A 3D isothermal model for nematic liquid crystals with delay terms

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Abstract

In this paper we consider a model describing the evolution of a nematic liquid crystals flow with delay external forces. We analyze the evolution of the velocity field $\boldsymbol{u}$ which is ruled by the 3D incompressible Navier-Stokes system containing a delay term and with a stress tensor expressing the coupling between the transport and the induced terms. The dynamics of the director field $\boldsymbol{d}$ is described by a modified Allen-Cahn equation with a suitable penalization of the physical constraint $|\boldsymbol{d}| = 1$. We prove the existence of global in time weak solutions under appropriate assumptions, which in some cases requires the delay term is small with respect to the viscosity parameter.

Key words: Liquid crystal, Navier-Stokes system, delay terms

AMS (MOS) subject classification: 35D30, 35Q30, 76A15

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1 Introduction

We consider a well known system modeling the flow of nematic liquid crystals when the stretching effects are taken into account (see [26] and [28]). The material occupies a bounded spatial domain $\Omega \subset \mathbb{R}^3$ and the evolution of the state variables $\boldsymbol{u}$ and $\boldsymbol{d}$ governing the dynamics is ruled by

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \text{in } \Omega_T, \tag{1.1}$$
$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \mathbb{T} + \boldsymbol{f}, \qquad \text{in } \Omega_T, \tag{1.2}$$
$$\partial_t \boldsymbol{d} + \boldsymbol{u} \cdot \nabla \boldsymbol{d} - \alpha \boldsymbol{d} \cdot \nabla \boldsymbol{u} + (1-\alpha) \boldsymbol{d} \cdot \nabla^T \boldsymbol{u} = \gamma(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})), \quad \text{in } \Omega_T, \tag{1.3}$$

where $\Omega_T := (0,T) \times \Omega$ and the tensors $\mathbb{T}$, $\mathbb{S}$ are defined as follows

$$\mathbb{T} = \mathbb{S} - \lambda (\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) - \alpha\lambda(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \otimes \boldsymbol{d} + (1-\alpha)\lambda \boldsymbol{d} \otimes (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})), \tag{1.4}$$
$$\mathbb{S} = \mu \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u} \right). \tag{1.5}$$

The system consists of two coupled equations, namely, the 3D incompressible Navier-Stokes equations for the velocities $\boldsymbol{u}$, and a modified Allen-Cahn equation for the director field $\boldsymbol{d}$.

In the system p represents the scalar hydrodynamic pressure, $\mathbb{T}$ and $\mathbb{S}$ are the Cauchy stress and the Newtonian viscous stress tensors, respectively, while $\boldsymbol{f}$ is a given external force. In addition, μ, λ and γ denote the viscosity, the competition between kinetic energy and potential energy, and the microscopic elastic relaxation time (Deborah number) for the molecular orientation field, respectively.

The role of the term function W consists in the penalization of the deviation of the length $|\boldsymbol{d}|$ from the value 1, which is due to liquid crystal molecules being of similar size (cf. [14]). As a typical example, we can consider a double well potential given by $W(\boldsymbol{d}) = (|\boldsymbol{d}|^2 - 1)^2$. Finally, $\alpha \in [0,1]$ is a parameter related to the shape of the liquid crystal molecules.

As for the notation we will use throughout this paper, $\nabla_{\boldsymbol{d}}$ denotes the gradient with respect to the variable $\boldsymbol{d}$. $\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}$ stands for the 3×3 matrix whose (i,j)-th entry is given by $\nabla_i \boldsymbol{d} \cdot \nabla_j \boldsymbol{d}$, for $i \leq i, j \leq 3$, and $\otimes$ indicates the usual Kronecker product, i.e., $(\boldsymbol{u} \otimes \boldsymbol{u})_{ij} := \boldsymbol{u}_i \boldsymbol{u}_j$, for $i,j = 1,2,3$. Finally, ∇^T will be used to denote the transpose of the gradient.

A detailed derivation of liquid crystal models and their importance in applications can be found, e.g., in [4], [5], [10], [12], [13] [14], [15], [24], [26], [28] (see also the references therein). In this context, the mathematical analysis of these models is quite wide. We recall here, for instance, the contributions contained in [2], [3], [6], [7], [9], [18], [23], [26].

In our opinion, the analysis of an evolution problem is more accurate if we take into account the history of the phenomenon, since the future evolution of the problem is influenced, in one or another way, by what has happened in the recent or long term history (finite or infinite delay, respectively). This justifies the consideration of delay or memory terms in the formulation of the equations. Moreover, in most control problems, the construction of feedback controls requires the use of delay terms (see, for instance, [1], [8], [19], [29], [30] where several physical models containing delay or memory have been studied).

Therefore, the model we will analyze in the current paper includes an additional forcing term taking into account some past history information of the system. Consequently, we replace equation (1.2) with the following one

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \mathbb{T} + \boldsymbol{f} + \boldsymbol{g}(t, \boldsymbol{u}_t), \qquad \text{in } \Omega_T, \tag{1.6}$$

where the expression for the delay term $\boldsymbol{g}$ is given in a functional way so that several types of delays can be considered in a unified formulation (see [1]). The notation $\boldsymbol{u}_t$ will be used to denote the segment of the solution in the time interval $[t-h, t]$, where $h > 0$ denotes the maximum delay of the problem. More precisely

$$\boldsymbol{u}_t(s) = \boldsymbol{u}(t+s), \text{ for } s \in [-h, 0].$$

Main object of our investigation in this paper is to generalize the results proved in [2] to the case in which different types of delay appear in the system. We underline that in [2] no restriction on the viscosity coefficient μ or on the norms of the data are assumed in order to prove in a rigorous way the existence of global well defined weak solutions in 3D (compare with the results contained in [3]). This result is obtained by means of an appropriate choice of the test functions in the variational formulation and of a suitable regularization procedure in order to treat the high order stretching terms in the momentum equation. For the details on this particular kind of regularization see [16], [17] and [11].

However, in the case we are going to analyze, due to the appearance of the delay terms, it is necessary to impose a smallness condition on $\boldsymbol{g}$ with respect to the viscosity parameter μ, when this delay term is allowed to contain second order partial derivatives (see Remark 3.2 for more details). This makes a difference with the non-delay case. Moreover, as in [2] we will analyze two meaningful cases of boundary conditions for the director field $\boldsymbol{d}$: homogeneous Neumann boundary conditions and non-homogeneous Dirichlet boundary conditions.

The structure of the paper is the following. In Section 2 we introduce the initial and boundary value problems in the two different cases of boundary conditions for $\boldsymbol{d}$ and their weak formulation. In Section 3, we enlist the assumptions on the data and state the two theorems regarding existence of global in time solutions. The proof of the main results is given in the two following Sections 4 and 5. In particular, in Section 4 some a priori estimates are shown, from which we deduce rigorously the approximated Faedo-Galerkin scheme presented in Section 5. Finally, in the Appendix we will furnish some meaningful examples of the delay term $\boldsymbol{g}$.

2 Formulation of the problems

Here we associate to system (1.1), (1.6), (1.3) two different sets of initial and boundary conditions: in the first case $\boldsymbol{d}$ satisfies a homogeneous Neumann boundary condition and in the second case a non-homogeneous Dirichlet boundary condition.

The *initial* condition for $\boldsymbol{u}$ is assigned accordingly to the presence of the delay term $\boldsymbol{g}$.

We assume that $\Gamma := \partial\Omega$ is smooth enough and we set $\Gamma_T := (0,T)\times\Gamma$, $\Gamma_{h,T} := (-h,T)\times\Gamma$.

We will analyze the following problems where, for the sake of simplicity, we have taken $\gamma = \lambda = 1$.

Problem (P1)

$$
\begin{aligned}
&\operatorname{div}\boldsymbol{u} = 0, \quad \text{in } \Omega_T, && (2.1)\\
&\partial_t\boldsymbol{u} + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \operatorname{div}\left(\mu\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\right) - \operatorname{div}(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) && (2.2)\\
&\;-\operatorname{div}(\alpha(\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}))\otimes\boldsymbol{d} - (1-\alpha)\boldsymbol{d}\otimes(\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d})))\\
&\;+\boldsymbol{f} + \boldsymbol{g}(t,\boldsymbol{u}_t), \quad \text{in } \Omega_T,\\
&\partial_t\boldsymbol{d} + \boldsymbol{u}\cdot\nabla\boldsymbol{d} - \alpha\boldsymbol{d}\cdot\nabla\boldsymbol{u} + (1-\alpha)\boldsymbol{d}\cdot\nabla^T\boldsymbol{u} = (\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d})), \quad \text{in } \Omega_T, && (2.3)\\
&\boldsymbol{u}(s,\cdot) = \boldsymbol{u}_0(s,\cdot),\; s\in[-h,0], \quad \text{in } \Omega && (2.4)\\
&\boldsymbol{d}(0,\cdot) = \boldsymbol{d}_0, \quad \text{in } \Omega, && (2.5)\\
&\boldsymbol{u} = \boldsymbol{0}, \quad \text{on } \Gamma_{h,T}, && (2.6)\\
&\partial_{\boldsymbol{n}}\boldsymbol{d} = \boldsymbol{0}, \quad \text{on } \Gamma_T, && (2.7)
\end{aligned}
$$

Problem (P2)

$$
\begin{aligned}
&\operatorname{div}\boldsymbol{u} = 0, \quad \text{in } \Omega_T, && (2.8)\\
&\partial_t\boldsymbol{u} + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \operatorname{div}\left(\mu\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\right) - \operatorname{div}(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) && (2.9)\\
&\;-\operatorname{div}(\alpha(\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}))\otimes\boldsymbol{d} - (1-\alpha)\boldsymbol{d}\otimes(\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d})))\\
&\;+\boldsymbol{f} + \boldsymbol{g}(t,\boldsymbol{u}_t), \quad \text{in } \Omega_T,\\
&\partial_t\boldsymbol{d} + \boldsymbol{u}\cdot\nabla\boldsymbol{d} - \alpha\boldsymbol{d}\cdot\nabla\boldsymbol{u} + (1-\alpha)\boldsymbol{d}\cdot\nabla^T\boldsymbol{u} = (\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d})), \quad \text{in } \Omega_T, && (2.10)\\
&\boldsymbol{u}(s,\cdot) = \boldsymbol{u}_0(s,\cdot),\; s\in[-h,0], \quad \text{in } \Omega && (2.11)\\
&\boldsymbol{d}(0,\cdot) = \boldsymbol{d}_0, \quad \text{in } \Omega, && (2.12)\\
&\boldsymbol{u} = \boldsymbol{0}, \quad \text{on } \Gamma_{h,T}, && (2.13)\\
&\boldsymbol{d}|_\Gamma = \boldsymbol{h}, \quad \text{on } \Gamma_T. && (2.14)
\end{aligned}
$$

We introduce now the weak formulation of Problems (P1) and (P2) in which the momentum and the incompressible constraint equations are replaced by a family of integral identities, while the equation for the director field holds in the strong sense, due to the regularity we will obtain for $\boldsymbol{d}$.

Here the appropriate choice of the test functions leads to a rigorous weak formulation of Problem (P1) and Problem (P2) and in addition it will allow us to treat the stretching term in the stress tensor (compare with the results contained in [3] and see [2]).

Problem (P1) - weak formulation

A weak solution of Problem (P1) is a pair $(\boldsymbol{u},\, \boldsymbol{d})$ satisfying

$$\int_\Omega \boldsymbol{u}(t,\cdot)\cdot\nabla\varphi = 0, \quad \text{for a.a. } t\in(0,T), \tag{2.15}$$

$$\langle \partial_t \boldsymbol{u}, \varphi\rangle - \int_\Omega \boldsymbol{u}\otimes\boldsymbol{u} : \nabla\varphi + \int_\Omega \mu\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right) : \nabla\varphi = \int_\Omega (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) : \nabla\varphi \tag{2.16}$$

$$+\,\alpha\int_\Omega (\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}))\otimes\boldsymbol{d} : \nabla\varphi - (1-\alpha)\int_\Omega \boldsymbol{d}\otimes(\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d})) : \nabla\varphi$$

$$+\int_\Omega (\boldsymbol{f} + \boldsymbol{g}(\cdot,\boldsymbol{u}.))\cdot\varphi, \quad \text{for a.a. } t\in(0,T), \quad \forall\,\varphi\in W_0^{1,3}(\Omega;\mathbb{R}^3), \text{ s.t. } \operatorname{div}\varphi = 0,$$

$$\partial_t\boldsymbol{d} + \boldsymbol{u}\cdot\nabla\boldsymbol{d} - \alpha\boldsymbol{d}\cdot\nabla\boldsymbol{u} + (1-\alpha)\boldsymbol{d}\cdot\nabla^T\boldsymbol{u} = \Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}), \quad \text{a.e. in } \Omega_T, \tag{2.17}$$

$$\partial_{\boldsymbol{n}}\boldsymbol{d} = \boldsymbol{0}, \quad \text{a.e. on } \Gamma_T, \tag{2.18}$$

$$\boldsymbol{d}(0,\cdot) = \boldsymbol{d}_0, \quad \text{a.e. in } \Omega, \tag{2.19}$$

$$\boldsymbol{u}(s,\cdot) = \boldsymbol{u}_0(s,\cdot), \quad \text{a.e. } s\in(-h,0), \text{ a.e. in}\,\Omega. \tag{2.20}$$

Problem (P2) - weak formulation

A weak solution of Problem (P2) is a pair $(\boldsymbol{u},\, \boldsymbol{d})$ satisfying

$$\int_\Omega \boldsymbol{u}(t,\cdot)\cdot\nabla\varphi = 0, \quad \text{for a.a. } t\in(0,T), \tag{2.21}$$

$$\langle \partial_t \boldsymbol{u}, \varphi\rangle - \int_\Omega \boldsymbol{u}\otimes\boldsymbol{u} : \nabla\varphi + \int_\Omega \mu\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right) : \nabla\varphi = \int_\Omega (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) : \nabla\varphi \tag{2.22}$$

$$+\,\alpha\int_\Omega (\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}))\otimes\boldsymbol{d} : \nabla\varphi - (1-\alpha)\int_\Omega \boldsymbol{d}\otimes(\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d})) : \nabla\varphi$$

$$+\int_\Omega (\boldsymbol{f} + \boldsymbol{g}(\cdot,\boldsymbol{u}.))\cdot\varphi, \quad \text{for a.a. } t\in(0,T), \forall\,\varphi\in W_0^{1,3}(\Omega;\mathbb{R}^3), \operatorname{div}\varphi = 0,$$

$$\partial_t\boldsymbol{d} + \boldsymbol{u}\cdot\nabla\boldsymbol{d} - \alpha\boldsymbol{d}\cdot\nabla\boldsymbol{u} + (1-\alpha)\boldsymbol{d}\cdot\nabla^T\boldsymbol{u} = \Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}), \text{ a.e. in } \Omega_T, \tag{2.23}$$

$$\boldsymbol{d}_{|}\Gamma = \mathbf{h} \quad \text{a.e. on } \Gamma_T, \tag{2.24}$$

$$\boldsymbol{d}(0,\cdot) = \boldsymbol{d}_0, \quad \text{a.e. in } \Omega, \tag{2.25}$$

$$\boldsymbol{u}(s,\cdot) = \boldsymbol{u}_0(s,\cdot), \quad \text{a.e. } s\in(-h,0), \text{ a.e. in}\,\Omega. \tag{2.26}$$

3 Assumptions and main results

Here we introduce the assumptions on the data and state our main results concerning the existence of global-in-time weak solutions, without any restriction imposed on the initial data or on μ.

3.1 Assumptions on the data

We enlist the hypotheses on the known data of the problem. In particular we will describe with full details the delay function $\boldsymbol{g}$ in which relies the novelty of this paper.

$$\mu > 0,\ \alpha \in [0,1], \tag{3.1}$$
$$W \in C^2(\mathbb{R}^3), \quad W \geq 0, \tag{3.2}$$
$$W = W_1 + W_2 \text{ s.t. } W_1 \text{ is convex and } W_2 \in C^1(\mathbb{R}^3),\ \nabla W_2 \in C^{0,1}(\mathbb{R}^3;\mathbb{R}^3) \tag{3.3}$$
$$\boldsymbol{f} \in L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^3)), \tag{3.4}$$
$$\boldsymbol{g}(\cdot,\cdot) : (0,T) \times L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3)) \to W^{-1,2}(\Omega;\mathbb{R}^3) \text{ satisfies} \tag{3.5}$$

$(g1)$ $\forall \boldsymbol{\xi} \in L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3))$ the mapping $t \in [0,T] \to \boldsymbol{g}(t,\boldsymbol{\xi}) \in W^{-1,2}(\Omega;\mathbb{R}^3)$ is measurable,
$(g2)$ for all $t \in [0,T], \boldsymbol{g}(t,0) = 0$,
$(g3)$ there exists $L_g > 0$ such that $\forall\, t \in [0,T], \quad \forall \boldsymbol{\xi},\boldsymbol{\eta} \in L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3))$

$$\|\boldsymbol{g}(t,\boldsymbol{\xi}) - \boldsymbol{g}(t,\boldsymbol{\eta})\|_{W^{-1,2}(\Omega;\mathbb{R}^3)} \leq L_g \, \|\boldsymbol{\xi} - \boldsymbol{\eta}\|_{L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3))},$$

$(g4)$ there exists $C_g > 0$ such that $\forall\, t \in [0,T],\ \forall\, \boldsymbol{u},\boldsymbol{v} \in L^2(-h,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$

$$\int_0^t \|\boldsymbol{g}(s,\boldsymbol{u}_s) - \boldsymbol{g}(s,\boldsymbol{v}_s)\|^2_{W^{-1,2}(\Omega;\mathbb{R}^3)}\, ds \leq C_g^2 \int_{-h}^t \|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\, ds,$$

$(g5)$ if the sequence $\boldsymbol{v}^m$ converges weakly to $\boldsymbol{v}$ in $L^2(-h,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$ and strongly in $L^2(-h,T;L^2(\Omega;\mathbb{R}^3))$, then $\boldsymbol{g}(\cdot,\boldsymbol{v}^m_\cdot)$ converges weakly to $\boldsymbol{g}(\cdot,\boldsymbol{v}_\cdot)$ in $L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))$,

$$\tilde{\mu} := 2\mu - C_g > 0, \tag{3.6}$$
$$\boldsymbol{u}_0(\cdot,\cdot) \in L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3)), \quad \operatorname{div}(\boldsymbol{u}_0(t,\cdot) = 0 \text{ in } L^2(\Omega), \forall t \geq 0, \tag{3.7}$$
$$\boldsymbol{d}_0 \in W^{1,2}(\Omega;\mathbb{R}^3), \quad W(\boldsymbol{d}_0) \in L^1(\Omega), \tag{3.8}$$
$$\boldsymbol{h} \in H^1(0,T;H^{-1/2}(\Gamma;\mathbb{R}^3)) \cap L^\infty(0,T;H^{3/2}(\Gamma;\mathbb{R}^3)), \qquad \boldsymbol{h}(0) = \boldsymbol{d}_{0|\Gamma} \tag{3.9}$$

3.2 Statement of the existence theorems

The first result related to Problem (P1) is the following.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary Γ of class $C^{1,1}$. Assume that hypotheses (3.1)–(3.8) are fulfilled. Then problem (2.15)–(2.20) admits a global in time weak solution $(\boldsymbol{u},\ \boldsymbol{d})$ such that*

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(-h,T;W_0^{1,2}(\Omega;\mathbb{R}^3)), \tag{3.10}$$

$$\partial_t \boldsymbol{u} \in L^2(0,T;W^{-1,3/2}(\Omega;\mathbb{R}^3)), \tag{3.11}$$

$$W(\boldsymbol{d}) \in L^\infty(0,T;L^1(\Omega)), \tag{3.12}$$

$$\boldsymbol{d} \in L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{2,2}(\Omega;\mathbb{R}^3)) \cap H^1(0,T;L^{3/2}(\Omega;\mathbb{R}^3)). \tag{3.13}$$

In addition, the following energy inequality holds true, for a.a. $t \in (0,T)$,

$$\int_\Omega \big(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d})\big)(t) - \int_\Omega \big(|\boldsymbol{u}_0(0,x)|^2 + |\nabla \boldsymbol{d}_0|^2 + 2W(\boldsymbol{d}_0)\big) \tag{3.14}$$
$$+ 2\int_0^t \big\|(-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))(s)\big\|^2_{L^2(\Omega;\mathbb{R}^3)}\,ds + \tilde{\mu}\int_0^t \|\nabla \boldsymbol{u}(s)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} ds$$
$$\le C\int_0^t \|\boldsymbol{f}(s)\|^2_{W^{-1,2}(\Omega;\mathbb{R}^3)}\,ds + C_g\|\boldsymbol{u}_0\|^2_{L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3))},$$

where C is a positive constant depending on Ω and C_g is as in (g4).

Remark 3.2. The assumption $\tilde{\mu} > 0$ imposes some kind of smallness of the delay term with respect to μ. This is necessary in the general case of having $\boldsymbol{g}$ taking values in $W^{-1,2}(\Omega;\mathbb{R}^3)$. However, in the particular case in which $\boldsymbol{g}$ takes values in $L^2(\Omega;\mathbb{R}^3)$, this assumption can be avoided (see Garcia-Luengo et al. [8] for a similar situation in the case of Navier-Stokes in 2D).

Remark 3.3. Thanks to (3.10), (3.12) and (3.13), we can deduce

$$\boldsymbol{u}\otimes\boldsymbol{u}, \quad \nabla \boldsymbol{d}\odot\nabla\boldsymbol{d}, \quad (\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}))\otimes\boldsymbol{d} \in L^2(0,T;L^{3/2}(\Omega;\mathbb{R}^{3\times 3})), \tag{3.15}$$

so that their (distributional) divergence belong to

$$L^2(0,T;W^{-1,3/2}(\Omega;\mathbb{R}^3)).$$

This justifies the choice of the regularity of the test function φ in (2.16).

Observe that this approach does not depend on the fact that we are considering the 3D case. Indeed, also in the 2D case, in order to obtain the existence of weak solutions we need the same kind of weak formulation (cf. also [26] and [28]).

As for Problem (P2), the main result reads as follows.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary Γ of class $C^{1,1}$. Assume that hypotheses (3.1)–(3.9) are satisfied. Then problem (2.21)–(2.26) admits a global in time weak solution $(\boldsymbol{u},\ \boldsymbol{d})$ satisfying (3.10)–(3.13). Moreover the following energy inequality holds, for a.a. $t \in (0,T)$,*

$$\int_\Omega \big(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d})\big)(t) - \int_\Omega \big(|\boldsymbol{u}_0(0,x)|^2 + |\nabla \boldsymbol{d}_0|^2 + 2W(\boldsymbol{d}_0)\big) \tag{3.16}$$
$$+ 2\int_0^t \big\|(-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))(s)\big\|^2_{L^2(\Omega;\mathbb{R}^3)}\,ds + \tilde{\mu}\int_0^t \|\nabla \boldsymbol{u}(s)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} ds$$
$$\le C\left(\int_0^t \Big(\|\boldsymbol{h}_t(s)\|^2_{H^{-1/2}(\Gamma;\mathbb{R}^3)} + \|\boldsymbol{h}(s)\|^2_{H^{3/2}(\Gamma;\mathbb{R}^3)} + \|\nabla_{\boldsymbol{d}}W(\boldsymbol{h}(s))\|_{L^2(\Gamma;\mathbb{R}^3)}\Big)\,ds\right.$$
$$\left. + \int_0^t \|\boldsymbol{f}(s)\|^2_{W^{-1,2}(\Omega;\mathbb{R}^3)} ds\right) + C_g\|\boldsymbol{u}_0\|^2_{L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3))},$$

where C is a positive constant depending on Ω and C_g is as in (g4).

4 A priori bounds

Following the steps contained in [2] (see also [6]), we show here several formal a priori estimates, as well as the energy inequalities (3.14) and (3.16). By means of the Faedo-Galerkin approximation scheme decribed in Section 5 below all these estimates can be validated in a rigourous way.

Let us consider first the weak formulation of Problem (P1). We take $\varphi = \boldsymbol{u}$ in (2.16) and test (2.17) by $-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d})$ on Ω. Summing up the two resulting equations, by means of the divergence theorem and using (2.15), we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \big(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d})\big) + \mu\int_\Omega |\nabla \boldsymbol{u}|^2 + \int_\Omega |-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d})|^2 = {}_{H^{-1}}\langle \boldsymbol{f} + \boldsymbol{g}(t, \boldsymbol{u}_t), \boldsymbol{u}\rangle_{W_0^{1,2}}. \tag{4.1}$$

Moreover, integrating in time on the interval $(0,t)$, taking into account assumption $(g1)-(g4)$, applying the Schwarz, Poincaré and Young inequalities on the right hand side, then, by straightforward computations, we infer the energy estimate (3.14), where we recall that the constant $\tilde{\mu}$ is defined as $\tilde{\mu} = 2\mu - C_g$.

Integrating again in time on $(0,t)$ equation (4.1), by means of assumptions (3.2)–(3.5), we deduce the a priori bounds

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap L^{10/3}((0,T)\times\Omega;\mathbb{R}^3), \tag{4.2}$$

$$\boldsymbol{d} \in L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{4.3}$$

$$-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d}) \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))\,. \tag{4.4}$$

Taking advantage of (2.7) and (3.2), from (4.4) we have

$$\int_0^T\int_\Omega |\Delta \boldsymbol{d}|^2 + \int_0^T\int_\Omega \nabla(\nabla_{\boldsymbol{d}} W(\boldsymbol{d}))\nabla \boldsymbol{d} = \int_0^T\int_\Omega \boldsymbol{m}\cdot\Delta \boldsymbol{d},$$

where the function $\boldsymbol{m} \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$, from which, recalling assumption (3.3), we deduce

$$\frac{1}{2}\int_0^T\int_\Omega |\Delta \boldsymbol{d}|^2 \le \int_0^T\int_\Omega |\nabla(\nabla_{\boldsymbol{d}} W_2(\boldsymbol{d}))||\nabla \boldsymbol{d}| + \frac{1}{2}\int_0^T\int_\Omega |\boldsymbol{m}|^2.$$

Thanks to (4.3), we have $|\nabla \boldsymbol{d}| \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$. Then, on account of (4.4) and (3.3), it holds

$$\boldsymbol{d} \in L^2(0,T;W^{2,2}(\Omega;\mathbb{R}^3)), \quad \nabla_{\boldsymbol{d}} W(\boldsymbol{d}) \in L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{4.5}$$

Going back to (2.17), on account of (4.5) and the fact that $\boldsymbol{u}\cdot\nabla \boldsymbol{d}$ and $\boldsymbol{d}\cdot\nabla \boldsymbol{u}$ belong to $L^2(0,T;L^{3/2}(\Omega;\mathbb{R}^3))$, by comparison we deduce

$$\partial_t \boldsymbol{d} \in L^2(0,T;L^{3/2}(\Omega;\mathbb{R}^3)). \tag{4.6}$$

Consider now $q(1-a)=2$ in the following interpolation inequality

$$\|\nabla \boldsymbol{d}\|^q_{L^s(\Omega;\mathbb{R}^{3\times 3})} \leq c_1 \|\nabla \boldsymbol{d}\|^{aq}_{L^2(\Omega;\mathbb{R}^{3\times 3})} \|\nabla \boldsymbol{d}\|^{(1-a)q}_{L^6(\Omega;\mathbb{R}^{3\times 3})}, \tag{4.7}$$

where

$$s,\, q \in [1,+\infty),\ a \in (0,1), \quad 1/s = (1-a)/6 + a/2. \tag{4.8}$$

Taking advantage of (4.3–4.5), we get

$$\nabla \boldsymbol{d} \in L^{4s/(3s-6)}(0,T; L^s(\Omega;\mathbb{R}^{3\times 3}))\,, \tag{4.9}$$

which gives, for $s = 10/3$, the crucial estimate

$$\nabla \boldsymbol{d} \in L^{10/3}(0,T; L^{10/3}(\Omega;\mathbb{R}^{3\times 3})). \tag{4.10}$$

A combination of the previous results implies

$$\begin{aligned} &(-(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) + \alpha(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \otimes \boldsymbol{d} - (1-\alpha)\boldsymbol{d} \otimes (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))) \\ &\in L^{5/3}((0,T)\times\Omega;\mathbb{R}^{3\times 3}) \end{aligned}$$

and

$$\begin{aligned} &(-(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) + \alpha(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \otimes \boldsymbol{d} - (1-\alpha)\boldsymbol{d} \otimes (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))) \\ &\in L^{2}(0,T; L^{3/2}(\Omega;\mathbb{R}^{3\times 3})). \end{aligned}$$

Taking into account the a priori estimates (4.2), (4.3) and (4.6) we can deduce that any solution satisfies the regularity conditions (3.10) and (3.13), from which it follows (3.15) and then (3.11).

Finally, it is possible to prove the weak stability of the solutions to problem (2.15)–(2.20) with respect to the a priori bounds, namely, taking any sequence of weak solutions satisfying the above uniform bounds then it admits a convergent subsequence. We omit here the details of the proof.

Consider now Problem (P2). We note that since $\boldsymbol{d}$ satisfies a non-homogeneous Dirichlet boundary condition, then we deduce

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \big(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d})\big) + \mu \int_\Omega |\nabla \boldsymbol{u}|^2 + \int_\Omega |-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d})|^2 \\ &\quad = {}_{H^{-1/2}(\Gamma)}\langle \boldsymbol{h}_t, \partial_{\boldsymbol{n}} \boldsymbol{d}\rangle_{H^{1/2}(\Gamma)} + {}_{H^{-1}}\langle \boldsymbol{f} + \boldsymbol{g}(t,\boldsymbol{u}_t), \boldsymbol{u}\rangle_{W^{1,2}_0}. \end{aligned} \tag{4.11}$$

Integrating (4.11) in time on $(0,t)$, by using assumption $(g1)-(g4)$ we can estimate the term containing the delay as in the case of Problem (P1). Hence it holds

$$\begin{aligned} &\int_\Omega \big(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d})\big)(t) - \int_\Omega \big(|\boldsymbol{u}_0(0,x)|^2 + |\nabla \boldsymbol{d}_0|^2 + 2W(\boldsymbol{d}_0)\big) \\ &+2\int_0^t \big\|(-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))(s)\big\|^2_{L^2(\Omega;\mathbb{R}^3)}\, ds + \tilde{\mu}\int_0^t \|\nabla \boldsymbol{u}(s)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} ds \\ &\leq C\int_0^t \|\boldsymbol{f}(s)\|^2_{W^{-1,2}(\Omega;\mathbb{R}^3)}\, ds + C_g \|\boldsymbol{u}_0\|^2_{L^2(-h,0;W^{1,2}_0(\Omega;\mathbb{R}^3))} \\ &+2\int_0^t {}_{H^{-1/2}(\Gamma)}\langle \boldsymbol{h}_s, \partial_{\boldsymbol{n}} \boldsymbol{d}\rangle_{H^{1/2}(\Gamma)} ds. \end{aligned} \tag{4.12}$$

On account of assumption (3.9), using standard trace theorems and regularity results for elliptic equations (cf., e.g., [20, Lemma 3.2, p. 263]), we can estimate the last term on the right-hand side of (4.12) as follows

$$
\begin{aligned}
&{}_{H^{-1/2}(\Gamma)}\langle \boldsymbol{h}_t, \partial_{\boldsymbol{n}} \boldsymbol{d}\rangle_{H^{1/2}(\Gamma)} \le C\|\boldsymbol{h}_t\|_{H^{-1/2}(\Gamma)}\|\boldsymbol{d}\|_{H^2(\Omega)} \\
&\le C\left(\|\boldsymbol{h}_t\|^2_{H^{-1/2}(\Gamma)} + \|\boldsymbol{h}\|^2_{H^{3/2}(\Gamma)}\right) + \frac{1}{4}\|\Delta \boldsymbol{d}\|^2_{L^2(\Omega)}\,.
\end{aligned} \tag{4.13}
$$

Consider now the following chain of inequalities

$$
\begin{aligned}
\|-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d})\|^2_{L^2(\Omega)} &= \|\Delta \boldsymbol{d}\|^2_{L^2(\Omega)} + \|\nabla_{\boldsymbol{d}} W(\boldsymbol{d})\|^2_{L^2(\Omega)} - 2(\Delta \boldsymbol{d}, \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \\
&\ge \|\Delta \boldsymbol{d}\|^2_{L^2(\Omega)} + \|\nabla_{\boldsymbol{d}} W(\boldsymbol{d})\|^2_{L^2(\Omega)} \\
&\quad + 2\int_\Omega \nabla(\nabla_{\boldsymbol{d}} W(\boldsymbol{d}))\nabla \boldsymbol{d} - 2\int_\Gamma \partial_n \boldsymbol{d}(\nabla_{\boldsymbol{d}} W(\boldsymbol{d}))_{|\Gamma} \\
&\ge \|\Delta \boldsymbol{d}\|^2_{L^2(\Omega)} - C\|\nabla \boldsymbol{d}\|^2_{L^2(\Omega)} \\
&\quad - \frac{1}{4}\|\Delta \boldsymbol{d}\|^2_{L^2(\Omega)} - C\|\boldsymbol{h}\|^2_{H^{3/2}(\Gamma)} - C\|\nabla_{\boldsymbol{d}} W(\boldsymbol{h})\|^2_{L^2(\Gamma)} \\
&\ge \frac{1}{2}\|\Delta \boldsymbol{d}\|^2_{L^2(\Omega)} - C\|\boldsymbol{h}\|^2_{H^{3/2}(\Gamma)} - C\|\nabla_{\boldsymbol{d}} W(\boldsymbol{h})\|^2_{L^2(\Gamma)}\,,
\end{aligned} \tag{4.14}
$$

where we have used assumptions (3.2), (3.3) and again standard elliptic estimates, trace theorems, and the Gagliardo Nirenberg inequality (cf. [21, p.125]). Then we can handle the last term in (4.13) and combining with (4.12) we deduce the energy inequality (3.16).

Integrating again on time equation (4.11), at light of assumptions (3.2), (3.4) and (3.9) together with (4.13–4.14), we can deduce the a priori bounds

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap L^{10/3}((0,T)\times\Omega;\mathbb{R}^3), \tag{4.15}$$

$$\boldsymbol{d} \in L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{4.16}$$

$$-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d}) \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.17}$$

$$\boldsymbol{d} \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \qquad \nabla_{\boldsymbol{d}} W(\boldsymbol{d}) \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))\,. \tag{4.18}$$

Oserve that here we have exploited the assumption $\boldsymbol{h} \in L^\infty(0,T;H^{3/2}(\Gamma;\mathbb{R}^3))$ in order to guarantee $\boldsymbol{h} \in L^\infty(0,T;C^0(\bar{\Gamma};\mathbb{R}^3))$, which, in combination with $W \in C^2(\mathbb{R})$, gives $\|\nabla_{\boldsymbol{d}} W(\boldsymbol{h})\|^2_{L^2(\Gamma)} \in L^2(0,T)$.

5 The approximation scheme

In this section we construct a suitable family of approximate problems whose solutions weakly converge (up to subsequences) to some limit functions solving the problems of Section 2. We will show in details the estimates and the approximation-passage to the limit procedure on system (2.15)–(2.20). The procedure for the construction of a solution to Problem (P2), i.e. the case of Dirichlet boundary conditions for $\boldsymbol{d}$, is analogous, hence it will be omitted.

The approximation scheme consists of a standard Faedo-Galerkin method for the Navier-Stokes system (2.15)–(2.16) coupled with a regularization of the convective terms and of the momentum equation. More precisely, in order to regularize the convective terms we follow the original approach by Leray [12] to the Navier-Stokes system (see also Temam [27]), while in the momentum equation we introduce an additional term given by an r-Laplacian operator acting on the velocities (see [16],[17], [11] and references therein).

To this aim, we introduce the Hilbert space

$$W^{1,2}_{0,\mathrm{div}} = \{\boldsymbol{v} \in W^{1,2}_0(\Omega;\mathbb{R}^3) \mid \operatorname{div}\boldsymbol{v} = 0, \text{ a.e. in } \Omega\}$$

and consider an orthonormal basis $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$. Fixing M, $N \in \mathbb{N}$ such that $M \le N$, we define the finite-dimensional space $X_N = \mathrm{span}\{\boldsymbol{v}_n\}_{n=1}^{N}$. Moreover, the symbol $[\boldsymbol{v}]_M$ denotes the orthogonal projection onto the space $X_M = \mathrm{span}\{\boldsymbol{v}_n\}_{n=1}^{M}$.

Then the approximate velocity field $\boldsymbol{u}_{N,M} \in C^1([0,T];X_N)$ solves the Faedo-Galerkin system

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \boldsymbol{u}_{N,M}\cdot\boldsymbol{v} &= \int_\Omega [\boldsymbol{u}_{N,M}]_M \otimes \boldsymbol{u}_{N,M} : \nabla\boldsymbol{v} - \frac{1}{M}\int_\Omega |\nabla\boldsymbol{u}_{N,M}|^{r-2}\nabla\boldsymbol{u}_{N,M}\cdot\nabla\boldsymbol{v} \qquad (5.1)\\ &-\int_\Omega \mu\Big(\nabla\boldsymbol{u}_{N,M} + \nabla^T\boldsymbol{u}_{N,M}\Big) : \nabla\boldsymbol{v} + \int_\Omega \nabla\boldsymbol{d}_{N,M}\odot\nabla\boldsymbol{d}_{N,M} : \nabla\boldsymbol{v}\\ &+\alpha\int_\Omega (\Delta\boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}_{N,M}))\otimes\boldsymbol{d}_{N,M} : \nabla\boldsymbol{v}\\ &-(1-\alpha)\int_\Omega \boldsymbol{d}_{N,M}\otimes(\Delta\boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}_{N,M})) : \nabla\boldsymbol{v} + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\\ &+\int_\Omega \boldsymbol{g}(t,(\boldsymbol{u}_t)_{N,M})\cdot\boldsymbol{v} \quad \text{for all } t\in[0,T],\end{aligned}$$

$$\int_\Omega \boldsymbol{u}_{N,M}(0,\cdot)\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{v}, \qquad (5.2)$$

for any $\boldsymbol{v}\in X_N$ and $r \in (3, 10/3)$.

Here the function $\boldsymbol{d}_{N,M} = \boldsymbol{d}_{N,M}[\boldsymbol{u}_{N,M}]$ is the unique solution to the system

$$\begin{aligned}\partial_t\boldsymbol{d}_{N,M} + \boldsymbol{u}_{N,M}\cdot\nabla\boldsymbol{d}_{N,M} - \alpha\boldsymbol{d}_{N,M}\cdot\nabla\boldsymbol{u}_{N,M} + (1-\alpha)\boldsymbol{d}_{N,M}\cdot\nabla^T\boldsymbol{u}_{N,M} + \nabla_{\boldsymbol{d}}W(\boldsymbol{d}_{N,M}) \qquad (5.3)\\ = \Delta\boldsymbol{d}_{N,M}, \qquad \text{in } (0,T)\times\Omega,\end{aligned}$$

$$\partial_{\boldsymbol{n}}\boldsymbol{d}_{N,M} = \boldsymbol{0}, \qquad \text{on } (0,T)\times\Gamma, \qquad (5.4)$$

$$\boldsymbol{d}_{N,M}(0,\cdot) = \boldsymbol{d}_{0,M}, \qquad \text{in } \Omega, \qquad (5.5)$$

$\boldsymbol{d}_{0,M}$ being a suitable smooth approximation of the initial datum $\boldsymbol{d}_0$ (cf. (2.17–2.19)). Observe that in (5.1) it has been introduced the additional term $\frac{1}{M}|\nabla\boldsymbol{u}_{N,M}|^{r-2}\nabla\boldsymbol{u}_{N,M}$ (cf. (2.16)) in order to regularize the velocity field in (5.3).

We notice that all the a priori bounds we derived formally in Section 4 still hold for the approximation problem (5.1)–(5.5). Hence, fixing $\boldsymbol{u}\in C([0,T];X_N)$ we can find $\boldsymbol{d} = \boldsymbol{d}[\boldsymbol{u}]$ solution to (5.3)–(5.5). Inserting $\boldsymbol{d}[\boldsymbol{u}]$ in system (5.1)–(5.2) we can define a mapping $\boldsymbol{u} \mapsto \mathcal{T}[\boldsymbol{u}]$, $\mathcal{T}[\boldsymbol{u}]$ being the solution of the system. Then, by means of the classical Schauder's argument, it is possible to prove that $\mathcal{T}$ admits a fixed point on $(0, T_0)$, with $0 < T_0 \leq T$. Finally, applying again the a priori estimates, we can conclude that the approximate solutions can be extended to the whole time interval $[0, T]$ (see [7, Chapter 3 and 6] for details).

Consider now, for any M, $N \in \mathbb{N}$ with $M \leq N$, the pair $(\boldsymbol{u}_{N,M}, \boldsymbol{d}_{N,M})$ solution to (5.1)–(5.5). In the following two subsections we will pass to the limit first for $N \to \infty$ and then for $M \to \infty$.

5.1 Passage to the limit as $N \to \infty$

The first step consists in passing to the limit as $N \to \infty$ in (5.1)–(5.5).

Recalling the regularizing term introduced in (5.1), from the corresponding energy estimate we obtain

$$M^{-1} \|\nabla \boldsymbol{u}_{N,M}\|^r_{L^r((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \leq C, \quad \text{for } r \in (3, 10/3), \tag{5.6}$$

from which we can deduce that, for any fixed M, the set of functions $|\nabla \boldsymbol{u}_{N,M}|^{r-2}\nabla \boldsymbol{u}_{N,M}$ is uniformly bounded in $L^{\frac{r}{r-1}}((0,T) \times \Omega; \mathbb{R}^{3\times 3})$. Observe that, since $r \in (3, 10/3)$, it holds $r/(r-1) \in (10/7, 3/2)$.

Passing to the limit as $N \to \infty$ in (5.1)–(5.3), where in (5.1) the projection on X_M is kept in the convective term, on account of (5.6) it is possible to prove the following convergence results

$$\boldsymbol{u}_{N,M} \to \boldsymbol{u}_M \text{ weakly-(*) in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^3)), \tag{5.7}$$

$$\nabla \boldsymbol{u}_{N,M} \to \nabla \boldsymbol{u}_M \text{ weakly in } L^r(0,T; L^r(\Omega;\mathbb{R}^3)), \tag{5.8}$$

$$\partial_t \boldsymbol{u}_{N,M} \to \partial_t \boldsymbol{u}_M \text{ weakly in } L^{\frac{r}{r-1}}(0,T; W^{-1,r/r-1}(\Omega;\mathbb{R}^3)), \tag{5.9}$$

$$\boldsymbol{d}_{N,M} \to \boldsymbol{d}_M \text{ weakly-(*) in } L^\infty(0,T; W^{1,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,2}(\Omega;\mathbb{R}^3)). \tag{5.10}$$

Moreover, by means of (5.10) and simple interpolation arguments, we get

$$\nabla \boldsymbol{d}_{N,M} \to \nabla \boldsymbol{d}_M \text{ strongly in } L^\eta(\Omega_T;\mathbb{R}^{3\times 3}), \quad \text{for } \eta \in [1, 10/3). \tag{5.11}$$

On account of (5.7) and (5.8), applying standard interpolation results, some Sobolev embedding theorems and the Aubin-Lions lemma, it is possible to deduce the convergence

$$\boldsymbol{u}_{N,M} \to \boldsymbol{u}_M \text{ strongly in } L^s(\Omega_T;\mathbb{R}^3), \quad \text{for some } s > 5. \tag{5.12}$$

So that, by means of (5.7) and (5.12), assumption $(g5)$ implies that

$$\boldsymbol{g}(t, (\boldsymbol{u}_t)_{N,M}) \to \boldsymbol{g}(t, (\boldsymbol{u}_t)_M) \text{ in } L^2(0,T; W^{-1,2}(\Omega;\mathbb{R}^3)). \tag{5.13}$$

Combining (5.12) with (5.11) and (5.8) with (5.10) we arrive at

$$\boldsymbol{u}_{N,M} \cdot \nabla \boldsymbol{d}_{N,M} \to \boldsymbol{u}_M \cdot \nabla \boldsymbol{d}_M \text{ strongly in } L^p(\Omega_T), \quad \text{for some } p > 2, \tag{5.14}$$

$$\boldsymbol{d}_{N,M} \cdot \nabla \boldsymbol{u}_{N,M} \to \boldsymbol{d}_M \cdot \nabla \boldsymbol{u}_M \text{ weakly in } L^p(\Omega_T), \quad \text{for some } p > 2.$$

By comparison, we deduce

$$\partial_t \boldsymbol{d}_{N,M} \to \partial_t \boldsymbol{d}_M \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.15}$$

moreover, it holds

$$|\nabla \boldsymbol{u}_{N,M}|_{r-2} \nabla \boldsymbol{u}_{N,M} \to \overline{|\nabla \boldsymbol{u}_M|^{r-2}\nabla \boldsymbol{u}_M} \text{ weakly in } L^{r/r-1}(\Omega_T;\mathbb{R}^{3\times 3}).$$

On account of the previous results, it is possible to prove that the limit pair $(\boldsymbol{u}_M, \boldsymbol{d}_M)$ solves the problem

$$\int_\Omega \boldsymbol{u}_M(t,\cdot)\cdot \nabla\varphi = 0, \quad \text{for a.a. } t\in(0,T), \tag{5.16}$$

$$\begin{aligned}
\int_0^t \langle \partial_t \boldsymbol{u}_M, \varphi\rangle - \int_0^t\int_\Omega \Big([\boldsymbol{u}_M]_M \otimes \boldsymbol{u}_M : \nabla\varphi\Big) + \int_0^t\int_\Omega \mu\left(\nabla \boldsymbol{u}_M + \nabla^T \boldsymbol{u}_M\right) : \nabla\varphi \\
= \int_0^t\int_\Omega \left(\nabla \boldsymbol{d}_M \odot \nabla \boldsymbol{d}_M + \alpha\left(\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)\right)\otimes \boldsymbol{d}_M\right) : \nabla\varphi \\
- \int_0^t\int_\Omega (1-\alpha)\boldsymbol{d}_M \otimes \left(\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)\right) : \nabla\varphi \\
-\frac{1}{M}\int_0^t\int_\Omega \overline{|\nabla \boldsymbol{u}_M|^{r-2}\nabla \boldsymbol{u}_M}\cdot\nabla\varphi \\
+ \int_0^t\int_\Omega \boldsymbol{f}\cdot\varphi + \int_0^t\int_\Omega \boldsymbol{g}(t,(\boldsymbol{u}_t)_M)\cdot\varphi \quad \text{for all } t\in(0,T),
\end{aligned} \tag{5.17}$$

for any $\varphi \in C^\infty(\overline{\Omega};\mathbb{R}^3)$ such that $\operatorname{div}\varphi = 0$.

Passing to the limit as $N\to\infty$ also in the system for $\boldsymbol{d}$, we have

$$\partial_t \boldsymbol{d}_M + \boldsymbol{u}_M\cdot\nabla \boldsymbol{d}_M - \alpha \boldsymbol{d}_M\cdot\nabla \boldsymbol{u}_M + (1-\alpha)\boldsymbol{d}_M\cdot\nabla^T \boldsymbol{u}_M = \Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M), \text{ a.e. in } \Omega_T \tag{5.18}$$

$$\partial_{\boldsymbol{n}} \boldsymbol{d}_M = \boldsymbol{0}, \quad \text{a.e. in } (0,T)\times\Gamma, \tag{5.19}$$

$$\boldsymbol{d}_M(0,\cdot) = \boldsymbol{d}_{0,M}, \quad \text{a.e. in } \Omega. \tag{5.20}$$

Taking $\boldsymbol{v} = \boldsymbol{u}_{N,M}$ in (5.1) and then integrating in time over $(0,t)$, we obtain

$$\begin{aligned}
\|\boldsymbol{u}_{N,M}(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \mu|\nabla \boldsymbol{u}_{N,M} + \nabla^T \boldsymbol{u}_{N,M}|^2 + \frac{2}{M}\int_0^t\int_\Omega |\nabla \boldsymbol{u}_{N,M}|^r \\
= \|\boldsymbol{u}_0\|^2_{L^2(\Omega)} + 2\int_0^t\int_\Omega \left(\nabla \boldsymbol{d}_{N,M}\odot\nabla \boldsymbol{d}_{N,M}\right) : \nabla \boldsymbol{u}_{N,M} \\
+2\alpha\int_0^t\int_\Omega \left(\Delta \boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_{N,M})\right)\otimes \boldsymbol{d}_{N,M} : \nabla \boldsymbol{u}_{N,M} \\
-2(1-\alpha)\int_0^t\int_\Omega \boldsymbol{d}_{N,M}\otimes\left(\Delta \boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_{N,M})\right) : \nabla \boldsymbol{u}_{N,M} \\
+ \int_0^t\int_\Omega \boldsymbol{f}\cdot \boldsymbol{u}_{N,M} + \int_0^t\int_\Omega \boldsymbol{g}(t,(\boldsymbol{u}_t)_{N,M})\cdot \boldsymbol{u}_{N,M},
\end{aligned} \tag{5.21}$$

for all $t \in (0,T)$.

Then, by means of (5.7)–(5.9), taking in (5.17) $\varphi = \boldsymbol{u}_M$ we can obtain

$$\|\boldsymbol{u}_M(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \mu |\nabla \boldsymbol{u}_M + \nabla^t \boldsymbol{u}_M|^2 + \frac{2}{M}\int_0^t \int_\Omega |\nabla \boldsymbol{u}_M|^r \tag{5.22}$$

$$= \|\boldsymbol{u}_0\|^2_{L^2(\Omega)} + 2\int_0^t \int_\Omega \ (\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)) \otimes \boldsymbol{d}_M : \nabla \boldsymbol{u}_M$$

$$+2\alpha \int_0^t \int_\Omega (\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)) \otimes \boldsymbol{d}_M : \nabla \boldsymbol{u}_M$$

$$-2(1-\alpha)\int_0^t \int_\Omega \boldsymbol{d}_M \otimes (\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)) : \nabla \boldsymbol{u}_M$$

$$+2\int_0^t \int_\Omega \boldsymbol{f}\cdot \boldsymbol{u}_M + \int_0^t \int_\Omega \boldsymbol{g}(t,(\boldsymbol{u}_t)_M)\cdot \boldsymbol{u}_M, \quad \text{for all } t \in (0,T).$$

Observe that at this point the L^r-regularity of $\nabla \boldsymbol{u}_M$ (cf. (5.8)) is essential since we do not know if the terms $(\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)) \otimes \boldsymbol{d}_M$ and $(\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)) \otimes \boldsymbol{d}_M$ belong to $L^2(\Omega_T;\mathbb{R}^3)$. Actually, we can just guarantee that they lie in $L^{5/3}(\Omega_T;\mathbb{R}^3)$, cf. (5.10) and (5.11).

Now, testing (5.3) by $\Delta \boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_{N,M})$ and (5.18) by $\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)$, and then integrating on $(0,t)$, we obtain for all $t \in (0,T)$,

$$\|\nabla \boldsymbol{d}_{N,M}(t)\|^2_{L^2(\Omega)} + 2\int_\Omega W(\boldsymbol{d}_{N,M})(t) + 2\int_0^t \int_\Omega |\Delta \boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_{N,M})|^2 \tag{5.23}$$

$$= \|\nabla \boldsymbol{d}_{0,M}\|^2_{L^2(\Omega)} + 2\int_\Omega W(\boldsymbol{d}_{0,M}) + 2\int_0^t (\boldsymbol{u}_{N,M}\cdot \nabla \boldsymbol{d}_{N,M}, \Delta \boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_{N,M}))$$

$$+2\int_0^t \left(-\alpha \boldsymbol{d}_{N,M}\cdot \nabla \boldsymbol{u}_{N,M} + (1-\alpha)\boldsymbol{d}_{N,M}\cdot \nabla^T \boldsymbol{u}_{N,M}, \Delta \boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_{N,M})\right),$$

$$\|\nabla \boldsymbol{d}_M(t)\|^2_{L^2(\Omega)} + 2\int_\Omega W(\boldsymbol{d}_M)(t) + 2\int_0^t \int_\Omega |\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)|^2 \tag{5.24}$$

$$= \|\nabla \boldsymbol{d}_{0,M}\|^2_{L^2(\Omega)} + 2\int_\Omega W(\boldsymbol{d}_{0,M})$$

$$+2\int_0^t \left(\boldsymbol{u}_M \cdot \nabla \boldsymbol{d}_M - \alpha \boldsymbol{d}_M \cdot \nabla \boldsymbol{u}_M + (1-\alpha)\boldsymbol{d}_M \cdot \nabla^T \boldsymbol{u}_M, \Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)\right).$$

Observe that, due to the higher regularity (5.8) and (5.12) of $\boldsymbol{u}_M$ and $\nabla \boldsymbol{u}_M$ given by the regularizing term $\frac{1}{M}|\nabla \boldsymbol{u}_M|^{r-2}\nabla \boldsymbol{u}_M$ in (5.1), then (5.18) in meaningful in $L^2(\Omega_T;\mathbb{R}^3)$ (cf. (5.14)–(5.15)).

Summing (5.21) with (5.23) and then (5.22) with (5.24), we can pass to the limit as $N \to \infty$ in both the resulting equations, obtaining

$$\int_0^T \int_\Omega |\nabla \boldsymbol{u}_{N,M}|^r \to \int_0^T \int_\Omega \overline{|\nabla \boldsymbol{u}_M|^{r-2}\nabla \boldsymbol{u}_M} : \nabla \boldsymbol{u}_M,$$

$$\int_0^T \int_\Omega |\Delta \boldsymbol{d}_{N,M} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_{N,M})|^2 \to \int_0^T \int_\Omega |\Delta \boldsymbol{d}_M - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}_M)|^2.$$

Hence, standard Minty's trick and monotonicity argument give the results

$$\nabla \boldsymbol{u}_{N,M} \to \nabla \boldsymbol{u}_M \text{ strongly in } L^r(\Omega_T; \mathbb{R}^{3\times 3}),$$

$$\Delta \boldsymbol{d}_{N,M} \to \Delta \boldsymbol{d}_M \text{ strongly in } L^2(\Omega_T; \mathbb{R}^3).$$

5.2 Passage to the limit as $M \to \infty$

In the second step we pass to the limit as $M \to \infty$ in (5.16)–(5.20).

First, we observe that the convergence results in (5.7), (5.12) and therefore (5.13) still hold when taking $M \to \infty$. Moreover, we can deduce

$$\partial_t \boldsymbol{u}_M \to \partial_t \boldsymbol{u} \text{ weakly in } L^{\frac{r}{r-1}}(0,T; W^{-1,r/r-1}(\Omega;\mathbb{R}^3))\,, \tag{5.25}$$

$$\boldsymbol{d}_M \to \boldsymbol{d} \text{ weakly-(*) in } L^\infty(0,T; W^{1,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,2}(\Omega;\mathbb{R}^3))\,, \tag{5.26}$$

$$\partial_t \boldsymbol{d}_M \to \partial_t \boldsymbol{d} \text{ weakly in } L^2(0,T; L^{3/2}(\Omega;\mathbb{R}^3))\,, \tag{5.27}$$

and in particular

$$M^{-1/(r-1)} \nabla \boldsymbol{u}_M \to 0 \text{ strongly in } L^{r-1}((0,T)\times\Omega; \mathbb{R}^{3\times 3}). \tag{5.28}$$

On account of the previous convergence results, we can pass to the limit as $M \to \infty$ in (5.16)–(5.20) and finally recover (2.15)–(2.19).

Appendix

In this section we would like to illustrate some examples of delay forcing terms fulfilling assumptions $(g1)$–$(g5)$. More details in the case of a Navier-Stokes problem can be seen in [1].

(a) **Variable delay case**

Let $\boldsymbol{G} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function satisfying $\boldsymbol{G}(t,0) = 0$ for all $t \in [0,T]$, and assume that there exists $L_G > 0$ such that

$$|\boldsymbol{G}(t,\boldsymbol{u}) - \boldsymbol{G}(t,\boldsymbol{v})|_{\mathbb{R}^3} \le L_G |\boldsymbol{u} - \boldsymbol{v}|_{\mathbb{R}^3}, \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3.$$

Consider a function $\rho(t)$, which is going to play the role of the delay function. We suppose that $\rho \in C^1([0,T])$, $\rho(t) \ge 0$ for all $t \in [0,T]$, $h = \max_{t\in[0,T]} \rho(t) > 0$ and $\rho_* = \max_{t\in[0,T]} \rho'(t) < 1$. Then, we define $\boldsymbol{g}(t,\boldsymbol{\xi})(x) = \boldsymbol{G}(t, \boldsymbol{\xi}(-\rho(t))(x))$ for each $\boldsymbol{\xi} \in L^2(-h,0; W_0^{1,2}(\Omega;\mathbb{R}^3))$, $x \in \Omega$ and $t \in [0,T]$. Notice that, in this case, the delayed term $\boldsymbol{g}$ in our problem turns to $\boldsymbol{g}(t,\boldsymbol{u}_t) = \boldsymbol{G}(t, \boldsymbol{u}(t-\rho(t)))$. Then, $\boldsymbol{g}$ satisfies hypotheses $(g1)-(g4)$. Indeed, $(g1)-(g3)$ follow immediately.

On the other hand, if $\boldsymbol{u}, \boldsymbol{v} \in L^2(-h,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$, using the change of variable $\tau = s - \rho(s)$ it is easy to see that

$$\int_0^t \|\boldsymbol{g}(s,\boldsymbol{u}_s) - \boldsymbol{g}(s,\boldsymbol{v}_s)\|^2_{L^2(\Omega;\mathbb{R}^3)}\, ds \le C_g^2 \int_{-h}^t \|\boldsymbol{u}(\tau) - \boldsymbol{v}(\tau)\|^2_{L^2(\Omega;\mathbb{R}^3)}\, d\tau \quad \forall\, t \in [0,T],$$

where $C_g^2 = \frac{L_G^2}{1-\rho_*}$ and, consequently, $(g4)$ is fulfilled on account of the continuous inclusions $W_0^{1,2}(\Omega;\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3) \subset W^{-1,2}(\Omega;\mathbb{R}^3)$.

(b) **Distributed delay case**

Let now $\boldsymbol{G} : [0,T]\times[-h,0]\times\mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function satisfying $\boldsymbol{G}(t,s,0) = 0$ for all $(t,s) \in [0,T]\times[-h,0]$ and such that there exists a function $\gamma \in L^2(-h,0)$ such that

$$|\boldsymbol{G}(t,s,\boldsymbol{u}) - \boldsymbol{G}(t,s,\boldsymbol{v})|_{\mathbb{R}^3} \leq \gamma(s)|\boldsymbol{u}-\boldsymbol{v}|_{\mathbb{R}^3}, \forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^3 \quad \forall\,(t,s) \in [0,T]\times[-h,0].$$

Then, we define $\boldsymbol{g}(t,\boldsymbol{\xi})(x) = \int_{-h}^0 \boldsymbol{G}(t,s,\boldsymbol{\xi}(s)(x))\,ds$ for each $\boldsymbol{\xi} \in L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3))$, $x\in\Omega$ and $t\in[0,T]$. In this case, the delayed term $\boldsymbol{g}$ in our problem becomes

$$\boldsymbol{g}(t,\boldsymbol{u}_t) = \int_{-h}^0 \boldsymbol{G}(t,s,\boldsymbol{u}(t+s))\,ds.$$

As in the case of variable delay, $\boldsymbol{g}$ satisfies hypotheses $(g1)-(g4)$.

Indeed, $(g1)$ and $(g2)$ can be deduced immediately. On the other hand, if $\boldsymbol{\xi},\boldsymbol{\eta} \in L^2(-h,0;L^2(\Omega;\mathbb{R}^3))$, for each $t\in[0,T]$ we obtain

$$\begin{aligned}
\|\boldsymbol{g}(t,\boldsymbol{\xi}) - \boldsymbol{g}(t,\boldsymbol{\eta})\|^2_{L^2(\Omega,\mathbb{R}^3)} &\leq \int_\Omega \left(\int_{-h}^0 |\boldsymbol{G}(t,s,\boldsymbol{\xi}(s)(x)) - \boldsymbol{G}(t,s,\boldsymbol{\eta}(s)(x))|_{\mathbb{R}^3}\,ds\right)^2 dx \\
&\leq \int_\Omega \left(\int_{-h}^0 \gamma(s)|\boldsymbol{\xi}(s)(x) - \boldsymbol{\eta}(s)(x)|_{\mathbb{R}^3}\,ds\right)^2 dx \\
&\leq \int_\Omega \|\gamma\|^2_{L^2(-h,0)} \left(\int_{-h}^0 |\boldsymbol{\xi}(s)(x) - \boldsymbol{\eta}(s)(x)|^2_{\mathbb{R}^3}\,ds\right) dx \\
&\leq \|\gamma\|^2_{L^2(-h,0)}\|\boldsymbol{\xi}-\boldsymbol{\eta}\|_{L^2(-h,0;L^2(\Omega;\mathbb{R}^3))},
\end{aligned}$$

which implies that $(g3)$ is fulfilled thanks again to the continuous inclusions $W_0^{1,2}(\Omega;\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3) \subset W^{-1,2}(\Omega;\mathbb{R}^3)$.

Finally, if $\boldsymbol{u},\boldsymbol{v} \in L^2(-h,T;L^2(\Omega;\mathbb{R}^3))$ then, for each $t\in[0,T]$ it follows

$$\int_0^t \|\boldsymbol{g}(\tau,\boldsymbol{u}_\tau) - \boldsymbol{g}(\tau,\boldsymbol{v}_\tau)\|^2_{L^2(\Omega;\mathbb{R}^3)}\,d\tau \leq \|\gamma\|^2_{L^2(-h,0)} \int_0^t \left(\int_{-h}^0 \|\boldsymbol{u}(s+\tau) - \boldsymbol{v}(s+\tau)\|^2_{L^2(\Omega;\mathbb{R}^3)}\,ds\right) d\tau,$$

and, with the change $r = s+\tau$,

$$\begin{aligned}
&\int_0^t \|\boldsymbol{g}(\tau,\boldsymbol{u}_\tau) - \boldsymbol{g}(\tau,\boldsymbol{v}_\tau)\|^2_{L^2(\Omega;\mathbb{R}^3)}\,d\tau \\
&\qquad \leq \|\gamma\|^2_{L^2(-h,0)} \int_0^t \left(\int_{\tau-h}^\tau \|\boldsymbol{u}(r) - \boldsymbol{v}(r)\|^2_{L^2(\Omega;\mathbb{R}^3)}\,dr\right) d\tau \\
&\qquad \leq T\|\gamma\|^2_{L^2(-h,0)} \int_{-h}^t \|\boldsymbol{u}(r) - \boldsymbol{v}(r)\|^2_{L^2(\Omega;\mathbb{R}^3)}\,dr,
\end{aligned}$$

which, at light of the previously mentioned continuous inclusions, guarantees that $(g4)$ holds.

(c) **Other delay terms**

Now, we shall exhibit a situation where certain delay can appear in terms containing partial derivatives with respect to the spatial variables.

Let $B(\cdot) \in L^\infty(0,T;\mathcal{L}(W_0^{1,2}(\Omega;\mathbb{R}^3);L^2(\Omega;\mathbb{R}^3))$ and $\rho \in C^1([0,T])$, such that $\rho(t) \geq 0$ for all $t \in [0,T]$, $h = \max_{t\in[0,T]} \rho(t) > 0$ and $\rho_* = \max_{t\in[0,T]} \rho'(t) < 1$. We now define $\boldsymbol{g}(t,\xi) = B(t)\boldsymbol{\xi}(-\rho(t))$ for each $\boldsymbol{\xi} \in L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3))$, and $t \in [0,T]$. Thus, in this case the delayed term $\boldsymbol{g}$ in our problems turns to $\boldsymbol{g}(t,\boldsymbol{u}_t) = B(t)\boldsymbol{u}(t-\rho(t))$. It is easy to see that g satisfies conditions $(g1)-(g4)$.

Also condition $(g5)$ is fulfilled. Indeed, if $\boldsymbol{v}^m$ converges to zero weakly in $L^2(-h,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$ and $\boldsymbol{\psi} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$ is given, we have

$$\int_0^T \langle \boldsymbol{g}(t,\boldsymbol{v}_t^m), \psi(t)\rangle \, dt = \int_0^T \langle B^*(t)\boldsymbol{\psi}(t), \boldsymbol{v}^m(t-\rho(t))\rangle \, dt,$$

with $B^*(\cdot) \in L^\infty(0,T;\mathcal{L}(L^2(\Omega;\mathbb{R}^3);W^{-1,2}(\Omega;\mathbb{R}^3))) \subset L^\infty(0,T;\mathcal{L}(W_0^{1,2}(\Omega;\mathbb{R}^3);W^{-1,2}(\Omega;\mathbb{R}^3)))$ the adjoint of $B(\cdot)$. Using the change of variables $\tau = t-\rho(t) = \omega(t)$, we obtain

$$\begin{aligned} \int_0^T \langle \boldsymbol{g}(t,\boldsymbol{v}_t^m), \boldsymbol{\psi}(t)\rangle \, dt &= \int_{\omega(0)}^{\omega(T)} \langle B^*(\omega^{-1}(\tau))\boldsymbol{\psi}(\omega^{-1}(\tau)), \boldsymbol{v}^m(\tau)\rangle \frac{1}{\omega'(\omega^{-1}(\tau))}\, d\tau \\ &= \int_{-h}^T \langle \boldsymbol{\Psi}(\tau), \boldsymbol{v}^m(\tau)\rangle \, d\tau, \end{aligned}$$

with

$$\boldsymbol{\Psi}(\tau) = \begin{cases} \dfrac{1}{\omega'(\omega^{-1}(\tau))} B^*(\omega^{-1}(\tau))\boldsymbol{\psi}(\omega^{-1}(\tau)) & \text{if } \tau \in [\omega(0),\omega(T)], \\ 0 & \text{if } \tau \in [-h,T] \setminus [\omega(0),\omega(T)]. \end{cases}$$

For this function $\boldsymbol{\Psi}$ it follows

$$\int_{-h}^T \|\boldsymbol{\Psi}(\tau)\|_*^2 \, d\tau = \int_{\omega(0)}^{\omega(T)} \frac{1}{(\rho'(\rho^{-1}(\tau)))^2} \|B^*(\rho^{-1}(\tau))\boldsymbol{\psi}(\rho^{-1}(\tau))\|_*^2 \, d\tau,$$

and thus, by means of the change $\tau = \omega(t) = t-\rho(t)$,

$$\int_{-h}^T \|\boldsymbol{\Psi}(\tau)\|_*^2 \, d\tau = \int_0^T \frac{1}{1-\rho'(t)} \|B^*(t)\boldsymbol{\psi}(t)\|_*^2 \, dt \leq \frac{b_0^2}{1-\rho_*} \int_0^T \|\boldsymbol{\psi}(t)\|^2 \, dt,$$

where $b_0 = \|B^*(\cdot)\|_{L^\infty(0,T;\mathcal{L}(W_0^{1,2}(\Omega;\mathbb{R}^3);W^{-1,2}(\Omega;\mathbb{R}^3)))}$.

Consequently, $\boldsymbol{\Psi} \in L^2(-h,T;W^{-1,2}(\Omega;\mathbb{R}^3))$ and

$$\lim_{m\to\infty} \int_0^T \langle \boldsymbol{g}(t,\boldsymbol{v}_t^m), \boldsymbol{\psi}(t)\rangle \, dt = \lim_{m\to\infty} \int_{-h}^T \langle \boldsymbol{\Psi}(\tau), \boldsymbol{v}^m(\tau)\rangle \, d\tau = 0.$$

Therefore, condition $(g5)$ is satisfied.

Let $K \in L^\infty(-h,T;\mathcal{L}(W_0^{1,2}(\Omega;\mathbb{R}^3);W^{-1,2}(\Omega;\mathbb{R}^3)))$ and consider a term of the form $\boldsymbol{g}(t,\boldsymbol{u}_t) = \int_{-h}^0 K(t+s)\boldsymbol{u}(t+s)\, ds$, defined for all $\boldsymbol{u} \in L^2(-h,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$. This term corresponds to the situation $g(t,\boldsymbol{\xi}) = \int_{-h}^0 K(t+s)\boldsymbol{\xi}(s)\, ds$ for each $t \in [0,T]$ and $\boldsymbol{\xi} \in L^2(-h,0;W_0^{1,2}(\Omega;\mathbb{R}^3))$. In this case, it is easy to see that $\boldsymbol{g}$ is well defined and satisfies $(g1)-(g3)$. In particular, if we denote $k = \|K(\cdot)\|_{L^\infty(-h,T;\mathcal{L}(W_0^{1,2}(\Omega;\mathbb{R}^3);W^{-1,2}(\Omega;\mathbb{R}^3)))}$, we can see that, for each $t \in [0,T]$ and each $\boldsymbol{u} \in L^2(-h,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, we have

$$\int_0^t \|\boldsymbol{g}(s,\boldsymbol{u}_s)\|_*^2 \, ds \le k^2 h \min(h,T) \int_{-h}^t \|\boldsymbol{u}(s)\|^2 \, ds,$$

and thus, $(g4)$ holds by setting $C_g = k^2 h \min(h,T)$.

On the other hand, let $\boldsymbol{v}^m$ be weakly converging to zero in $L^2(-h,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, and fix $\boldsymbol{\psi} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$. Then

$$\int_0^T \langle \boldsymbol{g}(t,\boldsymbol{v}_t^m), \boldsymbol{\psi}(t)\rangle \; dt = \int_0^T \left\langle \int_{t-h}^t K(\tau)\boldsymbol{v}^m(\tau)\, d\tau, \psi(t) \right\rangle \, dt,$$

and, by Fubini's theorem, it is easy to see that

$$\int_0^T \langle g(t,\boldsymbol{v}_t^m), \boldsymbol{\psi}(t)\rangle \; dt = \int_{-h}^T \langle \boldsymbol{\Sigma}(\tau), \boldsymbol{v}^m(\tau)\rangle \; d\tau,$$

with $\boldsymbol{\Sigma}(\tau) = K^*(\tau)\boldsymbol{\Psi}(\tau)$ and

$$\boldsymbol{\Psi}(\tau) = \begin{cases} \int_0^{\tau+h} \boldsymbol{\psi}(t)\, dt & \text{if } -h \le \tau < 0, \\ \int_\tau^{\tau+h} \boldsymbol{\psi}(t)\, dt & \text{if } 0 \le \tau < T-h, \\ \int_\tau^T \boldsymbol{\psi}(t)\, dt & \text{if } T-h \le \tau \le T, \end{cases}$$

in the case $h \le T$, and

$$\boldsymbol{\Psi}(\tau) = \begin{cases} \int_0^{\tau+h} \boldsymbol{\psi}(t)\, dt & \text{if } -h \le \tau < T-h, \\ \int_0^T \boldsymbol{\psi}(t)\, dt & \text{if } T-h \le \tau < 0, \\ \int_\tau^T \boldsymbol{\psi}(t)\, dt & \text{if } 0 \le \tau \le T, \end{cases}$$

in the case $h > T$. In both cases $\boldsymbol{\Psi} \in C^0([0,T];W_0^{1,2}(\Omega;\mathbb{R}^3))$, and in particular $\boldsymbol{\Sigma} \in L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))$.

Consequently, if $\boldsymbol{v}^m$ converges weakly to zero in $L^2(-h,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, then $\boldsymbol{g}(\cdot,\boldsymbol{v}_\cdot^m)$ converges weakly to zero in $L^2(-h,T;W^{-1,2}(\Omega;\mathbb{R}^3))$ and thus, $\boldsymbol{g}$ satisfies hypothesis $(g5)$.

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