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SPDEs with fractional noise in space: Continuity in law with respect to the Hurst index

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In this article, we consider the quasi-linear stochastic wave and heat equations on the real line and with an additive Gaussian noise which is white in time and behaves in space like a fractional Brownian motion with Hurst index $H \in (0, 1)$. The drift term is assumed to be globally Lipschitz. We prove that the solution of each of the above equations is continuous in terms of the index H, with respect to the convergence in law in the space of continuous functions.

Keywords: fractional noise; stochastic heat equation; stochastic wave equation; weak convergence

1. Introduction

We consider the following stochastic wave and heat equations on $[0, \infty) \times \mathbb{R}$, respectively:

$$\begin{cases} \dfrac{\partial^2 u^H}{\partial t^2}(t,x) = \dfrac{\partial^2 u^H}{\partial x^2}(t,x) + b\big(u^H(t,x)\big) + \dot{W}^H(t,x), \\ u^H(0,x) = u_0(x), \quad x \in \mathbb{R}, \\ \dfrac{\partial u^H}{\partial t}(0,x) = v_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{SWE}$$

and

$$\begin{cases} \dfrac{\partial u^H}{\partial t}(t,x) = \dfrac{\partial^2 u^H}{\partial x^2}(t,x) + b\big(u^H(t,x)\big) + \dot{W}^H(t,x), \\ u^H(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{SHE}$$

The initial conditions $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ are deterministic measurable functions which satisfy some regularity conditions specified below. The drift coefficient $b : \mathbb{R} \to \mathbb{R}$ is assumed to be globally Lipschitz.

The term $\dot{W}^H(t,x)$ stands for a random perturbation that is supposed to be a Gaussian noise which is white in time and has a spatially homogeneous correlation of fractional type. More precisely, on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the noise $\dot{W}^H$ is defined by a family of centered Gaussian random variables $\{W^H(\varphi), \varphi \in \mathcal{D}\}$, where $\mathcal{D} := C_0^\infty([0,\infty)\times\mathbb{R})$ is the space of infinitely differentiable functions with compact support, with covariance functional

$$\mathrm{E}\big[W^H(\varphi)W^H(\psi)\big] = \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\mu_H(d\xi)\,dt, \tag{1}$$

for all $\varphi, \psi \in \mathcal{D}$, where $\mathcal{F}$ denotes the Fourier transform in the space variable. For any $H \in (0,1)$, the *spectral* measure μ_H is given by

$$\mu_H(d\xi) := c_H|\xi|^{1-2H}\,d\xi, \qquad c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}. \tag{2}$$

The above covariance relation, as in [10], is used to construct an inner product on the space $\mathcal{D}$ defined in the following way:

$$\langle \varphi, \psi\rangle_H := \mathrm{E}\big[W^H(\varphi)W^H(\psi)\big], \quad \varphi, \psi \in \mathcal{D}.$$

Let $\mathcal{H}^H$ be the completion of $\mathcal{D}$ with respect to the inner product $\langle\cdot,\cdot\rangle_H$, which will be the natural space of deterministic integrands with respect to W^H. Indeed, our noise can be extended to a centered Gaussian family $\{W^H(g), g \in \mathcal{H}^H\}$ indexed on the Hilbert space $\mathcal{H}^H$ and satisfying

$$\mathrm{E}\big[W^H(g_1)W^H(g_2)\big] = \langle g_1, g_2\rangle_H.$$

As usual, for any $g \in \mathcal{H}^H$, we say that $W^H(g)$ is the Wiener integral of g and we denote it by

$$\int_0^\infty \int_{\mathbb{R}} g(t,x)W^H(dt,dx) := W^H(g).$$

The space $\mathcal{H}^H$ contains all functions g such that its Fourier transform in the space variable satisfies (see [4], Thm. 2.7, and [10], Prop. 2.9):

$$\int_0^\infty \int_{\mathbb{R}} \big|\mathcal{F}g(t,\cdot)(\xi)\big|^2|\xi|^{1-2H}\,d\xi\,dt < \infty.$$

In particular, the space $\mathcal{H}^H$ contains all elements of the form $1_{[0,t]\times[0,x]}$, with $t > 0$ and $x \in \mathbb{R}$. Then, the following random field is naturally associated to our noise W^H:

$$X^H(t,x) := W^H(1_{[0,t]\times[0,x]}).$$

As a consequence of the representation in law of the fractional Brownian motion as a Wiener type integral with respect to a complex Brownian motion (see, for instance, [17], p. 257), we have that

$$\begin{aligned}
\mathrm{E}\big[X^H(t,x)X^H(s,y)\big] &= \int_0^\infty \int_{\mathbb{R}} \mathcal{F}1_{[0,t]\times[0,x]}(r,\cdot)(\xi)\overline{\mathcal{F}1_{[0,s]\times[0,y]}(r,\cdot)(\xi)}\mu_H(d\xi)\,dt \\
&= \int_0^{t\wedge s} \int_{\mathbb{R}} \mathcal{F}1_{[0,x]}(\xi)\overline{\mathcal{F}1_{[0,y]}(\xi)}\mu_H(d\xi)\,dt \\
&= \frac{1}{2}(t\wedge s)\big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big).
\end{aligned}$$

This is the covariance of a standard Brownian motion in the time variable, while in the space variable we have obtained the covariance of a fractional Brownian motion with Hurst parameter H.

We denote by $(\mathcal{F}_t^H)_{t\geq 0}$ the filtration generated by W^H, namely

$$\mathcal{F}_t^H := \sigma\big(W^H(1_{[0,s]}\varphi), s \in [0,t], \varphi \in C_0^\infty(\mathbb{R})\big) \vee \mathcal{N}, \tag{3}$$

where $\mathcal{N}$ denotes the class of $\mathbb{P}$-null sets in $\mathcal{F}$.

The solution to equations (SWE) and (SHE) will be interpreted in the *mild* sense. That is, for any $T > 0$, we say that an adapted and jointly measurable process $u^H = \{u^H(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ solves (SWE) (resp. (SHE)) if, for all $(t,x) \in [0,T] \times \mathbb{R}$, it holds

$$\begin{aligned} u^H(t,x) = I_0(t,x) &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b\big(u^H(s,y)\big)\, dy\, ds, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{4}$$

Here, the function $G_t(x)$ is the fundamental solution of the wave (resp. heat) equation in $\mathbb{R}$, and $I_0(t,x)$ is the solution of the corresponding deterministic linear equation. These are given by

$$I_0(t,x) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} v_0(y)\, dy + \dfrac{1}{2}\big(u_0(x+t) - u_0(x-t)\big), & \text{wave equation}, \\ \displaystyle\int_{\mathbb{R}} G_t(x-y) u_0(y)\, dy, & \text{heat equation}, \end{cases} \tag{5}$$

and

$$G_t(x) = \begin{cases} \dfrac{1}{2} 1_{|x|<t}(x), & \text{wave equation}, \\ \dfrac{1}{(2\pi t)^{1/2}} \exp\left(-\dfrac{|x|^2}{2t}\right), & \text{heat equation}. \end{cases} \tag{6}$$

Our main objective consists in studying the continuity in law, in the space $C([0,T] \times \mathbb{R})$ of continuous functions, of the solution u^H to equations (SWE) and (SHE) with respect to the Hurst index $H \in (0,1)$. More precisely, we fix $H_0 \in (0,1)$ and we will provide sufficient conditions on the initial data under which, whenever $H \to H_0$, the $C([0,T] \times \mathbb{R})$-valued random variable u^H converges in law to u^{H_0} (cf. Theorem 4.1). Recall that the parameter H quantifies the regularity of the random perturbation W^H, and hence the level of noise in the system. So we will study the probabilistic behavior of the solution in terms of H, aiming at showing that the sensitivity in H implies the corresponding convergence of the solutions.

We note that continuity in law with respect to fractionality indices has been studied in other related contexts. We refer the reader to [14–16] for results involving symmetric, Wiener and multiple integrals with respect to fractional Brownian motion, respectively, while in [13,19] the convergence in law of the local time of the fractional Brownian motion and of anisotropic Gaussian random fields has been considered, respectively. Finally, in the recent paper [1], the continuity in law for some additive functionals of the sub-fractional Brownian motion has been studied.

In order to tackle our main objective, we start by focusing on the linear version of equations (SWE) and (SHE). That is, we consider the case where $b = 0$. Here, we first prove existence and uniqueness of solution, together with the existence of a continuous modification, for any $H \in (0, 1)$ (cf. Theorem 2.1). So, for the particular case of (SWE) and (SHE), this result puts together the more general ones of [3] (valid for $H \leq \frac{1}{2}$) and [9] (valid for $H > \frac{1}{2}$). The convergence in law of u^H to u^{H_0} reduces to analyze the convergence of the corresponding stochastic convolutions, which are centered Gaussian processes. For this, we first check that the corresponding family of probability laws is tight in the space $C([0, T] \times \mathbb{R})$, and then we identify the limit law by characterizing the underlying Gaussian candidate for the limit (see Theorem 2.8 for details). Finally, we point out that in the linear case, the proof of the main convergence result holds for both wave and heat equations.

We remark that there are several well-posedness results for equations (SWE) and (SHE) with $b = 0$ and a more general noise term, namely of the form $\sigma(u(t, x))\dot{W}^H(t, x)$, for some function $\sigma : \mathbb{R} \to \mathbb{R}$: if $H < \frac{1}{2}$, we refer the reader to, for example, [4,11], while the case $H \geq \frac{1}{2}$ falls in the general framework of Walsh and Dalang (see, for example, [9,10,18]). When $H < \frac{1}{2}$, most of the existing work focuses on the particular coefficient $\sigma(z) = z$, which corresponds to the so-called Hyperbolic Anderson Model (HAM) and the Parabolic Anderson Model (PAM), respectively (see [4,5,12] and references therein). In these cases, the fact that $H < \frac{1}{2}$ entails important technical difficulties in order to define stochastic integrals with respect to the noise W^H. Moreover, as proved in [5], Prop.3.7, the above equations admit a unique solution if and only if $H > \frac{1}{4}$. In the present article, we do not encounter such issues since the noise appears in the equations in an additive way. Indeed, we plan to address the convergence in law with respect to H for the HAM and PAM in a separate publication, where the underlying stochastic integrals are interpreted in the Skorohod sense.

We turn now to the study of the quasi-linear case, that is assuming that b is a general Lipschitz function. Here, we first prove that equations (SWE) and (SHE) admit a unique solution (see Theorem 3.1). This result holds for any $H \in (0, 1)$ and, as far as we know, is new for the case $H < \frac{1}{2}$ (if $H > \frac{1}{2}$, it follows from [10], Thm. 4.3). Moreover, we note that the proof of Theorem 3.1 can be built in a unified way for both wave and heat equations.

Nevertheless, the analysis of the weak convergence in the quasi-linear case does not admit a unified proof for wave and heat equations. More precisely, for the wave equation, the convergence in law of u^H to u^{H_0}, whenever $H \to H_0$, follows from a pathwise argument: we prove that, for almost all ω, the solution of (SWE) can be seen as the image of the stochastic convolution through a certain continuous functional $F : C([0, T] \times \mathbb{R}) \to C([0, T] \times \mathbb{R})$. In the case of the heat equation, this argument cannot be directly applied, for the associated deterministic equation which has to be solved in order to define the above-mentioned functional is not well-posed for a general coefficient b. We overcome this difficulty by first assuming that b is a bounded function and then using a truncation argument. As it will be exhibited in Section 4.3, this part of the paper contains most of the technical difficulties that we need to face. It is also worthy to point out that, in the analysis of the wave equation and the heat equation with bounded b, we have established ad hoc versions of Grönwall lemma which have been crucial to complete the corresponding proofs (see, respectively, Lemmas 4.2 and 4.4).

This article is organized as follows. Section 2 is devoted to study the convergence in law for equations (SWE) and (SHE) in the linear additive case (i.e., $b = 0$). In Section 3, existence, uniqueness and pathwise Hölder continuity in the quasi-linear additive case are established. Finally, the main result on weak convergence for the quasi-linear case is proved in Section 4: here we treat separately the case of the wave equation (Section 4.1), the heat equation with b bounded (Section 4.2) and the heat equation with general b (Section 4.3).

When we make use of the constant C, we are meaning that the value of that constant is not relevant for our computations, and also that it can change its value from line to line. When two constants (possibly different) appear on the same line, we will call them C_1, C_2. Sometimes we use C_p when we want to stress that the constant depends on some exponent p.

2. Weak convergence for the linear additive case

In this section, we consider equations (SWE) and (SHE) in the case where the drift term vanishes, that is $b = 0$. Then, the mild formulation (4) reads

$$u^H(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy), \tag{7}$$

where we recall that the term I_0 and the fundamental solution G have been defined in (5) and (6), respectively. Throughout this section, we assume that $H \in (0,1)$. Notice that (7) is now an explicit formula for the solution u^H. We consider the following hypotheses on the initial data:

Hypothesis A. *It holds that*

(a) Wave equation: u_0 *is continuous and* $v_0 \in L^1_{\text{loc}}(\mathbb{R})$.
(b) Heat equation: u_0 *is continuous and bounded*.

It can be easily verified that the above conditions on the initial data imply that $I_0 : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function. On the other hand, the stochastic convolution in (7) is a well-defined centered Gaussian random variable since, for any $(t,x) \in [0,T] \times \mathbb{R}$,

$$\begin{aligned} \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy)\right|^2\right] &= \int_0^t \int_{\mathbb{R}} \left|\mathcal{F}G_{t-s}(\xi)\right|^2 |\xi|^{1-2H}\, d\xi\, ds \\ &\le \int_0^T \int_{\mathbb{R}} \left|\mathcal{F}G_s(\xi)\right|^2 |\xi|^{1-2H}\, d\xi\, ds < \infty, \end{aligned}$$

where we have applied Lemma 2.4 below. Hence, we have the following result.

Theorem 2.1. *Assume that Hypothesis A holds and let $H \in (0,1)$. Then, there exists a unique solution $u^H = \{u^H(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ of equation (7). Moreover, the random field u^H admits a modification with continuous sample paths.*

Proof. We only need to prove that u^H has a modification with continuous paths. Indeed, since I_0 is deterministic and continuous, we check that the stochastic convolution $\tilde{u}^H(t,x) := u^H(t,x) - I_0(t,x)$ admits a continuous modification. This is a direct consequence of Step 1 in the proof of

Theorem 2.8 below. More precisely, for any $p \geq 2$, there exists a constant C (depending only on p) such that, for all $t, t' \in [0, T]$ and $x, x' \in \mathbb{R}$, it holds

$$\mathrm{E}\left[\left|\tilde{u}^H(t,x) - \tilde{u}^H\left(t',x'\right)\right|^p\right] \leq C\left\{\left|t-t'\right|^{\alpha p} + \left|x-x'\right|^{pH}\right\},$$

where $\alpha = H$ for the wave equation and $\alpha = \frac{H}{2}$ for the heat equation. An application of Kolmogorov's continuity criterion concludes the proof. □

Remark 2.2. In the case of the heat equation, the assumptions of Theorem 2.1 indeed imply that, for all $p \geq 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left|u^H(t,x)\right|^p\right] < \infty.$$

For the wave equation, this property can be obtained by slightly strengthening the hypotheses of u_0 and v_0, for example, assuming that they are bounded functions (see [10], Lem. 4.2).

Remark 2.3. The proof of Theorem 2.1 implies that the stochastic convolution in equation (7) has a modification which is (locally) β_1-Hölder continuous in time for any $\beta_1 \in (0, \alpha)$ and (locally) β_2-Hölder continuous in space for any $\beta_2 \in (0, H)$.

In the proof of the main result of the present section (cf. Theorem 2.8), we will need the following three technical lemmas (proved in [4]). They provide explicit estimates, depending on H, of the norm in the space $L^2(\mathbb{R}; \mu^H)$ of the Fourier transforms of the fundamental solutions of the deterministic wave and heat equations, where we recall that, respectively:

$$\mathcal{F}G_t(\xi) = \frac{\sin(t|\xi|)}{|\xi|} \quad \text{and} \quad \mathcal{F}G_t(\xi) = \exp\left(\frac{-t\xi^2}{2}\right), \qquad t > 0, \xi \in \mathbb{R}. \tag{8}$$

In the following three lemmas, we will denote either one of these two functions by $\mathcal{F}G_t(\xi)$. We recall that the spatial spectral measure is given by $\mu^H(d\xi) = c_H|\xi|^{1-2H}\,d\xi$ (see (2)).

Lemma 2.4 ([4], Lemma 3.1). *Let $T > 0$. Then, the integral*

$$A_T(\alpha) := \int_0^T \int_{\mathbb{R}} \left|\mathcal{F}G_t(\xi)\right|^2 |\xi|^\alpha \,d\xi\,dt$$

converges if and only if $\alpha \in (-1, 1)$. In this case, it holds:

$$A_T(\alpha) = \begin{cases} 2^{1-\alpha} C_\alpha \dfrac{1}{2-\alpha} T^{2-\alpha} & \textit{for the wave equation,} \\ \dfrac{2}{1-\alpha} \Gamma\left(\dfrac{\alpha+1}{2}\right) T^{(1-\alpha)/2} & \textit{for the heat equation,} \end{cases}$$

where the constant C_α *is given by*

$$C_\alpha = \begin{cases} \dfrac{\Gamma(\alpha)}{1-\alpha}\sin(\pi\alpha/2), & \alpha \in (-1,1)\setminus\{0\}, \\ \dfrac{\pi}{2}, & \alpha = 0. \end{cases}$$

Lemma 2.5 ([4], Lemma 3.4). *Let* $T>0$ *and* $\alpha \in (-1,1)$. *Then, for any* $h>0$, *it holds*:

$$\int_0^T \int_{\mathbb{R}} \big(1-\cos(\xi h)\big)\big|\mathcal{F}G_t(\xi)\big|^2 |\xi|^\alpha \, d\xi \, dt \le \begin{cases} C|h|^{1-\alpha} & \text{for the heat equation,} \\ CT|h|^{1-\alpha} & \text{for the wave equation,} \end{cases}$$

where $C = \int_{\mathbb{R}} (1-\cos\eta)|\eta|^{\alpha-2}\, d\eta$.

Lemma 2.6 ([4], Lemma 3.5). *Let* $T>0$ *and* $\alpha \in (-1,1)$. *Then, for any* $h>0$, *it holds*:

$$\int_0^T \int_{\mathbb{R}} \big|\mathcal{F}G_{t+h}(\xi) - \mathcal{F}G_t(\xi)\big|^2 |\xi|^\alpha \, d\xi \, dt \le \begin{cases} C_\alpha |h|^{(1-\alpha)/2} & \text{for the heat equation,} \\ C_\alpha T|h|^{1-\alpha} & \text{for the wave equation,} \end{cases}$$

where

$$C_\alpha = \int_{\mathbb{R}} \frac{(1-e^{-\eta^2/2})^2}{|\eta|^{2-\alpha}}\, d\eta \quad \text{for the heat equation,} \quad \text{and}$$

$$C_\alpha = 4\int_{\mathbb{R}} \frac{\min(1,|\eta|^2)}{|\eta|^{2-\alpha}}\, d\eta \quad \text{for the wave equation.}$$

We will also make use of the following tightness criterion in the plane (see [20], Prop. 2.3):

Theorem 2.7. *Let* $\{X_\lambda\}_{\lambda\in\Lambda}$ *be a family of random functions indexed on the set* Λ *and taking values in the space* $C([0,T]\times\mathbb{R})$, *in which we consider the metric of uniform convergence over compact sets. Then, the family* $\{X_\lambda\}_{\lambda\in\Lambda}$ *is tight if, for any compact set* $J\subset\mathbb{R}$, *there exist* $p', p > 0$, $\delta > 2$, *and a constant* C *such that the following holds for any* $t', t \in [0,T]$ *and* $x', x \in J$:

(i) $\sup_{\lambda\in\Lambda} \mathrm{E}[|X_\lambda(0,0)|^{p'}] < \infty$,
(ii) $\sup_{\lambda\in\Lambda} \mathrm{E}[|X_\lambda(t',x') - X_\lambda(t,x)|^p] \le C(|t'-t| + |x'-x|)^\delta$.

We are now in position to state and prove the main result of this section.

Theorem 2.8. *Consider a family* $\{u^{H_n}\}_{n\ge1}$ *of solutions of equation* (SWE) *or* (SHE), *and suppose that the Hurst indexes* $H_n \to H_0 \in (0,1)$, *as* $n\to\infty$. *Then* $u^{H_n} \xrightarrow{d} u^{H_0}$, *as* $n\to\infty$, *where the convergence holds in distribution in the space* $C([0,T]\times\mathbb{R})$, *where the latter is endowed with the metric of uniform convergence on compact sets.*

Proof. We split the proof in two steps. In the first one, we prove that the sequence of stochastic convolutions is tight in $C([0,T]\times\mathbb{R})$, while the second step is devoted to the identification of the limit law.

Step 1: Since $H_n \to H_0$, the sequence $\{H_n\}$ is contained in a compact set $K\subset(0,1)$. For a fixed $H\in(0,1)$, we have that the solution u^H is expressed as

$$u^H(t,x)=I_0(t,x)+\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W^H(ds,dy).$$

We will apply Theorem 2.7 to the family $\{\tilde{u}^H=u^H-I_0\}_{H\in K}$ of stochastic convolutions:

$$\tilde{u}^H(t,x)=u^H(t,x)-I_0(t,x)=\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W^H(ds,dy).$$

We write then, supposing without loss of generality that $t'\ge t$ and $x'\ge x$:

$$\begin{aligned}\tilde{u}^H(t',x')-\tilde{u}^H(t,x)=&\int_t^{t'}\int_{\mathbb{R}} G_{t'-s}(x'-y)W^H(ds,dy)\\&+\int_0^t\int_{\mathbb{R}}\big[G_{t'-s}(x'-y)-G_{t-s}(x-y)\big]W^H(ds,dy).\end{aligned}$$

Thus, we have

$$\mathrm{E}\big[\big|u(t,x)-u(t',x')\big|^p\big]\le C_p(I_1+I_2),$$

where I_1, I_2 are defined as:

$$\begin{aligned}I_1&:=\mathrm{E}\bigg[\bigg|\int_t^{t'}\int_{\mathbb{R}} G_{t'-s}(x'-y)W^H(ds,dy)\bigg|^p\bigg],\\ I_2&:=\mathrm{E}\bigg[\bigg|\int_0^t\int_{\mathbb{R}}\big[G_{t-s}(x-y)-G_{t'-s}(x'-y)\big]W^H(ds,dy)\bigg|^p\bigg].\end{aligned}$$

Since I_1 is the moment of order p of a centered Gaussian random variable, we have

$$\begin{aligned}I_1&=\mathrm{E}\bigg[\bigg|\int_0^T\int_{\mathbb{R}} 1_{[t,t']}(s)G_{t'-s}(x'-y)W^H(ds,dy)\bigg|^p\bigg]\\&=z_p c_H^{p/2}\bigg[\int_0^T 1_{[t,t']}(s)\int_{\mathbb{R}}\big|\mathcal{F}G_{t'-s}(x'-\cdot)(\xi)\big|^2|\xi|^{1-2H}\,d\xi\,ds\bigg]^{p/2}\\&=z_p c_H^{p/2}\bigg[\int_t^{t'}\int_{\mathbb{R}}\big|\mathcal{F}G_{t'-s}(x'-\cdot)(\xi)\big|^2|\xi|^{1-2H}\,d\xi\,ds\bigg]^{p/2}\\&=z_p c_H^{p/2}\bigg[\int_0^{t'-t}\int_{\mathbb{R}}\big|\mathcal{F}G_{s'}(\xi)\big|^2|\xi|^{1-2H}\,d\xi\,ds'\bigg]^{p/2}.\end{aligned}\tag{9}$$

Notice that we have used the standard properties of Fourier transform in the space variable, and we performed the change of variable $s' = t' - s$. The constant z_p is the p-order moment of a standard normal distribution and c_H is given by (2).

Now we apply Lemma 2.4 and obtain

$$I_1 \leq \begin{cases} z_p c_H^{p/2}\Big[2^{2H}\tilde{C}_{1-2H}\dfrac{1}{1+2H}\big(t'-t\big)^{1+2H}\Big]^{p/2}, & \text{wave equation,} \\ z_p c_H^{p/2}\Big[\dfrac{1}{H}\Gamma(1-H)\big(t'-t\big)^{H}\Big]^{p/2}, & \text{heat equation.} \end{cases} \tag{10}$$

The above constant $\tilde{C}_{1-2H}$ is the one of Lemma 2.4:

$$\tilde{C}_{1-2H} = \begin{cases} \dfrac{\Gamma(1-2H)}{2H}\sin\Big(\pi\dfrac{1-2H}{2}\Big), & H \in (0,1), H \neq \dfrac{1}{2}, \\ \dfrac{\pi}{2}, & H = \dfrac{1}{2}. \end{cases}$$

First, we observe that z_p is independent of H and

$$c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \leq \frac{\Gamma(3)}{2\pi} = \frac{1}{\pi}.$$

Next, as far as estimate (10) for the wave equation is concerned, we note that $2^{2H} \leq 4$ and $\frac{1}{1+2H} \leq 1$, for any $H \in (0,1)$. Thus, we concentrate on the constant $\tilde{C}_{1-2H}$, which we show that it is uniformly bounded in H. Clearly, the function $\tilde{C}_{1-2H} : (0,1) \to \mathbb{R}$ has, possibly, a singularity only in $H = \frac{1}{2}$, but since $\Gamma(x) \sim \frac{1}{x}$ as $x \to 0_+$, by simple calculations we have that the function $\tilde{C}_{1-2H}$ is continuous also at the point $H = \frac{1}{2}$. Therefore, $\tilde{C}_{1-2H}$ is bounded on the set K.

On the other hand, regarding estimate (10) for the heat equation, we have that $\frac{1}{H}\Gamma(1-H)$ defines a continuous function of H on the interval $(0,1)$, and thus it is bounded on K.

We now turn to the analysis of the term I_2. More precisely, we have

$$\begin{aligned} I_2 &= \mathrm{E}\Big[\Big|\int_0^T\int_{\mathbb{R}} 1_{[0,t]}(s)\big[G_{t-s}(x-y) - G_{t'-s}\big(x'-y\big)\big]W^H(ds,dy)\Big|^p\Big] \\ &= z_p c_H^{p/2}\Big[\int_0^T 1_{[0,t]}(s)\int_{\mathbb{R}}\big|\mathcal{F}\big(G_{t-s}(x-\cdot) - G_{t'-s}\big(x'-\cdot\big)\big)(\xi)\big|^2|\xi|^{1-2H}\,d\xi\,ds\Big]^{p/2} \\ &= z_p c_H^{p/2}\Big[\int_0^t\int_{\mathbb{R}}\big|\mathcal{F}G_{t-s}(x-\cdot)(\xi) - \mathcal{F}G_{t'-s}\big(x'-\cdot\big)(\xi)\big|^2|\xi|^{1-2H}\,d\xi\,ds\Big]^{p/2} \\ &\leq z_p c_H^{p/2} C_p\Big(\Big[\int_0^t\int_{\mathbb{R}}\big|\mathcal{F}G_{t'-s}\big(x'-\cdot\big)(\xi) - \mathcal{F}G_{t-s}\big(x'-\cdot\big)(\xi)\big|^2|\xi|^{1-2H}\,d\xi\,ds\Big]^{p/2} \end{aligned}$$

$$+\left[\int_0^t\int_{\mathbb{R}}\left|\mathcal{F}G_{t-s}\left(x'-\cdot\right)(\xi)-\mathcal{F}G_{t-s}(x-\cdot)(\xi)\right|^2|\xi|^{1-2H}\,d\xi\,ds\right]^{p/2}\Bigg)$$
$$= z_p c_H^{p/2} C_p(J_1+J_2),$$

where C_p denotes some constant depending on p. We estimate J_1 and J_2 using similar techniques as those used for the term I_1. Hence, via the change of variable $s'=t-s$, we have:

$$J_1=\left[\int_0^t\int_{\mathbb{R}}\left|\mathcal{F}G_{s'+(t'-t)}\left(x'-\cdot\right)(\xi)-\mathcal{F}G_{s'}\left(x'-\cdot\right)(\xi)\right|^2|\xi|^{1-2H}\,d\xi\,ds'\right]^{p/2}.$$

Thus, by Lemma 2.6,

$$J_1\le\begin{cases}M_H^{p/2}t^{p/2}\left(t'-t\right)^{pH}\le M_H^{p/2}T^{p/2}\left(t'-t\right)^{pH}, & \text{wave equation},\\ N_H^{p/2}\left(t'-t\right)^{pH/2}, & \text{heat equation}.\end{cases}$$

The above constants are the following:

$$\begin{aligned}\frac{1}{4}M_H&=\int_{\mathbb{R}}\frac{\min(1,|h|^2)}{|h|^{1+2H}}\,dh\\&=\int_{|h|>1}\frac{1}{|h|^{1+2H}}\,dh+\int_{|h|<1}\frac{1}{|h|^{2H-1}}\,dh\\&=\frac{1}{H}+\frac{1}{1-H},\end{aligned}$$

and

$$\begin{aligned}N_H&=\int_{\mathbb{R}}\frac{(1-e^{-\frac{h^2}{2}})^2}{|h|^{1+2H}}\,dh\\&\le\int_{\mathbb{R}}\frac{1-e^{-\frac{h^2}{2}}}{|h|^{1+2H}}\,dh\\&\le\int_{|h|>1}\frac{1}{|h|^{1+2H}}\,dh+\int_{|h|<1}\frac{1}{|h|^{2H-1}}\,dh\\&=\frac{1}{H}+\frac{1}{1-H}.\end{aligned}$$

The function $H\mapsto\frac{1}{H}+\frac{1}{1-H}$ is again continuous in $(0,1)$, and thus bounded for $H\in K$.

For the term J_2, we have:

$$J_2 = \left[\int_0^t \int_{\mathbb{R}} \left|\mathcal{F}G_{t-s}\left(x'-\cdot\right)(\xi) - \mathcal{F}G_{t-s}(x-\cdot)(\xi)\right|^2 |\xi|^{1-2H}\, d\xi\, ds\right]^{p/2}$$
$$= \left[\int_0^t \int_{\mathbb{R}} \left[1-\cos\left(\xi\left(x'-x\right)\right)\right] \left|\mathcal{F}G_{s'}(x-\cdot)(\xi)\right|^2 |\xi|^{1-2H}\, d\xi\, ds'\right]^{p/2},$$

and applying Lemma 2.5 we end up with

$$J_2 \le \begin{cases} C_H^{p/2} t^{p/2}\left(x'-x\right)^{pH} \le C_H^{p/2} T^{p/2}\left(x'-x\right)^{pH}, & \text{wave equation,} \\ C_H^{p/2}\left(x'-x\right)^{pH}, & \text{heat equation.} \end{cases}$$

Here, the constant C_H is

$$C_H = \int_{\mathbb{R}} \frac{1-\cos(h)}{|h|^{1+2H}}\, dh \le \frac{1}{H} + \frac{1}{1-H},$$

which again is a bounded function on the set K.

To sum up, we have proved that

$$\mathrm{E}\left[\left|\tilde{u}^H(t,x) - \tilde{u}^H\left(t',x'\right)\right|^p\right] \le C\left(\left(t'-t\right)^{\alpha p} + \left(x'-x\right)^{pH}\right),$$

where $\alpha = H$ for the wave equation and $\alpha = \frac{H}{2}$ for the heat equation, and the constant C depends only of p and T. Thus, choosing $p > \frac{4}{\min_{H\in K} H}$, we have that the hypotheses of Theorem 2.7 are fulfilled by the family $\{\tilde{u}^H\}_{H\in K}$, for both the solution to (SWE) and (SHE). This concludes the first step of the proof.

Step 2: In order to identify the limit law of the sequence $\{u^{H_n}\}_{n\ge 1}$, we proceed to prove the convergence of the finite dimensional distributions of $\tilde{u}^{H_n}$ when $n \to \infty$.

We recall that, for every $H \in (0,1)$, $\tilde{u}^H = u^H - I_0$ is a centered Gaussian process, so it suffices to analyze the convergence of the corresponding covariance functions.

Let $(t,x), (t',x') \in [0,T] \times \mathbb{R}$ and suppose that $t' \ge t$. Then,

$$\mathrm{E}\left[\tilde{u}^{H_n}(t,x)\tilde{u}^{H_n}\left(t',x'\right)\right] = c_{H_n} \int_0^t \int_{\mathbb{R}} \mathcal{F}G_{t-s}(x-\cdot)(\xi)\overline{\mathcal{F}G_{t'-s}\left(x'-\cdot\right)(\xi)} |\xi|^{1-2H_n}\, d\xi\, ds.$$

Let us first consider the case of the wave equation. Taking into account the explicit form of $\mathcal{F}G_t(\xi)$ (see (8)), we have

$$\mathrm{E}\left[\tilde{u}^{H_n}(t,x)\tilde{u}^{H_n}\left(t',x'\right)\right] = c_{H_n} \int_0^t \int_{\mathbb{R}} \frac{e^{-i\xi(x-x')} \sin((t-s)|\xi|)\sin((t'-s)|\xi|)}{|\xi|^{1+2H_n}}\, d\xi\, ds.$$

We clearly have that $c_{H_n} \to c_{H_0}$. The integrand function in the latter integral converges, as $n \to \infty$, to

$$\frac{e^{-i\xi(x-x')} \sin((t-s)|\xi|)\sin((t'-s)|\xi|)}{|\xi|^{1+2H_0}},$$

for almost every $(s,\xi)\in[0,t]\times\mathbb{R}$. Moreover, thanks to the fact that $|\sin(z)|\le z$ for all $z\in\mathbb{R}$, its modulus is dominated by the integrable function

$$\begin{cases} \dfrac{(t-s)(t'-s)}{|\xi|^{2\sup_n(H_n)-1}}, & s\in[0,t],\ |\xi|\le 1,\\ \dfrac{1}{|\xi|^{2\inf_n(H_n)+1}}, & s\in[0,t],\ |\xi|>1. \end{cases}$$

Then, by the dominated convergence theorem, we obtain that

$$\begin{aligned}\lim_{n\to\infty}\mathrm{E}\big[\tilde u^{H_n}(t,x)\tilde u^{H_n}\big(t',x'\big)\big] &= c_{H_0}\int_0^t\int_{\mathbb{R}}\frac{e^{-i\xi(x-x')}\sin((t-s)|\xi|)\sin((t'-s)|\xi|)}{|\xi|^{1+2H_0}}\,d\xi\,ds\\ &= \mathrm{E}\big[\tilde u^{H_0}(t,x)\tilde u^{H_0}\big(t',x'\big)\big].\end{aligned}$$

On the other hand, in the case of the heat equation, we have

$$\mathrm{E}\big[\tilde u^{H_n}(t,x)\tilde u^{H_n}\big(t',x'\big)\big]=c_{H_n}\int_0^t\int_{\mathbb{R}}\frac{e^{-i\xi(x-x')}e^{-\frac{(t-s)\xi^2}{2}}e^{-\frac{(t'-s)\xi^2}{2}}}{|\xi|^{2H_n-1}}\,d\xi\,ds. \tag{11}$$

The pointwise limit of the above integrand is given by

$$\frac{e^{-i\xi(x-x')}e^{-\frac{(t-s)\xi^2}{2}}e^{-\frac{(t'-s)\xi^2}{2}}}{|\xi|^{2H_0-1}},$$

for all $s\in[0,t]$ and $\xi\in\mathbb{R}$, and its modulus reads

$$\frac{e^{-\frac{(t+t'-2s)\xi^2}{2}}}{|\xi|^{2H_n-1}}.$$

Now, we use the bound

$$e^{-ax^2}<\frac{1}{ax^2},\quad \text{if } a>0,$$

with $a=(t+t'-2s)/2$ (which is always positive provided that $s\in[0,t]$). Thus,

$$\frac{e^{-\frac{(t+t'-2s)\xi^2}{2}}}{|\xi|^{2H_n-1}}\le\begin{cases}\dfrac{1}{|\xi|^{2\sup_n(H_n)-1}}, & |\xi|\le 1,\ s\in[0,t],\\ \dfrac{2}{(t'-t)|\xi|^{2\inf_n(H_n)+1}}, & |\xi|>1,\ s\in[0,t].\end{cases}$$

This covers all cases except $t=t'$. In this latter case, the modulus of the integrand appearing in (11) becomes

$$\frac{e^{-(t-s)\xi^2}}{|\xi|^{2H_n-1}}\le\begin{cases}\dfrac{1}{|\xi|^{2\sup_n(H_n)-1}}, & |\xi|\le 1,\ s\in[0,t],\\ \dfrac{\exp(-(t-s)\xi^2)}{|\xi|^{2\inf_n(H_n)-1}}, & |\xi|>1,\ s\in[0,t],\end{cases}$$

and the integrability of this function is an easy consequence of Lemma 2.4. Therefore, by the dominated convergence theorem, we also obtain that

$$\lim_{n\to\infty} \mathrm{E}\big[\tilde{u}^{H_n}(t,x)\tilde{u}^{H_n}\big(t',x'\big)\big] = \mathrm{E}\big[\tilde{u}^{H_0}(t,x)\tilde{u}^{H_0}\big(t',x'\big)\big],$$

which concludes Step 2 of the proof.

To finish the proof of the theorem, it remains to observe that, since the translation by I_0 is clearly a continuous mapping from $C([0,T]\times\mathbb{R})$ into itself, the convergence in distribution $\tilde{u}^{H_n} \stackrel{d}{\to} \tilde{u}^{H_0}$ implies the convergence in distribution $u^{H_n} \stackrel{d}{\to} u^{H_0}$, which was our statement. □

3. Quasi-linear additive case: Existence of solution

In this section, we consider equations (SWE) and (SHE) with a general drift coefficient b, where we assume that $b:\mathbb{R}\to\mathbb{R}$ is a globally Lipschitz function. Let $T>0$. Owing to (4), we recall that a solution to these equations is an adapted and jointly measurable process $\{u^H(t,x), (t,x)\in[0,T]\times\mathbb{R}\}$ such that, for all $(t,x)\in[0,T]\times\mathbb{R}$,

$$\begin{aligned} u^H(t,x) = I_0(t,x) &+ \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W^H(ds,dy) \\ &+ \int_0^t\int_{\mathbb{R}} b\big(u^H(s,y)\big)G_{t-s}(x-y)\,dy\,ds, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{12}$$

where the term I_0 and the fundamental solution G are specified in (5) and (6), respectively.

If $H>\frac{1}{2}$, the existence of a unique solution to (12) follows from [10], Thm. 4.3, assuming that the term I_0 satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \big|I_0(t,x)\big| < \infty.$$

The case $H=\frac{1}{2}$ was considered in [18]. Finally, we have not been able to find a proof of existence in the case $H<\frac{1}{2}$. This section is devoted to present a proof of existence and uniqueness of solution to (12) which holds for any $H\in(0,1)$ (cf. Theorem 3.1). Furthermore, we provide sufficient conditions on the initial data ensuring that the solution admits a Hölder-continuous version (cf. Theorem 3.2 below).

Along this section, we will require more restrictive conditions for the initial conditions. Concretely, we consider the following assumption:

Hypothesis B. *It holds that*

(a) Wave equation: u_0 *and* v_0 *are* H*-Hölder continuous and bounded.*
(b) Heat equation: u_0 *is* H*-Hölder continuous and bounded.*

Moreover, we recall that we are considering the filtration $(\mathcal{F}_t^H)_{t\geq 0}$ which is generated by our fractional noise W^H (see (3)).

Theorem 3.1. *Let $p \geq 2$ and assume that Hypothesis B is satisfied. Then, equation (12) has a unique solution u^H in the space of $L^2(\Omega)$-continuous and adapted stochastic processes satisfying*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\big[\big|u^H(t,x)\big|^p\big] < \infty.$$

Proof. We follow similar arguments as those used in [9]. We split the proof in four parts.

Step 1: We define the following Picard iteration scheme. For $n = 0$, we set

$$u_0^H(t,x) := I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy), \tag{13}$$

and for $n \geq 1$ we define

$$u_n^H(t,x) := u_0^H(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b\big(u_{n-1}^H(s,y)\big)\, dy\, ds. \tag{14}$$

Clearly, the process u_0^H is adapted and, by step 1 in Section 4.3, it is $L^2(\Omega)$-continuous. Then, u_0^H admits a jointly measurable modification (cf. [2], Prop. B.1), which will be denoted in the same way.

Owing to Lemma 3.3, we obtain that, for every $n \geq 0$, the Picard iteration u_n^H is $L^2(\Omega)$-continuous, and thus has a jointly measurable modification. Moreover, by Lemma 3.4 below, u_n^H is uniformly bounded in $L^p(\Omega)$, that is,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\big[\big|u_n^H(t,x)\big|^p\big] < \infty.$$

The above two facts imply that u_n^H is well-defined, for all $n \geq 0$. On the other hand, it is clear that any Picard iteration defines an adapted process.

Step 2: We prove that the Picard iteration scheme converges in the space of $L^2(\Omega)$-continuous, adapted and $L^p(\Omega)$-uniformly bounded processes, which is a complete normed space when endowed with the norm

$$\big\|u^H\big\|_p = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \big(\mathrm{E}\big[\big|u^H(t,x)\big|^p\big]\big)^{1/p}.$$

Indeed, it can be seen as the closed subset formed by adapted process of the space

$$L^\infty\big([0,T]\times\mathbb{R}; L^p(\Omega)\big),$$

which is a Banach space for any $p \geq 2$.

Then, it is sufficient to show that the sequence of Picard iterations is Cauchy with respect to $\|\cdot\|_p$ to infer the existence of a limit.

We use that b is Lipschitz and Minkowski inequality for integrals to obtain

$$\begin{aligned}
&\big(\mathrm{E}\big[\big|u_{n+1}^H(t,x) - u_n^H(t,x)\big|^p\big]\big)^{1/p} \\
&\quad = \left(\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\big[b\big(u_n^H(s,y)\big) - b\big(u_{n-1}^H(s,y)\big)\big]\, dy\, ds\right|^p\right]\right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\le C\bigg(\mathrm{E}\bigg[\bigg|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\big|u_n^H(s,y)-u_{n-1}^H(s,y)\big|\,dy\,ds\bigg|^p\bigg]\bigg)^{1/p} \\
&\le C\int_0^t\int_{\mathbb{R}}\big(\mathrm{E}\big[G_{t-s}(x-y)^p\big|u_n^H(s,y)-u_{n-1}^H(s,y)\big|^p\big]\big)^{1/p}\,dy\,ds \\
&\le C\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\sup_{\substack{y\in\mathbb{R},\\ s'\in[0,s]}}\big(\mathrm{E}\big[\big|u_n^H(s',y)-u_{n-1}^H(s',y)\big|^p\big]\big)^{1/p}\,dy\,ds \\
&= C\int_0^t\sup_{\substack{y\in\mathbb{R},\\ s'\in[0,s]}}\big(\mathrm{E}\big[\big|u_n^H(s',y)-u_{n-1}^H(s',y)\big|^p\big]\big)^{1/p}\,ds.
\end{aligned}$$

This inequality implies that

$$\begin{aligned}
&\sup_{\substack{x\in\mathbb{R},\\ s\in[0,t]}}\big(\mathrm{E}\big[\big|u_{n+1}^H(s,x)-u_n^H(s,x)\big|^p\big]\big)^{1/p} \\
&\qquad\le C\int_0^t\sup_{\substack{y\in\mathbb{R},\\ s'\in[0,s]}}\big(\mathrm{E}\big[\big|u_n^H(s',y)-u_{n-1}^H(s',y)\big|^p\big]\big)^{1/p}\,ds
\end{aligned}$$

If we define

$$f_n(t):=\sup_{\substack{x\in\mathbb{R},\\ s\in[0,t]}}\big(\mathrm{E}\big[\big|u_{n+1}^H(s,x)-u_n^H(s,x)\big|^p\big]\big)^{1/p},$$

we have that

$$f_n(t)\le C\int_0^t f_{n-1}(s)\,ds.$$

Thanks to Lemma 3.4, we have that f_0 is a bounded function on $[0,T]$, and thus integrable. Then, by Grönwall lemma, we can conclude that $\{u_n^H\}_{n\ge 0}$ defines a Cauchy sequence in the underlying space, and therefore it converges to a limit u^H, namely

$$\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}}\mathrm{E}\big[\big|u_n^H(t,x)-u^H(t,x)\big|^p\big]=0.$$

Since any u_n^H is $L^2(\Omega)$-continuous and adapted, u^H has the same properties. In particular, $L^2(\Omega)$-continuity implies the existence of a joint-measurable version of u^H.

Step 3: We check that the process u^H is a solution of (12). To do this, we take $n\to\infty$ with respect to the uniform $L^p(\Omega)$-norm in the expression

$$u_{n+1}^H(t,x)=u_0^H(t,x)+\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)b\big(u_n^H(s,y)\big)\,dy\,ds.$$

The left-hand side, by its definition, converges to u^H, while for the non-constant (with respect to n) part of the right-hand side, we argue as follows:

$$
\begin{aligned}
&\left(\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\big(b\big(u_n^H(s,y)\big)-b\big(u^H(s,y)\big)\big)\,dy\,ds\right|^p\right]\right)^{1/p}\\
&\quad\le C\left(\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\big|u_n^H(s,y)-u^H(s,y)\big|\,dy\,ds\right|^p\right]\right)^{1/p}\\
&\quad\le C\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\big(\mathrm{E}\big[\big|u_n^H(s,y)-u^H(s,y)\big|^p\big]\big)^{1/p}\,dy\,ds\\
&\quad\le C\int_0^t \sup_{(s,y)\in[0,T]\times\mathbb{R}} \big(\mathrm{E}\big[\big|u_n^H(s,y)-u^H(s,y)\big|^p\big]\big)^{1/p}\,ds\\
&\quad\le C \sup_{(s,y)\in[0,T]\times\mathbb{R}} \big(\mathrm{E}\big[\big|u_n^H(s,y)-u^H(s,y)\big|^p\big]\big)^{1/p}.
\end{aligned}
$$

We note that the latter term converges to zero as $n\to\infty$. Thus, we have that u^H satisfies (12).

Step 4: Uniqueness can be checked by using analogous arguments as those used in the previous steps. □

We have the following property of the sample paths of the solution u^H.

Theorem 3.2. *Let $p\ge 2$. Assume that Hypothesis B is fulfilled. Let u^H be the solution of (12). Then, for any $t,t'\in[0,T]$ and $x,x'\in\mathbb{R}$ such that $|t'-t|\le 1$ and $|x'-x|\le 1$, the following inequalities hold true*:

$$
\sup_{x\in\mathbb{R}} \mathrm{E}\big[\big|u^H\big(t',x\big)-u^H(t,x)\big|^p\big]\le C_p\big|t'-t\big|^{\gamma p} \tag{15}
$$

and

$$
\sup_{t\in[0,T]} \mathrm{E}\big[\big|u^H\big(t,x'\big)-u^H(t,x)\big|^p\big]\le C_p\big|x'-x\big|^{Hp}, \tag{16}
$$

where $\gamma=H$ for the wave equation and $\gamma=\frac{H}{2}$ for the heat equation. Hence, the process u^H has a modification whose trajectories are almost surely γ'-Hölder continuous in time, for all $\gamma'<\gamma$, and H'-Hölder continuous in space for all $H'<H$.

Proof. The bounds (15) and (16) are an easy corollary of the stronger results obtained in Step 1 of Section 4.3. Indeed, in that theorem, the same kind of estimates have been obtained uniformly with respect to the Hurst index H, when restricted on a compact set $[a,b]\subset(0,1)$. Nevertheless, here we need to obtain (15) and (16) only for a fixed $H\in(0,1)$. □

In order to conclude this section, we state and prove the two lemmas that we used in Step 1 of the proof of Theorem 3.1 above.

Lemma 3.3. *For each $n \geq 0$, the process u_n^H defined by (13) and (14) satisfies the following. There exists a constant $C = C(n, H)$ such that, for any $t \in [0, T]$ and $h \in \mathbb{R}$ with $t + h \leq T$, it holds*

$$\sup_{x\in\mathbb{R}} \mathrm{E}\big[\big|u_n^H(t+h,x) - u_n^H(t,x)\big|^2\big] \leq \begin{cases} Ch^{\min(2H,1)}, & \textit{wave equation,} \\ Ch^H, & \textit{heat equation.} \end{cases} \tag{17}$$

and, for any $x \in \mathbb{R}$ and $h \in \mathbb{R}$ with $|h| < 1$,

$$\sup_{t\in[0,T]} \mathrm{E}\big[\big|u_n^H(t,x+h) - u_n^H(t,x)\big|^2\big] \leq Ch^{2H}. \tag{18}$$

In particular, the process u_n^H is $L^2(\Omega)$-continuous.

Proof. We proceed by induction. In the case $n = 0$, first we study the time increments. We focus on the right continuity. The computations for the left continuity are analogous. We have

$$\mathrm{E}\big[\big|u_0^H(t+h,x) - u_0^H(t,x)\big|^2\big] \leq 2(A_1 + A_2),$$

where

$$\begin{aligned} A_1 &= \big|I_0(t+h,x) - I_0(t,x)\big|^2, \\ A_2 &= \mathrm{E}\bigg[\bigg|\int_0^t \int_{\mathbb{R}} \big[G_{t+h-s}(x-y) - G_{t-s}(x-y)\big] W^H(ds,dy) \\ &\quad + \int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y) W^H(ds,dy)\bigg|^2\bigg]. \end{aligned}$$

In Theorem 3.7 of [4], it is shown that

$$A_1 \leq \begin{cases} Ch^{2H} & \text{for the wave equation,} \\ Ch^H & \text{for the heat equation.} \end{cases}$$

Concerning the term A_2, we have

$$A_2 \leq 2(A_{2,1} + A_{2,2}),$$

where

$$\begin{aligned} A_{2,1} &= \mathrm{E}\bigg[\bigg|\int_0^t \int_{\mathbb{R}} \big[G_{t+h-s}(x-y) - G_{t-s}(x-y)\big] W^H(ds,dy)\bigg|^2\bigg], \\ A_{2,2} &= \mathrm{E}\bigg[\bigg|\int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y) W^H(ds,dy)\bigg|^2\bigg]. \end{aligned}$$

These terms have been studied in the proof of Theorem 2.8, concretely $A_{2,1}$ corresponds to term J_1 in that theorem and term $A_{2,2}$ corresponds to I_1. So,

$$A_{2,1} \leq \begin{cases} Ch^{1+2H}, & \text{for the wave equation,} \\ Ch^{\frac{1}{2}+H}, & \text{for the heat equation,} \end{cases}$$

and

$$A_{2,2} \leq \begin{cases} Ch^{1+2H}, & \text{for the wave equation,} \\ Ch^{\frac{1}{2}+H}, & \text{for the heat equation.} \end{cases}$$

Putting together the above estimates, we obtain the validity of (17) for $n = 0$.

Regarding the space increments, we have, for any $h \in \mathbb{R}$ with $|h| < 1$,

$$\mathrm{E}\big[\big|u_0^H(t, x+h) - u_0^H(t,x)\big|^2\big] \leq 2(B_1 + B_2),$$

where

$$B_1 = \big|I_0(t, x+h) - I_0(t,x)\big|^2,$$
$$B_2 = \mathrm{E}\bigg[\bigg|\int_0^t \int_{\mathbb{R}} \big[G_{t-s}(x+h-y) - G_{t-s}(x-y)\big] W^H(ds, dy)\bigg|^2\bigg].$$

As before, by [4], Thm. 3.7, we have

$$B_1 \leq Ch^{2H}$$

for both heat and wave equations. The term B_2 corresponds to J_2 in the proof of Theorem 2.8, hence

$$B_2 \leq C|h|^{1+2H}.$$

So, we have proved (18) for $n = 0$.

We suppose now by induction hypothesis that u_n^H satisfies (17) and (18). Let us compute the time increments of u_{n+1}^H, for $0 < h \ll 1$:

$$\mathrm{E}\big[\big|u_{n+1}^H(t+h, x) - u_{n+1}^H(t,x)\big|^2\big] \leq 3(D_1 + D_2 + D_3),$$

where

$$D_1 = \mathrm{E}\big[\big|u_0^H(t+h, x) - u_0^H(t,x)\big|^2\big],$$
$$D_2 = \mathrm{E}\bigg[\bigg(\int_0^t \int_{\mathbb{R}} G_s(y)\big|b\big(u_n^H(t+h-s, x-y)\big) - b\big(u_n^H(t-s, y)\big)\big|\, dy\, ds\bigg)^2\bigg],$$
$$D_3 = \mathrm{E}\bigg[\bigg(\int_t^{t+h} \int_{\mathbb{R}} G_s(y)\big|b\big(u_n(t+h-s, x-y)\big)\big|\, dy\, ds\bigg)^2\bigg].$$

We already showed that D_1 is bounded as the right-hand side of (17), so we only need to handle D_2 and D_3. As in Lemma 19 of [9], first we compute D_2. Namely, using that b is Lipschitz and applying Cauchy–Schwarz inequality and Fubini theorem, we have

$$\begin{aligned}
D_2 &\le C\left(\int_0^t \int_{\mathbb{R}} G_s(y)\,dy\,ds\right) \\
&\quad \times \mathrm{E}\left[\int_0^t \int_{\mathbb{R}} G_s(y)\left|u_n^H(t+h-s,x-y)-u_n^H(t-s,x-y)\right|^2 dy\,ds\right] \\
&\le C\mathrm{E}\left[\int_0^t \int_{\mathbb{R}} G_s(y)\left|u_n^H(t+h-s,x-y)-u_n^H(t-s,x-y)\right|^2 dy\,ds\right] \\
&= C\int_0^t \int_{\mathbb{R}} G_s(y)\mathrm{E}\left[\left|u_n^H(t+h-s,x-y)-u_n^H(t-s,x-y)\right|^2\right] dy\,ds \\
&\le \begin{cases} Ch^{2H}, & \text{wave equation,} \\ Ch^H, & \text{heat equation.} \end{cases}
\end{aligned}$$

Notice that in the last inequality we used the induction hypothesis.

Regarding D_3, we have

$$D_3 \le C\int_t^{t+h} \int_{\mathbb{R}} \left(1+\mathrm{E}\left[\left|u_n^H(t+h-s,x-y)\right|^2\right]\right) G_s(y)\,dy\,ds.$$

The uniform boundedness in $L^2(\Omega)$ of u_n^H (by Lemma 3.4) gives that

$$D_3 \le C\int_t^{t+h} \int_{\mathbb{R}} G_s(y)\,dy\,ds \le Ch,$$

for both wave and heat equations. Thus, taking into account the above estimates for J_1, J_2 and J_3, we obtain that u_{n+1}^H satisfies (17).

We are left to deal with the spatial increments of u_{n+1}^H. Indeed, we have

$$\mathrm{E}\left[\left|u_{n+1}^H(t,x+h)-u_{n+1}^H(t,x)\right|^2\right] \le 2(K_1+K_2),$$

where

$$\begin{aligned}
K_1 &= \mathrm{E}\left[\left|u_0^H(t,x+h)-u_0^H(t,x)\right|^2\right], \\
K_2 &= \mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}} \left|b\left(u_n^H(t-s,x+h-y)\right)-b\left(u_n^H(t-s,x-y)\right)\right| G_s(y)\,dy\,ds\right)^2\right].
\end{aligned}$$

The term K_1 has already been studied, and K_2 can be treated as the term J_2, obtaining that $K_2 \le C|h|^{2H}$. So we can infer that (18) is fulfilled for u_{n+1}^H. □

Lemma 3.4. *Let $p \geq 2$ and $[a,b] \subset (0,1)$. Let u_n^H, $n \geq 0$, be the Picard iteration scheme defined in* (13) *and* (14). *Then,*

$$\sup_{n\geq 0} \sup_{H\in[a,b]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\big[\big|u_n^H(t,x)\big|^p\big] < \infty.$$

Proof. First, we have

$$\mathrm{E}\big[\big|u_0^H(t,x)\big|^p\big] \leq C_p\left(\big|I_0(t,x)\big|^p + \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy)\right|^p\right]\right).$$

By [10], Lemma 4.2, we have that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \big|I_0(t,x)\big| < \infty,$$

and this is uniform in H, since we are considering the same initial conditions for every H. Regarding the stochastic term, arguing as in (9) and applying Lemma 2.4, we get

$$\begin{aligned}
&\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy)\right|^p\right] \\
&= z_p c_H^{p/2}\left[\int_0^t \int_{\mathbb{R}} \big|\mathcal{F}G_{t-s}(x-\cdot)(\xi)\big|^2 |\xi|^{1-2H}\, d\xi\, ds\right]^{p/2} \\
&\leq \begin{cases} C_p\big(t^{1+2H}\big)^{p/2}, & \text{wave equation,} \\ C_p\big(t^H\big)^{p/2}, & \text{heat equation.} \end{cases}
\end{aligned}$$

The last inequality comes from an estimate essentially identical to the one already computed in (10). All above constants which are dependent on H can be uniformly bounded, provided that H is in the compact interval $[a,b] \subset (0,1)$. The above considerations yield

$$\sup_{H\in[a,b]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\big[\big|u_0^H(t,x)\big|^p\big] < \infty.$$

Next, owing to (14) we can infer that

$$\mathrm{E}\big[\big|u_{n+1}^H(t,x)\big|^p\big] \leq C\left(1 + \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b\big(u_n^H(s,y)\big)\, dy\, ds\right|^p\right]\right).$$

If we apply Hölder inequality, we obtain

$$\begin{aligned}
&\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b\big(u_n^H(s,y)\big)\, dy\, ds\right|^p\right] \\
&\leq C\mathrm{E}\left[\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\big(1 + \big|u_n^H(s,y)\big|^p\big)\, dy\, ds\right]
\end{aligned}$$

$$
\begin{aligned}
&= C_1 + C_2 \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathrm{E}\big[\big|u_n^H(s,y))\big|^p\big]\, dy\, ds \\
&\le C_1 + C_2 \int_0^t \int_{\mathbb{R}} \sup_{H\in[a,b]} \sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\big[\big|u_n^H(s',y))\big|^p\big] G_{t-s}(x-y)\, dy\, ds \\
&\le C_1 + C_2 \int_0^t \sup_{H\in[a,b]} \sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\big[\big|u_n^H(s',y))\big|^p\big]\, ds. \qquad (19)
\end{aligned}
$$

The constants appearing in the previous calculations are clearly independent of H. Then, we have

$$
\begin{aligned}
&\sup_{H\in[a,b]} \sup_{(t',y)\in[0,t]\times\mathbb{R}} \mathrm{E}\big[\big|u_{n+1}^H(t',y)\big|^p\big] \\
&\quad \le C_1 + C_2 \int_0^t \sup_{H\in[a,b]} \sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\big[\big|u_n^H(s',y))\big|^p\big]\, ds.
\end{aligned}
$$

We conclude the proof by applying Grönwall lemma. □

4. Quasi-linear additive case: Weak convergence

This section is devoted to prove that the mild solution u^{H_n} of equation (SWE) (resp. (SHE)) converges in law in the space of continuous functions, as $H_n \to H_0$, to the solution u^{H_0} of (SWE) (resp. (SHE)) corresponding to the Hurst index H_0.

Throughout this section, we fix $H_0 \in (0,1)$ and any sequence $(H_n)_{n\ge1}$ converging to H_0. Then, we consider the following assumptions for the initial data:

Hypothesis C. *For some $\alpha > H_0$, it holds that*

(a) Wave equation: u_0 *and* v_0 *are α-Hölder continuous and bounded.*
(b) Heat equation: u_0 *is α-Hölder continuous and bounded.*

Without any loss of generality, we assume that $H_n \le \alpha$, for all $n \ge 1$. Hence, we will be able to apply the results of the previous section for all these Hurst indexes.

The main strategy to prove that u^{H_n} converges in law to u^{H_0} can be summarized as follows. Recall that b is assumed to be globally Lipschitz. Let η be a deterministic function in $C([0,T]\times\mathbb{R})$, and consider the (deterministic) integral equation

$$
z(t,x) = \int_0^t \int_{\mathbb{R}} b\big(z(s,y)\big) G_{t-s}(x-y)\, ds\, dy + \eta(t,x), \qquad (20)
$$

which is defined on the space $C([0,T]\times\mathbb{R})$, endowed with the metric of uniform convergence on compact sets.

We will prove that (20) admits a unique solution. This allows us to define the solution operator

$$
F : C\big([0,T]\times\mathbb{R}\big) \longrightarrow C\big([0,T]\times\mathbb{R}\big) \qquad (21)
$$

by $(F\eta)(t,x) := z(t,x)$. We will show that this operator is continuous. Note that $u^{H_n} = F(\bar{u}^{H_n})$ (almost surely), for all $n \geq 0$, where $\bar{u}^{H_n}$ denotes the solution in the linear additive case (i.e., $b = 0$). Moreover, by Theorem 2.8, $\bar{u}^{H_n}$ converges in law, in the space of continuous functions, to $\bar{u}^{H_0}$. Therefore, we can apply Theorem 2.7 of [7] to obtain the desired result.

Here is the main result of the paper.

Theorem 4.1. *Assume that Hypothesis C is fulfilled and b is globally Lipschitz. Then, $u^{H_n} \xrightarrow{d} u^{H_0}$, as $n \to \infty$, where the convergence holds in distribution in the space $C([0,T] \times \mathbb{R})$.*

The proof of the above theorem will be tackled in the following three subsections. Indeed, we need to distinguish the case of the wave equation from the one of the heat equation. Moreover, for the heat equation, we split the analysis in two subcases: bounded b and possibly unbounded b. As it will be made clear in the sequel, in the latter case, the above-explained strategy based on the *solution operator* cannot be applied, so the case b unbounded will be studied separately.

4.1. Wave equation

In this section, we provide the proof of Theorem 4.1 for the stochastic wave equation (SWE). For this, as already explained, it suffices to prove that equation (20) has a unique solution and that the solution operator (21) is continuous. These two facts will be proved in Theorem 4.3 below.

We recall that the fundamental solution G of the wave equation on $[0,\infty) \times \mathbb{R}$ is

$$G_t(x) = \frac{1}{2} 1_{\{|x| \leq t\}}.$$

We will make use of the following ad hoc version of Grönwall lemma ([6]). We give its proof for the sake of completeness. We remark that, using [8], Lem. 3.7, one could get a sharper version of this result.

Lemma 4.2. *Let $\{f_n, n \geq 0\}$ be a sequence of real-valued non-negative functions defined on $[0,T] \times [a-T, b+T]$, for some $a, b \in \mathbb{R}$ such that $a < b$, and $T > 0$. Suppose that there exist $\lambda, \mu > 0$ such that, for every $(t,x) \in [0,T] \times [a,b]$ and $n \geq 0$,*

$$f_{n+1}(t,x) \leq \lambda + \frac{\mu}{2} \int_0^t \int_{x-t+s}^{x+t-s} f_n(s,y)\, dy\, ds,$$

and that f_0 is bounded. Then, for every $n \geq 0$ and $(t,x) \in [0,T] \times [a,b]$, it holds that

$$f_n(t,x) \leq \lambda \sum_{k=0}^{n-1} \frac{(\mu t^2)^k}{k!} + \|f_0\|_\infty \frac{(\mu t^2)^n}{n!}, \tag{22}$$

which in particular implies that

$$\limsup_{n\to\infty} f_n(t,x) \leq \lambda \exp(\mu t^2).$$

Proof. We prove it by induction: the case $n = 1$ reduces to the inequality

$$f_1(t,x) \le \lambda + \mu t^2 \| f_0 \|_\infty,$$

that is clearly satisfied. We go on with the inductive step: if (22) holds true, then

$$\begin{aligned} f_{n+1}(t,x) &\le \lambda + \frac{\mu}{2} \int_0^t \int_{x-t+s}^{x+t-s} \left[\lambda \sum_{k=0}^{n-1} \frac{(\mu s^2)^k}{k!} + \| f_0 \|_\infty \frac{(\mu s^2)^n}{n!} \right] dy\, ds \\ &= \lambda + \frac{\mu}{2} \int_0^t 2(t-s) \left[\lambda \sum_{k=0}^{n-1} \frac{(\mu s^2)^k}{k!} + \| f_0 \|_\infty \frac{(\mu s^2)^n}{n!} \right] ds \\ &\le \lambda + \mu \int_0^t t \left[\lambda \sum_{k=0}^{n-1} \frac{(\mu s^2)^k}{k!} + \| f_0 \|_\infty \frac{(\mu s^2)^n}{n!} \right] ds \\ &= \lambda + \mu \left[\lambda \sum_{k=0}^{n-1} \frac{\mu^k (t^2)^{k+1}}{k!(2k+1)} + \| f_0 \|_\infty \frac{\mu^n (t^2)^{n+1}}{n!(2n+1)} \right] \\ &= \lambda + \lambda \sum_{k=0}^{n-1} \frac{\mu^{k+1} (t^2)^{k+1}}{k!(2k+1)} + \| f_0 \|_\infty \frac{\mu^{n+1} (t^2)^{n+1}}{n!(2n+1)} \\ &\le \lambda \sum_{k=0}^{n} \frac{\mu^k (t^2)^k}{k!} + \| f_0 \|_\infty \frac{\mu^{n+1} (t^2)^{n+1}}{(n+1)!}, \end{aligned}$$

which is our thesis. In the last two inequalities, we shifted by one the index of the sum and we used the fact that $4k^2 + 6k + 2 > k + 1$, for every $k \in \mathbb{N}$. If we take the lim sup as $n \to \infty$ in both sides of the inequality, we also obtain easily that

$$\limsup_{n \to \infty} f_n(t,x) \le \lambda \exp(\mu t^2). \qquad \square$$

We will use the above Grönwall-type lemma to prove the following theorem, proved also in [6].

Theorem 4.3. *Let $\eta \in C([0,T] \times \mathbb{R})$ and consider the deterministic equation (20) in the case where G is the fundamental solution of the wave equation. Then, (20) has a unique solution $z \in C([0,T] \times \mathbb{R})$. Moreover, the solution operator*

$$F : C([0,T] \times \mathbb{R}) \to C([0,T] \times \mathbb{R})$$

defined by $F(\eta) = z$ is continuous, if we endow $C([0,T] \times \mathbb{R})$ with the metric of uniform convergence on compact sets.

Proof. We define the Picard iteration scheme

$$\begin{aligned}
z_0(t,x) &:= \eta(t,x),\\
z_n(t,x) &:= \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b\big(z_{n-1}(s,y)\big)\,dy\,ds + \eta(t,x) \\
&= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} b\big(z_{n-1}(s,y)\big)\,dy\,ds + \eta(t,x), \quad n \geq 1.
\end{aligned} \tag{23}$$

Clearly, the above expressions of the Picard scheme are well-defined. Moreover, since b is Lipschitz continuous, if z_{n-1} is continuous then also $b \circ z_{n-1}$ is so. This gives by induction that z_n is a continuous function. Moreover, we will show that z_n converges uniformly on compact sets on $[0,T] \times \mathbb{R}$. More precisely, we prove that the sequence $\{z_n\}_{n\geq 0}$ is uniformly Cauchy on $[0,T] \times [-L,L]$, for every $L > 0$. Indeed, for all $(t,x) \in [0,T] \times [-L,L]$, we have

$$\begin{aligned}
\big|z_{n+1}(t,x) - z_n(t,x)\big| &= \left|\frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \big[b\big(z_n(s,y)\big) - b\big(z_{n-1}(s,y)\big)\big]\,dy\,ds\right| \\
&\leq C \int_0^t \int_{x-t+s}^{x+t-s} \big|z_n(s,y) - z_{n-1}(s,y)\big|\,dy\,ds.
\end{aligned}$$

We can apply Lemma 4.2 to the sequence of functions $f_n := |z_{n+1} - z_n|$ and with $\lambda = 0$ and $\mu = 2C$, obtaining that

$$\begin{aligned}
\big|z_{n+1}(t,x) - z_n(t,x)\big| &\leq \Big(\sup_{(s,y)\in[0,T]\times[-L-T,L+T]} \big|z_1(s,y) - z_0(s,y)\big|\Big) \frac{(2Ct^2)^n}{n!} \\
&\leq \Big(\sup_{(s,y)\in[0,T]\times[-L-T,L+T]} \big|z_1(s,y) - z_0(s,y)\big|\Big) \frac{(2CL^2)^n}{n!}.
\end{aligned}$$

Notice that the latter bound does not depend on t and x. This remark, together with the fact that the function $z_1 - z_0$ is bounded on any compact set, and that the sum $\sum_{k=0}^{\infty} \frac{(2CL^2)^n}{n!}$ is convergent, yield that the sequence $\{z_n(t,x)\}_{n\geq 0}$ is uniformly Cauchy on $[0,T] \times [-L,L]$. Let $z(t,x)$ denote its limit. Then, by the uniqueness of the pointwise limit, the fact that $C([0,T] \times \mathbb{R})$ is a complete metric space (with the underlying metric) and that z_n, $n \geq 0$, are continuous functions, we have that z is also a continuous function in $C([0,T] \times \mathbb{R})$.

Letting $n \to \infty$ in (24) and observing that $b \circ z_n \to b \circ z$ uniformly on compact sets, one easily gets that z solves equation (20).

The uniqueness of the solution comes from a simple remark: suppose we have two solutions z_1, z_2 relative to the same η. Then, for a fixed $L > 0$ and for any $(t,x) \in [0,T] \times [-L,L]$, we have

$$\begin{aligned}
\big|z_1(t,x) - z_2(t,x)\big| &\leq \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \big|b\big(z_1(s,y)\big) - b\big(z_2(s,y)\big)\big|\,dy\,ds \\
&\leq C \int_0^t \int_{x-t+s}^{x+t-s} \big|z_1(s,y) - z_2(s,y)\big|\,dy\,ds.
\end{aligned}$$

It remains to apply Lemma 4.2 to obtain the uniqueness for every $L > 0$, and thus for the equation on the whole space.

Let us now turn to the analysis of the solution operator $F : C([0,T]\times\mathbb{R}) \longrightarrow C([0,T]\times\mathbb{R})$, which is defined by $F(\eta)(t,x) := z(t,x)$. We need to prove that this operator is continuous with respect to the metric of uniform convergence on compact sets. That is, we show the continuity of the restricted mapping

$$F : C\big([0,T]\times[-L,L]\big) \longrightarrow C\big([0,T]\times[-L,L]\big),$$

for every $L > 0$.

We denote by $\|\cdot\|_{\infty,L}$ the supremum norm on $C([0,T]\times[-L,L])$. Let $z_1 := F(\eta_1)$ and $z_2 := F(\eta_2)$ for some $\eta_1, \eta_2 \in C([0,T]\times\mathbb{R})$. Then, for $(t,x)\in[0,T]\times[-L,L]$,

$$\begin{aligned}\big|z_1(t,x) - z_2(t,x)\big| &\le \int_0^t \int_{x-t+s}^{x+t-s} \big|b\big(z_1(s,y)\big) - b\big(z_2(s,y)\big)\big|\,dy\,ds + \big|\eta_1(t,x) - \eta_2(t,x)\big| \\ &\le C \int_0^t \int_{x-t+s}^{x+t-s} \big|z_1(s,y) - z_2(s,y)\big|\,dy\,ds + \|\eta_1 - \eta_2\|_{\infty,L}.\end{aligned}$$

Here, we apply again Lemma 4.2 to obtain that

$$\|z_1 - z_2\|_{\infty,L} \le C\|\eta_1 - \eta_2\|_{\infty,L}.$$ □

4.2. Heat equation: *b* bounded

In this section, we prove Theorem 4.1 for the stochastic heat equation (SHE) in the particular case where the drift b is assumed to be a bounded function. This is necessary in order to construct a Picard iteration scheme to solve equation (20).

Recall that the fundamental solution of the heat equation in $[0,\infty)\times\mathbb{R}$ is given by

$$G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x|^2}{2t}}.$$

As we did in the previous subsection, first we establish an ad hoc version of Grönwall lemma.

Lemma 4.4. *Let* $\{f_n\}_{n\ge1}$, $f_n : [0,T]\times\mathbb{R}\to\mathbb{R}$, *be a sequence of functions that satisfy, for every* $(t,x)\in[0,T]\times\mathbb{R}$, *the following inequality: for some* $\mu, \lambda > 0$,

$$\begin{aligned}&\big|f_{n+1}(t,x) - f_n(t,x)\big| \\ &\quad\le \mu \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|x-y|^2}{2(t-s)}} \big|b\big(f_n(s,y)\big) - b\big(f_{n-1}(s,y)\big)\big|\,dy\,ds + \lambda,\end{aligned}$$

where $b : \mathbb{R}\to\mathbb{R}$ *is bounded and Lipschitz continuous with Lipschitz constant* C. *Then, we have that, for any* $n \ge 1$ *and* $(t,x)\in[0,T]\times\mathbb{R}$,

$$\big|f_{n+1}(t,x) - f_n(t,x)\big| \le 2\|b\|_\infty \frac{C^{n-1}(\mu t)^n}{n!} + \sum_{k=0}^{n-1} \frac{\lambda t^k}{k!}.$$

As a consequence, we also have that

$$\limsup_{n\to\infty}\Big(\sup_{x\in\mathbb{R}}\big|f_{n+1}(t,x)-f_n(t,x)\big|\Big)\le \lambda e^t.$$

Proof. We prove it by induction. First, we compute

$$\begin{aligned}
\big|f_2(t,x)-f_1(t,x)\big| &\le \mu\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}\big|b\big(f_1(s,y)\big)-b\big(f_0(s,y)\big)\big|\,dy\,ds+\lambda\\
&\le 2\mu\|b\|_\infty\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}\,dy\,ds+\lambda\\
&\le 2\mu\|b\|_\infty\int_0^t 1\,ds+\lambda\\
&= 2\mu t\|b\|_\infty+\lambda.
\end{aligned}$$

For the inductive step, we have to exploit the Lipschitz continuity of b:

$$\begin{aligned}
&\big|f_{n+1}(t,x)-f_n(t,x)\big|\\
&\le \mu\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}\big|b\big(f_n(s,y)\big)-b\big(f_{n-1}(s,y)\big)\big|\,dy\,ds+\lambda\\
&\le \mu C\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}\big|f_n(s,y)-f_{n-1}(s,y)\big|\,dy\,ds+\lambda\\
&\le \mu C\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}\Bigg[2\|b\|_\infty\frac{C^{n-2}(\mu s)^{n-1}}{(n-1)!}\\
&\quad+\sum_{k=0}^{n-2}\frac{\lambda s^k}{k!}\Bigg]dy\,ds+\lambda\\
&=\int_0^t\Bigg[2\|b\|_\infty\frac{\mu^nC^{n-1}s^{n-1}}{(n-1)!}+\sum_{k=0}^{n-2}\frac{\lambda s^k}{k!}\Bigg]dy\,ds+\lambda\\
&=2\|b\|_\infty C^{n-1}\frac{(\mu t)^n}{n!}+\sum_{k=1}^{n-1}\frac{\lambda t^k}{k!}+\lambda.
\end{aligned}$$

A direct consequence of this fact is that

$$\limsup_{n\to\infty}\big|f_{n+1}(t,x)-f_n(t,x)\big|\le\lambda e^t,$$

which concludes the proof. □

The proof of Theorem 4.1 in our standing case follows from the following result.

Theorem 4.5. *Let $\eta \in C([0,T]\times\mathbb{R})$ and consider the deterministic equation* (20) *in the case where G is the fundamental solution of the heat equation, and such that b is Lipschitz and bounded. Then,* (20) *has a unique solution $z \in C([0,T]\times\mathbb{R})$. Moreover, the solution operator*

$$F : C\big([0,T]\times\mathbb{R}\big) \to C\big([0,T]\times\mathbb{R}\big)$$

defined by $F(\eta)=z$ is continuous, if we endow $C([0,T]\times\mathbb{R})$ with the metric of uniform convergence on compact sets.

Proof. As in the case of the wave equation, we consider the Picard iteration scheme

$$\begin{aligned}
z_0(t,x) &= \eta(t,x),\\
z_n(t,x) &= \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b\big(z_{n-1}(s,y)\big)\, dy\, ds + \eta(t,x)\\
&= \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|x-y|^2}{2(t-s)}} b\big(z_{n-1}(s,y)\big)\, dy\, ds + \eta(t,x), \quad n \geq 1.
\end{aligned}$$

We clearly have that z_0 is continuous. Assume that z_{n-1} is well-defined and continuous, and we check that z_n is so. The well-definiteness of z_n follows from the fact that b is bounded, which implies that the integral defining $z_n(t,x)$ is convergent for every $(t,x) \in [0,T]\times\mathbb{R}$. Regarding the continuity of z_n, let $(t,x) \in [0,T]\times\mathbb{R}$ and pick a sequence $(t_m, x_m) \to (t,x)$ as $m \to \infty$. Then,

$$\begin{aligned}
z_n(t_m,x_m) &= \int_0^{t_m} \int_{\mathbb{R}} G_{t_m-s}(x_m-y) b\big(z_n(s,y)\big)\, dy\, ds + \eta(t_m,x_m)\\
&= \int_0^{t_m} \int_{\mathbb{R}} G_{s'}\big(y'\big) b\big(z_{n-1}\big(t_m-s', x_m-y'\big)\big)\, dy'\, ds' + \eta(t_m,x_m)\\
&= \int_0^{\sup_m t_m} \int_{\mathbb{R}} 1_{[0,t_m]\times\mathbb{R}}\big(s',y'\big) G_{s'}\big(y'\big) b\big(z_{n-1}\big(t_m-s', x_m-y'\big)\big)\, dy'\, ds'\\
&\quad + \eta(t_m,x_m).
\end{aligned}$$

Thanks to the continuity of b and z_{n-1}, the latter integrand converges point-wise to

$$1_{[0,t]\times\mathbb{R}}\big(s',y'\big) G_{s'}\big(y'\big) b\big(z_{n-1}\big(t-s', x-y'\big)\big).$$

Since b is bounded and G has finite integral over $[0, \sup_m t_m]\times\mathbb{R}$, we can apply the dominated convergence theorem to obtain that

$$\lim_{m\to\infty} z_n(t_m,x_m) = z_n(t,x),$$

so z_n is continuous.

For every $(t,x) \in [0,T] \times \mathbb{R}$, we can infer that

$$\big|z_{n+1}(t,x) - z_n(t,x)\big| \le \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|x-y|^2}{2(t-s)}} \big|b\big(z_n(s,y)\big) - b\big(z_{n-1}(s,y)\big)\big|\, dy\, ds.$$

By Lemma 4.4, we get

$$\big|z_{n+1}(t,x) - z_n(t,x)\big| \le 2\|b\|_\infty \frac{C^{n-1}t^n}{n!} \le 2\|b\|_\infty \frac{C^{n-1}T^n}{n!}.$$

Since the rightmost term of this inequality is the general term of a converging series, and the series does not depend on (t,x), we can infer that the sequence $\{z_n(t,x)\}_{n\ge 0}$ is uniformly Cauchy in $C([0,T]\times\mathbb{R})$. This means that a limit z exists and, since $z_n \to z$ uniformly, $z \in C([0,T]\times\mathbb{R})$. Moreover, it is straightforward to verify that z is the solution to equation (20). Finally, uniqueness of solution can be easily checked by applying again Lemma 4.4.

As far as the continuity of the solution operator $F : C([0,T]\times\mathbb{R}) \to C([0,T]\times\mathbb{R})$ is concerned, where $F(\eta)(t,x) = z(t,x)$, this property can be verified similarly to the case of the wave equation, but applying Lemma 4.4. □

4.3. Heat equation: *b* general

In this section, we aim to verify the validity of Theorem 4.1 for the stochastic heat equation (SHE) in the case of a general globally Lipschitz coefficient b. Recall that the initial condition u_0 is assumed to satisfy Hypothesis C. In particular, u_0 is α-Hölder continuous for some $\alpha > H_0$.

We will use a truncation argument on the drift b: for every $m \ge 1$, set

$$b_m(x) := \begin{cases} b(x) \wedge m, & \text{if } b(x) \ge 0, \\ b(x) \vee -m, & \text{if } b(x) < 0. \end{cases}$$

We have that b_m is bounded and Lipschitz continuous, and converge pointwise to b, as $m \to \infty$. Moreover, a unique Lipschitz constant can be fixed for all functions b_m, $m \ge 1$, and b. We define $u_m^{H_n}$ to be the solution of (12) where b is replaced by b_m, and corresponding to the Hurst index H_n. An immediate consequence of Section 4.2 is that, for any $m \ge 1$,

$$u_m^{H_n} \xrightarrow[n\to\infty]{d} u_m^{H_0} \tag{24}$$

on $C([0,T]\times\mathbb{R})$.

Then, the proof of Theorem 4.1 is split in three steps.

Step 1: First, we check that the family of laws of $\{u^{H_n}\}_{n\ge 1}$ is tight in $C([0,T]\times\mathbb{R})$. For this, we will apply the criterion stated in Theorem 2.7. We point out that, indeed, the computations of this step are valid for both heat and wave equations.

Notice that condition (i) of Theorem 2.7 is clearly satisfied, since $u^{H_n}(0,0)$ is deterministic and does not depend on n. Regarding condition (ii), let $t, t' \in [0,T]$ and $x, x' \in \mathbb{R}$ with $t' \ge t$ and

$x' \geq x$, and we can suppose that $|x - x'| < 1$ and $|t - t'| < 1$. We aim to estimate

$$\begin{aligned} \mathrm{E}\big[\big|u^{H_n}\big(t', x'\big) - u^{H_n}(t, x)\big|^p\big] &\leq C_p\big(\mathrm{E}\big[\big|u^{H_n}\big(t', x'\big) - u^{H_n}\big(t, x'\big)\big|^p\big] \\ &\quad + \mathrm{E}\big[\big|u^{H_n}\big(t, x'\big) - u^{H_n}(t, x)\big|^p\big]\big) \\ &=: C_p(I + J). \end{aligned} \tag{25}$$

We will see that

$$I \leq C_1\big|t' - t\big|^{\beta_I p}, \qquad J \leq C_2\big|x' - x\big|^{\beta_J p}, \tag{26}$$

where $\beta_I, \beta_J > 0$ are two positive constants.

To start with, we have that

$$\begin{aligned} I \leq\ & C_p\Bigg(\big|I_0\big(t', x'\big) - I_0\big(t, x'\big)\big|^p \\ & + \mathrm{E}\bigg[\bigg|\int_0^{t'}\int_{\mathbb{R}} G_{t'-s}\big(x' - y\big) W^{H_n}(ds, dy) - \int_0^t \int_{\mathbb{R}} G_{t-s}\big(x' - y\big) W^{H_n}(ds, dy)\bigg|^p\bigg] \\ & + \mathrm{E}\bigg[\bigg|\int_0^{t'}\int_{\mathbb{R}} G_{t'-s}\big(x' - y\big) b\big(u^{H_n}(s, y)\big)\, dy\, ds \\ & - \int_0^t \int_{\mathbb{R}} G_{t-s}\big(x' - y\big) b\big(u^{H_n}(s, y)\big)\, dy\, ds\bigg|^p\bigg]\Bigg) \\ =:\ & C_p(I_1 + I_2 + I_3). \end{aligned}$$

Regarding I_1, it is known from [4], Theorem 3.7, that, for a α-Hölder continuous initial condition, it holds

$$I_1 \leq C\big|t' - t\big|^{\frac{\alpha p}{2}} \leq C\big|t' - t\big|^{\frac{(\inf_n H_n) p}{2}}. \tag{27}$$

Next, by step 1 in the proof of Theorem 2.8, we clearly obtain that

$$I_2 \leq C\big|t' - t\big|^{\frac{H_n p}{2}} \leq C\big|t' - t\big|^{\frac{(\inf_n H_n) p}{2}}. \tag{28}$$

It remains to estimate I_3. First, in the first summand of I_3 we perform the change of variables $s' = s - (t' - t)$, so that we obtain $I_3 \leq C_p(I_{3,1} + I_{3,2})$, where

$$I_{3,1} := \mathrm{E}\bigg[\bigg|\int_{-(t'-t)}^0 \int_{\mathbb{R}} G_{t-s'}\big(x' - y\big) b\big(u^{H_n}\big(s' + \big(t' - t\big), y\big)\big)\, dy\, ds'\bigg|^p\bigg]$$

and

$$I_{3,2} := \mathrm{E}\bigg[\bigg|\int_0^t \int_{\mathbb{R}} G_{t-s}\big(x' - y\big)\big(b\big(u^{H_n}\big(s + \big(t' - t\big), y\big)\big) - b\big(u^{H_n}(s, y)\big)\big)\, dy\, ds\bigg|^p\bigg].$$

Clearly, $I_{3,1} \leq C|t'-t|^p$ by Hölder inequality, Lemma 3.4 and the linear growth of b. For $I_{3,2}$, we have that

$$\begin{aligned}
I_{3,2} &= \mathrm{E}\bigg[\bigg|\int_0^t \int_{\mathbb{R}} G_{t-s}(x'-y)\big(b\big(u^{H_n}\big(s+(t'-t),y\big)\big)-b\big(u^{H_n}(s,y)\big)\big)\,dy\,ds\bigg|^p\bigg] \\
&\leq C\mathrm{E}\bigg[\int_0^t \int_{\mathbb{R}} G_{t-s}(x'-y)\big|u^{H_n}\big(s+(t'-t),y\big)\big)-u^{H_n}(s,y)\big|^p\,dy\,ds\bigg] \\
&\leq C\int_0^t \int_{\mathbb{R}} G_{t-s}(x'-y)\Big(\sup_{n\geq 1}\sup_{y\in\mathbb{R}} \mathrm{E}\big[\big|u^{H_n}\big(s+(t'-t),y\big)\Big)-u^{H_n}(s,y)\big|^p\big]\big)\,dy\,ds \\
&= C\int_0^t \sup_{n\geq 1}\sup_{y\in\mathbb{R}} \mathrm{E}\big[\big|u^{H_n}\big(s+(t'-t),y\big)\big)-u^{H_n}(s,y)\big|^p\big]\,ds.
\end{aligned}$$

This latter estimate, together with (27) and (28) and the very definition of I, let us infer that

$$\begin{aligned}
&\sup_{n\geq 1}\sup_{x\in\mathbb{R}} \mathrm{E}\big[\big|u^{H_n}\big(t+(t'-t),x\big)-u^{H_n}(t,x)\big|^p\big] \\
&\quad \leq C_1\big|t'-t\big|^{\beta_I p}+C_2\int_0^t \sup_{n\geq 1}\sup_{y\in\mathbb{R}} \mathrm{E}\big[\big|u^{H_n}\big(s+(t'-t),y\big)\big)-u^{H_n}(s,y)\big|^p\big]\,ds,
\end{aligned}$$

where the constants C_1 and C_2 do not depend on H_n and $\beta_I = \frac{1}{2}\inf_n H_n$. Hence, by Grönwall lemma, we obtain the desired estimate for I (see (26)).

Let us now deal with the term J in (25). Assume that $x' = x+h$, for some $h>0$. We have

$$\begin{aligned}
&\mathrm{E}\big[\big|u^{H_n}(t,x+h)-u^{H_n}(t,x)\big|^p\big] \leq C_p\bigg(\big|I_0(t,x+h)-I_0(t,x)\big|^p \\
&\quad +\mathrm{E}\bigg[\bigg|\int_0^t\int_{\mathbb{R}} G_{t-s}(x+h-y)W^{H_n}(ds,dy)-\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W^{H_n}(ds,dy)\bigg|^p\bigg] \\
&\quad +\mathrm{E}\bigg[\bigg|\int_0^t\int_{\mathbb{R}} G_{t-s}(x+h-y)b\big(u^{H_n}(s,y)\big)\,dy\,ds \\
&\quad -\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)b\big(u^{H_n}(s,y)\big)\,dy\,ds\bigg|^p\bigg]\bigg) \\
&=: J_1+J_2+J_3.
\end{aligned} \tag{29}$$

By [4], Theorem 3.7, and step 1 in the proof of Theorem 2.8, we get, respectively,

$$J_1 \leq Ch^{(\inf_n H_n)p} \quad \text{and} \quad J_2 \leq Ch^{(\inf_n H_n)p}. \tag{30}$$

In order to tackle the term J_3, we perform the change of variable $y' = y - h$ in its first summand, yielding

$$J_3 = \mathrm{E}\bigg[\bigg|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y') b\big(u^{H_n}(s, y'+h)\big)\, dy'\, ds \\ - \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b\big(u^{H_n}(s,y)\big)\, dy\, ds\bigg|^p\bigg].$$

Then, renaming the variable y' as y, we have

$$J_3 = \mathrm{E}\bigg[\bigg|\int_0^t \int_{\mathbb{R}} \big(b\big(u^{H_n}(s,y+h)\big) - b\big(u^{H_n}(s,y)\big)\big) G_{t-s}(x-y)\, dy\, ds\bigg|^p\bigg] \\ \leq C \int_0^t \sup_{n\geq 1} \sup_{y\in\mathbb{R}} \mathrm{E}\big[\big|u^{H_n}(s,y+h)) - u^{H_n}(s,y))\big|^p\big]\, ds.$$

Putting together this bound and those of (30), we get

$$\sup_{n\geq 1} \sup_{x\in\mathbb{R}} \mathrm{E}\big[\big|u^{H_n}(t,x+h) - u^{H_n}(t,x)\big|^p\big] \\ \leq C_1 h^{\beta_J p} + C_2 \int_0^t \sup_{n\geq 1} \sup_{y\in\mathbb{R}} \mathrm{E}\big[\big|u^{H_n}(s,y+h)) - u^{H_n}(s,y)\big|^p\big]\, ds,$$

where $\beta_J = \inf_n H_n$. By Grönwall lemma, we conclude that estimates (26) hold. Therefore, by Theorem 2.7, the family of laws of $\{u^{H_n}\}_{n\geq 1}$ is tight in $C([0,T]\times\mathbb{R})$.

Step 2: This part of the proof is devoted to show the following uniform $L^2(\Omega)$-convergence:

$$\sup_{H\in[a,b]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\big[\big|u_m^H(t,x) - u^H(t,x)\big|^2\big] \xrightarrow[m\to\infty]{} 0.$$

We remark that, indeed, the uniformity with respect to $(t,x)\in[0,T]\times\mathbb{R}$ will not be needed in step 3, but we obtain it for free thanks to our Grönwall-type argument exhibited below.

We argue as follows:

$$\mathrm{E}\big[\big|u_m^H(t,x) - u^H(t,x)\big|^2\big] \\ \leq C \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathrm{E}\big[\big|b_m\big(u_m^H(s,y)\big) - b\big(u^H(s,y)\big)\big|^2\big]\, dy\, ds \\ \leq C\bigg(\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathrm{E}\big[\big|b_m\big(u_m^H(s,y)\big) - b_m\big(u^H(s,y)\big)\big|^2\big]\, dy\, ds \\ + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathrm{E}\big[\big|b_m\big(u^H(s,y)\big) - b\big(u^H(s,y)\big)\big|^2\big]\, dy\, ds\bigg) \\ \leq C\bigg(\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathrm{E}\big[\big|u_m^H(s,y) - u^H(s,y)\big|^2\big]\, dy\, ds$$

$$
\begin{aligned}
&+\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[\big|b_m\big(u^H(s,y)\big)-b\big(u^H(s,y)\big)\big|^2 1_{\{|u^H(s,y)|>m\}}\Big]\,dy\,ds\Bigg)\\
&\le C\Bigg(\int_0^t \sup_{H\in[a,b]}\sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\Big[\big|u_m^H\big(s',y\big)-u^H\big(s',y\big)\big|^2\Big]\,ds\\
&+\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[\big|b_m\big(u^H(s,y)\big)-b\big(u^H(s,y)\big)\big|^4\Big]^{\frac{1}{2}}\\
&\times \mathbb{P}\big(\big|u^H(s,y)>m\big|\big)^{\frac{1}{2}}\,dy\,ds\Bigg),
\end{aligned}
\tag{31}
$$

where in the progress we used the fact that $|b_m(u^H(s,y)) - b(u^H(s,y))| = 0$, whenever $|u^H(s,y)| \le m$.

A direct consequence of Lemma 3.4 is that u^H is uniformly bounded in $L^p(\Omega)$, with respect to $H \in [a,b]$ and $(t,x) \in [0,T]\times\mathbb{R}$, for any $p \ge 2$, which means that there exists a constant M_p which depends only on p and T such that

$$
\sup_{H\in[a,b]}\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[\big|u^H(t,x)\big|^p\Big] \le M_p. \tag{32}
$$

Hence, by Markov inequality,

$$
\mathbb{P}\big(\big|u^H(s,y)\big|>m\big) \le \frac{\mathrm{E}[|u^H(s,y)|^2]}{m^2} \le \frac{M_2}{m^2}.
$$

Note that the latter estimate is again uniform with respect to $H \in [a,b]$ and $(s,y) \in [0,T]\times\mathbb{R}$. Thus, going back to (31) and using the linear growth of b and (32), we get

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[\big|b_m\big(u^H(s,y)\big)-b\big(u^H(s,y)\big)\big|^4\Big]^{\frac{1}{2}}\mathbb{P}\big(\big|u^H(s,y)>m\big|\big)^{\frac{1}{2}}\,dy\,ds\\
&\le \int_0^t \int_{\mathbb{R}} C\frac{M_2^{1/2}}{m} G_{t-s}(x-y)\,dy\,ds\\
&\le \int_0^t C\frac{M_2^{1/2}}{m}\,ds =: \frac{C}{m}.
\end{aligned}
\tag{33}
$$

We observe now that if on the left-hand side of (31) we replace t with any $t' \le t$, the inequality would still hold exactly in the same way (indeed, the integrand on the right-hand side is positive, so it is increasing as a function of t). Therefore, we can infer that

$$
\begin{aligned}
&\sup_{H\in[a,b]}\sup_{(t',x)\in[0,t]\times\mathbb{R}} \mathrm{E}\Big[\big|u_m^H\big(t',x\big)-u^H\big(t',x\big)\big|^2\Big]\\
&\le \frac{C_1}{m} + C_2\int_0^t \sup_{H\in[a,b]}\sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\Big[\big|u_m^H\big(s',y\big)-u^H\big(s',y\big)\big|^2\Big]\,ds.
\end{aligned}
$$

Then, Grönwall lemma implies that

$$\sup_{H\in[a,b]}\sup_{(t',x)\in[0,T]\times\mathbb{R}} \mathrm{E}\big[\big|u_m^H\big(t',x\big)-u^H\big(t',x\big)\big|^2\big] \leq \frac{C}{m} \xrightarrow[m\to\infty]{} 0,$$

which is what we wanted to show.

Step 3: We prove that the finite dimensional distributions of u^{H_n} converge to those of u^{H_0}. Given a finite dimensional vector $\{(t_1,x_1),\dots,(t_k,x_k)\}$ and $f\in C_b(\mathbb{R}^k)$, we can write

$$\begin{aligned}
&\big|\mathrm{E}\big[f\big(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k)\big)-f\big(u^{H_0}(t_1,x_1),\dots,u^{H_0}(t_k,x_k)\big)\big]\big| \\
&\quad\leq \big|\mathrm{E}\big[f\big(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k)\big)-f\big(u_m^{H_n}(t_1,x_1),\dots,u_m^{H_n}(t_k,x_k)\big)\big]\big| \\
&\qquad+\big|\mathrm{E}\big[f\big(u_m^{H_n}(t_1,x_1),\dots,u_m^{H_n}(t_k,x_k)\big)-f\big(u_m^{H_0}(t_1,x_1),\dots,u_m^{H_0}(t_k,x_k)\big)\big]\big| \\
&\qquad+\big|\mathrm{E}\big[f\big(u_m^{H_0}(t_1,x_1),\dots,u_m^{H_0}(t_k,x_k)\big)-f\big(u^{H_0}(t_1,x_1),\dots,u^{H_0}(t_k,x_k)\big)\big]\big| \\
&\quad=: I_1(m,n)+I_2(m,n)+I_3(m).
\end{aligned}$$

Assume that $f:\mathbb{R}^k\to\mathbb{R}$ is Lipschitz continuous with Lipschitz constant L_f (we can always restrict to the class of Lipschitz continuous functions to verify weak convergence). Then, for all $H\in[a,b]$,

$$\begin{aligned}
&\sup_{H\in[a,b]}\big|\mathrm{E}\big[f\big(u^H(t_1,x_1),\dots,u^H(t_k,x_k)\big)-f\big(u_m^H(t_1,x_1),\dots,u_m^H(t_k,x_k)\big)\big]\big| \\
&\quad\leq \sup_{H\in[a,b]}\mathrm{E}\big[\big|f\big(u^H(t_1,x_1),\dots,u^H(t_k,x_k)\big)-f\big(u_m^H(t_1,x_1),\dots,u_m^H(t_k,x_k)\big)\big|\big] \\
&\quad\leq \sup_{H\in[a,b]} L_f\mathrm{E}\Bigg[\Bigg(\sum_{j=1}^k\big|u_m^H(t_j,x_j)-u^H(t_j,x_j)\big|^2\Bigg)^{1/2}\Bigg] \\
&\quad\leq L_f\sup_{H\in[a,b]}\Bigg(\mathrm{E}\Bigg[\sum_{j=1}^k\big|u_m^H(t_j,x_j)-u^H(t_j,x_j)\big|^2\Bigg]\Bigg)^{1/2} \\
&\quad= L_f\sup_{H\in[a,b]}\Bigg(\sum_{j=1}^k\mathrm{E}\big[\big|u_m^H(t_j,x_j)-u^H(t_j,x_j)\big|^2\big]\Bigg)^{1/2} \\
&\quad\leq L_f k^{\frac{1}{2}}\Big(\sup_{H\in[a,b]}\sup_{(t,x)\in[0,T]\times\mathbb{R}}\mathrm{E}\big[\big|u_m^H(t,x)-u^H(t,x)\big|^2\big]\Big)^{1/2},
\end{aligned} \tag{34}$$

where the last term converges to 0 as $m\to\infty$ thanks to step 2, and taking into account that we are considering an arbitrary but fixed number of terms k. Hence, for any $\varepsilon>0$, there exists $m_0\geq 1$ such that, for all $m\geq m_0$, we have

$$\sup_{n\geq 1}\big(I_1(m,n)+I_3(m)\big)\leq\frac{\varepsilon}{2}.$$

In particular, we have

$$\left|\mathrm{E}\left[f\left(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k)\right)-f\left(u^{H_0}(t_1,x_1),\dots,u^{H_0}(t_k,x_k)\right)\right]\right|\le I_2(m_0,n)+\frac{\varepsilon}{2}.$$

Finally, it is sufficient to observe that the convergence (24) implies the corresponding convergence of the finite dimensional distributions, and thus for some $n_0 \geq 1$ we have that, for all $n \geq n_0$, it holds $I_2(m_0,n) < \frac{\varepsilon}{2}$. Therefore,

$$\left|\mathrm{E}\left[f\left(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k)\right)-f\left(u^{H_0}(t_1,x_1),\dots,u^{H_0}(t_k,x_k)\right)\right]\right|<\varepsilon,$$

where ε can be taken arbitrary small. This concludes the proof of Theorem 4.1 for the stochastic heat equation (SHE) in the case of a general Lipschitz continuous drift b.

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