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Functional Properties of p-de Branges Spaces

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Introduction

When the physicists realized that the observable quantities can be described by self-adjoint operators acting in abstract Hilbert spaces, mathematicians had begun to study this family of operators. Indeed, it is possible to associate to every self-adjoint operator a real measure μ so that the action of the self-adjoint operator A in the Hilbert space H is equivalent to the multiplication operator by the free variable x in $L^2(\mu)$, [68].

However, why should we limit ourselves to self-adjoint operators? Are there other families of operators for which there exists a spectral model?

The de Branges spaces, the main topic of this Ph.D. dissertation, were born as functional models to describe the simple, closed, symmetric operators with deficiency indexes (1,1), [1].

Let us consider for example the stationary Schroedinger equation with Neumann boundary conditions:

$$A := \begin{cases} -\ddot{u}(x) = zu(x) & x \in (0,\pi) \ , \\ \dot{u}(0) = 0 \ , & \dot{u}(\pi) = 0 \ . \end{cases}$$

This operator is self-adjoint in $L^2[0,\pi]$ and, if we introduce the measure μ defined as

$$\mu(\Omega) := \sum_{n\in\mathbb{N}} \frac{\delta_n(\Omega)}{\|\cos(n\ \cdot)\|^2_{L^2(0,\pi)}} \quad \text{where for every Borel set } \Omega \subset \mathbb{C}$$

$$\delta_n(\Omega) := \begin{cases} 1 & \text{if } n \in \Omega \ , \\ 0 & \text{if } n \notin \Omega \end{cases} ,$$

the corresponding spectral transform $\mathcal{U}$ is defined as

$$\begin{aligned} \mathcal{U}\ &: (A, L^2(0,\pi)) \longrightarrow (M_x, L^2(\mu)) \\ &f \longrightarrow \mathcal{U}[f](m) = \langle f, \cos(m\ \cdot)\rangle_{L^2(0,\pi)} \ , \quad m \in \mathbb{N} \ . \end{aligned}$$

However, what happens if we remove one of the two boundary conditions of A ? The operator stops being self-adjoint and we can no longer associate it to a spectral transform $\mathcal{U}$. Nevertheless, following the procedure of M. Krein [45] and L. de Branges, [28], [70] and [71], it is possible to associate it to a space of entire functions: the de Branges space.

These entire function spaces were first introduced in the West by L. de Branges, when he studied the inverse spectral theory for canonical systems, and in Russia by M. Krein, who studied functional models for closed, symmetric, simple operators with deficiency indexes (n, n).

In the early 2000s, some mathematicians start considering the de Branges spaces for other reasons. In chronological order, V. Havin and N. Nikolski, A. Aleksandrov, K. Seip and Y. Lyubarskii, and finally Y. Belov, A. Baranov and K. Dyakonov (and many others) aware of the connection between the model spaces and de Branges spaces, start studying them as spaces of entire functions. They looked at them independently of their connection with the operators, simply by investigating the analytical properties of their elements. In this Ph.D. dissertation with similar goal, I will focus on some analytical properties of de Branges spaces.

This dissertation is divided into three parts, in particular in eight chapters. In the rest of the introduction, I briefly describe all of them highlighting their relationships with the rest of the literature.

The first two chapters are preliminary to the rest of the thesis. Although a reader who is already familiar with de Branges spaces might skip these chapters, I still decided to include them. In fact, even if there are excellent books on de Branges spaces, one of the main difficulties I faced at the beginning of my Ph.D experience was studying the classical notions of this topic. As already anticipated, the de Branges spaces have been studied by different mathematicians with different interests, approaches and terminologies. Therefore, I think it is necessary explaining my notation and introducing, with the relative proofs, the elementary properties I need.

In the first chapter, I introduce the de Branges spaces $\mathcal{H}(E)$ using the Hermite Biehler functions $E(z)$, (1.9) and the model spaces, (1.18).

Indeed, it is useful considering the relationship between these two spaces of holomorphic functions, Proposition 1.12: although many theorems could be stated equivalently for the two spaces, it is easier to prove some of them by using meromorphic inner functions of $\mathbb{C}^+$ and some others by using Hermite Biehler functions. This approach, which for example was not explicitly used by L. de Branges, was taught to me by A. Baranov during his Ph.D. course on the de Branges spaces in Bologna.

There are at least four important classical theorems on the de Branges spaces that would deserve to be stated: the ordering theorem, the inverse spectral theory, the axiomatic definition and, finally, the existence of bases of reproducing kernels. In the second chapter I present the latter result. In fact, I show that every de Branges spaces have a basis of reproducing kernels centred at real points, Theorem 2.8 and Theorem 2.9. It is only in this chapter that I briefly talk about the multiplication operator M_z in the de Branges spaces and I show how the density of its domain is related to the properties of the Hermite Biehler function, [8].

The rest of the thesis gathers my research work.

The first thing I do, is extending some properties holding in the Paley-Wiener spaces, see Example 5 for the definition, to the general de Branges spaces. In the second part of the thesis, I study which conditions the Hermite Biehler function $E(z)$ must satisfy so that the translation operator T_ζ and the embedding operator $\iota_{p,q}$ are bounded in the corresponding de Branges space $\mathcal{H}(E)$.

In the third and fourth chapters, I focus my attention on the translation operator T_ζ. This operator has never been systematically studied in the de Branges spaces.

First of all, I describe an easy necessary condition that $E(z)$ must satisfy for the boundedness of T_ζ in $\mathcal{H}(E)$, Theorem 3.5.

My interest in this problem has got a double origin. I know that the vertical translation operator $T_{i\tau}$ is bounded in the Paley-Wiener spaces and more generally in the Bernstein spaces, see Definition 2.14. This fact makes me wonder if the vertical translation operator $T_{i\tau}$ is bounded also in other de Branges spaces as well.

Secondly, translation operator is closely related to the differentiation operator. The Bernstein inequality and the boundedness of the differentiation operator has been already widely studied by K.M. Dyakonov in [32], and by A. Baranov in [6], [9], [10], among other papers. For this reason I think that studying the translation operators makes sense as well.

To do this, I associate to $T_{i\tau}$ some Carleson measures of $K^2(\Theta)$ with an approach very similar to the one used in interpolation problems. The Carleson measures are an extremely useful instrument in studying analytic functions spaces and for this reason many mathematicians tried to characterize them. The first fundamental results on this topic were obtained by A.L. Vol'berg and S.R. Treil' in [78], where they gave a sufficient condition for a measure to be a Carleson measure in $K^2(\Theta)$ and a necessary and sufficient condition to be a Carleson measure in $K^2(\Theta)$ when $\Theta(z)$ is a connected level set inner function (CLS). Subsequently A. Aleksandrov gave a different proof of the same theorems in [2].

In the third chapter I study also a necessary condition for the horizontal translation operator T_σ, Theorem 3.9. Also in this case, I associate it a Carleson measure with real support.

In the fourth chapter I go on studying T_ζ, and in particular I present an original necessary and sufficient condition for the boundedness of T_ζ, Theorems 4.9 and 4.12. The approach I use recalls the standard techniques for studying multipliers in analytic function spaces.

Besides the Hilbert de Branges spaces $\mathcal{H}^2(E)$, there are also the p-de Branges spaces, see (1.27). These Banach spaces were first introduced by A. Baranov in [10] and then extensively studied by other authors. Therefore, it is natural looking for the properties that $E(z)$ needs to satisfy so that the embedding operator $\iota_{p,q}$ from $\mathcal{H}^p(E)$ into $\mathcal{H}^q(E)$ is continuous. This problem has already been widely studied in the literature. First of all the continuity of the embedding operator between different p-Bernstein spaces is a well known result, [5] and [56].

This problem has already been investigated and solved in [32], [30], [34] by K. Dyakonov, for $p > 1$. Dyakonov proved that the boundedness of the derivative of the phase function of the meromorphic inner function $\Theta(z)$ is a necessary and sufficient condition for the continuity of the embedding operator in $*$-invariant subspaces $K^p(\Theta)$ of the Hardy space. This condition can be adapted also to the de Branges spaces.

Instead of focusing on the case $p > 1$, I fix my attention on the case $p = 1$, which has already been studied by A. Baranov in [11]. I prove that

$$\|\phi'\|_{L^\infty} < \infty$$

if and only if the embedding operator

$$\iota : \mathcal{H}^1(E) \hookrightarrow \mathcal{H}^q(E)$$

is continuous. However, to obtain this theorem, Theorem 5.7, I add further conditions to $E(z)$, (5.9) and (5.10). I point out also that in general the boundedness of the derivative of the phase function is not necessary for the continuity of the embedding operator, as proved in [11]. Moreover Proposition 4.2 of [11] is similar to my Theorem 5.7. For a characterization of the Hermite Biehler functions which satisfy (5.9) and (5.10), I recall [48].

In the third part, I study a completely different problem: the characterization of the dual of the 1-de Branges spaces.

In order to describe it, I first recall what is known about the dual of $\mathcal{H}^p(E)$ when $1 < p < \infty$. Indeed, it has been already discovered that the duals of the p-de Branges spaces are isomorphic to the q-de Branges spaces where $q = p/(p-1)$, [24], [26] and Corollary 6.8. However, as I show in the sixth chapter, in order to prove this result, I need to use Toeplitz operator with anti-analytic symbols which are always unbounded when $p = 1$, Theorems 6.10 and 6.15. For this reason in order to describe $\mathcal{H}^1(E)^*$, a completely different approach is needed.

In the last two chapters I firstly describe the dual of the 1-Bernstein spaces and finally of some 1-de Branges spaces.

My starting point has been [18] where R. Bessonov described the dual of the 1-$*$ invariant subspaces of the 1-Hardy space of the unit disk. He proved that their duals can be identified with the quotient spaces of sequences BMO(μ) where μ is a discrete measure associate to the corresponding $*$-invariant subspaces.

Actually, also W. Cohn in [26] studied a subspace of BMOA, $K^*_\Theta := \mathrm{BMOA} \cap K^2(\Theta)$, which is closely related to $K^1(\Theta)^*$ as proved by R. Bessonov in Proposition 4.2 in [18], even if it is not isomorphic.

I study the dual of the 1-Bernstein space without using the Cayley transform, but introducing directly the atomic structure of $\mathcal{B}^1_\pi$. By doing this, I obtain the analogue of Bessonov's result for $\mathcal{B}^1_\pi$:

$$\mathcal{B}^{1\,*}_\pi = \mathrm{BMO}(\mathbb{Z}) \ .$$

I go further and I look for a characterization of $(\mathcal{B}^1_\pi)^*$ in terms of entire functions. Studying some properties of BMO($\mathbb{Z}$), I introduce a new space $\mathcal{X}$, Definition 7.16, made by entire functions, which is isomorphic to BMO($\mathbb{Z}$), Theorem 7.17.

Unfortunately the elements of $\mathcal{X}$ have a rigid structure and it is not easy to verify if, given an entire function f, it does belong to $\mathcal{X}$. Therefore I study a further description of $(\mathcal{B}^1_\pi)^*$, fixing my attention on the properties of its elements. By doing this, I introduce the space $\mathcal{Y}$, which is my definitive description of $(\mathcal{B}^1_\pi)^*$, Theorem 7.21.

Finally, in the last chapter, I characterize $\mathcal{H}^1(E)^*$. I use the atomic decomposition of some $\mathcal{H}^1(E)$ which is original for the Branges spaces, Theorem 8.1. Subsequently I give two different description of the dual of $\mathcal{H}^1(E)$, Theorems 8.2 and 8.23. The calculations in this case are much more complicated, since the Fourier transform is not available. Even if these two chapters partially overlap, I think that separating them makes sense. Indeed the computations in the Bernstein cases are simpler and they are useful to understand what is necessary to do for the de Branges spaces.

During my Ph.D. experience, I have also faced other relevant problems which should be mentioned here although they have not been solved. I fix my attention only on those related to the topics of this dissertation.

Even if the connection between the translation operator and the differentiation operator is clear, I do not know if it is possible to consider a de Branges space $\mathcal{H}(E)$ where the translation operator is bounded but the differentiation operator is not. Looking at Baranov's calculations in [6] and at my estimates in Chapter 4, this problem seems to be related to the derivative of the phase function and to the boundedness of its Hilbert transform. Furthermore, if this space exists, it is also an example of a semi-group of bounded operators, whose generator is an unbounded operator.

Another interesting question deals with the atomic decomposition of $\mathcal{H}^1(E)$. Although in the seventh chapter I show that when the Hermite Biehler function satisfies some particular conditions, the associated 1-de Branges space admits an atomic structure, I do not know what are the minimal conditions necessary for it. I think also that the atomic structure could be a powerful instrument for studying problems concerning $\mathcal{H}^1(E)$. Indeed, it might be used also for studying the boundedness of the embedding operator $\iota_{1,q}$.

Finally, in the last chapter, I study the dual of $\mathcal{H}^1(E)$. The characterizations I give, does not use entire functions. I believe that it would be important characterizing it also in this way, by using theorems about the zeros of the functions in the Smirnov's class.

I. Preliminaries

1. Hermite Biehler functions and de Branges spaces

These first two chapters are preliminary to the rest of the thesis: we clarify the notations and we introduce the p-de Branges spaces, explaining some of their well known properties.

In the first section we introduce the Hermite Biehler functions. We highlight their relationship with the meromorphic inner functions (1.7), and we show their canonical representation form, Theorem 1.7.

The Hermite Biehler functions are used for defining the de Branges spaces $\mathcal{H}(E)$. In the second section we show that $\mathcal{H}(E)$ is a Hilbert space of entire functions with reproducing kernels, Theorem 1.10. We also prove that the Hilbert de Branges spaces are unitarily equivalent to the model spaces of the upper half plane $K^2(\Theta)$, Theorem 1.12, and we provide two well-known examples of Hilbert de Branges spaces.

We are interested in some geometric properties of the space $\mathcal{H}(E)$ and for this reason in the fourth section we introduce the phase function $\phi_E(x)$ associated to the Hermite Biehler function $E(z)$. After having proved the analytic expression of the derivative of the phase function (1.24), we show the connection between the norm of the reproducing kernel at the real points and the derivative of the phase function (1.26).

Finally, in the fifth section, we introduce the p-de Branges spaces, when $p \neq 2$. Unlike the Hilbert case, these spaces have not been deeply investigated yet. A lot of properties which hold in the Hilbert case, are true also for the case $p \neq 2$. Nevertheless, in this thesis, we fix our attention on their differences and we face some questions whose answers in the non-Hilbert case are contrary to all the expectations.

1. Hermite Biehler functions

There are several ways to introduce the Hilbert de Branges space. Perhaps, the easiest one, that we will see in the next section, (1.9), uses the Hardy spaces of the upper half plane $\mathbb{C}^+$. In order to explain this technique, we need the Hermite Biehler functions, which we define and characterize in this section.

Definition 1.1 Let $f(z)$ be an entire function. The sharp operator, $\#$, is defined as

$$f^{\#}(z) := \bar{f}(\bar{z}) \ . \tag{1.1}$$

Definition 1.2 An entire function $E(z)$ is a Hermite Biehler function if

$$|E(z)| > \left|E^{\#}(z)\right| = |E(\bar{z})| \qquad \forall z \in \mathbb{C}^+ \ . \tag{1.2}$$

Obviously the Hermite Biehler function $E(z)$ has no zeros in the upper half plane. Moreover, in order to avoid some technical computations, we assume also that $E(x) \neq 0$ for every $x \in \mathbb{R}$.

Definition 1.3 An entire function $S(z)$ is said to be real if

$$S(z) = S^{\#}(z) \qquad \forall z \in \mathbb{C} \ . \tag{1.3}$$

We associate two important real functions to the Hermite Biehler function $E(z)$:

$$A(z) = \frac{E(z) + E^{\#}(z)}{2} \ , \qquad B(z) = \frac{-E(z) + E^{\#}(z)}{2i} \ . \tag{1.4}$$

Therefore, $E(z) = A(z) - iB(z)$.

Example 1 If $a \in \mathbb{R}$, the function e^{iaz^n} is a Hermite Biehler function if and only if $n = 1$ and $a < 0$. Indeed, if $z = re^{i\theta}$, then

$$\left|e^{iaz^n}\right| = e^{\Re iaz^n} = e^{-ar^n \sin n\theta} \ , \qquad \left|e^{ia\overline{z^n}}\right| = e^{\Re ia\overline{z^n}} = e^{ar^n \sin n\theta} \ .$$

The function e^{iaz^n} is in the Hermite Biehler class if and only if $|e^{ia\overline{z^n}}| < \left|e^{iaz^n}\right|$, that is,

$$0 < -2ar^n \sin n\theta \qquad \forall\theta \in (0, \pi) \ , \forall r > 0 \ .$$

This last condition is possible if and only if $n = 1$ and $a < 0$. We note also that

$$e^{iaz} = \cos(-az) - i \sin(-az) \ .$$

In the general case, if $a = x_a + iy_a \in \mathbb{C}$,

$$\left|e^{az^n}\right| = e^{x_a r^n \cos(n\theta) - y_a r^n \sin(n\theta)} \ , \qquad \text{while} \qquad \left|e^{a\bar{z}^n}\right| = e^{x_a r^n \cos(n\theta) + y_a r^n \sin(n\theta)} \ .$$

Thus, only the sign of the imaginary part of a does matter.

Example 2 The function $(1 - z/z_n)$ where $z_n = x_n + iy_n \in \mathbb{C}^-$ is in the Hermite Biehler class. Let us compute as above:

$$|1 - z/z_n|^2 = \left[(x - x_n)^2 + (y_n - y)^2\right] / |z_n|^2 \ ,$$
$$|1 - \bar{z}/z_n|^2 = \left[(x - x_n)^2 + (y_n + y)^2\right] / |z_n|^2 \ .$$

Therefore

$$|1 - z/z_n| > |1 - \bar{z}/z_n| \quad \text{if and only if}$$
$$\left[(x - x_n)^2 + (y_n - y)^2\right] / |z_n|^2 > \left[(x - x_n)^2 + (y_n + y)^2\right] / |z_n|^2 \ ,$$

which is true if and only if $y_n < 0$.

Example 3 It turns out that, if $z_n = x_n + iy_n \in \mathbb{C}^-$, the function $(1 - z/z_n)e^{z/z_n}$ is not in the Hermite Biehler class. Indeed we know that

$$(1 - z/z_n)e^{z/z_n} = (1 - z/z_n)e^{-i\frac{y_n}{|z_n|^2}z} e^{\frac{x_n}{|z_n|^2}z} \quad \text{where } -\frac{y_n}{|z_n|^2} > 0 \ .$$

Consequently

$$\begin{aligned} \left|(1 - z/z_n)e^{z/z_n}\right| &= |(1 - z/z_n)|\, e^{\frac{y_n}{|z_n|^2}y} e^{\frac{x_n}{|z_n|^2}x} \quad \text{and} \\ \left|(1 - \bar{z}/z_n)e^{\bar{z}/z_n}\right| &= |(1 - \bar{z}/z_n)|\, e^{-\frac{y_n}{|z_n|^2}y} e^{\frac{x_n}{|z_n|^2}x} \ . \end{aligned}$$

If $y_n < 0$, the condition (1.2) is not satisfied for y large enough.

We associate to every Hermite Biehler function $E(z)$ a meromorphic inner function $\Theta(z)$.

Definition 1.4 A meromorphic inner function $\Theta(z)$ is an inner function having a meromorphic extension to the whole $\mathbb{C}$.

As described in [29], [41], [63] a singular inner function $S(z)$ has the expression

$$S(z) = \exp\left\{ i \int_{\mathbb{R}} \frac{1 + tz}{t - z} dn(t) \right\}$$

where $n(t)$ is a Borel measure singular with respect to the Lebesgue measure, and it does not have any meromorphic extension to $\mathbb{C}$. Indeed, if $n(t)$ contains point mass measures, then $S(z)$ will have essential singularities. On the other side, if $n(t)$ is singularly continuous, the points in the support of the measure create singularities that are not isolated.

For this reason, the meromorphic inner functions are described just by the two parameters (Λ, a), the zeros set of $\Theta(z)$ and its *mean type*.

Definition 1.5 A function $\Theta(z) \in H^\infty(\mathbb{C}^+)$ is a meromorphic inner function if

$$\Theta(z) := e^{iaz} \prod_n e^{i\varphi_n} \frac{z - \lambda_n}{z - \overline{\lambda_n}} \quad \text{where } a > 0,\ \lambda_n = a_n + ib_n \in \mathbb{C}^+ \ , \tag{1.5}$$

where

$$\sum_n \frac{b_n}{a_n^2 + b_n^2} < \infty \ , \quad a \geq 0 \ , \tag{1.6}$$

and $\{\lambda_n\}$ does not have a finite accumulation point.

There is a deep relationship between meromorphic inner functions and Hermite Biehler functions.

Proposition 1.6 If $E(z)$ is a Hermite Biehler function, the associated meromorphic inner function $\Theta(z)$ is defined as

$$\Theta(z) := E^\#(z)/E(z) \ , \quad \forall z \in \overline{\mathbb{C}^+} \ . \tag{1.7}$$

PROOF. We know that

$$|E(\bar{z})| < |E(z)| \ , \ \forall z \in \mathbb{C}^+ \quad \text{which implies that} \quad |\Theta(z)| < 1 \ , \ \forall z \in \mathbb{C}^+ \ .$$

Moreover, the trace of the inner function $\Theta(x)$ on $\mathbb{R}$ has modulus equal to 1 for all $x \in \mathbb{R}$, since the Hermite Biehler function $E(z)$ is entire. □

We observe that the correspondence between the Hermite Biehler class and the meromorphic inner function is not one-to-one. Indeed let us consider a real entire function $S(z)$, (1.3). Then, $S(z)E(z)$ is still Hermite Biehler but $\Theta_{SE}(z) = \Theta_E(z)$ for $z \in \mathbb{C}^+$.

The condition (1.7) gives us a necessary condition that the zeros of $E(z)$ need to satisfy. Since

$$\Theta_E(z) = 0 \text{ if and only if } E^{\#}(z) = 0 \ ,$$

then $E(z_n) = 0$, where $z_n = x_n + iy_n$, if and only if $\Theta_E(\bar{z}_n) = 0$. Consequently, the zeros of the function $E(z)$, that is $\{z_n\}_n \subset \mathbb{C}^-$, satisfy the Blaschke condition, that is:

$$\sum_n \frac{-y_n}{1+|\bar{z}_n|^2} < \infty \quad \text{or equivalently} \quad \sum_n \Im\left(\frac{1}{z_n}\right) < \infty \ . \tag{1.8}$$

The Blaschke condition (1.6) does not imply any further condition on the rate of growth of the zeros of $E(z)$. Indeed,

$$\left|\Im \frac{1}{z_n}\right| \geq \frac{1}{|z_n|^q} \quad \text{if and only if} \quad -y_n \geq \frac{1}{|z_n|^{q-2}} \ .$$

For this reason, the Blaschke condition bounds the exponent of convergence of the zeros of $E(z)$ if and only if its zeros belong to the domain $\Omega := \{z \in \mathbb{C}^- \text{ such that } -y \geq |z|^{2-q}\}$.

To conclude this section, we recall the canonical expression for the functions in the Hermite Biehler class, see [55]. We use Levin's notation [55].

Let us consider the function $p(z) := \sum_k (a_k + ib_k) z^k$. Then $Rp := \sum_k a_k z^k$ and $Ip := \sum_k b_k z^k$. For sake of brevity, we set $P_j(w) := w + \cdots + \frac{1}{j} w^j$.

Theorem 1.7 Let $E(z)$ be a Hermite Biehler function and let $\{z_n\}_n \subset \mathbb{C}^-$ be its zeros. Then, according to Weierstrass factorization [56], we know that:

$$E(z) = C z^k e^{f(z)} \prod_n \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \cdots + \frac{1}{j_n}\left(\frac{z}{z_n}\right)^{j_n}\right)$$

where $\sum_n \frac{1}{|z_n|^{j_n+1}}$ converges and $f(z)$ is entire. Then the function $E(z)$ is equal to

$$E(z) = C z^k e^{g(z) + i(mz+d)} \prod_n \left(1 - \frac{z}{z_n}\right) \exp\left(RP_{j_n}(z/z_n)\right) \ ,$$

where $m < 0$ and $g(z)$ is a real function.

The above theorem is very important. For example, by using it we describe all the Hermite Biehler function of order at most 2.

Proposition 1.8 Let $\{z_n\}_n \subset \mathbb{C}^-$ satisfies the Blaschke condition (1.6) and $\sum_n \frac{1}{|z_n|^2} < \infty$. If $E(z)$ is Hermite Biehler of order 2 and $E(z_n) = 0$, then

$$E(z) = e^{i(az+d)+p_2z^2+p_1z+p_0} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z\Re\frac{1}{z_n}} \ ,$$

where $a < 0$ and $p_2, p_1, p_0, d \in \mathbb{R}$.

Proof. Since the exponent of convergence of the zeros of the Hermite Biehler function is less than or equal to 2, then

$$\prod_n \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}}$$

converges to an entire function. Moreover, thanks to the previous formula,

$$e^{ibz} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z\Re\frac{1}{z_n}}$$

converges in $\mathbb{C}$ where $b = \sum_n \Im \frac{1}{z_n}$. Consequently,

$$\begin{aligned} E(z) &= Ce^{p(z)} e^{ibz} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z\Re\frac{1}{z_n}} \\ &= e^{i(az+d)+p_2z^2+p_1z+p_0} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z\Re\frac{1}{z_n}} \ , \end{aligned}$$

and we have proved the statement. □

2. Hilbert de Branges spaces

We define the Hilbert de Branges spaces.

Definition 1.9 Let $E(z)$ be a Hermite Biehler function. The 2-de Branges space $\mathcal{H}(E)$ is the space of entire functions $f(z)$ such that

$$\frac{f(z)}{E(z)} \quad \text{and} \quad \frac{f^{\#}(z)}{E(z)} \in H^2(\mathbb{C}^+) \ . \tag{1.9}$$

It is useful to observe that

$$f^{\#}(z)/E(z) \in H^2(\mathbb{C}^+) \quad \text{if and only if} \quad f(z)/E^{\#}(z) \in H^2(\mathbb{C}^-) \ . \tag{1.10}$$

Indeed, if a function $g(z) \in H^2(\mathbb{C}^+)$, then $g^{\#}(z) \in \mathrm{Hol}\,(\mathbb{C}^-)$. Moreover the function $g^{\#}(z)$ belongs to $H^2(\mathbb{C}^-)$ since

$$\|g^{\#}\|^2_{H^2(\mathbb{C}^-)} = \sup_{y<0} \int_{\mathbb{R}} |g^{\#}(x+iy)|^2 dx = \sup_{t>0} \int_{\mathbb{R}} |g(x+it)|^2 dx = ||g||^2_{H^2(\mathbb{C}^+)} \ .$$

We introduce the inner product of $\mathcal{H}(E)$.

Theorem 1.10 The de Branges space $\mathcal{H}(E)$, endowed with the inner product

$$\langle g, f\rangle_{\mathcal{H}(E)} := \int_{\mathbb{R}} g(x)\overline{f(x)}\frac{dx}{|E(x)|^2}\ , \tag{1.11}$$

is a Hilbert space. Moreover, for every $z \in \mathbb{C}$, the function

$$k_z(w) = \frac{1}{2\pi i}\frac{\overline{E(z)}E(w) - E(\bar{z})\overline{E(\overline{w})}}{(\bar{z} - w)} \tag{1.12}$$

is the reproducing kernel at z, that is, for every $f \in \mathcal{H}(E)$

$$\langle f, k_z\rangle_{\mathcal{H}(E)} = f(z)\ .$$

Proof. If we fix $f \in \mathcal{H}(E)$, f/E and $f/E^{\#}$ satisfy the Cauchy integral formula since they are elements of $H^2(\mathbb{C}^+)$ and $H^2(\mathbb{C}^-)$ respectively. Consequently, it holds that

$$\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(x)}{E(x)}\frac{dx}{x-z} = \begin{cases}\frac{f(z)}{E(z)} & \text{if } z \in \mathbb{C}^+\\ 0 & \text{if } z \in \mathbb{C}^-\end{cases}\ ,$$

$$\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(x)}{E^{\#}(x)}\frac{dx}{x-z} = \begin{cases}0 & \text{if } z \in \mathbb{C}^+\\ -\frac{f(z)}{E^{\#}(z)} & \text{if } z \in \mathbb{C}^-\end{cases}\ .$$

Therefore, for every $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\begin{aligned}\langle f, k_z\rangle_{\mathcal{H}(E)} &= \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{E(z)\overline{E(x)} - \overline{E(\bar{z})}E(x)}{x-z}f(x)\frac{dx}{|E(x)|^2}\\ &= \frac{E(z)}{2\pi i}\int_{\mathbb{R}}\frac{f(x)}{E(x)}\frac{dx}{x-z} - \frac{E^{\#}(z)}{2\pi i}\int_{\mathbb{R}}\frac{f(x)}{E^{\#}(x)}\frac{dx}{x-z} = f(z)\ .\end{aligned}$$

The function $\langle f, k_z\rangle_{\mathcal{H}(E)}$ is an entire function in the variable z. Indeed, it is clearly holomorphic in $\mathbb{C}\setminus\mathbb{R}$. Let $z = x_0 \in \mathbb{R}$. First of all $\langle f, k_{x_0}\rangle_{\mathcal{H}(E)}$ is well defined since

$$\begin{aligned}|\langle f, k_{x_0}\rangle_{\mathcal{H}(E)}| &= \left|\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(x)}{|E(x)|^2}\frac{E(x_0)\overline{E(x)} - \overline{E(x_0)}E(x)}{x-x_0}dx\right|\\ &\leq \frac{1}{2\pi}\int_{x_0-\epsilon}^{x_0+\epsilon}\left|\frac{f(x)}{E(x)^2}\right|\left|E(x_0)\overline{\partial_z E(x_0)} - \overline{E(x_0)}\partial_z E(x_0)\right|dx\\ &\qquad + \frac{1}{2\pi}\int_{|x-x_0|\geq\epsilon}\left|\frac{f(x)}{E(x)}\right|\left|\frac{E(x_0)\overline{E(x)} - \overline{E(x_0)}E(x)}{(x-x_0)E(x)}\right|dx\\ &\leq M_{x_0}\left\|\frac{f}{E}\right\|_{L^2(\mathbb{R})} + \frac{1}{2\pi}\left\|\frac{f}{E}\right\|_{L^2(\mathbb{R})}\int_{|x-x_0|\geq\epsilon}\frac{M'_{x_0}}{(x-x_0)^2} < \infty\ .\end{aligned} \tag{1.13}$$

With similar computations, it is also possible to show that

$$\lim_{|h|\to 0}\frac{\langle f, k_{x_0+h}\rangle_{\mathcal{H}(E)} - \langle f, k_{x_0}\rangle_{\mathcal{H}(E)}}{h} \tag{1.14}$$

exists finite. Since $\langle f, k_z\rangle_{\mathcal{H}(E)}$ and $f(z)$ are entire functions and

$$\langle f, k_z\rangle_{\mathcal{H}(E)} = f(z)$$

for every $z \in \mathbb{C} \setminus \mathbb{R}$, also for $z = x \in \mathbb{R}$ it is true that

$$f(x) = \langle f, k_x\rangle_{\mathcal{H}(E)} \ .$$

The functions k_z belong to $\mathcal{H}(E)$. Indeed, they are clearly entire. In fact, k_z/E and $k^{\#}/E$ belong to the Hardy space $H^2(\mathbb{C}^+)$. To prove this last statement, let us consider the function in k_z/E. It turns out that if $z \in \mathbb{C}^+$, the function k_z/E is L^2-integrable on every horizontal line of the upper half plane $\Im(z) = y$. Indeed, arguing as in (1.13)

$$\begin{aligned}
(1.15)\qquad \int_{\mathbb{R}} \left|\frac{k_z(x+iy)}{E(x+iy)}\right|^2 dx &= \int_{\mathbb{R}} \left|\frac{\overline{E(z)}E(x+iy) - E(\bar z)\overline{E(x-iy)}}{2iE(x+iy)(\bar z - x - iy)}\right|^2 dx \\
&\le \int_{\Re z-\epsilon}^{\Re z+\epsilon} \left|\frac{\overline{E(z)}E(x+iy) - \overline{E(x-iy)}E(\bar z)}{CE(x+iy)(\bar z - x - iy)}\right|^2 dx \\
&\quad + \int_{|x-\Re z|\ge\epsilon} \left|\frac{\overline{E(z)}E(x+iy) - E(\bar z)\overline{E(x-iy)}}{2iE(x+iy)(\bar z - x - iy)}\right|^2 dx \\
&\le M + \int_{|x-\Re z|\ge\epsilon} \frac{M'}{(x-\Re z)^2} dx < \infty \ .
\end{aligned}$$

With an analogous reasoning, one can also prove that the function $k_z^{\#}(x)/E(x)$ in (**??**) belongs to $H^2(\mathbb{C}^+)$. Consequently, k_z are the reproducing kernels of $(\mathcal{H}(E), \langle\cdot,\cdot\rangle_{\mathcal{H}(E)})$.

The last thing that we have still to prove is that $(\mathcal{H}(E), \langle\cdot,\cdot\rangle_{\mathcal{H}(E)})$ is complete. Let $\{f_n\}$ be a Cauchy sequence of $\mathcal{H}(E)$. Since $H^2(\mathbb{C}^+)$ is a Hilbert space,

$$f_n/E \to g/E \in H^2(\mathbb{C}^+) \quad \text{and} \quad f_n^{\#}/E \to h/E \in H^2(\mathbb{C}^+) \ .$$

Consequently the functions f_n converge to $f_0 \in \mathrm{Hol}\,(\mathbb{C})$), where

$$f_0(z) := \begin{cases} g(z) & \text{if } z \in \overline{\mathbb{C}^+} \\ h^{\#}(z) & \text{if } z \in \mathbb{C}^- \end{cases} \ .$$

We note that $g(x) = \overline{h(x)}$ for every $x \in \mathbb{R}$ since

$$g(x) = \lim_{n\to\infty} f_n(x) = \overline{\lim_{n\to\infty} \overline{f_n(x)}} = \overline{h(x)} \ ,$$

and we have proved the theorem. □

It is worth observing that there are several other different ways to define the Hilbert de Branges spaces, see [28], [69], [70] and [71]. For example, the axiomatic approach, described in [28], is extemely useful for the spectral representation of closed, symmetric operators with deficiency indexes $(1,1)$, see [4], [45], [59] and [74].

The $\mathcal{H}(E)$-norm of the reproducing kernel is an important tool in the study of the geometric properties of $\mathcal{H}(E)$. For this reason, we observe that

$$k_x(x) = \lim_{w\to\ x} k_w(w) = |E(x)|^2 \lim_{w\to x} \frac{1-|\Theta(w)|^2}{4\pi\Im w} \ ,$$

where $\Theta(z) = e^{iaz}\prod_j b_j(z)$ is defined as in (1.5). Since,

$$\begin{aligned}(1.16)\quad 1-|\Theta(w)|^2 &= (1-\left|e^{iaw}\right|^2) + \sum_{n\geq 1}\left|e^{iaw}\right|^2\left[\prod_{j=0}^{n-1}|b_j(w)|^2\left(1-|b_n(w)^2\right)\right]\\ &= (1-\left|e^{iaw}\right|^2) + \sum_{n\geq 1}\left|e^{iaw}\right|^2\left[\prod_{j=0}^{n-1}|b_j(w)|^2\frac{4y_n\Im w}{|w-\bar{z}_n|^2}\right] \ ,\end{aligned}$$

we obtain that

$$(1.17)\quad k_x(x) = |E(x)|^2\frac{1}{\pi}\left(\frac{a}{2}+\sum_n\frac{y_n}{|x-\bar{z}_n|^2}\right) \ .$$

where $\{\bar{z}_n := x_n - iy_n\} \in \mathbb{C}^-$ are the zeros of $E(z)$.

There is an important relationship between the model spaces associated to meromorphic inner functions and the de Branges spaces. Let us first recall the definition of the model spaces. For a more complete description of these spaces, we refer to [24], [40], [61] and [63].

Definition 1.11 The model space, also known as the 2-$*$ invariant subspace of $H^2(\mathbb{C}^+)$, is defined as

$$(1.18)\quad K^2(\Theta) := \left\{f \in H^2(\mathbb{C}^+) \text{ such that } f(x) = \Theta(x)\overline{g(x)} \text{ for } x\in\mathbb{R}\ ,\ \text{ where } g\in H^2(\mathbb{C}^+)\right\}\ ,$$

where $\Theta(z)$, defined as in (1.5), is an inner function.

The subspace $K^2(\Theta)$ can be defined also as the orthogonal complement in H^2 of the subspace ΘH^2, [40]. As proved for example in [6], the space $K^2(\Theta)$ is isometrically isomorphic to $\mathcal{H}(E)$, when (1.7) holds.

Proposition 1.12 Let $E(z)$ be a Hermite Biehler function and let $\Theta(z)$ be the corresponding meromorphic inner function described in (1.7). The map $\mathcal{U}$,

$$\mathcal{U}:\ \mathcal{H}^2(E)\to K^2(\Theta)\ ,\qquad \text{where}\qquad \mathcal{U}(F)(z) := f(z) = \frac{F(z)}{E(z)}\ ,$$

is an isometric isomorphism from $\mathcal{H}^2(E)$ onto $K^2(\Theta)$.

PROOF. Let $F \in \mathcal{H}^2(E)$, then $f = F/E \in H^2(\mathbb{C}^+)$ and $\overline{\Theta(x)}f(x) = F(x)/\overline{E(x)} = \overline{F^\#(x)/E(x)}$, which is the boundary value of an element of $\overline{H^2(\mathbb{C}^+)}$.

On the other hand, if $f \in K^2(\Theta)$, let $F := fE$. Then $F/E \in H^2(\mathbb{C}^+)$, $F^\#/E \in H^2(\mathbb{C}^+)$. Furthermore the function $F = fE$ is entire. □

We observe that the model spaces are closed subspaces of $H^2(\mathbb{C}^+)$ and the functions

$$K_w(z) : \frac{i}{2\pi}\frac{1-\overline{\Theta(w)}\Theta(z)}{z-\overline{w}} \qquad w \in \overline{\mathbb{C}^+} \tag{1.19}$$

are the reproducing kernels of $K^2(\Theta)$.

3. Some examples

In this short section we provide two examples of Hilbert de Branges spaces [74].

Example 4 (Space of polynomials) We consider the linear space

$$\mathcal{P}_n := \{\text{polynomials of degree } < n\} \ .$$

It turns out to be a de Branges space by choosing as Hermite Biehler function any polynomial whose zeros are all in the lower half plane.

Lemma 1.13 Let $q(z)$ be a a polynomial whose zeros are all in the lower half plane. Then $q(z)$ is a Hermite Biehler function.

PROOF. It is clear that $q(z)$ is an entire function. Furthermore,

$$q(z) = \alpha(z-\lambda_1)\dots(z-\lambda_n) \ ,$$

where λ_i's are the zeros of $q(z)$, all contained in $\mathbb{C}^-$. Since, for every $z \in \mathbb{C}^+$,

$$|z-\lambda_i|^2 = (\Re z - \Re\lambda_i)^2 + (\Im z - \Im\lambda_i)^2 > (\Re\overline{z} - \Re\lambda_i)^2 + (\Im\overline{z} - \Im\lambda_i)^2 = |\overline{z}-\lambda_i|^2 \ ,$$

the condition

$$|q(z)| > |q(\overline{z})| \quad \forall z \in \mathbb{C}^+$$

is always satisfied. □

We check that $\mathcal{P}_n$ is a Hilbert de Branges space.

Proposition 1.14 Let $q(z)$ be a a polynomial of degree equal to n, whose zeros are all in the lower-half complex plane. Then

$$\mathcal{H}(q) = \mathcal{P}_n \ .$$

PROOF. It is clear that, for every $n \in \mathbb{N}$, $z^{n\#} = z^n$. Since

$$z^m/q(z) \in H^2(\mathbb{C}^+) \quad \text{if } m < n \ ,$$

$\mathcal{P}_n \subseteq \mathcal{H}(q)$. On the other hand, z^N cannot be an element of $\mathcal{H}(q)$ because z^N/q is not in $L^2(\mathbb{R})$ if $N \geq n$.

Moreover, if a function $f(z)$ belongs to $\mathcal{H}(q)$, then it needs to grow at most as a polynomial. Indeed f/q belongs to $H^2(\mathbb{C}^+ - i\epsilon)$, while $f/q^\#$ belongs to $H^2(\mathbb{C}^- + i\epsilon)$. These two conditions imply that $|f(z)| \leq c|q(z)|$ for every $z \in \mathbb{C}$, that is $f(z)$ is a polynomial. $\square$

Clearly, $\mathcal{H}(q)$ is finite dimensional. Furthermore, to describe $\mathcal{H}(q)$ only the zeros in the lower half plane are relevant. Indeed, it might be proved that $\mathcal{H}(q)$ is isometrically isomorphic to $\mathcal{H}(qp)$ where p is any polynomial with all its zeros on $\mathbb{R}$.

Example 5 We recall that the Paley-Wiener spaces, parametrized by $a > 0$, are defined as

$$\mathcal{PW}_a := \left\{ \begin{array}{c} f \text{ entire} : \int_{\mathbb{R}} |f(x)|^2 dx < \infty \\ \text{and } \forall \epsilon > 0 \ , \ \text{exists } C_{f,\epsilon} > 0 : |f(z)| \leq C_{f,\epsilon} e^{(a+\epsilon)|z|} \quad \forall z \in \mathbb{C} \end{array} \right\} .$$

These spaces are the most important example of de Branges space. Indeed,

Proposition 1.15 It holds that

$$\mathcal{PW}_a = \mathcal{H}(e^{-iaz}) \ .$$

PROOF. It is necessary to characterize the Paley-Wiener spaces by means of the Fourier transform. Indeed

$$\mathcal{PW}_a = \mathcal{F}^{-1} \ L^2(-a, a) \ ,$$

that is, $\mathcal{PW}_a$ coincides with the holomorphic extension of the inverse Fourier transform of the L^2-integrable functions on the symmetric interval $(-a, a)$. To prove that for every function $f \in \mathcal{PW}_a$, $f \in \mathcal{H}(e^{-iaz})$, we apply the definition (1.9). We check that

$$f(z)/e^{-iaz} \in H^2(C^+) \quad \text{and} \quad \overline{f(\overline{z})/e^{ia\overline{z}}} \in H^2(C^+) \ . \tag{1.20}$$

Considering the Fourier transform,

$$\text{supp}\ \mathcal{F}\left[\frac{f(\cdot)}{e^{-ia\cdot}}\right] \subseteq \mathbb{R}^+ \tag{1.21}$$

and we obtain the left inclusion in (1.20). With similar reasoning, also the right inclusion in (1.20) and the inclusion $\mathcal{H}(e^{-iaz}) \subseteq \mathcal{PW}_a$ are obtained. □

It is worth observing that $\langle\cdot,\cdot\rangle_{\mathcal{H}(e^{-iaz})}$ is equivalent to $\langle\cdot,\cdot\rangle_{\mathcal{PW}_a}$. Indeed, $|e^{-iaz}| = 1$, whenever $z \in \mathbb{R}$.

The Paley-Wiener spaces have been deeply studied. An important article on them is [64], where J. Ortega-Cerdà and K. Seip characterize the de Branges spaces which have got some sampling properties similar to those of the Paley-Wiener spaces.

For other examples of Hilbert de Branges spaces we cite the third part of [28]. We point out also the reference [12], where A.D. Baranov, Y. Belov and A. Poltoratski characterize the Hermite Biehler functions associated to Schrödinger operator with integrable potential.

4. The phase function ϕ_E

While studying de Branges space $\mathcal{H}(E)$, the phase function ϕ_E and its derivative are of fundamental importance. In this section we describe its expression and we highlight its relationship with the norm of the reproducing kernel and with the derivative of the associated meromorphic inner function $\Theta(z)$.

Definition 1.16 Let $E(z)$ be a Hermite Biehler function. The real valued function $\phi_E(t)$ such that

$$e^{i\phi_E(t)}E(t) = |E(t)| \quad \forall t \in \mathbb{R} \tag{1.22}$$

is the phase function of $E(z)$.

Every Hermite Biehler function $E(z)$ admits a phase function.

Theorem 1.17 For every Hermite Biehler function $E(z)$ there exists an associated phase function $\phi_E(t)$; moreover $\phi_E(t)$ is $\mathbb{C}^\infty(\mathbb{R})$.

Proof. For sake of clearness, let us call $\phi_E(t) = \phi(t)$. First of all, since we assume that $E(x) \neq 0$ for $x \in \mathbb{R}$, we define $\log(E(z))$ for $z \in \overline{\mathbb{C}}^+$. This function is holomorphic in a small neighborhood of $\mathbb{C}^+$. For this reason, we define

$$-i\phi(t) = i\Im \log E(t) \ .$$

The function $\phi(t)$ is differentiable. Furthermore, because of the Cauchy-Riemann equations,

$$0 > -\frac{\partial}{\partial y}\log|E(z)|\,|_{z=t} = \frac{\partial}{\partial x}\arg(E(z))\,|_{z=t}\ ,$$

where the first inequality is satisfied because of (1.2). Therefore $\phi(t)' > 0$ for every $t \in \mathbb{R}$. □

From (1.7), we recall that $\Theta(t) = \frac{E^{\#}(t)}{E(t)} = e^{2i\phi_E(t)}$ for every $t \in \mathbb{R}$. Therefore,

$$2i\phi_E(t)' e^{2i\phi_E(t)} = \Theta'(t) \quad \text{and} \quad \phi_E(t)' = \frac{|\Theta'(t)|}{2}\ . \tag{1.23}$$

The derivative of Θ' is well defined since the inner function admits an analytic continuation beyond the real line. Indeed, if (1.5) holds,

$$|\Theta(t)'| = \sum_n \frac{2y_n}{|t - z_n|^2} + a\ , \quad \forall t \in \mathbb{R}\ .$$

Consequently, because of (1.23),

$$\phi_E'(t) = \sum_n \frac{y_n}{|t - z_n|^2} + \frac{a}{2}\ . \tag{1.24}$$

Thanks to (1.22), (1.17) and (1.24) we obtain other equivalent expressions for $|k_t(x)|/|E(x)|$ and $\|k_t\|^2_{\mathcal{H}^2(E)}$, when $t, x \in \mathbb{R}$:

$$\left|\frac{k_t(x)}{E(x)}\right| = \frac{1}{\pi}|E(t)|\left|\frac{\sin(\phi(x) - \phi(t))}{(x - t)}\right|\ , \tag{1.25}$$

$$k_t(t) = \frac{|E(t)|^2 \phi_E'(t)}{\pi}\ . \tag{1.26}$$

5. p-de Branges spaces

Unlike the Hilbert case, the p-de Branges spaces have not been completely investigated yet. They were first formally introduced in [9] and other references on this subject are [10], [43], [44]; we recall the main definitions.

Definition 1.18 Let $0 < p \le \infty$. Given the Hermite Biehler function $E(z)$, the p-de Branges space $\mathcal{H}^p(E)$ is defined as:

$$\mathcal{H}^p(E) := \left\{f \text{ entire such that } \ f/E \in H^p(\mathbb{C}^+),\ f^{\#}/E \in H^p(\mathbb{C}^+)\right\}\ , \tag{1.27}$$

where $f^{\#}(z) := \overline{f(\bar{z})}$ and $H^p(\mathbb{C}^+)$ is the p-Hardy space of $\mathbb{C}^+$.

If $p \ge 1$, $\mathcal{H}^p(E)$ is a Banach space with the norm inherited from $H^p(\mathbb{C}^+)$

$$\|f\|_{\mathcal{H}^p} := \|f/E\|_{H^p} = \left(\int_{\mathbb{R}}\left|\frac{f(x)}{E(x)}\right|^p dx\right)^{1/p}\ . \tag{1.28}$$

If $0 < p < 1$, $\mathcal{H}^p(E)$ is just a complete metric space and the distance is described by

$$d(f,g) = \|f-g\|_{H^p} = \left(\int_R \left| \frac{f(x)-g(x)}{E(x)} \right|^p dx \right) . \tag{1.29}$$

As it happens for the case $p = 2$, Theorem 1.12, if $p \geq 1$, the p-de Branges spaces are isometrically isomorphic to the $*$-p invariant subspaces of $H^p(\mathbb{C}^+)$.

Definition 1.19 If $\Theta(z)$ is an inner function, the space $K^p(\Theta) \subset H^p(\mathbb{C}^+)$ is defined as

$$K^p(\Theta) := \left\{ f \in H^p(\mathbb{C}^+) \ \text{ such that } \ \langle f, g \rangle_{H^2} = 0, \quad \forall g \in \Theta H^q \right\} \ ,$$

where $1/p + 1/q = 1$ and $1 < p < \infty$. In other words, $K^p(\Theta)$ is the subspace of $H^p(\mathbb{C}^+)$ annihilated by $\Theta H^q(\mathbb{C}^+)$.

Proposition 1.20 Let $1 < p < \infty$. The space $(K^p(\Theta), \|\cdot\|_{H^p})$ is a closed subspace of $H^p(\mathbb{C}^+)$.

Proof. Let $\{f_n\} \in K^p(\Theta)$ such that $f_n \to f \in H^p(\mathbb{C}^+)$. Then

$$\langle f, g \rangle_{H^2} = \lim_{n \to \infty} \langle f_n, g \rangle_{H^2} = 0 \ , \quad \forall g \in \Theta H^q(\mathbb{C}^+) \ .$$

Consequently $f \in K^p(\Theta)$. $\square$

As shown in [63], the elements of $K^p(\Theta)$ can be described in a more direct way.

Proposition 1.21 Let $\Theta(z)$ be an inner function. It turns out that the trace on $\mathbb{R}$ of the elements of $K^p(\Theta)$ is equal to the trace on $\mathbb{R}$ of the elements in the intersection between $H^p(\mathbb{C}^+)$ and $\Theta H^p(\mathbb{C}^-)$, that is,

$$K^p(\Theta)|_{\mathbb{R}} = \left(H^p(\mathbb{C}^+) \cap \Theta H^p(\mathbb{C}^-) \right)|_{\mathbb{R}} \ ,$$

where $f|_{\mathbb{R}}$ is the trace of f on the real line.

Proof. For every $f \in H^p(\mathbb{C}^+)$, $f \in K^p(\Theta)$ if and only if

$$\langle f, \Theta h \rangle_{H^2} = 0 \quad \forall h \in H^q \ \text{ which is equivalent to saying that } \left\langle \overline{\Theta} f, h \right\rangle_{H^2} = 0 \ .$$

The above relation holds if and only if $\overline{\Theta(x)} f(x) \in H^p(\mathbb{C}^-)|_{\mathbb{R}}$, that is $f(x) \in \Theta(x) H^p(\mathbb{C}^-)\ |_{\mathbb{R}}$. We note that the fact that $\Theta(z)$ is inner has been used to guarantee that $\Theta(x)\overline{\Theta(x)} = 1$ almost everywhere on $\mathbb{R}$. $\square$

Proposition 1.21 is fundamental. Indeed it allows us to define the 1-$*$ invariant subspaces.

Definition 1.22 If $\Theta(z)$ is an inner function, the space $K^1(\Theta) \subset H^1(\mathbb{C}^+)$ is defined as

$$K^1(\Theta) := \left\{ f \in H^1(\mathbb{C}^+) \ \text{ such that } \overline{\Theta(x)} f(x) \in H^1(\mathbb{C}^-)|_{\mathbb{R}} \right\} \ ,$$

As it happens for the model spaces, Proposition 1.12, when $1 \leq p < \infty$, $K^p(\Theta)$ is isometrically isomorphic to $\mathcal{H}^p(E)$.

Proposition 1.23 If $1 \leq p < \infty$, then $\mathcal{H}^p(E) := EK^p(\Theta)$, where $\Theta(z)$ is defined according to (1.7).

PROOF. Let $f \in K^p(\Theta)$ and $F = Ef$. It is clear that $F/E \in H^p(\mathbb{C}^+)$. On the other hand $\Theta(x)\overline{f(x)} \in H^p(\mathbb{C}^+)|_{\mathbb{R}}$, which implies that $\Theta(z)f^{\#}(z) \in H^p(\mathbb{C}^+)$. This last inclusion is equivalent to saying that $F^{\#}/E \in H^p(\mathbb{C}^+)$, which implies that $F \in \mathcal{H}^p(E)$.

The reverse implication works in the same way. $\square$

If $p \geq 1$ and $f \in \mathcal{H}^p(E)$

$$f(z) = \langle f, k_z \rangle_{\mathcal{H}^2} \ ,$$

where the kernel k_z is defined as in (1.12) for the case $p = 2$. However if $p \leq 1$, $k_z(t)$ does not belong to $\mathcal{H}^p(E)$. This fact will create some complications in the discussion of the continuity of the embedding operator (Chapter 5) and it will force us to introduce an atomic structure, which we will discuss in details in the following chapters.

2. Orthonormal basis of reproducing kernels in Hilbert de Branges spaces

The 2-de Branges space $\mathcal{H}(E)$ is a reproducing kernel Hilbert space of entire functions, Theorem 1.10. In this chapter, we look for orthonormal systems of reproducing kernels $\{k_{t_n}(z)\}_{n\in\mathbb{Z}}$ with $t_n \in \mathbb{R}$.

In order to describe these families, we use the Herglotz functions. This class of meromorphic functions is identified with the Cayley transforms of the meromorphic inner functions $\Theta(z)$, (2.2).

At the same time, Herglotz functions are also defined as the Poisson integrals of positive, Borel measures with real support, Theorem 2.5. In particular, for the description of the orthonormal basis, we consider Herglotz functions associated to Clark measures, see [**23**], [**39**], [**67**] and [**73**].

In the second section, we prove the main theorem of this chapter, Theorem 2.8: we characterize the complete family of orthogonal reproducing kernels of $\mathcal{H}(E)$. This theorem is a well known result, maybe the best known theorem, concerning the de Branges spaces. The original proof, that we write down here, can be also found in [**28**].

In the third section, we face the problem of the density of the domain of the multiplication operator M_z in $\mathcal{H}(E)$. The multiplication operator in the de Branges space $\mathcal{H}(E)$ is closed, symmetric and unbounded, [**28**]. Consequently, because of the Hellinger–Toeplitz theorem [**68**], M_z cannot be everywhere defined in $\mathcal{H}(E)$. For this reason, we characterize the de Branges spaces for which the multiplication operator M_z has dense domain, Theorem 2.11. This problem is deeply connected to the completeness of the orthonormal systems of reproducing kernels and we characterize these spaces, following the reasonings in [**8**].

Finally in the fourth section, we study the Plancherel-Pólya inequality for the p-de Branges spaces, Theorem 2.16. This inequality is the natural generalization of the orthogonal decomposition possible only in the Hilbert case. It was first discovered for the p-Bernstein spaces, [**56**], and it describes some sampling and interpolating systems of $\mathcal{H}^p(E)$. In Theorem 2.16, the Hermite Biehler function $E(z)$ needs to satisfy several hypothesis which we better analyse in the following chapters.

1. Herglotz functions

The Hermite Biehler class of entire functions is strictly connected to other classes of meromorphic functions. As explained in (1.7), every meromorphic inner functions $\Theta(z)$ is associated to a Hermite Biehler function $E(z)$. In this section we introduce the class of Herglotz functions and we explain its relationship with the Hermite Biehler class.

Definition 2.1 A complex values function $m(z)$, defined on the whole complex plane $\mathbb{C}$, is a meromorphic Herglotz function if

- it is meromorphic in $\mathbb{C}$.
- $\Im[m](z) > 0 \quad$ for every $z \in \mathbb{C}^+$.
- $m(\bar{z}) = \overline{m(z)} \quad$ for every $z \in \mathbb{C}$, that is, $m(z) = m^{\#}(z)$.

The meromorphic Herglotz functions $m(z)$ are the Cayley transform of the meromorphic inner functions $\Theta(z)$, as we now see.

Proposition 2.2 If $m(z)$ is a meromorphic Herglotz function, then $\Theta(z)$ defined as

$$\Theta(z) := e^{i2\pi\alpha}\frac{m(z)-i}{m(z)+i} \tag{2.1}$$

is a meromorphic inner function, when $0 \leq \alpha < 1$. On the other hand, given a meromorphic inner function $\Theta(z)$, the corresponding meromorphic Herglotz function $m^\alpha(z)$ is defined as

$$m^\alpha(z) := i\frac{e^{i2\pi\alpha}+\Theta(z)}{e^{i2\pi\alpha}-\Theta(z)} \ . \tag{2.2}$$

Proof. We have just to prove that the expressions on the right side of (2.1) and of (2.2) belong to the required classes. Indeed $\Theta(z) \in \mathrm{Hol}\,(\mathbb{C}^+)$ since

$$m(z) + i \neq 0 \quad \text{for every } z \in \mathbb{C}^+ \ .$$

Moreover, for every $z \in \mathbb{C}^+$,

$$|\Theta(z)|^2 = \frac{(\Re m(z))^2 + (\Im m(z) - 1)^2}{(\Re m(z))^2 + (\Im m(z) + 1)^2} < 1$$

while $|\Theta(z)| = 1$ if $z = x \in \mathbb{R}$, since $m(x) \in \mathbb{R}$. Therefore, the function $\Theta(z)$ is inner. Finally, since the singularities are those points $z \in \mathbb{C}^-$ such that $m(z) = -i$, $\Theta(z)$ turns out to be a meromorphic inner function. On the other hand, in order to prove that the right term of (2.2) is a Herglotz function, we first check that

$$\Im\left(i\frac{e^{i2\pi\alpha}+\Theta(z)}{e^{i2\pi\alpha}-\Theta(z)}\right) = \Re\left(\frac{1+2i\Im\left(e^{i2\pi\alpha}\Theta(z)\right)-|\Theta(z)|^2}{|e^{i2\pi\alpha}-\Theta(z)|^2}\right) = \frac{1-|\Theta(z)|^2}{|e^{i2\pi\alpha}-\Theta(z)|^2} > 0$$

if $z \in \mathbb{C}^+$. Moreover, since

$$1/\overline{\Theta(z)} = \Theta(\bar{z}) \ , \tag{2.3}$$

we obtain that

$$\overline{\left(i\frac{1+\Theta(z)}{1-\Theta(z)}\right)} = i\frac{-e^{-i2\pi\alpha}-\overline{\Theta(z)}}{e^{-i2\pi\alpha}-\overline{\Theta(z)}} = i\frac{e^{i2\pi\alpha}+\Theta(\bar{z})}{e^{i2\pi\alpha}-\Theta(\bar{z})} \ ,$$

and consequently $m(z)$ is an Herglotz function. $\square$

We note that equation (2.3) makes sense since $\Theta(z) = B_\Lambda(z)e^{iaz}$, as stated in (1.5). Indeed

$$1/\overline{e^{iaz}} = 1/e^{-ia\bar{z}} = e^{ia\bar{z}} \ .$$

Analogously, manipulating separately every Blaschke factor $b_\lambda(z)$ of $B_\Lambda(z)$, it is also true that

$$1/\overline{b_\lambda(z)} = \frac{\overline{\lambda^2}+1}{|\lambda^2+1|}\,\frac{\bar{z}-\lambda}{\bar{z}-\bar{\lambda}} = \frac{|\lambda^2+1|}{\lambda^2+1}\,\frac{\bar{z}-\lambda}{\bar{z}-\bar{\lambda}} = b_\lambda(\bar{z}) \quad .$$

There exists another powerful description of the meromorphic Herglotz functions $m(z)$. This representation is related to Clark measures.

Definition 2.3 A positive, Borel measure $\sigma(t)$ supported on the real line is said to be Poisson integrable if

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{1+t^2} < \infty \ . \tag{2.4}$$

If $\sigma(t)$ satisfies (2.4), the Schwarz integral

$$\mathcal{S}[\sigma](z) := \frac{1}{\pi i}\int_{\mathbb{R}} \left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] d\sigma(t) \tag{2.5}$$

is well defined for every $z \in \mathbb{C}\setminus\mathbb{R}$. Indeed

$$|\mathcal{S}[\sigma](z)| \le \frac{1}{\pi}\left|\int_{\mathbb{R}} \frac{1+tz}{(t-z)(1+t^2)} d\sigma(t)\right| \le C(z)\int_{\mathbb{R}} \frac{d\sigma(t)}{1+t^2} < \infty \ . \tag{2.6}$$

It is worth observing that

$$\mathcal{S}[\sigma] = \mathcal{P}[\sigma] + i\mathcal{Q}[\sigma] \ ,$$

where

$$\mathcal{P}[\sigma](z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y\ d\sigma(t)}{(x-t)^2+y^2} \tag{2.7}$$

is the Poisson integral and

$$\mathcal{Q}[\sigma](z) = \frac{1}{\pi}\int_{\mathbb{R}} \left[\frac{x-t}{(x-t)^2+y^2} + \frac{t}{1+t^2}\right] d\sigma(t) \tag{2.8}$$

is the conjugate Poisson integral. If $z \in \mathbb{C}^+$, with computations similar to those in (2.6), $\mathcal{P}[\sigma](z)$ is well defined. Furthermore, if $z \in \mathbb{C}^+$, $|\mathcal{Q}[\sigma](z)|$ is also well defined. Indeed

$$|\mathcal{Q}[\sigma](z)| \le C\int_{\mathbb{R}} \frac{|(x-t)(1+tx)| + |t|\,y^2}{((x-t)^2+y^2)\,(1+t^2)} d\sigma(t) < \infty \ .$$

To describe the meromorphic Herglotz functions through Poisson integrable measures, we use Herglotz theorem, Theorem 4 in [28].

Theorem 2.4 Every meromorphic inner function $\Theta(z)$ is described by a positive, Poisson integrable Borel measure $\sigma(t)$. Indeed

$$\Re\left(\frac{1+\Theta(z)}{1-\Theta(z)}\right) = p_0 y + \frac{1}{\pi}\int_{\mathbb{R}} \frac{y\, d\sigma_0(t)}{(x-t)^2+y^2} \quad \text{for every } z = x+iy \in \mathbb{C}^+ \ , \tag{2.9}$$

where $p_0 > 0$ is the *weight* of the measure $\sigma_0(t)$ at infinity.

Considering equations (2.2) and (2.9), the representation formula for $m(z)$ through Poisson measure $\sigma_0(t)$ appears.

Theorem 2.5 If $m(z)$ is a meromorphic Herglotz function, then

$$m(z) = p_0 z + c + iS[\sigma_0](z) \tag{2.10}$$

where $p_0 \geq 0$, $c \in \mathbb{R}$ and $\sigma_0(t)$ is a positive, Poisson finite Borel measure with support in $\mathbb{R}$.

Proof. The equation (2.10) follows directly from Herglotz theorem, formula (2.9). Indeed

$$\frac{1+\Theta(z)}{1-\Theta(z)} - i\Im\left(\frac{1+\Theta(i)}{1-\Theta(i)}\right) = \mathcal{S}[\sigma_0](z) - ip_0 z \ .$$

Thus, from (2.2),

$$m(z) = p_0 z - \Im\left(\frac{1+\Theta(i)}{1-\Theta(i)}\right) + i\mathcal{S}[\sigma_0](z) \ ,$$

and we have proved the statement. □

Instead of using $\Theta(z)$, in (2.9) we consider also $e^{-i2\pi\alpha}\Theta(z)$, which is still an inner function. By doing this, we obtain other Herglotz functions:

$$\Re\left(\frac{e^{i2\pi\alpha}-\Theta(z)}{e^{i2\pi\alpha}-\Theta(z)}\right) = \Re\left(\frac{1+e^{-i2\pi\alpha}\Theta(z)}{1-e^{-i2\pi\alpha}\Theta(z)}\right) = p_\alpha y + \frac{1}{\pi}\int_{\mathbb{R}} \frac{y\, d\sigma_\alpha(t)}{(x-t)^2+y^2} \ ,$$

and

$$m^\alpha(z) = p_\alpha z - \Im\left(\frac{e^{2i\pi\alpha}+\Theta(i)}{e^{2i\pi\alpha}-\Theta(i)}\right) + i\mathcal{S}[\sigma_\alpha](z) \quad . \tag{2.11}$$

The measures $\sigma_\alpha(t) + p_\alpha\delta_\infty(t)$ are the Clark measures associated to the inner function $\Theta(z)$. It is also possible to compute $D[\sigma_\alpha](t)$, the Radon-Nikodym derivative of $\sigma_\alpha(t)$ with respect to the Lebesgue measure. Since

$$\frac{1-|\Theta(z)|^2}{|e^{2i\pi\alpha}-\Theta(z)|^2} = p_\alpha y + \frac{1}{\pi}\int_{\mathbb{R}} \frac{y\, d\sigma_\alpha(t)}{(x-t)^2+y^2} \ ,$$

by applying Fatou's theorem, we note that

$$D[\sigma_\alpha](t) = \frac{1-|\Theta(t)|^2}{|e^{2i\pi\alpha}-\Theta(t)|^2} \quad \text{a.e. on } \mathbb{R} \ . \tag{2.12}$$

Therefore, from equation (2.12), since $|\Theta(t)| = 1\ \forall t \in \mathbb{R}$, $\sigma_\alpha(t)$ is a singular discrete measure.

Proposition 2.6 The Clark measure $\sigma_\alpha(t) + p_\alpha\delta_\infty(t)$ defined in (2.9) is a discrete measure consisting of Dirac masses supported at the points of the set

$$I^\alpha := \left\{ t_n^\alpha \in \mathbb{R} : \Theta(t_n^\alpha) = e^{2i\pi\alpha} \right\} \ . \tag{2.13}$$

Proof. As $\Theta(z)$ is a meromorphic inner function, the set I^α has to be countable and discrete. We have already checked that $\sigma_\alpha(t)$ is a singular measure (2.12); however, we have still to verify that at every point $t_n^\alpha \in I^\alpha$, $\sigma_\alpha(t)$ has a non-zero Dirac mass. The function $(e^{2i\pi\alpha} + \Theta)/(e^{2i\pi\alpha} - \Theta)$ is a meromorphic function with poles exactly at I^α while the poles of $\mathcal{S}[\sigma_\alpha](z)$ coincide with the support of $\sigma_\alpha(t)$. Therefore, since

$$\frac{e^{2i\pi\alpha} + \Theta(z)}{e^{2i\pi\alpha} - \Theta(z)} = -ip_\alpha z + \mathcal{S}[\sigma_\alpha](z) - ic \ ,$$

the support of $\sigma_\alpha(t)$ is equal to I^α and it is made by discrete points. $\square$

We describe explicitly the relationship between the Herglotz functions $m^0(z)$, $m^{1/2}(z)$ and the associated Hermite Biehler function $E(z)$. It allows us to clarify also the value of the point mass measures which constitute $\sigma_{1/2}(t)$ in (2.11).

Let us consider the meromorphic inner function $\Theta_E(z)$, associated to $E(z)$. We compute the associated Herglotz functions $m_E^0(z)$ and $m_E^{1/2}(z)$. If $E(z) = A(z) - iB(z)$ as in (1.4), then

$$m_E^0 = i\frac{1+\Theta_E}{1-\Theta_E} = i\frac{1+\frac{E^\#}{E}}{1-\frac{E^\#}{E}} = i\frac{E+E^\#}{E-E^\#} = -\frac{A}{B} \ ,$$

and

$$m_E^{1/2} = i\frac{-1+\Theta_E}{-1-\Theta_E} = \frac{B}{A} \ .$$

Moreover,

$$m_E^{0\,\#}(z) = m_E^0(z) \ , \ m_E^{1/2\,\#}(z) = m_E^{1/2}(z) \qquad \text{and}$$
$$\Im\left(-A/B\right) > 0 \text{ if and only if } \Im\left(B/A\right) > 0 \ .$$

Theorem 2.7 Let us consider the entire function $E(z) = A(z) - iB(z)$ defined as in (1.4), where $A(z)$ and $B(z)$ are real functions. The following sentences are equivalent:

- $E(z)$ is in the Hermite Biehler class.
- $B(z)/A(z)$ is in the Herglotz class.

Proof. Let $E(z)$ be a Hermite Biehler function. Then $B(z)/A(z)$ is real. Furthermore, for every $z \in \mathbb{C}^+$,

$$\Im\left(B(z)/A(z)\right) = \frac{1}{4}\Re\frac{(E(z) - E^\#(z))(\overline{E(z)} + \overline{E^\#(z)})}{|A(z)|^2} \qquad = \frac{|E(z)|^2 - |E^\#(z)|^2}{4|A(z)|^2} \geq 0$$

since $E(z)$ is in the Hermite Biehler class. The reverse implication works exactly in the same way. $\square$

We describe explicitly the Clark measure $\sigma_{1/2}(t)$ of the Herglotz function $B(z)/A(z)$, as defined in (2.11). Indeed

$$\frac{B}{A}(z) = p_{1/2} z + c + \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma_{1/2}(t) \ .$$

The set $\operatorname{supp}(\sigma_{1/2})$ is equal to Z_A, that is, the zeros $A(z)$, and

$$\frac{B}{A}(z) = p_{1/2} z + c + \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \left(\frac{1}{t_n - z} - \frac{t_n}{1+t_n^2} \right) p_n^{1/2} \ , \text{ where } t_n \in Z_A \ . \tag{2.14}$$

We note that $p_{1/2} \geq 0$ and

$$\frac{p_n^{1/2}}{\pi} = B(t_n)/A'(t_n) > 0 \ . \tag{2.15}$$

In particular

$$\begin{aligned} \frac{p_n^{1/2}}{\pi} = \frac{B(t_n)}{A'(t_n)} &= \frac{\sin(\phi_E(t_n))\,|E(t_n)|}{\phi_E'(t_n)\sin(\phi_E(t_n))\,|E(t_n)| + \cos(\phi_E(t_n))\,|E(t_n)|'} \\ &= \frac{\sin(\phi_E(t_n))\,|E(t_n)|}{\phi_E'(t_n)\sin(\phi_E(t_n))\,|E(t_n)|} = \frac{1}{\phi_E'(t_n)} \ . \end{aligned} \tag{2.16}$$

Differentiating (2.14), we obtain

$$\left(\frac{B}{A} \right)'(z) = p_{1/2} + \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{p_n^{1/2}}{(t_n - z)^2} > 0 \ , \quad \forall z \in \mathbb{C}^+ \ .$$

We note that the zeros of $B(z)$ and $A(z)$ interlace with each others.

In order to describe the general expression of $\sigma_\alpha(t)$, we introduce

$$S_\alpha(z) := e^{i\pi\alpha} E(z) - e^{-i\pi\alpha} E^{\#}(z) \ . \tag{2.17}$$

The real function $S_\alpha(z)$ has got simple real zeros $\{t_n^\alpha\}$ which coincide with I^α (2.13). Moreover

$$\frac{S_{\alpha+1/2}(z)}{S_\alpha(z)} = i\frac{e^{2i\pi\alpha} + \Theta(z)}{e^{2i\pi\alpha} - \Theta(z)} = m^\alpha(z) \ . \tag{2.18}$$

Consequently with computations similar to those in (2.16), we obtain that

$$m^\alpha(z) = p_\alpha z + c + \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \left(\frac{1}{t_n^\alpha - z} - \frac{t_n^\alpha}{1+t_n^{\alpha 2}} \right) p_n^\alpha \ , \tag{2.19}$$

where

$$p_n^\alpha := \frac{\pi}{\phi_E'(t_n^\alpha)} \ . \tag{2.20}$$

2. Orthogonal bases of reproducing kernels and Clark measures

Let us consider the Hermite Biehler function $E(z) = A(z) - iB(z)$ defined as in (1.4). Let $\{t_n\} = Z_A$ be the zeros of the real function $A(z)$. We consider the reproducing kernels centred at t_n:

$$
\begin{aligned}
(2.21)\qquad k_{t_n}(z) &= \frac{1}{2\pi i}\frac{\overline{E(t_n)}E(z) - E(t_n)\overline{E(\bar{z})}}{(t_n - z)} = \frac{B(z)\overline{A}(t_n) - A(z)\overline{B}(t_n)}{\pi(z - t_n)} \\
&= \frac{-A(z)\overline{B}(t_n)}{\pi(z-t_n)} = \frac{-\overline{B}(t_n)}{\pi}\frac{A(z)}{(z-t_n)} \ .
\end{aligned}
$$

We remind the reader that the function $k_{t_n}(z)$ in (2.21) belongs to $\mathcal{H}(E)$, Theorem 1.10.

The set Z_A is of fundamental importance. The collection $\{k_{t_n}\}$ gives rise to an orthogonal system of $\mathcal{H}(E)$.

Theorem 2.8 If $t_n \in Z_A$, the set of reproducing kernel $\{k_{t_n}(z)\}$ is an orthogonal system of $\mathcal{H}(E)$. Furthermore, if $A(z) \notin \mathcal{H}(E)$, then $\{k_{t_n}(z)\}$ is an orthogonal basis. On the other hand, if $A(z) \in \mathcal{H}(E)$, then $\{k_{t_n}(z)\} \cup \{A(z)\}$ is an orthogonal basis.

Proof. First of all, let us check the orthogonality. Indeed

$$
\langle k_{t_n}, k_{t_m}\rangle_{\mathcal{H}} = -\frac{\overline{B(t_n)}}{\pi}\left\langle \frac{A}{\cdot - t_n}, k_{t_m}(\cdot)\right\rangle_{\mathcal{H}} = -\frac{\overline{B(t_n)}}{\pi}\frac{A(t_m)}{t_m - t_n} = 0 \ ,
$$

since the function $\frac{A(z)}{z-t_n} \in \mathcal{H}(E)$ because of Theorem 1.10. At the same time, if $A(z) \in \mathcal{H}(E)$, then

$$
\langle A(\cdot), k_{t_n}(\cdot)\rangle_{\mathcal{H}} = A(t_n) = 0 \ \text{ for every } t_n \in Z_A \ .
$$

Let us consider the Herglotz function (2.14)

$$
h(z) := \frac{B}{A}(z) = pz + c + \frac{1}{\pi}\sum_n \left(\frac{1}{z-t_n} - \frac{t_n}{1+t_n^2}\right)p_n
$$

where $\frac{p_n}{\pi} = B(t_n)/A'(t_n)$ and $\sum_n p_n/(1+t_n^2) < \infty$. Then,

$$
\frac{B(z)\overline{A}(w) - A(z)\overline{B}(w)}{(z-\overline{w})A(z)\overline{A(w)}} = \frac{h(z) - \overline{h(w)}}{z - \overline{w}} = p + \frac{1}{\pi}\sum_n \frac{p_n}{(z-t_n)(\overline{w}-t_n)} \ .
$$

Therefore,

$$
(2.22)\qquad \pi k_w(z) = pA(z)\overline{A(w)} + \frac{1}{\pi}\sum_n \frac{p_n\overline{A(w)}}{\overline{w}-t_n}\frac{A(z)}{z-t_n} \ .
$$

If $z = w$, then

$$
\pi k_w(w) = p\left|A(w)\right|^2 + \frac{1}{\pi}\sum_n p_n\left|\frac{A(w)}{w-t_n}\right|^2 < \infty
$$

because of the Poisson summability. Thanks to (1.26),

$$\left\|\frac{A(\cdot)}{\cdot - t_n}\right\|^2_{\mathcal{H}(E)} = \frac{\pi^2}{|B(t_n)|^2} k_{t_n}(t_n) = \frac{\pi}{|B(t_n)|^2} |E(t_n)|^2 \phi'_E(t_n) = \pi\phi'_E(t_n) \ ,$$

and from (2.22) we obtain that

$$\begin{aligned}\pi^2 \|k_w\|^2_{\mathcal{H}} =& p^2 |A(w)|^2 \|A\|^2_{\mathcal{H}} + \frac{1}{\pi^2}\sum_n p_n^2 \left|\frac{A(w)}{w-t_n}\right|^2 \left\|\frac{A(\cdot)}{\cdot - t_n}\right\|^2_{\mathcal{H}(E)} \\ =& p^2 |A(w)|^2 \|A\|^2_{\mathcal{H}} + \sum_n p_n \left|\frac{A(w)}{w-t_n}\right|^2 < \infty\end{aligned}$$

since, due to (2.16),

$$\frac{p_n}{\pi} = \frac{1}{\phi'_E(t_n)} \ .$$

Consequently, if $p = 0$, for every $w \in \mathbb{C}$, $k_w(z) \in \overline{\operatorname{Span}\{k_{t_n}(z)\}_n}^{\mathcal{H}}$. Otherwise, if $p \neq 0$, then $A(z) \in \mathcal{H}(E)$ and for every $w \in \mathbb{C}$, $k_w(z) \in \overline{\operatorname{Span}\{k_{t_n}(z)\}_n \cup \{A(z)\}}^{\mathcal{H}}$. $\square$

Theorem 2.8 can be generalized to Z_{S_α} and $\{k_{t_n^\alpha}\}$, the system of reproducing kernels centred at the real points $t_n^\alpha \in I^\alpha$, (2.13).

Theorem 2.9 The set $\{k_{t_n^\alpha}\}$ where $t_n^\alpha \in I^\alpha$ is an orthogonal bases for $\mathcal{H}(E)$, unless $S_\alpha(z) \in \mathcal{H}(E)$, defined as in (2.17).

For the proof of the above theorem see Theorem 22 in [28]. It is essentially the same of that of Theorem 2.8.

When we normalize the sets $\{k_{t_n^\alpha}(z)\}$, we obtain orthonormal basis of $\mathcal{H}(E)$.

Corollary 2.10 If $S_\alpha(z) \notin \mathcal{H}(E)$, the set $\left\{\sqrt{\pi}k_{t_n^\alpha}(z)/\left(|E(t_n^\alpha)|\sqrt{\phi'_E(t_n^\alpha)}\right)\right\}$ is an orthonormal bases. Furthermore, $\forall f(z) \in \mathcal{H}(E)$

$$f(z) = \sum_n f(t_n^\alpha)\frac{\pi k_{t_n^\alpha}(z)}{|E(t_n^\alpha)|^2 \phi'_E(t_n^\alpha)} \quad \text{and} \quad \|f\|^2_{\mathcal{H}} = \sum_n \left|\frac{f(t_n^\alpha)}{E(t_n^\alpha)}\right|^2 \frac{\pi}{\phi'_E(t_n^\alpha)} \ .$$

3. p_α different from zero

Let us consider the de Branges space $\mathcal{H}(E)$ and the corresponding meromorphic inner function $\Theta := E^\#/E := e^{iaz}B_\Lambda$, where $a > 0$ and B_Λ is a Blaschke product, i.e $\Lambda = \{\lambda_n := a_n + ib_n\} \subset \mathbb{C}^+$. In this section, we look for the characterization of those Hermite Biehler function $E(z)$ for which there exists an $0 \le \alpha < 1$, such that the corresponding system of reproducing kernel $\{k_{t_n^\alpha}\}$, as defined in Theorem 2.9 is not complete in $\mathcal{H}(E)$. In this section we provide the proof contained in [8].

Theorem 2.11 Let $\mathcal{H}(E)$ be a Hilbert de Branges space. There exists an α such that $\{k_{t_n^\alpha}\}$ is not complete for $\mathcal{H}(E)$ if and only if the two following conditions hold:

$$\sum_n b_n < \infty \quad \text{and} \quad a = 0 \; .$$

where b_n and a have been defined in (1.5).

Proof. By considering Theorem 2.8 and Theorem 2.9, we know that there exists an $\alpha \in [0,1)$ such that the system $\{k_{t_n^\alpha}\}$ is not complete if and only if there exists an α such that the corresponding value p_α of (2.19) is not zero.

The value p_α is given by

$$p_\alpha = \lim_{y\to\infty} \frac{\Im m^\alpha(iy)}{y} := \lim_{y\to\infty} \frac{1}{y} \Re\left(\frac{e^{2i\alpha\pi} + \Theta(iy)}{e^{2i\alpha\pi} - \Theta(iy)}\right) \; .$$

Therefore, p_α is different from zero if and only if

$$\lim_{y\to\infty} |e^{2i\alpha\pi} - \Theta(iy)| = 0 \; .$$

If this happens, the limit of $\Theta(iy)$ as $y \to \infty$ has to exist and has to be equal to $e^{2i\alpha\pi}$.

First of all,

$$|\Theta(x+iy)| = e^{-ya} \left| \prod_\lambda c_\lambda \frac{z-\lambda}{z-\bar{\lambda}} \right| \le e^{-ya} \to 0 \text{ as } y \to \infty \; .$$

Consequently, if $\lim_{y\to\infty} \Theta(iy) = e^{2i\alpha\pi}$, then $a = 0$. On the other hand, if $a = 0$, $\Theta(z) = B_\Lambda(z)$. Since $p_\alpha \neq 0$, then

$$p_\alpha = \lim_{y\to\infty} \frac{1}{y} \frac{1 - |\Theta(iy)|^2}{|e^{i2\alpha\pi} - \Theta(iy)|^2} \le \liminf_{y\to\infty} \frac{1}{y} \frac{1 + |\Theta(iy)|}{|e^{i2\alpha\pi} - \Theta(iy)|} \le \liminf_{y\to\infty} \frac{2}{y\,|e^{i2\alpha\pi} - \Theta(iy)|} \; .$$

Consequently

$$\limsup_{y\to\infty} y\left|e^{i2\alpha\pi} - \Theta(iy)\right| \le \frac{2}{p_\alpha} \; .$$

Since, due to (1.16),

$$\limsup_{y\to\infty} y(1 - |\prod_{|n|<N} b_{\lambda_n}(iy)|) = 2 \sum_{|n|<N} b_n \; ,$$

we obtain that it is necessary that $\sum_n b_n < \infty$.

In order to prove the reverse implication, we suppose that $a = 0$ and $\sum_n b_n < \infty$ hold. Therefore, with computations similar to those used above, as $y \to \infty$,

$$B_\Lambda(iy) = \Gamma \exp\left(-\frac{2\sum_n b_n}{y} + o\left(\frac{1}{y}\right)\right) \; ,$$

where $|\Gamma| = 1$ and, consequently

$$B_\Lambda(iy) = \Gamma\left(1 - \frac{2\sum_n b_n}{y} + o\left(\frac{1}{y}\right)\right) .$$

It follows that

$$\lim_{y\to\infty} y\,|\Gamma - \Theta(iy)| = 2\sum_n b_n \ ,$$

and consequently

$$p_\alpha = \lim_{y\to\infty} \frac{1}{y}\Re\left(\frac{\Gamma + \Theta(iy)}{\Gamma - \Theta(iy)}\right) = \frac{C}{\sum_n b_n} \ ,$$

where $\alpha := \arg(\Gamma)/2\pi$. □

As we said at the beginning of this chapter, Theorem 2.11 is used to describe the de Branges spaces for which the domain of the multiplication M_z is not dense.

Definition 2.12 Let $\mathcal{H}(E)$ be a Hilbert de Branges space. The multiplication operator M_z is defined as

$$\text{(2.23)} \quad \begin{aligned} \operatorname{dom}(M_z) \subset \mathcal{H}(E) &\to \mathcal{H}(E) \\ f(z) &\to M_z(f) := zf(z) \ . \end{aligned}$$

As proved in [70] and [71], M_z is symmetric, closed and with deficiency indexes (1,1). We describe those de Branges spaces for which the domain of M_z is dense in $\mathcal{H}(E)$. This characterization has been obtained in Theorem 29 of [28]

Theorem 2.13 A necessary and sufficient condition such that an element $S(z) \in \mathcal{H}(E)$ is orthogonal to $\operatorname{dom}(M_z)$, is that $S(z) = uA(z) + vB(z)$, where $u, v \in \mathbb{C}$.

Therefore, if there exists an α such that $\{k_{t_n^\alpha}\}$ is not complete in $\mathcal{H}(E)$, the multiplication operator M_z is not densely defined. Indeed

$$\overline{\operatorname{Dom} M_z}^{\mathcal{H}(E)} = \overline{\{k_{t_n^\alpha}(z)\}}^{\mathcal{H}(E)} = \mathcal{H}(E) \setminus \{S_\alpha(z)\} \ .$$

4. Plancherel-Pólya inequality for $\mathcal{H}^p(E)$

When one considers Banach spaces, it is clearly impossible finding orthogonal basis. However, there are other kinds of systems which deserve to be searched. In this section we look for some sampling and interpolating points for $\mathcal{H}^p(E)$.

According to [56], in the Bernstein spaces $\mathcal{B}^p_\pi$ it is possible to use the integers $\mathbb{Z}$ to interpolate $\ell^p(\mathbb{Z})$.

Definition 2.14 For every $0 < p \leq \infty$, the p-Bernstein space $\mathcal{B}^p_\pi$ are the p-de Branges space with Hermite Biehler function equal to $e^{-i\pi z}$.

The original Plancherel-Pólya inequality states that for every $\ell^p(\mathbb{Z})$ sequence $\{a_k\}$ when $1 < p < \infty$, there exists an entire function $f \in \mathcal{B}^p_\pi$ such that

$$f(k) = a_k \ , \quad \forall k \in \mathbb{Z} \ .$$

On the other hand, it states also that for every $f \in \mathcal{B}^p_\pi$ the sequence $\{f(k)\} \in \ell^p(\mathbb{Z})$ and $\|\{f(k)\}\|_{\ell^p} \asymp \|f\|_{\mathcal{B}^p_\pi}$.

Considering the proof of [56] for the Bernstein spaces, we extend these inequalities to some family of p-de Branges spaces.

Definition 2.15 For every p, we define the vertical translation operator $T_{i\tau}$ as

$$(2.24)\quad T_{i\tau}: \quad \mathrm{Dom}\{T_{i\tau}\} \subseteq \mathcal{H}^p(E) \to \mathcal{H}^p(E) \qquad f(z) \to T_{i\tau}[f](z) := f(z+i\tau) \quad z \in \mathbb{C},\ \tau > 0\,.$$

Theorem 2.16 Let $1 < p < \infty$. Let $E(z)$ be a Hermite Biehler function such that:

(1) $\sup_{x\in\mathbb{R}} \left|\frac{E(x+i\tau)}{E(x)}\right| < \infty$.

(2) $0 < \inf_{x\in\mathbb{R}} \phi'_E(x) \leq \sup_{x\in\mathbb{R}} \phi'_E(x) < \infty$.

(3) The translation operator $T_{i\tau}$ is bounded in $\mathcal{H}^p(E)$.

Then, for every sequence $\{c_k\}$, such that

$$(2.25)\qquad \|\{c_k\}\|^p_{\ell^p(\mathbb{Z},w)} := \sum_{k\in\mathbb{Z}} \left|\frac{c_k}{A'(t_k)}\right|^p < \infty \text{ with } t_k \in Z_A\ ,$$

the series

$$(2.26)\qquad f(z) := \sum_{k\in\mathbb{Z}} \frac{c_k}{k_{t_k}(t_k)} k_{t_k}(z)$$

converges to an element of $\mathcal{H}^p(E)$ so that

$$(2.27)\qquad \|f\|_{\mathcal{H}^p} \leq K\, \|\{c_k\}\|_{\ell^p(\mathbb{Z},w)}\ .$$

On the other hand, for every $f \in \mathcal{H}^p(E)$, the sequence $\{f(t_k)\}$ satisfies (2.25) and

$$(2.28)\qquad \|\{f(t_k)\}\|_{\ell^p(\mathbb{Z},w)} \leq K\, \|f\|_{\mathcal{H}^p}\ .$$

Observation We note that, if we define

$$(2.29)\qquad \ell^p(\mathbb{Z},w) := \left\{ \{c_n\}_{n\in\mathbb{Z}} \ :\ \|\{c_n\}\|^p_{\ell^p(\mathbb{Z},w)} := \sum_{n\in\mathbb{Z}} \left|\frac{c_n}{A'(t_n)}\right|^p < \infty\ ,\ \text{where } t_n \in Z_A \right\}\ ,$$

then Theorem 2.16 is stating that Z_A is a complete interpolating sequence for $\mathcal{H}^p(E)$ with values in $\ell^p(\mathbb{Z},w)$.

PROOF OF 2.16. First of all, we note that (2.26) is equivalent to

$$f(z) = \sum_{k\in\mathbb{Z}} \frac{c_k}{A'(t_k)} \frac{A(z)}{z-t_k} \ .$$

We define

$$\psi_{n,m}(z) := \sum_{k=n}^{m} \frac{c_k}{A'(t_k)} \frac{1}{z-t_k} \ , \qquad \text{and} \qquad \Psi_{n,m}(z) := \psi_{n,m}(z)A(z) \ .$$

Consequently

$$\begin{aligned}
\|\Psi_{n,m}\|_{\mathcal{H}^p}^p = \int_{\mathbb{R}} \left|\frac{\Psi_{n,m}(t)}{E(t)}\right|^p dt &\le \sup_{t\in\mathbb{R}} \left|\frac{E(t+i\tau)}{E(t)}\right|^p \int_{\mathbb{R}} \left|\frac{\Psi_{n,m}(t)}{E(t+i\tau)}\right|^p dt \\
&\le C \int_{\mathbb{R}} \left|\frac{\Psi_{n,m}(t+i\tau-i\tau)}{E(t+i\tau)}\right|^p dt \\
&= C \int_{\mathbb{R}} \left|\frac{(T_{i\tau}\Psi_{n,m})^{\#}(t+i\tau)}{E(t+i\tau)}\right|^p dt \\
&\le C \,\|T_{i\tau}(\Psi_{n,m})\|_{\mathcal{H}^p}^p \\
&= C \int_{\mathbb{R}} \left|\sum_{k=n}^{m} \frac{c_k}{A'(t_k)} \frac{A(t+i\tau)}{t+i\tau-t_k}\right|^p \frac{dt}{|E(t)|^p} \ .
\end{aligned}$$

Since

$$\left|\frac{A(t+i\tau)}{E(t)}\right| = \frac{1}{2}\left|\frac{E(t+i\tau)}{E(t)} + \frac{E^{\#}(t+i\tau)}{E(t)}\right| < \infty \ ,$$

then

$$\|\Psi_{n,m}\|_{\mathcal{H}^p}^p \le C' \,\|\psi_{n,m}(\cdot+i\tau)\|_{H^p}^p \ .$$

By using the duality between H^p and H^q when $1/p+1/q=1$, we obtain

$$\begin{aligned}
\|\Psi_{n,m}\|_{\mathcal{H}^p} &\le C' \sup_{g,\ \|g\|_{H^q}=1} \left|\int_{-\infty}^{\infty} \psi_{n,m}(x+i\tau)\overline{g(x)}dx\right| \\
&= C' \sup_{g,\ \|g\|_{H^q}=1} \left|\int_{-\infty}^{\infty} \sum_{k=n}^{m} \frac{\overline{c_k}}{A'(t_k)} \frac{1}{x-(t_k+i\tau)} g(x)dx\right| \\
&= C' \sup_{g,\ \|g\|_{H^q}=1} 2\pi \left|\sum_{k=n}^{m} \frac{\overline{c_k}}{A'(t_k)} g(t_k+i\tau)\right| \\
&\le C' \sup_{g,\ \|g\|_{H^q}=1} 2\pi \left(\sum_{k=n}^{m} \left|\frac{c_k}{A'(t_k)}\right|^p\right)^{1/p} \left(\sum_{k=n}^{m} |g(t_k+i\tau)|^q\right)^{1/q} \\
&\le C \,\|\{c_k\}\|_{\ell^p(\mathbb{Z},w)} \ ,
\end{aligned}$$

since, for every sequence $\{\lambda_k\} \subset \mathbb{C}^+$ such that $0 < \delta < h_1 < \Im\lambda_k < h_2$ and $|\lambda_n - \lambda_k| > 2\delta$, for every $h \in H^q$, we have

$$\sum_k |h(\lambda_k)|^q \le \frac{h_2 + \delta}{\pi\delta^2} \|h\|_{H^q}^q \ .$$

The sequence $\{t_k + i\tau\}$ satisfies these hypothesis since

$$|t_n - t_k| \ge |t_{k+1} - t_k| \ge \frac{\pi}{\|\phi_E'\|_\infty} \ .$$

Since the sequence $\{\Psi_{n,m}\}$ is uniformly bounded in $\mathcal{H}^p(E)$ norm, the series in (2.26) belongs to $\mathcal{H}^p(E)$ and, by replacing z with t_k, $f(t_k) = c_k$.

In order to prove the reverse implication, we note that for every $f \in \mathcal{H}^p(E)$,

$$\left|\frac{f(t_k)}{A'(t_k)}\right| = \left|\frac{f(t_k)}{E(t_k)}\right| \left|\frac{E(t_k)}{A'(t_k)}\right| = \left|\frac{f(t_k)}{E(t_k)}\right| \frac{1}{\phi_E'(t_k)} \ .$$

Therefore

$$\begin{aligned}
\sum_k \left|\frac{f(t_k)}{A'(t_k)}\right|^p \le C \sum_k \left|\frac{f(t_k)}{E(t_k)}\right|^p &\le C \sup_{x\in\mathbb{R}} \left|\frac{E(x+i\tau)}{E(x)}\right| \sum_k \left|\frac{f(t_k)}{E(t_k+i\tau)}\right|^p \\
&\le C' \sum_k \left|\frac{f(t_k)}{E(t_k+i\tau)}\right|^p \\
&\le C' \sum_k \left|\frac{f(t_k+i\tau-i\tau)}{E(t_k+i\tau)}\right|^p \\
&\le C' \sum_k \left|\frac{T_{i\tau}(f)^\#(t_k+i\tau)}{E(t_k+i\tau)}\right|^p \\
&\le C \left\|T_{i\tau}(f)^\#\right\|_{\mathcal{H}^p} \le C' \|f\|_{\mathcal{H}^p} \ .
\end{aligned}$$

$\square$

We highlight that the hypothesis assumed in Theorem 2.16 are far from being optimal. If for example, instead of the three conditions, we assume that the Hermite Biehler function $E(z)$ satisfies the CLS condition, see Definition 8.5, then (2.28) is already obtained in [25] and in [78]. It is also possible to verify that the second condition of Theorem 2.16 implies that the Hermite Biehler function $E(z)$ satisfies the CLS condition.

In the following sections we analyse in depth the vertical translation operator $T_{i\tau}$ and we see that the first and the third condition of Theorem 2.16 are deeply connected.

The characterization of complete interpolating sequences for spaces of entire functions is an active research area. This section, and more generally this dissertation, does not want describe the results obtained on this topic. However some important papers have to be cited:
[57], where the authors study problems of sampling and interpolation in a wide class of de Branges spaces. In particular, they completely characterize the class of Hermite Biehler functions for which results on sampling and interpolation related to the classical Paley-Wiener

spaces can be extended in a direct and natural way.

[64], where the authors consider and solve the classical problem of when a sequence of exponentials with real frequencies $\Lambda = \{\lambda_k \ : k \in \mathbb{Z}\}$ forms a frame.

[60], where an easier proof of the Beurling-Malliavin Theorem is provided.

There are several other more recent articles that should be cited. For a more complete list of them we refer to the bibliographies of [38] and [40].

II. Boundedness of translations and embedding operators in de Branges spaces

3. Necessary conditions for the boundedness of translation operators

The purpose of this chapter is looking for necessary conditions for the boundedness of the translation operators in the Hilbert de Branges spaces $\mathcal{H}(E)$.

In the first section, we associate to the vertical translation operator the following measure defined in $\mathbb{C}^+$

$$d\mu(z) := \sum_n \pi \frac{\delta_{t_n+i\tau}(z)}{\phi'(t_n)} \left| \frac{E(t_n+i\tau)}{E(t_n)} \right|^2 , \tag{3.1}$$

where $\{t_n\} = I^0$ are defined as in (2.13), that is

$$\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R} \text{ such that } \phi(t_n) \equiv_\pi 0 \ .$$

We prove that, if $T_{i\tau}$ is bounded in $\mathcal{H}(E)$, $d\mu(z)$ is a Carleson measure for the model space $K^2(\Theta)$, where $\Theta(z)$ satisfies (1.7). For this reason, the characterization of Carleson measures in model spaces is meaningful for the boundedness of $T_{i\tau}$.

In the second section we find a necessary condition for the boundedness of $T_{i\tau}$. We prove that if $T_{i\tau}$ is bounded in $\mathcal{H}(E)$, then the Hermite Biehler function $E(z)$ satisfies the following condition:

$$\sup_{x\in\mathbb{R}} \left| \frac{E(x+i\tau)}{E(x)} - \frac{\overline{E(x-i\tau)}}{E(x)} \right| \frac{1}{\tau\phi'(x)} \leq \tilde{C} \ .$$

In order to obtain the above condition, we link the measure $d\mu(z)$ to an infinite matrix Γ, which defines a bounded operator in $\ell^2(\mathbb{Z})$, Theorem 3.5.

Finally, in the third section, we study the horizontal translation operator T_σ; it is defined in the same way:

$$T_\sigma : \quad \mathrm{Dom}\{T_\sigma\} \subseteq \mathcal{H}(E) \to \mathcal{H}(E) \qquad f(z) \to T_\sigma[f](z) := f(z+\sigma) \quad z \in \mathbb{C},\ \sigma > 0 \ . \tag{3.2}$$

As already done for $T_{i\tau}$, we associate a Carleson measure to T_σ, which, this time, is supported on the real line. This fact allows us to give an easier necessary condition that the horizontal translation operator must satisfy in order to be bounded on $\mathcal{H}(E)$, (3.19). The Condition (3.19) holds only when the derivative of the phase of the Hermite Biehler function $E(z)$ is lower and upper bounded on $\mathbb{R}$.

The results of this chapter are original and they are published in [15].

1. Vertical translation, Carleson measures and two weighted Hilbert transform

In this section we show that the problem of the boundedness of the vertical translation operator $T_{i\tau}$ in the de Branges space $\mathcal{H}(E)$ can be reformulated as a problem concerning Carleson measures in the model space $K^2(\Theta)$. This problem is equivalent to the boundedness of the two weighted Hilbert transform. The measures involved will be purely atomic.

Theorem 3.1 If the operator $T_{i\tau}$ is bounded in $\mathcal{H}(E)$, the measure $d\mu(z)$ in (3.1) is a Carleson measure for $K^2(\Theta)$, the associated model space.

PROOF. Let us consider $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that $\phi(t_n) \equiv_\pi 0$. We make the following computations:

$$C\,\|f\|^2_{\mathcal{H}} \geq \|T_{i\tau}f\|^2_{\mathcal{H}} \geq \sum_{n\in\mathbb{Z}} \left| \left\langle T_{i\tau}f, \frac{k_{t_n}}{\|k_{t_n}\|_{\mathcal{H}}} \right\rangle_{\mathcal{H}} \right|^2 = \sum_{n\in\mathbb{Z}} \left| \frac{f(t_n+i\tau)}{\|k_{t_n}\|_{\mathcal{H}}} \right|^2$$
$$= \sum_{n\in\mathbb{Z}} \left| \left\langle f, \frac{k_{t_n+i\tau}}{\|k_{t_n}\|_{\mathcal{H}}} \right\rangle_{\mathcal{H}} \right|^2 = \int_{\mathbb{C}^+} |f(z)|^2 \left(\sum_n \frac{\delta_{t_n+i\tau}(z)}{\|k_{t_n}\|^2_{\mathcal{H}}} \right) .$$

Moving to the model space $K^2(\Theta) \ni F := f/E$ when $f \in \mathcal{H}(E)$,

$$C\,\|F\|^2_K \geq \int_{\mathbb{C}^+} |F(z)|^2 \left(\sum_n \pi \frac{\delta_{t_n+i\tau}(z)}{\phi'(t_n)} \left| \frac{E(t_n+i\tau)}{E(t_n)} \right|^2 \right) ,$$

that is

$$K^2(\Theta) \hookrightarrow L^2(d\mu) \quad \text{continuously} . \tag{3.3}$$

The above embedding tells us that $d\mu$ is a Carleson measure for $K^2(\Theta)$. $\square$

In the proof of the above theorem, if we consider the points

$$\{t_n^\alpha\}_{n\in\mathbb{Z}} \subset \mathbb{R} \text{ such that } \phi(t_n) \equiv_\pi \alpha\pi \text{ , for every } 0 \leq \alpha < 1 \text{ ,}$$

the measure

$$d\mu^\alpha(z) := \sum_n \pi \frac{\delta_{t_n^\alpha+i\tau}(z)}{\phi'(t_n^\alpha)} \left| \frac{E(t_n^\alpha+i\tau)}{E(t_n^\alpha)} \right|^2$$

is still a Carleson measure for $K^2(\Theta)$.

To simplify the notation, we say that $E(\Theta,\mu^\alpha)$ is the embedding operator from $K^2(\Theta)$ to $L^2(d\mu^\alpha)$. In Theorem 3.1, we have proved that the operator $E(\Theta,\mu^\alpha)$ is continuous.
Before providing the necessary condition for the boundedness of $T_{i\tau}$, in the rest of this section we show the necessary and sufficient condition that the measure $d\mu^\alpha$ must satisfy in order to be a Carleson measure for $K^2(\Theta)$ [52]. From now on, we omit the letter α as our calculations work for any $0 \leq \alpha < 1$. First of all let us show an equivalent description for $K^2(\Theta)$.

Theorem 3.2 Let

$$d\sigma(z) := \sum_{n\in\mathbb{Z}} \pi \frac{\delta_{t_n}(z)}{\phi'(t_n)} \ .$$

If $1-\Theta(z) \notin K^2(\Theta)$, then the operator H_0, defined as

$$H_0(f)(z) := (1-\Theta(z)) \int_{\mathbb{R}} \frac{f(t)}{t-z}\frac{1}{2\pi i} d\sigma(t) \tag{3.4}$$

is a unitary operator from $L^2(d\sigma)$ to $K^2(\Theta)$.

Proof. This theorem is the Nyquist–Shannon sampling theorem for model spaces, see [28] and [58]. Indeed for every $f \in K^2(\Theta)$

$$f(z) := \sum_{n\in\mathbb{Z}} \left\langle f, \frac{K_{t_n}}{\|K_{t_n}\|} \right\rangle \frac{K_{t_n}}{\|K_{t_n}\|} = \sum_{n\in\mathbb{Z}} \frac{\pi f(t_n)}{\phi'(t_n)} K_{t_n}(z) = \sum_{n\in\mathbb{Z}} \frac{\pi f(t_n)}{\phi'(t_n)} \frac{1-\Theta(z)}{2\pi i(t_n - z)} \ , \tag{3.5}$$

where $K_{t_n}(z)$ have been defined in (1.19) and t_n satisfies $\Theta(t_n) = 1$. □

The boundedness of $E(\Theta,\mu)$ is equivalent to the boundedness of the two-weights Hilbert transform H_τ.

Theorem 3.3 Let $1-\Theta(z) \notin K^2(\Theta)$. The embedding operator $E(\Theta,\mu)$ is bounded if and only if the two-weights Hilbert transform H_τ from $L^2(d\sigma) \to L^2(d\tau)$ is continuous, where

$$d\sigma(z) := \sum_{n\in\mathbb{Z}} \pi \frac{\delta_{t_n}(z)}{\phi'(t_n)} \ , \tag{3.6}$$

$$d\tau(z) := \sum_{n\in\mathbb{Z}} \pi \left|1-\Theta(t_n+i\tau)\right|^2 \left|\frac{E(t_n+i\tau)}{E(t_n)}\right|^2 \frac{\delta_{t_n+i\tau}(z)}{\phi'(t_n)} \ , \tag{3.7}$$

$$H_\tau(f)(z) := \int_{\mathbb{R}} \frac{f(t)}{t-z}\frac{1}{2\pi i} d\sigma(t) \ , \qquad f \in L^2(d\sigma) \ . \tag{3.8}$$

Proof. Let us suppose that $H_\tau : L^2(d\sigma) \to L^2(d\tau)$ is bounded. Then we have that

$$\int_{\mathbb{R}} |(1-\Theta(z))H_\tau(f)(z)|^2 \sum_{n\in\mathbb{Z}} \pi \left|\frac{E(t_n+i\tau)}{E(t_n)}\right|^2 \frac{\delta_{t_n+i\tau}(z)}{\phi'(t_n)}$$
$$= \int_{\mathbb{R}} |H_\tau(f)(z)|^2 \, d\tau(z) \leq \mathcal{C}^2 \, \|f\|^2_{L^2(\sigma)} \ .$$

For any $g \in K^2(\Theta)$, let $g = H_0 f$ according to (3.4); the above inequality says that

$$\int_{\overline{\mathbb{C}^+}} |g(z)|^2 \sum_{n\in\mathbb{Z}} \pi \left|\frac{E(t_n+i\tau)}{E(t_n)}\right|^2 \frac{\delta_{t_n+i\tau}(z)}{\phi'(t_n)} \leq \mathcal{C}^2 \, \|f\|^2_{L^2(\sigma)} = \mathcal{C}^2 \, \|g\|^2_{K_\Theta} \ .$$

Thus, we have that $\|H_\tau\|^2 \leq \mathcal{C}^2$. Since the argument is completely reversible, we can say that $\|H_\tau\| = \mathcal{C}$ which proves the theorem.

□

We point out that it is possible, by using the results of Lacey-Sawyer-Shen-Uriarte-Tuero-Wick, see [53], [49] and [52], completely characterize when the two-weights Hilbert transform H_τ is bounded.

2. Boundedness of $T_{i\tau}$ and the matrix Γ

In this section we prove a necessary condition that the Hemite Biehler function $E(z)$ must satisfy if the vertical translation operator $T_{i\tau}$ is bounded in $\mathcal{H}(E)$.

Indeed if $\|T_{i\tau}\| < C$, then, due to Theorem 3.1,

$$C^2 \left\|K_{t_n^\alpha}\right\|_K^2 \geq \sum_{l\in\mathbb{Z}} \left|\left\langle K_{t_n^\alpha}, K_{t_l^\alpha+i\tau}\right\rangle\right|^2 \frac{\left|E(t_l^\alpha+i\tau)\right|^2}{\left\|k_{t_l^\alpha}\right\|^2}$$
$$= \sum_{l\in\mathbb{Z}} \left|K_{t_n^\alpha}(t_l^\alpha+i\tau)\right|^2 \frac{\left|E(t_l^\alpha+i\tau)\right|^2}{\left\|k_{t_l^\alpha}\right\|^2} = \sum_{l\in\mathbb{Z}} \left|k_{t_n^\alpha}(t_l^\alpha+i\tau)\right|^2 \frac{1}{\left\|k_{t_l^\alpha}\right\|^2 \left|E(t_n^\alpha)\right|^2} .$$

This inequality implies that $\forall n \in \mathbb{Z}$ and for every $0 \leq \alpha < 1$

$$\sum_{l\in\mathbb{Z}} \left|k_{t_n^\alpha}(t_l^\alpha+i\tau)\right|^2 \frac{1}{\left\|k_{t_n^\alpha}\right\|^2 \left\|k_{t_l^\alpha}\right\|^2} \leq C^2 . \tag{3.9}$$

The above relation assures that there exists a family of uniformly bounded operators $\Gamma(\alpha)$ in $\ell^2(\mathbb{Z})$.

Proposition 3.4 For every $0 \leq \alpha < 1$, there exists an infinite matrix $\Gamma(\alpha)$

$$\Gamma(\alpha) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$$
$$\Gamma(\alpha)[(c_n)] := \left(\sum_{n\in\mathbb{Z}} \left\langle k_{t_n^\alpha}, k_{t_l^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_n^\alpha}\right\| \left\|k_{t_l^\alpha}\right\|} c_n\right)_l , \tag{3.10}$$

which acts in $\ell^2(\mathbb{Z})$, such that for every $0 \leq \alpha < 1$

$$\|\Gamma(\alpha)\| \leq C . \tag{3.11}$$

Proof. First of all, thanks to Shannon's sampling theorem for de Branges spaces [28], we know that for every $0 \leq \alpha < 1$,

$$\overline{\mathrm{Span}\left\{\tilde{k}_{t_n^\alpha}\right\}_{n\in\mathbb{Z}}}^{\mathcal{H}} \hookrightarrow \mathcal{H}(E) \quad \text{continuously} ,$$

where $\tilde{k}_w$ is the normalized reproducing kernel of $\mathcal{H}(E)$. We can now identify in the canonical way the space $\overline{\mathrm{Span}\left\{\tilde{k}_{t_n^\alpha}\right\}_{n\in\mathbb{Z}}}^{\mathcal{H}}$ with $\ell^2(\mathbb{Z})$ as $\left\langle \tilde{k}_{t_n}, \tilde{k}_{t_m}\right\rangle_{\mathcal{H}} = \delta_{n,m}$. Therefore we define the operator

$$\mathcal{T}(\alpha) : \overline{\text{Span}\left\{\tilde{k}_{t_n^\alpha}\right\}_{n\in\mathbb{Z}}}^{\mathcal{H}} \to \overline{\text{Span}\left\{\tilde{k}_{t_l^\alpha}\right\}_{l\in\mathbb{Z}}}^{\mathcal{H}}$$

$$\mathcal{T}(\alpha)(\tilde{k}_{t_n^\alpha})(z) := \sum_{l\in\mathbb{Z}} \left\langle k_{t_n^\alpha}, k_{t_l^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_n^\alpha}\right\|\left\|k_{t_l^\alpha}\right\|}\tilde{k}_{t_l^\alpha}(z)\ . \tag{3.12}$$

Thanks to estimate (3.9), for every $0 \le \alpha < 1$ the operator $\mathcal{T}(\alpha)$ is bounded. Consequently we can define the family of infinite matrices $\Gamma(\alpha)$ as in (3.10) such that $\|\Gamma(\alpha)\| \le C$. $\square$

We can explicitly described the matrix $\Gamma(\alpha)$. Indeed .

$$\Gamma(\alpha) :=$$

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \left\langle k_{t_{-1}^\alpha}, k_{t_{-1}^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_{-1}^\alpha}\right\|^2} & \left\langle k_{t_0^\alpha}, k_{t_{-1}^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_0^\alpha}\right\|\left\|k_{t_{-1}^\alpha}\right\|} & \left\langle k_{t_1^\alpha}, k_{t_{-1}^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_1^\alpha}\right\|\left\|k_{t_{-1}^\alpha}\right\|} & \cdots \\ \cdots & \left\langle k_{t_{-1}^\alpha}, k_{t_0^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_{-1}^\alpha}\right\|\left\|k_{t_0^\alpha}\right\|} & \left\langle k_{t_0^\alpha}, k_{t_0^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_0^\alpha}\right\|^2} & \left\langle k_{t_1^\alpha}, k_{t_0^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_0^\alpha}\right\|\left\|k_{t_1^\alpha}\right\|} & \cdots \\ \cdots & \left\langle k_{t_{-1}^\alpha}, k_{t_1^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_{-1}^\alpha}\right\|\left\|k_{t_1^\alpha}\right\|} & \left\langle k_{t_0^\alpha}, k_{t_1^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_0^\alpha}\right\|\left\|k_{t_1^\alpha}\right\|} & \left\langle k_{t_1^\alpha}, k_{t_1^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_1^\alpha}\right\|^2} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Of course, the diagonal elements are bounded. Indeed, for any $0 \le \alpha < 1$

$$\left|\left\langle k_{t_n^\alpha}, k_{t_n^\alpha+i\tau}\right\rangle \frac{1}{\left\|k_{t_n^\alpha}\right\|^2}\right| \le C \quad \forall n \in \mathbb{Z}\ . \tag{3.13}$$

We are now ready to state the necessary condition that the Hermite Biehler function $E(z)$ must satisfy when the vertical translation operator $T_{i\tau}$ is bounded in $\mathcal{H}(E)$.

Theorem 3.5 Let us assume that the vertical translation operator $T_{i\tau}$ is bounded in $\mathcal{H}(E)$. Then the Hermite Biehler function $E(z)$ satisfies

$$\sup_{x\in\mathbb{R}} \left|\frac{E(x+i\tau)}{E(x)} - \frac{\overline{E(x-i\tau)}}{E(x)}\right| \frac{1}{\tau\phi'(x)} \le \tilde{C}\ . \tag{3.14}$$

Proof . As already explained in the above proposition, relation (3.13) holds. We explicitly compute (3.13) in order to obtain (3.14). Indeed

$$\begin{aligned} \tilde{C} &\ge \left|E(t_n^\alpha+i\tau)\overline{E(t_n^\alpha)} - \overline{E(t_n^\alpha-i\tau)}E(t_n^\alpha)\right| \frac{1}{\tau\left|E(t_n^\alpha)\right|^2\phi'(t_n^\alpha)} \\ &= \left|\frac{E(t_n^\alpha+i\tau)}{E(t_n^\alpha)} - \frac{\overline{E(t_n^\alpha-i\tau)}}{E(t_n^\alpha)}\right| \frac{1}{\tau\phi'(t_n^\alpha)}\ . \end{aligned}$$

By varying $0 \le \alpha < 1$ and $n \in \mathbb{Z}$, we get the estimate (3.14). $\square$

We point also that the estimate (3.14) can be rewritten as

$$\sup_{x\in\mathbb{R}}\left|\frac{T_\tau E(x)}{E(x)}-\left(\frac{T_\tau E(x)}{E(x)}\right)^{\#}\right|\frac{1}{\tau\phi'(x)}\le\tilde{C}\ .$$

3. Horizontal translation and Carleson measures

In this section we prove a necessary condition for the boundedness of the horizontal translation T_σ. First of all we have the analogue of Theorem 3.1 with almost the same proof.

Theorem 3.6 If the operator T_σ is bounded in $\mathcal{H}(E)$, the measure

$$d\nu^\alpha(z):=\pi\sum_{n\in\mathbb{Z}}\frac{\delta_{t_n^\alpha+\sigma}(z)}{\phi'(t_n^\alpha)}\left|\frac{E(t_n^\alpha+\sigma)}{E(t_n^\alpha)}\right|^2 \tag{3.15}$$

is a Carleson measure for $K^2(\Theta)$.

Therefore if T_σ is bounded the measure $d\nu^\alpha(z)$ is a Carleson measure for $K^2(\Theta)$ and it is supported in $\mathbb{R}$.

The following theorem, proved in [6], will give us a necessary and sufficient condition the measure $d\nu^\alpha(z)$ of (3.15) must satisfy in order to be a Carleson measure for $K^2(\Theta)$.

Theorem 3.7 Let $E(z)$ be a Hermite Biehler function satisfying

$$\left\|\left(\frac{F}{E}\right)'\right\|_{H^2(\mathbb{C}^+)}\le\alpha\,\|F\|_{\mathcal{H}(E)}\quad\forall F\in\mathcal{H}(E) \tag{3.16}$$

and let μ be a Borel measure on the line $\mathbb{R}$. Then (1) implies (2), where

(1) There are constants $L, C_L>0$ such that $\mu(I)<C_L L$ for any interval I such that $|I|\le L$.

(2) The measure μ satisfies

$$\int_{\mathbb{R}}\left|\frac{F}{E}\right|^2 d\mu\le C\,\|F\|^2_{\mathcal{H}}\quad\forall F\in\mathcal{H}(E)\ . \tag{3.17}$$

Under the additional condition $|\phi'(t)|>0$, the condition (2) implies (1).

We can apply the above result to the measure $d\nu^\alpha(z)$ of (3.15).

Proposition 3.8 Suppose that $0<c_1\le|\phi'(x)|\le c_2<\infty$, $\forall x\in\mathbb{R}$. The measure

$$d\nu^\alpha(z):=\pi\sum_{n\in\mathbb{Z}}\frac{\delta_{t_n^\alpha+\sigma}(z)}{\phi'(t_n^\alpha)}\left|\frac{E(t_n^\alpha+\sigma)}{E(t_n^\alpha)}\right|^2$$

is a Carleson measure for $K^2(\Theta)$ if and only if

$$\sum_{t_n^\alpha+\sigma\in I}\frac{\pi}{\phi'(t_n^\alpha)}\left|\frac{E(t_n^\alpha+\sigma)}{E(t_n^\alpha)}\right|^2\le C_L L \tag{3.18}$$

for every interval $|I|<L$.

We can finally state the necessary condition that the Hermite Biehler function $E(z)$ has to satisfy so that T_σ is bounded.

Theorem 3.9 Suppose that $0 < c_1 \le |\phi'(x)| \le c_2 < \infty$, $\forall x \in \mathbb{R}$. If the horizontal translation operator T_σ is bounded, then

$$\sup_{x\in\mathbb{R}} \left| \frac{E(x+\sigma)}{E(x)} \right| \le C \ . \tag{3.19}$$

Proof. The Condition (3.19) comes directly from (3.18) by varying $\alpha \in [0,1)$ and $n \in \mathbb{Z}$. Indeed, for every $x \in \mathbb{R}$, $x = \alpha\pi + n\pi$ for some $0 \le \alpha < 1$ and $n \in \mathbb{Z}$. Consequently, since C_L in (3.18) does not depend on α,

$$\begin{aligned}\left| \frac{E(x+\sigma)}{E(x)} \right|^2 = \left| \frac{E(t_n^\alpha+\sigma)}{E(t_n^\alpha)} \right|^2 &\le \frac{\sup_{x\in\mathbb{R}} |\phi'(x)|}{\pi} \left(\frac{\pi}{\phi'(t_n^\alpha)} \left| \frac{E(t_n^\alpha+\sigma)}{E(t_n^\alpha)} \right|^2 \right) \\ &\le \frac{\sup_{x\in\mathbb{R}} |\phi'(x)|}{\pi} C_L L \ ,\end{aligned}$$

which proves the theorem. $\square$

4. Necessary and sufficient condition for boundedness of translation operator in de Branges spaces

In the previous pages we found some necessary conditions for the boundedness of the translation operator T_ζ in the de Branges space $\mathcal{H}(E)$. In that case we used the Carleson measures for the associated model space. In this chapter we start from the Pancherel-Polya inequality in the Paley-Wiener space and from the Bernstein inequality in the de Branges space. This different approach allows us to obtain new conditions, in some cases necessary and sufficient, for the boundedness of T_ζ in the de Branges space.

In the next sections we study the necessary and sufficient conditions for the boundedness of the vertical translation operator $T_{i\tau}$. After showing the general sufficient conditions (Section 1) and the general necessary conditions (Section 2), we present a necessary and sufficient condition (Section 3). However, to do this we require some additional properties for the Hermite Biehler function $E(z)$, (4.14) and (4.16).

Next, we move to the study of the horizontal translation operator T_σ (Section 4). As it happens for $T_{i\tau}$, in order to obtain the necessary and sufficient condition for the boundedness of T_σ, we are forced to impose stronger hypothesis on the Hermite Biehler function $E(z)$ (Section 5). In the final section (Section 6), adding some hypotheses, we prove that if T_σ is bounded, also T_{σ_1} is bounded when $0 < \sigma_1 < \sigma$.

The results of this chapter are original and they are presented in an article, [17].

1. Sufficient conditions for vertical translation

We first prove the sufficient conditions for the boundedness of $T_{i\tau}$, $\tau > 0$.

Theorem 4.1 Let $z = x + iy$. If the following three conditions are satisfied

$$\frac{E(z+i\tau)}{E(z)} \in H^\infty(\mathbb{C}^+) \ , \tag{4.1}$$

$$\sup_{0<y\leq\tau} \sup_x \left| \frac{(T_{i\tau}E)^\# (z)}{E(z)} \right| = \sup_{0<y\leq\tau} \sup_x \left| \frac{\overline{E(x-iy+i\tau)}}{E(x+iy)} \right| < \infty \ , \tag{4.2}$$

$$\sup_{\tau<y} \sup_x \left| \frac{T_{-i\tau}E(z)}{E(z)} \right| = \sup_{\tau<y} \sup_x \left| \frac{E(x+iy-i\tau)}{E(x+iy)} \right| < \infty \ , \tag{4.3}$$

then the operator $T_{i\tau}$ is bounded.

Proof. Let $f \in \mathcal{H}(E)$. Then

$$\begin{aligned} \sup_{y>0} \int_{\mathbb{R}} \left| \frac{f(x+iy+i\tau)}{E(x+iy)} \right|^2 dx &\leq \sup_{y>0} \left(\int_{\mathbb{R}} \left| \frac{f(x+iy+i\tau)}{E(x+iy+i\tau)} \right|^2 dx \right) \sup_{y>0,\ x\in\mathbb{R}} \left| \frac{E(x+iy+i\tau)}{E(x+iy)} \right|^2 \\ &\leq \|f\|^2_{\mathcal{H}} \, \|T_{i\tau}(E)/E\|^2_{H^\infty(\mathbb{C}^+)} \ , \end{aligned}$$

which is finite because of (4.1). It is also clear that $T_{i\tau}(f)(z)$ is entire and that $T_{i\tau}f/E(z)$ is holomorphic in $\mathbb{C}^+$.

We note that $\frac{(T_{i\tau}f)^\#}{E}(z) \in \mathrm{Hol}\,(\mathbb{C}^+)$. Finally, we check that $(T_{i\tau}f)^\#/E \in H^2(\mathbb{C}^+)$. Indeed,

$$\begin{aligned} \sup_{y>0} \int_{\mathbb{R}} \left| \frac{(T_{i\tau}f)^\# (z)}{E(z)} \right|^2 dx &= \sup_{y>0} \int_{\mathbb{R}} \left| \frac{f(x-iy+i\tau)}{E(x+iy)} \right|^2 dx \\ &\leq \max \left\{ \sup_{\tau\geq y} \left(\int_{\mathbb{R}} \left| \frac{f(x-iy+i\tau)}{E(x-iy+i\tau)} \right|^2 dx \right) \sup_{\tau\geq y} \left\| \frac{E(\cdot-iy+i\tau)}{E(\cdot+iy)} \right\|^2_\infty , \right. \\ &\qquad \left. \sup_{\tau<y} \left(\int_{\mathbb{R}} \left| \frac{f(x-iy+i\tau)}{E(x+iy-i\tau)} \right|^2 dx \right) \sup_{\tau<y} \left\| \frac{E(\cdot+iy-i\tau)}{E(\cdot+iy)} \right\|^2_\infty \right\} \\ &\leq \|f\|^2_{\mathcal{H}} \max \left\{ \sup_{0<y\leq\tau} \left\| \frac{E(\cdot-iy+i\tau)}{E(\cdot+iy)} \right\|^2_\infty , \sup_{\tau<y} \left\| \frac{E(\cdot+iy-i\tau)}{E(\cdot+iy)} \right\|^2_\infty \right\} \ , \end{aligned}$$

which is bounded because of (4.2) and (4.3). □

We note that the conditions (4.1) and (4.2) are related to the necessary condition of Theorem 3.5.

We observe that the previous calculations provide us with an upper estimate for the operator norm of $T_{i\tau}$. Indeed, it holds that

$$\|T_{i\tau}\| \leq \left\| \frac{T_{i\tau}E}{E} \right\|_{H^\infty} .$$

The sufficient condition may seem a bit artificial. Therefore, we find more natural ones.

Corollary 4.2 The translation operator $T_{i\tau}$ is bounded if

$$\frac{T_{i\tau}E}{E} \in H^\infty(\mathbb{C}^+), \quad \frac{(T_{i\tau}E)^\#}{E} \in H^\infty(\mathbb{C}^+), \quad \frac{T_{-i\tau}E}{E} \in H^\infty(\mathbb{C}^+) \ . \tag{4.4}$$

Corollary 4.3 The translation operator $T_{i\tau}$ is bounded if

$$\left\|\frac{T_{is}E}{E}\right\|_{H^\infty(\mathbb{C}^+)} < C < \infty \quad \forall s \in [-\tau, \tau] \quad . \tag{4.5}$$

Proof. We prove that the condition (4.2) is implied by the condition (4.5). Indeed

$$\begin{aligned}\sup_{0<y\le\tau}\sup_x \left|\frac{\overline{E(x-iy+i\tau)}}{E(x+iy)}\right| &= \sup_{0<y\le\tau}\sup_x \left|\frac{\overline{E(x+iy-i2y+i\tau)}}{E(x+iy)}\right| \\ &\le \sup_{0<y\le\tau}\left\|\frac{T_{i(\tau-2y)}E}{E}\right\|_{H^\infty} < C \ ,\end{aligned}$$

because of (4.5). □

The sufficient condition becomes simpler if we ask $E(z)$ to be more regular. For example, the condition (4.3) is automatically satisfied by the functions $E(z)$ in the Polya class [28].

Definition 4.4 An entire function $E(z)$ is said to be in the Polya class if it has no zeros in the upper half-plane, if $|E(x-iy)| < |E(x+iy)|$ for $y > 0$, and if $|E(x+iy)|$ is a nondecreasing function of $y > 0$ for each fixed x.

2. Necessary condition for vertical translation

We focus our attention on finding some necessary conditions.

First of all, let us recall the following lemma, whose proof can be found in [7] and [41].

Lemma 4.5 Let $\Theta(z)$ be an inner function and $h_1, h_2 > 0$. Then

$$\frac{1}{8}\min\left(\frac{h_1}{h_2}, \frac{h_2}{h_1}\right) \le \frac{1-|\Theta(x+ih_1)|}{1-|\Theta(x+ih_2)|} \le 8\max\left(\frac{h_1}{h_2}, \frac{h_2}{h_1}\right) \ .$$

We describe the necessary conditions.

Theorem 4.6 If $T_{i\tau}$ is bounded in $\mathcal{H}(E)$, then there exists $C(h,\tau) > 0$ such that the following three conditions hold:

$$\forall\ h > 0 \quad , \quad \sup_{x\in\mathbb{R}}\left|\frac{E(x+ih+i\tau)}{E(x+ih)}\right| \le C(h,\tau) \ , \tag{4.6}$$

$$\forall\ h > \tau \quad , \sup_{x\in\mathbb{R}} \left| \frac{E(x+ih-i\tau)}{E(x+ih)} \right| \leq C(h,\tau) \quad , \tag{4.7}$$

$$\forall\ 0 < h \leq \tau \quad , \sup_{x\in\mathbb{R}} \left| \frac{E(x-ih+i\tau)}{E(x+ih)} \right| \leq C(h,\tau) \quad . \tag{4.8}$$

PROOF. Let us prove that (4.6) holds. For the other conditions the reasoning is similar starting from different functions G. Let

$$G(z) := -2\pi i \frac{K_w(z)}{\overline{E(w)}} = \frac{E(z)\overline{E(w)} - E^{\#}(z)E(\overline{w})}{(z-\overline{w})\overline{E(w)}} \in \mathcal{H}(E) \quad .$$

Then

$$\begin{aligned} \frac{T_{i\tau}(G)(z)}{E(z)} = \frac{G(z+i\tau)}{E(z)} &= \frac{E(z+i\tau)}{E(z)(z+i\tau-\overline{w})} - \frac{E^{\#}(z+i\tau)E(\overline{w})}{E(z)\overline{E(w)}(z+i\tau-\overline{w})} \\ &= \frac{T_{i\tau}(E)(z)}{E(z)} \frac{1}{z+i\tau-\overline{w}} - \frac{E^{\#}(z+i\tau)}{E(z)}\overline{\Theta(w)} \frac{1}{z+i\tau-\overline{w}} \\ &= \frac{T_{i\tau}(E)(z)}{E(z)} \frac{1}{z+i\tau-\overline{w}} \left[1 - \frac{E^{\#}(z+i\tau)}{E(z+i\tau)}\overline{\Theta(w)}\right] \quad , \end{aligned}$$

where we set $\Theta := E^{\#}/E\ |_{\overline{\mathbb{C}^+}}$ and $w, z \in \mathbb{C}^+$. Thus,

$$\begin{aligned} \left|\frac{T_{i\tau}(G)(z)}{E(z)}\right| &= \left|\frac{T_{i\tau}(E)(z)}{E(z)}\right| \left|\frac{1}{z+i\tau-\overline{w}}\right| \left|1 - \Theta(z+i\tau)\overline{\Theta(w)}\right| \\ &\geq \left|\frac{T_{i\tau}(E)(z)}{E(z)}\right| \frac{1}{|z+i\tau-\overline{w}|} \left(1 - |\Theta(w)|\right) \quad . \end{aligned}$$

Therefore

$$\left|\frac{T_{i\tau}(E)(z)}{E(z)}\right| \leq \left|\frac{T_{i\tau}(G)(z)}{E(z)}\right| |z+i\tau-\overline{w}| \frac{1}{1-|\Theta(w)|} \quad . \tag{4.9}$$

We have

$$\left|\frac{T_{i\tau}(G)(z)}{E(z)}\right| = \frac{1}{|E(z)|} \left|\langle T_{i\tau}G, K_z\rangle\right| \leq \frac{1}{|E(z)|} \|K_z\|_{\mathcal{H}} \|T_{i\tau}G\|_{\mathcal{H}} \leq \frac{1}{|E(z)|} \|K_z\|_{\mathcal{H}} \|T_{i\tau}\| \|G\|_{\mathcal{H}} \quad , \tag{4.10}$$

and

$$\frac{\|K_z\|_{\mathcal{H}}}{|E(z)|} = \sqrt{\frac{K_z(z)}{|E(z)|^2}} = \sqrt{\frac{i}{2\pi} \frac{|E(z)|^2 - |E(\bar{z})|^2}{|E(z)|^2 (z-\bar{z})}} = \sqrt{\frac{1}{4\pi \Im z} \left(1 - |\Theta(z)|^2\right)} \tag{4.11}$$

and

$$\|G\|_{\mathcal{H}} = \sqrt{4\pi^2 \frac{K_w(w)}{|E(w)|^2}} = \sqrt{\frac{\pi}{\Im w} \left(1 - |\Theta(w)|^2\right)} \quad . \tag{4.12}$$

We substitute (4.11) and (4.12) into (4.10). Let $z = x+ih$ and $w = x+2ih$; then (4.9) becomes

$$\begin{aligned}\left|\frac{T_{i\tau}(E)(z)}{E(z)}\right| \leq & \|T_{i\tau}\|\,|z+i\tau-\overline{w}|\sqrt{\frac{\pi}{\Im w}\left(1-|\Theta(w)|^2\right)}\sqrt{\frac{1}{4\pi\Im z}\left(1-|\Theta(z)|^2\right)}\frac{1}{1-|\Theta(w)|} \\ = & \frac{\|T_{i\tau}\|}{2}\,|z+i\tau-\overline{w}|\left(\frac{1}{\Im w\Im z}\frac{\left(1-|\Theta(w)|^2\right)\left(1-|\Theta(z)|^2\right)}{(1-|\Theta(w)|)^2}\right)^{1/2} \\ \leq & \|T_{i\tau}\|(\tau+3h)\left(\frac{1}{2h^2}\frac{(1-|\Theta(z)|)}{(1-|\Theta(w)|)}\right)^{1/2} \\ \leq & \|T_{i\tau}\|(\frac{\tau}{h}+3)8^{1/2}\ .\end{aligned}$$

In the last inequality we have used Lemma 4.5.

In order to obtain (4.7) we proceed in the same way starting with the function

$$G(z) := 2\pi i\frac{K_w(z)}{E(\overline{w})} = -\frac{E(z)\overline{E(w)}}{(z-\overline{w})E(\overline{w})} + \frac{E^{\#}(z)}{z-\overline{w}}\ ,$$

where $w = x-2ih \in \mathbb{C}^-$ and $z = x+ih$ so that $z - i\tau \in \mathbb{C}^+$.

In order to obtain (4.8) we use the function

$$G(z) := -2\pi i\frac{K_w(z)}{\overline{E(w)}} = \frac{E(z)}{z-\overline{w}} - \frac{E^{\#}(z)E(\overline{w})}{\overline{E(w)}(z-\overline{w})}\ ,$$

where $w = x+2ih$ and $z = x+ih$ so that $\bar{z}+i\tau \in \mathbb{C}^+$. □

From the previous computations we note that we have proved that if $T_{i\tau}$ is bounded in $\mathcal{H}(E)$, then

$$\forall\ z \in \mathbb{C}^+\quad ,\quad \left|\frac{E(z+i\tau)}{E(z)}\right| \leq C\left(\frac{\tau}{\Im z}+c\right)\ . \tag{4.13}$$

3. The case of no zeros in the horizontal strip

We are not able to obtain a necessary and sufficient condition for the boundedness of $T_{i\tau}$ in the general case. However, this fact should not come as a surprise, since even for the boundedness of the differentiation operator in [6] the necessary condition usually does not coincide with the sufficient one. Nevertheless, adding some regularity we get a necessary and sufficient condition.

From now on, in this section, we assume that the zeros of the Hermite Biehler function $\{z_k\}$ are separated from the real line, that is

$$\inf_k |\Im z_k| = M > 0\ . \tag{4.14}$$

Proposition 4.7 If $E(z)$ satisfies condition (4.14), then the exponent of convergence of its zeros is less than or equal to 2.

PROOF. Let $\bar{z}_n = a_n - ib_n$ be the zeros of $E(z)$. Then

$$\sum_n \frac{1}{|\bar{z}_n|^2} = \sum_n \frac{1}{a_n^2 + b_n^2} = \sum_n \frac{1}{M} \frac{M}{a_n^2 + b_n^2} \leq \frac{1}{M} \sum_n \frac{b_n}{a_n^2 + b_n^2} < \infty \quad ,$$

because of the Blaschke condition (1.6). □

Using Krein's theorem for the representation of the Hermite Biehler functions, Theorem 1.7, we note that if $E(z)$ satisfies (4.14), then

$$E(z) := S(z)\tilde{E}(z) := E(0)S(z)e^{-iaz} \prod_n \left(1 - \frac{z}{\bar{z}_n}\right) e^{\frac{a_n}{|z_n|^2} z} \tag{4.15}$$

where $S(z) = S^{\#}(z)$ is a real function , $a > 0$ and $z_n = a_n + ib_n \in \mathbb{C}^+$.

We first analyze the case $S(z) = 1$, that is

$$E(z) := E(0)e^{-iaz} \prod_n \left(1 - \frac{z}{\bar{z}_n}\right) e^{\frac{a_n}{|z_n|^2} z} \ . \tag{4.16}$$

We describe the ratio $E(z + i\tau)/E(z)$ in a different way when $z = x + iy \in \mathbb{C}^+$ by using the following estimate:

$$\log(1 + t) \geq \frac{t}{k} \quad \text{if} \quad 0 \leq t \leq k - 1 \ .$$

Indeed

$$\begin{aligned} \left|\frac{E(z+i\tau)}{E(z)}\right|^2 &= e^{2a\tau} \left| \left[\prod_n \left(1 - \frac{z+i\tau}{\bar{z}_n}\right) e^{\frac{a_n}{a_n^2+b_n^2} z} \right] \Big/ \left[\prod_n \left(1 - \frac{z}{\bar{z}_n}\right) e^{\frac{a_n}{a_n^2+b_n^2} z} \right] \right|^2 \\ &= e^{2a\tau} \left| \prod_n \frac{a_n - x - i(b_n + y + \tau)}{a_n - x - i(b_n + y)} \right|^2 \\ &= e^{2a\tau} \prod_n \left(1 + \frac{\tau^2 + 2\tau(y + b_n)}{(a_n - x)^2 + (b_n + y)^2}\right) \\ &\geq e^{2a\tau} \exp \frac{1}{k} \left(\sum_n \frac{\tau^2 + 2\tau(y + b_n)}{(a_n - x)^2 + (b_n + y)^2} \right) \end{aligned} \tag{4.17}$$

with the choice of k so that

$$0 \leq \frac{\tau^2 + 2\tau(y + b_n)}{(a_n - x)^2 + (b_n + y)^2} \leq \frac{\tau^2 + 2\tau(y + b_n)}{(b_n + y)^2} \leq \mu \ .$$

If $T_{i\tau}$ is bounded in $\mathcal{H}(E)$, we note from (4.13) that

$$\tau \left(a + \frac{1}{2k} \sum_n \frac{\tau + 2(y + b_n)}{(a_n - x)^2 + (b_n + y)^2} \right) \leq \log C \left(\frac{\tau}{y} + c \right) \tag{4.18}$$

when $z = x + iy \in \mathbb{C}^+$. In particular,

$$\tau\left(a+\frac{1}{2k}\sum_n \frac{\sigma+2(\tau/4+b_n)}{(a_n-x)^2+(b_n+\tau/4)^2}\right) \le \tau\left(a+\frac{1}{2k}\sum_n \frac{\tau+2(\tau/4+b_n)}{(a_n-x)^2+(b_n+\tau/4)^2}\right) \le \log C\,(4+c) \tag{4.19}$$

when $\sigma \le \tau$.

Theorem 4.8 Let $E(z)$ satisfy (4.14) and (4.16). If $T_{i\tau}$ is bounded in $\mathcal{H}(E)$, then

$$\sup_{x\in\mathbb{R}}\left|\frac{E(x+i\tau)}{E(x)}\right| \le C(\tau)\ . \tag{4.20}$$

PROOF. Reasoning similarly to (4.17), we have

$$\begin{aligned}
\left|\frac{E(x+i\tau)}{E(x)}\right|^2 &= e^{2a\tau}\left|\left[\prod_n\left(1-\frac{x+i\tau}{\bar z_n}\right)e^{\frac{a_n}{a_n^2+b_n^2}x}\right]\Big/\left[\prod_n\left(1-\frac{x}{\bar z_n}\right)e^{\frac{a_n}{a_n^2+b_n^2}x}\right]\right|^2\\
&= e^{2a\tau}\prod_n\left(1+\frac{\tau^2+2\tau b_n}{(a_n-x)^2+b_n^2}\right)\\
&\le e^{2a\tau}\exp\left(\sum_n\frac{\tau^2+2\tau b_n}{(a_n-x)^2+b_n^2}\right)\\
&= e^{2a\tau}\exp\left(\tau\sum_n\frac{\tau/2+2(b_n+\tau/4)}{(a_n-x)^2+b_n^2}\right)\\
&\le e^{2a\tau}\exp\left[\tau\left(\sum_n\frac{\tau/2+2(b_n+\tau/4)}{(a_n-x)^2+(b_n+\tau/4)^2}\right)\ \sup_n\left(1+\frac{\tau^2/16+b_n\tau/2}{(a_n-x)^2+b_n^2}\right)\right]
\end{aligned}$$

where the last term is bounded since $b_n > M$. Therefore,

$$\sup_x\left|\frac{E(x+i\tau)}{E(x)}\right|^2 \le C'\ ,$$

thanks to (4.19). $\square$

Even if (4.20) is satisfied, we are looking for a necessary and sufficient condition when (4.14) holds and the Hermite Biehler function does not have real factors (4.16).

Theorem 4.9 Let $E(z)$ satisfy (4.14) and do not have real factor (4.16). Then

$$\frac{E(z+i\tau)}{E(z)} \in H^\infty(\mathbb{C}^+) \tag{4.21}$$

if and only if $T_{i\tau}$ is bounded in $\mathcal{H}(E)$.

PROOF NECESSARY CONDITION. We know that the function $\frac{T_{i\tau}E}{E}(z) \in \mathrm{Hol}\,(\mathbb{C}^+)$ and $\frac{T_{i\tau}E}{E}(z) \neq 0$ in $\mathbb{C}^+$. Consequently, from (4.17),

$$\Re\left(\log\left(\frac{T_{i\tau}E}{E}(z)\right)\right) \geq \tau\left[a+\frac{1}{2k}\left(\sum_n \frac{\tau^2+2\tau(y+b_n)}{(a_n-x)^2+(b_n+y)^2}\right)\right] \geq 0\ .$$

Therefore

$$\log\left(\frac{T_{i\tau}E}{E}(z)\right) \text{ is in the Smirnov class of the upper half plane .}$$

For the definition of the Smirnov class we refer to [28] and [29]. Since $\log\left(\frac{T_{i\tau}E}{E}(x)\right) \in L^\infty(\mathbb{R})$ because of Theorem 4.8, then $\log\left(\frac{T_{i\tau}E}{E}(z)\right) \in H^\infty(\mathbb{C}^+)$. In particular

$$\frac{T_{i\tau}E}{E}(z) \in H^\infty(\mathbb{C}^+)\ .$$

$\square$

Proof sufficient condition. We prove that if

$$\frac{T_{i\tau}E}{E}(z) \in H^\infty(\mathbb{C}^+) \text{ implies that } \frac{T_{is}E}{E}(z) \in H^\infty(\mathbb{C}^+) \quad \forall\ s \in [-\tau,\tau]\ ,$$

which is the sufficient condition obtained in (4.5). Indeed, because of (4.17), then

$$1 \leq \left|\frac{E(z+i\tau)}{E(z)}\right| \leq \left\|\frac{T_{i\tau}E}{E}\right\|_{H^\infty} = \mu \qquad z \in \mathbb{C}^+\ ,$$

and

$$0 \leq \frac{1}{2}\log\left|\frac{E(z+i\tau)}{E(z)}\right|^2 \leq \log\mu\ .$$

Let $z = x+iy \in \mathbb{C}^+$, then, arguing as in (4.17),

$$\begin{aligned}
\left|\frac{E(z+is)}{E(z)}\right|^2 &= e^{2as}\left|\left[\prod_n\left(1-\frac{z+is}{\bar z_n}\right)e^{\frac{a_n}{a_n^2+b_n^2}z}\right]\Big/\left[\prod_n\left(1-\frac{z}{\bar z_n}\right)e^{\frac{a_n}{a_n^2+b_n^2}z}\right]\right|^2\\
&= e^{2as}\exp\left(\sum_n \log\left(1+\frac{s^2+2s(y+b_n)}{(a_n-x)^2+(b_n+y)^2}\right)\right)\\
&\leq e^{2as}\exp\left(\sum_n \frac{s^2+2s(y+b_n)}{(a_n-x)^2+(b_n+y)^2}\right)\\
&\leq e^{2ka\tau}\exp k\left(\frac{1}{k}\sum_n \frac{\tau^2+2\tau(y+b_n)}{(a_n-x)^2+(b_n+y)^2}\right)\\
&\leq C\left|\frac{E(z+i\tau)}{E(z)}\right|^k \leq C\mu^k
\end{aligned}$$

for every $s \in [-\tau,\tau]$ and k fixed in (4.17). $\square$

When (4.16) holds, if $T_{i\tau}$ is bounded, then also T_{ik} is bounded for every $0 < k \leq \tau$, since the function $E(z)$ belongs to the Polya class, as shown in [51]. Moreover

$$\left\|\frac{T_{ik}E}{E}\right\|_{H^\infty} \leq \left\|\frac{T_{ik}E}{T_{i\tau}E}\right\|_{H^\infty} \left\|\frac{T_{i\tau}E}{E}\right\|_{H^\infty} < \infty$$

and consequently, when a vertical translation operator $T_{i\tau}$ is bounded, any other vertical translation operator T_{ik} with $0 < k < \tau$ is bounded.

From now until the end of this section, we consider Hermite Biehler functions with expression (4.15). Even if we will not characterize when the translation operator in $\mathcal{H}(SE)$, that is $T^{SE}_{i\tau}$, is bounded, we will highlight some connections between the boundedness of $T^{SE}_{i\tau}$ and the boundedness of $T^{E}_{i\tau}$. Until the end of this section, we assume that $E(z)$ satisfies (4.14).

First of all, if $S(z) = e^{Az}$, with $A \in \mathbb{R}$, and $T^{E}_{i\tau}$ is bounded, then $T^{SE}_{i\tau}$ is bounded. Indeed the sufficient condition (4.5) is satisfied by $S(z)E(z)$ since

$$\left|\frac{S(z+is)E(z+is)}{S(z)E(z)}\right| = \left|\frac{E(z+is)}{E(z)}\right| < \left\|\frac{T_{i\tau}E}{E}\right\|_{H^\infty} \quad \forall s \in [-\tau, \tau] \ ,$$

where the last inequality has been proved in Theorem 4.9. However it is not clear whether also the reverse implication is true: we do not know whether the boundedness of $T^{SE}_{i\tau}$ implies also the boundedness of $T^{E}_{i\tau}$ and the fact that $S(z) = e^{Az}$.

Nevertheless, we can prove something more about the relationship between $T^{SE}_{i\tau}$ and $T^{E}_{i\tau}$. First of all, if $S(z) = e^{Az}$, then if $T^{SE}_{i\tau}$ is bounded also $T^{E}_{i\tau}$ is bounded. Indeed we check that

$$\begin{aligned}\sup_{y>0}\int_{\mathbb{R}}\left|\frac{f(x+iy+i\tau)}{E(x+iy)}\right|^2 dx &= \sup_{y>0}\int_{\mathbb{R}}\left|\frac{e^{A(x+iy+i\tau)}f(x+iy+i\tau)}{e^{A(x+iy)}E(x+iy)}\right|^2 dx \\ &= \left\|T^{SE}_{i\tau}e^{A\cdot}f\right\|^2_{\mathcal{H}(SE)} \leq \left\|T^{SE}_{i\tau}\right\|^2 \|f\|^2_{\mathcal{H}(E)} \ ,\end{aligned}$$

and that

$$\begin{aligned}\sup_{y>0}\int_{\mathbb{R}}\left|\frac{f(x-iy+i\tau)}{E(x+iy)}\right|^2 dx &= \sup_{y>0}\int_{\mathbb{R}}\left|\frac{e^{A(x-iy+i\tau)}f(x-iy+i\tau)}{e^{A(x+iy)}E(x+iy)}\right|^2 dx \\ = \left\|\left(T^{SE}_{i\tau}e^{A\cdot}f\right)^{\#}\right\|^2_{\mathcal{H}(SE)} &= \left\|T^{SE}_{i\tau}e^{A\cdot}f\right\|^2_{\mathcal{H}(SE)} \leq \left\|T^{SE}_{i\tau}\right\|^2 \|f\|^2_{\mathcal{H}(E)} \ .\end{aligned}$$

Moreover, let us suppose that $T^{E}_{i\tau}$ is bounded and that S is of finite order. Then if $T^{SE}_{i\tau}$ is bounded, we obtain that $S(z) = e^{Az}$. Indeed the Hermite Biehler function $S(z)E(z)$ has to satisfy the necessary conditions (4.6) and (4.7). Since

$$\inf_{x}\left|\frac{E(x+ih+i\tau)}{E(x+ih)}\right| \geq 1 \ , \tag{4.22}$$

because of (4.7), then

$$\sup_{h>\tau}\sup_{x}\left|\frac{S(x+ih+i\tau)}{S(x+ih)}\right| < C \ , \tag{4.23}$$

due to (4.6) and (4.22). At the same time, since, when $h > \tau$,

$$\inf_x \left| \frac{E(x+ih-i\tau)}{E(x+ih)} \right| = \inf_x \left| \frac{E(x+iu)}{E(x+iu+i\tau)} \right| \geq \frac{1}{\|T_{i\tau}E/E\|_{H^\infty(\mathbb{C}^+)}} , \tag{4.24}$$

then

$$\sup_{h>\tau} \sup_x \left| \frac{S(x+ih-i\tau)}{S(x+ih)} \right| < C' , \tag{4.25}$$

because of (4.7) and (4.24). However, it is easy to see that the only real function of finite order which satisfies both (4.23) and (4.25) is $S(z) = e^{Az}$ with $A \in \mathbb{R}$.

4. Sufficient condition and necessary condition for horizontal translation

In this section we focus our attention on the horizontal translation operator T_σ. In particular, we first find some sufficient conditions and, subsequently, some necessary conditions. The proofs are similar to those of Theorems 4.1 and 4.6.

Theorem 4.10 Let $z = x + iy$. If

$$\frac{E(z+\sigma)}{E(z)} \in H^\infty(\mathbb{C}^+) , \tag{4.26}$$

then the operator T_σ is bounded.

Proof. We easily find some upper bounds:

$$\begin{aligned}
\|T_\sigma f\|^2_{\mathcal{H}(E)} &= \sup_{y>0} \int_{\mathbb{R}} \left| \frac{f(x+\sigma+iy)}{E(x+iy)} \right|^2 dx \\
&\leq \sup_{y>0} \left(\int_{\mathbb{R}} \left| \frac{f(x+iy+\sigma)}{E(x+iy+\sigma)} \right|^2 dx \right) \sup_{z\in\mathbb{C}^+} \left(\left| \frac{E(z+\sigma)}{E(z)} \right|^2 \right) \\
&= \|f\|^2_{\mathcal{H}(E)} \, \|T_\sigma(E)/E\|^2_{H^\infty(\mathbb{C}^+)} \quad .
\end{aligned}$$

Furthermore, $T_\sigma(f)(z)$ is entire and $T_\sigma f/E(z)$ is holomorphic in $\mathbb{C}^+$. We check also that

$$\begin{aligned}
\sup_{y>0} \int_{\mathbb{R}} \left| \frac{(T_\sigma f)^\#(x+iy)}{E(x+iy)} \right|^2 dx &= \sup_{y>0} \int_{\mathbb{R}} \left| \frac{f(x-iy+\sigma)}{E(x+iy)} \right|^2 dx \\
&\leq \sup_{y>0} \left(\int_{\mathbb{R}} \left| \frac{f(x-iy+\sigma)}{E(x+iy+\sigma)} \right|^2 dx \right) \sup_{z\in\mathbb{C}^+} \left\| \frac{E(z+\sigma)}{E(z)} \right\|^2_\infty \\
&= \|f\|^2_{\mathcal{H}(E)} \left\| \frac{T_\sigma E}{E} \right\|^2_{H^\infty(\mathbb{C}^+)} ,
\end{aligned}$$

which proves the statement. ∎

We point out that the previous computations provide us with a bound for the norm of T_σ. Indeed, it holds that $\|T_\sigma\| \leq \left\|\frac{T_\sigma E}{E}\right\|_{H^\infty}$.

We state now the necessary condition. As already done in Theorem 4.6, we will adapt the proof of Proposition 2.2 of [6] to the operator T_σ.

Theorem 4.11 If the horizontal translation operator T_σ is bounded in $\mathcal{H}(E)$, then

$$\forall\ h>0,\quad \sup_{x\in\mathbb{R}}\left|\frac{E(x+ih+\sigma)}{E(x+ih)}\right| \leq C(h,\sigma) \quad . \tag{4.27}$$

PROOF. Let

$$G(z) := -2\pi i\frac{K_w(z)}{\overline{E(w)}} = \frac{E(z)\overline{E(w)} - E^\#(z)E(\overline{w})}{(z-\overline{w})\overline{E(w)}} \in \mathcal{H}(E) \quad .$$

Then

$$\begin{aligned}\frac{T_\sigma(G)(z)}{E(z)} = \frac{G(z+\sigma)}{E(z)} &= \frac{E(z+\sigma)}{E(z)(z+\sigma-\overline{w})} - \frac{E^\#(z+\sigma)E(\overline{w})}{E(z)\overline{E(w)}(z+\sigma-\overline{w})}\\ &= \frac{T_\sigma(E)(z)}{E(z)}\frac{1}{z+\sigma-\overline{w}}\left[1-\frac{E^\#(z+\sigma)}{E(z+\sigma)}\overline{\Theta(w)}\right]\end{aligned}$$

where we set $\Theta := E^\#/E\mid_{\overline{\mathbb{C}^+}}$ and $w,z\in\mathbb{C}^+$. Thus

$$\begin{aligned}\left|\frac{T_\sigma(G)(z)}{E(z)}\right| &= \left|\frac{T_\sigma(E)(z)}{E(z)}\right|\left|\frac{1}{z+\sigma-\overline{w}}\right|\left|1-\Theta(z+\sigma)\overline{\Theta(w)}\right|\\ &\geq \left|\frac{T_\sigma(E)(z)}{E(z)}\right|\frac{1}{|z+\sigma-\overline{w}|}\left(1-|\Theta(z+\sigma)|\,|\Theta(w)|\right) \quad .\end{aligned}$$

Therefore

$$\left|\frac{T_\sigma(E)(z)}{E(z)}\right| \leq \left|\frac{T_\sigma(G)(z)}{E(z)}\right| |z+\sigma-\overline{w}|\,\frac{1}{1-|\Theta(z+\sigma)|\,|\Theta(w)|} \quad . \tag{4.28}$$

We have

$$\left|\frac{T_\sigma(G)(z)}{E(z)}\right| = \frac{1}{|E(z)|}\left|\langle T_\sigma G, K_z\rangle\right| \leq \frac{1}{|E(z)|}\|K_z\|_{\mathcal{H}}\|T_\sigma G\|_{\mathcal{H}} \leq \frac{1}{|E(z)|}\|K_z\|_{\mathcal{H}}\|T_\sigma\|\,\|G\|_{\mathcal{H}} \ . \tag{4.29}$$

We substitute (4.11) and (4.12) into (4.29). Let $z = x+ih$ and $w = x+ih$; Then (4.28) becomes,

$$\begin{aligned}\left|\frac{T_\sigma(E)(z)}{E(z)}\right| &\leq \|T_\sigma\|\,|z+\sigma-\overline{w}|\sqrt{\frac{\pi}{\Im w}\left(1-|\Theta(w)|^2\right)}\sqrt{\frac{1}{4\pi\Im z}\left(1-|\Theta(z)|^2\right)}\frac{1}{1-|\Theta(z+\sigma)|\,|\Theta(w)|}\\ &\leq \frac{\|T_\sigma\|}{2}|\sigma+2h|\left(\frac{1}{\Im w\Im z}\frac{\left(1-|\Theta(w)|^2\right)\left(1-|\Theta(z)|^2\right)}{(1-|\Theta(w)|)^2}\right)^{1/2}\\ &\leq \|T_\sigma\|(\sigma+2h)\left(\frac{1}{h^2}\frac{(1-|\Theta(z)|)}{(1-|\Theta(w)|)}\right)^{1/2} \leq \|T_\sigma\|\left(\frac{\sigma}{h}+2\right) \quad .\end{aligned}$$

□

5. Necessary and sufficient condition with no zeros in the horizontal strip

In order to obtain a necessary and sufficient condition for the boundedness of T_σ we assume that $E(z)$ satisfies condition (4.14). Consequently the Hermite Biehler function $E(z)$ has the expression (4.15).

Theorem 4.12 Let $E(z)$ satisfies (4.14) and $\sigma < M/2$. The horizontal translation operator is bounded in $\mathcal{H}(E)$ if and only if

$$\frac{T_\sigma E}{E} \in H^\infty(\mathbb{C}^+) \quad . \tag{4.30}$$

In Theorem 4.10, we have already proved that condition (4.30) is sufficient for the boundedness of the horizontal translation. We want to prove that, when (4.14) holds, (4.30) is also necessary. We need the following lemma.

Lemma 4.13 Let Θ be a meromorphic inner function which satisfies (4.14) and $\sigma < M/2$. Then, $\forall\ x \in \mathbb{R}$ and $h > 0$

$$\frac{1 - |\Theta(x+ih)|}{1 - |\Theta(x+\sigma+ih)|} \le 32 \ . \tag{4.31}$$

Proof. First of all let $z = x + ih$; then, because of (1.5),

$$\log|\Theta(z)| \le -\sum_n \frac{2hb_n}{(x-a_n)^2 + (h+b_n)^2} \ . \tag{4.32}$$

Indeed,

$$\log|\Theta(z)|^2 \le \sum_n \log\left|\frac{z - z_n}{z - \overline{z_n}}\right|^2 = \sum_n \log\left(1 - \frac{4hb_n}{(x-a_n)^2 + (h+b_n)^2}\right) . \tag{4.33}$$

Considering that $\log(1-t) < -t,\ t \in (0,1)$, then

$$\log|\Theta(z)| \le -\sum_n \frac{2hb_n}{(x-a_n)^2 + (h+b_n)^2} \ .$$

Let us come back to the proof of (4.31). If $1 - |\Theta(x+\sigma+ih)| \ge 1/2$, then

$$\frac{1 - |\Theta(x+ih)|}{1 - |\Theta(x+\sigma+ih)|} \le 2\,(1 - |\Theta(x+ih)|) \le 2 \quad .$$

On the other hand, when $1 - |\Theta(x+\sigma+ih)| < 1/2$, since $2(1-t) \ge -\log(t)$, if $1/2 < t \le 1$, then

$$2\,(1 - |\Theta(x+\sigma+ih)|) \ge -\log|\Theta(x+\sigma+ih)| \ge +ah + 2h\sum_n \frac{b_n}{(x+\sigma-a_n)^2 + (h+b_n)^2}$$

because of (4.32). Now

$$(x+\sigma-a_n)^2+(h+b_n)^2=(x-a_n)^2+(h+b_n)^2+\sigma^2+2\sigma(x-a_n)\ .$$

If $\sigma^2+2\sigma(x-a_n)\le 0$, then $(x+\sigma-a_n)^2+(h+b_n)^2\le(x-a_n)^2+(h+b_n)^2$. On the other side, when $\sigma^2+2\sigma(x-a_n)>0$, then

$$\begin{aligned}(x+\sigma-a_n)^2+(h+b_n)^2 \le&(x-a_n)(x-a_n+2\sigma)+2(h+b_n)^2\\ &\le|x-a_n|\left(|x-a_n|+2\sigma\right)+2(h+b_n)^2\ ,\end{aligned}$$

since $\sigma<M/2$. If $|x-a_n|<2\sigma$, then

$$\begin{aligned}(x+\sigma-a_n)^2+(h+b_n)^2 \le&2\sigma(2\sigma+2\sigma)+2(h+b_n)^2\\ \le&4(b_n+h)^2\\ \le&4\left((x-a_n)^2+(b_n+h)^2\right)\ .\end{aligned}$$

If, on the other hand, $|x-a_n|\ge 2\sigma$, then

$$(x+\sigma-a_n)^2+(h+b_n)^2\le 2(x-a_n)^2+2(h+b_n)^2\le 2\left((x-a_n)^2+(b_n+h)^2\right)\ .$$

Consequently in any case, we state that

$$\begin{aligned}2\left(1-|\Theta(x+\sigma+ih)|\right)\ge h\left(a+2\sum_n\frac{b_n}{(x+\sigma-a_n)^2+(h+b_n)^2}\right)\\ \ge h\left(a+\frac{1}{2}\sum_n\frac{b_n}{(x-a_n)^2+(h+b_n)^2}\right)\ .\end{aligned}$$

Therefore

$$\frac{1-|\Theta(x+ih)|}{1-|\Theta(x+\sigma+ih)|}\le 4\frac{1-|\Theta(x+ih)|}{h\left(2a+\sum_n\frac{b_n}{(x-a_n)^2+(h+b_n)^2}\right)}\ . \tag{4.34}$$

If $h\sum_n\frac{b_n}{(x-a_n)^2+(h+b_n)^2}\ge 1/8$, then (4.34) <32. Now, If $h\sum_n\frac{b_n}{(x-a_n)^2+(h+b_n)^2}<1/8$, then, since $1-t\le-\log(t),\ t\in(0,1)$, using (4.33)

$$\begin{aligned}1-|\Theta(x+ih)|\le&-\log|\Theta(x+ih)|\\ =&ah+\frac{1}{2}\sum_n\sum_{k\ge1}\frac{1}{k}\left(\frac{4hb_n}{(x-a_n)^2+(h+b_n)^2}\right)^k\end{aligned}$$

$$
\begin{aligned}
&\leq ah + \frac{1}{2}\sum_{k\geq 1}\left(4h\sum_n \frac{b_n}{(x-a_n)^2+(h+b_n)^2}\right)^k \\
&\leq ah + 4h\sum_n \frac{b_n}{(x-a_n)^2+(h+b_n)^2} \\
&\leq 2ah + 4h\sum_n \frac{b_n}{(x-a_n)^2+(h+b_n)^2}\ .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1-|\Theta(x+ih)|}{1-|\Theta(x+\sigma+ih)|} &\leq 4\frac{1-|\Theta(x+ih)|}{h\left(2a+\sum_n \frac{b_n}{(x-a_n)^2+(h+b_n)^2}\right)} \\
&\leq 4\frac{2a+4\sum_n \frac{b_n}{(x-a_n)^2+(h+b_n)^2}}{2a+\sum_n \frac{b_n}{(x-a_n)^2+(h+b_n)^2}} \leq 4+12\ .
\end{aligned}
$$

□

We now prove Theorem 4.12.

Proof of Theorem 4.12. As we did in the proof of Theorem 4.11, we know that

$$
\begin{aligned}
&\left|\frac{T_\sigma(E)(z)}{E(z)}\right| \\
&\leq \|T_\sigma\|\,|z+\sigma-\overline{w}|\sqrt{\frac{\pi}{\Im w}\left(1-|\Theta(w)|^2\right)}\sqrt{\frac{1}{4\pi\Im z}\left(1-|\Theta(z)|^2\right)}\frac{1}{1-|\Theta(z+\sigma)|\,|\Theta(w)|}\ .
\end{aligned}
$$

However, in this case we consider $z = x+ih$ and $w = x+\sigma+ih$. Consequently,

$$
\begin{aligned}
\left|\frac{T_\sigma(E)(z)}{E(z)}\right| \leq& \frac{\|T_\sigma\|}{2}2h\left(\frac{1}{\Im w\Im z}\frac{\left(1-|\Theta(z)|^2\right)}{\left(1-|\Theta(w)|^2\right)}\right)^{1/2} \\
\leq& \|T_\sigma\| h\left(\frac{2}{h^2}\frac{(1-|\Theta(x+ih)|)}{(1-|\Theta(x+\sigma+ih)|)}\right)^{1/2} \\
\leq& 8\|T_\sigma\|\quad .
\end{aligned}
$$

Consequently, if T_σ is bounded, then

$$
\sup_{z\in\mathbb{C}^+}\left|\frac{T_\sigma(E)(z)}{E(z)}\right| < \infty\ .
$$

□

We note that (4.30) implies the necessary condition of Theorem 3.9 of [**15**], without any request about the phase of $E(z)$.

Observation We strongly believe that Theorem 4.12 is true also without the condition $\sigma < M/2$. However, in this case, another argument would be needed.

6. Boundedness of T_{σ_1} from the boundedness of T_σ

As proved in Theorem 4.12, under the assumption (4.14), the condition (4.30), that is $\frac{T_\sigma E}{E} \in H^\infty(\mathbb{C}^+)$ is equivalent to the boundedness of T_σ.

Let the Hermite Biehler function $E(z)$ satisfy condition (4.14). In addition to that, we assume also that the Hermite Biehler function $E(z)$ has the following simplified expression:

$$(4.35) \qquad E(z) := E(0)e^{Az}e^{-iaz}\prod_n \left(1-\frac{z}{\overline{z_n}}\right) \qquad \text{where } z_n = a_n + ib_n \in \mathbb{C}^+ \ .$$

In fact, in (4.35) we are considering only Hermite Biehler functions of order less or equal than 1.

With this new hypothesis, we prove the boundedness of T_{σ_1} from that of T_σ, if $0<\sigma_1<\sigma$.

Theorem 4.14 Let $E(z)$ satisfy condition (4.14) and have the expression (4.35). If $0<\sigma_1<\sigma$, and $\sigma \le M/2$, then also T_{σ_1} is bounded.

Proof. To prove the theorem, we want to check that

$$(4.36) \qquad \frac{T_{\sigma_1}E}{E} \in H^\infty(\mathbb{C}^+) \ .$$

This is true since

$$\begin{aligned}
\left|\frac{T_{\sigma_1}E(z)}{E(z)}\right|^2 &= e^{2A\sigma_1}\left|\frac{\prod_n\left(1-\frac{z+\sigma_1}{\overline{z_n}}\right)}{\prod_n\left(1-\frac{z}{\overline{z_n}}\right)}\right|^2 \\
&= e^{2A\sigma_1}\prod_n \frac{(a_n-x-\sigma_1)^2+(y+b_n)^2}{(a_n-x)^2+(y+b_n)^2} \\
&\le e^{2A\sigma_1}\prod_n \frac{(a_n-x-\sigma)^2+(y+b_n)^2}{(a_n-x)^2+(y+b_n)^2}\ \sup_n \frac{(a_n-x-\sigma_1)^2+(y+b_n)^2}{(a_n-x-\sigma)^2+(y+b_n)^2} \ .
\end{aligned}$$

We check that the last factor is bounded. Indeed if $y+b_n \ge |a_n-x-\sigma_1|$, then

$$\sup_n \frac{(a_n-x-\sigma_1)^2+(y+b_n)^2}{(a_n-x-\sigma)^2+(y+b_n)^2} \le \sup_n \frac{2(y+b_n)^2}{(y+b_n)^2} = 2 \ .$$

On the other hand, if $y+b_n < |a_n-x-\sigma_1|$, then

$$\begin{aligned}
&M < |a_n-x-\sigma_1| \le |a_n-x-\sigma| + (\sigma-\sigma_1) \qquad \text{and} \\
&|a_n-x-\sigma| > M-(\sigma-\sigma_1) \ge \sigma+\sigma_1 \ .
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{(a_n-x-\sigma_1)^2+(y+b_n)^2}{(a_n-x-\sigma)^2+(y+b_n)^2} =& \frac{(a_n-x-\sigma+(\sigma-\sigma_1))^2+(y+b_n)^2}{(a_n-x-\sigma)^2+(y+b_n)^2} \\
\le& 1+\frac{(\sigma-\sigma_1)^2}{(a_n-x-\sigma)^2+(y+b_n)^2}+\frac{2(\sigma-\sigma_1)\,|a_n-x-\sigma|}{(a_n-x-\sigma)^2+(y+b_n)^2}
\end{aligned}$$

$$\leq \left(1 + \frac{\sigma^2}{M^2} + \frac{2(\sigma - \sigma_1)}{|a_n - x - \sigma| \left(1 + \frac{M^2}{(a_n - x - \sigma)^2}\right)}\right)$$

$$\leq \left(1 + \frac{\sigma^2}{M^2} + \frac{2\sigma}{\sigma + \sigma_1}\right) < 4.$$

Therefore

$$\left|\frac{T_{\sigma_1}E(z)}{E(z)}\right| \leq 2e^{A(\sigma_1 - \sigma)} \left|\frac{T_\sigma E(z)}{E(z)}\right|$$

and we have obtained the statement. □

Previously we proved a similar result for the vertical translation operator. However, in that case the computations were easier since the Hermite Biehler function in (4.16) belongs to the Polya class (4.4).

From Theorem 4.14, we highlight a condition sufficient for the boundedness of any translation T_σ.

Corollary 4.15 Let $E(z)$ satisfy condition (4.14) and have the expression (4.35). If there exists a $\sigma_0 \leq M/2$ such that T_{σ_0} is bounded, then any other horizontal translation T_σ is bounded.

Proof. Let us suppose $\sigma = n\sigma_0 + r$, with $r < \sigma_0$ and $n \in \mathbb{N}$. Then, thanks to the Theorem 4.14, also the horizontal translation T_r is bounded. Consequently any operator T_σ is bounded, since it is the composition of bounded operators, that is

$$T_\sigma := T_r \circ T_{\sigma_0}^n \quad .$$

□

In the above situation we can even estimate the norm of T_σ. Indeed, if $\sigma = n\sigma_0 + r$, then

$$\|T_\sigma\| \leq \left\|\frac{T_\sigma E}{E}\right\|_{H^\infty} \leq \left\|\frac{T_{\sigma_0}E}{E}\right\|_{H^\infty}^n \left\|\frac{T_r E}{E}\right\|_{H^\infty} \quad .$$

5. Continuous embedding between p-de Branges space

In this chapter, we study which conditions the Hermite Biehler function $E(z)$ must satisfy so that the embedding operator

$$\iota : \mathcal{H}^p(E) \hookrightarrow \mathcal{H}^q(E) \quad \text{when } 0 < p < q \leq \infty \tag{5.1}$$

is continuous.

This chapter is divided in three short sections. In the next section we prove Theorem 5.1, the necessary and sufficient condition for the continuity of the embedding ι when $p > 1$, and we highlight where our proof is different from that of [34]. In the third section, we study the necessary condition for $p = 1$, proving Theorem 5.7. Finally, in the last section, we prove some other partial results, which we write down for completeness. We also give an overview of what happens to the case $p < 1$, even if these spaces behave very differently (1.29).

The results of this chapter are published in [16].

1. Proof in the case $p > 1$

In this section we fix our attention on the necessary and sufficient condition for the continuity of the embedding operator when $p > 1$. The proof that we present works directly at the level of the de Branges spaces.

Theorem 5.1 Let $1 < p < q \leq \infty$ and let $E(z)$ be a Hermite Biehler function. The embedding operator $\iota : \mathcal{H}^p(E) \hookrightarrow \mathcal{H}^q(E)$ is continuous if and only if $\|\phi'\|_\infty < \infty$.

Let us start with the sufficiency. We need this preliminary lemma.

Lemma 5.2 Let $\phi(x)'$, the derivative of the Hermite Biehler function $E(z)$ (1.24) be bounded on $\mathbb{R}$ and let $p \geq 1$. Then for every $f \in \mathcal{H}^p(E)$ and for every $x \in \mathbb{R}$

$$\left|\frac{f(x)}{E(x)}\right| \leq C(E,p)\, \|f\|_{\mathcal{H}^p} \left(1 + \frac{1}{\inf_{\xi\in\mathbb{R}, 0<\eta<\tau} |\Theta(\xi + i\eta)|^p}\right)^{1/p}, \tag{5.2}$$

where $\Theta(z)$ is defined as in (1.7) and τ depends only on $E(z)$.

PROOF. Thanks to Lemma 1 of [34], if $\phi' \in L^\infty(\mathbb{R})$, then for some $\delta > 0$,

$$\inf\{|\Theta(z)| : 0 < \Im z < \delta\} > 0 \quad \text{where } \Theta = E^\#/E \;. \tag{5.3}$$

Due to the subharmonicity of $|f/E|^p$ in a strip which contains the real line, we know that

$$\left|\frac{f(x)}{E(x)}\right| \leq \left(\frac{1}{\pi\tau^2}\int_{B_\tau(x)}\left|\frac{f(\xi)}{E(\xi)}\right|^p d\xi\right)^{1/p} \qquad \text{if } \tau < \delta \;.$$

Thus, we obtain that

$$\begin{aligned}
\left|\frac{f(x)}{E(x)}\right| &\leq \left(\frac{1}{\pi\tau^2}\int_\tau^\tau\int_{\mathbb{R}}\left|\frac{f(x+\xi+i\eta)}{E(x+\xi+i\eta)}\right|^p d\xi d\eta\right)^{1/p} \\
&= \left[\frac{1}{\pi\tau^2}\left(\int_0^\tau\int_{\mathbb{R}}\left|\frac{f(\xi+i\eta)}{E(\xi+i\eta)}\right|^p d\xi d\eta + \int_{-\tau}^0\int_{\mathbb{R}}\left|\frac{f(\xi+i\eta)}{E(\xi+i\eta)}\right|^p d\xi d\eta\right)\right]^{1/p} \\
&\leq \left[\frac{1}{\pi\tau^2}\left(\tau\,\|f\|_{\mathcal{H}^p}^p + \int_{-\tau}^0\int_{\mathbb{R}}\left|\frac{f(\xi+i\eta)}{E(\xi-i\eta)}\right|^p\left|\frac{E(\xi-i\eta)}{E(\xi+i\eta)}\right|^p d\xi d\eta\right)\right]^{1/p} \\
&\leq \left[\frac{1}{\pi\tau^2}\left(\tau\,\|f\|_{\mathcal{H}^p}^p + \sup_{\xi,\eta}\left|\frac{E(\xi-i\eta)}{E(\xi+i\eta)}\right|^p\int_{-\tau}^0\int_{\mathbb{R}}\left|\frac{f^\#(\xi-i\eta)}{E(\xi-i\eta)}\right|^p d\xi d\eta\right)\right]^{1/p} \\
&\leq \left[\frac{1}{\pi\tau^2}\left(\tau\,\|f\|_{\mathcal{H}^p}^p + \sup_{\xi\in\mathbb{R},\ 0<-\eta<\tau}\frac{1}{\left|\frac{E^\#(\xi-i\eta)}{E(\xi-i\eta)}\right|^p}\tau\,\|f^\#\|_{\mathcal{H}^p}^p\right)\right]^{1/p} \\
&= \left(\frac{1}{\pi\tau}\right)^{1/p}\|f\|_{\mathcal{H}^p}\left(1+\frac{1}{\inf_{\xi\in\mathbb{R},\ 0<\eta<\tau}|\Theta(\xi+i\eta)|^p}\right)^{1/p} .
\end{aligned}$$

□

PROOF SUFFICIENCY 5.1. At this point, the proof is an easy consequence of (5.2). We recall that, thanks to Smirnov Theorem [29], if $f \in H^p(\mathbb{C}^+) \cap L^q(\mathbb{R})$, then $f \in H^q(\mathbb{C}^+)$. Consequently we have just to check the relation between the two norms. For every $f \in \mathcal{H}^p$, we note that

$$\begin{aligned}
\left(\int_{\mathbb{R}}\left|\frac{f(x)}{E(x)}\right|^q dx\right)^{1/q} &\leq \left(\int_{\mathbb{R}}\left|\frac{f(x)}{E(x)}\right|^p dx\right)^{1/q}\left(\sup_{x\in\mathbb{R}}\left|\frac{f(x)}{E(x)}\right|^{q-p} dx\right)^{1/q} \\
&\leq \|f\|_{\mathcal{H}^p}^{p/q}\, C(E,p,q)\,\|f\|_{\mathcal{H}^p}^{1-p/q} \leq C\,\|f\|_{\mathcal{H}^p} \;,
\end{aligned}$$

with the obvious changes if $q = \infty$. □

Observation We point out that the proof of the sufficiency condition is similar to that of [34]. The main difference is the use of Lemma 5.2, instead of some estimates on the reproducing kernel. We think that this lemma is more similar to the techniques used to prove the continuity of the embedding in Bernstein spaces.

We move on to the proof of the necessary condition for the continuity of the embedding operator $\mathcal{H}^p(E) \hookrightarrow \mathcal{H}^q(E)$ when $1 < p < q \le \infty$. We need some preliminaries estimates.

Proposition 5.3 Let $\{z_n = a_n - ib_n\}_{n\in\mathbb{Z}}$ be the zeros of the Hermite Biehler function $E(z)$. If $\iota : \mathcal{H}^p(E) \hookrightarrow \mathcal{H}^q(E)$ is continuous, then there exists $M > 0$ such that (4.14) holds.

PROOF. Let us consider

$$k_{\overline{z_n}}(t) = \frac{1}{2\pi i}\frac{E(t)E^\#(z_n) - E^\#(t)E(z_n)}{z_n - t} = \frac{1}{2\pi i}\frac{E(t)E^\#(z_n)}{z_n - t} \ ,$$

since $E(z_n) = 0$ for every $n \in \mathbb{Z}$. Since ι is continuous

$$\begin{aligned}\frac{\left|E^\#(z_n)\right|}{2\pi}\left(\int_{\mathbb{R}} \frac{1}{|z_n - t|^q}dt\right)^{1/q} &= \|k_{\overline{z_n}}\|_{\mathcal{H}^q} \\ &\le C\,\|k_{\overline{z_n}}\|_{\mathcal{H}^p} = C\frac{\left|E^\#(z_n)\right|}{2\pi}\left(\int_{\mathbb{R}} \frac{1}{|z_n - t|^p}dt\right)^{1/p}\end{aligned}$$

and consequently

$$b_n^{1/q-1} \le C' b_n^{1/p-1} \ , \qquad \text{that is} \qquad b_n^{1/q-1/p} \le C' \ ,$$

for every $n \in \mathbb{Z}$, which implies that (4.14) holds. $\square$

In the following estimates we will repeatedly use condition (4.14) and the constant M will be used for the lower bound of imaginary parts of the zeros of $E(z)$.

Lemma 5.4 Let $\iota : \mathcal{H}^p(E) \hookrightarrow \mathcal{H}^q(E)$ be continuous. It holds that

$$\phi'(s) \le 4\phi'(t) \qquad \text{for every } s \text{ such that } |s - t| \le M \ . \tag{5.4}$$

PROOF. We know form (1.24) that

$$\begin{aligned}\phi'(s) = \frac{a}{2} + \sum_n \frac{b_n}{(a_n - s)^2 + b_n^2} &\le \frac{a}{2} + \sum_n \frac{b_n}{(a_n - t)^2 + b_n^2}\ \sup_n \frac{(a_n - t)^2 + b_n^2}{(a_n - s)^2 + b_n^2} \\ &\le \frac{a}{2} + \sum_n \frac{b_n}{(a_n - t)^2 + b_n^2}\ \sup_n\left(1 + \frac{(s-t)^2 + 2\,|(s-t)(a_n - s)|}{(a_n - s)^2 + b_n^2}\right) \ ,\end{aligned}$$

where a is defined as in (1.24). If $|a_n - s| < b_n$

$$\sup_n\left(1 + \frac{(s-t)^2 + 2\,|(s-t)(a_n - s)|}{(a_n - s)^2 + b_n^2}\right) \le \sup_n\left(1 + \frac{M^2 + 2Mb_n}{b_n^2}\right) \le 4 \ .$$

On the other hand, if $|a_n - s| \ge b_n \ge M$

$$\sup_n\left(1 + \frac{(s-t)^2 + 2\,|(s-t)(a_n - s)|}{(a_n - s)^2 + b_n^2}\right)$$

$$\leq \sup_n \left(1 + \frac{M^2 + 2M\,|a_n - s|}{(a_n - s)^2}\right) \leq \sup_n \left(1 + \frac{(a_n - s)^2 + 2(a_n - s)_n^2}{(a_n - s)^2}\right) = 4\,.$$

Therefore in any cases, we have (5.4). ☐

Lemma 5.5 Let $\iota : \mathcal{H}^p(E) \hookrightarrow \mathcal{H}^q(E)$ be continuous. It holds that

$$\phi'(s) \geq \frac{1}{2}\phi'(t) \quad \text{for every } s \text{ such that } |s - t| \leq \frac{M}{4}. \tag{5.5}$$

Proof. We know from (1.24) that

$$\phi'(s) = \frac{a}{2} + \sum_n \frac{b_n}{(a_n - s)^2 + b_n^2} \geq \frac{a}{2} + \sum_n \frac{b_n}{(a_n - t)^2 + b_n^2} \inf_n \frac{(a_n - t)^2 + b_n^2}{(a_n - s)^2 + b_n^2}\,,$$

where a is defined as in (1.24). If $|a_n - s| \leq b_n$

$$\inf_n \frac{(a_n - t)^2 + b_n^2}{(a_n - s)^2 + b_n^2} \geq \inf_n \frac{b_n^2}{2b_n^2} = \frac{1}{2}\,.$$

On the other hand, if $|a_n - s| > b_n$, and we know also that $b_n \geq M > M/4 > |s - t|$ due to (4.14) and (5.5),

$$\begin{aligned}
\inf_n \left(1 + \frac{(s-t)^2}{(a_n - s)^2 + b_n^2} + \frac{2(s-t)(a_n - s)}{(a_n - s)^2 + b_n^2}\right) &\geq \inf_n \left(1 - \frac{2\,|s-t|\,|a_n - s|}{(a_n - s)^2 + b_n^2}\right)\\
&\geq \inf_n \left(1 - \frac{M\,|a_n - s|}{2(a_n - s)^2}\right)\\
&= \inf_n \left(1 - \frac{M}{2\,|a_n - s|}\right) \geq 1 - \frac{1}{2} = \frac{1}{2}\,.
\end{aligned}$$

Therefore in any cases we have (5.5). ☐

The two lemmas above are fundamental for the proof of the following proposition.

Proposition 5.6 Let $\iota : \mathcal{H}^p(E) \hookrightarrow \mathcal{H}^q(E)$ be continuous. If $\|\phi'\|_\infty = \infty$, there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ which goes to infinity such that

$$\int_{\mathbb{R}} \left|\frac{\sin(\phi(x) - \phi(t_n))}{x - t_n}\right|^p dx \asymp \phi'(t_n)^{p-1}\,. \tag{5.6}$$

Proof. Since $\|\phi'\|_\infty = \infty$, there exist $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \phi'(t_n) = \infty\,.$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}} \left|\frac{\sin(\phi(x) - \phi(t_n))}{x - t_n}\right|^p dx &= \int_{\mathbb{R}} \left|\frac{\sin(\phi(x + t_n) - \phi(t_n))}{x}\right|^p dx\\
&\geq \int_{-M/4}^{M/4} \left|\frac{\sin(\phi(x + t_n) - \phi(t_n))}{x}\right|^p dx\,.
\end{aligned}$$

If

$$(5.7) \qquad |x| \le \frac{\pi}{24\phi'(t_n)} \ ,$$

then

$$|\phi(x+t_n) - \phi(t_n)| \le \phi'(s)\,|x| \le 4\phi'(t_n)\,|x| \le \frac{\pi}{6} \quad \text{where } |s - t_n| \le |x| \le M/4 \ .$$

Therefore, since

$$|\sin(x)| \ge \frac{3}{\pi}\,|x| \ \text{ for } \ |x| \le \frac{\pi}{6} \ ,$$

we have that

$$|\sin(\phi(x+t_n) - \phi(t_n))| \ge \frac{3}{\pi}\,|\phi(x+t_n) - \phi(t_n)| \ \text{ when } x \text{ satisfies (5.7)} \ .$$

We can assume that $\frac{\pi}{24\phi'(t_n)} < M/4$ if t_n is large enough. Thus

$$\begin{aligned}
\int_{\mathbb{R}} \left|\frac{\sin(\phi(x) - \phi(t_n))}{x - t_n}\right|^p dx &\ge \int_{-\frac{\pi}{24\phi'(t_n)}}^{\frac{\pi}{24\phi'(t_n)}} \left|\frac{3}{\pi}\phi'(t_n + s(x))\right|^p dx \\
&\ge \int_{-\frac{\pi}{24\phi'(t_n)}}^{\frac{\pi}{24\phi'(t_n)}} \left|\frac{3}{2\pi}\phi'(t_n)\right|^p dx = \frac{1}{8}\left(\frac{3}{2\pi}\right)^{p-1} \phi'(t_n)^{p-1} \ .
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_{\mathbb{R}} \left|\frac{\sin(\phi(x) - \phi(t_n))}{x - t_n}\right|^p dx &= \int_{\mathbb{R}} \left|\frac{\sin(\phi(x+t_n) - \phi(t_n))}{x}\right|^p dx \\
&\le \int_{-\frac{\pi}{24\phi'(t_n)}}^{\frac{\pi}{24\phi'(t_n)}} \left|\frac{\sin(\phi(x+t_n) - \phi(t_n))}{x}\right|^p + 2\int_{\frac{\pi}{24\phi'(t_n)}}^{\infty} \frac{1}{x^p} \\
&\le (4\phi'(t_n))^p \frac{\pi}{12\phi'(t_n)} + \frac{(24)^{p-1}2}{(p-1)\pi^{p-1}}\phi'(t_n)^{p-1} \\
&\le C\phi'(t_n)^{p-1} \ ,
\end{aligned}$$

since we assumed that $\frac{\pi}{24\phi'(t_n)} < M$ for t_n large enough. $\square$

Proof Necessity 5.1. First of all we suppose that $\iota : \mathcal{H}^p(E) \hookrightarrow \mathcal{H}^\infty(E)$ is continuous. Assume towards a contradiction that $\|\phi'\|_\infty = \infty$. Then, as proved in Proposition 5.6 there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that (5.6) holds. Thanks to the boundedness of the embedding operator, we know that

$$\begin{aligned}
\|k_{t_n}\|_{\mathcal{H}^\infty} \le C\,\|k_{t_n}\|_{\mathcal{H}^p} &= C\left(\int_{\mathbb{R}} \left|\frac{\sin(\phi(x) - \phi(t_n))}{(x - t_n)}\right|^p dx\right)^{1/p} |E(t_n)| \\
&\le \tilde{C}(\phi'(t_n))^{1-1/p}\,|E(t_n)| \ ,
\end{aligned}$$

where the first equality has been proved in (1.25). Therefore, $\forall\, t_n$, because of (1.26)

$$\tilde{C}\phi'(t_n)^{1-1/p}\,|E(t_n)| \geq \|k_{t_n}\|_{\mathcal{H}^\infty} \geq \left|\frac{k_{t_n}(t_n)}{E(t_n)}\right| = \frac{\phi'(t_n)}{\pi}\,|E(t_n)| \ .$$

Consequently we obtain that

$$|\phi'(t_n)| \leq C \ ,$$

but it is impossible. For this reason $\|\phi'\|_\infty < \infty$.

Let us move on to the general statement for $q < \infty$. As done before, let us assume that $\|\phi'\|_\infty = \infty$. Then, as proved in Proposition 5.6 there exist $\{t_n\}_{n\in\mathbb{N}}$ such that (5.6) holds. We know that for every t_n, $\|k_{t_n}\|_{\mathcal{H}^q} \leq C\,\|k_{t_n}\|_{\mathcal{H}^p}$ and consequently,

$$\begin{aligned} C^*\phi'(t_n)^{1-1/q} &\leq \left(\int_{\mathbb{R}} \left|\frac{\sin(\phi(x)-\phi(t_n))}{x-t_n}\right|^q dx\right)^{1/q} \\ &\leq C\left(\int_{\mathbb{R}} \left|\frac{\sin(\phi(x)-\phi(t_n))}{x-t_n}\right|^p dx\right)^{1/p} \leq \tilde{C}\phi'(t_n)^{1-1/p} \end{aligned}$$

that is

$$\phi'(t_n)^{1/p-1/q} \leq \tilde{C}^* \ . \tag{5.8}$$

However this last inequality cannot hold if $\phi'(t_n)$ goes to infinity when n goes to infinity. For this reason $\|\phi'(t_n)\| < \infty$. ☐

Observation The proof of the necessity condition is more complicated than the same proof of [34]. However, it can be generalized to the case $1/2 < p \leq 1$ as we show in the next section.

2. Proof in the case $p = 1$

In this section we prove what happens when $p = 1$.

Theorem 5.7 Let $\{z_n = a_n - ib_n\}_{n\in\mathbb{Z}}$ be the zeros of the Hermite Biehler function $E(z)$. Let the derivative of the phase function $\phi'(x)$ be uniformly bounded away from zero, that is

$$\inf_{x\in\mathbb{R}} \phi'(x) \geq \delta > 0 \tag{5.9}$$

and let

$$\inf_{n\in\mathbb{Z}} |b_n| \geq M > 0 \ . \tag{5.10}$$

Let $1 < q \leq \infty$. Then, $\iota : \mathcal{H}^1(E) \hookrightarrow \mathcal{H}^q(E)$ is continuous if and only if $\|\phi'\|_\infty < \infty$.

The sufficiency of Theorem 5.7 is equal to the proof of the sufficiency of Theorem 5.1 and for this reason we omit the proof.

It is clear that (5.9) implies that for every $t,\ s \in \mathbb{R}$ such that $\phi(t) = \phi(s) + \pi$,

$$t - s = \pi/\phi'(\zeta) \leq \pi/\delta \ , \qquad \text{where} \qquad s \leq \zeta \leq t \ .$$

In the proof of the following lemma, we use the relations (5.4) and (5.5).

Lemma 5.8 Let us assume that the function $E(z)$ satisfy (5.9) and (5.10). Furthermore, let us consider

$$s,\ t \in \mathbb{R} \text{ so that } \phi(t) = \phi(s) + \pi \ .$$

Then

$$\int_{\mathbb{R}} \left| \frac{\sin(\phi(x) - \phi(s))(t-s)}{(x-t)(x-s)} \right| dx \leq C\left(|\log(\phi'(t))| + |\log(\phi'(s)| + K\right) \ . \tag{5.11}$$

In particular, if $\{s_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ and $\phi'(s_n) \geq \phi'(t_n) \to \infty$,

$$\int_{\mathbb{R}} \left| \frac{\sin(\phi(x) - \phi(s_n))(t-s_n)}{(x-t_n)(x-s_n)} \right| dx \leq C' \log(\phi'(s_n)) \ .$$

Proof. We have to consider three possible situations. First of all, let us assume that

$$\frac{\pi}{24}\frac{1}{\phi'(s)} < \frac{(t-s)}{2} \ , \qquad \frac{\pi}{24}\frac{1}{\phi'(t)} < \frac{(t-s)}{2} \ . \tag{5.12}$$

We split the integral in (5.11) in five parts:

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{\sin(\phi(x) - \phi(s))(t-s)}{(x-t)(x-s)} \right| dx &= \int_{\mathbb{R}} \left| \frac{\sin(\phi(s+\zeta) - \phi(s))(t-s)}{\zeta(t-s-\zeta)} \right| d\zeta \\ &= \int_{-\infty}^{-\frac{\pi}{24}\frac{1}{\phi'(s)}} + \int_{-\frac{\pi}{24}\frac{1}{\phi'(s)}}^{\frac{\pi}{24}\frac{1}{\phi'(s)}} + \int_{\frac{\pi}{24}\frac{1}{\phi'(s)}}^{t-s-\frac{\pi}{24}\frac{1}{\phi'(t)}} + \int_{t-s-\frac{\pi}{24}\frac{1}{\phi'(t)}}^{t-s+\frac{\pi}{24}\frac{1}{\phi'(t)}} + \int_{t-s+\frac{\pi}{24}\frac{1}{\phi'(t)}}^{\infty} \\ &= A + B + C + D + E \ . \end{aligned} \tag{5.13}$$

We know that

$$\begin{aligned} A = \int_{-\infty}^{-\frac{\pi}{24}\frac{1}{\phi'(s)}} \left| \frac{\sin(\phi(s+\zeta) - \phi(s))(t-s)}{\zeta(t-s-\zeta)} \right| d\zeta &\leq \int_{-\infty}^{-\frac{\pi}{24}\frac{1}{\phi'(s)}} \left| \frac{1}{\zeta} + \frac{1}{t-s-\zeta} d\zeta \right| \\ &\leq \left| \log\left(\frac{\pi}{24}\frac{1}{\phi'(s)} \right) \right| + |\log(C\pi/\delta)| \ . \end{aligned}$$

After a change of variables, we have the same estimate for E:

$$E \leq \log(C\phi'(t)) + |\log(C\pi/\delta)| \ .$$

Let us move to B; we know that

$$B = \int_{-\frac{\pi}{24}\frac{1}{\phi'(s)}}^{\frac{\pi}{24}\frac{1}{\phi'(s)}} \left| \frac{\sin(\phi(s+\zeta) - \phi(s))(t-s)}{\zeta(t-s-\zeta)} \right| d\zeta \le \frac{\pi}{12}\frac{1}{\phi'(s)} 4\phi'(s)2 = \pi\frac{2}{3} \ .$$

The same calculations work also for D, for which we obtain the same estimate. Finally

$$\begin{aligned} C &\le \int_{\frac{\pi}{24}\frac{1}{\phi'(s)}}^{t-s-\frac{\pi}{24}\frac{1}{\phi'(t)}} \left| \frac{1}{\zeta} + \frac{1}{t-s-\zeta} \right| d\zeta \\ &\le \left| \log\left(\frac{\pi}{24}\frac{1}{\phi'(s)} \right) \right| + \left| \log\left(t-s-\frac{\pi}{24}\frac{1}{\phi'(s)} \right) \right| \\ &\quad + \left| \log\left(\frac{\pi}{24}\frac{1}{\phi'(t)} \right) \right| + \left| \log\left(t-s-\frac{\pi}{24}\frac{1}{\phi'(t)} \right) \right| \\ &\le 2\log(C\phi'(t)) + 2\log(C'\phi'(s)) + 2|\log(\pi/\delta)| \ . \end{aligned}$$

Considering all the estimates, we proved relation (5.11).

Secondly, we assume that

$$\frac{\pi}{24}\frac{1}{\phi'(s)} \ge \frac{(t-s)}{2} \ , \qquad \frac{\pi}{24}\frac{1}{\phi'(t)} \ge \frac{(t-s)}{2} \ . \tag{5.14}$$

We split the integral in (5.11) in four parts:

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{\sin(\phi(x) - \phi(s))(t-s)}{(x-s)(x-t)} \right| dx &= \int_{\mathbb{R}} \left| \frac{\sin(\phi(s+\zeta) - \phi(s))(t-s)}{\zeta(t-s-\zeta)} \right| d\zeta \\ &= \int_{-\infty}^{-\frac{(t-s)}{2}} + \int_{-\frac{(t-s)}{2}}^{\frac{(t-s)}{2}} + \int_{\frac{(t-s)}{2}}^{\frac{3(t-s)}{2}} + \int_{\frac{3(t-s)}{2}}^{\infty} = A + B + C + D \ . \end{aligned} \tag{5.15}$$

We know that

$$\begin{aligned} A &\le \int_{-\infty}^{-\frac{(t-s)}{2}} \left| \frac{1}{\zeta} + \frac{1}{t-s-\zeta} \right| d\zeta \le \left| \log\left(\frac{1}{2}(t-s) \right) \right| + \left| \log\left(\frac{3}{2}(t-s) \right) \right| \\ &\le 2\left| \log\left(C\frac{\pi}{\phi'(p)} \right) \right| \le 2\left| \log\left(C\frac{3\pi}{\phi'(s)} \right) \right| \ . \end{aligned}$$

After a change of variables, we have the same estimate for D:

$$D \le 2\log\left(C\phi'(t)\right) \ .$$

Let us work on B; we know that

$$B = \int_{-\frac{(t-s)}{2}}^{\frac{(t-s)}{2}} \left| \frac{\sin(\phi(s+\zeta) - \phi(s))(t-s)}{\zeta(t-s-\zeta)} \right| d\zeta \le (t-s)\phi'(p)2 \le \frac{\pi}{12}\frac{1}{\phi'(s)} 4\phi'(s)2 = \frac{2}{3}\pi \ .$$

Analougsly we estimate

$$C \le \frac{\pi}{12}\frac{1}{\phi'(t)} 4\phi'(t)2 = \frac{2}{3}\pi \ .$$

Considering all the estimates, we proved relation (5.11).

Finally, we assume that

$$\text{(5.16)} \qquad \frac{\pi}{24}\frac{1}{\phi'(s)} < \frac{(t-s)}{2}\ , \qquad \frac{\pi}{24}\frac{1}{\phi'(t)} \geq \frac{(t-s)}{2}\ .$$

We split the integral in (5.11) in five parts:

$$\text{(5.17)} \qquad \int_{\mathbb{R}} \left| \frac{\sin(\phi(x)-\phi(s))(t-s)}{(x-t)(x-s)} \right| dx = \int_{\mathbb{R}} \left| \frac{\sin(\phi(s+\zeta)-\phi(s))(t-s)}{\zeta(t-s-\zeta)} \right| d\zeta$$
$$= \int_{-\infty}^{-\frac{\pi}{24}\frac{1}{\phi'(s)}} + \int_{-\frac{\pi}{24}\frac{1}{\phi'(s)}}^{\frac{\pi}{24}\frac{1}{\phi'(s)}} + \int_{\frac{\pi}{24}\frac{1}{\phi'(s)}}^{(t-s)/2} + \int_{(t-s)/2}^{3(t-s)/2} + \int_{3(t-s)/2}^{\infty} = A+B+C+D+E\ .$$

The only integral we have not already considered, is C:

$$\begin{aligned} C &\leq \int_{\frac{\pi}{24}\frac{1}{\phi'(s)}}^{(t-s)/2} \left| \frac{1}{\zeta} + \frac{1}{t-s-\zeta} \right| d\zeta \\ &\leq 2\log\left(\frac{t-s}{2}\right) + 2\left|\log\left(\frac{\pi}{24}\frac{1}{\phi'(s)}\right)\right| \left|\log\left(t-s-\frac{\pi}{24}\frac{1}{\phi'(s)}\right)\right| \\ &\leq 2\log\left(C\pi/\delta\right) + 2\left|\log\left(\frac{\pi}{24}\frac{1}{\phi'(s)}\right)\right|\ . \end{aligned}$$

Considering all the estimates, we proved relation (5.11) also in this final case. □

Proof Necessity 5.7. Towards a contradiction, let us assume that $\|\phi'\|_\infty = \infty$. Then, there exists a sequence of points $\{s_n\}$ such that $\phi'(s_n) \to \infty$ as n goes to infinity and $E(z)$ satisfies (5.9). First of all, we consider the function

$$\Lambda_n(x) := k_{s_n}(x) + \frac{|E(s_n)|}{|E(t_n)|}k_{t_n}(x) = \frac{|E(x)|\,|E(s_n)|}{\pi}\frac{\sin(\phi(x)-\phi(s_n))(t_n-s_n)}{(x-t_n)(x-s_n)}\ ,$$

where $s_n,\ t_n \in \mathbb{R}$ so that $\phi(t_n) = \phi(s_n)+\pi$. Thanks to Lemma 5.8,

$$\|\Lambda_n\|_{\mathcal{H}^1} \leq C\,|E(s_n)|\log(\phi'(s_n)),\ \text{when } \phi'(s_n) > \phi'(t_n)\ .$$

If, on the other hand, $\phi'(s_n) \leq \phi'(t_n)$, we consider

$$\tilde{\Lambda}_n(x) := k_{t_n}(x) + \frac{|E(t_n)|}{|E(s_n)|}k_{s_n}(x) = \frac{|E(x)|\,|E(t_n)|}{\pi}\frac{\sin(\phi(x)-\phi(t_n))(t_n-s_n)}{(x-t_n)(x-s_n)}\ ,$$

for which

$$\left\|\tilde{\Lambda}_n\right\|_{\mathcal{H}^1} \leq C\,|E(t_n)|\log(\phi'(t_n)) \text{ holds.}$$

For simplicity, from now on, we assume $\phi'(s_n) \geq \phi'(t_n)$.

We start by the case $q = \infty$. Thanks to the boundedness of the embedding operator, we know that

$$\|\Lambda_n\|_{\mathcal{H}^\infty} \leq C \|\Lambda_n\|_{\mathcal{H}^1} = C' |E(s_n)| \log(\phi'(s_n)) \; .$$

Therefore, $\forall \; s_n$,

$$C' |E(s_n)| \log(\phi'(s_n)) \geq \|\Lambda_n\|_{\mathcal{H}^\infty} \geq \left| \frac{\Lambda_n(s_n)}{E(s_n)} \right| = \frac{\phi'(s_n)}{\pi} |E(s_n)| \; ,$$

which is impossible. Consequently,

$$|\phi'(s_n)| \leq C \; , \text{ and } \|\phi'\|_\infty < \infty \; .$$

Let us move on to the general statement for $q < \infty$. We know that for every n, $\|\Lambda_n\|_{\mathcal{H}^q} \leq C \|\Lambda_n\|_{\mathcal{H}^1}$. We know also that, when n is large enough,

$$\begin{aligned}
\|\Lambda_n\|^q_{\mathcal{H}^q} &= |E(s_n)|^q \int_{\mathbb{R}} \left| \frac{\sin(\phi(x) - \phi(s_n))(s_n - t_n)}{(x - s_n)(x - t_n)} \right|^q dx \\
&\geq \int_{-\frac{\pi}{24\phi'(s_n)}}^{\frac{\pi}{24\phi'(s_n)}} \left| \frac{3}{\pi} \phi'(s_n + \zeta(x)) \right|^q dx \left| \frac{t_n - s_n}{t_n - s_n + \frac{\pi}{24\phi'(s_n)}} \right|^q \\
&\geq \int_{-\frac{\pi}{24\phi'(s_n)}}^{\frac{\pi}{24\phi'(s_n)}} \left| \frac{3}{4\pi} \phi'(s_n) \right|^q dx = \frac{1}{8} \left(\frac{3}{4\pi} \right)^{q-1} \phi'(s_n)^{q-1} \; .
\end{aligned}$$

Therefore

$$\begin{aligned}
C' |E(s_n)| \, \phi'(s_n)^{1-1/q} &\leq |E(s_n)| \int_{\mathbb{R}} \left| \frac{\sin(\phi(x) - \phi(s_n))(s_n - t_n)}{(x - s_n)(x - t_n)} \right| dx \\
&\leq C'' |E(s_n)| \log(\phi'(s_n))
\end{aligned}$$

However this last inequality cannot hold if $\phi'(s_n)$ goes to infinity when n goes to infinity. Consequently $\|\phi'\|_\infty < \infty$. □

Observation We strongly believe that Theorem 5.7 is true even without condition (5.9). However, in this case, completely different computations would be needed.

Observation We note that the previous proof with some natural changes shows also that Theorem 5.7 holds even when $1/2 < p < 1$.

3. More observations and further results

In this final section we give some different proofs of the preceding results and we collect some other results concerning the embedding operators. The following propositions hold even if (5.9) and (5.10) do not hold.

Let us start from some sufficiency. The following proposition is inspired by some results contained in [43].

Proposition 5.9 Let us suppose that $\phi' \in L^\infty(\mathbb{R})$; then $\mathcal{H}^q(E) \hookrightarrow \mathcal{H}^2(E) \hookrightarrow \mathcal{H}^p(E)$ when $1 < q < 2 < p \leq \infty$.

PROOF. First we show that $\mathcal{H}^2(E) \hookrightarrow \mathcal{H}^\infty(E)$. If $F \in \mathcal{H}^2(E)$, then

$$\left|\frac{F(t)}{E(t)}\right| \leq \frac{\|F\|_{\mathcal{H}^2}\,\|k_t\|_{\mathcal{H}^2}}{|E(t)|} \leq \|F\|_{\mathcal{H}^2}\sqrt{\frac{\|\phi'\|_\infty}{\pi}}\ .$$

Consequently

$$\|F\|_{\mathcal{H}^\infty} \leq C\,\|F\|_{\mathcal{H}^2}\ .$$

By using the log-convexity of the L^p norms, see [29], we state that

$$\|F\|_{\mathcal{H}^p} \leq \|F\|_{\mathcal{H}^2}^{1-\theta}\,\|F\|_{\mathcal{H}^\infty}^{\theta} \leq \sqrt{\frac{\|\phi'\|_\infty}{\pi}}\,\|F\|_{\mathcal{H}^2}\ ,$$

that is $\mathcal{H}^2(E) \hookrightarrow \mathcal{H}^p(E)$ continuously for every $p > 2$.

In order to prove that $\mathcal{H}^q(E) \hookrightarrow \mathcal{H}^2(E)$, when $1 < q < 2$, we note that this embedding operator is the adjoint of the embedding between $\mathcal{H}^2(E) \hookrightarrow \mathcal{H}^p(E)$, when p is the conjugate exponent of q and that the adjoint of a bounded operator in a locally convex space is still a bounded operator. □

What we are interested more is the reverse implication, that is, which conditions the Hermite Biehler function $E(z)$ must satisfy when $\mathcal{H}^2 \hookrightarrow \mathcal{H}^p$ continuously and $2 < p \leq \infty$ [43].

Proposition 5.10 Let $2 < p < \infty$. If the embedding operator from $\mathcal{H}^2(E)$ to $\mathcal{H}^p(E)$ is continuous, then

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} \left|\frac{\sin(\phi(x)-\phi(t))}{(x-t)\sqrt{\phi'(t)}}\right|^p dx < \infty\ . \tag{5.18}$$

Moreover, if $p = \infty$ and the embedding operator from $\mathcal{H}^2(E)$ to $\mathcal{H}^\infty(E)$ is continuous, again $\phi' \in L^\infty$.

PROOF. These conditions come from easy computations. Indeed, if the embedding operator ι is continuous, then

$$\|k_t\|_{\mathcal{H}^p} \leq C\,\|k_t\|_{\mathcal{H}^2} \quad \forall\, t \in \mathbb{R}\ . \tag{5.19}$$

Therefore, replacing $\|k_t\|_{\mathcal{H}^2}$ with its explicit expression, we get

$$\left(\int_{\mathbb{R}} \left|\frac{E(t)}{\pi}\frac{\sin(\phi(x)-\phi(t))}{x-t}\right|^p dx\right)^{1/p} \leq C\sqrt{\frac{\phi'(t)}{\pi}}\,|E(t)| \quad \forall\, t\in\mathbb{R}\ ,$$

which implies that

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} \left|\frac{\sin(\phi(x)-\phi(t))}{(x-t)\sqrt{\phi'(t)}}\right|^p dx \leq (\sqrt{\pi}C)^p\ .$$

If $p=\infty$, thanks to the boundedness of the embedding operator, we know that

$$\|k_t\|_{\mathcal{H}^\infty} \leq C\,\|k_t\|_{\mathcal{H}^2} = C\sqrt{\frac{\phi'(t)}{\pi}}\,|E(t)|\ .$$

Therefore, $\forall\, t\in\mathbb{R}$,

$$C\sqrt{\frac{\phi'(t)}{\pi}}\,|E(t)| \geq \|k_t\|_{\mathcal{H}^\infty} \geq \left|\frac{k_t(t)}{E(t)}\right| = \frac{\phi'(t)}{\pi}\,|E(t)|\ .$$

Consequently we can conclude that

$$\|\phi'\|_{L^\infty} \leq C\sqrt{\pi}\ ,$$

which is the statement. □

It is clear that (5.18) is equivalent to (5.19). Moreover, (5.19) implies that ι^*, the embedding operator between $\mathcal{H}^q(E)$ and $\mathcal{H}^2(E)$, is a closed operator, as we are going to prove in the following proposition.

Proposition 5.11 Let $E(z)$ be a Hermite Biehler function and $2<p$. If $\|k_t\|_{\mathcal{H}^p} \leq C\,\|k_t\|_{\mathcal{H}^2}$ for every $t\in\mathbb{R}$, then the embedding operator

$$\iota^* : \mathcal{H}^q(E) \hookrightarrow \mathcal{H}^2(E) \tag{5.20}$$

is closed, where $1/q+1/p=1$.

PROOF. Let us suppose that $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}^2(E)\cap\mathcal{H}^q(E)$ and

$$f_n \to f \text{ in } \mathcal{H}^2(E) \qquad \text{while} \qquad f_n \to g \text{ in } \mathcal{H}^q(E)\ .$$

We want to prove that $g=f$. Indeed

$$\begin{aligned} |f(t)-g(t)| &\leq |f(t)-f_n(t)| + |g(t)-f_n(t)| \\ &\leq \left(\|f-f_n\|_{\mathcal{H}^2} + C\,\|g-f_n\|_{\mathcal{H}^q}\right)\|k_t\|_{\mathcal{H}^2} \leq \epsilon\,\|k_t\|_{\mathcal{H}^2} \end{aligned} \tag{5.21}$$

for every ϵ arbitrarily small. Thus, by varying $t\in I$, a bounded interval of $\mathbb{R}$, we find that $f(t)=g(t)$ for every $t\in I$. Since f and g are entire functions, they coincide everywhere. □

III. Duality results

6. Duality and Toeplitz operator

In this chapter, we start studying the dual of p-de Branges spaces.

In the first section, we describe $\mathcal{H}^p(E)^*$ when $1 < p < \infty$. In order to obtain this classical result, Corollary 6.8, we use the relationship between the kernels of the Toeplitz operators and the p-$*$ invariant subspaces, Proposition 6.3.

Actually we are mainly interested in the description of the dual when $p = 1$, which is extremely more complicated and, for this reason, left for the following chapters. Nevertheless, in the second section, we show why the characterization of the dual of the p-de Branges spaces cannot be adapted to the case $p = 1$. The main obstacle is due to the fact that the Toeplitz operators $T_{\overline{\Theta}}$ are not continuous in $H^1(\mathbb{C}^+)$, Theorem 6.10 and Theorem 6.15.

The third section deals with a different topic: we describe the domain of the multiplication operator M_Θ in $\mathrm{BMO}(\mathbb{R})$. This problem, besides being related to the domain of the Toeplitz operators in $H^1(\mathbb{R})$, is interesting in itself and deserves further investigation.

1. The dual of $\mathcal{H}^p(E)$ when $p > 1$

We describe the dual of the p-de Branges spaces when $1 < p < \infty$. First of all, we remind the reader that $\mathcal{H}^p(E)$ are isomorphic to the p-$*$ invariant subspaces of the Hardy space $H^p(\mathbb{C}^+)$, Proposition 1.23. Therefore, in order to compute $\mathcal{H}^p(E)^*$, it is enough describing the dual of the p-$*$ invariant subspace $K^p(\Theta)$ and, subsequently, applying to these spaces the adjoint of the isomorphism described in Proposition 1.23.

We recall that the expression of the Cauchy projection from $L^p(\mathbb{R})$ onto the trace of the Hardy space, $H^p(\mathbb{C}^+)|_\mathbb{R}$, [76].

Definition 6.1 Let

$$P_+(f)(z) := \frac{1}{2\pi i}\int_\mathbb{R} \frac{f(t)}{t-z}dt \qquad \text{for } z \in \mathbb{C}^+$$

when $f \in L^p(\mathbb{R})$, $1 < p < \infty$. The Cauchy projection P_+ from $L^p(\mathbb{R})$ onto $H^p(\mathbb{C}^+)|_\mathbb{R}$ is defined as

$$(6.1) \qquad P_+(f)(x) := \lim_{y\to 0^+} P_+(f)(x+iy) \qquad \text{for almost all } x \in \mathbb{R}\ .$$

By using P_+, we can introduce the Toeplitz operators.

Definition 6.2 Let P_+ be the Cauchy projection. The Toeplitz operator with symbol $g \in L^\infty(\mathbb{R})$ is defined as

$$T_g : \; L^p(\mathbb{R}) \to H^p(\mathbb{C}^+)|_{\mathbb{R}} \;, \qquad T_g(f) := P_+(gf) \;, \tag{6.2}$$

for $1 < p < \infty$.

Let Θ be a meromorphic inner function. The Toeplitz operator $T_{\overline{\Theta}}$ is defined on the whole $H^p(\mathbb{C}^+)$ and it is bounded in $H^p(\mathbb{C}^+)$, $1 < p < \infty$. Indeed

$$\|T_{\overline{\Theta}}(f)\|_{H^p} = \left\|P_+(\overline{\Theta}f)\right\|_{H^p} = \left\|P_+(\overline{\Theta}f)\right\|_{L^p} \le C_p \left\|\overline{\Theta}f\right\|_{L^p} \le C_p \|f\|_{H^p} \;,$$

where we used the fact that the Cauchy projector is bounded in $L^p(\mathbb{R})$ if $p > 1$, [76].

We highlight the relationship between $K^p(\Theta)$ and the kernel of the Toeplitz operator with anti-analytic symbol $T_{\overline{\Theta}}$.

Proposition 6.3 Let Θ be a meromorphic inner function. Then $K^p(\Theta) = \ker(T_{\overline{\Theta}}) \cap H^p(\mathbb{C}^+)$, $1 < p < \infty$, where the Toeplitz operator $T_{\overline{\Theta}}$ is defined as

$$T_{\overline{\Theta}}(f) := P_+(\overline{\Theta}f) \;. \tag{6.3}$$

Proof. Let $f \in \ker(T_{\overline{\Theta}}) \cap H^p(\mathbb{C}^+)$. Then $f \in H^p(\mathbb{C}^+)$ and $P_+(\overline{\Theta}f) = 0$. This last condition implies that $\overline{\Theta(x)}f(x) \in H^p(\mathbb{C}^-)|_{\mathbb{R}}$. Therefore, according to Proposition 1.21, $f \in K^p(\Theta)$.

On the other hand, if $f \in K^p_\Theta$, then $f \in H^p(\mathbb{C}^+)$ and $P_+(\overline{\Theta}f) = 0$, because of Proposition 1.21. Therefore $f \in \ker(T_{\overline{\Theta}})$. □

Thanks to Proposition 6.3, $K^p(\Theta)^* = (\ker(T_{\overline{\Theta}}))^*$. Therefore, since $T_{\overline{\Theta}}$ is bounded, $\ker T_{\overline{\Theta}}$ is a closed subspace of $H^p(\mathbb{C}^+)$.

We compute the adjoint of $T_{\overline{\Theta}}$.

Proposition 6.4 Let $\Theta(z)$ be an inner function. When $1 < p < \infty$,

$$(T_{\overline{\Theta}})^* = T_\Theta \;.$$

Proof. The domain of the operator T_Θ is the whole $H^p(\mathbb{C}^+)$. On the other hand,

$$\langle T_{\overline{\Theta}}(f), g\rangle = \left\langle P_+(\overline{\Theta}(f)), g\right\rangle = \langle f, \Theta g\rangle = \langle f, T_\Theta g\rangle \;,$$

for every $f \in H^p(\mathbb{C}^+)$ and for every $g \in H^q(\mathbb{C}^+)$, when $1/p + 1/q = 1$. □

When Θ is an inner function of $H^\infty(\mathbb{C}^+)$, the subspace $T_\Theta H^q(\mathbb{C}^+) = \Theta H^q(\mathbb{C}^+)$ is a closed subspace of $H^q(\mathbb{C}^+)$, [19], [54], [63].

Before proving the duality result, we recall some well know theorems concerning the quotient spaces and their duals. For sake of completeness, we provide the proofs, which can be found also in any functional analysis textbook, for example [68], [72] and [79].

Theorem 6.5 If M is a closed subspace of the Banach space X, then M^* is isometrically isomorphic to $X^*/M^\perp$, where $M^\perp$ is the annihilator of M.

PROOF. Given $l \in M^*$, Hahn–Banach theorem tells us that there exists an extension $\tilde{l} \in X^*$ such that

$$\left\|\tilde{l}\right\|_{X^*} = \|l\|_{M^*} \ .$$

Consider the mapping

$$\iota : M^* \to X^*/M^\perp \ , \quad l \mapsto \tilde{l} \ ,$$

it is well-defined and ι is a bijection of M^* onto $X^*/M^\perp$. Since

$$\|l\|_{M^*} \leq \inf_{g \in M^\perp} \left\|\tilde{l} + g^\perp\right\|_{X^*} = \left\|\tilde{l} + M^\perp\right\|_{X^*/M^\perp}$$

and

$$\|l\|_{M^*} = \left\|\tilde{l}\right\|_{X^*} \geq \inf_{g \in M^\perp} \left\|\tilde{l} + g^\perp\right\|_{X^*} = \left\|\tilde{l} + M^\perp\right\|_{X^*/M^\perp} \quad ,$$

ι is an isometric isomorphism from M^* onto $X^*/M^\perp$. □

Theorem 6.6 Let U be a linear operator on the Banach space X whose dual space is X^*. If U^* is well defined, then $\ker(U) = {}^\perp\mathrm{ran}(U^*)$, the pre annihilator of the range of the adjoint operator U^*.

PROOF. If $x \in \ker(U)$, that is, $Ux = 0$, then $\langle Ux, y^* \rangle = 0$ for every $y^* \in X^*$. Consequently $\langle x, U^* y^* \rangle = 0$, which implies that $x \in {}^\perp\mathrm{ran}(U^*)$.

The reverse inclusion works in the same way □

Theorem 6.7 Let N be a subspace of X^*, the dual space of the Banach space X. Then $\left({}^\perp N\right)^\perp = \overline{N}^{w*}$, the weak-$*$ closure of N.

PROOF. If $x^* \in N$, then $\langle x, x^* \rangle = 0$,$\forall x \in {}^\perp N$. The subspace $\left({}^\perp N\right)^\perp$ contains the weak-$*$ closure of N. If $x^* \notin \overline{N}^{w*}$, the Hahn-Banach theorem implies the existence of an $x \in {}^\perp N$ such that $\langle x, x^* \rangle \neq 0$; thus $x^* \notin \left({}^\perp N\right)^\perp$, and we have obtained the result. □

At this point we are ready to prove the duality result.

Corollary 6.8 Let $1 < p < \infty$. Then

$$K^p(\Theta)^* = \left(\ker(T_{\overline{\Theta}}) \cap H^p(\mathbb{C}^+)\right)^* = H^q(\mathbb{C}^+)/\left(\Theta H^q(\mathbb{C}^+)\right) = K^q(\Theta) \ ,$$

where $1/p + 1/q = 1$.

PROOF. We have proved all the equalities but the last one: the first equivalence is proved in Proposition 6.3 while the second is due to the above theorems.

Let $f \in H^q(\mathbb{C}^+)$, then

$$\overline{\Theta} f = g_+ + g_- \ ,$$

where, since $\overline{\Theta} f \in L^q(\mathbb{R})$, $1 < q < \infty$, $g_+ := P_+(\overline{\Theta} f) \in H^q(\mathbb{C}^+)|_{\mathbb{R}}$ and $g_- := (\mathrm{id} - P_+)(\overline{\Theta} f) \in H^q(\mathbb{C}^-)|_{\mathbb{R}}$. Since, when $1 < q < \infty$,

$$L^q := H^q(\mathbb{C}^+)|_{\mathbb{R}} + H^q(\mathbb{C}^-)|_{\mathbb{R}} \ ,$$

we obtain

$$f = \Theta g_+ + \Theta g_- \in \Theta H^q(\mathbb{C}^+)|_{\mathbb{R}} + \Theta H^q(\mathbb{C}^-)|_{\mathbb{R}} \ .$$

From this last expression and Definition 1.19, we obtain $K^q(\Theta) = H^q(\mathbb{C}^+)|_{\mathbb{R}} / \Theta H^q(\mathbb{C}^+)|_{\mathbb{R}}$. $\square$

It is clear, thanks to Proposition 1.23, that $\mathcal{H}^p(E)^*$ is isomorphic to $\mathcal{H}^q(E)$, when $1/p+1/q = 1$ and $1 < p < \infty$.

2. Toeplitz operator in $H^1(\mathbb{C}^+)$

When $p = 1$ the reasoning of the above section cannot be applied and we have to put a much greater effort to characterize the dual of $K^1(\Theta)$. Indeed, when $p = 1$, the Toeplitz operator $T_{\overline{\Theta}}$ is unbounded on $H^1(\mathbb{C}^+)$ and its domain does not coincide with the whole $H^1(\mathbb{C}^+)|_{\mathbb{R}}$.

In this section we explain this problem: we prove that the Toeplitz operator $T_{\overline{\Theta}}$ with anti analytic symbol $\overline{\Theta}(z)$, is unbounded in $H^1(\mathbb{C}^+)|_{\mathbb{R}}$. We adapt to the upper half-plane some reasonings contained in [50], [65] and [75].

While manipulating and studying the Toeplitz operators, an extremely important instrument is the class of Hankel operators.

Definition 6.9 For every $f \in L^1(\mathbb{R})$ and $\Theta \in H^\infty(\mathbb{C}^+)$, the Hankel operator with symbol $\overline{\Theta}$ is defined as

$$H_{\overline{\Theta}}(f)(z) := P_- \left(\overline{\Theta} f\right)(z) \tag{6.4}$$

where

$$P_-(f)(z) := \frac{-1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - \bar{z}} dt \quad \text{ for } z \in \mathbb{C}^+$$

and, when it is well defined,

$$P_-(f)(x) := \lim_{y \to 0^+} P_-(f)(x + iy) \ . \tag{6.5}$$

We note that for every $f \in L^1(\mathbb{R})$ for which the two projectors P_+, see (6.1), and P_-, see (6.5), are well defined, it holds also that

$$P_+ + P_- = \mathrm{id}\ . \tag{6.6}$$

Indeed, for almost every $x \in \mathbb{R}$,

$$\begin{aligned} P_+(f)(x) + P_-(f)(x) &= \lim_{y\to 0^+} \left(P_+(f)(x+iy) + P_-(f)(x+iy)\right) \\ &= \lim_{y\to 0^+} \frac{1}{2\pi i}\left(\int_{\mathbb{R}} f(t)\left(\frac{1}{t-z} - \frac{1}{t-\bar{z}}\right)dt\right) \\ &= \lim_{y\to 0^+} \frac{1}{2\pi i}\left(\int_{\mathbb{R}} f(t)\left(\frac{t-\bar{z}}{|t-z|^2} - \frac{t-z}{|t-z|^2}\right)dt\right) \\ &= \lim_{y\to 0^+} \frac{1}{2\pi i}\left(\int_{\mathbb{R}} f(t)\frac{2iy}{|t-z|^2}dt\right) = f(x) \end{aligned}$$

Our interest in Hankel operators is justified by the following theorem.

Theorem 6.10 The operators $H_{\overline{\Theta}}$ is bounded in $H^1(\mathbb{C}^+)$ if and only if $T_{\overline{\Theta}}$ is bounded in $H^1(\mathbb{C}^+)$.

Proof. We control the $H^1(\mathbb{C}^+)$-norm of $T_{\overline{\Theta}}(f)$ with the sum of the $H^1(\mathbb{C}^+)$-norms of f and $H_{\overline{\Theta}}(f)$. Indeed

$$\begin{aligned} \|T_{\overline{\Theta}} f\|_{L^1} &= \left\| \lim_{y\to 0^+} P_+\left(\overline{\Theta} f\right)(\cdot + iy)\right\|_{L^1} \\ &= \left\| \lim_{y\to 0^+} (Id - P_-)\left(\overline{\Theta} f\right)(\cdot + iy)\right\|_{L^1} \\ &= \left\| \lim_{y\to 0^+} \overline{\Theta} f(\cdot + iy) - \lim_{y\to 0^+} P_-\left(\overline{\Theta} f\right)(\cdot + iy)\right\|_{L^1} \\ &\le \left\| \lim_{y\to 0^+} \overline{\Theta} f(\cdot + iy)\right\|_{L^1} + \left\| \lim_{y\to 0^+} P_-\left(\overline{\Theta} f\right)(\cdot + iy)\right\|_{L^1} \\ &\le \|f\|_{H^1} + \|H_{\overline{\Theta}} f\|_{L^1} \le \|f\|_{H^1}(C+1)\ . \end{aligned}$$

On the other hand, if we switch P_+ with P_-, with almost the same computations, we obtain

$$\|H_{\overline{\Theta}} f\|_{L^1} \le \left\|\overline{\Theta} f\right\|_{L^1} + \|T_{\overline{\Theta}} f\|_{L^1} \le \|f\|_{H^1}(C'+1)\ ,$$

which proves the theorem. □

Therefore, in order to prove that the operator $T_{\overline{\Theta}}$ is unbounded on $H^1(\mathbb{C}^+)$, we prove that the Hankel operator $H_{\overline{\Theta}}$ is unbounded on $H^1(\mathbb{C}^+)$.

The Cauchy projection P_+ (6.1) and its direct counterpart P_- (6.5) are strictly associated with the Hilbert transform, [**76**].

Definition 6.11 The Hilbert transform H is defined as

$$(6.7)\qquad Hf(x) := -i(P_+ - P_-)$$
$$= \lim_{y\to 0^+} \frac{-1}{2\pi}\left(\int_{\mathbb{R}} f(t)\left(\frac{t-\bar{z}}{|t-z|^2} + \frac{t-z}{|t-\bar{z}|^2}\right)dt\right) = p.v.\frac{1}{\pi}\left(\int_{\mathbb{R}} \frac{f(t)}{x-t}dt\right) ,$$

for every $f(x)$ which belongs to the domain of H.

In order to prove the unboundedness of $H_{\overline{\Theta}}$ we need another operator.

Definition 6.12 Let $M_{\overline{\Theta}}$ be the multiplication operator defined as

$$(6.8)\qquad M_{\overline{\Theta}}(f)(z) = \overline{\Theta} f(z) .$$

The commutator $[\overline{\Theta}, H]$ is defined as $M_{\overline{\Theta}}H - HM_{\overline{\Theta}}$. Our interest in $[\overline{\Theta}, H]$ is justified by the following lemma.

Lemma 6.13 The operator $[\overline{\Theta}, H]$ is bounded in $H^1(\mathbb{C}^+)|_{\mathbb{R}} + H^1(\mathbb{C}^-)|_{\mathbb{R}}$ if and only if the Hankel operator $H_{\overline{\Theta}}$ is bounded from $H^1(\mathbb{C}^+)|_{\mathbb{R}}$ to $H^1(\mathbb{C}^-)|_{\mathbb{R}}$.

Proof. Let $H^1(\mathbb{C}^+)|_{\mathbb{R}} \ni f = P_+ f$. Therefore

$$i[\overline{\Theta}, H]f := \overline{\Theta} f - iH(\overline{\Theta} f) = \overline{\Theta} f - P_+(\overline{\Theta} f) + P_-(\overline{\Theta} f) = 2P_-(\overline{\Theta} f) .$$

On the other hand, if $H^1(\mathbb{C}^-)|_{\mathbb{R}} \ni f = P_- f$, then

$$-i[\overline{\Theta}, H]f := \overline{\Theta} f + iH(\overline{\Theta} f) = -\overline{\Theta} f + P_+(\overline{\Theta} f) - P_-(\overline{\Theta} f) = -2P_-(\overline{\Theta} f) = -2\overline{\Theta} f ,$$

which is automatically bounded since $f \in H^1(\mathbb{C}^-)|_{\mathbb{R}}$. Combining these two equalities, we obtain the lemma. □

For $0 < p \leq 1$ the space $H^p + \overline{H^p}$ has a real-variable description in terms of atoms. However, in order to prove our result we need less: it is enough checking that the commutator $[\overline{\Theta}, H]$ is unbounded on the $H^1(\mathbb{R})$ atoms. The 1-atoms belong to $H^1 + \overline{H^1}$ since they belong to $H^1(\mathbb{R})$, [76].

Definition 6.14 The function $a(x)$ is an H^1-atom if there is a compact interval $I \subset \mathbb{R}$ such that $\mathrm{supp}(a) \subset I$, $|a| \leq |I|^{-1}$, and

$$\int_{\mathbb{R}} a(x)dx = 0 .$$

.

Theorem 6.15 If Θ is not a constant inner function, the commutator $[\overline{\Theta}, H]$ is unbounded on $H^1(\mathbb{C}^+)|_{\mathbb{R}} + H^1(\mathbb{C}^-)|_{\mathbb{R}}$.

PROOF. We prove this theorem by contradiction. We consider any atoms $a(x) \in H^1(\mathbb{R})$ with supporting interval equal to I. Let $\tilde{I}$ be the double of I with the same center x_0. If $[\overline{\Theta}, H]$ is bounded, then for any atoms $a(x)$

$$\int_{\mathbb{R}\setminus\tilde{I}} \left|[\overline{\Theta}, H](a)(x)\right| dx \leq \int_{\mathbb{R}} \left|[\overline{\Theta}, H](a)(x)\right| dx \leq K$$

where K is the norm of the commutator $[\overline{\Theta}, H]$. We prove that the left member of the above inequality is unbounded, thus giving a contradiction.

We know that

$$\begin{aligned}
[\overline{\Theta}, H]a(x) &= \frac{1}{\pi}\int_I \left(\overline{\Theta}(x) - \overline{\Theta}(t)\right) \frac{a(t)}{x-t} dt \\
&= \frac{1}{\pi}\int_I \left[\left(\overline{\Theta}(x) - \overline{\Theta}_I\right) + \left(\overline{\Theta}_I - \overline{\Theta}(t)\right)\right] \frac{a(t)}{x-t} dt = T_1(x) + T_2(x) \ ,
\end{aligned}$$

where

$$\Theta_I := \frac{1}{|I|}\int_I \Theta(x) dx \ .$$

Computing the first term, we note that

$$\begin{aligned}
|T_1(x)| &= \left|\frac{1}{\pi}\int_I \left(\overline{\Theta}(x) - \overline{\Theta}_I\right) a(t) \frac{1}{x-t} dt\right| \\
&= \left|\frac{1}{\pi}\int_I \left(\overline{\Theta}(x) - \overline{\Theta}_I\right) a(t) \left(\frac{1}{x-t} - \frac{1}{x-x_0}\right) dt\right| \\
&\leq \frac{1}{\pi}\int_I \left|\overline{\Theta}(x) - \overline{\Theta}_I\right| \frac{1}{|I|} \left(\frac{|t-x_0|}{|(x-t)(x-x_0)|}\right) dt \\
&\leq \frac{1}{\pi}\int_I \left|\overline{\Theta}(x) - \overline{\Theta}_I\right| \frac{C}{|I|} \left(\frac{|t-x_0|}{(x-x_0)^2}\right) dt \\
&\leq \frac{1}{\pi}\int_I \left|\overline{\Theta}(x) - \overline{\Theta}_I\right| \frac{C}{|I|} \frac{|I|}{(x-x_0)^2} dt \\
&\leq \frac{1}{\pi} \left|\overline{\Theta}(x) - \overline{\Theta}_I\right| \frac{C}{(x-x_0)^2} |I|
\end{aligned}$$

where $C \leq 3$. Therefore

$$\int_{\mathbb{R}\setminus\tilde{I}} |T_1(x)| dx \leq \frac{1}{\pi} 2C |I| \|\Theta\|_\infty \int_{\mathbb{R}\setminus\tilde{I}} \frac{1}{(x-x_0)^2} \leq \frac{1}{\pi} 4C |I| \|\Theta\|_\infty \ ,$$

which is bounded. For the other term,

$$\begin{aligned} T_2(x) =& \frac{1}{\pi}\int_I \left(\bar\Theta_I - \bar\Theta(t)\right)\frac{a(t)}{x-t}dt \\ =& \frac{1}{\pi}\int_I \left(\bar\Theta_I - \bar\Theta(t)\right)a(t)\left(\frac{1}{x-t}-\frac{1}{x-x_0}\right)dt \\ & \qquad + \frac{1}{\pi}\int_I \left(\bar\Theta_I - \bar\Theta(t)\right)a(t)\frac{1}{x-x_0}dt = T_{21}+T_{22}\ . \end{aligned}$$

We split this last expression into two parts, which we estimate separately. The first part behaves well; indeed

$$\begin{aligned} |T_{21}(x)| \leq& \frac{1}{\pi}\int_I \left|\bar\Theta_I - \bar\Theta(t)\right| |a(t)| \left|\frac{1}{x-t}-\frac{1}{x-x_0}\right| dt \\ \leq& \frac{1}{\pi}\frac{1}{|I|}\int_I \left|\bar\Theta_I - \bar\Theta(t)\right| dt\ C\frac{|I|}{(x-x_0)^2}\ , \end{aligned}$$

whose L^1-norm is bounded on $\mathbb{R}\setminus\tilde I$:

$$\begin{aligned} \int_{\mathbb{R}\setminus\tilde I} |T_{21}(x)|\,dx \leq& \frac{C}{\pi}\left\|\bar\Theta\right\|_{BMO}|I|\int_{\mathbb{R}\setminus\tilde I}\frac{1}{(x-x_0)^2}dx \\ \leq& \frac{C}{\pi}\left\|\bar\Theta\right\|_{BMO} 2\ . \end{aligned}$$

The second term has L^1-norm unbounded. Indeed

$$\begin{aligned} |T_{22}(x)| =& \left|\frac{1}{\pi}\int_I \left(\bar\Theta_I - \bar\Theta(t)\right)a(t)\frac{1}{x-x_0}dt\right| \\ =& \frac{1}{|x-x_0|}\frac{1}{\pi}\left|\int_I \left(\bar\Theta_I - \bar\Theta(t)\right)a(t)dt\right|\ , \end{aligned}$$

whose L^1-norm is

$$\int_{\mathbb{R}\setminus\tilde I} |T_{22}(x)|\,dx = \frac{1}{\pi}\left|\int_I \left(\bar\Theta_I - \bar\Theta(t)\right)a(t)dt\right|\int_{\mathbb{R}\setminus\tilde I}\frac{1}{|x-x_0|}dx\ .$$

The latter quantity is unbounded unless

$$\left|\int_I \left(\bar\Theta_I - \bar\Theta(t)\right)a(t)dt\right| = 0$$

However since the H^1-atom $a(x)$ can be chosen arbitrarily we have that for almost every $t\in\mathbb{R}$,

$$\bar\Theta_I = \bar\Theta(t)\ ,$$

which implies that $\Theta(z)$ has to be constant. □

Observation We note that our symbol $\bar\Theta$ is in BMO($\mathbb{R}$), [42], since it belongs to $L^\infty(\mathbb{R})$.

Thanks to Theorem 6.15, Lemma 6.13 and Theorem 6.10, we have proved that no Toeplitz operator $T_{\bar\Theta}$ is bounded in $H^1(\mathbb{C}^+)$.

3. Domain of Multiplication operator in BMO($\mathbb{R}$)

For every meromorphic inner function Θ, the Toeplitz operator $T_{\overline{\Theta}}$ is unbounded in $H^1(\mathbb{C}^+)|_{\mathbb{R}}$. For this reason, studying the domain of the operator $T_{\overline{\Theta}}$ makes sense.

In this section, instead of fixing our attention on the domain of the Toeplitz operators acting on $H^1(\mathbb{R})$ or on $H^1(\mathbb{C}^+)|_{\mathbb{R}}$, we describe the domain of the multiplication operator M_Θ on the space BMO($\mathbb{R}$) and of the Toeplitz operator T_Θ in BMOA($\mathbb{R}$). The reason why we are interested in this problem instead of the original one, is multifold. First of all, the two pairs of spaces $H^1(\mathbb{R})$-BMO($\mathbb{R}$) and $H^1(\mathbb{C}^+)|_{\mathbb{R}}$-BMOA($\mathbb{R}$) are linked to each others. Indeed, as proved for example in [41] and in [42], the space BMO($\mathbb{R}$) is identified with the topological dual of $H^1(\mathbb{R})$ and BMOA($\mathbb{R}$) is identified with the topological dual of $H^1(\mathbb{C}^+)|_{\mathbb{R}}$. Therefore, when Dom($T_{\overline{\Theta}}$) is dense in $H^1(\mathbb{C}^+)|_{\mathbb{R}}$, we are studying the domain of the adjoint operator ${T_{\overline{\Theta}}}^*$.

Even if Sundberg in [77] provides a complete description for the domain of M_Θ in BMO($\mathbb{R}$), we think that this subject is still incomplete. In this section besides characterizing the domain, see [77] and [31], we present also a still open problem that we face while approaching this subject.

For a preliminary description of the spaces BMO($\mathbb{R}$) and BMOA($\mathbb{R}$) we cite [22], [41], [42] and [46] as good references.

Let us describe the domain of M_Θ, when Θ is a meromorphic inner function. Given a function $f \in$ BMO($\mathbb{R}$), we consider its harmonic extension to $\mathbb{C}^+$, that is,

$$[f](z) := P_z * f = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} f(t)dt \qquad \text{when } z = x+iy \ ,$$

where P_z is the Poisson kernel, (2.7). The above extension makes sense for every function $f(x) \in L^1(\mathbb{R}, \frac{dt}{1+t^2})$, and, for this reason, for every function $f \in$ BMO($\mathbb{R}$), [41].

Given the almost everywhere defined function $f(x)$ on $\mathbb{R}$, we recall that

$$f_{\mathcal{I}} := \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} f(t)dt \ ,$$

the mean of the function $f(x)$ on the interval $\mathcal{I}$.

The characterization we write down was first discovered by C. Sundberg in [77].

Theorem 6.16 Let $f \in$ BMO($\mathbb{R}$) and Θ be a non-constant inner function of $\mathbb{C}^+$. The function $\Theta f \in$ BMO($\mathbb{R}$) if and only if

$$\text{(6.9)} \qquad \sup_{\mathcal{I}} \frac{1}{|\mathcal{I}|}\,|f_{\mathcal{I}}| \int_{\mathcal{I}} |\Theta - \Theta_{\mathcal{I}}| < \infty \ ,$$

where $\mathcal{I}$ is any compact interval of $\mathbb{R}$.

PROOF. This theorem follows from easy computations. Indeed, we already know that if $f \in \mathrm{BMO}(\mathbb{R})$, then $\Theta f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Therefore,

$$\begin{aligned}
\|f\Theta\|_{\mathrm{BMO}} =& \sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|}\int_{\mathcal{J}} |f\Theta - (f\Theta)_{\mathcal{J}}| \\
\leq& \sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|}\int_{\mathcal{J}} |f\Theta - (f)_{\mathcal{J}}\Theta| + |f_{\mathcal{J}}\Theta - f_{\mathcal{J}}\Theta_{\mathcal{J}}| + |f_{\mathcal{J}}\Theta_{\mathcal{J}} - (f\Theta)_{\mathcal{J}}| \\
\leq& \|f\|_{\mathrm{BMO}} + \sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|} |f_{\mathcal{J}}| \int_{\mathcal{J}} |\Theta - \Theta_{\mathcal{J}}| + \sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|}\int_{\mathcal{J}} |f\Theta - \Theta f_{\mathcal{J}}| \\
\leq& 2\,\|f\|_{\mathrm{BMO}} + \sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|} |f_{\mathcal{J}}| \int_{\mathcal{J}} |\Theta - \Theta_{\mathcal{J}}| \ .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|} |f_{\mathcal{J}}| \int_{\mathcal{J}} |\Theta - \Theta_{\mathcal{J}}| =& \sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|}\int_{\mathcal{J}} |f_{\mathcal{J}}\Theta - f_{\mathcal{J}}\Theta_{\mathcal{J}}| \\
\leq& \sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|}\int_{\mathcal{J}} |f\Theta - (f\Theta)_{\mathcal{J}} + f_{\mathcal{J}}\Theta - f\Theta + (f\Theta)_{\mathcal{J}} - f_{\mathcal{J}}\Theta_{\mathcal{J}}| \\
\leq& \|f\Theta\|_{\mathrm{BMO}} + 2\sup_{\mathcal{J}} \frac{1}{|\mathcal{J}|}\int_{\mathcal{J}} |f\Theta - f_{\mathcal{J}}\Theta| \\
\leq& \|f\Theta\|_{\mathrm{BMO}} + 2\,\|f\|_{\mathrm{BMO}} \ ,
\end{aligned}$$

which proves the theorem. □

There are two other similar description for $\mathrm{Dom}(M_\Theta)$ in BMO($\mathbb{R}$).

Theorem 6.17 Let $f \in \mathrm{BMO}(\mathbb{R})$ and let Θ be an inner function of $H^\infty(\mathbb{C}^+)$. The function $\Theta f \in \mathrm{BMO}(\mathbb{R})$ if and only if

$$\sup_{z\in\mathbb{C}^+} |[f](z)|^2 \int_{\mathbb{R}} |\Theta(t) - \Theta(z)|^2 P_z(t)dt < \infty \ . \tag{6.10}$$

PROOF. If $f \in \mathrm{BMO}(\mathbb{R})$ then

$$\int_{\mathbb{R}} \frac{|f(t)|^2}{1+t^2} dt < \infty \ ,$$

which implies that also

$$\int_{\mathbb{R}} \frac{|\Theta(t)f(t)|^2}{1+t^2} dt < \infty \ .$$

Therefore

$$\begin{aligned}
&\sup_{z\in\mathbb{C}^+} \int_{\mathbb{R}} |f(t)\Theta(t) - [f\Theta](z)|^2 P_z(t)dt \\
=& \sup_{z\in\mathbb{C}^+} \int_{\mathbb{R}} |f(t)\Theta(t) - [f](z)\Theta(t) + [f](z)\Theta(t) - [f](z)\Theta(z) + [f](z)\Theta(z) - [f\Theta](z)|^2 P_z(t)dt
\end{aligned}$$

$$
\begin{aligned}
\leq & 3 \sup_{z\in\mathbb{C}^+} \Big(\int_{\mathbb{R}} \left|f(t)\Theta(t) - [f](z)\Theta(t)\right|^2 P_z(t)dt + \\
& \qquad + \int_{\mathbb{R}} \left|[f](z)\Theta(t) - [f](z)\Theta(z)\right|^2 P_z(t)dt + \left|[f](z)\Theta(z) - [f\Theta(z)]\right|^2 \Big) \\
\leq & C\left\|f\right\|^2_{\mathrm{BMO}} + 3\sup_{z\in\mathbb{C}^+} \Big(\left|[f](z)\right|^2 \int_{\mathbb{R}} \left|\Theta(t)-\Theta(z)\right|^2 P_z(t)dt + \\
& \qquad \left| \int_{\mathbb{R}} \left([f](z)\Theta(t) - f(t)\Theta(t)\right) P_z(t)dt \right|^2 \Big) \\
\leq & C\left\|f\right\|^2_{\mathrm{BMO}} + 3\sup_{z\in\mathbb{C}^+} \left|[f](z)\right|^2 \int_{\mathbb{R}} \left|\Theta(t)-\Theta(z)\right|^2 P_z(t)dt \\
& \qquad + 3\sup_{z\in\mathbb{C}^+} \int_{\mathbb{R}} \left|[f](z)\Theta(t) - f(t)\Theta(t)\right|^2 P_z(t)dt \int_{\mathbb{R}} P_z t dt \\
\leq & C\left\|f\right\|^2_{\mathrm{BMO}} + 3\sup_{z\in\mathbb{C}^+} \left|[f](z)\right|^2 \int_{\mathbb{R}} \left|\Theta(t)-\Theta(z)\right|^2 P_z(t)dt + C\left\|f\right\|^2_{\mathrm{BMO}} \ .
\end{aligned}
$$

On the other hand

$$
\sup_{z\in\mathbb{C}^+} \left|[f](z)\right|^2 \int_{\mathbb{R}} \left|\Theta(t)-\Theta(z)\right|^2 P_z(t)dt = \sup_{z\in\mathbb{C}^+} \int_{\mathbb{R}} \left|[f](z)\Theta(t) - [f](z)\Theta(z)\right|^2 P_z(t)dt
$$

$$
\begin{aligned}
= & \sup_{z\in\mathbb{C}^+} \int_{\mathbb{R}} \left|f(t)\Theta(t) - [f\Theta](z) + [f](z)\Theta(t) - f(t)\Theta(t) + [f\Theta](z) - [f](z)\Theta(z)\right|^2 P_z(t)dt \\
\leq & 3\sup_{z\in\mathbb{C}^+} \Big(\int_{\mathbb{R}} \left|f(t)\Theta(t) - [f\Theta](z)\right|^2 P_z(t)dt + \\
& \qquad + \left|[f](z)\Theta(t) - f(t)\Theta(t)\right|^2 P_z(t)dt + \left|[f\Theta](z) - [f](z)\Theta(z)\right|^2 \Big) \\
\leq & C\left\|f\Theta\right\|^2_{\mathrm{BMO}} + C\left\|f\right\|^2_{\mathrm{BMO}} + \sup_{z\in\mathbb{C}^+} \left| \int_{\mathbb{R}} \left([f](z)\Theta(t) - f(t)\Theta(t)\right) P_z(t)dt \right|^2 \\
\leq & C\left\|f\Theta\right\|^2_{\mathrm{BMO}} + C\left\|f\right\|^2_{\mathrm{BMO}} + \sup_{z\in\mathbb{C}^+} \int_{\mathbb{R}} \left|[f](z)\Theta(t) - f(t)\Theta(t)\right|^2 P_z(t)dt \\
\leq & C\left\|f\Theta\right\|^2_{\mathrm{BMO}} + 2C\left\|f\right\|^2_{\mathrm{BMO}} \ ,
\end{aligned}
$$

which proves the theorem. □

With the help of Theorem 6.17, we are able to provide an extremely more practical characterization for the domain of M_Θ.

Theorem 6.18 Let $f \in \mathrm{BMO}(\mathbb{R})$ and let Θ be an inner function of $\mathbb{C}^+$. Then $\Theta f \in \mathrm{BMO}(\mathbb{R})$ if and only if

$$
\sup_{z\in\mathbb{C}^+} \left|[f](z)\right|^2 \left(1 - \left|\Theta(z)\right|^2\right) < \infty \ . \tag{6.11}
$$

Proof. This result is a quick consequence of Theorem 6.17. Indeed

$$\begin{aligned}\sup_{z\in\mathbb{C}^+} |[f](z)|^2 (1-|\Theta(z)|^2) &= \sup_{z\in\mathbb{C}^+} |[f](z)|^2 \int_{\mathbb{R}} (1-|\Theta(z)|^2) P_z(t)dt \\ &= \sup_{z\in\mathbb{C}^+} |[f](z)|^2 \int_{\mathbb{R}} (|\Theta(t)|^2-|\Theta(z)|^2) P_z(t)dt \\ &= \sup_{z\in\mathbb{C}^+} |[f](z)|^2 \int_{\mathbb{R}} |\Theta(t)-\Theta(z)|^2 P_z(t)dt \ .\end{aligned}$$

Since (6.10) is equivalent to the fact that $f \in \mathrm{Dom}(M_\Theta)$, we obtain the desired statement.

□

Theorem 6.18 provides us with a satisfactory condition. In [**31**], K. Dyakonov characterizes the domain of the Toeplitz operator T_Θ in BMOA(ℝ).

Theorem 6.19 If $f \in \mathrm{BMOA}(\mathbb{R})$ and Θ is inner, the following conditions are equivalent:

(i) $\overline{\Theta} f \in \mathrm{BMO}(\mathbb{R})$;

(ii) $f\Theta \in \mathrm{BMOA}(\mathbb{R})$;

(iii) $\sup_{\mathbb{C}^+} |f|^2 (1-|\Theta|^2) < \infty$;

(iv) $\sup_{\Omega(\Theta,\epsilon)} |f| < \infty$, for every $0 < \epsilon < 1$;

(v) $\sup_{\Omega(\Theta,\epsilon)} |f| < \infty$, for some $0 < \epsilon < 1$.

The set $\Omega(\Theta, \epsilon)$ is the level set of Θ and it is defined as

$$\Omega(\Theta,\epsilon) := \{z \in \mathbb{C}^+ \ |\Theta(z)| < \epsilon\} \ .$$

We omit the proof of Theorem 6.19. However we note that the third condition in Theorem 6.19 is equal to (6.11) of Theorem 6.18.

Before concluding this chapter and coming back to the study of p-de Branges spaces, we describe another possible approach which can be used to characterize the domain of M_Θ in BMO(ℝ).

To the best of our knowledge, this technique was first used by Dyakonov in [**33**] where the author describes the elements of the Lipschitz spaces with the help of the pseudocontinuation. Subsequently this approach was used again by Dyakonov and Girela in [**35**] in Q_p spaces when $0 < p < 1$. This family of spaces is deeply related to BMO(ℝ), since "BMO(ℝ) = Q_1". In [**35**], the two authors characterize the elements of these spaces with the help of the pseudocontinuation. Their approach seems useful also in BMO(ℝ) and, therefore, it seems possible to describe the elements of BMO(ℝ) by using it.

Unfortunately, we did not manage to obtain this description for BMO(ℝ) and, to the best of our knowledge, a characterization of the elements of the space BMO(ℝ) with the help of the pseudocontinuation is still unknown. It is clear that, once this description is available, we could have another possible characterization for the domain of M_Θ in terms of pseudocontinuation.

7. Dual of 1-Bernstein space

Even if in the second part of [56], B. Levin describes some properties of the p-Bernstein spaces, the description of the dual of the 1-Bernstein space $\mathcal{B}^1_\pi$ is still unknown. In this chapter we present three different but equivalent characterizations of its dual space.

This chapter is divided in four sections. In the first section, we recall the atomic description of the 1-Bernstein space; this characterization was first discovered by C. Eoff and by S. Boza and M.J. Carro in [37] and [20].

In the second section, we introduce the quotient space of sequences $\mathrm{BMO}(\mathbb{Z})$ and we prove that it is isomorphic to $(\mathcal{B}^1_\pi)^*$. This quotient space of sequences has many interesting properties, which we investigate and prove.

In the third section we introduce the space $\mathcal{X}$, a new quotient space in the space of entire functions, and, thanks to the results of the previous section, we prove that $\mathcal{X}$ is isomorphic to $\mathrm{BMO}(\mathbb{Z})$.

After having studied some properties of $\mathcal{X}$, in the fifth section, we introduce the space $\mathcal{Y}$ and we prove that it is isomorphic to $\mathcal{X}$.

In this work we fix our attention on functions of exponential type less than or equal to π. However, the properties we will prove hold in any Bernstein space $\mathcal{B}^1_\alpha$, $\alpha > 0$.

1. The 1-Bernstein space

In this preliminary section we describe the atomic structure of the 1-Bernstein space. We recall the definition.

Definition 7.1 The 1-Bernstein space, $\mathcal{B}^1_\pi$, is the space of the entire functions of exponential type less than or equal to π, whose restriction to the real line belongs to $L^1(\mathbb{R})$. The norm of $\mathcal{B}^1_\pi$ is the same as the norm of $L^1(\mathbb{R})$.

For our purpose, we need to introduce the spaces $H^1(\mathbb{Z})$ and $H^1_{\mathrm{at}}(\mathbb{Z})$.

Definition 7.2 The space $H^1(\mathbb{Z})$ is made up of all the ℓ^1-sequences for which the discrete Hilbert transform H is in $\ell^1(\mathbb{Z})$. Furthermore, $\forall\ \{a_n\} \in H^1(\mathbb{Z})$,

$$(7.1) \qquad \|\{a_n\}\|_{H^1(\mathbb{Z})} = \|\{a_n\}\|_{\ell^1} + \|\{(H\,(a_n))_k\}\|_{\ell^1} \ .$$

We recall that the discrete Hilbert transform is defined as

$$\{(H(a_n))_k\} := \left\{ \left(\sum_{n\neq k} \frac{a_n}{k-n} \right)_k \right\} . \tag{7.2}$$

Instead on considering H, one can describe $H^1(\mathbb{Z})$ by using the shifted discrete Hilbert transform H_c defined as

$$\{(H_c\{a_n\})_k\} := \left\{ \left(\sum_{n\in\mathbb{Z}} \frac{a_n}{k-n+c} \right)_k \right\}$$

for any fixed $c \in (0,1)$ and for any sequences $\{a_n\} \in \ell^1(\mathbb{Z})$. Through H_c one obtains an equivalent norm for $H^1(\mathbb{Z})$ since, as proved in [37], for every $0 < c < 1$

$$C_1 \|H(a_n)\|_{\ell^1} \leq \|H_c(a_n)\|_{\ell^1} \leq C_2 \|H(a_n)\|_{\ell^1} ,$$

where $C_1, C_2 > 0$. Together with $H^1(\mathbb{Z})$, we introduce the atomic space of sequences $H^1_{\text{at}}(\mathbb{Z})$.

Definition 7.3 A sequence $\{\alpha_n\}$ is an atom of $H^1(\mathbb{Z})$ if it satisfies these three conditions:

- The cardinality of the support of the sequence $\{\alpha_n\}$ is finite:

$$\#\{n \in \mathbb{Z} \text{ such that } \alpha_n \neq 0\} < \infty . \tag{7.3}$$

- The elements of the sequence $\{\alpha_n\}$ satisfy the growth condition:

$$|\alpha(n)| \leq \frac{1}{\#\ \mathrm{supp}(\{\alpha_n\})} . \tag{7.4}$$

- The sequence $\{\alpha_n\}$ satisfies the zero mean property:

$$\sum_{n\in\mathbb{Z}} \alpha_n = 0 . \tag{7.5}$$

The $H^1(\mathbb{Z})$ atoms are the building blocks of $H^1_{\text{at}}(\mathbb{Z})$.

Definition 7.4 We define the space of sequences $H^1_{\text{at}}(\mathbb{Z})$ as

$$H^1_{\text{at}}(\mathbb{Z}) := \left\{ \{g_n\} \text{ such that } g_n = \sum_j \lambda_j \alpha^j_n ,\, \forall n \in \mathbb{Z} \right\} , \tag{7.6}$$

where $\{\alpha^j_n\}$ are atoms of $H^1(\mathbb{Z})$ and $\lambda_j \in \mathbb{C}$. Furthermore,

$$\|\{g_n\}\|_{H^1_{\text{at}}} = \inf_{\{\lambda_j\}} \sum_j |\lambda_j| . \tag{7.7}$$

In theorems 3.10 and 3.14 of [20], S. Boza and M. Carro have proved that the two spaces $H^1(\mathbb{Z})$ and $H^1_{\text{at}}(\mathbb{Z})$ are isomorphic.

We note, and it is important for the following computations, that $H^1_{\mathrm{at}}(\mathbb{Z})$ embeds into $\ell^1(\mathbb{Z})$. Indeed, if $\{g_n\} \in H^1_{\mathrm{at}}(\mathbb{Z})$, then

$$\|\{g_n\}\|_{\ell^1} = \left\| \left\{ \sum_j \lambda_j \alpha_n^j \right\} \right\|_{\ell^1} \leq \sum_j |\lambda_j| \, \|\{\alpha_n\}^j\|_{\ell^1} \leq \sum_j |\lambda_j| \ , \tag{7.8}$$

since $\|\{\alpha_n\}\|_{\ell^1} = 1$ for every $H^1(\mathbb{Z})$ atom. Because (7.8) holds for any sequence $\{\lambda_j\}$, then $\|\{g_n\}\|_{\ell^1} \leq \|\{g_n\}\|_{H^1_{\mathrm{at}}(\mathbb{Z})}$.

As proved by C. Eoff in Theorems 6 and 7 of [37], the two spaces $\mathcal{B}^1_\pi$ and $H^1(\mathbb{Z})$ are isometrically isomorphic.

Theorem 7.5 The 1-Bernstein space is isomorphic to $H^1(\mathbb{Z})$. In particular, T, the isomorphism between the two spaces, is given by:

$$T : \mathcal{B}^1_\pi \to H^1(\mathbb{Z}) \ , \qquad f(x) \mapsto \{f_n\} := \{(-1)^n f(n)\} \tag{7.9}$$

and

$$T^{-1} : H^1(\mathbb{Z}) \to \mathcal{B}^1_\pi \ , \qquad \{a_n\} \mapsto \sum_{n\in\mathbb{Z}} (-1)^n a_n \mathrm{sinc}(\pi(z-n)) \ , \tag{7.10}$$

where $\mathrm{sinc}(x)$ is the cardinal sine.

Because $H^1(\mathbb{Z})$ is isomorphic to $H^1_{\mathrm{at}}(\mathbb{Z})$, we are able to describe the atoms of $\mathcal{B}^1_\pi$.

Definition 7.6 The "atoms" $a(z)$ of the space $\mathcal{B}^1_\pi$ are defined as

$$T^{-1}(\{\alpha_n\})(z) := \sum_n (-1)^n \alpha_n \ \mathrm{sinc}(\pi(z-n)) \ , \tag{7.11}$$

where $\{\alpha_n\}$ is an atom of $H^1(\mathbb{Z})$. With $\mathcal{A}$ we denote the set made of all the atoms of $\mathcal{B}^1_\pi$.

Thanks to the isomorphism T, we note that the subspace containing $\mathcal{A}$ is dense in $\mathcal{B}^1_\pi$. Indeed, let $f \in \mathcal{B}^1_\pi$. Therefore for the sequence $\{f_n\}$ we have that

$$\{(-1)^n f(n)\} \in H^1_{\mathrm{at}}(\mathbb{Z}) \ ,$$

which means that there exists a family of atoms $\alpha^j := \{\alpha^j_n\}$ such that

$$(-1)^n f(n) = \sum_j \lambda_j \alpha^j_n \quad \text{and} \quad 2\,\|\{(-1)^n f(n)\}\|_{H^1(\mathbb{Z})} \geq \sum_j |\lambda_j| \ .$$

Consequently,

$$\begin{aligned} \left\| f - \sum_{j=-N}^{N} \lambda_j \left(\sum_n (-1)^n \alpha^j_n \ \mathrm{sinc}(\cdot - n) \right) \right\|_{\mathcal{B}^1_\pi} &\leq C \left\| \{(-1)^n f(n)\} - \sum_{j=-N}^{N} \lambda_j \{\alpha_n\}^j \right\|_{H^1_{at}} \\ &\leq C \sum_{|j|\geq N} |\lambda_j| \ , \end{aligned}$$

which is small if N is large enough.

We note that if $f(z) \in \mathcal{B}^1_\pi$, then

$$\text{(7.12)} \quad \int_{\mathbb{R}} f(x)\cos(\pi x)dx := \int_{\mathbb{R}} f(x)\left(\frac{e^{i\pi x}+e^{-i\pi x}}{2}\right)dx = \frac{1}{2}\left(\mathcal{F}f(\pi)+\mathcal{F}f(-\pi)\right) = 0 \ ,$$

where $\mathcal{F}(f)$ is the Fourier transform of the L^1-integrable function f. The last sum has to be equal to zero since $\mathcal{F}f$ has to be continuous and its support has to be contained in $[-\pi,\pi]$.

2. BMO($\mathbb{Z}$): the dual of $H^1(\mathbb{Z})$

Our interest in the space $H^1_{\mathrm{at}}(\mathbb{Z})$ is justified by the fact that we are able to characterize its dual.

Definition 7.7 The space $\widetilde{\mathrm{BMO}}(\mathbb{Z})$ is made by all the sequence $\{g_n\}$ such that

$$\|\{g_n\}\|_{\mathrm{BMO}} := \sup_{A\subset\mathbb{Z}} \frac{1}{\#A}\sum_{n\in A}|g_n - \langle g_n\rangle_A| < \infty$$

where A is an interval of $\mathbb{Z}$ and

$$\langle g_n\rangle_A := \frac{1}{\#A}\sum_{n\in A} g_n \ .$$

The operator $\|\cdot\|_{\mathrm{BMO}}$ does not satisfy the conditions required to be the norm of $\widetilde{\mathrm{BMO}}(\mathbb{Z})$, since

$$\text{for every } \{c_n\} = \{k\} \ \text{ with } k \neq 0, \text{ it holds } \|\{c_n\}\|_{\mathrm{BMO}} = 0 \ .$$

For this reason, we have to introduce the quotient space

$$\mathrm{BMO}(\mathbb{Z}) := \widetilde{\mathrm{BMO}}(\mathbb{Z}) \Big/ \left\{\{f_n\} \in \widetilde{\mathrm{BMO}}(\mathbb{Z}) \ \text{ such that } \ \|\{f_n\}\|_{\mathrm{BMO}} = 0\right\} \ , \quad \text{that is}$$

$$\text{(7.13)} \quad \mathrm{BMO}(\mathbb{Z}) := \widetilde{\mathrm{BMO}}(\mathbb{Z}) \Big/ \{\{c_n\} = \{k\} \ , \ k \in \mathbb{C}\} \ ,$$

and the operator

$$\text{(7.14)} \quad \|\{f_n\}\|_{\mathrm{BMO}} := \sup_{A\subset\mathbb{Z}} \frac{1}{\#A}\left(\sum_{n\in A}|f(n) - \langle f_n\rangle_A|\right)$$

is the norm of BMO($\mathbb{Z}$).

We note that if there exist constants c_A such that

$$\sup_{A\subset\mathbb{Z}} \frac{1}{\#A}\sum_{n\in A}|b_n - c_A| < \infty$$

for every bounded interval of $\mathbb{Z}$, then

$$\sup_{A\subset\mathbb{Z}} \frac{1}{\#A}\sum_{n\in A}|b_n - \langle b_n\rangle_A| \le \sup_{A\subset\mathbb{Z}} \frac{2}{\#A}\sum_{n\in A}|b_n - c_A| < \infty \ ,$$

and consequently $\{b_n\} \in \widetilde{\mathrm{BMO}}(\mathbb{Z})$.

Proposition 7.8 Let $\{b_n\} \in \mathrm{BMO}(\mathbb{Z})$. Then $\{\Re b_n\} \in \mathrm{BMO}(\mathbb{Z})$ and $\{\Im b_n\} \in \mathrm{BMO}(\mathbb{Z})$.

Proof. We have to check that (7.14) is bounded for the sequences $\{\Re b_n\}$ and $\{\Im b_n\}$. Indeed

$$\|\{\Re b_n\}\|_{\mathrm{BMO}} \leq \frac{1}{2}\left(\|\{b_n\}\|_{\mathrm{BMO}} + \left\|\{\overline{b_n}\}\right\|_{\mathrm{BMO}}\right) = \|\{b_n\}\|_{\mathrm{BMO}} \ .$$

With the same computations we obtain also (7.14) for $\{\Im b_n\}$. □

The space $\ell^\infty(\mathbb{Z})$ is a proper subspace of $\mathrm{BMO}(\mathbb{Z})$. Indeed the sequence

$$\{b_n\} := \begin{cases} 0 & |n| = 0, 1 \\ \log|n| & \text{elsewhere} \end{cases} \in \mathrm{BMO}(\mathbb{Z}) \setminus \ell^\infty(\mathbb{Z}) \ .$$

In order to prove that $\{b_n\} \in \mathrm{BMO}(\mathbb{Z})$, we note that, for any constant $C_A \in \mathbb{R}$,

$$\frac{1}{\#A}\sum_{n\in A} |\log|n| - C_A| \leq 2\frac{1}{\left|\tilde{A}\right|}\int_{\tilde{A}} |\log|x| - C_A|\, dx \ . \tag{7.15}$$

Therefore

$$\|\{b_n\}\|_{\mathrm{BMO}} \leq 2\,\|\log|\cdot|\|_{\mathrm{BMO}(\mathbb{R})} < \infty \ .$$

In any case, any sequence $\{c_n\} \in \mathrm{BMO}(\mathbb{Z}) \setminus \ell^\infty(\mathbb{Z})$ can be approximated by sequences in $\ell^\infty(\mathbb{Z})$. We need this auxiliary proposition.

Proposition 7.9 Let $c_n \in \mathbb{R}$, for every $n \in \mathbb{Z}$. Then

$$\{c_n^{\mathcal{M}}\} := \begin{cases} -\mathcal{M} & \text{if } c_n < \mathcal{M}\ , \\ \mathcal{M} & \text{if } c_n > \mathcal{M}\ , \\ c_n & \text{otherwise} \end{cases} \in \mathrm{BMO}(\mathbb{Z}) \ .$$

Proof. This proposition is a consequence of the fact that the set of all the real sequences in $\mathrm{BMO}(\mathbb{Z})$ is a lattice. Indeed, if $\{f_n\} \in \mathrm{BMO}(\mathbb{Z})$ and $\{g_n\} \in \mathrm{BMO}(\mathbb{Z})$, then also

$$\|\{\min(f_n, g_n)\}\|_{\mathrm{BMO}} \leq 2\max\left(\|\{f_n\}\|_{\mathrm{BMO}}, \|\{g_n\}\|_{\mathrm{BMO}}\right) \ ,$$
$$\|\{\max(f_n, g_n)\}\|_{\mathrm{BMO}} \leq 2\max\left(\|\{f_n\}\|_{\mathrm{BMO}}, \|\{g_n\}\|_{\mathrm{BMO}}\right) \ .$$

This inclusion is justified by the fact that if $\{b_n\} \in \mathrm{BMO}(\mathbb{Z})$, then

$$\|\{|b_n|\}\|_{\mathrm{BMO}} \leq 2\,\|\{b_n\}\|_{\mathrm{BMO}} \ .$$

Indeed,

$$\|\{|b_m|\}\|_{\text{BMO}} = \sup_{A\subset\mathbb{Z}} \frac{1}{\#A}\sum_{n\in A} \left||b_n| - \langle |b_n|\rangle_A\right| \le \sup_{A\subset\mathbb{Z}} \frac{2}{\#A}\sum_{n\in A} \left||b_n| - |\langle b_n\rangle_A|\right|$$
$$\le 2\,\|\{b_n\}\|_{\text{BMO}} \ ,$$

which proves the theorem. □

Proposition 7.10 Let $\{b_n\} \in \text{BMO}(\mathbb{Z})$. There exists a family $\{c_n^j\}$ of $\ell^\infty(\mathbb{Z})$ sequences such that

$$\left\|\{c_n^j\}\right\|_{\text{BMO}} \le C\,\|\{b_n\}\|_{\text{BMO}} \quad \text{and} \quad \lim_{j\to\infty} c_n^j = b_n\ , \text{ for every fixed } n\in\mathbb{Z}\ .$$

Proof. According to proposition 7.8 and proposition 7.9, it is enough defining the sequence $\{c_n^j\}$ as

$$c_n^j = (\Re b_n)^j + i\,(\Im b_n)^j \ , \text{ as } j>0 \text{ and it goes to infinity.}$$

□

The following proposition describes another property of BMO($\mathbb{Z}$) that will be fundamental for the characterization of $(\mathcal{B}^1_\pi)^*$ described in the following section.

Proposition 7.11 Let $\{\phi_n\} \in \text{BMO}(\mathbb{Z})$. Then

$$\sum_{n\in\mathbb{Z}} \frac{|\phi_n|}{n^2+1} < \infty\ .$$

Proof. Assume $\{\phi_n\} \in \text{BMO}(\mathbb{Z})$ and let $z = m + i \in \mathbb{C}^+$ and $k \in \mathbb{N}_0$. Let I_0 be the interval $I_0 := \{n \in \mathbb{Z} \text{ such that } |n-m| < 1\}$ and let I_k be the interval $I_k := \left\{n\in\mathbb{Z} \text{ such that } |n-m| < 2^k\right\}$. Then

$$\#I_k = 2^{k+1} - 1$$

and

$$P_1^d(n-m) := \frac{1}{(n-m)^2+1^2} \le 1,\ n\in I_0\ , \text{ while} \quad P_1^d(n-m) \le \frac{1}{4^{k-1}},\ n \in I_k \setminus I_{k-1}\ .$$

Consequently,

$$\sum_n \left|\phi_n - \langle\phi_n\rangle_{I_0}\right| P_1^d(n-m)$$
$$\le \sum_{n\in I_0} \left|\phi_n - \langle\phi_n\rangle_{I_0}\right| + \sum_{k\ge1}\frac{4}{2^{2k}} \sum_{n\in I_k\setminus I_{k-1}} \left|\phi_n - \langle\phi_n\rangle_{I_k}\right| + \sum_{k\ge1}\frac{4}{2^{2k}} \sum_{n\in I_k\setminus I_{k-1}} \left|\langle\phi_n\rangle_{I_0} - \langle\phi_n\rangle_{I_k}\right|$$

$$\leq \|\{\phi_n\}\|_{\mathrm{BMO}} + 4\sum_k \frac{1}{2^k}\|\{\phi_n\}\|_{\mathrm{BMO}} + \sum_k \frac{k32}{2^k}\|\{\phi_n\}\|_{\mathrm{BMO}} \ ,$$

where we used the fact that

$$\left|\langle\phi_n\rangle_{I_k} - \langle\phi_n\rangle_{I_{k-1}}\right| \leq \frac{1}{\#I_{k-1}} \sum_{n\in I_{k-1}} \left|\phi_n - \langle\phi_n\rangle_{I_k}\right|$$
$$\leq \frac{\#I_k}{\#I_{k-1}} \frac{1}{\#I_k} \sum_{n\in I_k} \left|\phi_n - \langle\phi_n\rangle_{I_k}\right| \leq 4\|\{\phi_n\}\|_{\mathrm{BMO}} \ ,$$

and consequently

$$\left|\langle\phi_n\rangle_{I_k} - \langle\phi_n\rangle_{I_0}\right| \leq \sum_{i=1}^{k} \left|\langle\phi_n\rangle_{I_i} - \langle\phi_n\rangle_{I_{i-1}}\right| \leq 4k\|\{\phi_n\}\|_{\mathrm{BMO}} \ .$$

Therefore

$$\sum_n |\phi_n|\, P_1^d(n-m) := \sum_n |\phi_n| \frac{1}{(m-n)^2+1} < \infty \ .$$

If we fix $m = 0$, we obtain

$$\sum_n \frac{|\phi_n|}{n^2+1} < \infty \ ,$$

which proves the theorem. □

We note that the above proposition tells us that every $\{\phi_n\} \in \mathrm{BMO}(\mathbb{Z})$ is Poisson summable. However, we do not have any uniform bound for the Poisson sum and we could not even state that the Poisson sum of $\{\phi_n\}$ is controlled by the BMO norm of $\{\phi_n\}$. We recall that the same property holds also in BMO($\mathbb{R}$) [41].

We have introduced the quotient space BMO($\mathbb{Z}$) and we have described some of its first properties. With the help of all these propositions, we want to prove the main result of this section: we show that $(H^1_{\mathrm{at}}(\mathbb{Z}))^*$ is isometrically isomorphic to BMO($\mathbb{Z}$). However, in order to prove this theorem, we need to know the $H^1_{\mathrm{at}}(\mathbb{Z})$ norm of some particular sequences; we estimate them in the following propositions.

Proposition 7.12 Let $\{b_n\} \in \ell^2(\mathbb{Z})$ such that $\#\mathrm{supp}(\{b_n\}) < \infty$ and $\sum_n b_n = 0$. If

$$\left(\frac{1}{\#\mathrm{supp}(\{b_n\})}\sum_n |b_n|^2\right)^{1/2} \leq \frac{1}{\#\mathrm{supp}(\{b_n\})} \ , \tag{7.16}$$

then

$$\|\{b_n\}\|_{H^1(\mathbb{Z})} \leq C \ , \tag{7.17}$$

where the constant C is independent of $\{b_n\}$.

PROOF. First of all,

$$\|\{b_n\}\|_{\ell^1} \le (\#\mathrm{supp}(\{b_n\}))^{1/2} \|\{b_n\}\|_2 \le 1$$

due to (7.16). Let us now estimate $\|H_c\{b_n\}\|_{\ell^1}$ where $0 < c < 1$. Let B^* be the interval concentric with the supp($\{b_n\}$) but having the measure doubled. Then

$$\sum_{j\in B^*} \left|(H_c\{b_n\})_j\right| \le \left(\sum_{j\in B^*} \left|(H_c\{b_n\})_j\right|^2\right)^{1/2} (2\#\mathrm{supp}(\{b_n\})^{1/2} \le C \ ,$$

thanks to the continuity of the discrete Hilbert transform in $\ell^2(\mathbb{Z})$ and (7.16). On the other side, when $j \in B^{*c}$, then

$$\left|(H_c\{b_n\})_j\right| = C\left|\sum_n b_n \left(\frac{1}{j-n+c} - \frac{1}{j-M+c}\right)\right| = C\,\|\{b_n\}\|_{\ell^2}\,\|\{c_n\}\|_{\ell^2} \ ,$$

where M is the centre of B^* and

$$c_n := \begin{cases} \frac{-M+n}{(j-n+c)(j-M+c)} & \text{if } n \in \mathrm{supp}(\{b_n\}) \ , \\ 0 & \text{elsewhere} \end{cases} .$$

Therefore, going on with the previous estimate, we obtain

$$\begin{aligned} \left|(H_c\{b_n\})_j\right| \le& C' (\#\mathrm{supp}(\{b_n\}))^{-1/2} \frac{\#\mathrm{supp}(\{b_n\})}{|j-M|^2} (\#\mathrm{supp}(\{b_n\}))^{1/2} \\ =& C' (\#\mathrm{supp}(\{b_n\})) \frac{1}{|j-M|^2} \ , \end{aligned}$$

thanks to (7.16). Consequently

$$\begin{aligned} \sum_{j\in B^{*c}} \left|(H_c\{b_n\})_j\right| \le& C'\#\mathrm{supp}(\{b_n\}) \sum_{j\in B^{*c}} \frac{1}{|j-M|^2} \\ \le& C''\#\mathrm{supp}(\{b_n\}) \int_{|t|>\#\mathrm{supp}(\{b_n\})} \frac{1}{t^2} dt \le C \ , \end{aligned}$$

where C does not depend on the sequence $\{b_n\}$. □

Proposition 7.13 Let $\{b_n\} \in \ell^2(\mathbb{Z})$ such that $\#\mathrm{supp}(\{b_n\}) < \infty$. If $\sum_n b_n = 0$, then

$$\|\{b_n\}\|_{H^1(\mathbb{Z})} \le C\,\|\{b_n\}\|_{\ell^2} (\#\mathrm{supp}(\{b_n\}))^{1/2} \ . \tag{7.18}$$

PROOF. Given the sequence $\{b_n\}$, for every $n \in \mathbb{Z}$, we consider

$$a_n := \frac{b_n}{\|b_n\|_{\ell^2}} (\#\mathrm{supp}(\{b_n\}))^{-1/2} \ .$$

The sequence $\{a_n\}$ satisfies all the hypothesis of proposition 7.12 and consequently

$$\|\{a_n\}\|_{H^1(\mathbb{Z})} \leq C_2 \ ,$$

which implies that

$$\|\{b_n\}\|_{H^1(\mathbb{Z})} \leq C' \,\|\{b_n\}\|_{\ell^2} \left(\mathrm{supp}(\{b_n\})\right)^{1/2} \|\{a_n\}\|_{H^1(\mathbb{Z})} \leq C_2 \,\|\{b_n\}\|_{\ell^2} \left(\mathrm{supp}(\{b_n\})\right)^{1/2} \ ,$$

which proves the theorem. □

We are finally ready to prove that the dual of $\mathcal{B}^1_\pi$ is isomorphic to BMO($\mathbb{Z}$).

Theorem 7.14 The dual space of $\mathcal{B}^1_\pi$ is isomorphic to BMO($\mathbb{Z}$).

PROOF. In order to obtain this equivalence, we have to check that $(H^1(\mathbb{Z}))^*$ is isomorphic to BMO($\mathbb{Z}$). First of all, we prove that $(H^1(\mathbb{Z}))^* \subseteq \mathrm{BMO}(\mathbb{Z})$. Let us consider $\{a_n\} \in \ell^2(\mathbb{Z})_{A,0}$, where

$$\ell^2(\mathbb{Z})_{A,0} := \left\{ \{a_n\} \in \ell^2(\mathbb{Z}) \text{ such that } \mathrm{supp}(\{a_n\}) = A,\ \#A < \infty \text{ and } \sum_n a_n = 0 \right\} .$$

Let $\beta \in H^1(\mathbb{Z})^*$. It is clear that $\{a_n\} \in H^1_{\mathrm{at}}(\mathbb{Z})$, since it is a multiple of a $H^1(\mathbb{Z})$ atom. By using proposition 7.13 , when $a_n \in \ell^2(\mathbb{Z})_{A,0}$, we note that

$$|\beta(a_n)| \leq \|\beta\|_* \,\|\{a_n\}\|_{H^1(\mathbb{Z})} \leq C'\|\beta\|_*(\#A)^{1/2} \,\|\{a_n\}\|_2 \ .$$

Therefore, the functional β belongs to $\left(\ell^2_{A,0}\right)^*$, which means that there exists $\{b^A_n\} \in \ell^2_{A,0}$ such that

$$\beta(a_n) = \sum_{n\in A} a_n \overline{b^A_n} \quad \text{and} \quad \left\|\{b^A_n\}\right\|_{\ell^2_A} \leq C_2\|\beta\|_*(\#A)^{1/2} \ .$$

Since we want to consider $\{b_n\}$ for every $n \in \mathbb{Z}$, we define

$$\tilde{b}_m := b^A_m \ \text{ when } m \in A \ .$$

We check that $\{\tilde{b}_m\}$ is well defined. Let us consider $Q \subset A$. Then,

$$\left\langle f_n, b^Q_n - b^A_n \right\rangle_{\ell^2} = 0 \ \text{ for every } \{f_n\} \in \ell^2(\mathbb{Z})_{Q,0} \ .$$

Consequently, for every $\{f_n\} \in \ell^2(Q)$,

$$0 = \left\langle f_n - \langle f_n \rangle_Q \,, b^Q_n - b^A_n \right\rangle_{\ell^2} = \left\langle f_n, b^Q_n - b^A_n - \left\langle b^Q_n - b^A_n \right\rangle_Q \right\rangle_{\ell^2} \ ,$$

which means that $\{b^Q_n - b^A_n\}$ is a constant sequence when $n \in Q$ and for this reason $\{b^Q_n\}$ and $\{b^A_n\}$ represent the same element in the quotient space BMO($\mathbb{Z}$).

Finally,

$$\begin{aligned}\frac{1}{(\#A)}\sum_{n\in A}\left|\tilde{b_n}-\left\langle \tilde{b_n}\right\rangle_A\right| &= \frac{1}{(\#A)}\sum_{n\in A}\left|b_n^A-\left\langle b_n^A\right\rangle_A\right| \\ &\le (\#A)^{1/2-1}\left(\sum_{n\in A}\left|b_n^A-\left\langle b_n^A\right\rangle_A\right|^2\right)^{1/2} \\ &= (\#A)^{1/2-1}\left(\left\|\{b_n^A\}\right\|_{\ell^2(A)}+\left\|\{b_n^A\}\right\|_{\ell^2(A)}\right) \\ &\le (\#A)^{1/2-1}\left\|\beta\right\|_*(\#A)^{1/2}2 = 2\left\|\beta\right\|_* .\end{aligned}$$

On the other hand, let $\{b_n\}\in \mathrm{BMO}(\mathbb{Z})$ and let $\{\alpha_n\}$ be an H^1-atom, such that $\operatorname{supp}(\{\alpha_n\})=A$. Therefore

$$\left|\sum_n \alpha_n\overline{b_n}\right| = \left|\sum_n \alpha_n\overline{b_n}-\sum_n\alpha_n\overline{\langle b_n\rangle_A}\right| \le \frac{1}{\#A}\sum_{n\in A}\left|b_n-\langle b_n\rangle_A\right| \le \left\|\{b_n\}\right\|_{\mathrm{BMO}} .$$

If $g_n=\sum_j\lambda_j\alpha_n^j$ and $2\left\|\{g_n\}\right\|_{H^1_{\mathrm{at}}(\mathbb{Z})}\ge\sum_j|\lambda^j|$, then

$$\left|\sum_n g_nb_n\right| \le \sum_n\sum_j|\lambda_j|\left|\alpha_n^jb_n\right| = \sum_j|\lambda_j|\sum_n\left|\alpha_n^jb_n\right| \le 2\left\|\{g_n\}\right\|_{H^1_{\mathrm{at}}(\mathbb{Z})}\left\|\{b_n\}\right\|_{\mathrm{BMO}} ,$$

when $\{b_n\}\in\mathrm{BMO}(\mathbb{Z})\cap\ell^\infty(\mathbb{Z})$. To extend the above computation to $\{b_n\}\notin\ell^\infty(\mathbb{Z})$, we note that if $\{b_n\}\in\mathrm{BMO}(\mathbb{Z})$, there exists a bounded sequence $\{b_n^{\mathcal{M}}\}$ defined as in proposition 7.10, such that

$$\left\|\{b_n^{\mathcal{M}}\}\right\|_{\mathrm{BMO}}\le C\left\|\{b_n\}\right\|_{\mathrm{BMO}} .$$

Consequently

$$\lim_{\mathcal{M}\to\infty}\left\langle f_n,b_n^{\mathcal{M}}\right\rangle_{\ell^2}$$

defines a bounded functional of $H^1(\mathbb{Z})$ that we call L. Thanks to the first part of this theorem, there exists $\{\beta_n\}\in\mathrm{BMO}(\mathbb{Z})$ such that

$$L(f_n)=\langle f_n,\beta_n\rangle_{H^1(\mathbb{Z}),BMO(\mathbb{Z})} .$$

We conclude by observing that $\{b_n\}$ is a representative of the class of $\{\beta_n\}$. Indeed $\{b_n-\beta_n\}$ is equal to a constant sequence, since they act in the same way on the H^1 atoms, and

$$\begin{aligned}\langle f_n,b_n\rangle = L(f_n) = \lim_{\mathcal{M}\to\infty}\left\langle f_n,b_n^{\mathcal{M}}\right\rangle &\le \lim_{M\to\infty}2C\left\|\{b_n^{\mathcal{M}}\}\right\|_{\mathrm{BMO}}\left\|\{f_n\}\right\|_{H^1} \\ &\le 2C_2\left\|\{b_n\}\right\|_{\mathrm{BMO}}\left\|\{f_n\}\right\|_{H^1} .\end{aligned}$$

□

Let $b_n \in \widetilde{BMO}(\mathbb{Z})$; then

$$(7.19)\quad \tilde{T}^{-1}(\{b_n\})(z) := \sum_{0\neq n\in\mathbb{Z}} \left(\frac{(-1)^n b_n}{z-n} + \frac{(-1)^n b_n}{n}\right)\frac{\sin\pi(z-n)}{\pi} + b_0\frac{\sin\pi z}{z\pi}$$

is well defined. Indeed (7.19) is equal to

$$(7.20)\quad \tilde{T}^{-1}(\{b_n\})(z) := \sum_{0\neq n\in\mathbb{Z}} b_n\left(\frac{1}{z-n} + \frac{1}{n}\right)\frac{\sin\pi z}{\pi} + b_0\frac{\sin\pi z}{z\pi}\ ,$$

where the series converges because of Proposition 7.11. For the same reason, the function $\tilde{T}^{-1}(\{b_n\})(z)$ is entire, since it is the uniform limit of entire functions on compact subset of $\mathbb{C}$. Furthermore, if $m \in \mathbb{Z}$, we obtain that

$$(7.21)\quad \begin{aligned}\tilde{T}^{-1}(\{b_n\})(m) &:= (b_0+b_m)\frac{\sin\pi m}{m\pi} + \sum_{0,m\neq n\in\mathbb{Z}} b_n\left(\frac{1}{m-n}+\frac{1}{n}\right)\frac{\sin\pi m}{\pi} + b_m(-1)^m\\ &= (-1)^m b_m\ .\end{aligned}$$

The expression (7.20) is similar to the representation of Mittag-Leffler for the meromorphic functions [27] and sometimes it is called Tschakaloff's interpolation formula [21].

We note that the operator $\tilde{T}^{-1}$ is injective. Indeed if $\tilde{T}^{-1}(\{b_n\})(z)$ is the zero function, then $\tilde{T}^{-1}(\{b_n\})(m) = 0$ for every $m \in \mathbb{Z}$, which implies that $\{b_n\} = \{0\}_{n\in\mathbb{Z}}$ because of (7.21). We define

$$\left\|\tilde{T}^{-1}(\{b_n\})\right\|_{\mathcal{X}} := \left\|\left\{\left((-1)^m\tilde{T}^{-1}(\{b_n\})(m)\right)_m\right\}\right\|_{\mathrm{BMO}} = \|\{b_n\}\|_{\mathrm{BMO}}\ .$$

Thanks to the observation (7.21), $\|\cdot\|_{\mathcal{X}}$ is a seminorm in $\tilde{T}^{-1}\left(\widetilde{\mathrm{BMO}}\right)$.

The operator $\|\cdot\|_{\mathcal{X}}$ is not a norm in $\tilde{T}^{-1}\left(\widetilde{\mathrm{BMO}}\right)$. Indeed if we consider $\{b_n\} := \{c\}$, then $\left\|\tilde{T}^{-1}(\{b_n\})\right\|_{\mathcal{X}} = 0$, but $\tilde{T}^{-1}(\{b_n\})$ is not the zero function. For this reason we have to consider a quotient space.

We define the operator $\tilde{T}^{-1}$ in $\mathrm{BMO}(\mathbb{Z})$, checking that it is well defined on the equivalence classes. To do this, we use the following representation for the cosine function, see for example [28] or [55].

Lemma 7.15 The entire function $\cos(\pi z)$ is represented by the series

$$\cos(\pi z) = \sum_{0\neq n\in\mathbb{Z}}\left(\frac{1}{z-n}+\frac{1}{n}\right)\frac{\sin(\pi z)}{\pi} + \frac{\sin(\pi z)}{\pi z} \qquad \text{for every } z\in\mathbb{C}.$$

Let $\{b_n\} \in \widetilde{\mathrm{BMO}}(\mathbb{Z})$ and

$$\tilde{T}^{-1}(\{b_n\})(z) := \sum_{0\neq n\in\mathbb{Z}} b_n\left(\frac{1}{z-n}+\frac{1}{n}\right)\frac{\sin(\pi z)}{\pi} + b_0\frac{\sin\pi z}{z\pi}\ .$$

We consider $\{c_n\} \in \widetilde{\mathrm{BMO}}(\mathbb{Z})$ defined as

$$c_n := b_n + \gamma \qquad \forall n \in \mathbb{Z} \text{ and } \gamma \in \mathbb{C} \ ;$$

$\{c_n\}$ and $\{b_n\}$ represent the same element of BMO($\mathbb{Z}$). Furthermore, thanks to Lemma 7.15,

$$\begin{aligned}\tilde{T}^{-1}(\{c_n\})(z) :&= \tilde{T}^{-1}(\{b_n\})(z) + \gamma \left(\sum_{0\neq n\in\mathbb{Z}} \left(\frac{1}{z-n} + \frac{1}{n} \right) \frac{\sin(\pi z)}{\pi} + \frac{\sin \pi z}{z\pi} \right) \\ &= \tilde{T}^{-1}(\{b_n\})(z) + \gamma \cos(\pi z) \ .\end{aligned}$$

Therefore, the difference of the images of two representative elements of the same class in BMO($\mathbb{Z}$) is equal to a multiple of $\cos(\pi z)$. We are now ready to introduce the dual space $\mathcal{X}$.

Definition 7.16 We define the space

$$\mathcal{X} := \tilde{T}^{-1}(\widetilde{\mathrm{BMO}}(\mathbb{Z}))/ \langle \cos \pi z \rangle \ , \tag{7.22}$$

and, if $\mathcal{X} \ni g := \tilde{T}^{-1}(\{b_n\})(z)$ with $\{b_n\} \in \widetilde{\mathrm{BMO}}(\mathbb{Z})$, then

$$\|g\|_{\mathcal{X}} := \|\{b_n\}\|_{\mathrm{BMO}} \ .$$

The $\|\cdot\|_{\mathcal{X}}$ is well defined. Indeed, let $f, g \in [g]_{\mathcal{X}}$ and $f = \tilde{T}^{-1}(\{b_n\})$, $g = \tilde{T}^{-1}(\{c_n\})$. Since $f(z) - g(z) = k\cos(\pi z)$, then $b_n - c_n = k$ for every $n \in \mathbb{Z}$ and consequently $\{b_n\} = \{c_n\}$ in BMO($\mathbb{Z}$). This reasoning proves that $\|\cdot\|_{\mathcal{X}}$ is well defined.

We note also that $\|\cdot\|_{\mathcal{X}}$ is a norm. Indeed we have already shown that it is a seminorm due to observation (7.21). Furthermore, $\|f\|_{\mathcal{X}} = 0$ if and only if $f = \tilde{T}^{-1}(k)$, that is $f = k\cos(\pi z)$ which implies that $f \in [0]_{\mathcal{X}}$.

The space $\mathcal{X}$ is another isomorphic description of the dual of $\mathcal{B}^1_\pi$.

Theorem 7.17 The space $\mathcal{X}$ is isomorphic to $(\mathcal{B}^1_\pi)^*$.

Proof. In order to prove that $\mathcal{X}$ is isomorphic to $(\mathcal{B}^1_\pi)^*$, it is enough checking that $\mathcal{X}$ is isomorphic to BMO($\mathbb{Z}$).

Let us introduce the isomorphism V from BMO($\mathbb{Z}$) to $\mathcal{X}$ defined as

$$V(\{b_n\})(z) := \sum_{0\neq n\in\mathbb{Z}} b_n \left(\frac{1}{z-n} + \frac{1}{n} \right) \frac{\sin(\pi z)}{\pi} + b_0 \frac{\sin(\pi z)}{\pi z} \ . \tag{7.23}$$

The operator V is well defined in BMO($\mathbb{Z}$). Indeed if $\{b_n\}, \{c_n\} \in [\{b_n\}]_{\mathrm{BMO}}$, then, thanks to lemma 7.15, $V(\{b_n\}) - V(\{c_n\}) = k\cos(\pi z)$, that is $V(\{b_n\}), V(\{c_n\}) \in [V(\{b_n\})]_{\mathcal{X}}$. Due to the definition of $\mathcal{X}$, V is surjective. Finally, we check that V is injective. Indeed,

$$V^{-1}(\gamma \cos(\pi x)) = \tilde{T}^{-1}(\gamma\cos(\pi x)) = \{\gamma\}_{n\in\mathbb{Z}}$$

which corresponds to the zero element of BMO($\mathbb{Z}$). The operator V is clearly an isomorphism since

$$\|g\|_{\mathcal{X}} = \|\{b_n\}\|_{\mathrm{BMO}(\mathbb{Z})} \qquad \text{when } g = \tilde{T}^{-1}(\{b_n\}) \ ,$$

which proves the theorem. □

We define the duality product between $\mathcal{B}^1_\pi$ and $\mathcal{X}$. Let $f \in \mathcal{B}^1_\pi$ and $g \in \mathcal{X}$, then

$$\langle f, g\rangle_{\mathcal{B}^1_\pi,\mathcal{X}} = \big\langle T(f), V^{-1}(g)\big\rangle_{H^1(\mathbb{Z}),\mathrm{BMO}(\mathbb{Z})} \ ,$$

where T has been defined in (7.9) and V in (7.23). This definition does not depend on the choice of the representative element g, since

$$\int_{\mathbb{R}} f(x)\cos(\pi x)dx = 0$$

for every $f \in \mathcal{B}^1_\pi$ as shown in (7.12).

For every $a(z)$ atom of $\mathcal{B}^1_\pi$, the duality product $\langle a, g\rangle_{\mathcal{B}^1_\pi,\mathcal{X}}$ can be better described: if $g = V(\{b_n\})$, then

$$\langle a, V(\{b_n\})\rangle_{\mathcal{B}^1_\pi,\mathcal{X}} := \lim_{N\to\infty} \int_{\mathbb{R}} a(x) \overline{\left(\sum_{|n|\le N} b_n \left(\frac{1}{x-n} + \frac{1}{n}\right)\frac{\sin(\pi x)}{\pi} + b_0 \frac{\sin \pi x}{x\pi}\right)} dx \ . \tag{7.24}$$

Indeed, since

$$a(z) := \sum_{n\in A} (-1)^n \alpha_n \mathrm{sinc}(\pi(z-n)) \ , \ \text{ and } \ \#A < \infty \ ,$$

then

$$\begin{aligned}\langle a, V(\{b_n\})\rangle_{\mathcal{B}^1_\pi,\mathcal{X}} &= \lim_{N\to\infty} \sum_{n\in A} \sum_{|m|\le N} (-1)^n(-1)^m \alpha_n \overline{b_m} \delta_{mn} \\ &\quad + \lim_{N\to\infty} \sum_{0\ne n\in A} \sum_{0\ne |m|<N} (-1)^n \alpha_n \frac{\overline{b_m}}{\pi m} \int_{\mathbb{R}} \mathrm{sinc}(\pi(x-n))\overline{\sin(\pi x)}dx \\ &= \sum_{n\in A} a(n)\overline{b(n)} := \langle a_n, b_n\rangle_{H^1(\mathbb{Z}),\mathrm{BMO}(\mathbb{Z})} \ ,\end{aligned}$$

since

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{sinc}(\pi(x-n))\sin(\pi x)dx &:= \frac{1}{2i}\int_{\mathbb{R}} \mathrm{sinc}(\pi(x-n))(e^{i\pi x} - e^{-i\pi x})dx \\ &= C\left(e^{i\pi n}\chi_{[-\pi,\pi]}(\pi) - e^{-i\pi n}\chi_{[-\pi,\pi]}(-\pi)\right) = 0 \ .\end{aligned} \tag{7.25}$$

We conclude this section by describing some properties of the space $\mathcal{X}$. First of all, we study the type of the functions which belong to the equivalence classes of $\mathcal{X}$.

Theorem 7.18 The space $\mathcal{X} \subset \mathcal{E}_\pi$, that is, its elements are equivalence classes of entire functions of exponential type less than or equal to π.

PROOF. In order to prove this theorem, it is enough to check that

$$f(z) = \sin(\pi z) \sum_{0\neq n\in\mathbb{Z}} c_n \left(\frac{1}{z-n} + \frac{1}{n}\right) \tag{7.26}$$

is entire of exponential type less than or equal to π when $\{c_n\} \in \widetilde{\mathrm{BMO}}(\mathbb{Z})$. We notice that $f(z)$ in (7.26) is entire. By computing its exponential type, we obtain

$$\begin{aligned}
&\left|\sin(\pi z) \sum_{0\neq n\in\mathbb{Z}} c_n \left(\frac{1}{z-n} + \frac{1}{n}\right)\right| \leq \\
&\qquad \leq \sum_{0<|n|<2|z|} |c_n| \left|\frac{\sin \pi(z-n)}{z-n}\right| \frac{|z|}{|n|} + 2 \sum_{|n|\geq 2|z|} |c_n| \frac{|\sin(\pi z)|}{|n|} \frac{|z|}{|n|} \\
&\qquad \leq \frac{Ce^{\pi|\Im(z)|}}{|\Im z|} \sum_{0<|n|<2|z|} \frac{|c_n|\,|z|}{|n|} + 2C' e^{\pi|\Im(z)|} \sum_{|n|\geq 2|z|} \frac{|c_n|\,|z|}{n^2} \\
&\qquad \leq \frac{Ce^{\pi|\Im(z)|}}{|\Im z|} \sum_{0<|n|<2|z|} \frac{|c_n|\,|2z^2|}{n^2} + 8C' e^{\pi|\Im(z)|} \sum_{|n|\geq 2|z|} \frac{|c_n|\,|z|}{n^2+1} \\
&\qquad \leq \frac{2Ce^{\pi|\Im(z)|}}{|\Im z|} \sum_{n\in\mathbb{Z}} \frac{|c_n|\,|2z^2|}{n^2+1} + 8C' e^{\pi|\Im(z)|} \sum_{n} \frac{|c_n|\,|z|}{n^2+1}\,.
\end{aligned}$$

Consequently $f(z)$ in (7.26) is of exponential type less than or equal to π.

□

We can also compute the rate of growth on the real line of $f \in [f]_{\mathfrak{X}}$.

Theorem 7.19 Let f be any element of any equivalence class of $\mathfrak{X}$. Then

$$f(x) = o(x^3), \quad |x| \to \infty \text{ when } x \in \mathbb{R}\,. \tag{7.27}$$

PROOF. In order to prove this theorem, it is enough to check that (7.27) holds for

$$f(x) = \sin(\pi x) \sum_{0\neq n\in\mathbb{Z}} c_n \left(\frac{1}{x-n} + \frac{1}{n}\right) \tag{7.28}$$

when $\{c_n\} \in \widetilde{\mathrm{BMO}}(\mathbb{Z})$. Therefore

$$\begin{aligned}
\left|\sin(\pi x) \sum_{0\neq n\in\mathbb{Z}} \left(c_n \frac{1}{x-n} + \frac{1}{n}\right)\right| &\leq \sum_{0<|n|<2|x|} |c_n| \left|\frac{\sin \pi(x-n)}{x-n}\right| \frac{|x|}{|n|} + \\
&\qquad 2 \sum_{|n|\geq 2|x|} |c_n| \frac{|\sin(\pi x)|}{|n|} \frac{|x|}{|n|} \\
&\leq \sum_{0<|n|<2|x|} \frac{|c_n|\,|x|}{|n|} + 2 \sum_{|n|\geq 2|x|} \frac{|c_n|\,|x|}{n^2}
\end{aligned}$$

$$\leq \sum_{0<|n|<2|x|} \frac{|c_n|\, 2x^2}{n^2} + 8 \sum_{|n|\geq 2|x|} \frac{|c_n|\,|x|}{n^2+1}$$
$$\leq 8x^2 \sum_{n\in\mathbb{Z}} \frac{|c_n|}{n^2+1} + 8\,|x| \sum_{n} \frac{|c_n|}{n^2+1} \ .$$

Consequently $|f(x)| = o(x^3)$. □

When we sample the function $g \in \mathcal{X}$ on translated sets of the integers, that is $\{n+a\}$ when $0 < a < 1$, the resulting sequence is in $\mathrm{BMO}(\mathbb{Z})$.

Theorem 7.20 Let $g \in \mathcal{X}$, then

(7.29) $\quad \{((-1)^n g(n+a))_n\} \in \mathrm{BMO}(\mathbb{Z}) \qquad \text{for every} \quad 0 < a < 1 \ .$

In particular

(7.30) $\quad \|\{((-1)^n g(n+a))_n\}\|_{\mathrm{BMO}} \leq \|\{((-1)^n g(n))_n\}\|_{\mathrm{BMO}} = \left\|V^{-1}(g)\right\|_{\mathrm{BMO}} \ .$

Proof. Let L be the functional of $\mathcal{B}^1_\pi$ described by the sequence $\{b_n\}$ in $\mathrm{BMO}(\mathbb{Z})$, as proved in Theorem 7.14. Furthermore, let $g(z) = V(\{b_n\})(z)$, as described in theorem 7.17.

It is well known that the horizontal translation operator S_a is a surjective isometry in $\mathcal{B}^1_\pi$ [13], [56]:

$$S_a(f)(x) := f(x-a) \ \text{ and } \ \|f\|_{\mathcal{B}^1} = \|S_a(f)\|_{\mathcal{B}^1} \ , \forall f \in \mathcal{B}^1_\pi \ .$$

Therefore, we consider $S_a^*(L)$. Let f be a $\mathcal{B}^1_\pi$ atom. Then

$$\begin{aligned}
(S_a^* L)(f) :&= L(S_a(f)) \\
&= \lim_{k\to\infty} \int_{\mathbb{R}} \left(\sum_n f(n)\mathrm{sinc}(\pi(x-a-n)) \right) \overline{\left(\sum_{|j|\leq k} (-1)^j b_j \mathrm{sinc}(\pi(x-j)) \right)} dx \\
&= \lim_{k\to\infty} \sum_n \sum_{|j|\leq k} f(n)(-1)^j \overline{b_j}\, \langle \mathrm{sinc}(\pi(\cdot - a - n)), \mathrm{sinc}(\pi(\cdot - j)) \rangle_{L^2(\mathbb{R})} \\
&= \lim_{k\to\infty} \sum_n \sum_{|j|\leq k} f(n)(-1)^j \overline{b_j}\ \mathrm{sinc}(\pi(j-a-n)) \\
&= \sum_n (-1)^n f(n) \overline{\beta_n}
\end{aligned}$$

where

$$\beta_n := (-1)^n \lim_{k\to\infty} \sum_{|j|\leq k} b_j \left(\frac{1}{n+a-j} + \frac{1}{j} \right) \frac{\sin(\pi(n+a))}{\pi} \ .$$

Since $S_a^*(\mathrm{BMO}(\mathbb{Z})) \subseteq \mathrm{BMO}(\mathbb{Z})$ and $S_a^*(\{b_n\}) = \beta_n := (-1)^n g(n+a)$, we obtain (7.29). Furthermore, since S_a is bounded, and

$$S_a^*(\{((-1)^n g(n))_n\} = \{((-1)^n g(n+a))_n\} \ ,$$

we obtain (7.30). □

4. The analytic dual of $\mathcal{B}^1_\pi$: the space $\mathcal{Y}$

Untill now, we have described $(\mathcal{B}^1_\pi)^*$ by using the space of sequences $\mathrm{BMO}(\mathbb{Z})$. In this final section, we give an analytic description of the space $(\mathcal{B}^1_\pi)^*$.

Let us introduce two new spaces: let $\tilde{\mathcal{Y}}$ the space of the entire functions of exponential type less than or equal to π such that

$$\left\|\left\{(-1)^k f\left(k+\frac{1}{2}\right)\right\}\right\|_{\mathrm{BMO}} + \left\|\{(-1)^k f(k)\}\right\|_{\mathrm{BMO}} < \infty \text{ and}$$
$$|f(x)| = o(x^3) \text{ as } |x| \to \infty \ , \text{ and } x \in \mathbb{R} \ .$$

We define

$$\mathcal{Y} := \tilde{\mathcal{Y}} / \langle \sin(\pi(z)), \cos(\pi(z)) \rangle \ . \tag{7.31}$$

In the quotient space $\mathcal{Y}$, we consider

$$\|f\|_{\mathcal{Y}} := \left\|\left\{(-1)^k f\left(k+\frac{1}{2}\right)\right\}\right\|_{\mathrm{BMO}(\mathbb{Z})} + \left\|\{(-1)^k f(k)\}\right\|_{\mathrm{BMO}(\mathbb{Z})} \ .$$

The norm is well defined. Indeed if $f, g \in [g]_{\mathcal{Y}}$, then $f(z) - g(z) = A\cos(\pi z) + B\sin(\pi z)$. Therefore $\{(-1)^k f(k)\} = \{(-1)^k g(k) + A\}$, and $\left\|\{(-1)^k f(k)\}\right\|_{\mathrm{BMO}(\mathbb{Z})} = \left\|\{(-1)^k g(k)\}\right\|_{\mathrm{BMO}(\mathbb{Z})}$.

Furthermore the operator $\|\cdot\|_{\mathcal{Y}}$ is a norm. Indeed, let us assume that $\|f\|_{\mathcal{Y}} = 0$. Then

$$(-)^k f(k) = C \quad \forall k \in \mathbb{Z} \ ,$$

which implies that $f(z) = C\cos(\pi z) + \sin(\pi z) g(z)$, where $g(z)$ is an entire function of π-exponential type. However, we know also that

$$(-1)^k f(k+\frac{1}{2}) = G \quad \forall k \in \mathbb{Z} \ ,$$

which means that

$$(-1)^k f(k+\frac{1}{2}) = (-1)^k \sin(\pi(k+\frac{1}{2}))g(k+\frac{1}{2}) =$$
$$(-1)^k g(k+\frac{1}{2})\left(\sin(\pi k)\cos(\frac{\pi}{2}) + \cos(\pi k)\sin(\frac{\pi}{2})\right) = g(k+\frac{1}{2}) = G \ ,$$

that is

$$g(k+\frac{1}{2}) - G = 0 \quad \forall k \in \mathbb{Z} \ ,$$

which implies that $g(z) = G + \sin(\pi(x - \frac{1}{2}))h(z)$, where $h(z)$ is an entire function of π-exponential type. However $h(z)$ has to be zero. Indeed

$$f(z) = C\cos(\pi z) + G\sin(\pi z) + \sin(\pi z)\sin(\pi(z - \frac{1}{2}))h(z) \ .$$

The order of $h(z)$ is less than or equal to 1, since, otherwise $f(z)$ would not be of exponential type. Furthermore

$$\left|h(re^{-i\pi/2})\right|, \left|h(re^{i\pi/2})\right| \leq M \quad , \quad \left|h(re^{i\pi 5/4})\right| \left|h(re^{i\pi/4})\right| \leq M', \ r \in \mathbb{R}^+ \ .$$

Since the order of $h(z)$ is less than or equal 1, by applying Phragmén–Lindelöf theorem,

$$|h(z)| \leq \max\{M, M'\} \ ,$$

and consequently it has to be 0. Therefore

$$f(z) = C\cos(\pi z) + G\sin(\pi z) \ ,$$

that is

$$f(z) \in [0]_{\mathcal{Y}} \ .$$

We are now ready to prove that $\mathcal{Y}$ is isomorphic to $(\mathcal{B}^1_\pi)^*$. Indeed

Theorem 7.21 The quotient space $\mathcal{Y}$ is isomorphic to $(\mathcal{B}^1_\pi)^*$.

Proof. In order to prove the theorem, it is enough to check that $\mathcal{Y}$ is isomorphic to $\mathcal{X}$ and then applying theorem 7.17. Let us define the operator $\mathcal{W}$ from $\mathcal{X}$ to $\mathcal{Y}$ such that

$$\mathcal{W}([g]_{\mathcal{X}}) := [g]_{\mathcal{Y}} \ ,$$

which is well defined since if $f, g \in [f]_{\mathcal{X}}$, then $f(z) - g(z) = A\cos(\pi z)$ and for this reason $f, g \in [f]_{\mathcal{Y}}$. Moreover, thanks to theorem 7.18 and theorem 7.20, we know that $\mathcal{X} \subset \mathcal{Y}$.

Let us prove that $\mathcal{W}$ is an isomorphism. First of all, $\mathcal{W}$ is injective; indeed, $\forall g \in [0]_{\mathcal{Y}}$,

$$\|\{(-1)^n g(n)\}\|_{\mathrm{BMO}(\mathbb{Z})} = 0 \ ,$$

which implies that $(-1)^n g(n) = A$ for every $n \in \mathbb{Z}$ and in particular

$$\tilde{T}^{-1}(\{(-1)^n g(n)\}) \in [0]_{\mathcal{X}} \ .$$

The operator $\mathcal{W}$ is also surjective. Indeed, if $f \in [f]_{\mathcal{Y}} \in \mathcal{Y}$, there exists $g \in [g]_{\mathcal{X}} \in \mathcal{X}$ such that $g \in [f]_{\mathcal{Y}}$. Indeed we consider

$$g(x) := \tilde{T}^{-1}\left(\{(-1)^n f(n)\}\right)(x) \ .$$

Due to (7.21), we know that

$$g(n) = \tilde{T}^{-1}\left(\{(-1)^k f(k)\}\right)(n) = (-1)^{2n} f(n) = f(n) \ .$$

For this reason, $f(z) - g(z) = \sin(\pi z)h(z)$, where h is an entire function of finite exponential type. We prove that $h(z)$ has to be a constant. Since $f(z)$ and $g(z)$ are in $\tilde{Y}$, also the function $\sin(\pi z)h(z) \in \tilde{Y}$, which implies that

$$|\sin(\pi x)h(x)| = o(|x|^3) \ , \ \text{as} \ |x| \to \infty \ .$$

We need to use lemma 1 of [36] which says that if $f(z)$ be entire function of exponential type π and $f(x) = o(x^3)$, then

$$|f(z)| = o\left(|z|^3 e^{\pi|\Im(z)|}\right) \ , \ \text{as} \ |z| \to \infty \ .$$

Consequently

$$|\sin(\pi z)h(z)| = o\left(|z|^3 e^{\pi|\Im(z)|}\right) \ , \ \text{as} \ \mathbb{C} \ni |z| \to \infty \ ,$$

which implies that

$$|h(z)| = o\left(|z|^3\right) \ , \ \text{as} \ \mathbb{C} \ni |z| \to \infty \ .$$

The last condition forces $h(z)$ to be a polynomial of degree less or equal than 2. However, since $h(z)$ has to be Poisson integrable over $\mathbb{Z} + 1/2$ because of proposition 7.11, it has to be equal to a constant. $\square$

8. Dual of 1-de Branges space

In this chapter, we consider the 1-de Branges spaces $\mathcal{H}^1(E)$. We first prove the atomic description of $\mathcal{H}^1(E)$ and, in order to obtain it, we use the Clark measures of $K^1(\Theta)$.

Our first main result is the following theorem.

Theorem 8.1 Let $E(z)$ be a Hermite Biehler function such that $\mathcal{H}^1(E) \hookrightarrow \mathcal{H}^2(E)$ continuously. Let $\Theta(z)$ be the associated meromorphic inner function and let ∞ belong to the spectrum of $\Theta(z)$. If $\Theta(z)$ satisfies the connected level set condition and σ_α is one of its Clark measures, then $f \in H^1_{\text{at}}(\sigma_\alpha)$ if and only if Ef admits an analytic continuation to $\mathbb{C}$ as a function $F \in \mathcal{H}^1(E)$. Moreover, such a function F is unique and the norms $\|f\|_{H^1_{\text{at}}(\sigma_\alpha)}$, $\|F\|_{\mathcal{H}^1(E)}$ are comparable.

The above theorem introduces several objects which we use to describe the geometry of the $*$-invariant subspaces and in the following section we recall all of them. We note also that the embedding condition

$$\mathcal{H}^1(E) \hookrightarrow \mathcal{H}^2(E)\ , \tag{8.1}$$

has been already extensively analyzed in the fifth chapter.

Contrary to what one might think, for the proof of Theorem 8.1, it is not possible to use the Cayley transform and apply the results of [18]. In fact, if we do this, the atoms of $\mathcal{H}^1(E)$ would not have been equal to linear combinations of k_{t_n}.

In this chapter we characterize also the dual of the 1-de Branges space $\mathcal{H}^1(E)$.

Theorem 8.2 Let the meromorphic inner function $\Theta(z)$ satisfy the connected level set condition and let ∞ belong to the spectrum of $\Theta(z)$. If $\mathcal{H}^1(E) \hookrightarrow \mathcal{H}^2(E)$, then $\mathcal{H}^1(E)^*$ is isomorphic to the quotient space

$$\mathcal{X}(E) := \left\{ f(z) := B(z) \sum_{n\in\mathbb{Z}} b(t_n) \left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1} \right) \frac{1}{\phi'(t_n)} \right\} \Big/ \langle A(z) \rangle\ ,$$

where $E(z) = A(z) - iB(z)$, $\phi(t_n) = n\pi$, $\{k_{t_n}(z)\}_{n\in\mathbb{Z}}$ is an orthogonal basis of $\mathcal{H}^2(E)$ and $b(x) \in \mathrm{BMO}(\sigma_0)$.

This chapter is divided in six sections. After having recalled the definitions of the objects that appear in the above theorems, in the second section we describe some important properties of the Clark measures. In the third and fourth sections we prove Theorem 8.1. The fifth and sixth sections are devoted to the description of $\mathcal{H}^1(E)^*$. We need first to introduce the space $\mathrm{BMO}(\sigma_\alpha)$. We describe some of its properties and we explain in details the duality result. Finally in the sixth section we prove Theorem 8.2.

1. Preliminaries

Besides $K^1(\Theta)$ and $\mathcal{H}^1(E)$, the other important space that in this chapter we use several times is $H^1_{\text{at}}(\sigma)$, where σ is a positive, Borel measure.

Definition 8.3 Let σ be a positive Borel measure on $\mathbb{R}$. A function $a(x)$ is a σ-atom if it satisfies these three properties:

- $a(x)$ has σ mean zero:

$$\int a(x)d\sigma(x) = 0 \; . \tag{8.2}$$

- The σ measure of the support of $a(x)$ is finite:

$$\sigma(\text{supp}(a)) < \infty \; . \tag{8.3}$$

- $a(x)$ satisfies the growth condition:

$$|a(x)| \leq \frac{1}{\sigma(\text{supp}(a))} \; , \qquad \forall x \in \text{supp}(\sigma) \; . \tag{8.4}$$

The space $H^1_{\text{at}}(\sigma)$ is defined as

$$H^1_{\text{at}}(\sigma) := \left\{ f = \sum_k \lambda_k a_k \; : \; \text{where } a_k \text{ are } \sigma\text{-atoms, and } \sum_k |\lambda_k| < \infty \right\} \; , \tag{8.5}$$

and

$$\|f\|_{H^1_{\text{at}}(\sigma)} = \left\{ \inf_{\{\lambda_k\}} \sum_k |\lambda_k| \; : f = \sum_k \lambda_k a_k \; , \text{ where } a_k \text{ are } \sigma\text{-atoms} \right\} \; . \tag{8.6}$$

As stated in Theorem 8.1, it is possible to obtain an atomic decomposition of $\mathcal{H}^1(E)$ if the Hermite Biehler function $E(z)$ satisfies some additional conditions.

First of all, it is worth recalling the expression of the Clark measure σ_α of $\Theta(z)$, see [23] and [67].

Definition 8.4 Let $\Theta(z)$ be a meromorphic inner function. Given $\alpha \in [0, 1)$, we consider the set

$$I := \left\{ t^\alpha \in \mathbb{R} : \Theta(t^\alpha) = e^{i2\pi\alpha} \right\} \; . \tag{8.7}$$

Then the Clark measure σ_α is defined as

$$d\sigma_\alpha(x) := \sum_{n\in\mathbb{Z}} \frac{\pi}{\phi'(t^\alpha_n)} \delta_{t^\alpha_n}(x) \qquad \text{where } \Theta(x) = e^{2i\phi(x)} \tag{8.8}$$

and $\delta_t(x)$ denotes the dirac measure at $x = t$.

Another important instrument for the study of the $*$-invariant subspaces is the spectrum of the inner function Θ, see [24]. To every inner function we can associate its spectrum $\rho(\Theta)$, which is a closed subset of $\overline{\mathbb{C}^+}$. If $\Theta(z)$ is meromorphic, it coincides with its zeros and possibly with ∞.

$$\rho(\Theta) := \left\{ \zeta \in \overline{\mathbb{C}^+} \cup \{\infty\} \;:\; \text{there exists } \{z_n\} \subset \mathbb{C}^+, \lim_{n\to\infty} z_n = \zeta \;,\; \Theta(z_n) = 0 \right\} \;. \tag{8.9}$$

This closed set is linked to several properties of the elements of $K^2(\Theta)$. For example it is studied for the analytically extendability of the elements of $K^2(\Theta)$, see [39] and [59]. In the following sections, we call

$$G_\theta := \mathbb{C}^- \setminus \overline{\rho(\Theta)} \;, \text{ where } \overline{\rho(\Theta)} := \left\{ z \in \mathbb{C}^- \mid \bar{z} \in \rho(\Theta) \right\} \;. \tag{8.10}$$

Finally, the functions $\Theta(z)$ has to satisfy the connected level set condition (CLS).

Definition 8.5 The meromorphic inner function $\Theta(z)$ satisfies the connected level set condition (CLS) if there exists $1 > \epsilon > 0$ such that the set

$$\Omega(\Theta, \epsilon) := \left\{ z \in \mathbb{C}^+ \text{ such that } |\Theta(z)| < \epsilon \right\}$$

is connected.

The connected level set property is used for the description of the Carleson measure of $K^2(\Theta)$, [25]. Furthermore, as shown in [3] and[18], it is also related to Clark measures.

2. Properties of Clark measures

From now on, we assume that $\Theta(z)$ is a connected level set meromorphic inner function, with $\infty \in \rho(\Theta)$. We need the above hypothesis in order to estimate $|\Theta'(x)|^{-1}$, as done in (26) of [10]. Indeed, if $\Theta(z)$ is (CLS) and $\infty \in \rho(\Theta)$, then

$$B \text{ dist}(x, \rho(\Theta)) \leq |\Theta'(x)|^{-1} \leq C \text{ dist}(x, \rho(\Theta)) \qquad \forall x \in \mathbb{R} \tag{8.11}$$

where $B, C > 0$ do not depend on x.
We prove some preliminary lemmas similar to those contained in [14].

Lemma 8.6 Let $\Theta(z)$ be a connected level set meromorphic inner function and let $\infty \in \rho(\Theta)$. For every $x < y \in \mathbb{R}$ such that

$$\phi(y) - \phi(x) \leq \frac{\pi}{N} \;, \qquad \text{where } N \text{ satisfies } \frac{2\pi C}{N} < 1 \;,$$

and C has been defined in (8.11), then $C_1 |\Theta'(x)| \leq |\Theta'(y)| \leq C_2 |\Theta'(x)|$, where $C_1, C_2 > 0$.

Proof. We know that

$$\frac{\pi}{N} \geq \int_x^y \phi'(t)dt \geq \frac{1}{2} \inf_{t\in[x,y]} |\Theta'(t)| \, (y - x) = \frac{1}{2} |\Theta'(\tau)| \, (y - x) \;,$$

where

$$|\Theta'(\tau)| = \inf_{t\in[x,y]} |\Theta'(t)| \ ,$$

and, according to (1.5),

$$|\Theta'(x)| = 2\phi'(x) = \sum_n \frac{2b_n}{(x-a_n)^2+b_n^2} + a \ .$$

Thanks to (8.11),

$$(y-x) \le \frac{2\pi}{N}\frac{1}{|\Theta'(\tau)|} \le \frac{2\pi}{N} C \ \mathrm{dist}\,(\tau, \rho(\Theta)) \ .$$

Consequently, with the right choice of N,

$$\begin{aligned}
|\Theta'(\tau)| \le |\Theta'(x)| &= 2\sum_n \frac{b_n}{\left|x-\overline{\lambda_n}\right|^2} + a \\
&= 2\sum_n \frac{b_n}{\left|\tau-\overline{\lambda_n}\right|^2}\left|\frac{\tau-\lambda_n}{x-\lambda_n}\right|^2 + a \\
&\le 2\sum_n \frac{b_n}{\left|\tau-\overline{\lambda_n}\right|^2}\left(\frac{|\tau-\lambda_n|}{|\tau-\lambda_n|-|x-\tau|}\right)^2 + a \\
&\le K\left(2\sum_n \frac{b_n}{\left|\tau-\overline{\lambda_n}\right|^2} + a\right) = K\,|\Theta'(\tau)| \ .
\end{aligned}$$

Therefore, we have obtained that

$$\frac{1}{K}\,|\Theta'(y)| \le |\Theta'(x)| \le K\,|\Theta'(y)| \ ,$$

where K does not depend on x. □

In the following proposition, we compare two different Clark measures. We assume that $\mathcal{R} \subset \mathbb{R}$ is an interval containing at least two points of the $\mathrm{supp}(\sigma_\alpha)$.

Proposition 8.7 Let $\Theta(z)$ be a connected level set meromorphic inner function and let $\infty \in \rho(\Theta)$. Then, for any compact interval $\mathcal{R} \subset \mathbb{R}$,

$$\sigma_\alpha(\mathcal{R}) \asymp \sigma_\beta(\mathcal{R}) \ , \tag{8.12}$$

where σ_α and σ_β are two different Clark measures of $\Theta(z)$.

Proof. Let

$$T_\ell := \{\tau_m^\ell\} := \left\{x \in \mathbb{R} \ : \phi(x) \equiv_\pi \frac{\ell}{N}\pi\right\} \ , \quad T := \bigcup_{\ell=0}^{N-1} T_\ell = \{\tau_m\} \ ,$$

where N is defined as in Lemma 8.6. Analogously,

$$S_k := \{s^k_m\} := \left\{x \in \mathbb{R} \ : \phi(x) \equiv_\pi \left(\frac{1}{N+\pi} + \frac{k}{N}\right)\pi\right\} \ , \quad S := \bigcup_{\iota=0}^{N-1} S_k = \{s_m\} \ .$$

Thanks to Lemma 8.6, we know that if

$$x, y \in [\tau_m, \tau_{m+1}], \qquad \text{then } C_1\,|\Theta'(x)| \le |\Theta'(y)| \le C_2\,|\Theta'(x)|$$

and if

$$x', y' \in [s_m, s_{m+1}], \qquad \text{then } D_1\,|\Theta'(x')| \le |\Theta'(y')| \le D_2\,|\Theta'(x')| \ .$$

We observe also that $S \cap T = \emptyset$. Consequently, by repeatedly applying the estimate of Lemma 8.6, if

$$x, y \in [t^\alpha_n, t^\alpha_{n+1}], \qquad \text{then } \frac{1}{K^{2N+1}}\,|\Theta'(x)| \le |\Theta'(y)| \le K^{2N+1}\,|\Theta'(x)| \ , \tag{8.13}$$

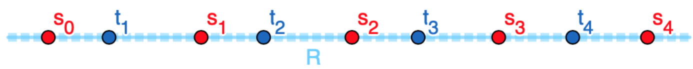
where $\{t^\alpha_n\}$ have been defined in (8.7). Therefore

$$\sigma_\alpha(\mathcal{R}) = \sum_{t^\alpha_n \in \mathcal{R}} \frac{\pi}{\phi'(t^\alpha_n)} \le 2K^{2N+1}\sigma_\beta(\mathcal{R})$$

and

$$\sigma_\alpha(\mathcal{R}) = \sum_{t^\alpha_n \in \mathcal{R}} \frac{\pi}{\phi'(t^\alpha_n)} \ge \frac{1}{2K^{2N+1}}\sigma_\beta(\mathcal{R}) \ ,$$

which proves the theorem. □



We define

$$D_{\sigma_\alpha}(k) := \bigcup_{x \ \in \ \mathrm{supp}(\sigma_\alpha)} D_x(k) \ , \tag{8.14}$$

where

$$D_x(k) := \{z \in \mathbb{C} \ , \ |\Im(z)| \le k\sigma_\alpha(x), \ |\Re z - x| \le k\sigma_\alpha(x)\} \ . \tag{8.15}$$

When $\Theta(z)$ satisfies (CLS), then the Clark measure σ_α of $\Theta(z)$ is *controlled* by the Lebesgue measure.

Lemma 8.8 Let $\Theta(z)$ satisfy (CLS) and let $\infty \in \rho(\Theta)$. There exists $A > 0$ such that for every interval $I \subset \mathbb{R}$ containing at least one atom of the measure σ_α, we have that

$$|I| \leq A\sigma_\alpha(I) \ . \tag{8.16}$$

On the other hand, there exists $B > 0$ such that for every I containing at least two atoms of σ_α, we have that

$$\sigma_\alpha(I) \leq 2B\,|I \setminus D_{\sigma_\alpha}(k)| \ . \tag{8.17}$$

The constant A depends only on $\Theta(z)$ while B depends on $\Theta(z)$ and k. With $|\cdot|$ we mean the Lebesgue linear measure on $\mathbb{R}$.

Proof. Thanks to Lagrange's theorem,

$$\pi = \frac{1}{2}\,|\Theta'(x)|\,(t^\alpha_{n+1} - t^\alpha_n) \qquad \text{where } x \in [t^\alpha_n, t^\alpha_{n+1}] \ .$$

Therefore, by using formula (8.13), we note that

$$\frac{2\pi}{K^{2N+1}}\frac{1}{|\Theta'(t^\alpha_n)|} \leq (t^\alpha_{n+1} - t^\alpha_n) \leq 2\pi K^{2N+1}\frac{1}{|\Theta'(t^\alpha_n)|} \ .$$

Consequently,

$$|I| \leq \sum_{k\in\mathcal{M}} \big|[t^\alpha_k, t^\alpha_{k+1}]\big| \leq 2K^{2N+1}\sum_{k\in\mathcal{M}} \sigma_\alpha(t^k_\alpha) = A\sigma_\alpha(I) \ ,$$

where $I \subseteq [t^\alpha_{\inf\mathcal{M}-1}, t^\alpha_{\sup\mathcal{M}+1}]$ and $A = 2K^{2N+1}$. On the other hand, by using (8.14) and (8.15), we obtain that

$$\sigma_\alpha(I) = \sum_{t^\alpha_k\in I}\sigma_\alpha(t^\alpha_k) \leq B\,|I| \leq B\,|I \setminus D_{\sigma_\alpha}(k)| + B2k\;\sigma_\alpha(D_{\sigma_\alpha}(k)) \ ,$$

where $B = 2K^{2N+1}$. If we chose $k < \frac{1}{4B}$, then

$$\sigma_\alpha(I) \leq 2B\,|I \setminus D_{\sigma_\alpha}(k)| \ ,$$

that is

$$2k\sigma_\alpha(I) \leq |I| \ , \tag{8.18}$$

which we will use in the following theorems. ∎

From (8.17) and (8.18), we note also that the sets $D_x(k)$, $x \in \mathrm{supp}(\sigma_\alpha)$ are disjoint if k is small enough.
Before describing the atomic decomposition of the de Branges space $\mathcal{H}^1(E)$, we state three more lemmas that we will use in the proof of Theorem 8.1.

Lemma 8.9 Let $\Theta(z)$ be a (CLS) inner function with $\infty \in \rho(\Theta)$. If $p > 0$, it holds that

$$\int_{-\infty}^{a-p} \frac{1}{(a-t)^2} d\sigma_\alpha(t) \le C \int_{-\infty}^{a-p} \frac{1}{(a-t)^2} dt \ ,$$
$$\int_{b+p}^{\infty} \frac{1}{(b-t)^2} d\sigma_\alpha(t) \le C \int_{b+p}^{\infty} \frac{1}{(b-t)^2} dt \ ,$$

where C depends only on $\Theta(z)$.

PROOF. It is clear that

$$\sum_{t_n^\alpha \le a-p} \frac{1}{(a-t_n^\alpha)^2} \frac{\pi}{\phi'(t_n^\alpha)} \le \sum_{t_n^\alpha \le a-p} \frac{1}{(a-t_n^\alpha)^2} C(t_{n+1}^\alpha - t_n^\alpha) \le C \int_{-\infty}^{a-p} \frac{1}{(a-t)^2} dt \ .$$

Anlogously ,

$$\sum_{t_n^\alpha \ge b+p} \frac{1}{(b-t_n^\alpha)^2} \frac{\pi}{\phi'(t_n^\alpha)} \le \sum_{t_n^\alpha \ge b+p} \frac{1}{(b-t_n^\alpha)^2} C(t_n^\alpha - t_{n-1}^\alpha) \le C \int_{b+p}^{\infty} \frac{1}{(b-t)^2} dt \ ,$$

which proves the theorem. □

Lemma 8.10 Let $\Theta(z)$ be a (CLS) meromorphic inner function with $\infty \in \rho(\Theta)$. Then, there exists $\delta > 0$ such that $|\alpha - \Theta(z)| \ge \delta$ for all $z \in \mathbb{C}^+ \setminus D_{\sigma_\alpha}(k)$.

PROOF. We define

$$d_0(x) := \operatorname{dist}(x, \rho(\Theta)) \quad \text{and} \quad d_\epsilon(x) := \operatorname{dist}(x, \Omega(\Theta, \epsilon)) \ , \tag{8.19}$$

where $\Omega(\Theta, \epsilon)$ has been specified in Definition 8.5. We consider $z \in \mathbb{C}^+ \setminus \bigcup_{x\in\mathbb{R}} D_x\left(\frac{k}{|\Theta'(x)|}\right)$. Hence,

$$\Im(z) \ge \frac{k}{|\Theta'(\Re z)|} \ . \tag{8.20}$$

There are two possibilities: if

$$\Im(z) > d_0(\Re(z)) \ge d_\epsilon(\Re(z)) \ , \quad \text{then} \quad |\Theta(z)| \le \epsilon \ .$$

Otherwise, $\Im z \le d_0(\Re(z))$, that is, $|z - \overline{z_n}| \le |z - \Re z| + |\Re z - \overline{z_n}| \le 2\,|\Re(z) - \overline{z_n}|$, where $\Theta(z_n) = 0$. Therefore, with computations similar to those in [10],

$$\begin{aligned}\log(|\Theta(z)|^2) \le & -2a\Im(z) - \sum_n 4\Im(z)\frac{\Im(z_n)}{|z-\overline{z_n}|^2} \\ & \le -\frac{2k}{|\Theta'(\Re(z))|}\,|\Theta'(\Re(z))| \inf_n \left|\frac{\Re z - \overline{z_n}}{z - \overline{z_n}}\right|^2 \le -\frac{k}{2} \ ,\end{aligned}$$

which implies that $|\Theta(z)| \leq e^{-k/4}$. Therefore, in any case, $|\Theta(z)| \leq \tilde{\epsilon}$ and we have $|\alpha - \Theta(z)| > \delta_1$. Now let $z \in \mathbb{C}^+ \cap D_x\left(\frac{k}{|\Theta'(x)|}\right)$, where $x \in \mathbb{R}$, and assume that $z \notin D_{\sigma_\alpha}(k)$. We have

$$|\Theta(\Re z) - \alpha| \geq \mathrm{dist}(\Re z, \mathrm{supp}(\sigma_\alpha)) \inf_{\Re z \leq t \leq \xi} |\Theta'(t)| \geq 2k \frac{\Theta'(\tau)}{\Theta'(\xi)} \geq 2\epsilon_2 \ ,$$

since $z \notin D_{\sigma_\alpha}(k)$, where $\Theta(\xi) = \alpha$, $|\Re z - \xi| = \mathrm{dist}(\Re z, \mathrm{supp}(\sigma_\alpha))$ and

$$\Theta'(\tau) = \inf_{\Re z \leq t \leq \xi} |\Theta'(t)| \ .$$

We note also that

$$\epsilon_2 = \frac{k}{K^{2N+1}}$$

thanks to (8.13). Using the fact that

$$|\Theta'(z)| \leq |\Theta'(\Re z)| \ ,$$

we obtain that

$$|\Theta(z) - \Theta(\Re z)| \leq \Im z \, |\Theta'(\Re z)| \leq k \frac{|\Theta'(\Re z)|}{|\Theta'(x)|} \leq \epsilon_2 \ .$$

Therefore, it follows that

$$|\Theta(z) - \alpha| > \delta := \min(\delta_1, \epsilon_2) \ ,$$

which proves the theorem. □

Lemma 8.11 Let $\Theta(z)$ be an inner function and let $g \in K^1(\Theta)$. Then, there exist functions $g_1(z)$, $g_2(z) \in K^1(\Theta)$ such that $g(z) = g_1(z) + i g_2(z)$, $g_j(x) = \Theta(x)\overline{g_j(x)}$ for $x \in \mathbb{R}$ with $\|g_j\|_{L^1(\mathbb{R})} \leq \|g\|_{L^1(\mathbb{R})}$, $j = 1, 2$.

Proof. Let us consider $g^*(x) := \Theta(x)\overline{g(x)}$. We know

$$g^* \in \Theta \overline{H^1(\mathbb{C}^+)} \cap H^1(\mathbb{C}^+) = K^1(\Theta) \ .$$

Now we define

$$g_1 := \frac{g + g^*}{2} \qquad \text{and} \qquad g_2 := \frac{g - g^*}{2i}$$

and the conclusion follows. □

3. Atomic decomposition: $H^1_{\mathbf{at}}(\sigma_\alpha) \subseteq \mathcal{H}^1(E)$

In this section we will show that there exists a continuous embedding from the space $H^1_{\mathrm{at}}(\sigma_\alpha)$ into $\mathcal{H}^1(E)$, when σ_α is a Clark measure for $\Theta(z) := E^\#/E$.

Let us first introduce the reverse Clark transform, see [23]. Given a function $f \in L^1(\sigma_\alpha)$, we define

$$(8.21) \quad V_\alpha^{-1}(f)(z) := \frac{i}{2\pi} \int_{\mathbb{R}} f(t) \frac{1-\Theta(z)\overline{\Theta(t)}}{(t-z)} d\sigma_\alpha(t) \;\text{ when } z \notin \mathrm{supp}(\sigma_\alpha) \;.$$

Since $f \in L^1(\sigma_\alpha)$ and $\frac{1-\Theta(z)\overline{\Theta(\cdot)}}{(\cdot - z)} \in L^\infty(\sigma_\alpha)$ for every $z \notin \mathrm{supp}(\sigma_\alpha)$, $V_\alpha^{-1}(f)(z)$ is well defined. When $\Theta(z)$ is a meromorphic inner function, (8.21) can be rewritten as

$$\begin{aligned}
V_\alpha^{-1}(f)(z) =& \frac{i}{2\pi} \int_{\mathbb{R}} f(t) \frac{1-\Theta(z)\overline{\Theta(t)}}{t-z} d\sigma_\alpha(t) \\
=& \frac{i}{2\pi} \int_{\mathbb{R}} f(t) \frac{1-\Theta(z)\overline{\Theta(t)}}{t-z} \sum_n \delta_{t_n^\alpha}(t) \frac{\pi}{\phi'(t_n^\alpha)} \\
=& \frac{i}{2} \sum_n f(t_n^\alpha) \frac{1-\Theta(z)\overline{\Theta(t_n^\alpha)}}{t_n^\alpha - z} \frac{1}{\phi'(t_n^\alpha)} \\
=& \frac{i}{2} \sum_n \frac{E(t_n^\alpha) f(t_n^\alpha)}{E(t_n^\alpha)} \frac{E(z)\overline{E(t_n^\alpha)} - E^\#(z)E(t_n^\alpha)}{\overline{E(t_n^\alpha)}E(z)(t_n^\alpha - z)} \frac{1}{\phi'(t_n^\alpha)} \\
=& \frac{1}{E(z)} \sum_n F(t_n^\alpha) \frac{k_{t_n^\alpha}(z)}{k_{t_n^\alpha}(t_n^\alpha)} \;,
\end{aligned}$$

where $F/E = f$ and $k_t(z)$ has been defined in (1.12). We introduce the isometry from $H^1_{\mathrm{at}}(\sigma_\alpha)$ into $\mathcal{H}^1(E)$.

Definition 8.12 Let $f = \sum_k \lambda_k a_k$ where $a_k(x)$ are the atoms of $H^1_{\mathrm{at}}(\sigma_\alpha)$, see Definition 8.3. We define

$$\mathcal{V}_\alpha^{-1}(f)(z) := \sum_k \lambda_k \mathcal{V}_\alpha^{-1}(a_k)(z) \;.$$

If $a(x) \in H^1_{\mathrm{at}}(\sigma_\alpha)$ is an atom, then

$$(8.22) \quad \mathcal{V}_\alpha^{-1}(a)(z) := \sum_n A(t_n^\alpha) \frac{k_{t_n^\alpha}(z)}{k_{t_n^\alpha}(t_n^\alpha)} \;,$$

where $A(t_n^\alpha) := a(t_n^\alpha)E(t_n^\alpha)$ and $k_x(z)$ is the reproducing kernel of $\mathcal{H}^2(E)$.

We note that the sum in (8.22) is finite and for this reason $\mathcal{V}_\alpha^{-1}(a)(z) \in \mathcal{H}^2(E)$.

In the rest of this section we prove that $\mathcal{V}_\alpha^{-1}(f) \in \mathcal{H}^1(E)$ and that

$$(8.23) \quad \left\|\mathcal{V}_\alpha^{-1}(f)\right\|_{\mathcal{H}^1(E)} \le C \left\|f\right\|_{H^1_{\mathrm{at}}(\sigma_\alpha)} \;, \forall f \in H^1_{\mathrm{at}}(\sigma_\alpha) \;.$$

Let us first consider $a(x)$, an atom of $H^1_{\mathrm{at}}(\sigma_\alpha)$ as described in Definition 8.3. It is clear that $\mathcal{V}_\alpha^{-1}(a) \in \mathcal{H}^1(E)$. Indeed, since $\mathcal{V}_\alpha^{-1}(a)$ is in the Smirnov class, [29], we have to check only the integrability condition; We write (8.22) in the following way:

$$\begin{aligned}
\mathcal{V}_\alpha^{-1}(a)(x) =& \sum_n E(t_n^\alpha)a(t_n^\alpha)\frac{k_{t_n^\alpha}(x)}{k_{t_n^\alpha}(t_n^\alpha)} \\
=&\frac{i}{2}\sum_n E(t_n^\alpha)a(t_n^\alpha)\frac{|E(x)|\,|E(t_n^\alpha)|}{|E(t_n^\alpha)|^2\,\phi'(t_n^\alpha)}\frac{\sin(\phi(x)-\phi(t_n^\alpha))}{x-t_n^\alpha} \\
=&\frac{i\,|E(x)|}{2}\sin(\phi(x)-\alpha\pi)\sum_n a(t_n^\alpha)\frac{e^{-i\phi(t_n^\alpha)}(-1)^n}{\phi'(t_n^\alpha)}\frac{1}{x-t_n^\alpha} \\
=&\frac{i\,|E(x)|}{2}e^{-i\alpha\pi}\sin(\phi(x)-\alpha\pi)\sum_n\frac{a(t_n^\alpha)}{\phi'(t_n^\alpha)}\frac{1}{x-t_n^\alpha}\ .
\end{aligned}$$

Consequently,

$$\left\|\mathcal{V}_\alpha^{-1}(a)\right\|_{\mathcal{H}^1(E)} \le \int_{\mathbb{R}}\left|\sum_n\frac{a(t_n^\alpha)}{\phi'(t_n^\alpha)}\frac{1}{x-t_n^\alpha}\right|dx<\infty$$

thanks to the zero mean property of the atom $a(x)$.

At this point we prove that

$$\left\|\mathcal{V}_\alpha^{-1}(a)\right\|_{\mathcal{H}^1(E)} \le C\,\|a\|_{H^1_{\rm at}(\sigma_\alpha)} = C \qquad \text{for every atom } a\in H^1_{\rm at}(\sigma_\alpha)\ . \tag{8.24}$$

We need the Aleksandrov-Clark disintegration formula, [66], that we write down for sake of completeness.

Theorem 8.13 For every $f\in L^1(\mathbb{R})$, it holds

$$\|f\|_{L^1(\mathbb{R})} = \int_0^1\int_{\mathbb{R}}|f(t)|\,d\sigma_\beta(t)d\beta\ . \tag{8.25}$$

Therefore, by applying (8.25), we know that

$$\left\|\mathcal{V}_\alpha^{-1}(a)\right\|_{\mathcal{H}^1(E)} = \left\|V_\alpha^{-1}(a)\right\|_{L^1(\mathbb{R})} = \int_0^1\int_{\mathbb{R}}\left|V_\beta\left(V_\alpha^{-1}(a)\right)(x)\right|d\sigma_\beta(x)d\beta\ .$$

Now

$$\begin{aligned}
&\int_{\mathbb{R}}\left|V_\beta\left(V_\alpha^{-1}(a)\right)(x)\right|d\sigma_\beta(x) \\
&\qquad = \int_{\mathbb{R}\setminus 2\mathcal{R}}\left|V_\beta\left(V_\alpha^{-1}(a)\right)(x)\right|d\sigma_\beta(x) + \int_{2\mathcal{R}}\left|V_\beta\left(V_\alpha^{-1}(a)\right)(x)\right|d\sigma_\beta(x)\ ,
\end{aligned}\tag{8.26}$$

where $\mathcal{R}$ is an interval containing $\operatorname{supp}(a)$. We start from estimating the second term:

$$\begin{aligned}
\int_{2\mathcal{R}}\left|V_\beta\left(V_\alpha^{-1}(a)\right)(x)\right|d\sigma_\beta(x) &= \sum_{t_n^\beta\in 2\mathcal{R}}\left|V_\alpha^{-1}(a)(t_n^\beta)\right|\frac{\pi}{\phi'(t_n^\beta)} \\
&\le \sqrt{\sigma_\beta(2\mathcal{R})}\left(\sum_{t_n^\beta\in 2\mathcal{R}}\left|V_\alpha^{-1}(a)(t_n^\beta)\right|^2\frac{\pi}{\phi'(t_n^\beta)}\right)^{1/2}\ .
\end{aligned}$$

Consequently, thanks to (8.25),

$$\begin{aligned}
&\int_0^1 \sqrt{\sigma_\beta(2\mathcal{R})} \left(\sum_{t_n^\beta \in 2\mathcal{R}} \left| V_\alpha^{-1}(a)(t_n^\beta) \right|^2 \frac{\pi}{\phi'(t_n^\beta)} \right)^{1/2} d\beta \\
&\le \left(\int_0^1 \sigma_\beta(2\mathcal{R}) d\beta \right)^{1/2} \left[\int_0^1 \left(\sum_{t_n^\beta \in 2\mathcal{R}} \left| V_\alpha^{-1}(a)(t_n^\beta) \right|^2 \frac{\pi}{\phi'(t_n^\beta)} \right) d\beta \right]^{1/2} \\
&\le |2\mathcal{R}|^{1/2} \left(\int_0^1 \int_{\mathbb{R}} \left| V_\alpha^{-1}(a)(x) \right|^2 d\sigma_\beta(x)\ d\beta \right)^{1/2} \\
&\le |2\mathcal{R}|^{1/2} \left(\int_{\mathbb{R}} \left| V_\alpha^{-1}(a)(x) \right|^2 dx \right)^{1/2} \\
&\le |2\mathcal{R}|^{1/2} \left\| V_\alpha^{-1}(a) \right\|_{H^2} \\
&\le |2\mathcal{R}|^{1/2} \left(\sum_{t_n^\alpha \in 2\mathcal{R}} \left| V_\alpha^{-1}(a)(t_n^\alpha) \right|^2 \frac{\pi}{\phi'(t_n^\alpha)} \right)^{1/2} .
\end{aligned}$$

Since

$$\sum_{t_n^\alpha \in 2\mathcal{R}} \left| V_\alpha^{-1}(a)(t_n^\alpha) \right|^2 \frac{\pi}{\phi'(t_n^\alpha)} \le \sum_{t_n^\alpha} |a(t_n^\alpha)|^2 \frac{\pi}{\phi'(t_n^\alpha)} \le \frac{1}{\sigma_\alpha(\mathcal{R})} ,$$

we obtain, due to Lemma 8.8, that

$$\int_0^1 \int_{2\mathcal{R}} \left| V_\beta\left(V_\alpha^{-1}(a)\right)(x) \right| d\sigma_\beta(x)\ d\beta \le \sqrt{2} \sqrt{\frac{|\mathcal{R}|}{\sigma_\alpha(\mathcal{R})}} \le C' .$$

We estimate now the first term in (8.26); let $\mathcal{R}$ be $[a,b]$ and $c = \frac{b-a}{2}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}\setminus 2\mathcal{R}} \left| V_\beta\left(V_\alpha^{-1}(a)\right)(x) \right| d\sigma_\beta(x) \\
&= \int_{\mathbb{R}\setminus 2\mathcal{R}} \left| \int_{\mathbb{R}} a(t) \frac{1 - \overline{\Theta(t)}\Theta(x)}{2\pi i(t-x)} d\sigma_\alpha(t) \right| d\sigma_\beta(x) \\
&\le \int_{\mathbb{R}\setminus 2\mathcal{R}} \left| \int_{\mathbb{R}} a(t) \left(1 - e^{-2i\pi\alpha + 2i\pi\beta}\right) \left(\frac{1}{2\pi i(t-x)} - \frac{1}{2\pi i(c-x)} \right) d\sigma_\alpha(t) \right| d\sigma_\beta(x) \\
&\le \frac{1}{2\pi^2} \int_{\mathbb{R}\setminus 2\mathcal{R}} \int_{\mathbb{R}} |a(t)| \left| \frac{c-t}{(t-x)(c-x)} \right| d\sigma_\alpha(t) d\sigma_\beta(x) ,
\end{aligned}$$

where we have used condition (8.2). Changing the order of integration and using (8.25), we obtain that

$$
\begin{aligned}
&\int_0^1 \int_{\mathbb{R}\setminus 2\mathcal{R}} \left|V_\beta\left(V_\alpha^{-1}(a)\right)(x)\right| d\sigma_\beta(x)\, d\beta \\
&\leq \frac{(b-a)/2}{2\pi^2} \int_{\mathbb{R}} |a(t)| \int_0^1 \int_{\mathbb{R}\setminus 2\mathcal{R}} \left|\frac{1}{(t-x)(c-x)}\right| d\sigma_\beta(x) d\beta\, d\sigma_\alpha(t) \\
&\leq \frac{b-a}{4\pi^2} \int_{\mathbb{R}} |a(t)| \left(\int_0^1 \int_{-\infty}^{a-\frac{b-a}{2}} \frac{1}{(a-x)^2} d\sigma_\beta(x)\, d\beta \right. \\
&\qquad \left. + \int_0^1 \int_{b+\frac{b-a}{2}}^{\infty} \frac{1}{(x-b)^2} d\sigma_\beta(x)\, d\beta \right) d\sigma_\alpha(t) \\
&= \frac{b-a}{4\pi^2} \int_{\mathbb{R}} |a(t)| \left(\int_{-\infty}^{a-\frac{b-a}{2}} \frac{1}{(a-x)^2} dx + \int_{b+\frac{b-a}{2}}^{\infty} \frac{1}{(x-b)^2} dx \right) d\sigma_\alpha(t) \\
&\leq \frac{C2}{\pi^2} \|a\|_{L^1(\sigma_\alpha)} = \frac{2C}{\pi^2} ,
\end{aligned}
$$

since a satisfies (8.4). Considering the two estimates together, we have obtained (8.24).

If $f \in H^1_{\mathrm{at}}(\sigma_\alpha)$, we know that there exist atoms $\{a_k\}$ and coefficients $\{\lambda_k\}$ such that

$$
f = \sum_k \lambda_k a_k \quad \text{and} \quad \sum_k |\lambda_k| \leq 2 \|f\|_{H^1_{\mathrm{at}}(\sigma_\alpha)} .
$$

Consequently we note that

$$
\mathcal{V}_\alpha^{-1}(f)(z) := \sum_k \lambda_k \mathcal{V}_\alpha^{-1}(a_k)(z)
$$

and

$$
\left\|\mathcal{V}_\alpha^{-1}(f)\right\|_{\mathcal{H}^1(E)} = \left\|V_\alpha^{-1}(f)\right\|_{L^1(\mathbb{R})} \leq \sum_k |\lambda_k| \left\|V_\alpha^{-1}(a_k)\right\|_{L^1(\mathbb{R})} \leq C \sum_k |\lambda_k| \leq C' \|f\|_{H^1_{\mathrm{at}}} .
$$

With this computations we have obtained that $\mathcal{V}_\alpha^{-1}(f) \in \mathcal{H}^1(E)$ and (8.23), and therefore $H^1_{\mathrm{at}}(\sigma_\alpha) \hookrightarrow H^1(E)$.

4. Atomic decomposition: $\mathcal{H}^1(E) \subseteq H^1_{\mathrm{at}}(\sigma_\alpha)$

Thanks to condition (8.1), $\forall F \in \mathcal{H}^1(E)$, $F/E \in K^2(\Theta)$. We denote

$$
f(x) := \frac{F(x)}{E(x)} \quad \text{when } x \in \operatorname{supp} \sigma_\alpha .
$$

Our aim is to proving that $f \in H^1_{\mathrm{at}}(\sigma_\alpha)$ and

$$
\|f\|_{H^1_{\mathrm{at}}(\sigma_\alpha)} \leq C \|F/E\|_{L^1(\mathbb{R})} = C \|F\|_{\mathcal{H}^1(E)} . \tag{8.27}
$$

First of all, let us assume that

$$
F(x) = \overline{F(x)} , \text{ that is } f(x) = \Theta(x)\overline{f(x)} \quad \text{when } x \in \mathbb{R} . \tag{8.28}
$$

At the end, we will briefly consider the general case.

We denote by S_x the angular region

(8.29) $$S_x := \left\{z \in \overline{\mathbb{C}^+} \text{ such that } \pi/4 \leq \arg(z-x) \leq 3\pi/4\right\}$$

and we define the maximal function

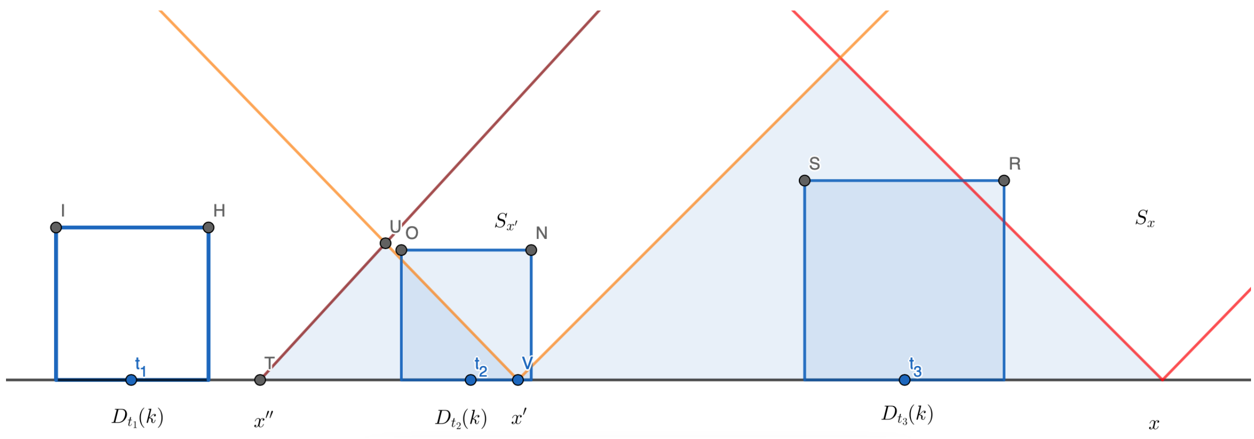
(8.30) $$f^*(x) := \sup_{z \in S_x} |F(z)/E(z)| \qquad \text{when } x \in \mathbb{R}\ .$$

We consider

(8.31) $$S_f(\lambda) := \overline{\mathbb{C}^+} \setminus \left\{z \in \overline{\mathbb{C}^+} \text{ such that there exists } x \in \mathbb{R}\ , z \in S_x \text{ and } f^*(x) < \lambda\right\}\ .$$

We denote

(8.32) $$R_f(\lambda) := \bigcup_i E_i \qquad \text{where } E_i \text{ are the connected components of } S_f(\lambda) \cup D_{\sigma_\alpha}(k)$$
$$\text{such that } E_i \cap S_f(\lambda) \neq \emptyset\ , E_i \cap D_{\sigma_\alpha}(k) \neq \emptyset\ .$$



We first prove some properties of the set $R_f(\lambda)$. First of all, $R_f(\lambda)$ is closed. Secondly, if $\lambda_1 < \lambda_2$, then $R_f(\lambda_2) \subseteq R_f(\lambda_1)$. Furthermore,

Lemma 8.14 $|f(x)| < \lambda$ for σ_α-almost all points $x \in \mathbb{R} \setminus R_f(\lambda)$.

Proof. It is clear that if $x \in \mathrm{supp}(\sigma_\alpha)$, then $x \in D_{\sigma_\alpha}(k)$. Therefore $x \notin R_f(\lambda)$ if and only if $D_x(k) \cap S_f(\lambda) = \emptyset$, that is $f^*(x) < \lambda$. □

Lemma 8.15 $|f(z)| \leq \lambda$ and $|\alpha - \Theta(z)| \geq \epsilon$ for all $z \in \partial R_f(\lambda) \cap \mathbb{C}^+$.

Proof. The second inequality is a direct consequence of Lemma 8.10. Furthermore, if $f(z) > \lambda$, z could not be a boundary point of $S_f(\lambda)$, since, in this case, $z \in S_x$ and $f^*(x) > \lambda$. □

In the following lemma, we will prove some other properties of the set $R_f(\lambda)$.

Lemma 8.16 Let E be one of the connected components of $R_f(\lambda)$. Put $\gamma = \partial E \cap \mathbb{C}^+$ and $R = \partial E \cap \mathbb{R}$. There exist constants depending only on $\Theta(z)$ such that

(1) γ is a rectifiable curve and $|\gamma| \le C\sigma_\alpha(R)$;
(2) $\sigma_\alpha(R) \le G\,|R \cap S_f(\lambda)|$ if E contains at least two atoms of σ_α;
(3) $\left|\int_R f d\sigma_\alpha\right| \le K\lambda\sigma_\alpha(R)$.

Proof. First of all, by the construction and Lemma 8.8, we know that

$$|\gamma| \le C'\,|R| \le C\sigma_\alpha(R) \ .$$

Furthermore, if R contains at least two points of $\operatorname{supp}(\sigma_\alpha)$, then

$$\sigma_\alpha(R) \le B_{\sigma_\alpha}\,|R \setminus D_{\sigma_\alpha}(k)| \le G\,|R \cap S_f(\lambda)| \ .$$

Let us now prove the third condition. We define $\gamma^* := \{z \in \mathbb{C} \text{ such that } \bar{z} \in \gamma\}$ and $\Gamma = \gamma \cup \gamma^*$. It is clear that $|\Gamma| \le 3\,|\gamma|$. Therefore, due to Lemma 8.15 ,

$$\left|\frac{f(z)}{1-\bar{\alpha}\Theta(z)}\right| \le \frac{\lambda}{\epsilon} \ , \quad z \in \Gamma \cap \mathbb{C}^+.$$

On the other hand, the function

$$z \to \overline{f(\bar{z})/\Theta(\bar{z})}$$

is analytic in G_θ defined in (8.10) and coincides with the function f on $\mathbb{R}$. By the uniqueness of the analytic continuation, we know that $f(z) = \overline{f(\bar{z})/\Theta(\bar{z})}$ for all $z \in G_\theta$. Now take a point $z \in G_\theta$; we compute

$$\frac{f(z)}{1-\bar{\alpha}\Theta(z)} = \frac{\overline{f(\bar{z})/\Theta(\bar{z})}}{1-\overline{\alpha/\Theta(\bar{z})}} = \overline{\left(\frac{f(\bar{z})}{\Theta(\bar{z})-\alpha}\right)} \ .$$

Therefore

$$\left|\frac{f(z)}{1-\bar{\alpha}\Theta(z)}\right| \le \frac{\lambda}{\epsilon} \ , \quad z \in \Gamma \cap G_\theta.$$

It is also true that

$$\begin{aligned}\oint_\Gamma \frac{f(z)}{1-\bar{\alpha}\Theta(z)}dz &= \frac{i}{2\pi}\oint_\Gamma \frac{1}{1-\bar{\alpha}\Theta(z)}\int_{\mathbb{R}} f(t)\frac{1-\bar{\alpha}\Theta(z)}{z-t}d\sigma_\alpha(t)dz \\ &= \frac{i}{2\pi}\int_{\mathbb{R}} f(t)\oint_\Gamma \frac{1}{z-t}dz d\sigma_\alpha(t) = \int_{\mathbb{R}} f(t)\chi_R(t)d\sigma_\alpha(t) \ ,\end{aligned}$$

because $f \in K^2(\Theta)$ and the function is absolutely integrable. Consequently

$$\left|\int_R f(t)d\sigma_\alpha(t)\right| \le \frac{\lambda}{2\pi\epsilon}\,|\Gamma| \le \frac{\lambda}{\epsilon}\frac{3}{2\pi}\,|\gamma| \le \frac{\lambda}{\epsilon}\frac{3}{2\pi}C\sigma_\alpha(R) \ .$$

This reasoning gives us property (3) when $\gamma \cap \rho(\theta) = \emptyset$. The general case can be reduced to the just considered one by a small perturbation of the contour γ. $\square$

For each $\lambda > 0$ the set $\mathcal{R}_f(\lambda) = R_f(\lambda) \cap \mathbb{R}$ is a union of closed disjoint interval, that is,

$$\mathcal{R}_f(\lambda) = \bigcup_{k \in I_\lambda} \mathcal{R}^k_f(\lambda) \ .$$

We consider the functions

$$G_\lambda(x) := \begin{cases} f(x) & x \in \mathbb{R} \setminus \mathcal{R}_f(\lambda) \\ \langle f \rangle_{\mathcal{R}^k_f(\lambda),\sigma_\alpha} & \text{otherwise} \end{cases} , \ B_\lambda(x) := \begin{cases} 0 & x \in \mathbb{R} \setminus \mathcal{R}_f(\lambda) \\ f(x) - \langle f \rangle_{\mathcal{R}^k_f(\lambda),\sigma_\alpha} & \text{otherwise} \end{cases} , \tag{8.33}$$

where

$$\langle f \rangle_{\mathcal{R}^k_f(\lambda),\sigma_\alpha} = \frac{1}{\sigma_\alpha(\mathcal{R}^k_f(\lambda))} \int_{\mathcal{R}^k_f(\lambda)} f(t) d\sigma_\alpha(t) \ .$$

Thanks to Lemma 8.14 and to property (3) of Lemma 8.16, we know that

$$|G_\lambda(x)| \le \max(K, 1)\lambda \quad \forall x \in \operatorname{supp}(\sigma_\alpha) \ , \tag{8.34}$$

and, from an easy computation, that

$$\langle B_\lambda(x) \rangle_{\mathcal{R}^k_f(\lambda),\sigma_\alpha} = 0 \ . \tag{8.35}$$

We define $g_n := G_{2^n}$ and $b_n := B_{2^n}$ where $-\infty < n < \infty$. It is clear that

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} (g_{n+1}(x) - g_n(x)) \ , \quad x \in \operatorname{supp}(\sigma_\alpha) \ . \tag{8.36}$$

Indeed, if we consider N large enough, we have that

$$\left| f(x) - \sum_{-N}^{N} (g_{n+1}(x) - g_n(x)) \right| = |g_{-N}(x)| \le K 2^{-N} \ .$$

Furthermore, we note that

$$f = b_n + g_n \quad \text{and consequently} \quad g_{n+1} - g_n = b_n - b_{n+1} \ .$$

Let I'_{2^n} be the set of indices $k \in I_{2^n}$ such that the set $\mathcal{R}^k_f(2^n)$ contains at least two atoms of the measure σ_α. The function $g_{n+1} - g_n$ vanishes σ_α-almost everywhere on each of the sets $\mathcal{R}^k_f(2^n)$ where $k \in I_{2^n} \setminus I'_{2^n}$. Indeed, for such index k, we have

$$\mathcal{R}^k_f(2^n) \cap \operatorname{supp}(\sigma_\alpha) = \{x\} \quad \text{and} \quad g_n(x) = g_{n+1}(x) = f(x) \ .$$

We define

$$a_{n,k}(x) = \chi_{\mathcal{R}^k_f(2^n)}(x)\,(b_n(x) - b_{n+1}(x)) \ , \quad \text{where } k \in I'_{2^n} \ . \tag{8.37}$$

The function $a_{n,k}(x)$ has got zero σ_α-mean. Indeed

$$\int_{\mathbb{R}} a_{n,k}(x) d\sigma_\alpha(x) = \int_{\mathcal{R}^k_f(2^n)} b_n(x) - b_{n+1}(x)\ d\sigma_\alpha(x) = -\sum_{m \in I} \int_{\mathcal{R}^m_f(2^{n+1})} b_{n+1}(x) d\sigma_\alpha(x) = 0 \ ,$$

where I denotes the set of indexes m such that $\mathcal{R}_f^m(2^{n+1}) \subset \mathcal{R}_f^k(2^n)$. We note also that

$$|a_{n,k}(x)| \le (|g_{n+1}(x)| + |g_n(x)|) \le 2\max(K,1)2^{n+1} \ .$$

Now we consider

$$A_{n,k}(x) = \frac{a_{n,k}(x)}{2\max(1,K)2^{n+1}\sigma_\alpha(\mathcal{R}_f^k(2^n))}, \quad k \in I'_{2^n} \ , \tag{8.38}$$

and we observe that $A_{n,k}(x)$ are atoms with respect to the measure σ_α. From (8.36), we note that

$$f(x) = \sum_{n\in\mathbb{Z}} \sum_{k\in I'_{2^n}} \lambda_{n,k} A_{n,k}(x) \ , \quad x \in \mathrm{supp}(\sigma_\alpha) \ , \tag{8.39}$$

where

$$\lambda_{n,k} := 2^{n+2}\max(1,K)\sigma_\alpha(\mathcal{R}_f^k(2^n)) \ . \tag{8.40}$$

It remains to check (8.27), that is,

$$\sum_n \sum_{k\in I'_{2^n}} \lambda_{n,k} \le C \|f\|_{L^1(\mathbb{R})} \ . \tag{8.41}$$

Indeed, by using property (2) of Lemma 8.16 and the fact that $f \in H^1(\mathbb{C}^+)$, we obtain that

$$\begin{aligned}
\sum_n \sum_{k\in I'_{2^n}} 2^n \sigma_\alpha(\mathcal{R}_f^k(2^n)) &\le \sum_n \sum_{k\in I'_{2^n}} G2^n \left|\mathcal{R}_f^k(2^n) \cap S_f(2^n)\right| \\
&\le \sum_n G2^n \left|\mathbb{R} \cap S_f(2^n)\right| \\
&= G\sum_n 2^n \left|\{x \in \mathbb{R} \mid f^*(x) \ge 2^n\}\right| \\
&\le G\sum_n 2^n \sum_{l=0}^{\infty} \left|\{x \ : 2^{n+l} < f^*(x) \le 2^{n+1+l}\}\right| \\
&\le \sum_{m\in\mathbb{Z}} 2^m \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) \left|\{x \ : 2^m < f^*(x) \le 2^{m+1}\}\right| \\
&\le 2\|f^*\|_{L^1(\mathbb{R})} \le C' \|f\|_{L^1(\mathbb{R})} \ ,
\end{aligned}$$

from which we have proved (8.41).

Untill this point, we have studied the problem assuming that the function f satisfies (8.28). In the general case, let f_1, f_2, be the functions from Lemma 8.11 associated to f. We apply the above argument to f_i and since $f = f_1 + if_2$, we obtain also the atomic decomposition for f.

5. The dual of $H^1_{\mathbf{at}}(\sigma_\alpha)$

In the previous sections we have proved that the two spaces $\mathcal{H}^1(E)$ and $H^1_{\mathrm{at}}(\sigma_\alpha)$ are isomorphic with equivalence of norms. Indeed when the meromorphic inner function $\Theta(z) := E^\#(z)/E(z)$

satisfies (8.5) and $\infty \in \rho(\Theta)$, if $\mathcal{H}^1(E) \hookrightarrow \mathcal{H}^2(E)$, then the space $\mathcal{H}^1(E)$ is isomorphic to $H^1_{\rm at}(\sigma_\alpha)$. In particular, the surjective isomorphism between the two spaces is given by:

$$\mathcal{V}_\alpha \ : \ \mathcal{H}^1(E) \to H^1_{\rm at}(\sigma_\alpha) \ , \qquad F(x) \mapsto f(x)|_{\mathrm{supp}(\sigma_\alpha)} := \left\{ \frac{F(t_n^\alpha)}{E(t_n^\alpha)} \right\} \ , \tag{8.42}$$

and

$$\mathcal{V}_\alpha^{-1} : H^1_{\rm at}(\sigma_\alpha) \ \to \mathcal{H}^1(E) \ , \qquad f(x)|_{\mathrm{supp}(\sigma_\alpha)} \mapsto \sum_n f(t_n^\alpha) E(t_n^\alpha) \frac{k_{t_n^\alpha}(z)}{k_{t_n^\alpha}(t_n^\alpha)} \ , \tag{8.43}$$

where $k_{t_n}(z)$ are the reproducing kernels of $\mathcal{H}^2(E)$. Furthermore

$$C_1 \left\| \mathcal{V}_\alpha(F) \right\|_{H^1_{\rm at}(\sigma_\alpha)} \le \|F\|_{\mathcal{H}^1(E)} \le C_2 \left\| \mathcal{V}_\alpha(F) \right\|_{H^1_{\rm at}(\sigma_\alpha)} .$$

We are interested in the space of sequences $H^1_{\rm at}(\sigma_\alpha)$ since we are able to characterize its dual.

Definition 8.17 The space $\widetilde{\mathrm{BMO}}(\sigma_\alpha)$ is made by all the function $g(x)$ such that

$$\|g\|_{\mathrm{BMO}(\sigma_\alpha)} := \sup_{A \subset \mathbb{R}} \frac{1}{\sigma_\alpha(A)} \int_A \left| g(x) - \langle g \rangle_{A,\sigma_\alpha} \right| d\sigma_\alpha(x) < \infty \tag{8.44}$$

where $A \subset \mathbb{R}$ is connected and

$$\langle g \rangle_{A,\sigma_\alpha} := \frac{1}{\sigma_\alpha(A)} \int_A g(x) d\sigma_\alpha(x) \ .$$

We note that the operator $\|\cdot\|_{\mathrm{BMO}(\sigma_\alpha)}$ is not a norm in $\widetilde{BMO}(\sigma_\alpha)$, since

$$c(t_n^\alpha) := c \neq 0 \text{ but } \ \|c\|_{\mathrm{BMO}(\sigma_\alpha)} = 0 \ .$$

Therefore, we have to consider the quotient space. We define

$$\mathrm{BMO}(\sigma_\alpha) := \widetilde{\mathrm{BMO}}(\sigma_\alpha) \Big/ \left\{ f \in \widetilde{\mathrm{BMO}}(\sigma_\alpha) \ \text{ such that } \ \|f\|_{\mathrm{BMO}(\sigma_\alpha)} = 0 \right\} \ ,$$

that is

$$\mathrm{BMO}(\sigma_\alpha) := \widetilde{\mathrm{BMO}}(\sigma_\alpha) / \left\{ c(x) : \ c(t_n^\alpha) = k \ , \ k \in \mathbb{C} \right\} \ . \tag{8.45}$$

With the help of this quotient, the operator

$$\|f\|_{\mathrm{BMO}(\sigma_\alpha)} := \sup_A \frac{1}{\sigma_\alpha(A)} \int_A \left| f(x) - \langle f \rangle_{A,\sigma_\alpha} \right| d\sigma_\alpha(x) \tag{8.46}$$

defines a norm in $\mathrm{BMO}(\sigma_\alpha)$.

We note that almost all the properties of $\mathrm{BMO}(\mathbb{R})$ hold also for $\mathrm{BMO}(\sigma_\alpha)$. First of all, if there exist constants c_A such that

$$\sup_{A \subset \mathbb{R}} \frac{1}{\sigma_\alpha(A)} \int_A |b(x) - c_A| \, d\sigma_\alpha(x) < \infty$$

for every bounded interval of $\mathbb{R}$, then

$$\sup_{A \subset \mathbb{R}} \frac{1}{\sigma_\alpha(A)} \int_A \left| b(x) - \langle b \rangle_{A,\sigma_\alpha} \right| d\sigma_\alpha(x) \le \sup_{A \subset \mathbb{R}} \frac{2}{\sigma_\alpha(A)} \int_A |b(x) - c_A| \, d\sigma_\alpha(x) < \infty \ ,$$

and consequently $b(x) \in \widetilde{\mathrm{BMO}}(\sigma_\alpha)$.

Proposition 8.18 Let $b \in \mathrm{BMO}(\sigma_\alpha)$. Then $\Re b(x), \Im b(x) \in \mathrm{BMO}(\sigma_\alpha)$.

PROOF. We have to check that (8.46) is bounded for $\Re b$ and $\Im b$. Indeed

$$\|\Re b\|_{\mathrm{BMO}(\sigma_\alpha)} \leq \frac{1}{2}\left(\|b\|_{\mathrm{BMO}(\sigma_\alpha)} + \|\bar{b}\|_{\mathrm{BMO}(\sigma_\alpha)}\right) = \|b\|_{\mathrm{BMO}(\sigma_\alpha)} \ .$$

With the same computations we obtain also (8.46) for $\Im b(x)$. □

Any function $c(x) \in \mathrm{BMO}(\sigma_\alpha)$ can be approximated by sequences in $L^\infty(\sigma_\alpha)$. We need this auxiliary proposition.

Proposition 8.19 Let $c(x)$ be a real function. Then

$$c^{\mathcal{M}}(x) := \begin{cases} -\mathcal{M} & \text{if } c(x) < \mathcal{M}\ , \\ \mathcal{M} & \text{if } c(x) > \mathcal{M}\ , \\ c(x) & \text{otherwise} \end{cases} \quad \in \mathrm{BMO}(\sigma_\alpha)\ .$$

PROOF. This proposition is a consequence of the fact that the set made up of all the real sequences in $\mathrm{BMO}(\sigma_\alpha)$ is a lattice. Indeed, if $f(x), g(x) \in \mathrm{BMO}(\sigma_\alpha)$, then also

$$\|\{\min(f(x), g(x))\}\|_{\mathrm{BMO}(\sigma_\alpha)} \leq 2\max\left(\|f\|_{\mathrm{BMO}(\sigma_\alpha)}, \|g\|_{\mathrm{BMO}(\sigma_\alpha)}\right)\ ,$$

$$\|\{\max(f(x), g(x))\}\|_{\mathrm{BMO}(\sigma_\alpha)} \leq 2\max\left(\|f\|_{\mathrm{BMO}(\sigma_\alpha)}, \|g\|_{\mathrm{BMO}(\sigma_\alpha)}\right)\ .$$

The above inequalities are justified by the fact that if $b(x) \in \mathrm{BMO}(\sigma_\alpha)$, then

$$\|\{|b|\}\|_{\mathrm{BMO}(\sigma_\alpha)} \leq 2\,\|\{b\}\|_{\mathrm{BMO}(\sigma_\alpha)}\ ,$$

which proves the theorem. □

Proposition 8.20 Let $b(x) \in \mathrm{BMO}(\sigma_\alpha)$. There exists a family $c(x)^j$ of $L^\infty(\sigma_\alpha)$ function such that

$$\left\|c^j\right\|_{\mathrm{BMO}(\sigma_\alpha)} \leq C\,\|b\|_{\mathrm{BMO}(\sigma_\alpha)} \qquad \text{and} \qquad \lim_{j\to\infty} c(x)^j = b(x)\ , \text{ for every fixed } x \in \mathrm{supp}(\sigma_\alpha)\ .$$

PROOF. According to Proposition 8.18 and Proposition 8.19, it is enough defining the function $c^j(x)$ as

$$c^j(x) = (\Re b)^j\,(x) + i\,(\Im b)^j\,(x)\ , \text{ as } j > 0 \text{ and goes to infinity.}$$

□

The following proposition describes a fundamental property of the elements of $\mathrm{BMO}(\sigma_\alpha)$, which will allows us to provide an analytic description of $\mathcal{H}^1(E)^*$.

Proposition 8.21 Let $\Theta(x)$ be a meromorphic (CLS) inner function and $\infty \in \rho(\Theta)$. If $\phi(x) \in \mathrm{BMO}(\sigma_\alpha)$, then

$$\int_{\mathbb{R}} |\phi(x)| \frac{d\sigma_\alpha(x)}{1+x^2} < \infty \ .$$

Proof. Let us consider $\phi(x) \in \mathrm{BMO}(\sigma_\alpha)$ and let $z = w + i \in \mathbb{C}^+$. I_0 is the interval

$$I_0 := \{t_n^\alpha \in \mathbb{R} \quad : |w - t_n^\alpha| < s\}$$

where we choose s so that $\#\left(\mathrm{supp}(\sigma_\alpha) \cap I_0\right) \geq 2$. On the other hand, I_k is the interval

$$I_k = \left\{t_n^\alpha \in \mathbb{R} \ : |w - t_n^\alpha| < 2^k s\right\}$$

where $k \in \mathbb{N}_0$. Then, because of Lemma 8.8, $\sigma_\alpha(I_k) \leq Ks2^{k+1}$, and $|I_k| \leq B\sigma_\alpha(I_k)$. Moreover

$$P_1(w - t_n^\alpha) := \frac{1}{(t_n^\alpha - w)^2 + 1} \leq 1 \text{ when } t_n^\alpha \in I_0 \ ,$$
$$P_1(t_n^\alpha - w) \leq 4/s^2 2^{2k} \text{ when } t_n^\alpha \in I_k \setminus I_{k-1} \ .$$

Consequently, if we define $I_{-1} := \emptyset$, then

$$\begin{aligned}
&\int \left|\phi(x) - \langle\phi\rangle_{I_0,\sigma_\alpha}\right| P_1(w-x) d\sigma_\alpha(x) \\
&= \sum_{k\in\mathbb{N}} \int_{I_k \setminus I_{k-1}} \left|\phi(x) - \langle\phi\rangle_{I_0,\sigma_\alpha}\right| P_1(w-x) d\sigma_\alpha(x) \\
&\leq \frac{1}{s^2} \int_{I_0} \left|\phi(x) - \langle\phi\rangle_{I_0,\sigma_\alpha}\right| d\sigma_\alpha(x) + \sum_{k\in\mathbb{N}_0} \frac{4}{s^2 2^{2k}} \int_{I_k\setminus I_{k-1}} \left|\phi(x) - \langle\phi\rangle_{I_k,\sigma_\alpha}\right| d\sigma_\alpha(x) \\
&\qquad + \sum_{k\in\mathbb{N}_0} \frac{4}{s^2 2^{2k}} \int_{I_k\setminus I_{k-1}} \left|\langle\phi\rangle_{I_k,\sigma_\alpha} - \langle\phi\rangle_{I_0,\sigma_\alpha}\right| d\sigma_\alpha(x) \\
&\leq \frac{2K}{s\sigma_\alpha(I_0)} \int_{I_0} \left|\phi(x) - \langle\phi\rangle_{I_0,\sigma_\alpha}\right| d\sigma_\alpha(x) + \sum_{k\in\mathbb{N}_0} \frac{4K}{s2^k} \frac{1}{\sigma_\alpha(I_k)} \int_{I_k} \left|\phi(x) - \langle\phi\rangle_{I_k,\sigma_\alpha}\right| d\sigma_\alpha(x) \\
&\qquad + \sum_{k\in\mathbb{N}_0} \frac{4}{s2^k} \left|\langle\phi\rangle_{I_k,\sigma_\alpha} - \langle\phi\rangle_{I_0,\sigma_\alpha}\right| \\
&\leq \frac{2K}{s} \|\phi\|_{\mathrm{BMO}(\sigma_\alpha)} + \frac{4K}{s} \|\phi\|_{\mathrm{BMO}(\sigma_\alpha)} + \sum_{k\in\mathbb{N}_0} \frac{kBK}{s2^k} \|\phi\|_{\mathrm{BMO}(\sigma_\alpha)} \\
&\leq C \|\phi\|_{\mathrm{BMO}(\sigma_\alpha)} \ ,
\end{aligned}$$

where we used the fact that

$$\begin{aligned}\left|\langle\phi\rangle_{I_k,\sigma_\alpha}-\langle\phi\rangle_{I_{k-1},\sigma_\alpha}\right| &\le \frac{1}{\sigma_\alpha(I_{k-1})}\int_{I_{k-1}}\left|\phi(x)-\langle\phi\rangle_{I_k,\sigma_\alpha}\right|d\sigma_\alpha(x)\\ &\le \frac{\sigma_\alpha(I_k)}{\sigma_\alpha(I_{k-1})}\frac{1}{\sigma_\alpha(I_k)}\int_{I_k}\left|\phi(x)-\langle\phi\rangle_{I_k,\sigma_\alpha}\right|d\sigma_\alpha(x)\\ &\le \frac{BKs2^{k+1}}{2^{k+1}s}\|\phi\|_{\mathrm{BMO}(\sigma_\alpha)}\end{aligned}$$

and consequently,

$$\left|\langle\phi\rangle_{I_k,\sigma_\alpha}-\langle\phi\rangle_{I_0,\sigma_\alpha}\right|\le\sum_{i=1}^{k}\left|\langle\phi\rangle_{I_i,\sigma_\alpha}-\langle\phi\rangle_{I_{i-1},\sigma_\alpha}\right|\le BKk\,\|\phi\|_{\mathrm{BMO}(\sigma_\alpha)}\ .$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}}\frac{|\phi(x)|}{1+x^2}d\sigma_\alpha(x) &= \int|\phi(x)|\,P_1(0-x)d\sigma_\alpha(x)\\ &\le C\,\|\phi\|_{\mathrm{BMO}(\sigma_\alpha)}+\left|\langle\phi\rangle_{I_0,\sigma_\alpha}\right|\int_{\mathbb{R}}\frac{d\sigma_\alpha(x)}{1+x^2}<\infty\ ,\end{aligned}$$

which proves the theorem. □

Before characterizing $H^1_{\rm at}(\sigma_\alpha)^*$, we need another preliminar lemma.

Lemma 8.22 Let $b(x)\in L^2(\sigma_\alpha)$ such that $\sigma_\alpha(\mathrm{supp}(b))<\infty$. If

$$\int_{\mathbb{R}}b(x)d\sigma_\alpha(x)=0\ ,$$

then

$$\|b\|_{H^1_{\rm at}(\sigma_\alpha)}\le C\,\|b\|_{L^2(\sigma_\alpha)}\left[\sigma_\alpha(\mathrm{supp}(b))\right]^{1/2}\ . \tag{8.47}$$

Proof. First of all, we note that $b(x)\in H^1_{\rm at}(\sigma_\alpha)$, since it is a multiple of an $H^1_{\rm at}(\sigma_\alpha)$-atom. To obtain (8.47), we have to use Aleksandrov-Clark disintegration formula, (8.25). Indeed

$$\begin{aligned}\|b\|_{H^1_{\rm at}(\sigma_\alpha)}=\left\|\mathcal{V}_\alpha^{-1}(b)\right\|_{\mathcal{H}^1(E)} &= \left\|V_\alpha^{-1}(b)\right\|_{L^1(\mathbb{R})}\\ &= \int_0^1\int_{\mathbb{R}}\left|V_\beta\left(V_\alpha^{-1}(b)\right)(x)\right|d\sigma_\beta(x)d\beta\ .\end{aligned}$$

With the same computations we did in the previous section, we obtain (8.47), since

$$\begin{aligned}\|b\|_{H^1_{\rm at}(\sigma_\alpha)} &\le C\sigma_\alpha(\mathrm{supp}(b))^{1/2}\left\|V_\alpha^{-1}(b)\right\|_{L^2(\mathbb{R})}+C'\,\|b\|_{L^1(\sigma_\alpha)}\\ &\le C\sigma_\alpha(\mathrm{supp}(b))^{1/2}\left\|\mathcal{V}_\alpha^{-1}(b)\right\|_{\mathcal{H}^2(E)}+C'\sigma_\alpha(\mathrm{supp}(b))^{1/2}\,\|b\|_{L^2(\sigma_\alpha)}\\ &= \tilde{C}\sigma_\alpha(\mathrm{supp}(b))^{1/2}\,\|b\|_{L^2(\sigma_\alpha)}\ ,\end{aligned}$$

which proves the theorem. □

Theorem 8.23 Let $\Theta(z)$ be a meromorphic (CLS) inner function, and let $\infty \in \rho(\Theta)$. If $\mathcal{H}^1(E) \hookrightarrow \mathcal{H}^2(E)$, then the dual space of $\mathcal{H}^1(E)$ is isomorphic to $\mathrm{BMO}(\sigma_\alpha)$.

Proof. Thanks to Theorem 8.1, in order to prove this result, it is enough checking that the dual of $H^1_{\mathrm{at}}(\sigma_\alpha)$ is isomorphic with equivalence of norms to $\mathrm{BMO}(\sigma_\alpha)$.

First of all, we prove that $(H^1_{\mathrm{at}}(\sigma_\alpha))^* \subseteq \mathrm{BMO}(\sigma_\alpha)$. Let us consider $a(x) \in L^2(\sigma_\alpha)_{A,0}$, where

$$L^2(\sigma_\alpha)_{A,0} := \left\{ a(x) \in L^2(\sigma_\alpha) : \mathrm{supp}(a) \subset A,\ \#(A \cap \mathrm{supp}(\sigma_\alpha)) < \infty \text{ and } \int a(x) d\sigma_\alpha(x) = 0 \right\} . \tag{8.48}$$

Let $\beta \in H^1_{\mathrm{at}}(\sigma_\alpha)^*$. Then, since the elements of $L^2(\mathbb{R}, \sigma_\alpha)_{A,0}$ are multiples of atoms of $H^1_{\mathrm{at}}(\sigma_\alpha)$, we obtain that

$$|\beta(a)| \leq \|\beta\|_* \|a\|_{H^1_{\mathrm{at}}(\sigma_\alpha)} \leq \|\beta\|_* \|a\|_{L^2(\mathbb{R},\sigma_\alpha)} (\sigma_\alpha(A))^{1/2} ,$$

where the second inequality is justified by (8.47). Therefore the functional β belongs to ${L^2(\sigma_\alpha)_{A,0}}^*$ and there exists $b_A \in L^2(\sigma_\alpha)_{A,0}$ such that

$$\int_A |b_A(x)|^2 \, d\sigma_\alpha(x) \leq \|\beta\|_*^2 \, \sigma_\alpha(A)$$

and such that

$$\beta(a) = \int_A a(x) \overline{b_A(x)} d\sigma_\alpha(x) .$$

Hence for every set A the function $b_A(x)$ is well defined. However we want a single function $\tilde{b}$ such that $\tilde{b}|_A - b_A$ is constant. To construct it, we consider that if $A_1 \subset A_2$, $b_{A_1} - b_{A_2}$ is constant in A_1. Therefore, we modify b_A, defining $\tilde{b}_A(x) := b_A(x) - b_A(t_0^\alpha)$. It follows that if $A_1 \subset A_2$, then $\tilde{b}_{A_1} = \tilde{b}_{A_2}$ on A_1. We can therefore consider unambiguously $\tilde{b}_A(x)$ when $x \in A$. Furthermore

$$\frac{1}{\sigma_\alpha(A)} \int_A \left| \tilde{b_A}(x) + b_A(t_0^\alpha) \right| d\sigma_\alpha(x) = \left(\frac{1}{\sigma_\alpha(A)} \int_A \left| \tilde{b_A}(x) + b_A(t_0^\alpha) \right|^2 d\sigma_\alpha(x) \right)^{1/2}$$

$$= \left(\frac{1}{\sigma_\alpha(A)} \int_A |b_A(x)|^2 \, d\sigma_\alpha(x) \right)^{1/2} \leq \|\beta\|_* \ .$$

The representative for the functional β is $\tilde{b}$, which does not depend on the choice of the set A.

For the other inclusion, we consider $a(x)$, an $H^1_{\mathrm{at}}(\sigma_\alpha)$-atom, such that $\mathrm{supp}(a) = A$ and let $b(x) \in \mathrm{BMO}(\sigma_\alpha)$. Then

$$\begin{aligned}\left|\int a(x)\overline{b(x)}d\sigma_\alpha(x)\right| &= \left|\int a(x)\left(\overline{b(x)-\langle b\rangle_{A,\sigma_\alpha}}\right)d\sigma_\alpha(x)\right|\\ &\le \frac{1}{\sigma_\alpha(A)}\int_A \left|b(x)-\langle b\rangle_{A,\sigma_\alpha}\right| d\sigma_\alpha(x) \le \|b\|_{\mathrm{BMO}(\sigma_\alpha)} \ .\end{aligned}$$

If $g(x)=\sum_j \lambda_j a^j(x)$, $2\|g\|_{H^1_{\mathrm{at}}(\sigma_\alpha)} \ge \sum_j |\lambda^j|$ and $b(x)\in \mathrm{BMO}(\sigma_\alpha)\cap L^\infty(\sigma_\alpha)$, then

$$\begin{aligned}\left|\int_{\mathbb{R}} g(x)\overline{b(x)}d\sigma_\alpha(x)\right| &\le \int_{\mathbb{R}}\sum_j |\lambda_j|\left|a^j(x)b(x)\right| d\sigma_\alpha(x)\\ &= \sum_j |\lambda_j|\int_{\mathbb{R}}\left|a^j(x)b(x)\right| d\sigma_\alpha(x) \le 2\|g\|_{H^1_{\mathrm{at}}(\sigma_\alpha)}\|b\|_{L^\infty(\sigma_\alpha)} \ ,\end{aligned}$$

and consequently

$$\begin{aligned}\left|\int_{\mathbb{R}} g(x)\overline{b(x)}d\sigma_\alpha(x)\right| &= \left|\sum_j \lambda_j\int_{\mathbb{R}} a^j(x)\overline{b(x)}d\sigma_\alpha(x)\right|\\ &\le \sum_j |\lambda_j|\,\|b\|_{\mathrm{BMO}(\sigma_\alpha)} \le 2\|g\|_{H^1_{\mathrm{at}}(\sigma_\alpha)}\|b\|_{\mathrm{BMO}(\sigma_\alpha)} \ .\end{aligned}$$

To extend the above computations to $b(x)\notin L^\infty(\sigma_\alpha)$, we observe that if $b(x)\in\mathrm{BMO}(\sigma_\alpha)$, there exists a sequence $b^{\mathcal{M}}(x)\in L^\infty(\sigma_\alpha)$, such that

$$\left\|b^{\mathcal{M}}\right\|_{\mathrm{BMO}(\sigma_\alpha)} \le C\|b\|_{\mathrm{BMO}(\sigma_\alpha)} \ .$$

Therefore

$$\left\|b^{\mathcal{M}}\right\|_{H^1_{\mathrm{at}}(\sigma_\alpha)^*} \le C'\left\|b^{\mathcal{M}}\right\|_{\mathrm{BMO}(\sigma_\alpha)} \le C\|b\|_{\mathrm{BMO}(\sigma_\alpha)} \ .$$

Consequently

$$L(f) := \lim_{\mathcal{M}\to\infty}\left\langle f, b^{\mathcal{M}}\right\rangle_{L^2(\sigma_\alpha)} \ , \quad \forall f\in H^1_{\mathrm{at}}(\sigma_\alpha)$$

and $L\in H^1_{\mathrm{at}}(\sigma_\alpha)^*$. Thanks to the first part of the theorem, when f is an $H^1_{\mathrm{at}}(\sigma_\alpha)$-atom

$$L(f) = \langle f,\beta\rangle_{L^2(\sigma_\alpha)} \ , \text{ where } \beta\in\mathrm{BMO}(\sigma_\alpha) \ .$$

Consequently $b(x)$ is a representative of $\beta(x)$. Indeed $b(x)-\beta(x)$ is equal to a constant and

$$\begin{aligned}L(f) = \lim_{\mathcal{M}\to\infty}\left\langle f, b^{\mathcal{M}}\right\rangle_{L^2(\mathbb{R},\sigma_\alpha)} &\le \lim_{\mathcal{M}\to\infty} 2C\left\|b^{\mathcal{M}}\right\|_{\mathrm{BMO}(\sigma_\alpha)}\|f\|_{H^1(\sigma_\alpha)}\\ &\le 2C_2\|b\|_{\mathrm{BMO}(\sigma_\alpha)}\|f\|_{H^1(\sigma_\alpha)} \ ,\end{aligned}$$

which proves the theorem. □

6. The dual of $\mathcal{H}^1(E)$: the space $\mathcal{X}(E)$

In this section we give another equivalent description of $\mathcal{H}^1(E)^*$: we associate to every $b(x) \in \widetilde{\mathrm{BMO}}(\sigma_\alpha)$ an entire function.

If $b(x) \in \widetilde{\mathrm{BMO}}(\sigma_\alpha)$, we consider $\tilde{T}_\alpha^{-1}(b)(z)$, defined as:

$$\tilde{T}_\alpha^{-1}(b)(z) := \sum_{n\in\mathbb{Z}} \left(\frac{E(t_n^\alpha)b(t_n^\alpha)}{t_n^\alpha - z} - \frac{E(t_n^\alpha)b(t_n^\alpha)t_n^\alpha}{t_n^{\alpha 2}+1} \right) \frac{E(z)\overline{E(t_n^\alpha)} - E^\#(z)E(t_n^\alpha)}{2\pi i} \frac{\pi}{\phi'(t_n^\alpha)\left|E(t_n^\alpha)\right|^2} . \tag{8.49}$$

When $z \in \mathbb{R}$, we write (8.49) in a different way:

$$\begin{aligned}
&\tilde{T}_\alpha^{-1}(b)(x) \\
&= \sum_{n\in\mathbb{Z}} E(t_n^\alpha)b(t_n^\alpha) \left(\frac{1}{t_n^\alpha - x} - \frac{t_n^\alpha}{t_n^{\alpha 2}+1} \right) \frac{|E(t_n^\alpha)|\,|E(x)| \left(e^{i(\phi(t_n^\alpha)-\phi(x))} - e^{-i(\phi(t_n^\alpha)-\phi(x))} \right)}{2i\phi'(t_n^\alpha)\left|E(t_n^\alpha)\right|^2} \\
&= \sum_{n\in\mathbb{Z}} E(t_n^\alpha)b(t_n^\alpha) \left(\frac{1}{t_n^\alpha - x} - \frac{t_n^\alpha}{t_n^{\alpha 2}+1} \right) \frac{|E(x)| \sin(\phi(t_n^\alpha) - \phi(x))}{\phi'(t_n^\alpha)\,|E(t_n^\alpha)|} \\
&= |E(x)| \sum_{n\in\mathbb{Z}} E(t_n^\alpha)b(t_n^\alpha) \left(\frac{1}{t_n^\alpha - x} - \frac{t_n^\alpha}{t_n^{\alpha 2}+1} \right) \frac{\sin(\alpha\pi + n\pi - \phi(x))}{\phi'(t_n^\alpha)\,|E(t_n^\alpha)|} \\
&= |E(x)| \left(\sin(\alpha\pi)\cos(\phi(x)) - \cos(\alpha\pi)\sin(\phi(x)) \right) \sum_{n\in\mathbb{Z}} b(t_n^\alpha) \left(\frac{1}{t_n^\alpha - x} - \frac{t_n^\alpha}{t_n^{\alpha 2}+1} \right) \frac{E(t_n^\alpha)(-1)^n}{|E(t_n^\alpha)|\,\phi'(t_n^\alpha)} .
\end{aligned}$$

Since,

$$\begin{aligned}
&|E(x)| \left(\sin(\alpha\pi)\cos(\phi(x)) - \cos(\alpha\pi)\sin(\phi(x)) \right) \\
&= \frac{|E(x)|}{2i} \left(e^{i\alpha\pi}\cos(\phi(x)) - e^{-i\alpha\pi}\cos(\phi(x)) - i\sin(\phi(x))e^{i\alpha\pi} - i\sin(\phi(x))e^{-i\alpha\pi} \right) \\
&= \frac{1}{2i} \left(e^{i\alpha\pi}E(x) - e^{-i\alpha\pi}E^\#(x) \right) ,
\end{aligned}$$

we obtain that

$$\begin{aligned}
\tilde{T}_\alpha^{-1}(b)(x) &= \frac{e^{-i\alpha\pi}}{2i} \left(e^{i\alpha\pi}E(x) - e^{-i\alpha\pi}E^\#(x) \right) \sum_{n\in\mathbb{Z}} b(t_n^\alpha) \left(\frac{1}{t_n^\alpha - x} - \frac{t_n^\alpha}{t_n^{\alpha 2}+1} \right) \frac{1}{\phi'(t_n^\alpha)} \\
&= \frac{e^{-i\alpha\pi}}{2i} S_\alpha(x) \sum_{n\in\mathbb{Z}} b(t_n^\alpha) \left(\frac{1}{t_n^\alpha - x} - \frac{t_n^\alpha}{t_n^{\alpha 2}+1} \right) \frac{1}{\phi'(t_n^\alpha)} ,
\end{aligned} \tag{8.50}$$

where S_α

$$S_\alpha(z) := e^{i\alpha\pi}E(z) - e^{-i\alpha\pi}E^\#(z) .$$

From now on, we consider the case $\alpha = 0$ and we do not write α unless two indexes would be used at the same time. We assume that $\{k_{t_n}\}_{n\in\mathbb{Z}}$ is a basis of $\mathcal{H}^2(E)$.

We note that when $\alpha = 0$, (8.49) becomes

$$\tilde{T}^{-1}(b)(z) := -B(z)\sum_{n\in\mathbb{Z}} b(t_n)\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right)\frac{1}{\phi'(t_n)} \ . \tag{8.51}$$

The function $\tilde{T}^{-1}(b)(z)$ is entire, since it is the uniform limit of entire functions on every compact subset of $\mathbb{C}$. Indeed, due to Proposition 8.21, the series converges uniformly on the compact subset of $\mathbb{C}$. Furthermore, if $z = t_n$, then

$$\begin{aligned} &\tilde{T}^{-1}(b)(t_n) \\ &\quad = b(t_n)\frac{B'(t_n)}{\phi'(t_n)} + B(t_n)\frac{t_n b(t_n)}{\phi'(t_n)(t_n^2+1)} - B(t_n)\sum_{n\neq k\in\mathbb{Z}}\frac{b(t_k)}{\phi'(t_k)}\left(\frac{1}{t_n - t_k} - \frac{t_k}{t_k^2+1}\right) \\ &\quad = b(t_n)\left|E(t_n)\right|(-1)^n \ , \end{aligned} \tag{8.52}$$

where we used the fact that $B(x) = \sin(\phi(x))\left|E(x)\right|$. Consequently,

$$\tilde{T}^{-1}(b)(t_n) := b(t_n)\left|E(t_n)\right|\cos(\phi(t_n)) = b(t_n)A(t_n) = b(t_n)E(t_n) \ . \tag{8.53}$$

We note that the operator $\tilde{T}^{-1}$ is injective. Indeed, if $\tilde{T}^{-1}(b)(z)$ is the zero function, then $\tilde{T}^{-1}(b)(t_n) = 0$ for every $n \in \mathbb{Z}$. Consequently $b(x) = 0$, $\forall x \in \mathrm{supp}(\sigma_0)$.

We define

$$\left\|\tilde{T}^{-1}(b)\right\|_{\mathrm{BMO}(\sigma_\alpha),\mathbb{R}} := \left\|\frac{\tilde{T}^{-1}(b)(x)}{E(x)}\right\|_{\mathrm{BMO}(\sigma_\alpha)} \ , \tag{8.54}$$

and therefore $\|\cdot\|_{\mathrm{BMO}(\sigma_\alpha),\mathbb{R}}$ is a seminorm in $\tilde{T}^{-1}(\tilde{\mathrm{BMO}}(\sigma_\alpha))$.

For every $F(z) \in \mathcal{H}^1(E)$, the duality product is defined as

$$\left\langle F, \tilde{T}^{-1}(b)\right\rangle := \lim_{\mathcal{M}\to\infty}\lim_{N\to\infty}\int_{\mathbb{R}} F\,\overline{\left(-B(x)\sum_{|n|\leq N} b^{\mathcal{M}}(t_n)\left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1}\right)\frac{1}{\phi'(t_n)}\right)}\frac{dx}{|E(x)|^2} \ , \tag{8.55}$$

where $b^{\mathcal{M}}(x)$ has been defined in (8.19). In particular, if $\mathcal{A}(z)$ is a $\mathcal{H}^1(E)$-atom, then

$$\mathcal{A}(z) := \sum_{n\in A}\mathcal{A}(t_n)\frac{k_{t_n}(z)}{k_{t_n}(t_n)} \qquad \text{and} \qquad \sigma_\alpha(\mathrm{supp}(\mathcal{A})) < \infty \ ,$$

and, since $k_{t_n}(z) = B(z)A(t_n)/(\pi(z - t_n))$,

$$\begin{aligned} \left\langle \mathcal{A}, \tilde{T}^{-1}(b)\right\rangle &= \lim_{\mathcal{M}\to\infty}\lim_{N\to\infty}\sum_{n\in\mathrm{supp}(\mathcal{A})}\ \sum_{|m|\leq N}\mathcal{A}(t_n)\overline{\left(\frac{b^{\mathcal{M}}(t_m)}{A(t_m)}\right)}\frac{\pi}{\phi'(t_m)}\delta_{m,n} + \\ &\qquad + \lim_{\mathcal{M}\to\infty}\lim_{N\to\infty}\int_{\mathbb{R}}\sum_{n\in\mathrm{supp}(\mathcal{A})}\ \sum_{|m|\leq N}\frac{\mathcal{A}(t_n)}{k_{t_n}(t_n)}\frac{\overline{b^{\mathcal{M}}(t_m)}}{\phi'(t_m)}\frac{t_m}{t_m^2+1}k_{t_n}(x)B(x)\frac{dx}{|E(x)|^2} \\ &= \lim_{\mathcal{M}\to\infty}\lim_{N\to\infty}\sum_{n\in\mathrm{supp}(\mathcal{A})}\ \sum_{|n|\leq N}\frac{\mathcal{A}(t_n)}{E(t_n)}\overline{b^{\mathcal{M}}(t_n)}\frac{\pi}{\phi'(t_n)} \end{aligned}$$

$$= \int_{\mathbb{R}} \frac{\mathcal{A}(x)}{E(x)} \overline{b(x)} d\sigma_0(x) = \langle \mathcal{V}_0(\mathcal{A}), b \rangle_{[H^1_{\text{at}}, \text{BMO}(\sigma_0)]} \ .$$

Indeed

$$\lim_{\mathcal{M}\to\infty} \lim_{N\to\infty} \int_{\mathbb{R}} \sum_{n\in\text{supp}(\mathcal{A})} \sum_{|m|<N} \frac{\mathcal{A}(t_n)}{k_{t_n}(t_n)} \frac{\overline{b^{\mathcal{M}}(t_m)}}{\phi'(t_m)} \frac{t_m}{t_m^2+1} k_{t_n}(x) B(x) \frac{dx}{|E(x)|^2}$$

$$= \lim_{\mathcal{M}\to\infty} \lim_{N\to\infty} \sum_{|m|<N} \frac{\overline{b^{\mathcal{M}}(t_m)}}{\phi'(t_m)} \frac{t_m}{t_m^2+1} \int_{\mathbb{R}} \sum_{n\in\text{supp}(\mathcal{A})} \frac{\mathcal{A}(t_n)}{k_{t_n}(t_n)} k_{t_n}(x) B(x) \frac{dx}{|E(x)|^2}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \sum_{n\in\text{supp}(\mathcal{A})} \frac{\mathcal{A}(t_n)}{k_{t_n}(t_n)} k_{t_n}(x) B(x) \frac{dx}{|E(x)|^2} &= \frac{1}{\pi} \int_{\mathbb{R}} \sum_{n\in\text{supp}(\mathcal{A})} \frac{a(t_n)}{x-t_n} B^2(x) \frac{1}{\phi'(t_n)} \frac{dx}{|E(x)|^2} \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \sum_{n\in\text{supp}(\mathcal{A})} \sin^2(\phi(x)) \frac{a(t_n)}{x-t_n} \frac{1}{\phi'(t_n)} dx \ , \end{aligned}$$

where $a(x) := \mathcal{A}(x)/E(x)$. If $\#\text{supp}(\mathcal{A}) = V$, we call

$$G(z) := \frac{B(z)}{x-t_1} \in \mathcal{H}^2(E) \ , \quad C(z) := \frac{B(z)\sum_{j=2}^{V} x^{j-2}\overline{c_j}}{\prod_{n\in\text{supp}(\mathcal{A}), n\neq 1}(x-t_n)} \in \mathcal{H}^2(E) \ ,$$

where c_j will be specified later. Then

$$\begin{aligned} \int_{\mathbb{R}} \sum_{n\in\text{supp}(\mathcal{A})} \sin^2(\phi(x)) \frac{a(t_n)}{x-t_n} \frac{1}{\phi'(t_n)} dx &= \int_{\mathbb{R}} \sin^2(\phi(x)) \frac{\sum_{j=2}^{V} x^{j-2} c_j}{\prod_{n\in\text{supp}(\mathcal{A})}(x-t_n)} dx \\ &= \int_{\mathbb{R}} G(x)\overline{C(x)} \frac{dx}{|E(x)|^2} = \langle G, C \rangle_{\mathcal{H}^2(E)} \ . \end{aligned}$$

However, since $G(z) = \frac{\pi}{A(t_1)} k_{t_1}(z)$, then

$$\int_{\mathbb{R}} \sum_{n\in\text{supp}(\mathcal{A})} \sin^2(\phi(x)) \frac{a(t_n)}{x-t_n} \frac{1}{\phi'(t_n)} dx = \frac{\pi}{A(t_1)} \frac{B(t_1)\sum_{j=2}^{V} t_1^{j-2}\overline{c_j}}{\prod_{n\in\text{supp}(\mathcal{A}), n\neq 1}(t_1-t_n)} = 0 \ . \tag{8.56}$$

We summarize what we have just proved by saying that

$$\langle \mathcal{A}, B \rangle = 0 \quad \forall\ \mathcal{H}^1(E)\text{-atoms and therefore for any elements in } \mathcal{H}^1(E) \ . \tag{8.57}$$

In general, if $\sum_k \lambda_k a^k = F \in \mathcal{H}^1(E) \subset \mathcal{H}^2(E)$ and $\|F\|_{\mathcal{H}^1(E)} = 2\sum_k |\lambda_k|$. Then

$$\begin{aligned} \left\langle F, \tilde{T}^{-1}(b) \right\rangle &:= \lim_{\mathcal{M}\to\infty} \lim_{N\to\infty} \int_{\mathbb{R}} F(x) \sum_{|m|\le N} \frac{\overline{b^{\mathcal{M}}(t_m)}}{A(t_m)} \frac{\pi}{\phi'(t_m)} \overline{k_{t_m}(x)} \frac{dx}{|E(x)|^2} \\ &= \lim_{\mathcal{M}\to\infty} \lim_{N\to\infty} \int_{\mathbb{R}} \sum_k \lambda_k a^k(x) \sum_{|m|\le N} \frac{\overline{b^{\mathcal{M}}(t_m)}}{A(t_m)} \frac{\pi}{\phi'(t_m)} \overline{k_{t_m}(x)} \frac{dx}{|E(x)|^2} \end{aligned}$$

$$= \lim_{\mathcal{M}\to\infty} \lim_{N\to\infty} \sum_{|m|\le N} \frac{\overline{b^{\mathcal{M}}(t_m)}}{A(t_m)} \frac{\pi}{\phi'(t_m)} \langle F, k_{t_m}\rangle_{\mathcal{H}^2(E)}$$

$$= \lim_{\mathcal{M}\to\infty} \lim_{N\to\infty} \sum_{|m|\le N} F(t_m) \frac{\overline{b^{\mathcal{M}}(t_m)}}{A(t_m)} \frac{\pi}{\phi'(t_m)} = \langle \mathcal{V}_0(F), b\rangle_{[H^1_{\text{at}},\text{BMO}(\sigma_0)]} \ .$$

The duality product between $\mathcal{H}^1(E)$ and the space $\tilde{T}^{-1}(\widetilde{\text{BMO}}(\sigma_0))$ is therefore equivalent to the duality product between $H^1(\sigma_0)$ and $\text{BMO}(\sigma_0)$;
We want to extend the operator $\tilde{T}^{-1}$ from $\widetilde{\text{BMO}}(\sigma_0)$ to $\text{BMO}(\sigma_0)$, checking that it is well defined on the equivalence classes. First of all we need the following well known identity, [28], [69], [71].

Lemma 8.24 Let $E(z) = A(z) - iB(z)$ be a Hermite Biehler function, as defined in (1.4). The entire function $A(z)$ can be also represented as

$$A(z) = -B(z) \sum_{n\in\mathbb{Z}} \left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1} \right) \frac{\pi}{\phi'(t_n)} \qquad z \in \mathbb{C} \ , \tag{8.58}$$

where $\{t_n\}$ are the zeros of $B(z)$.

First of all, let us observe that the operator $\tilde{T}^{-1}$ is well defined in $\text{BMO}(\sigma_\alpha)$. Let $b \in \widetilde{\text{BMO}}(\sigma_\alpha)$ and

$$\tilde{T}^{-1}(b)(z) := -B(z) \sum_{n\in\mathbb{Z}} b(t_n) \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) \frac{1}{\phi'(t_n)} \ .$$

We consider $c(x) \in \widetilde{\text{BMO}}(\sigma_\alpha)$ defined as

$$c(x) := b(x) + \gamma \ , \quad \forall x \in \text{supp}(\sigma_\alpha), \ \text{where } \gamma \in \mathbb{C} \ ;$$

It is clear that $c(x)$ and $b(x)$ represent the same element in $\text{BMO}(\sigma_\alpha)$. Furthermore,

$$\begin{aligned} \tilde{T}^{-1}(c)(z) : &= \tilde{T}^{-1}(b)(z) - \gamma B(z) \sum_{n\in\mathbb{Z}} \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) \frac{1}{\phi'(t_n)} \\ &= \tilde{T}^{-1}(b)(z) + \gamma A(z) \ . \end{aligned} \tag{8.59}$$

We are now ready to introduce the analytic description of $\mathcal{H}^1(E)$.

Definition 8.25 We define the space

$$\mathcal{X}(E) := \tilde{T}^{-1}(\widetilde{\text{BMO}}(\sigma_0))/\langle A(z)\rangle \ , \tag{8.60}$$

where if $\mathcal{X}(E) \ni g := \tilde{T}^{-1}(b)(z)$ with $b \in \widetilde{\text{BMO}}(\sigma_0)$, then

$$\|g\|_{\mathcal{X}(E)} := \|b\|_{\text{BMO}(\sigma_0)} \ .$$

The $\|\cdot\|_{\mathcal{X}(E)}$ is well defined. Indeed, let $f, g \in [g]_{\mathcal{X}(E)}$ and $f(z) = \tilde{T}^{-1}(b)(z)$, $g = \tilde{T}^{-1}(c)(z)$. Since $f(z) - g(z) = kA(z)$, then $b(x) - c(x) = k$ where $x \in \text{supp}(\sigma_0)$ and consequently $b(x) = c(x)$ in $\text{BMO}(\sigma_0)$.

We note also that $\|\cdot\|_{\mathcal{X}(E)}$ is a norm. We already know that it is a seminorm. Furthermore, it is a norm since $\|f\|_{\mathcal{X}(E)} = 0$ if and only if $f = \tilde{T}^{-1}(k)$, that is $f = kA(z)$ and consequently $f \in [0]_{\mathcal{X}(E)}$.

Let us now prove Theorem 8.2, that is, that $\mathrm{BMO}(\sigma_0)$ is isomorphic to $\mathcal{X}(E)$.

PROOF THEOREM 8.2. In order to prove this theorem it is enough checking that $\mathrm{BMO}(\sigma_0)$ is isomorphic to $\mathcal{X}(E)$.

Let us introduce the map V from $\mathrm{BMO}(\sigma_0)$ to $\mathcal{X}(E)$:

$$V(b)(z) := -B(z) \sum_{0 \neq n \in \mathbb{Z}} b(t_n) \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) \frac{1}{\phi(t_n)} \ . \tag{8.61}$$

Thanks to (8.59) the operator V is well defined in $\mathrm{BMO}(\sigma_0)$. Indeed if $b, c \in [b]_{\mathrm{BMO}(\sigma_0)}$, then $V(b) - V(c) = kA(z)$, that is $V(b), V(c) \in [V(b)]_{\mathcal{X}(E)}$. Due to the definition of $\mathcal{X}(E)$, V is surjective. Finally, we check that V is injective. Indeed,

$$V^{-1}(\gamma A)(x) = \tilde{T}(\gamma A)(x) = \gamma \quad \forall x \in \operatorname{supp}(\sigma_0) \ ,$$

which corresponds to the zero element of $\mathrm{BMO}(\sigma_0)$. It is clear that the operator V is an isomorphism since

$$\|g\|_{\mathcal{X}(E)} = \|b\|_{\mathrm{BMO}(\sigma_0)} \qquad \text{when } g = \tilde{T}^{-1}(b) \ ,$$

which proves the theorem. $\square$

The dualitiy product between $\mathcal{H}^1(E)$ and $\mathcal{X}(E)$ is defined as in (8.55) so that

$$\langle \mathcal{A}, g \rangle_{[\mathcal{H}^1(E), \mathcal{X}(E)]} = \sum_n \frac{\mathcal{A}(t_n)}{E(t_n)} \overline{b(t_n)} \frac{1}{\phi'(t_n)}$$

where $b = V^{-1}(g)$ and $\mathcal{A}$ is a $H^1(E)$-atom. Again this definition does not depend on the choice of the representative g, since

$$\int_{\mathbb{R}} f(x) A(x) \frac{dx}{|E(x)|^2} = 0$$

for every $f \in \mathcal{H}^1(E)$-atom and consequently for every element of $\mathcal{H}^1(E)$. Indeed

$$\begin{aligned} \int_{\mathbb{R}} f(x) A(x) \frac{dx}{|E(x)|^2} &= \int_{\mathbb{R}} \sum_{n \in \operatorname{supp}(f)} \frac{f(t_n)}{k_{t_n}(t_n)} k_{t_n}(x) A(x) \frac{dx}{|E(x)|^2} \\ &= \int_{\mathbb{R}} \sum_{n \in \operatorname{supp}(f)} \frac{F(t_n)}{x - t_n} A(x) B(x) \frac{1}{\phi'(t_n)} \frac{dx}{|E(x)|^2} \end{aligned}$$

$$= \int_{\mathbb{R}} \sum_{n\in \operatorname{supp}(f)} \cos(\phi(x)) \sin(\phi(x)) \frac{F(t_n)}{x-t_n} \frac{1}{\phi'(t_n)} dx$$

where $F := f/E$. If $\#\operatorname{supp}(f) = N$, we call

$$G(z) := \frac{A(z)}{x - s^{1/2}} \in \mathcal{H}^2(E) \ , \quad C(z) := \frac{(x-s^{1/2})B(z)\sum_{i=2}^{N} x^{i-2}\overline{w_i}}{\prod_{n\in\operatorname{supp}(f)}(x-t_n)} \in \mathcal{H}^2(E) \ ,$$

where $A(s^{1/2}) = 0$ and w_i will be specified later. Then

$$\begin{aligned}
&\int_{\mathbb{R}} \sum_{n\in \operatorname{supp}(f)} \cos(\phi(x)) \sin(\phi(x)) \frac{F(t_n)}{x-t_n} \frac{1}{\phi'(t_n)} dx \\
&= \int_{\mathbb{R}} \cos(\phi(x)) \sin(\phi(x)) \frac{\sum_{i=2}^{N} x^{i-2} w_i}{\prod_{n\in\operatorname{supp}(f)}(x-t_n)} dx \\
&= \int_{\mathbb{R}} G(x)\overline{C(x)} \frac{dx}{|E(x)|^2} = \langle G, C\rangle_{\mathcal{H}^2(E)} \ .
\end{aligned}$$

However, since $G(z) = -\frac{\pi}{B(s^{1/2})} k_{s^{1/2}}(z)$, then

$$\int_{\mathbb{R}} \sum_{n\in \operatorname{supp}(f)} \cos(\phi(x)) \sin(\phi(x)) \frac{F(t_n)}{t_n - x} \frac{1}{\phi'(t_n)} dx = 0 \ .$$

Summarizing, we say that

$$\langle f, A\rangle = 0 \quad \forall\ \mathcal{H}^1(E)\text{-atoms and therefore for every elements in } \mathcal{H}^1(E) \ . \tag{8.62}$$

We note that all the computations that we have made in these last two sections are similar to those needed to characterize the dual of $\mathcal{B}^1_\pi$. Nevertheless, we have included them since there are some huge differences: for example we cannot use the Fourier transform in order to simplify the estimates.

We note also that an analytic characterization of $\mathcal{H}^1(E)^*$ is still unknown. Indeed we did not manage to prove that the space $\mathcal{X}(E)$ is isomorphic to some space $\mathcal{Y}(E)$ similar to (7.31). We will face this last problem in the future.

Bibliography

[1] Akhiezer, N. I; Glazman, I. M. *Theory of linear operators in Hilbert space/ Two volumes bound as one*, Dover Publications, Inc. New York, 1993. pp. 147+218.

[2] Aleksandrov, A.B.*A simple proof of a theorem of Vol'berg and Treil' on the embedding of coinvariant subspaces of the shift operator*, J. Math. Sci. (New York) 85 (1997), no. 2, 1773–1778.

[3] Aleksandrov, A.B. *Embedding theorems for coinvariant subspaces of the shift operator. II*, J. Math. Sci. (New York) 110 (2002), no. 5, 2907–2929.

[4] Aleman, A; Martin, R.T.W; Ross, W.T. *On a theorem of Livsic*, J. Funct. Anal. 264 (2013), no. 4, 999–1048.

[5] Andersen, N.B. *Entire Lp-functions of exponential type*, Expo. Math. 32 (2014), no. 3, 199–220.

[6] Baranov, A. *Differentiation in the Branges spaces and embedding theorems*, Nonlinear equations and mathematical analysis. J. Math. Sci. (New York) 101 (2000), no. 2, 2881–2913.

[7] Baranov, A. *Bernstein's inequality in the de Branges spaces and embedding theorems*, AMS Translations, Ser. 2, 209 (2003), 21-49.

[8] Baranov, A. *Isometric embeddings of the spaces $K(\Theta)$ in the upper half-plane*, Function theory and partial differential equations. J. Math. Sci. (New York) 105 (2001), no. 5, 2319–2329.

[9] Baranov, A. *On estimate of the L^p-norms of derivatives in spaces of entire functions*, J. Math. Sci. (N.Y.) 129 (2005), no. 4, 3927–3943.

[10] Baranov, A. *Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings*, J. Funct. Anal. 223 (2005), no. 1, 116–146.

[11] Baranov, A. *On L^1-norms of meromorphic functions with fixed poles*, Proc. Amer. Math. Soc. 134 (2006), no. 10, 3003–3013.

[12] Baranov, A; Belov, Y; Poltoratski, A. *De Branges functions of Schroedinger equations*, Collect. Math. 68 (2017), no. 2, 251–263.

[13] Baranov, A; Bommier-Hato, H. *De Branges spaces and Fock spaces*, Complex Var. Elliptic Equ. 63 (2018), no. 7-8, 907–930.

[14] Baranov, A; Dyakonov, K.M. *The Feichtinger conjecture for reproducing kernels in model subspaces*, J. Geom. Anal. 21 (2011), no. 2, 276–287.

[15] Bellavita, C. *Necessary conditions for boundedness of translation operator in de Branges spaces*, Complex Anal. Oper. Theory 15 (2021), no. 6, Paper No. 96.

[16] Bellavita, C. *Continuous embedding between p–de Branges spaces*, Concr. Oper. 8 (2021), no. 1, 125–134.

[17] Bellavita, C. *Necessary and sufficient condition for boundedness of translation operator in de Branges spaces*, submitted.

[18] Bessonov, R.V. *Duality theorems for coinvariant subspaces of* H^1, Adv. Math. 271 (2015), 62–90.

[19] Beurling, A. *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1948), 239–255.

[20] Boza, S; Carro, M.J. *Discrete Hardy spaces*, Studia Math. 129 (1998), no. 1, 31–50.

[21] Cartwright, M.L. *On certain integral functions of order one*, Q. J. Math. 7 (1936), no.1, 46–55.

[22] Chang, D; Sadosky, C. *Functions of bounded mean oscillation*, Taiwanese J. Math. 10 (2006), no. 3, 573–601.

[23] Cima, J.A; Matheson, A.L; Ross, W.T. *The Cauchy transform*, Mathematical Surveys and Monographs, 125. American Mathematical Society, Providence, RI, 2006. pp. 272.

[24] Cima, J.A; Ross W.T. *The backward shift on the Hardy space*, Mathematical Surveys and Monographs, 79. American Mathematical Society, Providence, RI, 2000. pp. 199.

[25] Cohn, B. *Carleson measures for functions orthogonal to invariant subspaces*, Pacific J. Math. 103 (1982), no. 2, 347–364.

[26] Cohn, B. *Radial limits and star invariant subspaces of bounded mean oscillation*, Amer. J. Math. 108 (1986), no. 3, 719-749.

[27] Conway, J.B. *Functions of one complex variable*, Second edition. Graduate Texts in Mathematics, 11. Springer-Verlag, New York-Berlin, 1978. pp. 317.

[28] de Branges, L. *Hilbert spaces of entire functions*, Prentice-Hall, Inc. Englewood Cliffs, N.J. 1968. pp. 326.

[29] Duren, P.L. *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38 Academic Press, New York-London 1970. pp. 258.

[30] Dyakonov, K.M. *Moduli and arguments of analytic functions from subspaces of H^p that are invariant for the backward shift operator*, Siberian Math. J. 31 (1990), no. 6, 926–939.

[31] Dyakonov, K.M. *Division and multiplication by inner functions and embedding theorems for star-invariant subspaces*, Amer. J. Math. 115 (1993), no. 4, 881–902.

[32] Dyakonov, K.M. *Entire functions of exponential type and model subspaces in H^p*, J. Math. Sci. 71 (1994), no. 1, 2222–2233.

[33] Dyakonov, K.M. *Equivalent norms on Lipschitz-type spaces of holomorphic functions*, Acta Math. 178 (1997), no. 2, 143–167.

[34] Dyakonov, K.M. *Continuous and compact embeddings between star-invariant subspaces*, Complex analysis, operators, and related topics, Oper. Theory Adv. Appl. 113 (2000), 65–76.

[35] Dyakonov, K.M; Girela, D. *On Q_p spaces and pseudoanalytic extension*, Ann. Acad. Sci. Fenn. Math. 25 (2000), no. 2, 477–486.

[36] Dryanov, D. *Sobolev type decomposition of Paley-Wiener-Schwartz space with application to sampling theory*, Serdica Math. J. 33 (2007), no. 4, 411–432.

[37] Eoff, C. *The discrete nature of the Paley-Wiener spaces*, Proc. Amer. Math. Soc. 123 (1995), no. 2, 505–512.

[38] Fricain, E; Mashreghi, J. *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*, New Mathematical Monographs, 20. Cambridge University Press, Cambridge, 2016. pp. 681.

[39] Fricain, E; Mashreghi, J. *The theory of $\mathcal{H}(b)$ spaces. Vol. 2*, New Mathematical Monographs, 21. Cambridge University Press, Cambridge, 2016. pp. 619.

[40] Garcia, S.R; Mashreghi, J; Ross, W.T. *Introduction to model spaces and their operators*, Cambridge Studies in Advanced Mathematics, 148. Cambridge University Press, Cambridge, 2016. pp. 322.

[41] Garnett, J.B. *Bounded analytic functions* Revised first edition. Graduate Texts in Mathematics, 236. Springer, New York, 2007. pp. 459.

[42] Girela, D. *Analytic functions of bounded mean oscillation*, Complex function spaces (Mekrijärvi, 1999), 61–170, Univ. Joensuu Dept. Math. Rep. Ser. 4, Univ. Joensuu, Joensuu, 2001.

[43] Gonçalves, F. *Interpolation formulas with derivatives in de Branges spaces*, Trans. Amer. Math. Soc. 369 (2017), no. 2, 805–832.

[44] Gonçalves, F; Littmann, F. *Mean convergence of entire interpolations in weighted space*, Complex Anal. Oper. Theory 15 (2021), no. 5, Paper No. 90.

[45] Gorbachuk, M. L; Gorbachuk, V. I. M. G. *Krein's lectures on entire operators*, Operator Theory: Advances and Applications, 97. Birkhäuser Verlag, Basel, 1997.pp. 220.

[46] Grafakos, L. *Modern Fourier analysis*, Second edition. Graduate Texts in Mathematics, 250. Springer, New York, 2009. pp.504.

[47] Havin, V; Mashreghi, J. *Admissible majorants for model subspaces of H^2. I. Slow winding of the generating inner function*, Canad. J. Math. 55 (2003), no. 6, 1231–1263.

[48] Havin, V; Mashreghi, J. *Admissible majorants for model subspaces of H^2. II. Fast winding of the generating inner function*, Canad. J. Math. 55 (2003), no. 6, 1264–1301.

[49] Hytönen, T.P. *The two-weight inequality for the Hilbert transform with general measures*, Proc. Lond. Math. Soc. (3) 117 (2018), no. 3, 483–526.

[50] Janson, S; Peetre, J; Semmes, S. *On the action of Hankel and Toeplitz operators on some function spaces*, Duke Math. J. 51 (1984), no. 4, 937–958.

[51] Kaltenbäck, M; Woracek, H. *Pólya class theory for Hermite-Biehler functions of finite order*, J. London Math. Soc. (2) 68 (2003), no. 2, 338–354.

[52] Lacey, M.T; Sawyer, E.T; Shen, C.Y; Uriarte-Tuero, I. *Two-weight inequality for the Hilbert transform: a real variable characterization, I*, Duke Math. J. 163 (2014), no. 15, 2795–2820.

[53] Lacey, M.T; Sawyer, E.T; Shen, C.Y; Uriarte-Tuero, I; Wick, B. *Two Weight Inequalities for the Cauchy Transform from $\mathbb{R}$ to $\mathbb{C}^+$*, arXiv: Complex Variables, (2013).

[54] Lax, P.D. *Translation invariant spaces*, Acta Math. 101 (1959), 163–178.

[55] Levin, B.I.A. *Distribution of zeros of entire functions*, American Mathematical Society, Providence, RI, 1964. pp. 493.

[56] Levin, B.I.A. *Lectures on entire functions*, In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko. Translations of Mathematical Monographs, 150. American Mathematical Society, Providence, RI, 1996. pp. 248.

[57] Lyubarskii, Y.I; Seip, K. *Weighted Paley-Wiener spaces*, J. Amer. Math. Soc. 15 (2002), no. 4, 979–1006.

[58] Makarov, N; Poltoratski, A. *Meromorphic inner functions, Toeplitz kernels and the uncertainty principle*, Perspectives in analysis, Math. Phys. Stud. 27, Springer, Berlin (2005), 185–252.

[59] Martin, R.T.W. *Symmetric operators and reproducing kernel Hilbert spaces*, Complex Anal. Oper. Theory 4 (2010), no. 4, 845–880.

[60] Mashregi, D; Nazarov, F.L; Khavin, V.P. *The Beurling-Malliavin multiplier theorem: the seventh proof*, St. Petersburg Math. J. 17 (2006), no. 5, 699–744.

[61] Sz.-Nagy, B; Foias, C; Bercovici, H; Kérchy, L. *Harmonic analysis of operators on Hilbert space*, Second edition. Revised and enlarged edition. Universitext. Springer, New York, 2010. pp. 474.

[62] Nazarov, F; Treil, S; Volberg, A. *Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures*, arXiv: Analysis of PDEs, (2010).

[63] Nikolski, N.K. *Operators, functions, and systems: an easy reading. Vol. 2. Model operators and systems*, Mathematical Surveys and Monographs, 93. American Mathematical Society, Providence, RI, 2002. pp. 439.

[64] Ortega-Cerdà, J; Seip, K. *Fourier frames*, Ann. of Math. (2) 155 (2002), no. 3, 789–806.

[65] Papadimitrakis, M; Virtanen, J.A. *Hankel and Toeplitz transforms on H^1: continuity, compactness and Fredholm properties*, Integral Equations Operator Theory 61 (2008), no. 4, 573–591.

[66] Poltoratski, A. *Canonical systems and finite rank perturbations of spectra*, arXiv: Spectral Theory, 1995.

[67] Poltoratski, A; Sarason, D. *Aleksandrov-Clark measures*, Recent advances in operator-related function theory, Contemp. Math., 393, Amer. Math. Soc. Providence, RI (2006), 1–14.

[68] Reed, M; Simon, B. *Methods of modern mathematical physics. I*, Functional analysis. Second edition. Academic Press, Inc. New York, 1980. pp. 400.

[69] Remling, C. *Schrödinger operators and de Branges spaces*, J. Funct. Anal. 196 (2002), no. 2, 323–394.

[70] Remling, C. *Spectral theory of canonical systems*, De Gruyter Studies in Mathematics, 70. De Gruyter, Berlin, 2018. pp. 194.

[71] Romanov, R. *Canonical systems and de Branges spaces*, arXiv: Complex variables, 2014.

[72] Rudin, W. *Functional analysis*, Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc. New York, 1991. pp. 424.

[73] Saksman, E. *An elementary introduction to Clark measures*, Topics in complex analysis and operator theory, 85–136, Univ. Málaga, Málaga, 2007.

[74] Silva, L.O; Toloza, J.H. *de Branges Spaces and Kreĭn's Theory of Entire Operators*, In: Alpay D. (eds) Operator Theory. Springer, Basel (2015), 549-580.

[75] Stegenga, D.A. *Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation* Amer. J. Math. 98 (1976), no. 3, 573–589.

[76] Stein, E.M. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, With the assistance of T.S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993. pp. 695.

[77] Sundberg, C. *Truncations of BMO functions*, Indiana Univ. Math. J. 33 (1984), no. 5, 749–771.

[78] Vol'berg, A.L; Treil', S.R. *Embedding theorems for invariant subspaces of backward shift operator*, J. Soviet Math. 42 (1988), no. 2, 1562–1572.

[79] Yosida, K. *Functional analysis*, Reprint of the sixth (1980) edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. pp. 501.

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