|^2}\frac{\big(|\omega_n|-|b(\omega_n)|\big)^2}{1-|b(\omega_n)|^2}.\end{aligned}$$

Since $\lim_n \omega_n = \zeta$ and $\lim_n |b(\omega_n)| = \beta \in [0,1)$, we conclude that

$$\liminf_n \frac{\mathcal{D}_\zeta(k_n)}{\|k_n\|^2_{H(b)}} \geq \liminf_n \frac{(1-\beta)^2}{|1-\overline{\omega_n}\zeta|^2}\frac{(1-\beta)^2}{1-\beta^2} = +\infty,$$

contradicting the uniform bound in (3.4). □

The following result is contained in the proof of Theorem 3.3.

Corollary 3.4. *Let b be an analytic function on $\mathbb{D}$ with $\|b\|_{H^\infty} \leq 1$ and let $\zeta \in \mathbb{T}$. If b admits ADC at a point $\zeta \in \mathbb{T}$, then for all $\omega \in \mathbb{D}$ the reproducing kernel k_ω^b belongs to $\mathcal{D}_\zeta$.*

Proof. From (3.5), it follows that

$$\mathcal{D}_\zeta(k_\omega^b) = \left\|\frac{k_\omega^b - k_\omega^b(\zeta)}{\cdot - \zeta}\right\|^2_{H^2} = \|\overline{\omega}k_\omega^b(\zeta)c_\omega - \overline{b(\omega)}b(\zeta)\overline{\zeta}c_\omega k_\zeta^b\|^2_{H^2} < \infty. \quad \square$$

We have proved a positive result, that is, that $H(b) \hookrightarrow \mathcal{D}_\zeta$ when $\zeta \notin \overline{\sigma(b)}^{\mathrm{cl}}$, and a negative one, that is, that if $\zeta \in \sigma(b)$, $H(b) \not\subseteq \mathcal{D}_\zeta$. We now present some examples to show that, for the remaining case $\zeta \in \overline{\sigma(b)}^{\mathrm{cl}} \setminus \sigma(b)$, anything can happen.

Example. Set

$$s_0 := \frac{3-\sqrt{5}}{2}, \tag{3.6}$$

let $\zeta \in \mathbb{T}$ and define the function

$$b_\zeta(z) = \frac{(1-s_0)\overline{\zeta}z}{1-s_0\overline{\zeta}z}.$$

By Proposition 2 in [55], $H(b_\zeta) = \mathcal{D}_\zeta$ with equality of norms, guaranteeing the embedding. Since b_ζ is continuous up to the boundary $\mathbb{T}$, it holds

$$\sigma(b_\zeta) = \{\lambda \in \mathbb{T} \colon |b_\zeta(\lambda)| < 1\}.$$

Writing $\zeta = e^{i\eta}$ and $\lambda = e^{i\theta}$, if $e^{i\theta} \neq e^{i\eta}$, we have that

$$|1-s_0\overline{\zeta}\lambda|^2 = 1 - 2s_0\cos(\theta-\eta) + s_0^2 > 1 - 2s_0 + s_0^2 = |(1-s_0)\overline{\zeta}\lambda|^2.$$

On the other hand, when $e^{i\theta} = e^{i\eta}$,

$$|1-s_0\overline{\zeta}\lambda|^2 = |(1-s_0)\overline{\zeta}\lambda|^2.$$

This means that $\sigma(b_\zeta) = \mathbb{T} \setminus \{\zeta\}$, providing a function in H^∞ such that $\zeta \in \overline{\sigma(b_\zeta)}^{\mathrm{cl}} \setminus \sigma(b_\zeta)$ while the embedding $H(b_\zeta) \hookrightarrow \mathcal{D}_\zeta$ holds.

Now we provide an example of an extreme function b with $1 \in \overline{\sigma(b)}^{\mathrm{cl}} \setminus \sigma(b)$ such that $H(b) \hookrightarrow \mathcal{D}_1$ doesn't hold. We use Proposition 3.1 as a criterion for the inclusion.

Example. Let

$$b_1(z) := \frac{(1-s_0)z}{1-s_0 z}, \tag{3.7}$$

where s_0 is the constant in (3.6), so that $H(b_1) = \mathcal{D}_1$, and we construct an outer function b_2 as follows. We start by considering the function φ defined on $\mathbb{T}$ as

$$\varphi(\lambda) = \begin{cases} \log\left(\sqrt{1-|1-\lambda|^{\frac{3}{2}}}\right), & \text{if } \quad |\arg(\lambda)| \leq \frac{\pi}{6}, \\ 0, & \text{elsewhere.} \end{cases}$$

The function φ is in $L^\infty(\mathbb{T})$ and real-valued, and this allows us to define the outer function

$$b_2(z) := \exp\left\{ \int_{\mathbb{T}} \frac{\lambda+z}{\lambda-z}\, \varphi(\lambda)\, dm(\lambda) \right\},$$

that satisfies $|b_2| = e^\varphi$ a.e. on $\mathbb{T}$. The first condition of (ii) in Proposition 3.1 is trivially true, since b_2 is outer and therefore $\sigma(\theta_2) = \emptyset$. For the second condition of (ii), it holds that

$$1 - |b_2(\lambda)|^2 = |1-\lambda|^{\frac{3}{2}}, \qquad \text{for a.e. } \lambda \in \mathbb{T} \text{ with } |\arg(\lambda)| < \frac{\pi}{6}.$$

Since for all $\lambda \in \mathbb{T}$ it holds

$$1 - |b_1(\lambda)|^2 = \frac{(1-s_0)^2|1-\lambda|^2}{|1-s_0\lambda|^2},$$

it follows that in proximity of the point 1 the condition (ii) of Proposition 3.1 fails, meaning that the inclusion $H(b_2) \subset H(b_1) = \mathcal{D}_1$ cannot hold. Finally, from a classical argument with Poisson kernels found in the proof of [30, Theorem 1.9], it follows that for every $\lambda \in \mathbb{T}$ with $|\arg \lambda| < \frac{\pi}{6}$, it holds that

$$\lim_{z\to\lambda} |b_2(z)| = e^{\varphi(\lambda)} = \sqrt{1 - |1-\lambda|^{\frac{3}{2}}},$$

since φ is continuous on such λ's and bounded on $\mathbb{T}$. It follows that

$$\sigma(b) \cap \{\lambda \in \mathbb{T} \colon |\arg(\lambda)| < \pi/6\} = \{\lambda \in \mathbb{T} \colon |\arg(\lambda)| < \pi/6\} \setminus \{1\},$$

so that $1 \in \overline{\sigma(b_2)}^{\mathrm{cl}} \setminus \sigma(b_2)$ while $H(b_2) \not\subset \mathcal{D}_1$.

3.1.1 Embedding $H(b) \hookrightarrow \mathcal{D}_\mu$

In this section, we deal with the embedding $H(b) \hookrightarrow \mathcal{D}_\mu$. Again, we provide a sufficient condition and a necessary one for the embedding to hold, involving the support of the measure μ and the boundary spectrum $\sigma(b)$. We have the following theorem.

Theorem 3.5. *Let μ be a finite positive Borel measure on $\mathbb{T}$ and let $b \in H^\infty_1$. If $supp(\mu) \cap \overline{\sigma(b)}^{\mathrm{cl}} = \emptyset$, then the embedding $H(b) \hookrightarrow \mathcal{D}_\mu$ holds.*

Proof. By assumption, $supp(\mu)$ and $\overline{\sigma(b)}^{\mathrm{cl}}$ are disjoint compact sets, therefore

$$\delta := \operatorname{dist}\left(\operatorname{supp}(\mu), \overline{\sigma(b)}^{\mathrm{cl}}\right) > 0.$$

We consider the open set

$$U := \bigcup_{x \in \overline{\sigma(b)}^{\mathrm{cl}}} \left\{ z \in \mathbb{C} \colon |z - x| < \frac{\delta}{2} \right\}.$$

Notice that $U \supseteq \overline{\sigma(b)}^{\mathrm{cl}}$ and $\operatorname{dist}(U, \operatorname{supp}(\mu)) \geq \delta/2$. For $f \in H(b)$, we split the harmonically weighted Dirichlet integral of f into the sum

$$\mathcal{D}_\mu(f) = \frac{1}{\pi}\int_{\mathbb{D}\cap U} |f'|^2 P\mu \, dA + \frac{1}{\pi}\int_{\mathbb{D}\setminus U} |f'|^2 P\mu \, dA.$$

For the first summand, we use a classical Littlewood-Paley estimate, see Proposition 3.2 in [31]:

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{D}\cap U}|f'|^2 P\mu\,dA &= \frac{1}{\pi}\int_{\mathbb{D}\cap U}|f'(z)|^2(1-|z|^2)\left(\int_{\operatorname{supp}(\mu)}\frac{d\mu(\zeta)}{|z-\zeta|^2}\right)dA(z)\\ &\leq \frac{4}{\pi\delta^2}\mu(\mathbb{T})\int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)\,dA(z)\\ &\leq \frac{2}{\delta^2}\mu(\mathbb{T})\int_{\mathbb{T}}|f(\lambda)-f(0)|^2\,dm(\lambda)\\ &\leq \frac{8}{\delta^2}\mu(\mathbb{T})\|f\|_{H^2}^2.\end{aligned}$$

For the second summand, we recall that every function in the space $H(b)$ admits an analytic extension across $\mathbb{T}\setminus\overline{\sigma(b)}^{\mathrm{cl}}$. Hence, we have

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{D}\setminus U}|f'(z)|^2 P\mu(z)\,dA(z) &\leq \max_{\overline{\mathbb{D}}^{\mathrm{cl}}\setminus U}|f'|\,\frac{1}{\pi}\int_{\mathbb{D}}\int_{\mathbb{T}}\frac{1-|z|^2}{|z-\zeta|^2}\,d\mu(\zeta)\,dA(z)\\ &= \max_{\overline{\mathbb{D}}^{\mathrm{cl}}\setminus U}|f'|\int_{\mathbb{T}}d\mu(\zeta)\\ &= \max_{\overline{\mathbb{D}}^{\mathrm{cl}}\setminus U}|f'|\,\mu(\mathbb{T}).\end{aligned}$$

We have proved that for every $f\in H(b)$ it holds

$$\mathcal{D}_\mu(f)\leq\frac{8}{\delta^2}\mu(\mathbb{T})\|f\|_{H^2}^2+\max_{\overline{\mathbb{D}}^{\mathrm{cl}}\setminus U}|f'|\mu(\mathbb{T})<\infty.$$

The boundedness of the embedding $K_u\hookrightarrow\mathcal{D}_\mu$ follows from the closed graph theorem. □

Notice that Theorem 3.5 could also be proved using the analyticity of the mapping $\zeta\in\mathbb{T}\setminus\overline{\sigma(b)}^{\mathrm{cl}}\mapsto Q_\zeta^b\in\mathcal{B}\big(H(b)\big)$, where Q_ζ^b is the difference quotient operator. Now, we give a necessary condition for the embedding $H(b)\hookrightarrow\mathcal{D}_\mu$.

Theorem 3.6. *Let μ be a finite positive Borel measure on $\mathbb{T}$ and let $b\in H_1^\infty$. If the embedding $H(b)\hookrightarrow\mathcal{D}_\mu$ holds, then $\mu\big(\sigma(b)\big)=0$.*

Proof. For the proof, we recall the function $V_\mu\colon\mathbb{C}\to[0,+\infty]$ defined as

$$V_\mu(\omega):=\int_{\mathbb{T}}\frac{1}{|\zeta-\omega|^2}\,d\mu(\zeta),\qquad\omega\in\mathbb{C},$$

that was introduced in Section 1.1. We show that V_μ is finite on the boundary spectrum $\sigma(b)\cap\mathbb{T}$, which we can assume to be non-empty without loss of generality. Let $C>0$ be a constant such that

$$\mathcal{D}_\mu(f)\leq C\|f\|_{H(b)}^2,\qquad f\in H(b).$$

One could also take C to be the norm of such embedding. Let $\lambda \in \sigma(b)$ and, as we did in the proof of Theorem 3.3, let us consider a sequence $(\omega_n)_n$ in $\mathbb{D}$ such that $|b(\omega_n)| \to \beta \in [0,1)$ as $n \to \infty$. By the disintegration formula in (1.18) and the lower estimate for $\mathcal{D}_\zeta(k_n)$ obtained in the proof of Theorem 3.3, we have that

$$\begin{aligned} C\|k_n\|^2_{H(b)} \geq \mathcal{D}_\mu(k_n) &= \int_{\mathbb{T}} \mathcal{D}_\zeta(k_n)\, d\mu(\zeta) \\ &\geq \int_{\mathbb{T}} \|k_n\|^2_{H(b)} \frac{\big(1-|b(\omega_n)|\big)^2\big(|\omega_n|-|b(\omega_n)|\big)^2}{|\zeta-\omega_n|^2\big(1-|b(\omega_n)|^2\big)}\, d\mu(\zeta) \\ &= \|k_n\|^2_{H(b)} \frac{\big(1-|b(\omega_n)|\big)\big(|\omega_n|-|b(\omega_n)|\big)^2}{1+|b(\omega_n)|} \int_{\mathbb{T}} \frac{1}{|\zeta-\omega_n|^2}\, d\mu(\zeta). \end{aligned}$$

Hence, by Fatou's Lemma, it holds that

$$\begin{aligned} C &\geq \liminf_n \frac{\big(1-|b(\omega_n)|\big)\big(|\omega_n|-|b(\omega_n)|\big)^2}{1+|b(\omega_n)|} \int_{\mathbb{T}} \frac{1}{|\zeta-\omega_n|^2}\, d\mu(\zeta) \\ &\geq \frac{(1-\beta)^3}{1+\beta} \int_{\mathbb{T}} \frac{1}{|\zeta-\lambda|^2}\, d\mu(\zeta) \\ &= \frac{(1-\beta)^3}{1+\beta} V_\mu(\lambda). \end{aligned}$$

In particular,

$$V_\mu(\lambda) \leq C\frac{1+\beta}{(1-\beta)^3} < \infty, \qquad \lambda \in \sigma(b).$$

The theorem follows from the fact that $V_\mu = \infty$ μ-a.e. on $\mathbb{T}$ (see Proposition 1.5) and therefore, necessarily, we have that $\mu\big(\sigma(b)\big) = 0$. □

Remark. We note that, if there exists an $\varepsilon > 0$ such that for every $\lambda \in \sigma(b)$ we have $\beta < 1-\varepsilon$, then the potential V_μ is uniformly bounded on $\sigma(b)$. This happens, for example, when b is inner. In this case, we can always choose $\beta = 0$. In general, however, V_μ can be unbounded on $\sigma(b)$. As an example, take the usual function b_1 as in (3.7) that realizes the equality $H(b_1) = \mathcal{D}_1$. Clearly, the embedding holds, but the potential associated to the measure $\mu = \delta_1$,

$$V_\mu(z) = \frac{1}{|z-1|^2},$$

is not bounded on $\sigma(b) = \mathbb{T} \setminus \{1\}$.

We have another necessary condition for the embedding $H(b) \hookrightarrow \mathcal{D}_\mu$.

Proposition 3.7. *Let μ be a finite positive Borel measure on $\mathbb{T}$ and let $b \in H^\infty_1$. If the embedding $H(b) \hookrightarrow \mathcal{D}_\mu$ holds, then b belongs to $\mathcal{D}_\mu$.*

Notice that, when b is extreme, $b \notin H(b)$, so that in general the membership $b \in \mathcal{D}_\mu$ does not trivially follow from the embedding.

Proof of Proposition 3.7. Consider $\omega \in \mathbb{D}$ such that $b(\omega) \neq 0$. One has

$$b(z) = \frac{1}{\overline{b(\omega)}} \left[1 - (1 - \overline{\omega} z)\, k_\omega^b(z)\right], \qquad z \in \mathbb{D},$$

so that

$$b = \overline{b(\omega)}^{-1} \left(1 - (I - \overline{\omega} S)\, k_\omega^b\right) \in \mathcal{D}_\mu,$$

since $k_\omega^b \in \mathcal{D}_\mu$ and the forward shift S is well defined on $\mathcal{D}_\mu$, by Corollary 1.53. □

3.2 The case $b = u$ inner

In this part, we choose $b = u$ to be inner. In this case, Theorems 3.2 and 3.3 provide an equivalent characterization of the embedding $K_u \hookrightarrow \mathcal{D}_\mu$.

Corollary 3.8. *Let b be an analytic function with $\|b\|_{H^\infty} = 1$ with closed boundary spectrum, and let $\zeta \in \mathbb{T}$. Then, the embedding $H(b) \hookrightarrow \mathcal{D}_\zeta$ holds if and only if $\zeta \notin \sigma(b)$. In particular, if u is an inner function, then the embedding $K_u \hookrightarrow \mathcal{D}_\zeta$ holds if and only if $\zeta \notin \sigma(u)$.*

Proof. The result follows using Theorems 3.2 and 3.3 and the fact that the spectrum of b is closed. □

We can rewrite the embedding $K_u \hookrightarrow \mathcal{D}_\zeta$ in terms of the boundedness of the derivative operator, providing a corollary which is somehow related to the results of Baranov about the boundedness of the differentiation operator acting on model spaces, see [7].

Corollary 3.9. *Let u be an inner function and $\zeta \in \mathbb{T}$. Let D be the derivative operator*

$$D\colon K_u \to L^2\big(P\delta_\zeta dA\big), \quad f \mapsto f',$$

acting from the model space to the Lebesgue space $L^2(\mathbb{D}, P\delta_\zeta dA)$. Then, D is bounded if and only if $\zeta \notin \sigma(u)$.

Proof. It follows at once from Corollary 3.8, for

$$\|f'\|^2_{L^2\left(P\delta_\zeta dA\right)} = \int_{\mathbb{D}} |f'(z)|^2 \frac{1 - |z|^2}{|z - \zeta|^2}\, dA(z) = \pi \mathcal{D}_\zeta(f). \quad \square$$

The embedding $K_u \hookrightarrow \mathcal{D}_\zeta$ allows to find some Carleson measures for K_u. First, let us recall the definition.

Definition 3.10. Let H be a Hilbert space of holomorphic functions on $\mathbb{D}$. We say that a positive Borel measure ν on $\mathbb{D}$ is a *Carleson measure* for H if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f|^2\, d\nu \leq C\|f\|_H^2, \qquad f \in H. \tag{3.8}$$

Carleson measures for the Hardy space H^2 appeared in a very natural and powerful way in the proof of the Corona Theorem for H^∞, see [31]. Such measures have been well studied, and they admit a nice geometric characterization in terms of *Carleson boxes*.

Proposition 3.11. *Let ν be a finite positive Borel measure on $\mathbb{D}$. Given an arc $I \subseteq \mathbb{T}$, the Carleson box associated to I is*

$$S(I) := \{re^{i\theta} \colon e^{i\theta} \in I,\, 1 - |I| < r < 1\},$$

where $|I|$ denotes the arc length of I. Then, ν is Carleson for H^2 if and only if there exists a constant $C > 0$ such that

$$\nu(S(I)) \leq C|I|, \qquad I \subset \mathbb{T}. \tag{3.9}$$

Carleson measures of $\mathcal{D}_\zeta$ have been characterized in [14] in terms of Carleson measures of H^2, as follows:

Proposition 3.12. *Let ν be a finite positive Borel measure on $\mathbb{D}$. Then, ν is a Carleson measure for $\mathcal{D}_\zeta$ if and only if the measure $|z - \zeta|^2\, d\nu(z)$ is Carleson for H^2.*

Note that every Carleson measure of $\mathcal{D}_\zeta$ has to be finite, since $1 \in \mathcal{D}_\zeta$. Having mentioned these preliminary facts, we can state our result.

Corollary 3.13. *Let u be an inner function and ν a finite positive Borel measure on $\mathbb{D}$. If there exists $\zeta \in \mathbb{T} \setminus \sigma(u)$ such that $|z - \zeta|^2\, d\nu(z)$ is a Carleson measure for H^2, then ν is a Carleson measure for the model space K_u.*

Proof. Since $\zeta \notin \sigma(u)$, by Theorem 3.8 the embedding $K_u \hookrightarrow \mathcal{D}_\zeta$ holds. Also, by Proposition 3.12, the measure ν is a Carleson measure for $\mathcal{D}_\zeta$. Then, for every $f \in K_u$ it holds

$$\int_{\mathbb{D}} |f|^2\, d\nu \leq C\|f\|^2_{\mathcal{D}_\zeta} \leq C'\|f\|^2_{K_u},$$

for some positive constants C, C', meaning that ν is a Carleson measure for K_u. □

We conclude this part with an example of a Carleson measure for $\mathcal{D}_1$ (and thus for every model space K_u with $1 \notin \sigma(u)$) which is not Carleson for H^2.

Example. Let ν be the measure defined on Borel sets of $\mathbb{D}$ as

$$\nu(A) := \int_{A \cap [0,1]} \frac{1}{\sqrt{1-s}}\, ds.$$

We use the characterization in Proposition 3.11 to prove that ν is not a Carleson measure for H^2. For $\delta > 0$, consider the arc I_δ centered at 1 with arc length δ. One can compute the measure of the Carleson boxes $S(I_\delta)$ and obtain

$$\nu\big(S(I_\delta)\big) = \int_{1-\delta}^{1} \frac{1}{\sqrt{1-s}}\, ds = 2\sqrt{\delta},$$

showing that the bound in (3.9) cannot hold as $\delta \to 0$. However, the measure ν is a Carleson measure for the local Dirichlet space $\mathcal{D}_1$. We use Proposition 3.12, and because of the definition of ν it suffices to consider only the arcs that contain 1, and one can show that the measure $|z-1|^2\, d\nu(z)$ satisfies (3.9).

We move now to the description of multipliers from the model space K_u into the local Dirichlet space.

Definition 3.14. Let H_1, H_2 be Hilbert spaces of holomorphic functions on $\mathbb{D}$. The multipliers from H_1 to H_2 are defined as

$$M(H_1, H_2) := \{\phi \in \mathrm{Hol}(\mathbb{D}) \colon \phi H_1 \subseteq H_2\}.$$

When $H_1 = H_2$ we simply write $M(H_1)$.

The multiplier algebra $M(\mathcal{D}_\zeta)$ of the local Dirichlet space is characterized as follows. This result follows from Proposition 3.1 of [27]. For sake of completeness, we provide an explicit proof.

Lemma 3.15. *For $\zeta \in \mathbb{T}$, the multiplier algebra of $\mathcal{D}_\zeta$ is $\mathcal{D}_\zeta \cap H^\infty$.*

Proof. The fact that the multipliers of $\mathcal{D}_\zeta$ are in $\mathcal{D}_\zeta \cap H^\infty$ follows from the standard argument which holds for many other reproducing kernel Hilbert spaces of analytic functions, see for example Proposition 3.1 in [26]. Let us move to the other inclusion: let $\phi \in \mathcal{D}_\zeta \cap H^\infty$, and let $f \in \mathcal{D}_\zeta$. In light of the characterization in Theorem 1.46, there exist functions $\eta, g \in H^2$ such that

$$\phi(z) = \phi(\zeta) + (z-\zeta)\eta(z), \qquad f(z) = f(\zeta) + (z-\zeta)g(z), \qquad z \in \mathbb{D}. \tag{3.10}$$

Then, for $z \in \mathbb{D}$ it holds that

$$\begin{aligned}\phi(z)f(z) &= \big(\phi(\zeta) + (z-\zeta)\eta(z)\big)\big(f(\zeta) + (z-\zeta)g(z)\big)\\ &= \phi(\zeta)f(\zeta) + (z-\zeta)[\phi(\zeta)g(z) + \eta(z)f(\zeta) + (z-\zeta)\eta(z)g(z)].\end{aligned}$$

Again by Theorem 1.46, membership of the product ϕf in $\mathcal{D}_\zeta$ is equivalent to the membership in H^2 of the function

$$\phi(\zeta)g(z) + \eta(z)f(\zeta) + (z-\zeta)\eta(z)g(z).$$

Since $\eta, g \in H^2$, it suffices to show that $(z-\zeta)\eta(z)g(z)$ belongs to H^2, and this follows from (3.10) and the assumption that $\phi \in H^\infty$, for

$$(z-\zeta)\eta(z)g(z) = \big(\phi(z) - \phi(\zeta)\big)g(z). \quad \square$$

In [26], multipliers between model spaces are studied. It is shown that $M(K_u) = \mathbb{C}$, meaning that every function multiplying any model space into itself must be constant. Furthermore, multipliers from model spaces to the Hardy space H^2 are characterized in terms of a Carleson condition on the unit circle. More precisely, $\phi \in M\big(K_u, H^2\big)$ if and only if the measure $|\phi|^2 dm$ is a Carleson measure for K_u, i.e. there exists a constant $C > 0$ such that

$$\int_{\mathbb{T}} |f\phi|^2\, dm \le C\|f\|^2_{K_u}, \qquad f \in K_u.$$

Assuming the inclusion $K_u \subseteq \mathcal{D}_\zeta$, the local Dirichlet space $\mathcal{D}_\zeta$ is an intermediate space between K_u and H^2. This is reflected in our following multiplier theorem.

Theorem 3.16. *Let u be an inner function, $\zeta \setminus \sigma(u)$ and $\phi \in Hol(\mathbb{D})$. Then, ϕ is a multiplier from K_u to $\mathcal{D}_\zeta$ if and only if the measure $|\phi|^2 dm$ is Carleson for K_u and ϕ belongs to $\mathcal{D}_\zeta$.*

Proof. Let us assume that $\phi \in M(K_u, \mathcal{D}_\zeta)$. Then, in particular, the measure $|\phi|^2\, dm$ is a Carleson measure for K_u, so it suffices to show that every multiplier from K_u to $\mathcal{D}_\zeta$ belongs to $\mathcal{D}_\zeta$. If $u(0) = 0$, then $1 \in K_u$, implying that the multiplier ϕ belongs to $\mathcal{D}_\zeta$. If $u(0) \neq 0$, we consider the kernel

$$k_0^u = 1 - \overline{u(0)}u.$$

Using Theorem 1.47, one can check that $1/k_0^u \in H^\infty \cap \mathcal{D}_\zeta$, so that by Lemma 3.15 the function $1/k_0^u$ is a multiplier of $\mathcal{D}_\zeta$. Thus,

$$\phi = \frac{1}{k_0^u}\, \phi k_0^u \in \mathcal{D}_\zeta$$

which implies the statement. Let us now prove the other implication. We assume that $|\phi|^2\, dm$ is a Carleson measure for K_u and that ϕ belongs to $\mathcal{D}_\zeta$. Since $\phi \in M\big(K_u, H^2\big)$, for every $f \in K_u$ the product ϕf belongs to H^2. We compute the local Dirichlet integral. We have

$$\begin{aligned}
\mathcal{D}_\zeta(f\phi) &= \int_{\mathbb{T}} \left| \frac{f(\lambda)\phi(\lambda) - f(\zeta)\phi(\zeta)}{\lambda - \zeta} \right|^2 dm(\lambda) \\
&= \int_{\mathbb{T}} \left| \frac{f(\lambda)\phi(\lambda) - \phi(\lambda)f(\zeta) + \phi(\lambda)f(\zeta) - f(\zeta)\phi(\zeta)}{\lambda - \zeta} \right|^2 dm(\lambda) \\
&\le 2 \int_{\mathbb{T}} |\phi(\lambda)|^2 \left| \frac{f(\lambda) - f(\zeta)}{\lambda - \zeta} \right|^2 dm(\lambda) + 2|f(\zeta)|^2 \int_{\mathbb{T}} \left| \frac{\phi(\lambda) - \phi(\zeta)}{\lambda - \zeta} \right|^2 dm(\lambda) \\
&= 2 \left\| \frac{f - f(\zeta)}{\cdot - \zeta} \right\|^2_{L^2(|\phi|^2 dm)} + 2|f(\zeta)|^2 \left\| \frac{\phi - \phi(\zeta)}{\cdot - \zeta} \right\|^2_{H^2} \\
&\le C \left\| \frac{f - f(\zeta)}{\cdot - \zeta} \right\|^2_{K_u} + 2|f(\zeta)|^2 \mathcal{D}_\zeta(\phi) \\
&\le \big(C + 2\mathcal{D}_\zeta(\phi)\big) \|f\|^2_{\mathcal{D}_\zeta},
\end{aligned}$$

concluding the proof. □

It is natural to ask whether the condition in Theorem 3.16 guarantees the boundedness of the multipliers, in other words, whether $M(K_u, \mathcal{D}_\zeta)$ is contained or not in H^∞. The answer to this question is negative. Considering the simplest case $u(z) = z$, one has that $K_u = \mathbb{C}$, and therefore $M(K_u, \mathcal{D}_\zeta) = \mathcal{D}_\zeta$, which contains unbounded functions.

3.2.1 Embedding $K_u \hookrightarrow \mathcal{D}_\mu$

Given μ a finite positive Borel measure on $\mathbb{T}$ and an inner function u, Theorems 3.5 and 3.6 provide a sufficient condition and a necessary condition for the embedding $K_u \hookrightarrow \mathcal{D}_\mu$, respectively $\operatorname{supp}(\mu) \cap \sigma(u) = \emptyset$ and $\mu\big(\sigma(u)\big) = 0$. If $\mu = \delta_\zeta$, both these conditions are equivalent to $\zeta \notin \sigma(u)$, leading to the characterization

$$K_u \hookrightarrow \mathcal{D}_\zeta \iff \zeta \notin \sigma(u).$$

Considering the Lebesgue measure $\mu = m$, the two conditions do not coincide, but the sufficient one is also necessary. This is because, if the inclusion $K_u \hookrightarrow \mathcal{D} = \mathcal{D}_m$ holds, then by Proposition 3.7 the function u belongs to $\mathcal{D}$. However, by Corollary 1.51, the only inner functions in the classical Dirichlet space are the finite Blaschke products, resulting in the boundary spectrum $\sigma(u) \cap \mathbb{T}$ being empty. Therefore, we have the characterization

$$K_u \hookrightarrow \mathcal{D} \iff \operatorname{supp}(m) \cap \sigma(u) = \emptyset \iff \sigma(u) = \emptyset.$$

As a consequence of Theorem 2.10, we know that in general the sufficient condition $\operatorname{supp}(\mu) \cap \sigma(u) = \emptyset$ is not necessary for $K_u \hookrightarrow \mathcal{D}_\mu$.

Example. Let u be the one-component inner function considered in Proposition 2.13,

$$u(z) = \exp\left(\frac{z+1}{z-1}\right), \qquad z \in \mathbb{D}.$$

Consider a measure μ such that $\mu(\{1\}) = 0$ and

$$\int_{\mathbb{T}} \frac{1}{|\zeta - 1|^2}\, d\mu(\zeta) < \infty.$$

Then, the embedding $K_u \hookrightarrow \mathcal{D}_\mu$ holds. This is because, for every $f \in K_u$ and $\zeta \neq 1$,

$$\mathcal{D}_\zeta(f) = \|Q_\zeta^u f\|_{H^2}^2 \le \|Q_\zeta^u\|^2 \|f\|_{H^2}^2 \le \frac{C_u}{|\zeta - 1|^2} \|f\|_{H^2}^2,$$

so that

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}\setminus\{1\}} \mathcal{D}_\zeta(f)\, d\mu(\zeta) \le C_u \|f\|_{H^2}^2 \int_{\mathbb{T}} \frac{1}{|\zeta - 1|^2}\, d\mu(\zeta) < \infty.$$

We conclude this section discussing the compactness of the embeddings $K_u \hookrightarrow \mathcal{D}_\mu$. Due to the trivial norm inequality $\|\cdot\|_{H^2} \le \|\cdot\|_{\mathcal{D}_\mu}$, the compactness of the embedding $K_u \hookrightarrow \mathcal{D}_\mu$ implies the compactness of the identity map I_{K_u}. Therefore, it is easy to see that the embedding $K_u \hookrightarrow \mathcal{D}_\mu$ is compact if and only if K_u is finite dimensional, that is, if and only if u is a finite Blaschke product [30, Proposition 5.19].

3.3 The case b non-extreme

This section comes from original results of the author. We consider a non-extreme function $b \in H_1^\infty$, i.e., such that $\log(1-|b|)$ is integrable, and its Pythagorean mate a. We recall that a is the unique outer function such that $a(0) > 0$ and

$$|b|^2 + |a|^2 = 1 \qquad \text{a.e. on } \mathbb{T}. \tag{3.11}$$

This function a is defined by

$$a(z) := \exp\left(\int_{\mathbb{T}} \frac{\lambda+z}{\lambda-z} \log(1-|b(\lambda)|^2)^{\frac{1}{2}}\, dm(\lambda)\right), \qquad z \in \mathbb{D}.$$

We say that (b,a) is a pair. An application of the maximum principle for subharmonic functions yields that, in fact, if (b,a) is a pair, then

$$|b(z)|^2 + |a(z)|^2 \le 1 \qquad z \in \mathbb{D}. \tag{3.12}$$

Check also [29, Exercise 23.1.1]. We consider the following boundary zero set of the function a,

$$\mathcal{Z}(a) = \{\lambda \in \mathbb{T} \colon \lim_{r\to 1} a(r\lambda) = 0\}.$$

This set is related to the spectrum of b, in particular to the boundary part $\sigma(b)\cap\mathbb{T}$. Notice that

$$\mathbb{T}\setminus\sigma(b) = \{\lambda \in \mathbb{T} \colon \lim_{z\in\mathbb{D}, z\to\lambda} |b(z)| = 1\}.$$

Therefore, if $\lambda \in \mathbb{T}\setminus\sigma(b)$, then by (3.12) it follows that

$$\limsup_{z\to\lambda} |a(z)| \le \limsup_{z\to\lambda} \sqrt{1-|b(z)|^2} = 0.$$

In particular, we have the set inequality

$$\mathbb{T}\setminus\sigma(b) \subseteq \mathcal{Z}(a).$$

We will show that a converse to this inequality does not hold, in general. If the function b is continuous up to the boundary, then also the function a is continuous up to the boundary, and in this case we have the set identity

$$\mathcal{Z}(a) = \mathbb{T}\setminus\sigma(b), \tag{3.13}$$

and the set $\mathcal{Z}(a)$ is the usual zero set for the boundary function a. In particular, if b is continuous up to the boundary, then $\mathcal{Z}(a)$ is closed. However, this is not true in general.

Example. Consider the singular inner function associated to the Dirac measure δ_1,

$$u(z) = \exp\left(\frac{z+1}{z-1}\right), \qquad z \in \mathbb{D}.$$

Since $\sigma(u) = \{1\}$, the function u admits an analytic extension across any open arc $\Delta \subseteq \mathbb{T}\setminus\{1\}$ with $|u| = 1$ on Δ. Therefore, if we consider the non-extreme function

$$b = \frac{1+u}{2}$$

as in Proposition 1.35, we have that its mate is $a = (1-u)/2$ and

$$\mathcal{Z}(a)\setminus\{1\} = \{\zeta \in \mathbb{T}\setminus\{1\} \colon u(\zeta) = 1\} = \left\{\frac{2k\pi i + 1}{2k\pi i - 1} \colon k \in \mathbb{Z}\right\}.$$

The set $\mathcal{Z}(a)$ accumulates at 1, however a computation shows that

$$\lim_{r\to 1} u(r) = 0,$$

proving that $1 \notin \mathcal{Z}(a)$.

The next example shows that the equality in (3.13) does not always hold. This was suggested by T. Ransford, while the author was visiting Laval University, QC, through private communication.

Example. We consider again the singular inner function u associated to the Dirac measure δ_1, and we set

$$b(z) := \frac{1+z}{2}u(z), \qquad z \in \mathbb{D}.$$

Since $|u| = 1$ on $\mathbb{T}\setminus\{1\}$ and $\liminf_{z\to 1}|u(z)| = 0$, we have that $\sigma(b)\cap\mathbb{T} = \mathbb{T}$. Its Pythagorean mate is $a(z) := (1-z)/2$. Indeed, a is an outer function with $a(0) > 0$, and for $z \in \mathbb{T}\setminus\{1\}$ we have

$$|b(z)|^2 + |a(z)|^2 = \frac{1}{4}(|1+z|^2 + |1-z|^2) = 1.$$

In this case, we have that $\mathcal{Z}(a) = \{1\}$ and $\mathbb{T}\setminus\sigma(b) = \emptyset$.

Given μ a finite positive Borel measure on $\mathbb{T}$, we recall that its potential is defined as

$$V_\mu(w) := \int_{\mathbb{T}} \frac{1}{|\lambda - w|^2}\, d\mu(\lambda), \qquad w \in \mathbb{C},$$

and it satisfies $V_\mu = \infty$ $\mu-$a.e. on $\mathbb{T}$. We need some technical results from [17].

Lemma 3.17. *Let*

$$c_w(z) = \frac{1}{1 - \overline{w}z}, \qquad w, z \in \mathbb{D},$$

be the Cauchy-Szegö kernel. Then, if (b,a) is a pair,

$$\|c_w\|^2_{H(b)} = \frac{1 + |b(w)/a(w)|^2}{1 - |w|^2}, \tag{3.14}$$

and if μ is a finite positive Borel measure on $\mathbb{T}$, then

$$\|c_w\|^2_{\mathcal{D}_\mu} = \frac{1 + |w|^2 V_\mu(w)}{1 - |w|^2}. \tag{3.15}$$

Proof. See Lemmas 3.4 and 3.5 of [17]. □

In the following, we will use the notation $F \lesssim G$ to denote that F/G is a bounded function, and we will write $F \asymp G$ when $F \lesssim G$ and $G \lesssim F$.

Proposition 3.18. *Let (b, a) be a pair and μ a finite positive Borel measure on $\mathbb{T}$. If the set inequality $H(b) \subseteq \mathcal{D}_\mu$ holds, then $\mathcal{Z}(a)$ is a carrier for the measure μ. In particular, it holds the set inclusion supp$(\mu) \subseteq \overline{\mathcal{Z}(a)}^{\mathrm{cl}}$.*

Proof. By the closed graph theorem and Lemma 3.17,

$$1 + V_\mu(w) \lesssim 1 + \frac{|b(w)|^2}{|a(w)|^2}, \qquad w \in \mathbb{D}.$$

Let ζ be in $\mathbb{T} \setminus \mathcal{Z}(a)$. In particular, there exist $\varepsilon > 0$ and a sequence r_k in $(0, 1)$ converging to 1 as $k \to \infty$ such that $\lim_k |a(r_k\zeta)| > \varepsilon$. By Fatou's Lemma,

$$\begin{aligned} V_\mu(\zeta) &\leq \liminf_k V_\mu(r_k\zeta) \\ &\lesssim \liminf_k \frac{|b(r_k\zeta)|^2}{|a(r_k\zeta)|^2} \\ &\leq \frac{1}{\lim_k |a(r_k\zeta)|^2} < \infty. \end{aligned}$$

We have the set inclusion

$$\mathbb{T} \setminus \mathcal{Z}(a) \subseteq \{\zeta \in \mathbb{T} \colon V_\mu(\zeta) < \infty\}.$$

Since $V_\mu = \infty$ $\mu-$a.e. on $\mathbb{T}$, then $\mu(\mathbb{T} \setminus \mathcal{Z}(a)) = 0$, meaning that $\mathcal{Z}(a)$ is a carrier for the measure μ. Notice that by monotonicity it also holds

$$\mu\left(\mathbb{T} \setminus \overline{\mathcal{Z}(a)}^{\mathrm{cl}}\right) = 0,$$

thus by definition of support we have that $\mathrm{supp}(\mu) \subseteq \overline{\mathcal{Z}(a)}^{\mathrm{cl}}$. □

We study the reverse inclusion $\mathcal{D}_\mu \subseteq H(b)$. First of all, we note that, if the set inclusion $\mathcal{D}_\mu \subseteq H(b)$ holds, then the space $H(b)$ contains all the polynomials, and therefore the function b is non-extreme. As usual, we will denote by a the Pythagorean mate of b. We will make the assumption that (b, a) is a Corona pair, i.e., that there exists $\delta > 0$ such that

$$\inf_{z \in \mathbb{D}} (|b(z)| + |a(z)|) \geq \delta.$$

This is one of the two equivalent conditions in the Corona Theorem 1.26.

Proposition 3.19. *Let μ be a measure and (b, a) a Corona pair. If the embedding $\mathcal{D}_\mu \hookrightarrow H(b)$ holds, then $\mathcal{Z}(a) \subseteq$ supp(μ). Also, for every point $\zeta \in \mathcal{Z}(a)$ that is an isolated atom for μ, it holds the estimate*

$$\limsup_{w \to \zeta} \frac{|w - \zeta|}{|a(w)|} < \infty. \tag{3.16}$$

Notice that, a priori, there might not be any point ζ in $\mathcal{Z}(a)$ that is also an isolated atom for μ.

Proof. To prove the set inclusion $\mathcal{Z}(a) \subseteq \operatorname{supp}(\mu)$, we prove the opposite inclusion between the complements of the sets. Let $\zeta \in \mathbb{T} \setminus \operatorname{supp}(\mu)$. By definition of V_μ, one can check that $V_\mu(\zeta) < \infty$ and that

$$V_\mu(w) \le \frac{\mu(\mathbb{T})}{\operatorname{dist}(w, \operatorname{supp}(\mu))^2}, \qquad w \in \mathbb{C}. \tag{3.17}$$

In particular, $V_\mu(\zeta) < \infty$. Also, again by the closed graph theorem and Lemma 3.17,

$$\limsup_{r\to 1} \frac{|b(r\zeta)|^2}{|a(r\zeta)|^2} \lesssim \limsup_{r\to 1} r^2 V_\mu(r\zeta) < \infty.$$

With the assumption that (b, a) form a Corona pair, this shows that $\zeta \notin \mathcal{Z}(a)$. Now, fix a point $\zeta \in \mathcal{Z}(a)$ that is an isolated atom for μ, if there are any. By (3.17), we have that

$$\frac{|b(w)|^2}{|a(w)|^2} \lesssim |w|^2 V_\mu(w) \le \frac{|w|^2\mu(\mathbb{T})}{\operatorname{dist}(w, \operatorname{supp}(\mu))^2}, \qquad w \in \mathbb{D}.$$

Thus, in an appropriate neighbourhood U_ζ of ζ, since (b, a) form a Corona pair, it holds

$$\frac{|w-\zeta|}{|a(w)|} \lesssim 1, \qquad w \in U_\zeta,$$

concluding the proof. □

Notice that the growth condition (3.16) gives information on how fast the function $a(r\zeta)$ tends to 0 as $r \to 1^-$. In particular, if a is analytic in ζ, then (3.16) implies that a has a 0 of order 1 in ζ.

Confronting the two different inclusions, we obtain the following.

Theorem 3.20. *Let (b, a) be a Corona pair and μ a finite positive Borel measure on $\mathbb{T}$. If $H(b) = \mathcal{D}_\mu$, then*

$$\operatorname{supp}(\mu) = \overline{\mathcal{Z}(a)}^{\mathrm{cl}}.$$

Proof. It follows from Propositions 3.18 and 3.19. □

Notice that, if b is a rational function or if b has the form $(1+u)/2$ with u inner, then (b, a) forms a Corona pair.

3.3.1 μ finite sum of atoms

In [17], the authors proved the following theorem.

Theorem 3.21. *Let (b, a) be a pair such that a is rational, and let μ be a finite positive measure on $\mathbb{T}$. Then $H(b) = \mathcal{D}_\mu$ if and only if the following conditions hold:*

1. *the zeros of a on $\mathbb{T}$ are all simple;*
2. *the support of μ is exactly equal to this set of zeros;*
3. *none of these zeros lie in the spectrum $\sigma(b_i)$, where b_i is the inner factor of b.*

Using this theorem, we prove a similar result, but with a different point of view. We consider a measure μ that is a finite sum of atoms, i.e.

$$\mu = \sum_{j=1}^{N} \alpha_j \delta_{\zeta_j}, \tag{3.18}$$

where $\alpha_j > 0$ and $\zeta_j \in \mathbb{T}$, for $j = 1, \dots, N$. We characterize the conditions on the pair (b, a) in order to have the equality $H(b) = \mathcal{D}_\mu$. As a corollary of Theorem 3.21, in [17] the authors showed that $\mathcal{D}_\mu = H(b_\mu)$, where the function b_μ can be chosen as the non-extreme function of H_1^∞ having for Pythagorean mate a_μ a polynomial with simple zeros in the atoms of μ. We can write

$$a_\mu(z) = C \prod_{j=1}^{N} (z - \zeta_j), \qquad z \in \mathbb{D}, \tag{3.19}$$

where $C \in \mathbb{C}$ is an appropriate constant such that $a_\mu(0) > 0$ and $\|a_\mu\|_{H^\infty} \leq 1$. Also, in [17] the function b_μ is constructed starting from the function a_μ defined in (3.19) using the Fejér-Riesz Theorem. In particular, b_μ can be chosen to be a polynomial whose zeros lie outside the disk $\mathbb{D}$.

The choice of the pair (b_μ, a_μ) is not unique, for example in [55] the equality $H(b) = \mathcal{D}_\zeta$ is realized with the pair (b_ζ, a_ζ) given by

$$b_\zeta(z) = \frac{(1 - s_0)\overline{\zeta} z}{1 - s_0 \overline{\zeta} z}, \qquad a_\zeta(z) = \frac{(1 - s_0)(1 - \overline{\zeta} z)}{1 - s_0 \overline{\zeta} z}, \qquad z \in \mathbb{D},$$

where

$$s_0 = \frac{3 - \sqrt{5}}{2}.$$

Being that $\mathcal{D}_\mu = H(b_\mu)$, to study the equalities $\mathcal{D}_\mu = H(b)$ for a general b we can appeal to the following result from J. Ball and T. Kriete [6]. See also Theorem 27.15 of [29].

Theorem 3.22. *Let b_2 and b_1 be two functions in the closed unit ball of H^∞, with b_1 non-extreme. The following assertions are equivalent.*

(i) $H(b_2) \subseteq H(b_1)$;

(ii) *There exist $v, w \in H^\infty$ and $\gamma > 0$ such that*

- *(A)* $b_1 + v a_1 = b_2 w$
- *(B)* $1 - |b_2|^2 \leq \gamma(1 - |b_1|^2)$ *a.e. on* $\mathbb{T}$.

Using this theorem, we prove the following embedding result.

Theorem 3.23. *Let b be a non-extreme function in H_1^∞, a its Pythagorean mate, and μ an atomic measure as in* (3.18)*. Then, we have the embedding $H(b) \hookrightarrow \mathcal{D}_\mu$ if and only if the following conditions hold:*

(i) There exists $g \in H^\infty$ such that a has the form

$$a = \left(\prod_{j=1}^{N}(z-\zeta_j)\right) g;$$

(ii) $\{\zeta_1, \dots, \zeta_N\} \cap \sigma(b) = \emptyset$.

Proof. We assume that $H(b) \hookrightarrow \mathcal{D}_\mu$. Since

$$\mathcal{D}_\mu = \bigcap_{j=1}^{N} \mathcal{D}_{\zeta_j},$$

then by Theorem 3.6, necessarily $\zeta_j \notin \sigma(b)$ for each $j = 1, \dots, N$. This proves (ii). Now, applying (B) of Theorem 3.22 to our pair (b, a) and the pair (b_μ, a_μ) defined in (3.19), there exists a positive constant γ such that

$$1 - |b(\lambda)|^2 \le \gamma(1 - |b_\mu(\lambda)|^2), \qquad \text{a.e. } \lambda \in \mathbb{T}.$$

Since (b, a) and (b_μ, a_μ) form two Pythagorean pairs and a_μ has the expression (3.19), we have that

$$|a(\lambda)|^2 \le \gamma |C|^2 \prod_{j=1}^{N} |\lambda - \zeta_j|^2, \qquad \text{a.e. } \lambda \in \mathbb{T}.$$

In particular, the function defined for a.e. $\lambda \in \mathbb{T}$

$$g(\lambda) := \frac{a(\lambda)}{\prod_{j=1}^{N}(\lambda - \zeta_j)}$$

belongs to $L^\infty(\mathbb{T})$. We recall that the binomial $z - \zeta_j$ is outer. Being g a quotient of outer functions, and therefore in the Smirnov class, by Theorem 1.15 we have that

$$g(z) := \frac{a(z)}{\prod_{j=1}^{N}(z - \zeta_j)}, \qquad z \in \mathbb{D},$$

belongs to H^∞, proving (i). Now, we assume that (i) and (ii) hold. We recall that $\mathcal{D}_\mu = H(b_\mu)$, where the pair (b_μ, a_μ) is described in (3.19). Our goal is to apply Theorem 3.22. Condition (B) is $1 - |b|^2 \le \gamma(1 - |b_\mu|^2)$, a.e. on $\mathbb{T}$, and this follows from (i) and the fact that $g \in H^\infty$. On the other hand, by assumption we have that $\{\zeta_1, \dots, \zeta_N\} \cap \sigma(b) = \emptyset$, meaning that

$$\lim_{z \in \mathbb{D}, z \to \zeta_j} |b(z)| = 1$$

for every $j = 1, \dots, N$. By the definition (3.19) of a_μ, it follows that b and a_μ form a Corona pair. An application of the Corona Theorem 1.26 proves that condition (A) holds. We conclude that $H(b) \subseteq H(b_\mu) = \mathcal{D}_\mu$. $\square$

As an application of this theorem to the special case of one atom, $N = 1$, we obtain a complete characterization for the embedding in the local Dirichlet space $H(b) \hookrightarrow \mathcal{D}_\zeta$, completing Theorems 3.2 and 3.3, for a non-extreme b.

Corollary 3.24. *Let b be a non-extreme function in H_1^∞, a its Pythagorean mate, and $\zeta \in \mathbb{T}$. Then, it holds the embedding $H(b) \hookrightarrow \mathcal{D}_\zeta$ if and only if the two following conditions hold.*

- *$\zeta \notin \sigma(b)$.*
- *There exists $g \in H^\infty$ such that $a = (z - \zeta)g$.*

Now, we move to the reverse inclusion $\mathcal{D}_\mu \subseteq H(b)$.

Lemma 3.25. *Let μ be an atomic measure as in* (3.18)*, and (b, a) be a pair such that a has the form*

$$a = \left(\prod_{j=1}^{N} (z - \zeta_j) \right) g,$$

with $g \in H^\infty$. Then, we have the inclusion $\mathcal{D}_\mu \subseteq H(b)$ if and only if $\inf_{\mathbb{D}} |g| > 0$.

Proof. This is again a consequence of Theorem 3.22. Since a is outer and so is the product of the binomials $(z - \zeta_j)$, g is outer as well, for it cannot have inner factors. Also, the condition (B) of Theorem 3.22 in this context can be rewritten as

$$|a_\mu|^2 \leq \gamma |a|^2 \qquad \text{a.e. on } \mathbb{T},$$

which is equivalent to saying that $1 \lesssim |g|$ a.e. on $\mathbb{T}$. By Smirnov's Theorem 1.15, this happens if and only if the function $1/g$ belongs to H^∞, or, equivalently, if and only if $\inf_{\mathbb{D}} |g| > 0$. Now, being b_μ, a_μ polynomials, necessarily the relation $|b|^2 + |a|^2 = 1$ holds for every point on the boundary, and in particular $|b(\zeta_j)| = 1$ for every j. Therefore, assuming that $\inf_{\mathbb{D}} |g| > 0$, then b_μ and a form a Corona pair and condition (A) of Theorem 3.22 is satisfied. This concludes the proof. □

Theorem 3.26. *Let b be a non-extreme function in H_1^∞, a its Pythagorean mate, and μ an atomic measure as in* (3.18)*. Then, we have the equality $H(b) = \mathcal{D}_\mu$ if and only if the following conditions hold:*

(i) There exists $g \in H^\infty$ with $\inf_{\mathbb{D}} |g| > 0$ *such that a has the form*

$$a = \left(\prod_{j=1}^{n} (z - \zeta_j) \right) g;$$

(ii) $\{\zeta_1, \ldots, \zeta_N\} \cap \sigma(b) = \emptyset$.

Proof. We assume that $H(b) = \mathcal{D}_\mu$. By Theorem 3.23, since $H(b) \subseteq \mathcal{D}_\mu$, the function a has the form

$$a = \left(\prod_{j=1}^{n} (z - \zeta_j) \right) g$$

for some $g \in H^\infty$ and it holds (ii). Now, applying Lemma 3.25 we conclude that $\inf_{\mathbb{D}} |g| > 0$.

On the other hand, if conditions (i) and (ii) hold, then we automatically have both the inclusions, concluding the proof. □

To conclude this part, we show that neither one of the two conditions of Theorem 3.26 implies the other. In the following examples, $N = 1$ and $\zeta_1 = 1$. These examples were suggested to the author by Thomas Ransford through private communication.

Example. Let b be the function defined by

$$b(z) = \frac{1+z}{2} \exp\left(\frac{z+1}{z-1}\right), \qquad z \in \mathbb{D}.$$

We considered the same function in a previous example, at the beginning of Section 3.3. In this case, the Pythagorean mate of b is $a(z) := (1-z)/2$. Clearly, condition (i) holds with $g = -1/2$. However, because of the presence of the singular inner factor in b,

$$\liminf_{z \to 1} |b(z)| = 0,$$

so that $1 \in \sigma(b)$.

Example. Consider

$$b(z) = \frac{1+z^2}{2}, \qquad z \in \mathbb{D}.$$

Then, its Pythagorean mate is $a(z) := (1-z^2)/2$. Condition (ii) is easily satisfied: since

$$\lim_{z \to 1} |b(z)| = 1,$$

then $1 \notin \sigma(b)$. However, since by Proposition 1.35

$$a(z) = \frac{1-z^2}{2} = (z-1)\left(-\frac{1+z}{2}\right),$$

condition (i) cannot be satisfied.

Chapter 4

Generalized Cesàro means on Dirichlet spaces

In this chapter we present some recent results obtained in collaboration with J. Mashreghi, M. Nasri and W. Verreault (a preprint is now ready and currently under review). The content of this chapter differs from the main topic of the previous ones. So far, we have conducted a broad analysis on topics that cross the two different realms of de Branges-Rovnyak spaces and harmonically weighted Dirichlet spaces. However, in this last part of the thesis, the results that we will present are aimed to answer a very natural and specific question about the convergence of certain means on weighted Dirichlet spaces.

One of the fundamental facts in complex analysis is that every holomorphic function f on the open unit disk $\mathbb{D}$ has the Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad z \in \mathbb{D}.$$

Therefore, in any polynomial approximation scheme in a Banach space $\mathcal{X}$ consisting of such analytic functions, it is natural to consider the Taylor polynomials

$$(S_n f)(z) := \sum_{k=0}^{n} a_k z^k, \qquad n \geq 0, \tag{4.1}$$

and explore if $S_n f \to f$ in the ambient space $\mathcal{X}$. For example, it is straightforward to see that this holds in the Hardy space H^2 and in the classical Dirichlet space $\mathcal{D}$. Moreover, it is a deep result of Hardy–Littlewood [33, 34] that it is also a valid scheme in H^p-spaces for $1 < p < \infty$. However, this natural approximation method fails in some settings such as the disk algebra $\mathcal{A}$ and the Hardy space H^1.

As a first alternative to Taylor polynomials (4.1), the weighted versions

$$(\sigma_n f)(z) := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k z^k, \qquad n \geq 0, \tag{4.2}$$

known as Cesàro means or Fejér polynomials, were considered. In fact, Hardy-Littlewood demonstrated that $\sigma_n f \to f$ holds in both the disk algebra $\mathcal{A}$ and the

Hardy space H^1. They even considered more sophisticated means which we do not discuss in this work.

We consider weighted Dirichlet spaces. For this part, we allow a greater generality for the weights: we consider *superharmonic* functions. Standard references for this topic are [31, 47]. In literature, it is more common to define first the *subharmonic* functions, rather than the superharmonic ones.

Definition 4.1. Let $U \subset \mathbb{C}$ be an open set. A function $u\colon U \to [-\infty, +\infty)$ is called *subharmonic* if it is upper semicontinuous and it satisfies the local submean inequality, i.e., given $w \in U$, there exists $\rho > 0$ such that the closed ball $\{z \in \mathbb{C}\colon |z-w| \le \rho\}$ is contained in U and

$$u(w) \le \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})\, d\theta, \qquad (0 \le r < \rho).$$

We say that a function $v\colon U \to (-\infty, +\infty]$ is *superharmonic* if $-v$ is subharmonic.

Given a function ω that is positive and superharmonic on $\mathbb{D}$, we define the superharmonically weighted Dirichlet space

$$\mathcal{D}_\omega := \{f \in \mathrm{Hol}(\mathbb{D})\colon \int_{\mathbb{D}} |f'(z)|^2\, \omega(z)\, dA(z) < \infty\}.$$

These spaces were introduced by Alexandru Aleman in 1993 [2] as an interesting generalization of the harmonically weighted ones. This generalization in the weight comes with some differences in the structure and the properties of the space, with respect to the harmonic case. For example, the forward shift S acts as a 2-isometric operator on $\mathcal{D}_\omega$ if and only if the weight ω is harmonic. For more details, see [2].

In 2019, Mashreghi–Ransford [42] studied problems related to polynomial approximation on the spaces $\mathcal{D}_\omega$. More explicitly, it was demonstrated that while there are cases where $\|S_n f - f\|_{\mathcal{D}_w} \not\to 0$, the approximation scheme $\|\sigma_n f - f\|_{\mathcal{D}_w} \to 0$ remains valid. Later in this chapter, we will give more details about this initial work of Mashreghi–Ransford, to provide the reader with more context. In [41], the generalized Cesàro means

$$(\sigma_n^\alpha f)(z) = \binom{n+\alpha}{\alpha}^{-1} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} a_k z^k,$$

were considered on superharmonically weighted Dirichlet spaces. Here, α is a parameter in the interval $[0,1]$, and notice that for $\alpha = 0$ and $\alpha = 1$ one recovers, respectively, S_n and σ_n. It was shown that the approximation $\|\sigma_n^\alpha f - f\|_{\mathcal{D}_w} \to 0$ is valid for every superharmonic weight ω if and only if $\alpha > \frac{1}{2}$. In this chapter, we are interested in the asymptotic behavior of the norm of the operator σ_n^α associated to the generalized Cesàro means on the spaces $\mathcal{D}_\omega$, as $n \to \infty$. To tackle this topic, we will introduce the Hadamard multipliers and we will need several technical estimates.

4.1 Hadamard multipliers and main results

Definition 4.2. Given two formal power series $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \sum_{k=0}^{\infty} b_k z^k$, their *Hadamard product* is the formal power series given by the formula

$$(f * g)(z) := \sum_{k=0}^{\infty} (a_k b_k) z^k.$$

It is trivial that if either f or g is a polynomial, then $f * g$ is also a polynomial. In passing, we note that if $f, g \in H^2$, the coefficients $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ are the Fourier coefficients of their boundary functions. Then, the coefficients of the Hadamard product $f * g$ are the coefficients of the convolution of the boundary functions associated to f and g.

Definition 4.3. We say that an analytic function g on $\mathbb{D}$ is a *Hadamard multiplier* for $\mathcal{D}_\omega$ if the product $f * g$ belongs to $\mathcal{D}_\omega$ for every $f \in \mathcal{D}_\omega$.

The study of Hadamard multipliers on weighted Dirichlet spaces started in 2019 with a work by Mashreghi–Ransford [42]. The property of being a Hadamard multiplier was associated to the boundedness of a special infinite matrix on ℓ^2. For a sequence of complex numbers $(c_k)_{k\in\mathbb{N}}$, write T_c for the infinite matrix

$$T_c := \begin{pmatrix} c_1 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \dots \\ 0 & c_2 & c_3 - c_2 & c_4 - c_3 & \cdots \\ 0 & 0 & c_3 & c_4 - c_3 & \dots \\ 0 & 0 & 0 & c_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.3}$$

If this matrix is a bounded operator on ℓ^2, we denote its operator norm by $\|T_c\|_{\mathcal{B}(\ell^2)}$. For convenience of notation, if $(c_k)_{k\in\mathbb{N}}$ is the sequence of the Taylor coefficients of an analytic function h, or even of a formal power series $h(z) = \sum_{k=0}^{\infty} c_k z^k$, we also write T_h in place of T_c. Note that the term c_0 does not appear in the matrix T_c: this is consistent with the fact that the Dirichlet integrals annihilate the constants. The following central result is needed in our discussion.

Theorem 4.4 ([42])**.** *The function h is a Hadamard multiplier of $\mathcal{D}_\omega$, for any superharmonic weight ω, if and only if T_h defined as in* (4.3) *acts as a bounded operator on ℓ^2. Moreover, in this case, it holds the estimate*

$$\mathcal{D}_\omega(h * f) \leq \|T_h\|^2_{\mathcal{B}(\ell^2)}\, \mathcal{D}_\omega(f). \tag{4.4}$$

The constant $\|T_h\|^2_{\mathcal{B}(\ell^2)}$ is optimal. To be more precise, we introduce the quantity

$$\|T_h\|^2_{\mathcal{D}_\omega \to \mathcal{D}_\omega} := \sup_f \frac{\mathcal{D}_\omega(h * f)}{\mathcal{D}_\omega(f)}, \tag{4.5}$$

where ω is a superharmonic weight and the supremum is taken over all non-constant $f \in \mathcal{D}_\omega$. Hence, we know that, for each weight ω,

$$\|T_h\|_{\mathcal{D}_\omega \to \mathcal{D}_\omega} \leq \|T_h\|_{\mathcal{B}(\ell^2)},$$

and the sharpness of the constant $\|T_h\|_{\mathcal{B}(\ell^2)}$ means that, when we take another supremum with respect to weights ω, we obtain

$$\sup_\omega \|T_h\|_{\mathcal{D}_\omega \to \mathcal{D}_\omega} = \|T_h\|_{\mathcal{B}(\ell^2)}.$$

In [42], it is shown that this supremum is attained by choosing the harmonic weight

$$\omega_1(z) := \frac{1-|z|^2}{|1-z|^2}, \qquad z \in \mathbb{D}.$$

The corresponding Dirichlet space is the local Dirichlet space $\mathcal{D}_1$. Our goal is to further analyze

$$\|T_h\| := \|T_h\|_{\mathcal{B}(\ell^2)} = \sup_f \frac{\mathcal{D}_1(h*f)}{\mathcal{D}_1(f)},$$

for the special class of polynomials h that give rise to the generalized Cesàro means. Proper estimation of $\|T_h\|$ is crucial in applications.

After proving Theorem 4.4, the authors in [42] studied sufficient and necessary conditions for the matrix T_c to be bounded on ℓ^2. They also provided estimates for the norm $\|T_c\|_{\mathcal{B}(\ell^2)}$, with important consequences for the function spaces $\mathcal{D}_\omega$. For example, interpreting the radial approximation $f_r(z) = f(rz)$ as the Hadamard product $f * P_r$, where P_r is the Poisson kernel given by

$$P_r(z) = \sum_{k=0}^{\infty} r^k z^k,$$

they proved the following result: given ω a superharmonic weight and $f \in \mathcal{D}_\omega$, then for every $r \in (0,1)$ the function f_r belongs to $\mathcal{D}_\omega$ and

$$\mathcal{D}_\omega(f_r) \le r^2(2-r)\mathcal{D}_\omega(f). \tag{4.6}$$

Similar results were already known, but with these Hadamard multipliers techniques the authors managed to give a direct proof with a better constant. Actually, the estimate in (4.6) is the best one to date.

Now, we introduce the object of interest for this chapter: the generalized Cesàro means. For the rest of this work, $n > 1$ will be a fixed natural number and α a real number in the interval $[0,1]$.

Definition 4.5. For a power series $f(z) = \sum_{k=0}^\infty a_k z^k$, the generalized Cesàro operator σ_n^α acts on f as

$$(\sigma_n^\alpha f)(z) = \binom{n+\alpha}{\alpha}^{-1} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} a_k z^k,$$

where the generalized binomial coefficient is defined for a pair of real numbers x, y with $x > y > -1$ as

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}.$$

Here, Γ denotes the Gamma function. We set

$$c_k = c_{k,n}^{\alpha} = \binom{n+\alpha}{\alpha}^{-1}\binom{n-k+\alpha}{\alpha}, \qquad k = 0,\dots,n, \tag{4.7}$$

and $c_k = 0$ otherwise. In the language of Hadamard products, writing $h_n^{\alpha}(z) = \sum_{k=0}^{n} c_k z^k$ and assuming that $T_{h_n^{\alpha}}$ acts on the formal power series, we see that

$$T_{h_n^{\alpha}} = \sigma_n^{\alpha}.$$

We set $\|\sigma_n^{\alpha}\| := \|T_{h_n^{\alpha}}\|$, that is the quantity of interest for this chapter. Let us pay more attention to two special cases.

1. For $\alpha = 0$ and $n \in \mathbb{N}$, the coefficients c_k are equal to 1 for $k = 0,\dots,n$ and then they jump to zero for $k > n$. Hence, σ_n^0 is precisely equal to the n-th partial sum operator

$$S_n f(z) = \sum_{k=0}^{n} a_k z^k,$$

which is the Hadamard product of the Dirichlet kernel D_n with f. It is known that $\|\sigma_n^0\|^2 = n+1$ with the maximizing function $f(z) = nz^{n+1} - (n+1)z^n + 1$.

2. For $\alpha = 1$, we have that $\sigma_n^1 = \sigma_n$ is the Cesàro operator

$$\sigma_n(f) = \sum_{k=0}^{n}\left(1 - \frac{k}{n+1}\right) a_k z^k,$$

which satisfies $\sigma_n(f) = K_n * f$, where K_n is the classical Fejér kernel. It is known that $\|\sigma_n^1\|^2 = n/(n+1)$ with maximizing function $f(z) = z^{n+1} - (n+1)z + n$.

See [38, 39, 42] for further details.

Our main concern in this work is to estimate, as precisely as possible, the quantity $\|\sigma_n^{\alpha}\|$ for different values of the parameter α. As mentioned above, the value $\alpha = 1/2$ is a threshold point, and that is why we have three different theorems with different flavors about the behavior of $\|\sigma_n^{\alpha}\|$ corresponding to whether $\alpha > 1/2$, $\alpha = 1/2$, or $\alpha < 1/2$. In the following, the notation $f(n) \sim g(n)$ means that

$$\lim_{n} \frac{f(n)}{g(n)} = 1.$$

We state our main results.

Theorem 4.6. *Let $\alpha < \frac{1}{2}$. Then*

$$\|\sigma_n^{\alpha}\| \sim C_{\alpha} n^{\frac{1}{2}-\alpha},$$

where

$$C_{\alpha} := \Gamma(\alpha+1)\frac{\Gamma(1-2\alpha)^{1/2}}{\Gamma(1-\alpha)}$$

is a finite positive constant.

Theorem 4.7. *Let $\alpha = \frac{1}{2}$. Then*

$$\|\sigma_n^{\frac{1}{2}}\| \sim \frac{1}{2}\log^{1/2} n.$$

Theorem 4.8. *Let $\alpha \in (\frac{1}{2}, 1)$. Then*

$$\max\left\{1, \frac{\alpha}{(2\alpha-1)^{1/2}}\frac{(2\alpha-1)^{\alpha-1/2}}{(2\alpha)^\alpha}\right\} \leq \liminf_{n\to\infty}\|\sigma_n^\alpha\| \leq \limsup_{n\to\infty}\|\sigma_n^\alpha\| \leq \frac{\alpha}{(2\alpha-1)^{1/2}}.$$

We add to the main theorems the following comments.

1. Theorems 4.6, 4.7 and 4.8 reaffirm that $\sigma_n^\alpha(f) \to f$ in $\mathcal{D}_\omega$ if and only if $\alpha > 1/2$, as it was established in [41]. Additionally, the behavior at the critical case $\alpha = 1/2$ was conjectured and observed numerically in the doctoral dissertation of P. Parisé.

2. For $\alpha < \frac{1}{2}$, by Theorem 4.6 we have that $\|\sigma_n^\alpha\| \sim C_\alpha n^{\frac{1}{2}-\alpha}$, with

$$C_\alpha = \Gamma(\alpha+1)\frac{\Gamma(1-2\alpha)^{1/2}}{\Gamma(1-\alpha)}.$$

   For each fixed $n \geq 1$, interpreting $C_\alpha n^{\frac{1}{2}-\alpha}$ as a continuous function of α, we have that

$$\lim_{\alpha\to 0^+} C_\alpha n^{\frac{1}{2}-\alpha} = n^{\frac{1}{2}},$$

   which is coherent with the fact that the norm of $S_n = \sigma_n^0$ asymptotically behaves as $\sqrt{n}$.

3. Handling the case $\alpha > 1/2$ presented some challenges. For $\alpha \leq 1/2$, the norm $\|\sigma_n^\alpha\|$ diverged as $n \to \infty$, allowing us to isolate a dominant term that represented its growth at infinity. This simplification enabled us to disregard other terms that were overshadowed by the principal one. However, as established and reaffirmed above, whenever $\alpha > 1/2$, the norm $\|\sigma_n^\alpha\|$ remains bounded as n grows. Consequently, all contributing terms become significant, making it more complex to determine the asymptotic behavior of $\|\sigma_n^\alpha\|$ as n increases. The only case for which precise formulas have been derived is $\alpha = 1$ [38].

4.2 Proofs

We give some technical results that will be needed in the proofs of the three theorems previously stated. Here is a simple observation about the eigenvalues of T_c. As a matter of fact, we will just use a special case of (4.8), corresponding to $k = 1$, which also follows from more substantial results in [42].

Lemma 4.9. *Let* $(c_k)_{k\geq 1}$ *be a sequence of complex numbers. Then* c_k, $k \geq 1$, *is an eigenvalue of the matrix* T_c, *with corresponding eigenvector*

$$\mathbf{v}_k = \sum_{i=1}^{k} \mathbf{e}_i,$$

where $\mathbf{e}_i$ *is the sequence given by* $\mathbf{e}_i(j) = \delta_{ij}$, *the Kronecker delta, for* $j \in \mathbb{N}$. *In particular, if* T_c *is bounded on* ℓ^2,

$$\sup_{k\geq 1} |c_k| \leq \|T_c\|_{\mathcal{B}(\ell^2)}. \tag{4.8}$$

Proof. We prove by induction that $T_c\mathbf{v}_k = c_k\mathbf{v}_k$. For $k = 1$, the result is trivial. Assume that it holds for $k \in \mathbb{N}$. Then

$$\begin{aligned} T_c\mathbf{v}_{k+1} = T_c\mathbf{v}_k + T_c\mathbf{e}_{k+1} &= c_k\mathbf{v}_k + \sum_{i=1}^{k}(c_{k+1} - c_k)\mathbf{e}_i + c_{k+1}\mathbf{e}_{k+1} \\ &= c_k\mathbf{v}_k + (c_{k+1} - c_k)\mathbf{v}_k + c_{k+1}\mathbf{e}_{k+1} = c_{k+1}\mathbf{v}_{k+1}. \quad \square \end{aligned}$$

In [42], it was shown that for a polynomial h of degree n,

$$\|T_h\|^2 \leq (n+1)\sum_{k=1}^{n} |c_{k+1} - c_k|^2. \tag{4.9}$$

Additionally, this estimate was complemented in [41] by the lower bounds

$$\|T_h\|^2 \geq m\sum_{k=m}^{n} |c_{k+1} - c_k|^2, \tag{4.10}$$

that hold for every $m \in \{1, \ldots, n\}$. Using these results, we derive explicit asymptotic expressions for generalized Cesàro means. We begin with the following pair of inequalities, originally due to W. Gautschi [32]. For the reader's convenience, a sketch of the proof is provided.

Lemma 4.10 (Gautschi's Inequality)**.** *Let* x *be a positive real number and* $\alpha \in (0, 1)$. *Then*

$$(x+1)^{\alpha-1} < \frac{\Gamma(x+\alpha)}{\Gamma(x+1)} < x^{\alpha-1}. \tag{4.11}$$

Proof. Write $\Gamma(x+\alpha) = \Gamma\big((1-\alpha)x + \alpha(x+1)\big)$. By the strict log-convexity of the Gamma function on the positive real axis [5], we have

$$\Gamma(x+\alpha) < \Gamma(x)^{1-\alpha}\Gamma(x+1)^{\alpha}$$

for $x > 0$ and $\alpha \in (0, 1)$. Hence, by the well-known multiplication formula for the Gamma function, i.e., $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we deduce that

$$\Gamma(x+\alpha) < x^{\alpha-1}\Gamma(x+1),$$

proving the second inequality in (4.11).

For the first inequality, using similar arguments, we see that

$$\begin{aligned}\Gamma(x+1) &= \Gamma\big(\alpha(x+\alpha)+(1-\alpha)(x+\alpha+1)\big)\\ &< \Gamma(x+\alpha)^\alpha\Gamma(x+\alpha+1)^{1-\alpha}\\ &= \Gamma(x+\alpha)(x+\alpha)^{1-\alpha},\end{aligned}$$

concluding the proof. □

Notice that, following the proof, we have actually obtained as a lower bound the quantity $(x+\alpha)^{\alpha-1}$. However, since we are only concerned about the asymptotic behavior as $x\to\infty$, for the sake of simplicity we stick to the original statement. We can now show an optimal upper estimate for the quantity appearing in (4.9).

Lemma 4.11. *Let $\alpha\in(0,1)$, $n>1$, and let c_k be as in (4.7). Let*

$$S := \sum_{k=1}^{n} |c_{k+1}-c_k|^2.$$

Then

$$S \le \Gamma(\alpha+1)^2\frac{(n+1)^{2-2\alpha}}{(n+\alpha)^2}\left(1+\frac{(n-1)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)}+\frac{2\alpha-2}{\Gamma(\alpha)^2(2\alpha-1)}\right), \quad \text{if} \quad \alpha\neq\frac{1}{2}, \tag{4.12}$$

and

$$S \le \frac{\pi}{4}\frac{n+1}{(n+\frac{1}{2})^2}\left(1+\frac{1}{\pi}(\log(n-1)+1)\right), \quad \text{if} \quad \alpha=\frac{1}{2}. \tag{4.13}$$

Proof. Since $c_{n+1}=0$ and all the c_k's are real, then

$$S = c_n^2+\sum_{k=1}^{n-1}(c_{k+1}-c_k)^2.$$

For $k=1,\dots,n-1$,

$$\begin{aligned}c_{k+1}-c_k &= \binom{n+\alpha}{\alpha}^{-1}\left(\binom{n-k-1+\alpha}{\alpha}-\binom{n-k+\alpha}{\alpha}\right)\\ &= \binom{n+\alpha}{\alpha}^{-1}\frac{1}{\Gamma(\alpha+1)}\left(\frac{\Gamma(n-k+\alpha)}{\Gamma(n-k)}-\frac{\Gamma(n-k+\alpha+1)}{\Gamma(n-k+1)}\right)\\ &= c_{k+1}\left(1-\frac{n-k+\alpha}{n-k}\right)\\ &= -\frac{\alpha}{n-k}c_{k+1}.\end{aligned}$$

Hence,

$$\begin{aligned}S &= c_n^2\left(1+\frac{\alpha^2}{\Gamma(\alpha+1)^2}\sum_{k=1}^{n-1}\frac{1}{(n-k)^2}\frac{\Gamma(n-k+\alpha)^2}{\Gamma(n-k)^2}\right)\\ &= c_n^2\left(1+\frac{1}{\Gamma(\alpha)^2}\sum_{k=1}^{n-1}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2}\right).\end{aligned}$$

We start estimating c_n. We have

$$c_n^2 = \binom{n+\alpha}{\alpha}^{-2} = \frac{\Gamma(\alpha+1)^2\Gamma(n+1)^2}{\Gamma(n+\alpha+1)^2} < \Gamma(\alpha+1)^2 \frac{(n+1)^{2-2\alpha}}{(n+\alpha)^2}, \tag{4.14}$$

where we used Gautschi's inequality (4.11). On the other hand, once more by (4.11),

$$\sum_{k=1}^{n-1} \frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} < \sum_{k=1}^{n-1} k^{2\alpha-2}.$$

Now, estimating the sum with the corresponding integral, we see that

$$\sum_{k=1}^{n-1} \frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} < 1 + \sum_{k=2}^{n-1} k^{2\alpha-2} \leq 1 + \int_1^{n-1} x^{2\alpha-2}\, dx.$$

At this point, we have to distinguish between two cases. If $\alpha \neq \frac{1}{2}$, then

$$\sum_{k=1}^{n-1} \frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} < \frac{(n-1)^{2\alpha-1}}{2\alpha-1} + \frac{2\alpha-2}{2\alpha-1},$$

and therefore (4.12) is proved. If $\alpha = \frac{1}{2}$, we have

$$\sum_{k=1}^{n-1} \frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} < \log(n-1) + 1,$$

which gives (4.13). We also exploit the well-known identities

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

□

Similarly, we provide a lower estimate for the quantity in (4.10).

Lemma 4.12. *Let $\alpha \in (0,1)$, $n > 1$, let c_k be as in (4.7), and let m be a natural number with $1 \leq m < n$. Let*

$$\tilde{S}_m := \sum_{k=m}^{n} |c_{k+1} - c_k|^2.$$

Then

$$\tilde{S}_m > \Gamma(\alpha+1)^2 \frac{n^{2-2\alpha}}{(n+\alpha)^2} \left(1 + \frac{(n-m+2)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)} - \frac{2^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)}\right), \quad \textit{if} \quad \alpha \neq \frac{1}{2}, \tag{4.15}$$

and

$$\tilde{S}_m > \frac{\pi}{4} \frac{n}{(n+\frac{1}{2})^2} \left(1 + \frac{1}{\pi}\big(\log(n-m+2) - \log 2\big)\right), \quad \textit{if} \quad \alpha = \frac{1}{2}. \tag{4.16}$$

Proof. This proof has the same flavor as the previous proof, but we are interested in the opposite inequalities. For $m < n$,

$$\begin{aligned}\tilde{S}_m &= c_n^2\left(1+\frac{1}{\Gamma(\alpha)^2}\sum_{k=m}^{n-1}\frac{\Gamma(n-k+\alpha)^2}{\Gamma(n-k+1)^2}\right)\\ &= c_n^2\left(1+\frac{1}{\Gamma(\alpha)^2}\sum_{k=1}^{n-m}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2}\right).\end{aligned}$$

For the coefficient c_n, we have

$$c_n^2 = \frac{\Gamma(\alpha+1)^2\Gamma(n+1)^2}{\Gamma(n+\alpha+1)^2} > \Gamma(\alpha+1)^2\frac{n^{2-2\alpha}}{(n+\alpha)^2}. \tag{4.17}$$

Note that in (4.10) we were allowed to pick $m = n$. However, in this case $\tilde{S}_n = c_n^2$, and the required estimate is precisely the established inequality (4.17). Then

$$\sum_{k=1}^{n-m}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} > \sum_{k=1}^{n-m}(k+1)^{2\alpha-2} \geq \int_1^{n-m+1}(x+1)^{2\alpha-2}\,dx,$$

and we conclude the proof evaluating the integrals in the cases $\alpha \neq \frac{1}{2}$ and $\alpha = \frac{1}{2}$. □

Note that comparing (4.14) and (4.17), we obtain the asymptotic

$$c_n^2 \sim \frac{\Gamma(\alpha+1)^2}{n^{2\alpha}}. \tag{4.18}$$

Proof of Theorem 4.7. By (4.9) and (4.13), we have

$$\frac{\|\sigma_n^{\frac{1}{2}}\|^2}{\log n} \leq \frac{\pi}{4}\frac{(n+1)^2}{(n+\frac{1}{2})^2}\left(\frac{1}{\log n}+\frac{\log(n-1)}{\pi\log n}+\frac{1}{\pi\log n}\right),$$

revealing that

$$\limsup_n \frac{\|\sigma_n^{\frac{1}{2}}\|^2}{\log n} \leq \frac{1}{4}.$$

Now, let $\gamma \in (0,1)$, and set

$$m := \left[\frac{n-1}{2^\gamma}\right],$$

where $[x] := \max\{k \in \mathbb{Z} \colon k \leq x\}$ denotes the integer part of the real number x. Then we have

$$m \leq \frac{n-1}{2^\gamma} < n-1$$

and that, at least for $n \geq 7$,

$$m \geq \frac{n-1}{2^\gamma} - 1 \geq \frac{n-1}{2} - 1 = \frac{n-3}{4} \geq 1.$$

Therefore, for every $n \geq 7$, equations (4.10) and (4.16) yield

$$\begin{aligned}\|\sigma_n^{\frac{1}{2}}\|^2 &> \frac{\pi}{4}\frac{mn}{(n+\frac{1}{2})^2}\left(1+\frac{1}{\pi}\big(\log(n-m+2)-\log 2\big)\right)\\ &\geq \frac{\pi}{4}\left(\frac{n-1}{2^\gamma}-1\right)\frac{n}{(n+\frac{1}{2})^2}\left(1+\frac{1}{\pi}\left(\log\left(\frac{n(2^\gamma-1)+1+2^{\gamma+1}}{2^\gamma}\right)-\log 2\right)\right).\end{aligned}$$

In particular, for every $\gamma \in (0,1)$,

$$\liminf_n \frac{\|\sigma_n^{\frac{1}{2}}\|^2}{\log n} \geq \frac{1}{2^\gamma}\frac{1}{4},$$

and the theorem follows taking the limit as $\gamma \to 0^+$.

□

Proof of Theorem 4.6. By Gautschi's inequality (4.11),

$$\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} < k^{2(\alpha-1)}, \qquad k \geq 1. \tag{4.19}$$

Since in this case $0 < \alpha < \frac{1}{2}$, we conclude that

$$\sum_{k=1}^{\infty}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} < \infty.$$

In fact, we can go further and observe that

$$C_\alpha := \frac{1}{\Gamma(\alpha)^2}\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} = \sum_{k=0}^{\infty}\binom{k+\alpha-1}{\alpha-1}^2 = \sum_{k=0}^{\infty}\binom{k+\alpha-1}{k}^2$$

is the H^2-norm of the function $\sum_{k=0}^{\infty}\binom{k+\alpha-1}{k}z^k = (1-z)^{-\alpha}$ ($|z|<1$), so by Parseval's theorem we have that

$$\frac{1}{\Gamma(\alpha)^2}\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|1-e^{i\theta}|^{-2\alpha}\,d\theta.$$

This can be rewritten as the double integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|1-e^{i\theta}|^{-2\alpha}\,d\theta = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|e^{i\theta_1}-e^{i\theta_2}|^{-2\alpha}\,d\theta_1 d\theta_2,$$

by rotational invariance in the θ_2 coordinate. We obtained the Morris integral [44] for $n = 2$ (or a version of the Selberg integral [56, 57, 58]), that equals

$$\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|e^{i\theta_1}-e^{i\theta_2}|^{-2\alpha}\,d\theta_1 d\theta_2 = \frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)^2}.$$

Now, on the one hand, by (4.9),

$$\|\sigma_n^\alpha\|^2 \le (n+1)\sum_{k=1}^{n}|c_{k+1}-c_k|^2 = (n+1)c_n^2\left(1+\frac{1}{\Gamma(\alpha)^2}\sum_{k=1}^{n-1}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2}\right).$$

By the asymptotic (4.18), we have

$$\limsup_n \frac{\|\sigma_n^\alpha\|^2}{n^{1-2\alpha}} \le \Gamma(\alpha+1)^2\left(1+\frac{1}{\Gamma(\alpha)^2}\sum_{k=1}^{\infty}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2}\right) = \Gamma(\alpha+1)^2\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)^2}.$$

On the other hand, choose

$$m := \left[\frac{n-1}{2^\gamma}\right],$$

where $[\cdot]$ denotes the integer part and $\gamma \in (0,1)$. Then, by (4.10), we obtain

$$\|\sigma_n^\alpha\|^2 \ge m\tilde{S}_m = mc_n^2\left(1+\frac{1}{\Gamma(\alpha)^2}\sum_{k=1}^{n-m}\frac{\Gamma(k+\alpha)^2}{\Gamma(k+1)^2}\right).$$

Notice that $m \ge \frac{n-1}{2^\gamma} - 1$ and

$$n-m \ge n - \frac{n-1}{2^\gamma} = \frac{(2^\gamma-1)n+1}{2^\gamma},$$

so that $n-m \to +\infty$ as $n \to +\infty$, and for every $\gamma \in (0,1)$ it holds

$$\liminf_n \frac{\|\sigma_n^\alpha\|^2}{n^{1-2\alpha}} \ge \frac{\Gamma(\alpha+1)^2}{2^\gamma}\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)^2}.$$

Taking the limit as $\gamma \to 0^+$, we conclude the proof.

□

Proof of Theorem 4.8. By (4.9) and (4.12),

$$\begin{aligned}\|\sigma_n^\alpha\|^2 &\le (n+1)\sum_{k=1}^{n}|c_{k+1}-c_k|^2 \\ &\le \Gamma(\alpha+1)^2\frac{(n+1)^{3-2\alpha}}{(n+\alpha)^2}\left(1+\frac{(n-1)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)}+\frac{2\alpha-2}{\Gamma(\alpha)^2(2\alpha-1)}\right).\end{aligned}$$

In particular, since $\frac{1}{2} < \alpha < 1$,

$$\limsup_n \|\sigma_n^\alpha\|^2 \le \frac{\alpha^2}{2\alpha-1}.$$

For the other direction, by Lemma 4.9,

$$\|\sigma_n^\alpha\|^2 \ge |c_1|^2 = \frac{\Gamma(n+1)^2}{\Gamma(n+\alpha+1)^2}\frac{\Gamma(n+\alpha)^2}{\Gamma(n)^2} = \frac{n^2}{(n+\alpha)^2},$$

and thus $\liminf_n \|\sigma_n^\alpha\|^2 \geq 1$. Moreover, by (4.10) and (4.15), we have

$$\|\sigma_n^\alpha\|^2 \geq m\tilde{S}_m > m\Gamma(\alpha+1)^2 \frac{n^{2-2\alpha}}{(n+\alpha)^2}\left(1 + \frac{(n-m+2)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)} - \frac{2^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)}\right),$$

for all $m \in \{1,\dots,n\}$. Put $m = \left[\frac{n-1}{2\alpha}\right]$. This particular choice yields

$$\|\sigma_n^\alpha\|^2 \geq \frac{n-1-2\alpha}{2\alpha}\Gamma(\alpha+1)^2 \frac{n^{2-2\alpha}}{(n+\alpha)^2}\left(1 + \frac{\left(\frac{(2\alpha-1)n+3}{2\alpha}\right)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)} - \frac{2^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)}\right),$$

so that

$$\liminf_n \|\sigma_n^\alpha\|^2 \geq \frac{1}{2\alpha}\Gamma(\alpha+1)^2\left(\frac{2\alpha-1}{2\alpha}\right)^{2\alpha-1}\frac{1}{\Gamma(\alpha)^2(2\alpha-1)} = \frac{\alpha^2(2\alpha-1)^{2\alpha-2}}{(2\alpha)^{2\alpha}}.$$

□

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