NON-SYMPLECTIC AUTOMORPHISMS OF ODD PRIME ORDER ON MANIFOLDS OF $K3^{[n]}$-TYPE

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Abstract. We contribute to the classification of non-symplectic automorphisms of odd prime order on irreducible holomorphic symplectic manifolds which are deformations of Hilbert schemes of any number n of points on $K3$ surfaces, extending results already known for $n = 2$. In particular, we study the properties of the invariant lattice of the automorphism (and its orthogonal complement) inside the second cohomology lattice of the manifold. We also explain how to construct automorphisms with fixed action on cohomology: in the cases $n = 3, 4$ the examples provided realize all admissible actions in our classification. For $n = 4$, we present a construction of non-symplectic automorphisms on the Lehn–Lehn–Sorger–van Straten eightfold, which come from automorphisms of the underlying cubic fourfold.

1. Introduction

The study of automorphisms of $K3$ surfaces has been a very active research field for decades. The global Torelli theorem allows to reconstruct automorphisms of a $K3$ surface Σ from Hodge isometries of $H^2(\Sigma, \mathbb{Z})$ preserving the intersection product; this link, together with the seminal works of Nikulin [56], [55], provided the instruments to investigate finite groups of automorphisms on $K3$'s. In recent years, the interest in automorphisms has extended from $K3$ surfaces to manifolds which generalize them in higher dimension, namely irreducible holomorphic symplectic (IHS) varieties. Results by Huybrechts [36], Markman [47] and Verbitsky [65], which provide an analogous of the Torelli theorem for these manifolds, allow us to use similar methods, studying the action of an automorphism on the second cohomology group with integer coefficients (which carries again a lattice structure, provided by the Beauville–Bogomolov–Fujiki quadratic form; see [34, Part III] for further references).

A great number of results are known for automorphisms of prime order on IHS fourfolds that are deformations of the Hilbert scheme of two points on a $K3$ surface (so-called manifolds of $K3^{[2]}$-type). The symplectic case (i.e. automorphisms which preserve the symplectic form) is covered in [20] and [51]; in turn, the study of non-symplectic automorphisms was started by Beauville [8] and has seen many relevant contributions, culminating in a complete classification of the associated sublattices in cohomology ([12], [11], [62]). Explicit constructions of automorphisms realizing all cases in this classification have been exhibited throughout the years (see [57],

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[12], [53], [22]), with the exception of the automorphism of order 23 whose existence is proved in [11, Theorem 1.1].

Far less is known, in general, about non-symplectic automorphisms of manifolds of $K3^{[n]}$-type when $n \geq 3$. In [41, Chapter 4], Joumaah studies moduli spaces of manifolds X of $K3^{[n]}$-type with non-symplectic involutions $\iota : X \to X$, providing also a classification for the invariant lattice $H^2(X,\mathbb{Z})^{\iota^*} = \left\{v \in H^2(X,\mathbb{Z}) : \iota^*(v) = v\right\}$ in [41, Theorem 5.0.1]. This classification, however, is not entirely correct: the recent paper [21], by the two authors and Andrea Cattaneo, rectifies these inaccuracies (see [21, §2]) and provide some additional insight on non-symplectic involutions.

In this paper we construct a general theory of non-symplectic automorphisms of odd prime order p of manifolds of $K3^{[n]}$-type, for any $n \geq 2$.

In the case of fourfolds of $K3^{[2]}$-type, the authors of [12] discovered that the classification of non-symplectic automorphisms is more complex with respect to the case of automorphisms of $K3$ surfaces, and also fundamentally richer for $p = 2$ rather than for odd p. In the more general setting of manifolds of $K3^{[n]}$-type, we show that many additional cases appear whenever p divides $2(n-1)$. This is also one of the reasons why non-symplectic involutions deserve to be discussed separately, since for all $n \geq 2$ this divisibility condition becomes vacuous if $p = 2$.

If X is of $K3^{[n]}$-type, an automorphism $\sigma \in \mathrm{Aut}(X)$ is uniquely determined by the pull-back $\sigma^* \in O(H^2(X,\mathbb{Z}))$, since the homomorphism $\mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z}))$, $\sigma \mapsto \sigma^*$ is injective (see [6, Proposition 10] and [52, Lemma 1.2]). In turn, it is possible to describe σ^* by means of the invariant lattice $T = H^2(X,\mathbb{Z})^{\sigma^*}$ and its orthogonal complement $S = T^\perp \subset H^2(X,\mathbb{Z})$. Since σ^* is the unique extension to $H^2(X,\mathbb{Z})$ of the isometry $\mathrm{id}_T \oplus \sigma^*|_S \in O(T \oplus S)$ (see [56, Corollary 1.5.2]), we conclude that the automorphism $\sigma \in \mathrm{Aut}(X)$ is determined by the sublattices $T, S \subset H^2(X,\mathbb{Z})$ and by the isometry $\sigma^*|_S \in O(S)$ (which cannot be chosen arbitrarily). One can therefore classify non-symplectic automorphisms of manifolds of $K3^{[n]}$-type by studying their action on cohomology, and this requires a classification of the pairs of lattices (T, S).

As a lattice, $H^2(X,\mathbb{Z})$ is isometric to the abstract lattice $L := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$, therefore we can fix an isometry $\eta : H^2(X,\mathbb{Z}) \to L$ and consider T and S as orthogonal primitive sublattices of L. After recalling, in §2, some well-known results which we use throughout the paper and fixing the notation, in §3 we study the pairs of lattices (T, S) in the non-symplectic case. For any automorphism of odd prime order p we consider two numerical invariants m, a defined by the properties $\mathrm{rk}(S) = (p-1)m$ and $\frac{H^2(X,\mathbb{Z})}{T \oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$.

As it will be explained in §3, a triple (p, m, a) is *admissible* for a certain $n \geq 2$ if it satisfies a set of *necessary* (but not sufficient, a priori) conditions for the existence of a non-symplectic automorphism of order p on a manifold of $K3^{[n]}$-type with numerical invariants m, a as above. The exact definition is the following, where for any lattice V we denote by $l(A_V)$ the *length* of the discriminant group $A_V := V^\vee/V$, i.e. the minimal number of generators of A_V.

Definition (Definition 3.10)**.** Let (p, m, a) be a triple of integers with $3 \leq p \leq 23$ prime, $m \geq 1$, $(p-1)m \leq 22$ and $0 \leq a \leq \min\{m, 23 - (p-1)m\}$. Let $n \geq 2$ and write $2(n-1) = p^\alpha \beta$ with $(p, \beta) = 1$. The triple (p, m, a) is *admissible* for n if there exist two orthogonal sublattices $T, S \subset L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ such that

- $\mathrm{sign}(T) = (1, 22 - (p-1)m)$ and $\mathrm{sign}(S) = (2, (p-1)m - 2)$;

- $\frac{L}{T\oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$;
- $l(A_S) \equiv m \pmod 2$;

and the discriminant groups A_T and A_S satisfy one of the following:

(i) $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$;

(ii) $\alpha = 1$, $a = 0$, $A_S \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, $A_T \cong \frac{\mathbb{Z}}{\beta\mathbb{Z}}$;

(iii) $\alpha \geq 1$, $a \geq 1$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$, $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$.

A pair of lattices (T, S) as in the definition is also said to be admissible: by the previous discussion, such pairs are to be considered as good candidates for the invariant/co-invariant lattices of non-symplectic automorphisms, but it is not guaranteed that all of them can actually be realized in this way. By using classical results in lattice theory, mainly by Nikulin [56], it is possible to find a list of all admissible triples (p, m, a) for each value of n. Our first main result, which is purely lattice-theoretic, concerns the classification of pairs (T, S) corresponding to a given admissible triple, as in Definition 3.10.

Theorem (Theorem 3.12)**.** *If $p^2 \nmid 2(n-1)$, an admissible triple (p, m, a) for $n \geq 2$ uniquely determines the isometry class of the lattice S. Its orthogonal complement $T \subset L$ is also uniquely determined (up to isometries of L) by (p, m, a), assuming that $l(A_T) \leq 21 - (p-1)m$.*

Notice that, if p is odd, the first instance where $p^2 \mid 2(n-1)$ is for $n = 10, p = 3$. Combining the theorem with the previous study of the action of a non-symplectic automorphism on cohomology, we provide in §3.4 a complete classification of the admissible triples (p, m, a) and of the corresponding pairs of lattices (T, S) when $n = 3$ and $n = 4$, which are the cases of most immediate interest. While for $p \geq 5$ the admissible triples are essentially the same which appear for $n = 2$ (see [12, Appendix]), for $p = 3$ several new cases arise, especially when $n = 4$ (because $p = 3$ divides $2(n-1) = 6$).

Proposition 1.1. *The admissible triples $(3, m, a)$ for $n = 3, 4$ and the unique pairs of lattices (T, S) associated to them are the ones listed in Table 1 and Table 2 of Appendix A.*

The remaining part of the paper is dedicated to constructing examples of non-symplectic automorphisms of odd prime order, in order to understand which admissible triples are actually realized by automorphisms. For manifolds of $K3^{[2]}$-type, in [12] it is proved that *natural* automorphisms of Hilbert schemes of points (i.e. those which come from automorphisms of $K3$ surfaces) realize all but a few admissible triples. The residual cases (except for the aforementioned automorphism of order 23) are constructed as automorphisms of Fano varieties of lines on cubic fourfolds. For $n \geq 3$, it is necessary to expand our pool of tools. *Induced* automorphisms on moduli spaces of (possibly twisted) sheaves on $K3$ surfaces, studied in [53] and [22] (where they are used to construct examples of non-symplectic involutions of manifolds of $K3^{[2]}$-type), directly generalize natural automorphisms and allow us to realize many new pairs (T, S). We show, in §5, how to apply these constructions when $n = 3, 4$.

Theorem 1.2. *For $n = 3, 4$, all admissible pairs of lattices (T, S) with $\mathrm{rk}(T) \geq 2$ are realized by natural or (possibly twisted) induced automorphisms.*

Admissible pairs (T,S) where T has rank one require special attention. Indeed, these are the pairs where the co-invariant lattice S has largest rank and this implies that automorphisms realizing such pairs deform in families of maximum dimension, as we will explain in §3.3. There are only four distinct triples (p,m,a) which determine pairs of lattices (T,S) with $\mathrm{rk}(T)=1$: two for $p=3$ and two for $p=23$. However, for a fixed n at most two of them are admissible (no more than one for each value of $p\in\{3,23\}$). We study these four cases in Proposition 3.15, providing the corresponding isometry classes of the pairs (T,S): even though they can never be realized by natural or induced non-symplectic automorphisms (Lemma 4.3), we prove the following result.

Theorem (Theorem 4.5)**.** *Let (T,S) be a pair of lattices corresponding to a triple (p,m,a) which is admissible for a certain $n\geq 2$. If $\mathrm{rk}(T)=1$, there exists a manifold X of $K3^{[n]}$-type and a non-symplectic automorphism $f\in\mathrm{Aut}(X)$ of order p with invariant lattice $H^2(X,\mathbb{Z})^{f^*}\cong T$ and $\left(H^2(X,\mathbb{Z})^{f^*}\right)^{\perp}\cong S$.*

The proof of this statement is not constructive, since it employs the global Torelli theorem for IHS manifolds. However, in specific cases it is possible to provide a geometric construction of the automorphism. In §6 we focus on one of these pairs of lattices (T,S) with $\mathrm{rk}(T)=1$, corresponding to the admissible triple $(3,11,0)$ for $n=4$.

Theorem (Theorem 6.6)**.** *The pair of lattices $(T,S)=(\langle 2\rangle, U^{\oplus 2}\oplus E_8^{\oplus 2}\oplus A_2)$ is admissible for $n=4$ and it is realized by a non-symplectic automorphism of order three on a ten-dimensional family of Lehn–Lehn–Sorger–van Straten eightfolds, obtained from an automorphism of the underlying family of cyclic cubic fourfolds.*

We remark that this is the first known geometric construction of a non-induced, non-symplectic automorphism of odd order on a manifold of $K3^{[4]}$-type. Moreover, thanks to it we are able to complete the list of examples of automorphisms of odd prime order $p<23$ which realize all admissible pairs (T,S) for $n=3,4$.

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2. Preliminary notions

2.1. **Lattices.** We recall in this section the fundamental definitions and results of lattice theory which we will need. Our main reference for these topics is the seminal paper [56] by Nikulin; an overview of the subject can also be found in [28], [49, Chapter VIII] and [37, Chapter 14].

A *lattice* L is a free abelian group endowed with a symmetric, non-degenerate bilinear form $(\cdot,\cdot): L\times L\to\mathbb{Z}$. The lattice is *even* if the associated quadratic form is even on all elements of L. If t is a non-zero integer, $L(t)$ denotes the lattice having as bilinear form the one of L multiplied by t. Examples of lattices, which we will often use, are the *negative* definite lattices A_h, E_r corresponding to the Dynkin diagrams of the same names, for $h \geq 1$ and $r \in \{6, 7, 8\}$. We also define the two following lattices:

$$H_5 := \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}; \qquad K_{23} := \begin{pmatrix} -12 & 1 \\ 1 & -2 \end{pmatrix}.$$

For any integer $k \neq 0$ we denote by $\langle k \rangle$ the rank one lattice generated by an element of square k.

The *dual lattice* of L is $L^\vee := \mathrm{Hom}_\mathbb{Z}(L, \mathbb{Z})$, which admits the following description:

$$L^\vee = \{u \in L \otimes \mathbb{Q} : (u, v) \in \mathbb{Z} \ \ \forall v \in L\}.$$

Clearly, L can be seen as a subgroup of $L^\vee$ of maximal rank, thus the quotient $A_L := L^\vee/L$ is a finite group, called the *discriminant group* of L. We denote by $\mathrm{discr}(L)$ the order of the discriminant group, while the *length* $l(A_L)$ is defined as the minimal number of generators of A_L. If $A_L = \{0\}$, the lattice L is said to be *unimodular*: examples of unimodular lattices are the lattice E_8 and the (unique) even hyperbolic lattice U of rank two. If $A_L \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$ for a prime number p and a non-negative integer k, then the lattice L is said to be *p-elementary*; in this case, $l(A_L) = k$.

If A is a finite abelian group, a *finite quadratic form* is a map $q : A \to \mathbb{Q}/2\mathbb{Z}$ such that:

(*i*) $q(ka) = k^2 q(a)$ for all $k \in \mathbb{Z}$ and $a \in A$;

(*ii*) $q(a + a') - q(a) - q(a') = 2b(a, a')$ in $\mathbb{Q}/2\mathbb{Z}$, where $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ is a symmetric bilinear form (called the finite bilinear form associated to q).

A finite quadratic form $q : A \to \mathbb{Q}/2\mathbb{Z}$ is said to be non-degenerate if the associated finite bilinear form b is non-degenerate (i.e. for all $a \in A$, $a \neq 0$, there exists $a' \in A$ such that $b(a, a') \neq 0 \in \mathbb{Q}/\mathbb{Z}$). By using b we define the orthogonal complement $H^\perp \subset A$ for any subgroup $H \subset A$. The isometry group $O(q)$ is the group of isomorphisms of A which preserve the finite quadratic form q.

For an even lattice L, we define a finite quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$, called the *discriminant quadratic form*, as

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}, \qquad q_L(x + L) := (x, x) \pmod{2\mathbb{Z}} \quad \text{for any } x \in L^\vee.$$

If A_L is a finite direct sum of cyclic groups A_i, we write $A_L = \bigoplus_i A_i(\alpha_i)$ if the discriminant form q_L takes value $\alpha_i \in \mathbb{Q}/2\mathbb{Z}$ on the generator of the summand A_i. We will sometimes use the following result.

Proposition 2.1. [56, Proposition 1.2.1]. *Let A be an abelian group, q a finite quadratic form on A and $H \subset A$ a subgroup. If the restriction $q|_H$ is non-degenerate, then $q = q|_H \oplus q|_{H^\perp}$.*

A lattice isometry of L induces in a natural way an isometry of (A_L, q_L), as explained in [28, §1.2]: in this way it is possible to define a canonical homomorphism between the orthogonal groups $O(L) \to O(q_L)$. We will denote by $\overline{\psi} \in O(q_L)$ the image of $\psi \in O(L)$ under this homomorphism. Similarly, an isomorphism of lattices $\varphi : L_1 \to L_2$ induces an isomorphism of discriminant forms $\overline{\varphi} : q_{L_1} \to q_{L_2}$ ([56, §1.4]).

The *signature* of a lattice L is the signature of the $\mathbb{R}$-linear extension of the bilinear form $(\cdot, \cdot)$ to $L \otimes_\mathbb{Z} \mathbb{R}$; together with the discriminant quadratic form q_L, it defines the *genus* of L (see [56, §1]).

Theorem 2.2. [28, Proposition 1.4.7], [54, Theorem 2.2]. *An even, indefinite lattice L with $l(A_L) \leq \mathrm{rk}(L) - 2$ is uniquely determined, up to isometries, by its signature and its discriminant form q_L. Moreover, the natural homomorphism $O(L) \to O(q_L)$ is surjective.*

By [56, Theorem 1.3.1], two even lattices L_1, L_2 have isomorphic discriminant forms $q_{L_1} \cong q_{L_2}$ if and only if there exist unimodular lattices V_1, V_2 such that $L_1 \oplus V_1 \cong L_2 \oplus V_2$. Moreover, by [56, Theorem 1.1.1(a)] the signature $(v_{(+)}, v_{(-)})$ of an unimodular lattice V satisfies $v_{(+)} - v_{(-)} \equiv 0 \pmod 8$. It is therefore possible to define the *signature modulo* 8 of a finite quadratic form q: $\mathrm{sign}(q) = l_{(+)} - l_{(-)} \pmod 8$, where $(l_{(+)}, l_{(-)})$ is the signature of an even lattice L such that $q_L = q$.

We adopt the notation of [19]. Let p be an odd prime; by [56, Proposition 1.8.1], there are only two non-isometric, non-degenerate discriminant forms on $\frac{\mathbb{Z}}{p^\alpha \mathbb{Z}}$ $(\alpha \geq 1)$: they are denoted by $w^\epsilon_{p,\alpha}$, with $\epsilon \in \{-1, +1\}$. The quadratic form $w^{+1}_{p,\alpha}$ has generator value $q(1) = \frac{a}{p^\alpha} \pmod{2\mathbb{Z}}$, where a is the smallest positive even number which is a quadratic residue modulo p. In turn, for $w^{-1}_{p,\alpha}$ we have $q(1) = \frac{a}{p^\alpha}$ with a the smallest positive even number that is not a quadratic residue modulo p. Thus, a non-degenerate quadratic form q on $\frac{\mathbb{Z}}{p^\alpha \mathbb{Z}}$ such that $q(1) = \frac{x}{p^\alpha}$ is isometric to $w^\epsilon_{p,\alpha}$, with $\epsilon = (\frac{x}{p})$ (using Legendre symbol).

Any non-degenerate quadratic form on $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$, $k \geq 1$, is isomorphic to a direct sum of forms of type $w^{+1}_{p,1}$ and $w^{-1}_{p,1}$, with $w^{+1}_{p,1} \oplus w^{+1}_{p,1} \cong w^{-1}_{p,1} \oplus w^{-1}_{p,1}$ (see [56, Proposition 1.8.2]). This means that, if S is a p-elementary lattice with discriminant group of length k, the form q_S on A_S can only be of two types, up to isometries:

$$q_S = \begin{cases} \left(w^{+1}_{p,1}\right)^{\oplus k} \\ \left(w^{+1}_{p,1}\right)^{\oplus k-1} \oplus w^{-1}_{p,1} \end{cases}$$

Remark 2.3. The signatures (mod 8) of the discriminant forms $w^\epsilon_{p,\alpha}$ are listed in [56, Proposition 1.11.2]). For odd p we have $\mathrm{sign}(w^{+1}_{p,1}) \equiv 1 - p \pmod 8$ and $\mathrm{sign}(w^{-1}_{p,1}) \equiv 5 - p \pmod 8$. Hence, if S is p-elementary and $\mathrm{sign}(S) = (s_{(+)}, s_{(-)})$, the quadratic form on the discriminant group $A_S = \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$ is

$$q_S = \begin{cases} \left(w^{+1}_{p,1}\right)^{\oplus k} & \text{if } s_{(+)} - s_{(-)} \equiv k(1-p) \pmod 8 \\ \left(w^{+1}_{p,1}\right)^{\oplus k-1} \oplus w^{-1}_{p,1} & \text{if } s_{(+)} - s_{(-)} \equiv k(1-p) + 4 \pmod 8 \end{cases} \tag{1}$$

This means that the quadratic form of a p-elementary lattice ($p \neq 2$) is uniquely determined by its signature (see [59, §1] for additional details).

We recall that a sublattice $M \subset L$ is *primitive* if the quotient L/M is free. Analogously, an embedding of lattices $i : S \hookrightarrow L$ is primitive if $i(S) \subset L$ is a primitive sublattice.

Definition 2.4. Two primitive embeddings $i : S \hookrightarrow M$, $j : S \hookrightarrow M'$ define *isomorphic primitive sublattices* if there exists an isomorphism $\varphi : M \to M'$ such that $\varphi(i(S)) = j(S)$.

The following fundamental result, proved by Nikulin in [56, Proposition 1.15.1], provides a characterization of primitive embeddings.

Theorem 2.5. *Let S be an even lattice of signature $(s_{(+)}, s_{(-)})$ and discriminant form q_S. For an even lattice L of invariants $(m_{(+)}, m_{(-)}, q_L)$ unique in its genus, primitive embeddings $i : S \hookrightarrow L$ are determined by quintuples $\Theta_i := (H_S, H_L, \gamma, T, \gamma_T)$ such that:*

- *H_S is a subgroup of A_S, H_L is a subgroup of A_L and $\gamma : H_S \to H_L$ is an isometry $q_S|_{H_S} \cong q_L|_{H_L}$;*
- *T is a lattice of signature $(m_{(+)} - s_{(+)}, m_{(-)} - s_{(-)})$ and discriminant form $q_T = ((-q_S) \oplus q_L)|_{\Gamma^\perp/\Gamma}$, where $\Gamma \subset A_S \oplus A_L$ is the graph of γ and $\Gamma^\perp$ is its orthogonal complement in $A_S \oplus A_L$ with respect to the finite bilinear form associated to $(-q_S) \oplus q_L$;*
- *$\gamma_T \in O(q_T)$.*

The lattice T is isomorphic to the orthogonal complement of $i(S)$ in L. Moreover, two quintuples Θ and Θ' define isomorphic primitive sublattices if and only if $\overline{\mu}(H_S) = H'_S$ for $\mu \in O(S)$ and there exist isometries $\phi \in O(q_L)$, $\nu : T \to T'$ such that $\gamma' \circ \overline{\mu} = \phi \circ \gamma$ and $\overline{\nu} \circ \gamma_T = \gamma'_{T'} \circ \overline{\nu}$.

2.2. **Monodromies and global Torelli theorem for manifolds of $K3^{[n]}$-type.** An *irreducible holomorphic symplectic* (IHS) manifold is a complex, smooth, compact, Kähler manifold X such that $H^{2,0}(X) = \mathbb{C}\omega_X$, for an everywhere non-degenerate two-form ω_X. Examples of IHS manifolds are provided by $K3$ surfaces and, for any $n \geq 2$, by Hilbert schemes of n points on them, as well as by their IHS deformations, which are known as *manifolds of $K3^{[n]}$-type.* If X is an IHS manifold, the second cohomology group $H^2(X,\mathbb{Z})$ admits a lattice structure, by means of the non-degenerate bilinear form of signature $(3, b_2(X) - 3)$ due to Beauville–Bogomolov–Fujiki (see [31, Theorem 4.7]). An automorphism $\sigma \in \mathrm{Aut}(X)$ of prime order p is *non-symplectic* if $\sigma^*\omega_X = \xi\omega_X$, where ξ is a primitive p-th root of unity. By [6, Proposition 6], the existence of a non-symplectic automorphism on X guarantees that X is projective.

The global Torelli theorem for IHS manifolds gives the following Hodge theoretic result, due to Markman (see [47, Theorem 1.3]). This version involves the definition of parallel transport operator, which we do not recall and can be found in [47, Definition 1.1].

Theorem 2.6. *Let X, Y be two deformation equivalent irreducible holomorphic symplectic manifolds and let $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ be an isomorphism of integral Hodge structures and a parallel transport operator. There exists an isomorphism $\sigma : Y \to X$ such that $f = \sigma^*$ if and only if f maps a Kähler class of X to a Kähler class of Y.*

In the case $X = Y$, parallel transport operators $f : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ are called *monodromy operators*, which form a subgroup $\mathrm{Mon}^2(X) \subset O(H^2(X,\mathbb{Z}))$. If X is a manifold of $K3^{[n]}$-type, $n \geq 2$, the second cohomology lattice $H^2(X,\mathbb{Z})$ has rank 23 and it is isometric to $L := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ by [7, Proposition 6]. Let (X, η) be a marked holomorphic symplectic manifold of $K3^{[n]}$-type, i.e. X is IHS of $K3^{[n]}$-type and $\eta : H^2(X,\mathbb{Z}) \to L$ is a lattice isomorphism. We denote by $\mathrm{Mon}^2(L) := \eta \circ \mathrm{Mon}^2(X) \circ \eta^{-1} \subset O(L)$ the monodromy group of L: it is an arithmetic subgroup of $O(L)$ and it is independent on the choice of the marking, inside a connected component of the moduli space of marked IHS manifolds (X, η) (see [47, §9]).

We recall that, for an even lattice G, the *real spinor norm* $\mathrm{sn}_{\mathbb{R}}^G : O(G_{\mathbb{R}}) \to \mathbb{R}^*/\left(\mathbb{R}^*\right)^2 \cong \{\pm 1\}$ is defined as

$$\mathrm{sn}_{\mathbb{R}}^G(\gamma) = \left(-\frac{v_1^2}{2}\right) \dots \left(-\frac{v_r^2}{2}\right)$$

if $\gamma = \rho_{v_1} \circ \ldots \circ \rho_{v_r}$ as a product of reflections with respect to vectors $v_i \in G_{\mathbb{R}}$ (in particular, $r \leq \mathrm{rk}(G)$ by the Cartan–Dieudonné theorem [60, Theorem 5.4]).

Remark 2.7. Since $\mathrm{sn}_{\mathbb{R}}^G : O(G_{\mathbb{R}}) \to \mathbb{R}^*/\left(\mathbb{R}^*\right)^2 \cong \{\pm 1\}$ is a group homomorphism (see for instance [49, Theorem 10.2]), if $\gamma \in O(G_{\mathbb{R}})$ is an isometry of odd order then $\mathrm{sn}_{\mathbb{R}}^G(\gamma) = 1$.

We have a very explicit description of monodromy operators on L. Let $\mathcal{N}$ be the subgroup of $O(L)$ generated by reflections with respect to classes of square -2 and by the negative of reflections with respect to classes of square 2. Then, by combining results of Markman ([46, Lemma 9.2]) and Kneser ([42, Satz 4]) we obtain the following description.

Theorem 2.8. $\mathrm{Mon}^2(L) = \mathcal{N} = \left\{ g \in O(L) \mid \overline{g} = \pm\,\mathrm{id}_{A_L}, \mathrm{sn}_{\mathbb{R}}^L(g) = 1 \right\}$.

In the statement of the theorem, as usual, $\overline{g}$ is the isometry induced by g on the discriminant group A_L.

3. Isometries induced by automorphisms of odd prime order

The aim of this section is to study the action of non-symplectic automorphisms of odd prime order on the second cohomology lattice of manifolds of $K3^{[n]}$-type. We will focus our attention on determining the properties of the invariant sublattice and of its orthogonal; we will show how to classify, for any n, their isometry classes by use of numerical parameters related to their signatures and lengths. This classification is explicitly discussed for $n = 3, 4$ in §3.4. Moreover, in §3.3 we study in greater depth the cases where the invariant lattice has rank one.

3.1. **Discriminant groups of invariant and co-invariant sublattices.** Let X be a manifold of $K3^{[n]}$-type with an action of a finite group $G = \langle \sigma \rangle$, where σ is a non-symplectic automorphism of prime order $p \geq 3$. In particular, p can be at most 23 as a consequence of [6, Proposition 5], because $H^2(X, \mathbb{Z}) \cong L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle$ has rank 23. The group G acts by pullback on $H^2(X, \mathbb{Z})$. Following the notation of [15], we will denote by $T := H^2(X, \mathbb{Z})^{\sigma^*}$ the invariant sublattice of $H^2(X, \mathbb{Z})$ and by $S := T^\perp$ its orthogonal complement (the co-invariant lattice, as we will refer to it): they are both primitive sublattices of $H^2(X, \mathbb{Z})$ by [15, Remark 5.2].

Remark 3.1. If we choose a marking $\eta : H^2(X, \mathbb{Z}) \to L$, then the invariant and co-invariant lattices of an automorphism of X can also be regarded as primitive sublattices $T, S \subset L$. We point out that a different marking η' will produce a pair of sublattices (T', S') of L which is isomorphic to (T, S) in the sense of Definition 2.4. For this reason, we are interested in classifying the pairs (T, S) only up to isomorphisms of primitive sublattices in L.

We collect in the next proposition several results proved by Boissière–Nieper-Wißkirchen–Sarti [15, §5-6] and Tari [62, §2.1.3] (see also [17, §2]).

Proposition 3.2. *Let X be a manifold of $K3^{[n]}$-type and $G = \langle\sigma\rangle$ a group of prime order $p \geq 3$ acting non-symplectically on X. Then:*

- *there exists a positive integer m such that $\mathrm{rk}(S) = (p-1)m$;*
- *S has signature $(2, (p-1)m - 2)$ and T has signature $(1, 22 - (p-1)m)$;*
- *$\frac{H^2(X,\mathbb{Z})}{T\oplus S}$ is a p-torsion group, i.e. $\frac{H^2(X,\mathbb{Z})}{T\oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ for some non-negative integer a;*
- *$a \leq m$.*

If we consider T and S as sublattices of L, then $T \oplus S$ is a sublattice of maximal rank. The sequence of inclusions

$$T \oplus S \subset L \subset L^\vee \subset (T\oplus S)^\vee \cong T^\vee \oplus S^\vee$$

provides an identification of $\frac{L}{T\oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ with an isotropic subgroup $M \subset A_T \oplus A_S$ such that $M^\perp/M \cong A_L$ (see [56, §5]). Denoting by p_T and p_S the two projections from $A_T \oplus A_S$ to A_T and A_S respectively, their restrictions to M are injective (because $T \hookrightarrow L$ and $S \hookrightarrow L$ are primitive; see again [56, §5]). Their isomorphic images are $M_T := p_T(M) \subset A_T$ and $M_S := p_S(M) \subset A_S$. Since the discriminant groups A_T, A_S are finite, this implies $p^a \mid \mathrm{discr}(T)$ and $p^a \mid \mathrm{discr}(S)$. Moreover, the isomorphism $\gamma := p_S \circ (p_T)^{-1}|_{M_T} : M_T \to M_S$ is an anti-isometry, as a consequence of the isotropy of M: this means that $q_T(x) = -q_S(\gamma(x))$ for all $x \in M_T$.

Lemma 3.3. *Let σ be a non-symplectic automorphism of prime order $p \geq 3$ of a manifold of $K3^{[n]}$-type and let $\psi = \sigma^* \in \mathrm{Mon}^2(L)$. Then:*

(i) the action of ψ on $M^\perp \subset A_T \oplus A_S$ is trivial;

(ii) the co-invariant lattice $S = \left(L^\psi\right)^\perp$ is p-elementary.

Proof.

(i) As in the previous section, for an isometry g of a lattice V we denote by $\overline{g}$ the induced isometry of the discriminant group A_V. The monodromy operator ψ induces $\overline{\psi} = \mathrm{id}$ on $A_L \cong M^\perp/M$ by Theorem 2.8 and because $\psi^p = \mathrm{id}$ with odd p (therefore $\overline{\psi}$ cannot be $-\mathrm{id}$). This implies that for any element $(x, y) \in M^\perp \subset A_T \oplus A_S$ we have $\overline{(\psi|_{T\oplus S})}(x, y) - (x, y) \in M$. Moreover, ψ acts trivially on the discriminant group A_T (because $\psi|_T = \mathrm{id}$), thus $\overline{(\psi|_{T\oplus S})}(x, y) - (x, y) = (0, \overline{(\psi|_S)}(y) - y)$ (the natural inclusions of A_T and A_S in $A_{T\oplus S} \cong A_T \oplus A_S$ are $\overline{\psi}$-equivariant). Since M is the graph in $A_T \oplus A_S$ of the anti-isometry $\gamma : M_T \to M_S$, we deduce that $\overline{(\psi|_S)}(y) = y$ for any $y \in M_S = p_S(M^\perp)$. This means that the action of ψ is trivial on $M^\perp$, not only on the quotient $M^\perp/M$.

(ii) For any $n \geq 2$, the lattice $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ can be primitively embedded inside the *Mukai lattice* $\Lambda_{24} := U^{\oplus 4} \oplus E_8^{\oplus 2}$ as the orthogonal complement of a primitive element of square $2(n-1)$ belonging to one of the summands U of Λ_{24} (see [47, Corollary 9.5]). As we remarked in the previous point of the proof, the action of $\overline{\psi}$ on the discriminant A_L is trivial: this allows us to extend ψ to an isometry $\rho \in O(\Lambda_{24})$ such that $\rho|_{L^\perp} = \mathrm{id}$, by [56, Corollary 1.5.2]. The lattice Λ_{24} is unimodular, therefore both the invariant lattice $T_\rho := \Lambda_{24}^\rho \subset \Lambda_{24}$ and the co-invariant lattice $S_\rho := (T_\rho)^\perp \subset \Lambda_{24}$ are p-elementary (see for instance [62, Lemme

2.10]). Since $L^\perp \subset T_\rho$, passing to the orthogonal complements we have $S_\rho \subset L$, and therefore $S = S_\rho$ is p-elementary. $\square$

For fixed values of $n \geq 2$ and $p \geq 3$ prime, we write $2(n-1) = p^\alpha \beta$ with α, β integers, $\alpha \geq 0$ and $(p, \beta) = 1$. Then $A_L \cong \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \cong \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$ is an orthogonal splitting (see [56, Proposition 1.2.2]): in particular, we can show that there exists a subgroup of A_T isomorphic to the summand $\frac{\mathbb{Z}}{\beta\mathbb{Z}}$.

Lemma 3.4. *Let $(A_T)_p$ and $(A_S)_p$ be the Sylow p-subgroups of A_T and A_S respectively. Then*

$$A_T = (A_T)_p \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}, \qquad A_S = (A_S)_p.$$

Moreover, $|A_T| = p^a \beta t$ and $|A_S| = p^a s$ for some positive integers t, s such that $ts = p^\alpha$.

Proof. As $A_L \cong M^\perp/M$ and $|M| = p^a$, we deduce $\left|M^\perp\right| = p^{a+\alpha}\beta$. As a consequence, there exists a subgroup $N \subset M^\perp$ of order β, and it is unique since $|N|$ is coprime with $\left[M^\perp : N\right]$. Moreover, the restriction to N of the projection $M^\perp \to M^\perp/M$ is injective ($N \cap M = \{0\}$ because the orders of N and M are relatively prime). By the fact that there is also a unique subgroup of order β inside A_L, we conclude that N is isomorphic to the component $\frac{\mathbb{Z}}{\beta\mathbb{Z}}$ of A_L. By Lemma 3.3, the action of the automorphism σ on $N \subset M^\perp$ is trivial and any element of $p_S(N)$ is of p-torsion: we are lead to conclude $p_S(N) = 0$, because $(p, \beta) = 1$. Thus, N is contained in A_T.

Since $M_T \subset (A_T)_p$ and $M_S \subset (A_S)_p$ we can write $|A_T| = p^a \beta t$ and $|A_S| = p^a s$, with t, s positive integers. From

$$[L : (T \oplus S)]^2 = \frac{\mathrm{discr}(T) \cdot \mathrm{discr}(S)}{\mathrm{discr}(L)} = \frac{|A_T|\,|A_S|}{|A_L|}$$

(see [56, §4]) we get $ts = p^\alpha$. The two integers t, s are therefore powers of p with non-negative exponents. $\square$

We are now ready to describe the structures of the two discriminant groups A_T and A_S.

Proposition 3.5. *Let X be a manifold of $K3^{[n]}$-type and $G = \langle \sigma \rangle$ a group of odd prime order p acting non-symplectically on X. Then one of the following cases holds:*

(*i*) $A_S = M_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, $A_T \cong M_T \oplus A_L \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$;

(*ii*) $\alpha = 1$, $a = 0$, $A_S \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, $A_T \cong \frac{\mathbb{Z}}{\beta\mathbb{Z}}$;

(*iii*) $\alpha \geq 1$, $a \geq 1$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$, $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$.

Proof. If $a = 0$, the group M is trivial and $A_L \cong A_T \oplus A_S$. By Lemma 3.4 we deduce that there are only two possibilities: $A_S = 0$, $A_T \cong A_L$ or $A_S \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, $A_T \cong \frac{\mathbb{Z}}{\beta\mathbb{Z}}$. The second case, though, is admissible only for $\alpha = 1$, because we know that S is p-elementary by Lemma 3.3.

From now on we will assume $a \geq 1$. Let us first consider the case $\alpha = 0$: this implies $\beta = 2(n-1)$ and $t = s = 1$. Then, by using Lemma 3.4 we conclude $A_S = M_S$ and $A_T = M_T \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}} \cong M_T \oplus A_L$.

If $\alpha = 1$ we have $2(n-1) = p\beta$ and $ts = p$. There are two possibilities:

- $t = p, s = 1$. In this case, $A_S = M_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, therefore $q_S|_{M_S} = q_S$ is non-degenerate and the same holds for $q_T|_{M_T}$, since $q_T|_{M_T} \cong -q_S|_{M_S}$. Then, by Proposition 2.1 we can write $A_T = M_T \oplus M_T^{\perp}$, which implies $(A_T)_p = M_T \oplus \frac{\mathbb{Z}}{p\mathbb{Z}} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$ since $|(A_T)_p| = p^{a+1}$. Hence, $A_T = (A_T)_p \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}} = \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$ and therefore $A_T \cong M_T \oplus A_L$ as $p\beta = 2(n-1)$.
- $t = 1, s = p$. Now $A_T = M_T \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, since $(A_T)_p = M_T$. Hence $q_T|_{M_T}$ is non-degenerate, which again implies that also $q_S|_{M_S}$ is non-degenerate, i.e. $A_S = M_S \oplus M_S^{\perp}$. We are lead to conclude $A_S = M_S \oplus \frac{\mathbb{Z}}{p\mathbb{Z}} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$.

Therefore, if $\alpha = 1$ (and $a \geq 1$) both cases (i), (iii) appearing in the statement can occur.

Now assume $\alpha \geq 2$. Set $H := (A_T)_p \oplus A_S \subset A_T \oplus A_S$ and let $H[p] \subset H$ be the p-torsion subgroup. Since $M^{\perp}/M \cong A_L \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, there exists an element $x \in H$ of order at least p^α: the quotient $\langle x\rangle/(\langle x\rangle \cap H[p])$ has then order at least $p^{\alpha-1}$, which shows that $[H : H[p]] \geq p^{\alpha-1}$. On the other hand, $[H : H[p]] \leq p^\alpha$: indeed, $|H[p]| \geq p^{2a}$, because $M_T \oplus M_S \subset H[p]$, and $|H| = p^a t \cdot p^a s = p^{2a+\alpha}$ (by Lemma 3.4). We conclude that the index $[H : H[p]]$ is either p^α or $p^{\alpha-1}$.

If $[H : H[p]] = p^\alpha$, then $H[p] = M_T \oplus M_S$. By construction $H = H_p$, therefore:

$$H \cong \bigoplus_{i=1}^{2a+\alpha} \left(\frac{\mathbb{Z}}{p^i\mathbb{Z}}\right)^{\oplus m_i}, \qquad H[p] \cong \bigoplus_{i=1}^{2a+\alpha} \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus m_i}, \tag{2}$$

for suitable integers $m_i \geq 0$ such that $\sum_i im_i = 2a + \alpha$ and $\sum_i m_i = 2a$. Thus, the integers m_i must satisfy $\alpha = \sum_i (i-1)m_i$. Furthermore, since we know that H contains an element of order at least p^α, there exists $j \geq \alpha$ such that $m_j \geq 1$. This leaves us with two possibilities for the choice of the coefficients m_i.

- $H \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus 2a-1} \oplus \frac{\mathbb{Z}}{p^{\alpha+1}\mathbb{Z}}$. Then, after recalling that $A_S \supset M_S$ and $A_T \supset M_T$ with $M_S \cong M_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, we have either $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^{\alpha+1}\mathbb{Z}}$, $(A_T)_p = M_T$ or $A_S = M_S, (A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^{\alpha+1}\mathbb{Z}}$. Both cases are not admissible: by Proposition 2.1 (as we remarked discussing $\alpha = 1$) we would need to be able to write, respectively, $A_S = M_S \oplus M_S^{\perp}$ and $A_T = M_T \oplus M_T^{\perp}$, but now this is not possible.
- $H \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus 2a-2} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$. Disregarding the cases where $A_S = M_S$ or $(A_T)_p = M_T$ (which can be excluded as in the previous point) we are left with two alternatives:
  - $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}$;
  - $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$.

  In both cases, though, the lattice S is not p-elementary, contradicting Lemma 3.3.

We conclude that $[H : H[p]] = p^{\alpha-1}$. We can again write H and $H[p]$ as in (2), where now $\sum_i im_i = 2a + \alpha$, $\sum_i m_i = 2a + 1$ and as before there exists $j \geq \alpha$ such that $m_j \geq 1$. We then deduce $H \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus 2a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, which gives rise to four possible conclusions:

- $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, meaning $A_T \cong M_T \oplus A_L$ and $A_S = M_S$;
- $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$;
- $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$;
- $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$.

The last two cases are excluded because S is p-elementary by Lemma 3.3. □

Remark 3.6. We can make some additional remarks on the structures of the discriminant groups A_T, A_S after recalling the following result.

Theorem 3.7. [17, Theorem 2.2] *Let M be an even lattice and $\psi \in O(M)$ be an isometry of prime order $p \neq 2$ with co-invariant lattice S. Then $p^m \operatorname{discr}(S)$ is a square in $\mathbb{Z}$, where $m = \frac{\operatorname{rk}(S)}{p-1}$.*

Let X be a manifold of $K3^{[n]}$-type and $\psi \in \operatorname{Mon}^2(L)$ the isometry induced on L by an automorphism $\sigma \in \operatorname{Aut}(X)$ of prime order $p \geq 3$. From Proposition 3.5 we know that $\operatorname{discr}(S) = |A_S|$ is either p^a or p^{a+1}. In particular:

- if $p \nmid 2(n-1)$ (i.e. $\alpha = 0$), the groups A_T, A_S are as in Proposition 3.5 case (i), therefore a and m must be of same parity by Theorem 3.7.
- If $p \mid 2(n-1)$ (i.e. $\alpha \geq 1$), a and m are not required to have same parity: the structures of A_T and A_S are the ones given in Proposition 3.5 case (i) if a and m have same parity, the ones of cases (ii) or (iii) if a and m have different parity.

3.2. **Admissible triples.** We are now interested in studying primitive embeddings of lattices $T, S \hookrightarrow L$ satisfying Proposition 3.2 and Proposition 3.5, assuming $p \geq 3$. For the purposes of this work, we restrict to $\alpha \leq 1$: notice that, since $2(n-1) = p^\alpha\beta$, the first instance with $\alpha \geq 2$ occurs for $n = 10$, i.e. on manifolds of dimension 20.

Our main result is Theorem 3.12, in which we show that the values (p, m, a) defined in Proposition 3.2, under suitable hypotheses, uniquely determine the isometry classes of T and S. To do so we first need to provide a characterization of primitive embeddings $S \hookrightarrow L$ for lattices S as above (Lemma 3.8 and Proposition 3.9). Finally, in Proposition 3.14 we describe all possible structures, up to isometries, for the discriminant quadratic forms q_S and q_T.

We recall that, by Proposition 3.2, the lattice S has signature $(2, (p-1)m-2)$ and it is p-elementary by Lemma 3.3, with discriminant group $A_S = \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$, where k is the length of A_S. Then there are only two non-isometric possible forms q_S, the ones in (1) (see Remark 2.3).

Since $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle$, the quadratic form q_L on $A_L = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}$ is such that $q_L(1) = -\frac{1}{2(n-1)} \in \mathbb{Q}/2\mathbb{Z}$. If we write $2(n-1) = p^\alpha\beta$, with $(p, \beta) = 1$, then a trivial computation shows:

$$q_L = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}\left(-\frac{1}{2(n-1)}\right) \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}\left(-\frac{\beta}{p^\alpha}\right) \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}\left(-\frac{p^\alpha}{\beta}\right).$$

In §2.1 we defined the two non-isomorphic finite quadratic forms $\omega_{p,\alpha}^{\pm 1}$ on $\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$. Denoting by $q_{\alpha,\beta}$ the quadratic form $\frac{\mathbb{Z}}{\beta\mathbb{Z}}\left(-\frac{p^\alpha}{\beta}\right)$ and by $\left(\frac{-\beta}{p}\right)$ the Legendre symbol modulo p, we conclude:

$$q_L = \begin{cases} w_{p,\alpha}^{+1} \oplus q_{\alpha,\beta} & \text{if } \left(\frac{-\beta}{p}\right) = +1 \\ w_{p,\alpha}^{-1} \oplus q_{\alpha,\beta} & \text{if } \left(\frac{-\beta}{p}\right) = -1 \end{cases}. \tag{3}$$

Lemma 3.8. *Let S be an even lattice with discriminant group $A_S = \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$, $k \geq 0$, of genus $(2, (p-1)m - 2, q_S)$. Let $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ and let $e \in A_L$ be the generator of the component $\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$ of $A_L \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}\left(-\frac{\beta}{p^\alpha}\right) \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}\left(-\frac{p^\alpha}{\beta}\right)$. Then:*

- (i) *If $\alpha = 0$, primitive embeddings of S in L compatible with Proposition 3.5 are determined by pairs (T, γ_T), with T a lattice of signature $(1, 22-(p-1)m)$, $q_T = (-q_S) \oplus q_L$ and $\gamma_T \in O(q_T)$. Two pairs (T, γ_T) and $(T', \gamma'_{T'})$ determine isomorphic sublattices in L if and only if there exists an isometry $\nu : T \to T'$ such that $\overline{\nu} \circ \gamma_T = \gamma'_{T'} \circ \overline{\nu}$.*
- (ii) *If $\alpha = 1$, primitive embeddings of S in L compatible with Proposition 3.5 are determined by triples (x, T, γ_T), with T of signature $(1, 22-(p-1)m)$, $\gamma_T \in O(q_T)$ and either:*
  - (a) *$x = 0$, $q_T = (-q_S) \oplus q_L$, or*
  - (b) *$x \in A_S[p]$ with $q_S(x) = -\frac{\beta}{p}$ (mod $2\mathbb{Z}$) and $\Gamma^\perp/\Gamma \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k-1} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, where $\Gamma \subset A_S \oplus A_L$ is the subgroup generated by (x, e) and $\Gamma^\perp$ is its orthogonal complement with respect to the form $(-q_S) \oplus q_L$; moreover, $q_T = ((-q_S) \oplus q_L)|_{\Gamma^\perp/\Gamma}$.*

  Two triples (x, T, γ_T) and $(x', T', \gamma'_{T'})$ determine isomorphic sublattices in L if and only if there exists $\mu \in O(S)$ and an isometry $\nu : T \to T'$, such that $\overline{\mu}(x) = x'$ and $\overline{\nu} \circ \gamma_T = \gamma'_{T'} \circ \overline{\nu}$.

Proof. Each primitive embedding $i : S \hookrightarrow L$ is determined by a quintuple $\Theta_i = (H_S, H_L, \gamma, T, \gamma_T)$ as in Theorem 2.5, since the lattice L is unique in its genus by Theorem 2.2. Recalling that T is the orthogonal complement of $i(S)$ in L, we ask $\mathrm{sign}(T) = (1, 22-(p-1)m)$. We will discuss separately the cases $\alpha = 0$ and $\alpha = 1$.

- (i) $\alpha = 0$. Since p and β are coprime, the only possibility is: $H_S = \{0\}$, $H_L = \{0\}$ and $\gamma = \mathrm{id}$. The embedding $S \hookrightarrow L$ is therefore determined by the pair (T, γ_T). In particular, we have $\Gamma = \{(0,0)\}$, $\Gamma^\perp = A_S \oplus A_L$, thus $A_T = A_S \oplus A_L$ and the discriminant form is $q_T = (-q_S) \oplus q_L$. This is coherent with case (i) of Proposition 3.5.
- (ii) $\alpha = 1$. We have again the case $H_S = \{0\}, H_L = \{0\}$, $\gamma = \mathrm{id}$ (which means that S and T are as in case (i) of Proposition 3.5, hence $l(A_T) = l(A_S) + 1 = k + 1$). This case corresponds to the triples where $x = 0$ and it is described as for $\alpha = 0$. Alternatively, provided that there exists an element $x \in A_S$ of order p such that $q_S(x) = q_L(e)$, with e as in the statement, we can also choose $H_S = \langle x\rangle, H_L = \langle e\rangle$, $\gamma : x \mapsto e$. Such an element x does not exist only if $k = 0$ or if $q_S = w_{p,1}^{\xi}, q_L = w_{p,1}^{-\xi} \oplus q_{1,\beta}$, with $\xi \in \{\pm 1\}$: in all other cases, by using the isomorphism $w_{p,1}^{+1} \oplus w_{p,1}^{+1} \cong w_{p,1}^{-1} \oplus w_{p,1}^{-1}$ we can write the form q_S as in (1), where at least one of the direct

summands is of the same type as the $w^{\epsilon}_{p,1}$ appearing in q_L (the component corresponding to the subgroup H_L). In this setting, the graph Γ of γ is the subgroup of $A_S \oplus A_L$ generated by (x, e). In particular, since $\Gamma \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, the quotient $\Gamma^{\perp}/\Gamma$ cannot be isomorphic to $A_S \oplus A_L$, which implies that we are not in case (i) of Proposition 3.5. Nevertheless, if $\alpha = 1$ and $k \geq 1$ the structures of the discriminant groups can also be as in cases (ii) or (iii) of Proposition 3.5, where $l(A_T) = \max\{1, k-1\}$: the embedding is admissible if $\Gamma^{\perp}/\Gamma \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k-1} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, and if so the quadratic form on A_T is $q_T = ((-q_S) \oplus q_L)|_{\Gamma^{\perp}/\Gamma}$.

Finally, for both values of α the stated results about isomorphic sublattices follow directly from Theorem 2.5. $\square$

Lemma 3.8 allows us to list all possible primitive embeddings $i : S \hookrightarrow L$ satisfying Proposition 3.5 for a given lattice S. We now prove that, adding some extra hypotheses, the number of distinct isometry classes for $i(S)^{\perp}$ is actually very limited.

Proposition 3.9. *Let S and L be as in Lemma 3.8, with $\alpha \leq 1$ and $k \leq 21 - \alpha - (p-1)m$.*

- (i) *If $\alpha = 0$ or $k = 0$, or if $\alpha = 1$ and $q_S = w^{\xi}_{p,1}, q_L = w^{-\xi}_{p,1} \oplus q_{1,\beta}$ for $\xi \in \{\pm 1\}$, all primitive embeddings of S in L compatible with Proposition 3.5 define isomorphic sublattices. In particular, the isometry class of the orthogonal complement T is uniquely determined by the genus of S.*
- (ii) *Otherwise, provided that the natural homomorphism $O(S) \to O(q_S)$ is surjective, there are at most two distinct isometry classes for the orthogonal complement T of the image of a compatible embedding $S \hookrightarrow L$, one with $l(A_T) = k + 1$ and one with $l(A_T) = \max\{1, k-1\}$.*

Proof. If $\alpha = 0$ or $k = 0$, or if $\alpha = 1$ and $q_S = w^{\xi}_{p,1}, q_L = w^{-\xi}_{p,1} \oplus q_{1,\beta}$ for $\xi \in \{\pm 1\}$, by Lemma 3.8 a (compatible) primitive embedding of S in L is characterized by a pair (T, γ_T), with T a lattice of signature $(1, 22-(p-1)m)$, $q_T = (-q_S) \oplus q_L$ and $\gamma_T \in O(q_T)$. In this case, then, $l(A_T) = \max\{1, k+\alpha\}$, because $(p, \beta) = 1$. If such an indefinite lattice T exists and if $l(A_T) \leq \mathrm{rk}(T) - 2$ (i.e. if $k \leq 21 - \alpha - (p-1)m$), then the isometry class of T is uniquely determined, by Theorem 2.2. This assumption also guarantees that the natural morphism $O(T) \to O(q_T)$ is surjective (Theorem 2.2), therefore different choices of γ_T give isomorphic primitive sublattices S in L.

Assume now that we are not in one of the cases of point (i) (in particular, let $\alpha = 1$ and $k \geq 1$); moreover, suppose that $O(S) \to O(q_S)$ is surjective and $k \leq 20 - (p-1)m$. A compatible embedding $i : S \hookrightarrow L$ is determined by a triple (x, T, γ_T) as in Lemma 3.8. We make the following distinction.

- Triples $(0, T, \gamma_T)$ correspond to embeddings where $q_T = (-q_S) \oplus q_L$, so $l(A_T) = k + 1$. Then, as before, from the assumption $k \leq 20 - (p-1)m$ it follows that all these embeddings define isomorphic sublattices in L and that the isometry class of T is uniquely determined.
- If $x \neq 0$, the triple (x, T, γ_T) was obtained, in the proof of Lemma 3.8, from a quintuple $\Theta_i = (H_S, H_L, \gamma, T, \gamma_T)$, with $H_S = \langle x \rangle \subset A_S$, $H_L = \langle e \rangle \subset A_L$. If we now consider a different quintuple $\Theta_{i'}$, with $H'_S = \langle x' \rangle$ and $x' \neq 0$, the embeddings i, i' will define isomorphic sublattices of L. This follows from

Lemma 3.8 and Theorem 2.5, because, under our assumptions, two different subgroups $H_S, H'_S \subset A_S$ as above are conjugated by an automorphism of S. In fact, the restrictions $q_S|_{H_S}$ and $q_S|_{H'_S}$ are non-degenerate and isomorphic, since they are both isomorphic to $\frac{\mathbb{Z}}{p\mathbb{Z}}\left(-\frac{\beta}{p}\right)$; hence, by Proposition 2.1 and the classification of p-elementary forms (see §2.1), also the restrictions of the forms on $H_S^\perp$ and $(H'_S)^\perp$ will coincide. This implies that there exists an automorphism of A_S which exchanges H_S and H'_S: by the surjectivity of $O(S) \to O(q_S)$, this automorphism is induced by an automorphism of S. We conclude that the isometry class of S as a primitive sublattice of L does not depend on the choice of $x \neq 0$; it does not depend either on the choice of T or of γ_T, since $k \leq 20-(p-1)m$ and here $l(A_T) = \max\{1, k-1\}$, so $l(A_T) \leq \mathrm{rk}(T) - 2$ by Theorem 2.2. □

Adopting the terminology used in [12, §3.3], we provide the following definition.

Definition 3.10. Let (p, m, a) be a triple of integers, with $3 \leq p \leq 23$ prime, $m \geq 1$, $(p-1)m \leq 22$ and $0 \leq a \leq \min\{m, 23-(p-1)m\}$. The triple is said to be *admissible* for a given integer $n \geq 2$ if there exist two orthogonal sublattices $T, S \subset L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ such that: $\mathrm{sign}(T) = (1, 22-(p-1)m)$, $\mathrm{sign}(S) = (2, (p-1)m-2)$, $\frac{L}{T\oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, $l(A_S) \equiv m \pmod 2$, and the discriminant groups A_T and A_S are as in Proposition 3.5.

We also say that a pair (T, S) of sublattices of L as in Definition 3.10 is *admissible*.

Remark 3.11. The condition $\frac{L}{T\oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ implies that all admissible triples of the form $(p, m, 0)$ define orthogonal sublattices $T, S \subset L$ such that $L = T \oplus S$.

We can now rephrase Proposition 3.9 in the following way, taking into account the uniqueness of S too.

Theorem 3.12. *Let $2(n-1) = p^\alpha\beta$, with $(p, \beta) = 1$ and $\alpha \leq 1$. If (p, m, a) is an admissible triple, there exists a unique even p-elementary lattice S as in Definition 3.10, up to isometries of S. Its primitive embedding in L and its orthogonal complement $T \subset L$ are uniquely determined (up to isometries of L) by (p, m, a), assuming that $l(A_T) \leq 21-(p-1)m$.*

Proof. Let S be a lattice as in Definition 3.10, corresponding to an admissible triple (p, m, a). By Proposition 3.5 and Remark 3.6 the discriminant group A_S is $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ if m and a have same parity, otherwise $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$. Moreover, since the triple (p, m, a) determines the signature $\mathrm{sign}(S) = (2, (p-1)m-2)$, it also fixes the quadratic form on A_S, as we explained in Remark 2.3. Thus, if $\mathrm{rk}(S) \geq 3$ the isometry class of the lattice S is unique in its genus, by [12, Theorem 2.2]. This is also true for the remaining cases, i.e. the triples $(3, 1, 0)$ and $(3, 1, 1)$, where S is positive definite of rank two: by [25, Table 15.1] the only possible isometry class is $S \cong A_2(-1)$.

We now show that the homomorphism $O(S) \to O(q_S)$ is surjective. Since $\mathrm{sign}(S) = (2, (p-1)m-2)$, the two triples $(3, 1, 0)$ and $(3, 1, 1)$ are the only ones where S is not indefinite: for them, the surjectivity of $O(S) \to O(q_S)$ follows from [56, Remark 1.14.6], because $S \cong A_2(-1)$. For all other admissible triples, S is indefinite, and by Theorem 2.2 a sufficient condition is $\mathrm{rk}(S) \geq l(A_S) + 2$. From the hypotheses $p \geq 3$ and $a \leq m$ it follows

$$\mathrm{rk}(S) = (p-1)m \geq 2m \geq 2a.$$

Then, if $a \geq 3$ we have $\mathrm{rk}(S) \geq 2a \geq a + 3 \geq l(A_S) + 2$ for any m. On the other hand, if $a \leq 2$ and $m \geq 3$ then $\mathrm{rk}(S) \geq 2m \geq a + 3 \geq l(A_S) + 2$.

The cases left are $m \in \{1, 2\}$ and $0 \leq a \leq \min\{m, 2\}$.

- $m = 2$. For $a = 0$ and $a = 1$ we have $\mathrm{rk}(S) = 2(p-1) \geq 4 \geq a + 3 \geq l(A_S) + 2$. The inequality also holds for $a = 2$ whenever $p \geq 5$; the only remaining triple is $(3, 2, 2)$, where $l(A_S) = a = 2$ since a and m have the same parity, therefore $\mathrm{rk}(S) = 4 \geq l(A_S) + 2$.
- $m = 1$. Here either $a = 0$ or $a = 1$. As in the previous point, provided that $p \geq 5$ we have $\mathrm{rk}(S) = p - 1 \geq a + 3 \geq l(A_S) + 2$. We already discussed all the remaining triples with $p = 3$, where $S \cong A_2(-1)$.

Thus, the map $O(S) \to O(q_S)$ is surjective for any lattice S corresponding to an admissible triple. The statement follows then from Proposition 3.9 under the assumption $l(A_T) \leq 21 - (p-1)m$. $\square$

Remark 3.13. If both triples (p, m, a), $(p, m, a+1)$ are admissible, with m and a of different parity, then they determine the same lattice S, up to isometries. Indeed, in both cases the signature of S is $(2, (p-1)m-2)$ and, by Remark 3.6, its discriminant group is $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$. Notice, however, that the lattices T corresponding to the two triples are non-isometric, because their discriminant groups have different lengths by Proposition 3.5.

To conclude this subsection, we use our results to list all possible quadratic forms q_S, q_T, up to isometries, on the discriminant groups A_S, A_T: by Lemma 3.8, we will need to discuss separately the cases $\alpha = 0$ and $\alpha = 1$ and to distinguish on whether $-\beta$ is a quadratic residue modulo p. This classification of quadratic forms is needed for listing admissible pairs of lattices (T, S) for specific values of n and p.

Proposition 3.14. *Let $2(n-1) = p^\alpha \beta$, with $(p, \beta) = 1$ and $\alpha \leq 1$. Let (p, m, a) be an admissible triple as in Definition 3.10 and (T, S) a corresponding admissible pair of sublattices of L. Then one of the following holds:*

(*i*) $q_T = (-q_S) \oplus q_L$, *with* $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a}$ *or* $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a-1} \oplus w_{p,1}^{-1}$;

(*ii*) $\alpha = 1$, $-\beta$ *is a quadratic residue modulo* p *and*
  - (a) $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a+1}, q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a} \oplus q_{1,\beta}$, *or*
  - (b) $a \geq 1, q_S = \left(w_{p,1}^{+1}\right)^{\oplus a} \oplus w_{p,1}^{-1}, q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a-1} \oplus \left(-w_{p,1}^{-1}\right) \oplus q_{1,\beta}$.

(*iii*) $\alpha = 1$, $-\beta$ *is not a quadratic residue modulo* p *and*
  - (a) $a \geq 1, q_S = \left(w_{p,1}^{+1}\right)^{\oplus a+1}, q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a-1} \oplus \left(-w_{p,1}^{-1}\right) \oplus q_{1,\beta}$, *or*
  - (b) $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a} \oplus w_{p,1}^{-1}, q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a} \oplus q_{1,\beta}$.

Proof. As explained in the proof of Lemma 3.8, case (i) corresponds to embeddings $S \hookrightarrow L$ determined by quintuples $(H_S, H_L, \gamma, T, \gamma_T)$ with $H_S = 0, H_L = 0$. Moreover, the quadratic form q_S is as in (1), with $k = l(A_S) = a$, by Proposition 3.5. This is the only possibility when $\alpha = 0$. On the other hand, if $\alpha = 1$ there may also be compatible embeddings $S \hookrightarrow L$ corresponding to quintuples with $H_S \neq 0$ (see again Lemma 3.8): in this case, $O(S) \to O(q_S)$ is surjective (see Theorem 3.12), so, as we showed in the proof of Proposition 3.9, the subgroup H_S can be regarded as one of the direct summands in the representation (1) of the quadratic form q_S, up to changing the generators of A_S. Given $T \subset L$ the orthogonal complement of S, the discriminant quadratic form on A_T is then $q_T = ((-q_S) \oplus q_L)|_{\Gamma^\perp/\Gamma}$, with q_L as in (3) and q_S as in (1), where now $k = l(A_S) = a+1$ by Proposition 3.5.

Let us assume that $-\beta$ is a quadratic residue modulo p, so that $q_L = w_{p,1}^{+1} \oplus q_{1,\beta}$, and suppose $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a+1}$. Adopting the same notations used in the previous proofs, let $x \in A_S$ be the generator of the subgroup corresponding to one of the summands $w_{p,1}^{+1}$ in q_S and e be the generator of $\mathbb{Z}/p\mathbb{Z} \subset A_L$. Then $H_S = \langle x \rangle$, $H_L = \langle e \rangle$, $\gamma : x \mapsto e$ and the graph of γ is $\Gamma = \langle (x,e) \rangle \subset A_S \oplus A_L$. A direct computation shows that, with respect to the quadratic form $(-q_S) \oplus q_L$ on $A_S \oplus A_L$, the orthogonal of Γ is

$$\Gamma^\perp = \left(H_S^\perp \oplus H_L^\perp\right) + \Gamma.$$

This implies that the quadratic form $q_T \cong ((-q_S) \oplus q_L)|_{\Gamma^\perp/\Gamma}$ is isometric to the restriction of $(-q_S) \oplus q_L$ to $H_S^\perp \oplus H_L^\perp$, therefore $q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a} \oplus q_{1,\beta}$.

In turn, if $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a} \oplus w_{p,1}^{-1}$ we need to ask $a \geq 1$, otherwise it is not possible to find subgroups $H_S \subset A_S$ and $H_L \subset A_L$ such that $q_S|_{H_S} \cong q_L|_{H_L}$. As in the previous case, we can assume $H_S = \langle x \rangle$, $H_L = \langle e \rangle$, $\gamma : x \mapsto e$, where again $x \in A_S$ is the generator of one of the components $w_{p,1}^{+1}$ in q_S and $e \in A_L$ is the generator of the summand $w_{p,1}^{+1}$ of q_L. Since Γ, $\Gamma^\perp$ are the same as above, the form q_T still arises as the restriction of $(-q_S) \oplus q_L$ to $H_S^\perp \oplus H_L^\perp$, and therefore $q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a-1} \oplus (-w_{p,1}^{-1}) \oplus q_{1,\beta}$.

The two cases where $q_L = w_{p,1}^{-1} \oplus q_{1,\beta}$ (i.e. $-\beta$ is not a quadratic residue modulo p) can be discussed in an analogous way. □

3.3. **A special case:** $\mathrm{rk}(T) = 1$**.** In this subsection we focus on the cases where the invariant lattice T has rank one. As explained in [12, §4], if X is an IHS manifold of $K3^{[n]}$-type and $f \in \mathrm{Aut}(X)$ is a non-symplectic automorphism of prime order $p \geq 3$, then the deformation space of the pair (X, f) (in the sense of [50, Definition 1.1]) has dimension $\dim(H^{1,1}(X)^{f^*}) = m-1$, where m is the integer such that the co-invariant lattice of f has rank $(p-1)m$. As a consequence, for a fixed order p, the deformation families of maximal dimension correspond to the actions on cohomology whose invariant lattice has smallest rank. This explains why we are interested in the cases where $\mathrm{rk}(T) = 1$ and $\mathrm{rk}(S) = (p-1)m = 22$. For odd p, this can only happen if $p = 3, m = 11$ or $p = 23, m = 1$. As before, we write $2(n-1) = p^\alpha \beta$, with $(p, \beta) = 1$.

If $\alpha = 0$, then a must be odd, because it needs to be of the same parity as m (Remark 3.6); in particular, $a \geq 1$. Moreover $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta \mathbb{Z}}$ by Proposition 3.5. Since the length of A_T cannot exceed $\mathrm{rk}(T) = 1$, then necessarily $\alpha = 0$ and $a = 1$. We conclude $T \cong \langle 2p(n-1) \rangle$, since $\alpha = 0$ means that p and $2(n-1)$ are coprime.

If $\alpha \geq 1$, there are two possibilities:

- $a \equiv 1 \pmod 2$. Then $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta \mathbb{Z}}$ with $\alpha \geq 1$ and $a \geq 1$. As a consequence $l(A_T) \geq 2$, so T cannot have rank one.

- $a \equiv 0 \pmod 2$. By the classification provided in Proposition 3.5, T cannot be of rank one if $a > 0$. Hence $a = 0$, $\alpha = 1$, $T \cong \langle \beta \rangle = \langle \frac{2(n-1)}{p} \rangle$.

Moreover, we need to impose conditions on the orthogonal lattice S, using again Proposition 3.5. Since $\mathrm{rk}(T) = 1$, we can also use [33, Proposition 3.6] to determine the existence and the structure of such primitive sublattices $T, S \subset L$. We do it separately for the two possible cases that we found.

- $\alpha = 0, a = 1, T$ spanned by a primitive element $h \in L$ of square $h^2 = 2p(n-1)$.
  By [33, Proposition 3.6], the orthogonal lattice S has discriminant $\frac{4p(n-1)^2}{q^2}$, where $q > 0$ is the generator of the ideal $(h, L) \subset \mathbb{Z}$. By Proposition 3.5 we know that $A_S \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, therefore $\mathrm{discr}(S) = p$ and we need $q = 2(n-1)$. By applying again [33, Proposition 3.6] we can conclude that such a T exists if and only if $-p$ is a quadratic residue modulo $4(n-1)$.
- $\alpha = 1, a = 0, T$ spanned by a primitive element $h \in L$ of square $h^2 = \frac{2(n-1)}{p}$.
  We have $A_S \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, by Proposition 3.5, and $p = \mathrm{discr}(S) = \frac{4(n-1)^2}{pq^2}$, so $q = \frac{2(n-1)}{p}$. Here $p^2 \nmid 4(n-1)$, so p is invertible modulo $\frac{4(n-1)}{p}$, hence by [33, Proposition 3.6] such a T exists if and only if $-p$ is a quadratic residue modulo $\frac{4(n-1)}{p}$.

We rephrase these results as follows.

Proposition 3.15. *Let $p \geq 3$ be a prime and $2(n-1) = p^\alpha \beta$ with $(p, \beta) = 1$. A triple (p, m, a), with $(p-1)m = 22$, is admissible if and only if $\alpha \in \{0, 1\}$, $a = 1 - \alpha$ and $-p$ is a quadratic residue modulo $\frac{4(n-1)}{p^\alpha}$.*

If this happens, then one of the following holds:

(1) $\alpha = 0, p = 3, m = 11, a = 1, T \cong \langle 6(n-1) \rangle, S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$;
(2) $\alpha = 1, p = 3, m = 11, a = 0, T \cong \langle \beta \rangle = \langle \frac{2(n-1)}{3} \rangle, S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$;
(3) $\alpha = 0, p = 23, m = 1, a = 1, T \cong \langle 46(n-1) \rangle, S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$;
(4) $\alpha = 1, p = 23, m = 1, a = 0, T \cong \langle \beta \rangle = \langle \frac{2(n-1)}{23} \rangle, S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$.

Proof. The explicit description of the lattice S in the four cases is obtained by combining [33, Proposition 3.6] (where S is represented as $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus B$ for a negative definite, even lattice B of rank 2 depending on p, n, q) with the results on lattice isomorphisms given in [56, Corollary 1.13.5] and [59, §1], which guarantee the uniqueness, up to isometries, of p-elementary lattices of signature $(2, 20)$ and length one, when $p = 3$ or $p = 23$. The lattice K_{23} was defined in §2.1. $\square$

From Proposition 3.15 it follows, for instance, that the triple $(3, 11, 1)$ is admissible for $n = 2$, as already observed in [12, §3.3], because -3 is a quadratic residue modulo 4: in this case, we have $T = \langle 6 \rangle$. Similarly, $(3, 11, 0)$ is admissible when $n = 4$ (here $\alpha = 1$ and -3 is a quadratic residue modulo 4), with $T = \langle 2 \rangle$. On the other hand, $(3, 11, 1)$ is not admissible when $n = 3$, because -3 is not a quadratic residue modulo 8.

The triple $(23, 1, 1)$ was already found to be admissible for $n = 2$ in [11, §3] (where the isomorphism classes of T, S are also given). By our proposition, this triple is admissible for $n = 3, 4$ too, since $-23 \equiv 1$ both modulo 8 and modulo 12. Finally, the smallest value of n for which $(23, 1, 0)$ is admissible is $n = 24$, since $2(n-1) = 46 = 23 \cdot 2$ and -23 is a quadratic residue modulo 4.

3.4. **Admissible triples for $n = 3, 4$.** In this section we provide a complete classification of admissible triples (p, m, a) for $n = 3, 4$. In both cases, for any odd prime number p we have $\alpha \leq 1$, therefore Theorem 3.12 allows us to exhibit the lattices T, S (up to isometries) for each triple. This classification of admissible pairs of sublattices is achieved by direct computation for all possible triples (p, m, a), checking for each of them if lattices T, S as in Definition 3.10 exist or not; once we get existence, uniqueness follows from the previous results. We apply Theorem [56, Theorem 1.10.1], which provides necessary and sufficient conditions for the existence of an even lattice with given signature and discriminant form.

Manifolds of $K3^{[3]}$-type.

- For $p = 23$ there is only one admissible triple, namely $(23, 1, 1)$, as we already observed in §3.3: the isometry classes of S and T are given in Proposition 3.15, case (3).
- For all primes $5 \leq p \leq 19$, the admissible triples and the lattices S are the ones listed for $n = 2$ in the tables of [12, Appendix A], while the lattices T can be obtained from the corresponding ones in the tables by switching $\langle -2 \rangle$ with $\langle -4 \rangle$ in their description, since now $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -4 \rangle$. Indeed, this is true because in all these cases S is embedded in the K3 lattice $U^{\oplus 3} \oplus E_8^{\oplus 2} \subset L$, which is orthogonal to $\langle -2(n-1) \rangle$ for all $n \geq 2$. Notice that, with respect to [12, Table 5], by Remark 3.6 we can now say that the triple $(13, 1, 0)$ is not admissible, neither for $n = 2$ nor for $n = 3$: in fact, for these values of n we have $\alpha = 0$ for all possible primes p, hence m and a need to have the same parity.

  Example: $(p, m, a) = (5, 5, 3)$. This triple is not admissible for $n = 2$ and it is checked to be still not admissible for $n = 3$. In fact, for these values of p, m, a the lattice S would be isomorphic to $U(5) \oplus E_8^{\oplus 2} \oplus H_5$ (by Proposition 3.5 and Theorem 2.2), whose discriminant group is easily computed to be $A_S \cong A_{U(5)} \oplus A_{H_5} \cong \frac{\mathbb{Z}}{5\mathbb{Z}} \left(\frac{2}{5}\right)^{\oplus 3}$ (recall the definition of the lattice H_5 from §2.1). As we pointed out at the beginning of §3.2, for $n = 3$ we have $A_L \cong \frac{\mathbb{Z}}{4\mathbb{Z}} \left(-\frac{1}{4}\right)$, therefore if S admitted an embedding in L the quadratic form on T would be $q_T = \frac{\mathbb{Z}}{5\mathbb{Z}} \left(\frac{8}{5}\right)^{\oplus 3} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \left(-\frac{1}{4}\right)$ by Lemma 3.8 and $\mathrm{sign}(T) = (1, 2)$. By [56, Theorem 1.10.1], a lattice T with these invariants exists only if its 5-adic completion $T_5 := T \otimes_{\mathbb{Z}} \mathbb{Z}_5$ is such that $|A_T| \equiv \mathrm{discr}(K) \bmod (\mathbb{Z}_5^*)^2$, where K is the unique 5-adic lattice of rank $l(A_{T_5})$ and discriminant form q_{T_5} (see [56, Theorem 1.9.1]). In our case, since $A_{T_5} \cong (A_T)_5 \cong \frac{\mathbb{Z}}{5\mathbb{Z}} \left(\frac{8}{5}\right)^{\oplus 3}$, by using [56, Proposition 1.8.1] we compute $K = \langle 5 \cdot \frac{1}{8} \rangle^{\oplus 3}$, where $\frac{1}{8} \in \mathbb{Z}_5^*$. Thus, $|A_T| = 4 \cdot 5^3$ and $\mathrm{discr}(K) = \left(\frac{5}{8}\right)^3$: these two values do not satisfy the relation $|A_T| \equiv \mathrm{discr}(K) \bmod (\mathbb{Z}_5^*)^2$, because $2^{11} \notin (\mathbb{Z}_5^*)^2$. We conclude that a lattice T with such signature and quadratic form does not exist.
- For $p = 3$, Table 1 in Appendix A lists all admissible triples, with the corresponding isomorphism classes for T, S: as for larger primes, we can find many similarities with the analogous table for $n = 2$ in [12, Table 1]. However, there are also some significant differences.
  - As we observed in §3.3, there are no admissible triples with $m = 11$.
  - The triple $(3, 9, 5)$ is now admissible: here $S = U(3)^{\oplus 2} \oplus E_6 \oplus E_8$, while $\mathrm{sign}(T) = (1, 4)$ and $q_T = -q_S \oplus q_L \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \left(\frac{4}{3}\right)^{\oplus 5} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \left(-\frac{1}{4}\right)$. The

existence of a lattice T with these invariants is proved by applying [56, Theorem 1.10.1] and there is a unique isometry class in the genus of T by [25, Chapter 15, Theorem 21]. In particular, we can take $T = U(3) \oplus \Omega$, where Ω is the even lattice of rank three whose bilinear form is defined by the matrix

$$\Omega := \begin{pmatrix} -6 & 0 & -3 \\ 0 & -6 & 9 \\ -3 & 9 & -18 \end{pmatrix}.$$

We have $\mathrm{sign}(\Omega) = (0,3)$ and $q_\Omega = \frac{\mathbb{Z}}{3\mathbb{Z}}\left(\frac{4}{3}\right)^{\oplus 2} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\left(\frac{2}{3}\right) \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\left(-\frac{1}{4}\right)$, therefore $q_{U(3)\oplus\Omega} \cong -q_S \oplus q_L$ (by [56, Proposition 1.8.2]).

- ◦ An additional new admissible triple is $(p, m, a) = (3, 8, 6)$: here we compute $S = U(3)^{\oplus 2} \oplus E_6^{\oplus 2}$, therefore $\mathrm{sign}(T) = (t_{(+)}, t_{(-)}) = (1, 6)$ and $q_T = \frac{\mathbb{Z}}{3\mathbb{Z}}\left(\frac{4}{3}\right)^{\oplus 6} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\left(-\frac{1}{4}\right)$. In this case, the strict inequality $t_{(+)} + t_{(-)} > l(A_T)$ holds: since moreover $t_{(+)} - t_{(-)} \equiv \mathrm{sign}(q_T) \pmod 8$, such a lattice T exists by [56, Corollary 1.10.2] and again it is unique (up to isometries) by [25, Chapter 15, Theorem 21]. A representative of this genus is $T = U(3) \oplus A_2 \oplus \Omega$.

Manifolds of $K3^{[4]}$-type.

- For $p = 23$ we have that $(23, 1, 1)$ is the only admissible triple (see §3.3): the isomorphism classes of T, S are described in Proposition 3.15.
- For primes $5 \le p \le 19$, again the lattices T, S and all admissible triples are the ones listed in the tables of [12, Appendix A] (apart from $(13, 1, 0)$, which is not admissible), up to replacing the $\langle -2 \rangle$ summand with a $\langle -6 \rangle$ summand in T.
- The last prime we need to consider is $p = 3$. This is the first case we encounter where an odd p divides $2(n-1)$: in particular, $2(n-1) = 6 = 3^\alpha \beta$ with $\alpha = 1$ and $\beta = 2$. Since we have $\alpha = 1$, by Lemma 3.8 and Proposition 3.9 we know that we can expect to have many more admissible triples than the ones which we found for $p = 3$ and $n = 2, 3$: in fact, the same lattice S might be embedded in L in two non-isomorphic ways by Proposition 3.9 (iii). Table 2 (Appendix A) contains the list of all admissible triples and of the corresponding isomorphism classes for the lattices T, S. In particular, the triple $(3, 11, 0)$ is admissible thanks to Proposition 3.15; some other triples, such as $(3, 8, 6)$ and $(3, 8, 7)$, are excluded by using [56, Theorem 1.10.1], in a way completely analogous to what has been done previously for $n = 3$.

4. Existence of automorphisms

The classification of admissible lattices T, S presented in §3 does not tell us which cases can be realized by actual automorphisms. In this section we pass in review several tools to construct non-symplectic automorphisms of odd prime order on manifolds of $K3^{[n]}$-type, which are valid for any $n \ge 2$. In particular, we are interested in two types of manifolds: Hilbert schemes of points on $K3$ surfaces and moduli spaces of (possibly twisted) sheaves on $K3$'s. Moreover, in §4.3 we show that the existence of automorphisms which realize admissible pairs (T, S) where T has rank one can always be proved by using the global Torelli theorem for IHS manifolds.

4.1. **Natural automorphisms.** Let Σ be a smooth $K3$ surface. An automorphism $\varphi \in \mathrm{Aut}(\Sigma)$ induces an automorphism $\varphi^{[n]}$ on the Hilbert scheme $\Sigma^{[n]}$, by mapping a zero-dimensional subscheme $\xi \subset \Sigma$ of length n (i.e. a point in $\Sigma^{[n]}$) to its schematic image $\varphi(\xi)$. Such an automorphism $\varphi^{[n]}$ is said to be *natural* (see [10]).

By [7, Proposition 6], we have an injection $i : H^2(\Sigma, \mathbb{C}) \hookrightarrow H^2(\Sigma^{[n]}, \mathbb{C})$ compatible with the Hodge structures and such that

$$H^2(\Sigma^{[n]}, \mathbb{C}) = i\left(H^2(\Sigma, \mathbb{C})\right) \oplus \mathbb{C}[E]$$

where $[E]$ is the class of the exceptional divisor of the Hilbert-Chow morphism $\Sigma^{[n]} \to \mathrm{Sym}^n(\Sigma)$. In particular, if $\varphi \in \mathrm{Aut}(\Sigma)$ is a non-symplectic automorphism, then $\varphi^{[n]}$ still acts non-symplectically on $\Sigma^{[n]}$: indeed, if $\omega \in H^{2,0}(\Sigma)$ then $i(\omega) \in H^{2,0}(\Sigma^{[n]})$ and $\left(\varphi^{[n]}\right)^* (i(\omega)) = i\left(\varphi^*(\omega)\right)$ by [16, Theorem 1].

Moreover, $H^2(\Sigma^{[n]}, \mathbb{Z}) = i\left(H^2(\Sigma, \mathbb{Z})\right) \oplus \mathbb{Z}\delta$, where $H^2(\Sigma, \mathbb{Z}) \cong L_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}$ and $\delta \in H^2(\Sigma^{[n]}, \mathbb{Z})$ is a class of square $-2(n-1)$ such that $2\delta = [E]$. As observed in [16, §3], the action of the natural automorphism $\varphi^{[n]}$ on $H^2(\Sigma^{[n]}, \mathbb{Z})$ can be decomposed as $\left(\varphi^{[n]}\right)^* = (\varphi^*, \mathrm{id}_{\mathbb{Z}\delta})$. Hence, if $T_\varphi := (L_{K3})^\varphi$ is the invariant lattice of φ and $S_\varphi := (T_\varphi)^\perp \subset L_{K3}$ is the co-invariant lattice, and $T_{\varphi^{[n]}}, S_{\varphi^{[n]}} \subset L$ are the invariant and co-invariant lattices of $\varphi^{[n]}$, then

$$T_{\varphi^{[n]}} = i(T_\varphi) \oplus \mathbb{Z}\delta \cong T_\varphi \oplus \langle -2(n-1) \rangle, \quad S_{\varphi^{[n]}} = i(S_\varphi) \cong S_\varphi.$$

We conclude that all admissible pairs of lattices (T, S) with $T \cong T_{K3} \oplus \langle -2(n-1) \rangle$ and $S \cong S_{K3}$, where T_{K3} and S_{K3} are respectively the invariant lattice and its orthogonal complement for the action of a non-symplectic automorphism on a $K3$ surface, are realized by natural automorphisms. All possible isomorphism classes for the pairs (T_{K3}, S_{K3}) can be found in [3, Table 2] (order $p = 3$) and [4, Tables 2–7] (prime order $5 \leq p \leq 19$), therefore it is immediate to check for any n which admissible cases have a natural realization. We point out the different notation used in the two references: the invariant lattice T_{K3} is denoted as N in [3] and as S in [4], while the co-invariant lattice S_{K3} is denoted as T.

In the tables of Appendix A we mark with the symbol ♣ the triples realized by natural automorphisms. For $n = 4$ (Table 2), it may not always be immediate to recognize the invariant lattices of [3, Table 2] as direct summands in the lattices T that we provide, since we often choose different representatives in the same isomorphism classes. These isomorphisms become clear after observing the following isometries: $U \oplus E_6 \oplus A_2 \cong U(3) \oplus E_8$; $U \oplus A_2^{\oplus 3} \cong U(3) \oplus E_6$; $U \oplus A_2^{\oplus 2} \oplus E_8 \cong U \oplus E_6^{\oplus 2}$ (they can all be proved by using Theorem 2.2). The reason why we adopt different genus representatives for these lattices will become clear in §5.1 (Lemma 5.1).

4.2. **Induced automorphisms.** A direct generalization of the notion of natural automorphisms is given by *induced automorphisms*, which were first introduced and studied in [58, §3], [53] and later extended to the case of twisted $K3$ surfaces in [22, §3].

We recall here the fundamental definitions and results (see [38] and [22, §2.3, §3] for additional details and references). A *twisted K3 surface* is a pair (Σ, α), where Σ is a smooth K3 surface and $\alpha \in \mathrm{Br}(\Sigma) := H^2\left(\Sigma, \mathcal{O}_\Sigma^*\right)_{\mathrm{tor}}$ is a Brauer class. By [64, §2], if α has order k then it can be identified with a surjective homomorphism $\alpha : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/k\mathbb{Z}$, where $\mathrm{Tr}(\Sigma) := \mathrm{Pic}(\Sigma)^\perp \subset H^2(\Sigma, \mathbb{Z})$ is the transcendental lattice of the surface. A B-field lift of α is a class $B \in H^2(\Sigma, \mathbb{Q})$ (which can be determined via the exponential sequence) such that $kB \in H^2(\Sigma, \mathbb{Z})$ and $\alpha(v) = (kB, v)$ for all $v \in \mathrm{Tr}(\Sigma)$ (see [38, §3]). Notice that B is defined only up to an element in $H^2(\Sigma, \mathbb{Z}) + \frac{1}{k}\operatorname{Pic}(\Sigma)$.

The full cohomology $H^*(\Sigma, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z}) \oplus H^4(\Sigma, \mathbb{Z})$ admits a lattice structure, with pairing $(r, H, s) \cdot (r', H', s') = H \cdot H' - rs' - r's$. As a lattice, $H^*(\Sigma, \mathbb{Z})$ is isometric to the Mukai lattice $\Lambda_{24} = U^{\oplus 4} \oplus E_8^{\oplus 2}$. A Mukai vector $v = (r, H, s)$ is *positive* if $H \in \operatorname{Pic}(\Sigma)$ and either $r > 0$, or $r = 0$ and $H \neq 0$ effective, or $r = H = 0$ and $s > 0$. Starting from a primitive, positive vector $v = (r, H, s) \in H^*(\Sigma, \mathbb{Z})$ and a B-field lift B of α we can define the twisted Mukai vector $v_B := (r, H + rB, s + B \cdot H + r\frac{B^2}{2})$. Then, if v_B is primitive and positive, for a suitable choice of a polarization D of Σ the coarse moduli space $M_{v_B}(\Sigma, \alpha)$ of α-twisted Gieseker D-stable sheaves with Mukai vector v_B is a projective irreducible holomorphic symplectic manifold of $K3^{[n]}$-type, with $n = \frac{v_B^2}{2} + 1$. Moreover, we have a canonical isomorphism $H^2(M_{v_B}(\Sigma, \alpha), \mathbb{Z}) \cong v_B^\perp$ in the Mukai lattice (see [67, Theorems 3.16, 3.19]). For the sake of readability, we do not specify the ample divisor D in the notation for $M_{v_B}(\Sigma, \alpha)$, even though the construction depends on it: we will always assume that a choice of a polarization (generic with respect to the Mukai vector v_B, in the sense of [67, Definition 3.5]) has been made.

Now, let φ be an automorphism of Σ: the invariant Brauer classes α are exactly those such that $\alpha \circ \varphi^*|_{\mathrm{Tr}(\Sigma)} = \alpha$. The following result holds (see [53, Proposition 2.32] and [22, §3]).

Proposition 4.1. *Let (Σ, α) be a twisted $K3$ surface, φ an automorphism of Σ, v a positive Mukai vector and B a B-field lift of α such that v_B is primitive and positive. If v_B and α are φ-invariant, then φ induces (via pullback of sheaves) an automorphism $\widehat{\varphi}$ of $M_{v_B}(\Sigma, \alpha)$.*

The automorphisms $\widehat{\varphi}$ arising in this way are called *twisted induced* (or just *induced* in the non-twisted case, i.e. if $\alpha = 0$). As an application of the twisted version of the global Torelli theorem for $K3$ surfaces (see [35, Corollary 5.4]), it is possible to characterize twisted induced automorphisms by studying their action on the Mukai lattice.

Proposition 4.2. [22, Theorem 3.4] *Let σ be an automorphism of finite order on a manifold X of $K3^{[n]}$-type acting trivially on A_L. Then the following are equivalent:*

(1) *There exist a twisted K3 surface (Σ, α), an automorphism φ of Σ such that α is φ-invariant, a positive Mukai vector v and a B-field lift B of α such that v_B is primitive, positive and φ-invariant; in this case, $X = M_{v_B}(\Sigma, \alpha)$ and σ is twisted induced by φ.*
(2) *The invariant lattice of the extension of σ to the Mukai lattice contains primitively a copy of $U(d)$.*

When this occurs, the integer d is some multiple of the order of the Brauer class α.

Let $v = (r, H, s)$ be a positive Mukai vector. If $B \in H^2(\Sigma, \mathbb{Q})$ is a B-field lift of α such that v_B is primitive and positive, then the transcendental lattice of the moduli space $M_{v_B}(\Sigma, \alpha)$ is isomorphic to $\ker(\alpha) \subset \mathrm{Tr}(\Sigma)$, which is a sublattice (proper if $\alpha \neq 0$) of the same rank and of index equal to the order of α (see for instance [38, §2]). By [67, §3], $\operatorname{Pic}(M_{v_B}(\Sigma, \alpha)) \cong v_B^\perp \cap \operatorname{Pic}(\Sigma, \alpha)$ inside $H^*(\Sigma, \mathbb{Z})$, where $\operatorname{Pic}(\Sigma, \alpha) \cong \operatorname{Pic}(\Sigma) \oplus U$ if $\alpha = 0$, otherwise $\operatorname{Pic}(\Sigma, \alpha)$ is generated by $\operatorname{Pic}(\Sigma)$ and the vectors $(0, 0, 1), (k, kB, 0)$ by [45, Lemma 3.1], assuming the order of α is k. As a consequence, $\operatorname{rk}\operatorname{Pic}(M_{v_B}(\Sigma, \alpha)) = \operatorname{rk}\operatorname{Pic}(\Sigma) + 1$.

Lemma 4.3. *Let (Σ, α) be a twisted $K3$ surface, φ an automorphism of Σ, v a positive Mukai vector and $B \in H^2(\Sigma, \mathbb{Q})$ a B-field lift of a φ-invariant Brauer class α such that v_B is primitive, positive and φ-invariant; let $\widehat{\varphi}$ be the twisted induced automorphism of $M_{v_B}(\Sigma, \alpha)$. Then the invariant lattice T of $\widehat{\varphi}$ has rank ≥ 2.*

Proof. By construction $\mathrm{Pic}(\Sigma, \alpha) \otimes \mathbb{Q}$ contains at least a rank three sublattice W invariant for the action of φ: it is the sublattice spanned by $(0, 0, 1), (1, B, 0), (0, h, 0)$, where $h \in \mathrm{Pic}(\Sigma)$ is a φ-invariant ample class.

Indeed, φ acts as the identity on $H^4(\Sigma, \mathbb{Q})$, therefore $(0, 0, 1)$ is fixed. Moreover, it maps $(1, B, 0)$ to $(1, \varphi^*(B), 0)$, but these two classes coincide in $H^2(M_v(\Sigma, \alpha), \mathbb{Q})$: if $\widetilde{H}(\Sigma, B, \mathbb{Q})$ and $\widetilde{H}(\Sigma, \varphi^*(B), \mathbb{Q})$ are the two Hodge structures on Σ defined by the B-field lifts $B, \varphi^*(B)$ of α, the classes $(1, B, 0) = \exp(B)(1, 0, 0)$ and $(1, \varphi^*(B), 0) = \exp(\varphi^*(B))(1, 0, 0)$ correspond to each other via the Hodge isometry

$$\exp(\varphi^*(B) - B) : \widetilde{H}(\Sigma, B, \mathbb{Q}) \to \widetilde{H}(\Sigma, \varphi^*(B), \mathbb{Q})$$

(see [38, §2] and [22, Remark 2.4]).

As a consequence, $\mathrm{rk}(T) = \mathrm{rk}(T \otimes \mathbb{Q}) \geq \mathrm{rk}(W) \cap v_B^{\perp} = 2$. □

In the case $\alpha = 0$, which was already studied in [53], it is possible to provide some additional details on the action of induced automorphisms. Let $v \in H^*(\Sigma, \mathbb{Z})$ be a primitive positive Mukai vector. Then $M_v(\Sigma, 0)$ is isomorphic to the moduli space $M_\tau(v)$ of τ-stable objects of Mukai vector v, for $\tau \in \mathrm{Stab}(\Sigma)$ a suitable v-generic Bridgeland stability condition on the derived category $D^b(\Sigma)$ (see [18], [63, Theorem 1.3 and §3] and [5, Theorems 1.3 and 6.7] for details). By our previous discussion, the transcendental lattice of $M_\tau(v)$ coincides with $\mathrm{Tr}(\Sigma)$, while its Picard lattice is isomorphic to $v^\perp \cap (\mathrm{Pic}(\Sigma) \oplus U)$. In particular, the summand U in $\mathrm{Pic}(\Sigma) \oplus U$ is just $H^0(\Sigma, \mathbb{Z}) \oplus H^4(\Sigma, \mathbb{Z})$, which is the orthogonal complement of $H^2(\Sigma, \mathbb{Z}) \cong L_{K3}$ inside $H^*(\Sigma, \mathbb{Z}) \cong \Lambda_{24}$. Since L_{K3} is unimodular, the action of an automorphism $\varphi \in \mathrm{Aut}(\Sigma)$ on L_{K3} extends to an action on Λ_{24} which is trivial on $(L_{K3})^\perp$ (by [53, Lemma 1.4]). Let $T_{K3}, S_{K3} \subset L_{K3}$ and $\widehat{T}, \widehat{S} \subset \Lambda_{24}$ be the invariant and co-invariant lattices of these two actions: by what we stated, $\widehat{T} = T_{K3} \oplus U$ and $\widehat{S} = S_{K3}$. The induced automorphism $\widehat{\varphi}$ acts on $H^2(M_\tau(v), \mathbb{Z}) \cong L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle$: its invariant lattice is $T \cong \left(v^\perp\right)^\varphi = \widehat{T} \cap v^\perp$ (see [53, Lemma 1.34]). We rephrase the results of [53, §2-3] as follows.

Proposition 4.4. *Let (p, m, a) be an admissible triple for a certain $n \geq 2$, with (T, S) a corresponding admissible pair of lattices. Consider the canonical primitive embeddings $S \hookrightarrow L \hookrightarrow \Lambda_{24}$ and define $\widehat{T} := S^\perp \subset \Lambda_{24}$. Then the triple (p, m, a) is realized by an induced automorphism if $\widehat{T} \cong U \oplus T_{K3}$, $S \cong S_{K3}$, with (T_{K3}, S_{K3}) the invariant lattice and its orthogonal complement for the action of a non-symplectic automorphism on a $K3$ surface, and there exists a primitive vector $v \in \widehat{T}$ of square $2(n-1)$ such that $T \cong v^\perp \cap \widehat{T}$.*

In particular, all natural automorphisms can be considered as induced, since $\langle -2(n-1) \rangle$ is the orthogonal in U of an element of square $2(n-1)$ (see [53, §6]).

In §5 we will apply the theory of induced (and twisted induced) automorphisms to construct geometric realizations of several admissible triples for manifolds of type $K3^{[3]}$ and $K3^{[4]}$.

4.3. **Existence for** $\mathrm{rk}(T) = 1$**.** Theorem 2.6 can be applied to prove the existence of automorphisms of manifolds of $K3^{[n]}$-type realizing the admissible pairs of lattices (T, S) classified in Proposition 3.15, i.e. for $\mathrm{rk}(T) = 1$.

Theorem 4.5. *Let (p, m, a) be an admissible triple as in Proposition 3.15, for a certain $n \geq 2$, and let T, S be the lattices associated to it. Then, there exists a manifold X of $K3^{[n]}$-type and a non-symplectic automorphism $f \in \mathrm{Aut}(X)$ of order p such that $H^2(X,\mathbb{Z})^{f^*} \cong T$ and $\left(H^2(X,\mathbb{Z})^{f^*}\right)^{\perp} \cong S$.*

Proof. We discuss separately the four possible cases classified in Proposition 3.15, keeping the same numbering.

Case (2): $(3, 11, 0)$. Here we have $\alpha = 1$ and $2(n-1) = 3\beta$, with $(3, \beta) = 1$. The invariant and co-invariant lattices are $T = \langle \beta \rangle$ and $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$, which by Proposition 3.15 can be seen as orthogonal sublattices of $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$. We first construct a monodromy of the lattice L having invariant lattice T and co-invariant lattice S. The triple has $a = 0$, therefore $L = T \oplus S$ (see Remark 3.11): an isometry $\phi \in O(L)$ can then be represented as $\phi = \gamma \oplus \psi$, with $\gamma \in O(T)$ and $\psi \in O(S)$. Moreover, since we want ϕ to be of order three with invariant lattice T, we will need $\gamma = \mathrm{id}_T$ and ψ of order three with no non-zero fixed points.

By [3, Theorem 3.3], there exist a $K3$ surface Σ and a non-symplectic automorphism $\varphi \in \mathrm{Aut}(\Sigma)$ of order three with invariant lattice $T_{K3} = U$ and co-invariant lattice $S_{K3} = U^{\oplus 2} \oplus E_8^{\oplus 2}$. Thus, the natural automorphism $\varphi^{[n]}$ on the Hilbert scheme $\Sigma^{[n]}$ will have invariant lattice $T' = U \oplus \langle -2(n-1)\rangle$ and co-invariant lattice $S' = S_{K3} = U^{\oplus 2} \oplus E_8^{\oplus 2}$ (see §4.1). Notice that $T' \oplus S' = L$, meaning that the triple $(3, 10, 0)$ is realized by a natural automorphism for all $n \geq 2$. Moreover, since $\varphi^{[n]}$ has odd order, it induces a monodromy of L which acts as $+\mathrm{id}$ on the discriminant group $A_L \cong A_{T'} \oplus A_{S'}$ (Theorem 2.8). The restriction of this monodromy to S' is therefore an isometry $\mu \in O(S')$ of order three, with no non-zero fixed vectors, such that $\overline{\mu} = \mathrm{id}_{A_{S'}}$. On our original lattice $S = S' \oplus A_2$ we now consider the isometry $\psi = \mu \oplus \rho_0$, where ρ_0 acts on $A_2 = \left(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \left(\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right)\right)$ as

$$\rho_0(e_1) = e_2, \qquad \rho_0(e_2) = -e_1 - e_2.$$

It is easy to check that ρ_0 is an isometry of order three without non-zero fixed points, inducing the identity on the discriminant group A_{A_2} (this isometry was also used in [29, §6.6]). Notice that, since A_2 is negative definite, $\mathrm{sn}_{\mathbb{R}}^{A_2}(\rho_0) = 1$. We then conclude that $\psi = \mu \oplus \rho_0$ is an isometry of S of order three with no non-zero fixed points, which induces the identity on the discriminant group. Moreover, since the order of ψ is odd, $\mathrm{sn}_{\mathbb{R}}^{S}(\psi) = 1$ (see Remark 2.7). By the same reasoning, $\mathrm{sn}_{\mathbb{R}}^{L}(\phi) = 1$ and $\overline{\phi} = \mathrm{id}_{A_L}$. Thus, ϕ is a monodromy operator by Theorem 2.8, with invariant lattice T and co-invariant lattice S. By generalizing [12, Proposition 5.3], there exists a manifold X of $K3^{[n]}$-type and a marking $\eta : H^2(X,\mathbb{Z}) \to L$ such that $\eta(\mathrm{NS}(X)) = T$. The monodromy ϕ is an Hodge isometry, since it preserves $H^{2,0}(X) = \mathbb{C}\omega_X$ (because $\mathrm{NS}(X) = \omega_X^{\perp} \cap H^2(X,\mathbb{Z})$). Moreover, since $\mathrm{rk}(T) = 1$, ϕ fixes a Kähler class (the generator of $\eta(\mathrm{NS}(X)) = T$). The Hodge-theoretic consequence of the global Torelli theorem (Theorem 2.6) allows us to conclude that there exists an automorphism $f \in \mathrm{Aut}(X)$ such that $\eta \circ f^* \circ \eta^{-1} = \phi$.

Case (1): $(3, 11, 1)$. In this case, $T = \langle 6(n-1)\rangle$ and $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$. Now $T \oplus S$ is a proper sublattice of L, because $a = 1$; however, we can still consider the isometry $\phi = \mathrm{id}_T \oplus \psi \in O(T \oplus S)$ defined above. Since $\overline{\psi} = \mathrm{id}_{A_S}$, the isometry

ϕ can be extended to $\Phi \in O(L)$ by [56, Corollary 1.5.2]. As recalled in §3.1, $A_L \cong M^\perp/M$, with $M, M^\perp$ subgroups of $A_T \oplus A_S$, meaning that $\overline{\Phi} = \mathrm{id}_{A_L}$, since $\overline{\phi} = \mathrm{id} \in O(q_{T\oplus S})$. Moreover, we also have $\mathrm{sn}^L_{\mathbb{R}}(\Phi) = 1$ (again by Remark 2.7). Thus, $\Phi \in \mathrm{Mon}^2(L)$ and it still has invariant lattice T and co-invariant lattice S. We can now apply Theorem 2.6 in the same way as before to conclude that, also in this case, there exists an automorphism of a suitable manifold of $K3^{[n]}$-type which induces Φ on the second cohomology lattice.

Cases (3),(4): $(23,1,0)$ *and* $(23,1,1)$. These two cases can be realized by generalizing [11, Theorem 6.1], which proves the existence of an automorphism of order 23 on a manifold of $K3^{[2]}$-type with invariant lattice $T = \langle 46\rangle$ and co-invariant lattice $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$. In Proposition 3.15 we showed that, if a triple (p,m,a) with $p = 23$ is admissible, then $m = 1$ and $a \in \{0,1\}$; moreover, in this case the two orthogonal sublattices of L are $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$ and either $T = \langle 46(n-1)\rangle$ if $a = 1$ (as we have for $n = 2$), or $T = \langle \frac{2(n-1)}{23}\rangle$ if $a = 0$. We notice in particular that S does not depend on n and in [11, Proposition 5.3] it was proved that this lattice admits an isometry ψ of order 23 inducing the identity on A_S. Thus, $\mathrm{id}_T \oplus \psi \in O(T \oplus S)$ can be extended to an isometry $\phi \in O(L)$ such that $\overline{\phi} = \mathrm{id}_{A_L}$ (if $a = 0$ we have $L = T \oplus S$, so $\mathrm{id}_T \oplus \psi$ is already an isometry of L with this property; otherwise, if $a = 1$, we apply again [56, Corollary 1.5.2]). As the order of ϕ is odd, $\mathrm{sn}^L_{\mathbb{R}}(\phi) = 1$ by Remark 2.7. Following the same proof of [11, Theorem 6.1], there exists an automorphism realizing the triple. We point out that, while for $n = 2$ the monodromies of L are just the isometries preserving the positive cone, for $n \geq 3$ the isometry also needs to induce $\pm\,\mathrm{id}$ on A_L (see [47, Lemma 9.2]). This, however, is not a problem since we know that $\overline{\phi} = \mathrm{id}_{A_L}$. □

We remarked in §3.3 that the triple $(3,11,0)$ is admissible for $n = 4$, therefore we can now conclude that it is realized by an automorphism: we mark this case with the symbol ★ in the corresponding table of Appendix A. We will see an explicit geometric realization of it in §6.1.

5. Induced automorphisms for $n = 3, 4$, $p = 3$

The new admissible triples $(3,m,a)$ that appear passing from $n = 2$ to $n = 3$ and, more significantly, to $n = 4$ cannot be realized by natural automorphisms, since the corresponding pairs of lattices (T,S) are not of the form $T \cong T_{K3} \oplus \langle -2(n-1)\rangle$, $S \cong S_{K3}$ for the invariant lattice T_{K3} and its orthogonal complement S_{K3} of a non-symplectic automorphism of order three on a $K3$ surface (see §3.4, §4.1 and Appendix A). However, in this section we will show that all of these triples but one admit a realization using (possibly twisted) induced automorphisms, which were discussed in §4.2. The only exception is the triple $(3,11,0)$: by Lemma 4.3 this cannot be realized by a twisted induced automorphism since $\mathrm{rk}(T) = 1$.

5.1. Induced automorphisms for $n = 4$.

We observe the following general result.

Lemma 5.1. *Let φ be a non-symplectic automorphism of order three on a $K3$ surface, with invariant lattice T_{K3} and co-invariant lattice S_{K3}, and assume that $T_{K3} \cong U(3) \oplus W$ for some even lattice W. If (T,S) is an admissible pair for a certain $n \equiv 1 \pmod 3$ such that $T \cong U \oplus W \oplus \langle -2(n-1)\rangle$ and $S \cong S_{K3}$, then it* *is realized by the automorphism induced by φ on a suitable moduli space $M_\tau(v)$ of dimension $2n$.*

Proof. If $T_{K3} \cong U(3) \oplus W$, then $\widehat{T} \cong U \oplus U(3) \oplus W$ is the invariant lattice of the extended action of φ to Λ_{24}. Since $n \equiv 1 \pmod 3$, there exists a primitive Mukai vector v of square $2(n-1)$ in the summand $U(3)$ of $\widehat{T}$. Then $v^\perp \cap \widehat{T} \cong U \oplus W \oplus \langle -2(n-1)\rangle$ and Proposition 4.4 allows us to conclude. □

Theorem 5.2. *For $n = 4$, all admissible triples $(3, m, a) \neq (3, 11, 0), (3, 10, 3), (3, 9, 4), (3, 8, 5)$ admit a geometric realization via non-twisted induced automorphisms.*

Proof. Except for the four cases excluded in the statement, the only admissible triples in Table 2 of Appendix A which cannot be realized by a natural automorphism and do not satisfy the hypotheses of Lemma 5.1 are $(3, 8, 1), (3, 7, 0), (3, 4, 1)$ and $(3, 3, 0)$.

Consider the triple $(3, 8, 1)$. Here it is easy to check that the corresponding admissible pair is $(T, S) = (\langle 2\rangle \oplus E_6, U^{\oplus 2} \oplus E_6^{\oplus 2})$. By [3, Theorem 3.3] there exists a $K3$ surface Σ and a non-symplectic automorphism of order three $\varphi \in \mathrm{Aut}(\Sigma)$ with $S_{K3} \cong S$ and $T_{K3} \cong U \oplus A_2^{\oplus 2}$: in order to show that the triple $(3, 8, 1)$ is realized by an automorphism induced by φ we need to prove the existence of a primitive Mukai vector $v \in \widehat{T} = U^{\oplus 2} \oplus A_2^{\oplus 2}$ of square six and orthogonal complement $v^\perp \cap \widehat{T}$ isometric to T (see Proposition 4.4). We describe primitive embeddings $\langle 6\rangle \hookrightarrow \widehat{T}$ by using Theorem 2.5, since $\widehat{T}$ is unique in its genus by Theorem 2.2. The discriminant groups of the two lattices $\langle 6\rangle$ and $\widehat{T}$ are:

$$A_{\langle 6\rangle} = \langle s\rangle \cong \frac{\mathbb{Z}}{6\mathbb{Z}}\left(\frac{1}{6}\right); \qquad A_{\widehat{T}} = \langle t_1, t_2\rangle \cong \frac{\mathbb{Z}}{3\mathbb{Z}}\left(\frac{4}{3}\right) \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\left(\frac{4}{3}\right).$$

We consider the isometric subgroups $H := \langle 2s\rangle \subset A_{\langle 6\rangle}$ and $H' := \langle t_1 + t_2\rangle \subset A_{\widehat{T}}$. Let $\gamma : H \to H'$ be the isomorphism which maps the chosen generator of H to the chosen generator of H' (both these elements have order three and quadratic form $\frac{2}{3}$ mod $2\mathbb{Z}$). The graph of γ is the subgroup $\Gamma = \langle 2s + t_1 + t_2\rangle \subset A_{\langle 6\rangle}(-1) \oplus A_{\widehat{T}}$ and its orthogonal complement is $\Gamma^\perp = \langle s + t_1, s + t_2\rangle$. Passing to the quotient $\Gamma^\perp/\Gamma$, the class of the element $s + t_2$ becomes the opposite of the class of $s + t_1$, meaning that

$$\frac{\Gamma^\perp}{\Gamma} = \langle [s + t_1]\rangle \cong \frac{\mathbb{Z}}{6\mathbb{Z}}\left(\frac{7}{6}\right).$$

This quotient coincides with the discriminant group of $T = \langle 2\rangle \oplus E_6$: by Theorem 2.5, this implies that there exists a primitive embedding $\langle 6\rangle \hookrightarrow \widehat{T}$ with orthogonal complement T, thus the triple $(3, 8, 1)$ has an induced realization by Proposition 4.4. Moreover, this computation guarantees that the triple $(3, 4, 1)$ is also realized by an induced automorphism, since in this case both $T = \langle 2\rangle \oplus E_6 \oplus E_8$ and $T_{K3} = U \oplus A_2^{\oplus 2} \oplus E_8$ differ from the ones of $(3, 8, 1)$ only for an additional copy of the unimodular lattice E_8.

With a similar approach it is possible to show that the admissible triples $(3, 7, 0)$ and $(3, 3, 0)$ are realized by induced automorphisms too: here $T = \langle 2\rangle \oplus E_8$, $T_{K3} = U \oplus E_6$ and $T = \langle 2\rangle \oplus E_8^{\oplus 2}, T_{K3} = U \oplus E_6 \oplus E_8$ respectively. □

All the cases which can be realized by non-natural, non-twisted induced automorphisms are marked with the symbol ♮ in Table 2 of Appendix A.

5.2. **Twisted induced automorphisms for $n = 3, 4$.** Both for $n = 3$ and $n = 4$, in §3.4 we found admissible triples for $p = 3$ where the lattice S is different from all possible co-invariant lattices S_{K3} of non-symplectic automorphisms of order three on $K3$ surfaces, classified in [3, Table 2]. Thus, we cannot realize these cases in a natural way, nor using induced automorphisms on moduli spaces of ordinary sheaves on $K3$'s (Proposition 4.4). However we prove that they all admit a geometric realization using twisted induced automorphisms (see §4.2) except for $(3, 11, 0)$ when $n = 4$ (which will be discussed in §6.1).

We are interested in the following triples (p, m, a): $(3, 9, 5)$ and $(3, 8, 6)$ for $n = 3$; $(3, 10, 3), (3, 9, 4), (3, 8, 5)$ for $n = 4$. For each of these cases, let (T, S) be the corresponding pair of admissible lattices in Table 1 ($n = 3$) or Table 2 ($n = 4$) of Appendix A. Notice that S is always of the form $S = U(3)^{\oplus 2} \oplus W$, where W is one of the lattices $E_8^{\oplus 2}$, $E_6 \oplus E_8$, $E_6^{\oplus 2}$.

Let Σ be a $K3$ surface with transcendental lattice $\mathrm{Tr}(\Sigma) = S_{K3} \cong U \oplus U(3) \oplus W$, where S_{K3} is the co-invariant lattice of a non-symplectic automorphism $\varphi \in \mathrm{Aut}(\Sigma)$ of order three: the existence of (Σ, φ) is guaranteed, in all cases, by [3, Theorem 3.3] and [3, Table 2]. This $K3$ surface has $\mathrm{Pic}(\Sigma) = T_{K3} \cong U(3) \oplus M$, for an even lattice M which is either $0, A_2, A_2^{\oplus 2}$.

Proposition 5.3. *Let Σ be a $K3$ surface with a non-symplectic automorphism $\varphi \in \mathrm{Aut}(\Sigma)$ of order three whose co-invariant lattice is $\mathrm{Tr}(\Sigma) \cong U \oplus U(3) \oplus W$, where $W \in \left\{E_8^{\oplus 2}, E_6 \oplus E_8, E_6^{\oplus 2}\right\}$. Then there exists a φ-invariant Brauer class $\alpha \in \mathrm{Br}(\Sigma)$ of order three whose kernel in $\mathrm{Tr}(\Sigma)$ is isomorphic to $S = U(3)^{\oplus 2} \oplus W$.*

Proof. As we recalled in §4.2, a Brauer class $\alpha \in \mathrm{Br}(\Sigma)$ of order three corresponds to a surjective homomorphism $\alpha : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/3\mathbb{Z}$, which is φ-invariant if and only if $\alpha \circ \varphi^*|_{\mathrm{Tr}(\Sigma)} = \alpha$.

Let $\{e_1, e_2\}$ be a basis for the summand U in $\mathrm{Tr}(\Sigma) \cong U \oplus U(3) \oplus W$ and consider $\alpha := (e_1, -) : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/3\mathbb{Z}$. The kernel of this homomorphism is $\ker(\alpha) = K \oplus U(3) \oplus W$, with $K = \{v \in U : (e_1, v) \equiv 0 \,(\mathrm{mod}\ 3\mathbb{Z})\}$. In particular $K = \langle e_1, 3e_2\rangle \cong U(3)$, thus $\ker(\alpha) \cong S$.

We now want to check that $\alpha \circ \varphi^*|_{\mathrm{Tr}(\Sigma)} = \alpha$. By [3, Examples 1.1], since W is a direct sum of copies of E_6 and E_8 the action of the automorphism φ on $\mathrm{Tr}(\Sigma) \cong U \oplus U(3) \oplus W$ can be expressed as

$$\varphi^*|_{\mathrm{Tr}(\Sigma)} = \pi \oplus \rho$$

where ρ is a suitable isometry of order three of W with no fixed points and π is the isometry of $U \oplus U(3)$ which, with respect to a basis $\{e_1, e_2, f_1, f_2\}$, is given by:

$$e_1 \mapsto e_1 - f_1, \qquad e_2 \mapsto -2e_2 - f_2,$$

$$f_1 \mapsto -2f_1 + 3e_1, \qquad f_2 \mapsto f_2 + 3e_2.$$

We have $(e_1, \pi(e_i)) \equiv (e_1, e_i) \pmod{3\mathbb{Z}}$ and $(e_1, \pi(f_i)) \equiv (e_1, f_i) \equiv 0 \pmod{3\mathbb{Z}}$, for $i = 1, 2$, therefore the Brauer class α is invariant with respect to φ. □

Theorem 5.4. *The admissible triples $(p, m, a) = (3, 9, 5), (3, 8, 6)$ for $n = 3$ and $(p, m, a) = (3, 10, 3), (3, 9, 4), (3, 8, 5)$ for $n = 4$ admit a geometric realization by twisted induced automorphisms.*

Proof. Given a triple (p, m, a) as in the statement, let T, S be the invariant and co-invariant lattices associated to it and Σ, φ, α as in Proposition 5.3. We want to construct a moduli space $M_v(\Sigma,\alpha)$ having T as Picard lattice and S as transcendental lattice, and on which φ induces an automorphism.

We are considering α of the form $(e_1,-):\mathrm{Tr}(\Sigma)\to\mathbb{Z}/3\mathbb{Z}$, where e_1 is a generator of U inside $\mathrm{Tr}(\Sigma)\cong U\oplus U(3)\oplus W$. As a consequence, recalling §4.2, the element $B=\frac{e_1}{3}\in\mathrm{Tr}(\Sigma)\otimes_{\mathbb{Z}}\frac{1}{3}\mathbb{Z}\subset H^2(X,\mathbb{Q})$ is a B-field lift of α, with the properties $B^2=0$ and $B\cdot L=0$ for any $L\in\mathrm{Pic}(\Sigma)$.

Assume first that (p,m,a) is one of the three admissible triples for $n=4$. We already remarked that $\mathrm{Pic}(\Sigma)=T_{K3}\cong U(3)\oplus M$: this means that we can find a primitive divisor H in the summand $U(3)$ of $\mathrm{Pic}(\Sigma)$ with $H^2=6$. Moreover, up to taking its opposite we can assume that H is effective (by Riemann–Roch). Let $v=(0,H,0)\in H^*(\Sigma,\mathbb{Z})$ be the primitive positive Mukai vector defined by H, and $B=\frac{e_1}{3}$ the selected B-field lift of α: by the properties of B the twisted Mukai vector v_B (defined in §4.2) coincides with v, and therefore it has square six and it is invariant with respect to φ. By Proposition 4.1, φ induces a non-symplectic automorphism of order three on the moduli space of twisted sheaves $M_v(\Sigma,\alpha)$, which is a manifold of $K3^{[4]}$-type. The transcendental lattice of $M_v(\Sigma,\alpha)$ is $\ker(\alpha)\cong S$ (Proposition 5.3), while its Picard group is isomorphic to the intersection $v_B^\perp\cap\langle\mathrm{Pic}(\Sigma),(0,0,1),(3,3B,0)\rangle$, as we recalled in §4.2. Since $3B=e_1\in\mathrm{Tr}(\Sigma)$, the lattice generated by $(0,0,1)$ and $(3,3B,0)$ is orthogonal to $\mathrm{Pic}(\Sigma)$; moreover, it is isomorphic to $U(3)$, by the fact that $B^2=0$. Thus

$$\mathrm{Pic}\left(M_v(\Sigma,\alpha)\right)\cong\left(H^\perp\cap\mathrm{Pic}(\Sigma)\right)\oplus U(3)\cong\langle -6\rangle\oplus M\oplus U(3)$$

which is exactly the lattice T corresponding to (p,m,a) (see Table 2 of Appendix A).

Consider now the case where (p,m,a) is one of the admissible triples $(3,9,5)$, $(3,8,6)$ for $n=3$. In this case $\mathrm{Pic}(\Sigma)\cong U(3)\oplus A_2\oplus M'$, where M' is 0 for $(3,9,5)$ and A_2 for $(3,8,6)$. Therefore, if $\{e_1,e_2\}$ is a basis for the summand $U(3)$ of $\mathrm{Pic}(\Sigma)$ and $\{\delta_1,\delta_2\}$ is a basis for a summand A_2, we can take the primitive element of square four $\widetilde{H}=e_1+e_2+\delta_1\in\mathrm{Pic}(\Sigma)$. Let H be the effective divisor between $\widetilde{H}$ and $-\widetilde{H}$. As before, $v=v_B=(0,H,0)$ is a primitive positive Mukai vector, invariant with respect to φ. By Proposition 4.1, φ induces an automorphism on $M_v(\Sigma,\alpha)$, which is a manifold of $K3^{[3]}$-type with transcendental lattice $\ker(\alpha)\cong S$ and

$$\mathrm{Pic}(M_v(\Sigma,\alpha))\cong v_B^\perp\cap\langle\mathrm{Pic}(\Sigma),(0,0,1),(3,3B,0)\rangle\cong(H^\perp\cap\mathrm{Pic}(\Sigma))\oplus U(3).$$

A computation shows that the orthogonal complement of $e_1+e_2+\delta_1$ in $U(3)\oplus A_2$ is isomorphic to the lattice Ω defined in §3.4, thus $\mathrm{Pic}(M_v(\Sigma,\alpha))\cong\Omega\oplus M'\oplus U(3)$, which is the lattice T corresponding to the triple (p,m,a) in Table 1 of Appendix A.

To conclude the proof, we need to show that the automorphism induced by φ on $M_v(\Sigma,\alpha)$ leaves the whole Picard lattice invariant. Both for $n=3$ and $n=4$, the direct summand $U(3)$ in $\mathrm{Pic}\left(M_v(\Sigma,\alpha)\right)$ is the lattice $\langle(0,0,1),(3,3B,0)\rangle$. Notice that φ acts as the identity on $H^4(\Sigma,\mathbb{Z})$, therefore $(0,0,1)$ is fixed. Moreover, it maps $(3,3B,0)$ to $(3,3\varphi^*(B),0)$, but these two classes coincide in $H^2(M_v(\Sigma,\alpha),\mathbb{Z})$, as observed in the proof of Lemma 4.3. Since φ^* also fixes $\mathrm{Pic}(\Sigma)$, we get the result. □

In Table 1 and Table 2 of Appendix A we use the symbol ♢ to mark the five admissible triples of Theorem 5.4, which are realized by twisted induced automorphisms, but not by ordinary induced ones.

6. Automorphisms on the LLSvS eightfold

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. Moduli spaces of rational curves on Y and their compactifications were first studied by de Jong and Starr in [26]. If Y does not contain a plane, let $M_3(Y) = \mathrm{Hilb}^{gtc}(Y)$ be the irreducible component of $\mathrm{Hilb}^{3t+1}(Y)$ containing twisted cubic curves on Y. The manifold $M_3(Y)$ is smooth, projective of dimension ten and it is called the *Hilbert scheme of generalized twisted cubics on* Y (see [43, Theorem A]). In [43, Theorem B], Lehn, Lehn, Sorger, van Straten proved that there exist an irreducible holomorphic symplectic manifold Z_Y of dimension eight, a closed Lagrangian embedding $j : Y \hookrightarrow Z_Y$ and a morphism $u : M_3(Y) \to Z_Y$ which factors as $\Phi \circ a$, where $a : M_3(Y) \to Z'_Y$ is a $\mathbb{P}^2$-bundle to an eight-dimensional manifold Z'_Y and $\Phi : Z'_Y \to Z_Y$ is an extremal contraction (in the sense of [27, §6.5]), which contracts a divisor $D \subset Z'_Y$ to the image $j(Y) \subset Z_Y$. Moreover, by work of Addington and Lehn [1] Z_Y is a manifold of $K3^{[4]}$-type.

We recall some details about the construction. For any generalized twisted cubic curve C on Y we denote by $[C]$ the corresponding point in $M_3(Y)$. The linear span $\langle C\rangle \subset \mathbb{P}^5$ is a $\mathbb{P}^3$; in particular, C lies on the cubic surface $S_C = Y \cap \langle C\rangle$, which is integral since Y does not contain any plane. A point $p \in D \subset Z'_Y$ is defined by the datum $(y, \mathbb{P}(W))$, with $y \in Y$ and $\mathbb{P}(W) \subset \mathbb{P}^5$ a three-dimensional linear subspace through y contained in the tangent space T_yY (here and in the following $W \in \mathrm{Grass}(\mathbb{C}^6, 4)$). The generalized twisted cubics on Y parametrized by this datum are non-Cohen–Macaulay: an element $[C]$ in the fiber $a^{-1}(p)$ is given by a singular cubic curve C^0 cut out on Y by a plane through y contained in $\mathbb{P}(W) \subset T_yY$, together with an embedded point at y. The contraction $\Phi|_D : D \to j(Y)$ sends $p = (y, \mathbb{P}(W))$ to $j(y)$.

In turn, a point $p \in Z'_Y \setminus D$ corresponds to the choice of the following data:

- a three-dimensional linear subspace $\mathbb{P}(W) \subset \mathbb{P}^5$;
- a linear determinantal representation for the surface $S = \mathbb{P}(W) \cap Y$, i.e. the orbit $[A]$ of a 3×3-matrix A with coefficients in W^* such that $\det(A) = 0$ is an equation for S in $\mathbb{P}(W)$. Here the orbit is taken with respect to the action of $(\mathrm{GL}_3 \times \mathrm{GL}_3)/\Delta$, where $\Delta := \{(tI_3, tI_3) : t \in \mathbb{C} \setminus \{0\}\}$ (see [43, §3]).

Then, any curve C such that $[C] \in a^{-1}(p)$ lies on S and is arithmetically-Cohen–Macaulay (aCM for short). The generators of the homogeneous ideal $I_{C/S}$ are the three minors of a 3×2-matrix A_0, whose columns are independent linear combinations of the columns of A. The morphism Φ maps $Z'_Y \setminus D$ isomorphically to $Z_Y \setminus j(Y)$.

As showed in [30, §3] (see also [12, §6.2]), one can construct non-symplectic automorphisms of the Fano variety of lines $F(Y)$ (which is a manifold of $K3^{[2]}$-type, by [9, Proposition 2]) starting from automorphisms of the cubic fourfold Y. It is therefore natural to ask whether a similar approach can be used to produce automorphisms on Z_Y: the answer is positive, we will show how to do so and how to choose Y in order to construct a non-symplectic automorphism of Z_Y realizing the admissible triple $(3, 11, 0)$ for $n = 4$.

By [48, Theorem 2], automorphisms of a cubic hypersurface $Y \subset \mathbb{P}^5$ are restrictions of linear automorphisms of $\mathbb{P}^5$. The list of all automorphisms of prime order on smooth cubic fourfolds is provided in [32, Theorem 3.8].

Lemma 6.1. *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold not containing a plane and $\sigma \in \mathrm{PGL}(6)$ an automorphism such that $\sigma(Y) = Y$. Then, σ induces an automorphism $\check{\sigma}$ of $M_3(Y)$ such that $a(\check{\sigma}([C])) = a(\check{\sigma}([C']))$ if $a([C]) = a([C'])$.*

Proof. The universal property of the Hilbert scheme (see [2, §IX.4]) guarantees that σ induces an automorphism $\check{\sigma}$ of $M_3(Y)$ whenever $\sigma(Y) = Y$. In order to describe this action, we begin by looking at points in the fibers of a over $D \subset Z'_Y$. Let $p \in D$ be a point corresponding to $(y, \mathbb{P}(W))$, and $[C_1], [C_2] \in a^{-1}(p)$: as explained above, each $[C_i]$ consists of a plane cubic curve C_i^0, singular in y, together with an embedded point at y. In particular, there exist two subspaces π_1, π_2 of dimension two inside $\mathbb{P}(W)$ which are tangent to Y in y and such that $C_i^0 = \pi_i \cap Y$. Then, $\sigma(C_i^0)$ are again plane cubic curves, cut out on Y by two planes through $\sigma(y)$ inside $\sigma(\mathbb{P}(W)) \subset T_{\sigma(y)}Y$. Let $\check{\sigma}([C_i])$ consist of the curve $\sigma(C_i^0)$, with the unique non-reduced structure at $\sigma(y)$: then $\check{\sigma}([C_1]), \check{\sigma}([C_2])$ are elements of $M_3(Y)$ in the fiber $a^{-1}(p')$, with p' defined by $(\sigma(y), \sigma(\mathbb{P}(W)))$.

Consider now a point $p \in Z'_Y \setminus D$, corresponding to $\mathbb{P}(W) \subset \mathbb{P}^5$ and the orbit of a 3×3-matrix $A = (w_{i,j})$, with $w_{i,j} \in W^*$. Denote $\mathbb{P}(W') := \sigma(\mathbb{P}(W))$ and let S be the integral cubic surface $\mathbb{P}(W) \cap Y$, which is the vanishing locus in $\mathbb{P}(W)$ of $g := \det(A) \in S^3W^*$. Then, the surface $\sigma(S) \subset \mathbb{P}(W')$ is the vanishing locus of $g \circ \sigma^{-1}$, which is the determinant of the matrix $\sigma^* A := \left(w_{i,j} \circ \sigma^{-1}\right)$ with coefficients in $(W')^*$.

Two elements $[C_1], [C_2] \in a^{-1}(p)$ are aCM cubic curves on S: the generators of $I_{C_i/S}$ are given by the three minors of a 3×2-matrix A_i whose two columns are in the span of the columns of A, as shown in [43, §3.1]. Then, $\sigma(C_1), \sigma(C_2)$ are aCM curves on $\sigma(S)$: by pullback, the generators of $I_{\sigma(C_i)/\sigma(S)}$ are the minors of $\sigma^* A_i$, whose columns are again linear combinations of the columns of $\sigma^* A$. Thus, $\check{\sigma}([C_i]) = [\sigma(C_i)] \in M_3(Y)$, for $i = 1, 2$, belongs to the fiber of a over the point defined by $\mathbb{P}(W')$ and $[\sigma^* A]$. □

Proposition 6.2. *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold not containing a plane and $\sigma \in \mathrm{PGL}(6)$ an automorphism such that $\sigma(Y) = Y$. Then, σ induces an automorphism $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$. Moreover, if σ has finite order d, then $\widetilde{\sigma}$ is also of order d and it is non-symplectic if σ acts non-symplectically on $F(Y)$.*

Proof. As a consequence of Lemma 6.1, there exists an automorphism σ' of the manifold Z'_Y such that $\sigma' \circ a = a \circ \check{\sigma}$, and σ' leaves the divisor D invariant. Consider now the divisorial contraction $\Phi : Z'_Y \to Z_Y$; by [43, §4.5], we have $\Phi_*(\mathcal{O}_{Z'_Y}) = \mathcal{O}_{Z_Y}$. From the proof of the previous lemma, if $p, p' \in D$ are points which parametrize $(y, \mathbb{P}(W))$, $(y', \mathbb{P}(W'))$ respectively and $\Phi(p) = \Phi(p')$, i.e. $y = y'$, then $\Phi(\sigma'(p)) = \Phi(\sigma'(p')) = j(\sigma(y))$. The equality $\Phi(\sigma'(p)) = \Phi(\sigma'(p'))$ when $\Phi(p) = \Phi(p')$ clearly holds also if $p, p' \in Z'_Y \setminus D$, since $\Phi : Z'_Y \setminus D \to Z_Y \setminus j(Y)$ is an isomorphism. Then, by applying the rigidity lemma [27, Lemma 1.15(b)] to Φ and $\Phi \circ \sigma'$ we conclude that σ' descends to an automorphism $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$ such that $\widetilde{\sigma} \circ \Phi = \Phi \circ \sigma'$ (see also [44, Lemma 3.2]).

Assume that σ has finite order d. By [43, §4.1], the surjective morphism $s : M_3(Y) \to \mathrm{Grass}(\mathbb{C}^6, 4)$, $[C] \mapsto \langle C \rangle$ factors as

$$M_3(Y) \xrightarrow{a} Z'_Y \xrightarrow{b} \mathrm{Grass}(\mathbb{C}^6, 4)$$

where b is generically finite of degree 72. The morphism s is clearly equivariant with respect to $\check{\sigma} \in \mathrm{Aut}(M_3(Y))$ and the natural action of σ on the Grassmannian, both of which have the same order of σ (recall that $\mathrm{Aut}(\mathrm{Grass}(\mathbb{C}^6,4)) \cong \mathrm{Aut}(\mathbb{P}^5)$ by [23, Theorem I]). Hence, d is also the order of σ' and $\widetilde{\sigma}$.

Let $\omega_{F(Y)}$ and ω_{Z_Y} be suitably chosen symplectic forms on $F(Y)$ and Z_Y respectively, and let $\mathrm{pr}_i : F(Y) \times F(Y) \to F(Y)$ for $i = 1, 2$ be the projections on the two factors. By [66, Proposition 4.8], there exists a dominant rational map of degree six

$$\psi : F(Y) \times F(Y) \dashrightarrow Z_Y$$

such that

$$\psi^*(\omega_{Z_Y}) = \mathrm{pr}_1^*(\omega_{F(Y)}) - \mathrm{pr}_2^*(\omega_{F(Y)}). \tag{4}$$

The rational map ψ is defined as follows. Let $([l],[l']) \in F(Y) \times F(Y)$ be a generic element, so that the span $\langle l, l' \rangle$ is a $\mathbb{P}^3$, and let x be a point on l: the plane $\langle x, l' \rangle$ intersects the cubic fourfold Y along the union of the line l' and a conic Q passing through x. Then $C := l \cup_x Q$ is a rational cubic curve contained in Y: we set $\psi([l],[l']) := u([C]) \in Z_Y$, which is well-defined since all reducible cubic curves C arising from different choices of the point $x \in l$ belong to the same fiber of u (see the proof of [66, Proposition 4.8]). In order to conclude the proof of Proposition 6.2, we need the following:

Lemma 6.3. *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold not containing a plane, $\sigma \in \mathrm{PGL}(6)$ such that $\sigma(Y) = Y$ and $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$ the automorphism induced by σ on Z_Y. Then $\psi([\sigma(l)],[\sigma(l')]) = \widetilde{\sigma}(\psi([l],[l']))$ for generic $([l],[l']) \in F(Y) \times F(Y)$.*

Proof. As we recalled, $\psi([l],[l']) = u([C])$ with $C = l \cup_x Q$, $x \in l$ and $Y \cap \langle x, l' \rangle = l' \cup Q$. Moreover, $\widetilde{\sigma}(\psi([l],[l'])) = u(\check{\sigma}([C]))$ by Lemma 6.1. In turn, $\psi([\sigma(l)],[\sigma(l')]) = u([C'])$, where $C' = \sigma(l) \cup_{\sigma(x)} Q'$ and $Y \cap \langle \sigma(x), \sigma(l') \rangle = \sigma(l') \cup Q'$. However, the intersection $Y \cap \langle \sigma(x), \sigma(l') \rangle$ coincides with $\sigma(Y \cap \langle x, l' \rangle)$; as a consequence, $Q' = \sigma(Q)$ and so $[C'] = \check{\sigma}([C])$. □

End of proof of Proposition 6.2. Thanks to the equivariance of the map ψ and the relation (4) we deduce that, if σ acts non-symplectically on $F(Y)$, then $\widetilde{\sigma}$ is also non-symplectic. □

Proposition 6.4. *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold not containing a plane. The transcendental lattices of $F(Y)$ and Z_Y have the same rank.*

Proof. Let $\Gamma_\psi \subset F(Y) \times F(Y) \times Z_Y$ be the closure of the graph of the map $\psi : F(Y) \times F(Y) \dashrightarrow Z_Y$ and let V be a desingularization of Γ_ψ. We consider the projections $\pi_F : V \to F(Y) \times F(Y)$, $\pi_Z : V \to Z_Y$ which arise from the inclusion $\Gamma_\psi \subset F(Y) \times F(Y) \times Z_Y$. Let $\mathrm{Tr}_{\mathbb{C}}(F(Y)) \subset H^2(F(Y),\mathbb{C})$ and $\mathrm{Tr}_{\mathbb{C}}(Z_Y) \subset H^2(Z_Y,\mathbb{C})$ be the complexifications of the transcendental lattices of $F(Y)$ and Z_Y respectively. If we define $\mathcal{T} := (\pi_F)_*(\pi_Z^*(\mathrm{Tr}_{\mathbb{C}}(Z_Y)))$, using (4) we deduce:

$$\mathcal{T} \subset \mathrm{Tr}_{\mathbb{C}}(F(Y)) \oplus \mathrm{Tr}_{\mathbb{C}}(F(Y)) \subset H^2(F(Y) \times F(Y),\mathbb{C}).$$

In particular, since $\psi^*(\omega_{Z_Y}) \in \mathcal{T}$ and $\mathrm{Tr}_{\mathbb{C}}(F(Y)) \subset H^2(F(Y),\mathbb{C})$ is the minimal Hodge substructure containing holomorphic two-forms, $(\mathrm{pr}_i)_*(\mathcal{T}) = \mathrm{Tr}_{\mathbb{C}}(F(Y))$ for $i = 1$ or $i = 2$. This implies that the ranks of $\mathrm{Tr}(Z_Y)$ and $\mathrm{Tr}(F(Y))$ coincide. □

6.1. **The case of cyclic cubic fourfolds.** Let $\sigma \in \mathrm{PGL}(6)$ be the following automorphism of order three:

$$\sigma(x_0 : \ldots : x_5) = (x_0 : \ldots : x_4 : \xi x_5) \tag{5}$$

with $\xi = e^{\frac{2\pi i}{3}}$. We consider the ten-dimensional family $\mathcal{C}$ of smooth cubic hypersurfaces $Y \subset \mathbb{P}^5$ of equations

$$Y : x_5^3 + F_3(x_0, \ldots, x_4) = 0$$

with a homogeneous polynomial F_3 of degree three. Cubic fourfolds $Y \in \mathcal{C}$ are called *cyclic*: they arise as triple coverings of $\mathbb{P}^4$ ramified along the smooth cubic threefold of equation $F_3 = 0$. Any $Y \in \mathcal{C}$ is invariant with respect to σ, thus $\sigma|_Y \in \mathrm{Aut}(Y)$.

Remark 6.5. In [12, Example 6.4] it is proved that σ induces a non-symplectic automorphism of order three on the Fano variety of lines $F(Y)$, whose invariant lattice is $\langle 6 \rangle$. In particular, as observed in [14, Remark 3.6], we know that $\mathrm{Pic}(F(Y)) \cong \langle 6 \rangle$ for a very general choice of $Y \in \mathcal{C}$.

This allows us to deduce that a very general Y in the family $\mathcal{C}$, such that $\mathrm{Pic}(F(Y)) \cong \langle 6 \rangle$, does not contain any plane. In fact, if there existed a plane $\pi \subset Y$, it would define an algebraic class in $H^{2,2}(Y)$. In particular, the second Néron–Severi group $\mathrm{NS}_2(Y) = H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$ would contain $\langle H^2, \pi \rangle$, where H is an ample line bundle on Y, thus $\mathrm{rk}(\mathrm{NS}_2(Y)) \geq 2$ (references in [45, §3.2]). By applying the Abel–Jacobi map $H^{2,2}(Y) \to H^{1,1}(F(Y))$ (see [9, §3]), the Picard group of $F(Y)$ would also have at least rank 2, while we just observed that this is false for the chosen very general Y.

Summing up all the results in this section, we have obtained the following:

Theorem 6.6. *Let Y be a cyclic cubic fourfold in the family $\mathcal{C}$ not containing a plane. Then there exists a non-symplectic automorphism $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$ of order three, induced by the automorphism $\sigma|_Y \in \mathrm{Aut}(Y)$ of the form (5).*

Moreover, the invariant lattice of $\widetilde{\sigma}$ is $T \cong \langle 2 \rangle$.

Proof. The first part of the statement follows from Proposition 6.2.

As explained in Remark 6.5, the very general cubic fourfold $Y \in \mathcal{C}$ is such that $F(Y)$ has transcendental lattice of rank 22. This together with Proposition 6.4 allows us to conclude that the invariant lattice of $\widetilde{\sigma}$ has the same rank of the invariant lattice of the automorphism induced by σ on $F(Y)$, namely one. Therefore $T \cong \langle 2 \rangle$ by Proposition 3.15. □

At the end of this section we will present a more geometric proof of the second part of Theorem 6.6 by using Theorem 6.10. In order to do so, we first need to study the fixed locus of the automorphism $\widetilde{\sigma}$.

Let $H \subset \mathrm{Fix}(\sigma)$ be the hyperplane $\{x_5 = 0\} \subset \mathbb{P}^5$. The intersection $Y_H := Y \cap H$ is the smooth cubic threefold defined by $F_3(x_0, \ldots, x_4) = 0$ inside H. We denote by Z_H the image via the map $u : M_3(Y) \to Z_Y$ of the set of points parametrizing twisted cubics contained in Y_H: in [61, Proposition 2.9] it is proved that Z_H is a Lagrangian subvariety of Z_Y.

Lemma 6.7. *Let Y be a cubic fourfold in the family $\mathcal{C}$ not containing a plane. Then Z_H is contained in the fixed locus of $\widetilde{\sigma}$ and $j(Y)^{\widetilde{\sigma}} = Z_H \cap j(Y)$.*

Proof. Let $j(y)$ be a point in the image of the embedding $j : Y \hookrightarrow Z_Y$ such that $\widetilde{\sigma}(j(y)) = j(y)$. In the proof of Lemma 6.1 we showed that $\check{\sigma} \in \mathrm{Aut}(M_3(Y))$ maps the fiber of $u : M_3(Y) \to Z_Y$ over the point $j(y)$ to the fiber over $j(\sigma(y))$. Therefore, since $\widetilde{\sigma} \circ u = u \circ \check{\sigma}$, we need $\sigma(y) = y$, i.e. $y \in Y_H$. We conclude $\mathrm{Fix}(\widetilde{\sigma}) \cap j(Y) = j(Y_H)$. Clearly, since $H \subset \mathrm{Fix}(\sigma)$, we have $Z_H \subset \mathrm{Fix}(\widetilde{\sigma})$. Moreover, $Z_H \cap j(Y) \cong Y_H$ (see [61, §3]), thus $Z_H \cap j(Y) = j(Y_H)$. □

Proposition 6.8. *For Y in the family $\mathcal{C}$ not containing a plane, the fixed locus of the automorphism $\widetilde{\sigma}$ is Z_H.*

Proof. By Lemma 6.7, we need to prove that there are no fixed points outside Z_H, i.e. points $p \in Z_Y \setminus j(Y)$ which are fixed by $\widetilde{\sigma}$ and such that the curves parametrized by the fiber $u^{-1}(p)$ are not contained in H. Notice that a point p of this type corresponds to $(\mathbb{P}(W), [A])$, with $\sigma(\mathbb{P}(W)) = \mathbb{P}(W)$ but $\sigma|_{\mathbb{P}(W)} \neq \mathrm{id}$. A vector space $W \in \mathrm{Grass}(\mathbb{C}^6, 4)$ is σ-invariant if and only if it can be written as $W = W_1 \oplus W_\xi$, where we set $W_t := \{w \in W \mid \sigma(w) = tw\}$. The condition $\sigma|_{\mathbb{P}(W)} \neq \mathrm{id}$ implies $W_\xi \neq 0$, therefore W_ξ is the whole one-dimensional eigenspace of $\mathbb{C}^6$ with respect to the eigenvalue ξ of σ, while W_1 is a three-dimensional subspace of the eigenspace of $\mathbb{C}^6$ where σ acts as the identity. Let $y_0, y_1, y_2 \in W^*$ be the dual elements of a basis of W_1. Then, we can take y_0, y_1, y_2, x_5 as coordinates on $\mathbb{P}(W)$, so that the action of σ on it is $\sigma(y_0 : y_1 : y_2 : x_5) = (y_0 : y_1 : y_2 : \xi x_5)$.

We showed in the proof of Lemma 6.1 that, for a point p as above, we have $\widetilde{\sigma}(p) = (\mathbb{P}(W), [\sigma^* A])$. Therefore, p is fixed if and only if the matrices A and $\sigma^* A$ define the same $\mathbb{P}^2$ of generalized aCM twisted cubics on the surface $S = \mathbb{P}(W) \cap Y$, whose equation in $\mathbb{P}(W)$ is of the form $g := x_5^3 + f(y_0, y_1, y_2) = 0$, where f is the restriction of F_3 to $\mathbb{P}(W_1)$. Fix a curve C such that $[C] \in u^{-1}(p)$: its equations in $\mathbb{P}(W)$ are given by the three minors of a 3×2-matrix A_0 with linear entries in W^*. The matrix A_0, up to a change of basis, can only be of eight different types, listed in [43, §1]. Since the curve C lies on S, the polynomial g which defines the surface belongs to $I_{C/S}$, i.e. it is a combination of the minors of A_0 (see [43, §3.1]). We recall that A is a linear determinantal representation of the surface S therefore, without loss of generality, it is of the form

$$A = \left(\begin{array}{c|c} A_0 & \begin{matrix} * \\ * \\ * \end{matrix} \end{array} \right)$$

where the last column is uniquely determined by g (up to a combination of the columns of A_0; see [43, §3.1]). Again by [43, §3.1], the matrices A, $\sigma^* A$ define the same $\mathbb{P}^2$ of cubics on S if and only if the columns of $\sigma^* A_0$ belong to the span of the columns of A.

Assume A_0 is of the most general form, i.e. the form $A^{(1)} = \begin{pmatrix} w_0 & w_1 & w_2 \\ w_1 & w_2 & w_3 \end{pmatrix}^t$ of [43, §1], where $w_0, \ldots, w_3$ are suitable coordinates for $\mathbb{P}(W)$: in this case C is a smooth twisted cubic curve.

Let $M := (a_{i,j} \mid b_i)_{\substack{i=0,1,2,3 \\ j=0,1,2}} \in \mathrm{GL}_4(\mathbb{C})$ be the matrix defining the change of coordinates from $\{w_i\}_{i=0}^3$ to $\{y_0, y_1, y_2, x_5\}$. Then

$$A = \left(\begin{array}{cc|c} \sum_{j=0}^2 a_{0,j} y_j + b_0 x_5 & \sum_{j=0}^2 a_{1,j} y_j + b_1 x_5 & \sum_{j=0}^2 c_{0,j} y_j + d_0 x_5 \\ \sum_{j=0}^2 a_{1,j} y_j + b_1 x_5 & \sum_{j=0}^2 a_{2,j} y_j + b_2 x_5 & \sum_{j=0}^2 c_{1,j} y_j + d_1 x_5 \\ \sum_{j=0}^2 a_{2,j} y_j + b_2 x_5 & \sum_{j=0}^2 a_{3,j} y_j + b_3 x_5 & \sum_{j=0}^2 c_{2,j} y_j + d_2 x_5 \end{array} \right)$$

where the parameters $c_{i,j}$ and d_i are determined by g. Once we apply the automorphism we get:

$$\sigma^* A = \left(\begin{array}{cc|c} \sum_{j=0}^2 a_{0,j} y_j + \xi^2 b_0 x_5 & \sum_{j=0}^2 a_{1,j} y_j + \xi^2 b_1 x_5 & \sum_{j=0}^2 c_{0,j} y_j + \xi^2 d_0 x_5 \\ \sum_{j=0}^2 a_{1,j} y_j + \xi^2 b_1 x_5 & \sum_{j=0}^2 a_{2,j} y_j + \xi^2 b_2 x_5 & \sum_{j=0}^2 c_{1,j} y_j + \xi^2 d_1 x_5 \\ \sum_{j=0}^2 a_{2,j} y_j + \xi^2 b_2 x_5 & \sum_{j=0}^2 a_{3,j} y_j + \xi^2 b_3 x_5 & \sum_{j=0}^2 c_{2,j} y_j + \xi^2 d_2 x_5 \end{array} \right)$$

where the first two columns form the matrix $\sigma^* A_0$, whose minors define the curve $\sigma(C) \subset S$.

The first column of $\sigma^* A_0$ is a $\mathbb{C}$-linear combination of the columns of A if and only if the following linear system of twelve equations admits a solution $(h, k, t) \in \mathbb{C}^3$:

$$\begin{cases} a_{0,j} = h a_{0,j} + k a_{1,j} + t c_{0,j} & \text{for } j = 0,1,2 \\ a_{1,j} = h a_{1,j} + k a_{2,j} + t c_{1,j} & \text{for } j = 0,1,2 \\ a_{2,j} = h a_{2,j} + k a_{3,j} + t c_{2,j} & \text{for } j = 0,1,2 \\ \xi^2 b_0 = h b_0 + k b_1 + t d_0 \\ \xi^2 b_1 = h b_1 + k b_2 + t d_1 \\ \xi^2 b_2 = h b_2 + k b_3 + t d_2 \end{cases} \tag{6}$$

Notice that the system made of the last three equations always admits a unique solution, namely $(h, k, t) = (\xi^2, 0, 0)$. In fact, the determinant of its matrix of coefficients is different from zero, because it coincides with the coefficient of x_5^3 in the expression of the determinant of A (and $\sigma^* A$), which needs to be a (non-zero) scalar multiple of g. Now, the triple $(h, k, t) = (\xi^2, 0, 0)$ is a solution for the whole system (6) only if $a_{0,j} = a_{1,j} = a_{2,j} = 0$ $\forall j = 0, 1, 2$, which is not possible since the matrix M needs to be invertible. We conclude that the columns of $\sigma^* A_0$ can never be combinations of the columns of A if A_0 is of the form $A^{(1)}$. The remaining cases, i.e. A_0 of the forms $A^{(2)}, \ldots, A^{(8)}$ of [43, §1], can be discussed in an entirely similar way. □

Remark 6.9. Let Z_H be the fixed locus of the automorphism $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$, for $Y \in \mathcal{C}$ not containing a plane. By [61, Theorem 3.3] and [39, §6.3], Z_H also arises as resolution of the unique singular point of the theta divisor in the intermediate Jacobian $\mathrm{J}(Y_H)$ of the cubic threefold Y_H. This implies that Z_H is a variety of maximal Albanese dimension and $\mathrm{Alb}(Z_H) \cong \mathrm{J}(Y_H)$ (see for instance [40, §1] and references therein).

Let us fix a cubic fourfold $Y \in \mathcal{C}$ not containing a plane and choose a marking $\eta_0 : H^2(Z_Y, \mathbb{Z}) \to L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -6 \rangle$. We define $\rho := \eta_0 \circ (\widetilde{\sigma})^* \circ \eta_0^{-1} \in O(L)$. Following [13] and [14], a $(\rho, \langle 2 \rangle)$-*polarization* of an IHS manifold X of $K3^{[4]}$-type is given by a marking $\eta : H^2(X, \mathbb{Z}) \to L$ and an automorphism $g \in \mathrm{Aut}(X)$ of order three such that $g^*_{\mathbb{C}}|_{H^{2,0}(X)} = \xi \,\mathrm{id}$ and $\eta \circ g^* = \rho \circ \eta$ (in particular, the invariant lattice of g is isometric to $\langle 2 \rangle$, by Theorem 6.6). We consider the following equivalence relation: two $(\rho, \langle 2 \rangle)$-polarized eightfolds (X, η, g), (X', η', g') are equivalent if there exists an isomorphism $f : X \to X'$ such that $\eta' = \eta \circ f^*$ and $g' = f \circ g \circ f^{-1}$. Let $\mathcal{M}^{\rho,\xi}_{\langle 2 \rangle}$ be the set of equivalence classes of $(\rho, \langle 2 \rangle)$-polarized manifolds of $K3^{[4]}$-type and $\mathcal{U} \subset \mathcal{M}^{\rho,\xi}_{\langle 2 \rangle}$ be the subset which parametrizes manifolds $(Z_Y, \eta, \widetilde{\sigma})$, where Y is a cyclic cubic fourfold not containing a plane and σ is as in (5).

For any smooth cubic threefold $\mathcal{J} \subset \mathbb{P}^4$, we denote by $Y(\mathcal{J})$ the cubic fourfold which arises as triple covering of $\mathbb{P}^4$ ramified along $\mathcal{J}$. By using Proposition 6.8 we can prove the following result.

Theorem 6.10. *Let $\mathcal{J}, \mathcal{J}'$ be smooth cubic threefolds such that $Y(\mathcal{J}), Y(\mathcal{J}')$ do not contain a plane. If $(Z_{Y(\mathcal{J})}, \eta, \widetilde{\sigma})$, $(Z_{Y(\mathcal{J}')}, \eta', \widetilde{\sigma}')$ are equivalent as $(\rho, \langle 2\rangle)$-polarized manifolds, then $\mathcal{J} \cong \mathcal{J}'$. In particular, $\mathcal{U} \subset \mathcal{M}^{\rho,\xi}_{\langle 2\rangle}$ has dimension ten.*

Proof. Consider $(Z_Y, \eta, \widetilde{\sigma}) \in \mathcal{U}$ and let $Z_H \subset Z_Y$ be the fixed locus of $\widetilde{\sigma}$. Since $\mathrm{Alb}(Z_H) \cong \mathrm{J}(Y_H)$ (Remark 6.9), the Torelli theorem for cubic threefolds [24, Theorem 13.11] implies that the eightfold Z_Y and the action of the automorphism $\widetilde{\sigma}$ uniquely determine the threefold $\mathcal{J} = Y_H$, up to isomorphisms. The moduli space $\mathcal{C}_3^{sm}$ of smooth cubic threefolds is ten-dimensional and, for a very general $\mathcal{J} \in \mathcal{C}_3^{sm}$, the cubic fourfold $Y(\mathcal{J})$ does not contain a plane (see Remark 6.5). Since $\mathcal{M}^{\rho,\xi}_{\langle 2\rangle}$ is ten-dimensional, too, by [13, Corollary 6.5], we conclude that ten is also the dimension of the subset $\mathcal{U} \subset \mathcal{M}^{\rho,\xi}_{\langle 2\rangle}$. □

Theorem 6.10 allows us to provide the following alternative proof of the second part of Theorem 6.6. The automorphism $\widetilde{\sigma}$ corresponds to an admissible triple $(3, m, a)$, where $m-1$ coincides with the dimension of the moduli space $\mathcal{U}$ by [12, §4]. Since the dimension of $\mathcal{U}$ is ten, we can use Proposition 3.15 to deduce $m = 11, a = 0$. Hence the invariant lattice of $\widetilde{\sigma}$ is $T \cong \langle 2\rangle$.

Appendix A. Invariant and co-invariant lattices for $n = 3, 4,\ p = 3$

The two tables in this appendix list all admissible triples (p, m, a) (see Definition 3.10) and the corresponding isometry classes for the co-invariant lattice $S \subset H^2(X, \mathbb{Z})$ and the invariant lattice $T \subset H^2(X, \mathbb{Z})$ of non-symplectic automorphisms of order $p = 3$ on manifolds X of $K3^{[n]}$-type, for $n = 3, 4$. This classification is discussed in §3.4.

The symbol ♣ denotes the cases which can be realized by natural automorphisms (see §4.1). The cases marked with ♮ (respectively, ♢) correspond to admissible triples that admit a realization by induced automorphisms on moduli spaces of ordinary (respectively, twisted) sheaves on $K3$ surfaces, but not by natural automorphisms (see §5). Finally, the admissible triple $(3, 11, 0)$ for $n = 4$ (marked with the symbol ★) is realized by the automorphism constructed in §6.1 on a ten-dimensional family of Lehn–Lehn–Sorger–van Straten eightfolds (see also §4.3).

Finally, we recall that in the following tables, as in the rest of the paper, the root lattices A_2, E_6, E_8 are defined as negative definite.

p	m	a	S	T
♣ 3	10	0	$U^{\oplus 2} \oplus E_8^{\oplus 2}$	$U \oplus \langle -4 \rangle$
♣ 3	10	2	$U \oplus U(3) \oplus E_8^{\oplus 2}$	$U(3) \oplus \langle -4 \rangle$
♣ 3	9	1	$U^{\oplus 2} \oplus E_6 \oplus E_8$	$U \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	9	3	$U \oplus U(3) \oplus E_6 \oplus E_8$	$U(3) \oplus A_2 \oplus \langle -4 \rangle$
♢ 3	9	5	$U(3)^{\oplus 2} \oplus E_6 \oplus E_8$	$U(3) \oplus \Omega$
♣ 3	8	2	$U^{\oplus 2} \oplus E_6^{\oplus 2}$	$U \oplus A_2^{\oplus 2} \oplus \langle -4 \rangle$
♣ 3	8	4	$U \oplus U(3) \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2^{\oplus 2} \oplus \langle -4 \rangle$
♢ 3	8	6	$U(3)^{\oplus 2} \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2 \oplus \Omega$
♣ 3	7	1	$U^{\oplus 2} \oplus A_2 \oplus E_8$	$U \oplus E_6 \oplus \langle -4 \rangle$
♣ 3	7	3	$U \oplus U(3) \oplus A_2 \oplus E_8$	$U \oplus A_2^{\oplus 3} \oplus \langle -4 \rangle$
♣ 3	7	5	$U^{\oplus 2} \oplus A_2^{\oplus 5}$	$U(3) \oplus A_2^{\oplus 3} \oplus \langle -4 \rangle$
♣ 3	7	7	$U \oplus U(3) \oplus A_2^{\oplus 5}$	$U(3) \oplus E_6^{\vee}(3) \oplus \langle -4 \rangle$
♣ 3	6	0	$U^{\oplus 2} \oplus E_8$	$U \oplus E_8 \oplus \langle -4 \rangle$
♣ 3	6	2	$U \oplus U(3) \oplus E_8$	$U \oplus E_6 \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	6	4	$U^{\oplus 2} \oplus A_2^{\oplus 4}$	$U \oplus A_2^{\oplus 4} \oplus \langle -4 \rangle$
♣ 3	6	6	$U \oplus U(3) \oplus A_2^{\oplus 4}$	$U(3) \oplus A_2^{\oplus 4} \oplus \langle -4 \rangle$
♣ 3	5	1	$U^{\oplus 2} \oplus E_6$	$U \oplus E_8 \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	5	3	$U \oplus U(3) \oplus E_6$	$U \oplus A_2^{\oplus 2} \oplus E_6 \oplus \langle -4 \rangle$
♣ 3	5	5	$U \oplus U(3) \oplus A_2^{\oplus 3}$	$U \oplus A_2^{\oplus 5} \oplus \langle -4 \rangle$
♣ 3	4	2	$U^{\oplus 2} \oplus A_2^{\oplus 2}$	$U \oplus E_6^{\oplus 2} \oplus \langle -4 \rangle$
♣ 3	4	4	$U \oplus U(3) \oplus A_2^{\oplus 2}$	$U \oplus E_6 \oplus A_2^{\oplus 3} \oplus \langle -4 \rangle$
♣ 3	3	1	$U^{\oplus 2} \oplus A_2$	$U \oplus E_6 \oplus E_8 \oplus \langle -4 \rangle$
♣ 3	3	3	$U \oplus U(3) \oplus A_2$	$U \oplus E_6^{\oplus 2} \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	2	0	$U^{\oplus 2}$	$U \oplus E_8^{\oplus 2} \oplus \langle -4 \rangle$
♣ 3	2	2	$U \oplus U(3)$	$U \oplus E_6 \oplus E_8 \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	1	1	$A_2(-1)$	$U \oplus E_8^{\oplus 2} \oplus A_2 \oplus \langle -4 \rangle$

Table 1. $n = 3$, $p = 3$. See §3.4 for the definition of the lattice Ω.

p	m	a	S	T
★ 3	11	0	$U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$	$\langle 2 \rangle$
♣ 3	10	0	$U^{\oplus 2} \oplus E_8^{\oplus 2}$	$U \oplus \langle -6 \rangle$
♮ 3	10	1	$U \oplus U(3) \oplus E_8^{\oplus 2}$	$U \oplus \langle -6 \rangle$
♣ 3	10	2	$U \oplus U(3) \oplus E_8^{\oplus 2}$	$U(3) \oplus \langle -6 \rangle$
♢ 3	10	3	$U(3)^{\oplus 2} \oplus E_8^{\oplus 2}$	$U(3) \oplus \langle -6 \rangle$
♣ 3	9	1	$U^{\oplus 2} \oplus E_6 \oplus E_8$	$U \oplus A_2 \oplus \langle -6 \rangle$
♮ 3	9	2	$U \oplus U(3) \oplus E_6 \oplus E_8$	$U \oplus A_2 \oplus \langle -6 \rangle$
♣ 3	9	3	$U \oplus U(3) \oplus E_6 \oplus E_8$	$U(3) \oplus A_2 \oplus \langle -6 \rangle$
♢ 3	9	4	$U(3)^{\oplus 2} \oplus E_6 \oplus E_8$	$U(3) \oplus A_2 \oplus \langle -6 \rangle$
♮ 3	8	1	$U^{\oplus 2} \oplus E_6^{\oplus 2}$	$\langle 2 \rangle \oplus E_6$
♣ 3	8	2	$U^{\oplus 2} \oplus E_6^{\oplus 2}$	$U \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
♮ 3	8	3	$U \oplus U(3) \oplus E_6^{\oplus 2}$	$U \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
♣ 3	8	4	$U \oplus U(3) \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
♢ 3	8	5	$U(3)^{\oplus 2} \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
♮ 3	7	0	$U^{\oplus 2} \oplus A_2 \oplus E_8$	$\langle 2 \rangle \oplus E_8$
♣ 3	7	1	$U^{\oplus 2} \oplus A_2 \oplus E_8$	$U \oplus E_6 \oplus \langle -6 \rangle$
♮ 3	7	2	$U \oplus U(3) \oplus A_2 \oplus E_8$	$U \oplus E_6 \oplus \langle -6 \rangle$
♣ 3	7	3	$U \oplus U(3) \oplus A_2 \oplus E_8$	$U(3) \oplus E_6 \oplus \langle -6 \rangle$
♮ 3	7	4	$U^{\oplus 2} \oplus A_2^{\oplus 5}$	$U \oplus A_2^{\oplus 3} \oplus \langle -6 \rangle$
♣ 3	7	5	$U^{\oplus 2} \oplus A_2^{\oplus 5}$	$U(3) \oplus A_2^{\oplus 3} \oplus \langle -6 \rangle$
♮ 3	7	6	$U \oplus U(3) \oplus A_2^{\oplus 5}$	$U \oplus E_6^{\vee}(3) \oplus \langle -6 \rangle$
♣ 3	7	7	$U \oplus U(3) \oplus A_2^{\oplus 5}$	$U(3) \oplus E_6^{\vee}(3) \oplus \langle -6 \rangle$
♣ 3	6	0	$U^{\oplus 2} \oplus E_8$	$U \oplus E_8 \oplus \langle -6 \rangle$
♮ 3	6	1	$U \oplus U(3) \oplus E_8$	$U \oplus E_8 \oplus \langle -6 \rangle$
♣ 3	6	2	$U \oplus U(3) \oplus E_8$	$U(3) \oplus E_8 \oplus \langle -6 \rangle$
♮ 3	6	3	$U^{\oplus 2} \oplus A_2^{\oplus 4}$	$U \oplus E_6 \oplus A_2 \oplus \langle -6 \rangle$
♣ 3	6	4	$U^{\oplus 2} \oplus A_2^{\oplus 4}$	$U(3) \oplus E_6 \oplus A_2 \oplus \langle -6 \rangle$
♮ 3	6	5	$U \oplus U(3) \oplus A_2^{\oplus 4}$	$U \oplus A_2^{\oplus 4} \oplus \langle -6 \rangle$
♣ 3	6	6	$U \oplus U(3) \oplus A_2^{\oplus 4}$	$U(3) \oplus A_2^{\oplus 4} \oplus \langle -6 \rangle$
♣ 3	5	1	$U^{\oplus 2} \oplus E_6$	$U \oplus E_8 \oplus A_2 \oplus \langle -6 \rangle$
♮ 3	5	2	$U \oplus U(3) \oplus E_6$	$U \oplus E_8 \oplus A_2 \oplus \langle -6 \rangle$
♣ 3	5	3	$U \oplus U(3) \oplus E_6$	$U(3) \oplus E_8 \oplus A_2 \oplus \langle -6 \rangle$
♮ 3	5	4	$U \oplus U(3) \oplus A_2^{\oplus 3}$	$U \oplus E_6 \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
♣ 3	5	5	$U \oplus U(3) \oplus A_2^{\oplus 3}$	$U(3) \oplus E_6 \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
♮ 3	4	1	$U^{\oplus 2} \oplus A_2^{\oplus 2}$	$\langle 2 \rangle \oplus E_6 \oplus E_8$
♣ 3	4	2	$U^{\oplus 2} \oplus A_2^{\oplus 2}$	$U \oplus A_2^{\oplus 2} \oplus E_8 \oplus \langle -6 \rangle$
♮ 3	4	3	$U \oplus U(3) \oplus A_2^{\oplus 2}$	$U \oplus E_6^{\oplus 2} \oplus \langle -6 \rangle$
♣ 3	4	4	$U \oplus U(3) \oplus A_2^{\oplus 2}$	$U(3) \oplus E_6^{\oplus 2} \oplus \langle -6 \rangle$
♮ 3	3	0	$U^{\oplus 2} \oplus A_2$	$\langle 2 \rangle \oplus E_8^{\oplus 2}$
♣ 3	3	1	$U^{\oplus 2} \oplus A_2$	$U \oplus E_6 \oplus E_8 \oplus \langle -6 \rangle$
♮ 3	3	2	$U \oplus U(3) \oplus A_2$	$U \oplus E_6 \oplus E_8 \oplus \langle -6 \rangle$
♣ 3	3	3	$U \oplus U(3) \oplus A_2$	$U(3) \oplus E_6 \oplus E_8 \oplus \langle -6 \rangle$
♣ 3	2	0	$U^{\oplus 2}$	$U \oplus E_8^{\oplus 2} \oplus \langle -6 \rangle$
♮ 3	2	1	$U \oplus U(3)$	$U \oplus E_8^{\oplus 2} \oplus \langle -6 \rangle$
♣ 3	2	2	$U \oplus U(3)$	$U(3) \oplus E_8^{\oplus 2} \oplus \langle -6 \rangle$
♣ 3	1	1	$A_2(-1)$	$U \oplus E_8^{\oplus 2} \oplus A_2 \oplus \langle -6 \rangle$

TABLE 2. $n = 4$, $p = 3$.

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