Intuition in Mathematics: from Racism to Pluralism

Miriam Franchella[1] [ORCID]

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Abstract
In the nineteenth and twentieth centuries many mathematicians referred to intuition as the indispensable research tool for obtaining new results. In this essay we will analyse a group of mathematicians (Felix Klein, Henri Poincaré, Ludwig Bieberbach, Arend Heyting) who interacted with Luitzen Egbertus Jan Brouwer (the father of the intuitionist foundational school) in order to compare their conceptions of intuition. We will see how to the same word "intuition" (in German *Anschauung*) very different meanings corresponded: they varied from geometrical vision, to a unitary view of a demonstration, to the perception of time, to the faculty (shared by everybody) of considering concepts that habitually occur in our thinking separately. Furthermore, we will discover that these different meanings had a philosophical, very relevant counterside: they passed from a racial characterization of mathematics to a pluralistic view of it.

Keywords Foundation of mathematics · History of mathematics · Intuition · Logic · Philosophy of mathematics · Racism

1 Introduction

1912: Bertus Brouwer read his inaugural address, "Intuitionism and Formalism", in which he named his foundational school as neo-intuitionism, in (partial) continuity with previous intuitionists, among whom he mentioned Poincaré. 1914: Brouwer thanked Felix Klein, editor-in-chief of the prestigious journal *Mathematische Annalen*, for the gratifying news that he had been accepted as an editor. In the same year Ludwig Bieberbach praised Brouwer for "Intuitionism and Formalism". 1925: Arend Heyting received his doctorate with a thesis on the axiomatization of intuitionist projective geometry, with Brouwer as supervisor.

✉ Miriam Franchella
miriam.franchella@unimi.it

1 Dipartimento Di Studi Storici, Università Degli Studi Di Milano, Via Festa del Perdono 7, 20122 Milano, MI, Italy

⚓ Springer

These are only some examples of personal or intellectual acquaintances between Brouwer on the one hand and Klein, Poincaré, Bieberbach, Heyting on the other. This article is concerned with the nuances of meaning with their philosophical and social repercussions.

2 Kant

Kant's philosophy of mathematics represented the background of all the authors that we are going to mention. Therefore, I will outline here its main aspects, without entering into the debates that a careful analysis of his writings has triggered during the years regarding various problematic details.[1]

In *his Critique of pure reason*, aiming at establishing whether metaphysics could be considered a science, Kant invoked mathematical truth as a paradigm of necessary and universal truths. In both the Preamble to the *Prolegomena to Any Future Metaphysics* and the B-Introduction to the *Critique of Pure Reason*, Kant introduced the analytic/synthetic distinction: judgments the predicates of which belong to or are contained in the subject concept are called analytic; judgments the predicates of which are connected to but go beyond the subject concept are called synthetic. Mathematical judgments are *a priori* because they are necessary, but are not analytic, because their predicates add something to their subjects: such judgements do not cut the subject into parts to find out the predicate, but they express a synthesis between subject and predicate. In the case of arithmetic, Kant presented his famous example "7 + 5 = 12" and claimed that "no matter how long I analyze my concept of such a possible sum [of seven and five] I will still not find twelve in it […] One must go beyond these concepts [of seven and five], seeking assistance in the intuition that corresponds to one of the two […] and one after another add the units of the five given in the intuition to the concept of seven…and thus see the number 12 arise". (Kant, 1838, B15).

In the case of geometry[2] Kant explained that all its principles – even those that Euclides would have called "axiomata" (i.e. those that are common to other aspects of reality like a = a) express relations *among basic geometric concepts* inasmuch as these can be exhibited in intuition:

> 'A straight line between two points is the shortest,' is a synthetical proposition. For my conception of straight contains no notion of quantity, but is merely qualitative. The conception of the shortest is therefore for wholly an addition, and by no analysis can it be extracted from our conception of a straight line. Intuition must therefore here lend its aid, by means of which, and thus only, our synthesis is possible. Some few principles preposited by geometricians are, indeed, really analytical, and depend on the principle of contradiction. […] for example, a = a, the whole is equal to itself, or (a+b) —> a, the whole

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[1] See https://plato.stanford.edu/entries/kant-mathematics/#KanAnsHisQueHowPurMatPos.

[2] As an example of problematic detail in Kant's philosophy of mathematics, we can quote here the question whether Kant was committed merely to the syntheticity of the axioms of mathematics or was also committed to the syntheticity of mathematical inference itself.

is greater than its part. And yet even these principles themselves, though they derive their validity from pure conceptions, are only admitted in mathematics because they can be presented in intuition. What causes us here commonly to believe that the predicate of such apodeictic judgements is already contained in our conception, and that the judgement is therefore analytical, is merely the equivocal nature of the expression. We must join in thought a certain predicate to a given conception, and this necessity cleaves already to the conception. But the question is, not what we must join in thought to the given conception, but what we really think therein, though only obscurely, and then it becomes manifest that the predicate pertains to these conceptions, necessarily indeed, yet not as thought in the conception itself, but by virtue of an intuition, which must be added to the conception. (Kant, 1838, B17)

Kant's geometrical framework was Euclidean geometry. Therefore, after the discovery of non-Euclidean geometries, the synthetic a priori status of Euclidean geometrical statements came into question.

As for numbers, we find their concept in the *Transcendental Analytic*. There Kant deduced the table of twelve categories, or pure concepts of the understanding, divided into "mathematical" (the first six) and "dynamical"(the other six) (Kant, 1838, B110). The concept of number was included in the category of totality, which resulted at its turn from the combination of the concepts of unity and plurality. Furthermore, Kant specified that the categories must be "schematized" in order to be connected to the objects of experience, because they have a non-empirical origin in pure understanding. Transcendental schemata mediate between pure concepts and appearances. In particular, the number[3] was the scheme of temporal intuition:

For the external sense the pure image of all quantities (quantorum) is space; the pure image of all objects of sense in general, is time. But the pure schema of quantity (quantitatis) as a conception of the understanding, is number, a representation which comprehends the successive addition of one to one (homogeneous quantities). Thus, number is nothing else than the unity of the synthesis of the manifold in a homogeneous intuition, by means of my generating time itself in my apprehension of the intuition. (Kant, 1838, B182)

2.1 Klein

As we shall see below, Felix Klein has been repeatedly cited as the 'discoverer' of the existence of two racial attitudes, a Teutonic one, characterized by reliance on 'intuition', and a Latin-Jewish one, which is intrinsically logical-analytical.

His interest in the distinction between the two types of intuition certainly emerges on several occasions in his work, which he defined in various ways: one, which guides research but is imprecise, is called (depending on the context) spatial or

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[3] According to Ruvidotti 2011, in order to better understand what Kant meant here, we should point out of Kant's theoretical description two kinds of numbers as media: a number *n* linking each image of a single quantitas (for instance each image of '7') to its concept (for instance to the concept of '7') and a number s linking each *n* to the concept of *quantitas*.

immediate or naïve; the other, which consists in the correct definition of mathematical objects by means of axiomatization, is called idealized, abstract or refined.

In his Leipzig *inaugural lecture* delivered in 1880 but published in 1895, he presented a set of mathematical models, compiled together with Alexander von Brill at the Technische Hochschule in Munich, both for teaching purposes and for his own research, in order to stress the importance of 'intuition' in geometry not only in order to make mathematics more accessible but also to bring forth new ideas for abstract research (Klein, 1895, pp. 539–540). Furthermore, he stated that results achieved via arithmetical approaches should be reconnected with spatial intuition.

In his 1895 lecture, delivered at the public session of the Königliche Gesellschaft der Wissenschaften zu Göttingen and entitled *Über Arithmetisierung der Mathematik*, he re-examined the role of intuition (*Anschauung*) in mathematics. He blamed the fact that Gauß's incautious use of *spatial intuition* (*Raumanschauung*) as proof of the universal validity of propositions that were not at all universally valid had led to demands for exclusively arithmetical reasoning in mathematics.[4]. This was a pity, because mathematics is certainly not exhausted in logical deduction but that, alongside the latter, intuition completely retains its specific importance" (Klein, 1896, p. 144, translated by Martin Mattheis). He spoke of a

> naïve intuition, which is in large part an inherited talent and emerges unconsciously from the in-depth study of this or that field of science. The word 'Anschauung' has not perhaps been suitably chosen. I would like to include here the motoric sensation with which an engineer assesses the distribution of forces in something he is designing, and even that vague feeling possessed by the experienced number cruncher about the convergence of infinite processes with which he is confronted. I am saying that, in its fields of application, mathematical intuition understood in this way rushes ahead of logical thinking and in each moment has a wider scope than the latter. (Klein, 1896, p. 147, translated by Martin Mattheis).

Here Klein used the adjective 'naïve' to express an 'imprecise faculty' (a sort of sixth sense) that prompts scholars during their research activities. He stated that it *largely* inherited, and it can be assumed (on the basis of the above quote) that in a smaller part, to a more limited extent, it is due to experience in a specific field.[5]. He then argued that imprecise spatial intuition should first be *idealized* in the axioms in order to proceed with a mathematical approach. Therefore, he introduced a second kind of intuition, here called 'idealized'. He specified that such an approach should also be pursued in mechanics and mathematical physics (Klein, 1896, p. 146).

In his University of Göttingen courses, in his 1898/90 winter semester lectures on non-Euclidian geometry, published handwritten in 1892, Klein described once again the interplay between axioms and *spatial intuition* as countering the inexactness of intuition:

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[4] Allmendinger 2014, pp. 47–50 stresses that, although Klein rejected in principle the idea of accepting proof from intuition, in Klein 1892a, 1892b, p. 359 he derives just such proofs, showing that an algebraic question can be resolved intuitively purely by graphic geometric presentation.

[5] "Thus, Felix Klein subsumed under the general term of *Anschauung* a certain degree of intuition gained through experience. (Mattheis 2019, p.97).

Rather, in true geometric thinking, spatial intuition accompanies us at every step we take. […] I assign the axioms the role that they represent postulations with the aid of which we transcend the inaccuracy of intuition or the limitations of intuition in order to achieve unlimited accuracy. (Klein, 1892a, b, p. 354, translated by Martin Mattheis)

With our notion of the essence of intuition, an intuitive treatment of figurative representations will tend to yield a certain general guide on which mathematical laws apply and how their general proof may be structured. However, true proof will only be obtained if the given figures are replaced with figures generated by laws based on the axioms and these are then taken to carry through the general train of thought in an explicit case. Dealing with sensate objects gives the mathematician an impetus and an idea of the problems to be tackled, but it does not pre-empt the mathematical process itself. (Klein, 1892a, b, pp. 359–360, translated by Martin Mattheis)

In his 1908 lecture series on "Elementary Mathematics from a Higher Standpoint", he distinguished between spatial intuition and 'idealizing spatial intuition' (*idealisierende Raumanschauung*) which addressed the abstract notion of geometrical objects, i.e. the mathematical idea freed from the error-prone inexactness of real objects (Klein, 1908, p. 88).

This is the proper place to say a word about the nature of space intuition. It is variously ascribed to two different sources of knowledge. One the sensibly immediate, the empirical intuition of space, which we can control by means of measurement. The other is quite different, and consists in a subjective idealizing intuition, one might say, perhaps, our inherent idea of space, which goes beyond the inexactness of sense observation. (Klein, 1908, p. 88)

Such a distinction between sensately immediate intuition and idealizing inner intuition goes back to the respective concepts developed by Kant (Allmendinger, 2014, pp. 52–53).

In 1908 Felix Klein introduced a further term to the circle of concepts differentiating 'intuition'. In this context, he placed 'sensate intuition' alongside the notion of 'abstract intuition' and gave a clear example of what he meant by 'sense' intuition (or 'immediate' intuition) as opposed to 'abstract' intuition:

It is precisely in the discovery and in the development of the infinitesimal calculus that this inductive process, built up without compelling logical steps, played such a great role; and the *most effective heuristic aid was very often sense intuition*. And I mean here the *immediate* sense intuition, with all its inexactness, for which a curve is a stroke of definite width, *not* the *abstract* intuition, which postulates a completed passage to the limit, yielding a one-dimensional line. (Klein, 1908, pp. 455–456)

It is clear that what he meant by *abstract* intuition was the same as the meaning he had already assigned to concept of *idealizing* intuition. Furthermore, his allusion to the existence of different mathematical *personalities*, that are more or less fond for an intuitive approach (than to an abstract approach) to the discipline dates

to 1908. He stressed that these respective approaches continued to play an important role in the development of new mathematical ideas in mathematical physics, mechanics and differential geometry:

> The *force of conviction* inherent in such naïve guiding reflections is, of course, different for different individuals. Some – and I include myself here – find them very satisfying. Others, again, who are gifted only on the purely logical side, find them thoroughly meaningless and are unable to see how anyone can consider them as a basis for mathematical thought. (Klein, 1908, p. 460)

However, in reference to David Hilbert's lectures on the *Foundations of Arithmetic and Logic* (Hilbert, 2013) he argued that even at the highest level of abstraction, when attempting to free oneself from any form of intuition, for example in number theory considered in a purely formal way, some minimal amount of intuition remains all the same, even if it is only to recognize the symbols with which one is operating merely in accordance with axiomatic rules. (Klein, 1908, pp. 32–35).

Klein considered intuition within an educational context in Volume II of *Elementary Mathematics* from a Higher Standpoint, writing[6]:

> in schools you will always have to connect teaching at first with vivid concrete intuition and then only gradually bring logic elements to the fore; in general, the genetic method alone will provide a legitimate means slowly to develop a full understanding of concepts. (Klein, 1909, pp. 435–436)

He had alluded to the genetic teaching method in the first volume of Elementary Mathematics from a Higher Standpoint, proposing a principle resembling 'philogenesis recapitulates the ontogenesis':

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[6] His educational viewpoint played a key role in drawing up the 1905 Breslau Teaching Commission of the Gesellschaft Deutscher Naturforscher und Ärzte (GDNÄ)'s Meran Curriculum Proposal, which essentially formulated two key demands regarding mathematics teaching at secondary schools: "Strengthening the capacity to think in three dimensions (= geometrische Anachauung) and training the habit of functional reasoning." (Mattheis 2019, p. 93) Klein had already expressed his opinions concerning the need for reform of the teaching of mathematics in both secondary schools and universities, which he felt were too oriented towards humanistic education, in his Erlanger *Antrittsrede*. He was convinced that training of the mind through mathematical exercises was necessary both for natural scientists and for doctors. He realized the difficulties involved in adding further courses to medical courses, therefore he preferred to change things in secondary schools by modifying the educational methods used by mathematics teachers. What he required from them was 'a more spirited treatment' of the subject. Namely, the 'mathematics that is often presented possesses little that is of real educational value. Instead of developing a proper feeling for mathematical operations, or promoting an intuitive grasp of geometry, the class time is spent learning mindless formalities or practicing triivial tricks that exhibit no underlying principle. One learns to reduce with virtuosity long expressions that are devoid of meaning […]'. (Rowe 1985, p. 138) Hence, he suggested introducing more practical mathematics courses: '[…] we wish to hold exercises in drawing and building models' (Rowe 1985, p. 139) similar to what was done in polytechnics. A comparison between his initial and later viewpoints on teaching is given by Klein himself (1923, 18), Manegold (1970, 92), Pyenson (1979), and by (Rowe 1985). It is interesting to stress that, while at the beginning what he required of future mathematics teachers was research autonomy at high mathematics levels, in later life he stated: 'I would now suggest that teaching candidates of average talent should confine themselves to such studies as will be of fundamental importance in the later exercise of their profession, while everything beyond this should be reserved for those with unusual talent or favorable circumstances'. (Rowe 1985, p. 128) In any case, his stress on intuition by teaching remained constant.

In order to give precise expression to my own view on this point, I should like to bring forward the *biogenetic fundamental law*, according to which the individual in his development goes through, in an abridged series, all the stages in the development of the species. […] Now, I think that instruction in mathematics, as well as in everything else, should follow this law, at least in general. *Taking into account the native ability of youth, instruction should guide it slowly to higher things, and finally to abstract formulations; and in doing this it should follow the same road along which the human race has striven from its naïve original state to higher forms of knowledge*. (Klein, 1908, pp. 588–589)

The reference to different approaches to mathematics on a racial basis is found in the famous VI *Evanston lecture* of 1893 (p. 42) where he expressed the following views:

Finally, it must be said that the degree of exactness of the intuition of space may be different in different individuals, perhaps even in different races. It would seem as if a strong naive space-intuition were an attribute pre-eminently of the Teutonic race, while the critical, purely logical sense is more fully developed in the Latin and Hebrew races. A full investigation of this subject, somewhat on the lines suggested by Francis Galton in his researches on heredity, might be interesting.

Needless to say, the distinction between naïve and refined intuition echoed earlier and later ones.[7] Namely, Klein wrote 'the naïve intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact.' The naïve intuition is described as follows: 'in our naïve intuition, when thinking of a point we do not picture to our mind an abstract mathematical point, but substitute something concrete for it. In imagining a line, we do not picture ourselves 'length without breadth,' but a strip of a certain width'. (p. 39) He added that 'there actually are many cases where the conclusions derived by purely logical reasoning from exact definitions can no more be verified by intuition.'(p.39) He stressed the utility of a method based on geometrical intuition:

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[7] In his first *Evanston lecture* (Klein 1894, p. 2), Klein placed mathematicians in three main categories: logicians, formalists and intuitionists, stating: "the word *logician* is here used, of course, without reference to the mathematical logic of Boole, Peirce, etc.; it is only intended to indicate that the main strength of the men belonging to this class lies in their logical and critical power, in their ability to give strict definitions, and to derive rigid deductions therefrom.' But he immediately set such categorization aside, splitting mathematicians into only two categories.

He cited Karl Weierstrass as an example and argued that the formalists mainly excelled in the formal usage of a given question, 'in devising for it an 'algorithm'' using Paul Gordan, Arthur Cayley and James Sylvester as examples. Finally, he wrote (Klein 1894, p. 3): 'To the intuitionists belong those who lay particular stress on geometrical intuition ('Anschauung'), not in pure geometry only, but in all branches of mathematics. What Benjamin Peirce has called 'geometrizing a mathematical question' seems to express the same idea'. Lord Kelvin and Karl von Staudt were examples. Finally, he affirmed: 'Clebsch must be said to belong both to the second and third of these categories, while I should class myself with the third, and also the first.' It is clear that these categories are different from the dicotomic categories of mathematical approaches that he presented in other texts (and even in the Evanston lectures).

On the three categories of mathematicians, see Franchella 2019.

I am of the opinion that, certainly, for the purposes of research it is always necessary to combine the intuition with the axioms. I do not believe, for instance, that it would have been possible to derive the results discussed in my former lectures, the splendid researches of Lie, the continuity of the shape of algebraic curves and surfaces, or the most general forms of triangles, without the constant use of geometrical intuition (p. 40)

and expressed his opinion that only a limited amount of abstract mathematics (based on refined intuition) is used in the applied sciences[8]: '[…] it must not be forgotten that mathematical developments transcending the limit of exactness of the science are of no practical value. It follows that a large portion of abstract mathematics remains without finding any practical application' (p. 44). But he immediately specified, unequivocally:

I hope that what I have here said concerning the use of mathematics in the applied sciences will not be interpreted as in any way prejudicial to the cultivation of abstract mathematics as a pure science. Apart from the fact that pure mathematics cannot be supplanted by anything else as a means for developing the purely logical powers of the mind, there must be considered here as elsewhere the necessity of the presence of a few individuals in each country developed in a far higher degree than the rest, for the purpose of keeping up and gradually raising the general standard. Even a slight raising of the general level can be accomplished only when some few minds have progressed far ahead of the average (p. 44)

thus implicitly praising the few individuals in each nation who manage to cultivate abstract mathematics. He thus praises (non-Teutonic) abstract mathematics as the pinnacle of human achievement (even if it is not, in itself sufficient, for research progress).

Rowe, 1986 points out that such a racial distinction (intuition vs. logic) was a feature of the culture of the day, citing a letter from Weierstrass to Sonia Kowaleskaja as proof:

Kronecker is different [from their mutual colleague Ernst Kummer]. He quickly makes himself familiar with everything that is new; his ready ability to grasp enables him to do so, but not in a penetrating manner. He does not possess the talent to engage himself in a good, but unfamiliar work with the

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[8] He added a hierarchy of the sciences, according to the use that can be made of abstract mathematics in them:'I believe that the more or less close relation of any applied science to mathematics might be characterized by the degree of exactness attained, or attainable, in its numerical results. Indeed, a rough classification of these sciences could be based simply on the number of significant figures averaged in each. Astronomy (and some branches of physics) would here take the first rank; the number of significant figures attained may here be placed as high as seven, and functions higher than the elementary transcendental functions can be used to advantage. Chemistry would probably be found at the other end of the scale, since in this science rarely more than two or three significant figures can be relied upon. Geometrical drawing, with perhaps 3 to 4 figures, would rank between these extremes; and so we might go on.' (Klein 1894, p. 43).

same scientific interest that he pursues his own studies. Beyond this he shares the shortcoming that one finds in many intelligent people, especially those of Semitic stock: he does not possess sufficient fantasy (intuition I would prefer to say). And it is true, a mathematician who is not something of a poet will never be a complete mathematician. Comparisons are instructive: an all-embracing vision focused on the loftiest of ideals distinguishes [Niels Henrik] Abel from Jacobi, Riemann from his contemporaries ([Gotthold] Eisenstein, [Johann Georg] Rosenhain [both of Jewish descent]), and [Hermann von] Helmholtz from [Gustav Robert] Kirchhoff (although the latter is without a drop of Semitic blood) in an altogether splendid manner". (Rowe, 1986, p. 442)

It should also be noted that Klein's racial references were intended positively. For example, he first praised James Joseph Sylvester with numerous adjectives: 'Sylvester was extremely engaging, witty and effervescent. He was a brilliant orator and often distinguished himself by his pithy, agile poetic skill, to the mirth of everyone' and then considered Sylvester's best traits to be typical of his 'race': 'By his brilliance and agility of mind he was a genuine representative of his race; he hailed from a purely Jewish family, which, having been nameless before, had adopted the [sur]name Sylvester only in his generation.' (Klein, 1926, p. 163; transl. Rowe, 1986, p. 440).

He also expressed positive views of Jewish mathematician Kronecker:

In that he was mainly concerned with arithmetic and algebra, in later years however setting up definite intellectual norms for all mathematical work, he appears as the specifically *Jewish talent,* but in a special, individual enhancement. For he has foreseen many relationships of a fundamental nature in his fields of work, without being able to work them out clearly yet. (Klein, 1926, p. 281, transl. Rowe, 1986, p. 442)

Klein reserved the same treatment for Jewish Jacobi:

As is well known, the year 1812 brought with it the emancipation of the Jews in Prussia. Jacobi was the first Jewish mathematician to take a leading place in Germany, and in so doing he was again at the forefront of a great, and for our science significant, development. This measure opened up a large reservoir of new mathematical talent for our country, whose powers, along with those of the French immigrants, very soon bore fruit. It appears to me that our science has won a *strong stimulant through this type of blood replenishment*. Along with the already mentioned law regarding shifts of productivity from country to country, I would like to designate this phenomenon as the effect of national infiltration. (Klein, 1926, p. 114; transl. Rowe, 1986, p. 440)

Klein spoke of infiltration by a member alien to the German nation, but one which brought a beneficial transfusion of blood.

As Bair et al. 2017 (p. 200) note, the term 'racist' can be understood in (at least) two distinct ways: (1) someone interested in analysing differences in intellectual outlook between distinct ethnicities (racist1); (2) someone who believes in the

inferiority of one ethnicity to another based on such differences, and advocates corrective action (racist2). Klein's discussions of ethnic differences possibly make him racist1.

The letter written to Jewish mathematician Otto Toepliz,[9] who had been appointed full professor in Kiel and was himself recruiting new teachers, similarly cannot be considered racist in the second sense. In it, Klein (who had also himself hired Jewish Godon in Göttingen) recommends a certain prudence in recruiting only teachers of Jewish origin. Namely, Klein noted a 'tribal' solidarity between Jews which nurtured views that their efficiency was in fact inspiring a new anti-Semitism in response. Excessive enlistment of Jews could, therefore, act as grist to the mill in an unpredictable future. Siegmund-Schultze commented: 'Der Brief zeigt den grossen alten Organisator der Göttinger Mathematiker Klein in seinem Urteil unsicher. Er schwankt zwischen überkommenen antisemitischen Klischees, Anerkennung der Begabung jüdischer Mathematiker, Einräumung antisemitischer Diskriminierung und Befürchtungen für die zukünftige Entwicklung des Faches.' (Siegmund-Schultze, 2016, p. 26) It seems to me that the letter contains a concern for the fate potentially awaiting the Jewish community whose influence seemed (as in ancient Egypt) to be growing because of its gifts and robust sense of community. The latter could be considered a cliché about Jews but his language is cautious rather than aggressive.

However, this distinction was taken up by others, starting with Erich Rudolf Jaensch, a former student of his who stated that Klein was intrigued by the 'conflict between the German spirit and the preponderance of a completely different type of thinking in mathematics' and continually returned to this theme in his seminar 'despite the fact that it was intentionally repressed by several of the participants.' (Jaensch-Althoff 1939, p. 32; transl. Rowe, 1986, p. 440). According to Rowe (1986, p. 441), it is doubtful that Jaensch ever attended this seminar, as his name does not appear in the protocol book written by Klein himself. Furthermore, the protocol book supplies us with the fact that the racial issue came up in the seminar on one occasion, when a student named Steckel spoke about his experiences teaching in

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[9] Nun komme ich, um nichts zurückzuhalten, zur Frage des Antisemitismus. Sie wissen, wie ich es selbst immer gehalten habe, seit ich 1874 die Berufung von Gordan nach Erlangen veranlasste: mir war der einzelne Jude willkommen, indem ich voraussetzte, dass er mit den übrigen Mitgliedern der Universität kooperieren werde. Aber nun haben sich im Laufe der Zeit die Gegensätze prinzipiell verschärft. Wir haben auf der einen Seite nicht nur ein ungeheures, der merkwürdigen Leistungsfähigkeit entsprechendes Vordrängen des Judentums, sondern das Hervorkommen der jüdischen Solidarität (welche dem Stammesgenossen auf alle Weisen in erster Linie zu helfen strebt). Dazu nun als Rückwirkung den starren Antisemitismus. Das Problem ist ein allgemeines, bei dem Deutschland, soweit nicht gerade die moderne östliche Einwanderung in Betracht kommt, nur eine sekundäre Rolle spielt. Niemand kann sagen, wie sich das Ding [sie] weiter entwickelt. Aber ich mache darauf aufmerksam, dass die sämmtlichen fünf Gelehrten, die Sie für Ihr Ordinariat in Aussicht nehmen, jüdischen Ursprungs sind. Ist dies eine zweckmässige Politik? Ich nehme von vornherein an, dass Sie das nicht beabsichtigt haben. Man kann auch beinahe so argumentieren: dass der an allen Universitäten etc. latent vorhandene Antisemitismus die christlichen Kandidaten so bevorzugt habe, dass nur noch jüdische zur Verfügung stehen. Aber ich bitte doch darüber nachzudenken. Wir treiben möglicherweise in Gegensätze hinein, die für unsere gesammten Zustände unheilvoll werden können. (Quoted from Siegmund-Schultze 2016, p. 26).

Eastern Europe, arguing that Jews and Germans think differently when performing calculations.[10]

Later, Theodor Vahlen, who was an executive official in the ministry in 1933, and a professor in Berlin, gave an address on assuming his office as rector of University of Greifswald on 15 May 1923 entitled "Wesen und Wert der Mathematik" and, after presenting mathematics as a search for beauty, observed that this means that it depends on 'Geschmack' (taste) which varies from person to person, and from people to people. He stressed a clear difference between the 'ideal' Greeks, who preferred pure mathematics and the 'practical' Romans, who limited themselves to measuring fields and commercial calculations. In more recent times, he noted the difference between the Anglo-Saxons and the Germans seeing the Anglo-Saxons as more inclined to applied mathematics while the Germans focused on number and space foundations. Then, Vahlen, added, if peoples are grouped into races, there is a dividing line between West and East in mathematics with the latter preferring arithmetic and the former geometry. At this point, Vahlen quoted Klein ('one of our greatest geometers') stating that 'modern people have a strongly developed, fertile view of space, which is a particular advantage of the Teutonic Race' and that a 'purely logical, sharply critical sense' characterizes the Jews, generating a 'disintegrating criticalism' (Vahlen, 1923, p. 21). Furthermore Vahlen recalled that Klein deplored a lack of spatial sense in Jews that leads them 'to a disdain for and underestimation of the visual, to a preference for the unapparent, even to a taste for the unapparent, for the paradoxical' (Ibid.). Cited in this way, Klein's distinctions within mathematics culminated in open Antisemitism.

Curiously and somewhat ironically, according to Segal, 'in 1929 there appeared a set of books by Phillipp Stauff called *Sigilla veri* which purported to reveal the pernicious influence of Jews in German intellectual and cultural life. The mathematician Felix Klein was mentioned[11] in Volume 3, on page 552'. (Segal, 1986, pp. 127–128) Furthermore, as Rowe, 1986 narrates (p. 422), in November 1933 Hugo Dingler, in a memorandum sent to the Bavarian Ministry of Culture consisting of a twenty-page historical synopsis of the Jewish invasion of the fields of mathematics and physics after they were granted legal equality in 1869, claimed that Klein was half-Jewish, had filled Göttingen with Jews and foreigners and intended to re-make German mathematics along Jewish lines.

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[10] Rowe quotes the example that Steckel gave: 'To subtract 3/4 from 7 1/4, for example, the Germans would first reduce each by 1/4, and then calculate 7-1/2 = 6 1/2. The Jews, on the other hand, would convert 7 1/4 into 29/4, and then subtract to obtain 26/4.' Rowe adds that 'This example was, no doubt, meant to illustrate the usual stereotype: that Jews excelled in logical thinking, whereas Germans thought intuitively'. He also recalls that 'in another meeting, Felix Bernstein stressed that environment and training, even at an advanced level, play a more important role than heredity in determining a mathematician's outlook and style'. (Rowe 1986 p. 441).

[11] For an explanation of this error, see Segal 2003, p. 372, n.149.

A quarter of a century after writing his thesis under Klein, Ludwig Bieberbach (a Nazi mathematician who will be examined later), intervened to defend Klein[12] and transformed[13] this one in a pure German mathematician inside his own classification of mathematical types. Thus Bieberbach attributed his own views concerning German mathematics to Klein himself.

2.2 Poincaré

Henri Poincaré identified two types of mathematicians: so-called 'analysts' and so-called 'geometers', also referring to them to as 'logicians' and 'intuitives' respectively. His definitions of intuition vary from text to text. In the psychological reconstruction he gave of the process of mathematical creation,[14] that is, of finding proof, he stated that man has two selves, one unconscious and one conscious. Then he proceeded by metaphor seeing concepts ('mathematical facts') as atoms which stand still when the mind is resting, attached to one of its 'walls'. Then, during unconscious work, carried out on the basis of what had been reflected consciously, some atoms will detach themselves from the wall and thus be able to meet (and hook up in combination) with others set in motion or still hooked to the wall.[15] "The mobilized atoms are therefore not any atoms whatsoever; they are those from which we might reasonably expect the desired solution. The mobilized atoms, are those from which the solution sought can reasonably be expected". (Poincaré, 2014, p. 393) "All goes on as if the inventor were an examiner for the second degree who would only have to question the candidates who had passed a previous examination". (Poincaré, 2014, p. 386–387). The combinations whose beauty, elegance and harmony have the greatest impact on us, and are capable of arousing in us an intense aesthetic emotion, emerge into consciousness. The sensitivity to such emotions (typical of mathematicians) "once aroused, will call our attention to them, and thus give them occasion to become conscious". (Poincaré, 2014, p. 392) Beauty, elegance and harmony emerge from that specific combination, because "the mind without effort can embrace their totality while realizing the details". (Poincaré, 2014, p. 391).

This is (one of) Poincare's definitions of intuition: the ability to grasp the unity of a demonstration, its being ordered in a certain way, at a glance.

> A mathematical demonstration is not a simple juxtaposition of syllogisms, it is syllogisms *placed in a certain order,* and the order in which these elements are placed is much more important than the elements themselves. If I have the feeling, the intuition, so to speak, of this order, so as to perceive at a glance the reasoning as a whole, I need no longer fear lest I forget one of the elements, for

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[12] A student of Bieberbach's, Eva Manger, also defended Klein with the article "Felix Klein in Semi-Kürschner" (see Manger 1934), where Semi-Kürschner was a well-known appellation for the encyclopedia *Sigilla Veri*. On this topic, see Segal 2003, pp. 371–373.

[13] See Bieberbach 1934.

[14] Poincaré's description of creativity and its Darwinian explanation inspired much thinking. For a critical overview of the literature, see Kronfeldner (2011: esp. 64–65). For a comparison between Poincaré's description and that made by other scientists, see Miller (1984, 1992, 1996, 1997).

[15] Marie-Louise von Franz, at the end of the book *Man and His Symbols*, referred to this quote as an example of the fact that 'Our representations are 'ordered' before we become aware of them.' (von Franz 1964, p. 306).

each of them will take its allotted place in the array, and that without any effort of memory on my part […] this intuition of mathematical order, that makes us divine hidden harmonies and relations, can not be possessed by every one. (Poincaré, 2014, p. 385)

The author affirms that people can be divided into the following three categories: 1) those lacking in 'this delicate and difficult to define sensibility' and above average mnemonic and concentration strength – i.e. the majority; 2) those who possess this sensibility to a limited extent but have an extraordinary memory; 3) those whose sensibility is prodigious and accompanied by a non-significant memory advantage. The former are seen as incapable of understanding or creating mathematics, the second group can only understand it and the latter can also create it and will be its 'inventors'. With the data obtained from the unconscious (or 'subliminal') self, the inventor proceeds to make calculations, because these are made at the conscious level.

Poincaré repeatedly emphasized that, for man, the beautiful and the useful (i.e. what is mathematically fruitful) coincide, and muses about the origin of this. He cautiously ventures the hypothesis that this is the result of human evolution and selection: those who prevail by nature have a greater appreciation of what is intellectually beautiful (rather than what is beautiful to the senses, such as flashy colours and deafening sounds). After all, "this disinterested quest of the true for its own beauty is sane also and able to make man better". (Poincaré, 2014, p. 368) Without intuition there be no invention, mathematical novelty, and without a (minimal) intuition one cannot understand the mathematical demonstrations of others. Therefore, even 'logicians', in the inventive stage, must appeal to intuition. "Pure logic could never lead us to anything but tautologies" (Poincaré, 2014, p. 214). "This shows us that logic is not enough; that the science of demonstration is not all science and that intuition must retain its role as complement, I was about to say as counterpoise or as antidote of logic." (Poincaré, 2014, p. 217).

But what apparent differences are there between logicians and geometers? The battle seems to be played out around the acceptance of the reference to imagination, to the vision of the mathematical object as a sure basis for demonstration. Logicians do not accept it. Poincaré listed three types of intuition: "first, the appeal to the senses and the imagination; next generalization by induction, copied, so to speak, from the procedures of the experimental sciences; finally, we have the intuition of pure number, whence arose the second of the axioms just enunciated, which is able to create the real mathematical reasoning". [16] (Poincaré, 2014, pp. 215–216) Geometers' rely on the first type of intuition, 'analysts' on the third. "The analysts, […]

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[16] Poincaré's trust in intuition led him naturally to oppose Russell and Frege's newborn foundational school (later called 'logicism'). Still, Detlefsen stresses that Poincaré criticised 'not just logicism but, rather, the 'logicization' of mathematical proof—by which we mean the reduction of all inferences occurring within a mathematical proof to logical inferences. He believed that there are distinctively mathematical forms of inference, of which perhaps the clearest and most important is mathematical induction' (Detlefsen 1992, p.356). Namely, Detlefsen (1992), like Heinzmann later (1995), claimed that Poincaré had pointed out an *epistemological* difference between the mathematical prover and his/her logician counterpart. He replied to the Boutroux and Goldfarb viewpoint, i.e. Poincaré's argument against the 'logical' point of view was that it did not provide psychological conviction (Boutroux 1914 p. 39; Goldfarb 1985 p. 64). Heinzmann-Stump (2017) add that in any case 'It would be false to believe that he thus conflates logic and psychology'.

in order to be inventors, must, without the aid of the senses and imagination, have a direct sense of what constitutes the unity of a piece of reasoning, of what makes, so to speak, its soul and inmost life." (Poincaré, 2014, p.220).

Such intuitions, though very different, nevertheless see intuition as a unified vision. Namely, reasoning by recurrence (i.e. mathematical induction[17]), 'contains, as it were, condensed into a single formula, an infinity of hypothetical syllogisms' arranged in cascade: the theorem is true for the number 1; if it is true for 1, then it is also true for 2; therefore it is true for number 2; if it is true for 2, it is also true for 3, and so on. The conclusion of each syllogism is the premise of the next syllogism. In mathematical induction, we simply lay down the minor premise of the first syllogism (i.e. 'the theorem holds for number 1') and the general formula that contains all the major ones as special cases (i.e. 'if the theorem holds for n-1, then it holds for n'). Thus a unitary look at a sequence that would be infinite is provided. This rule cannot be proved over others, because in the end we will end up with utterances equivalent to it, so it is 'irreducible to the principle of contradiction'. (Poincaré, 2014, p. 37) Nor can it come from experience, because it would have to be an experience of infinite utterances and is thus impossible for human beings. Nor can it be understood as a convention, analogous to some geometrical postulates.[18] Therefore, the only

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[17] On the purpose of pure number intuition, Heinzmann-Stump (2017) pointed out the differences between this and Kant's: 'Poincaré's is intellectual in character and does not in the least solve the problem of the unity of spontaneity and receptivity by the introduction of a pure sensibility.' Another difference between Poincaré and Kant consists of the fact that Poincaré did not share Kant's opinion that space was an apriori form of our sensibility. As Boutroux explained (1914, p. 8), the philosophical discussion which the Euclidean postulate gave rise excludes this hypothesis ipso facto: the postulate is not an a priori judgment; for if it were a priori, it would have a character of necessity. Nor is the postulate an empirical judgment; for if Euclidean geometry is true for the physicist, non-Euclidean geometry cannot be less true. Finally, Poincaré thought he had proved against Russell that the properties of space cannot be analytically deduced from our belief in the possibility of experience. It remained for him to explain in turn the genesis of geometric space *via* introspection. He analyzed our sensations in detail and sought an explanation as to how man can gradually form this notion in his mind. The experiments that legitimize this geometry are above all physiological experiments. It is observing the order in which our sensations follow one another that ultimately gives rise to the notion of space. However the various spaces, visual, tactile or motor, are not yet geometrical, and they can have more than three dimensions. We give three dimensions to space not by necessity but because it is *convenient* for us. Therefore, in Kantian type judgements, *conventional* judgements are to be added.

[18] Dunlop 2016 recalls that some commentators have suggested that the intuition that grounds the use of induction in arithmetic also underlies the conception of a continuum whilst stating that this was not Poincaré's view. Poincaré, he argues, meant that 'we need arithmetic *and further things* in order to work in geometry. We still need to add, subtract, multiply, and divide when we are doing geometry, but we will be doing more than what we do in arithmetic'. (Dunlop 2016 p. 274) Folina specifies that Poincaré's continuum was guaranteed by *geometric* intuition: it was knowable 'via the form of 'outer' experience', that enables us to link our sense experience and to possess a concept of enduring object. (Folina 1992, p. 190) Therefore, 'although with the platonist he accepts the existence of classical continuum, that is, the determinacy of domain of all real numbers, in opposition to the platonism, the continuum is not a fundamentally 'arithmetical' object. [...]' That is, it is neither an ordinary set (it cannot be obtained only by applying set-theoretic axioms) nor an object upon which further set-theoretic operations can automatically be performed. (Folina 1992, p. 191) Boutroux recalled that Poincaré had defined the continuum as follows:' l'esprit a la faculté de créer des symboles, et c'est ainsi qu'il a construit le continu mathématique qui n'est qu'un système de symboles. Sa puissance n'est limitée que par la nécessité d'éviter toute contradiction; mais l'esprit n'en use que si l'experience lui en fournit une raison'. (Boutroux 1914, p. 17)

option thus remains that it is (using Kantian terminology) a synthetic a priori judgment, which asserts itself with irresistible evidence as an affirmation of the power of human intelligence: intelligence 'knows', it directly intuits its power when experience provides it with an occasion to become aware of it. Therefore, mathematical induction differs from scientific induction, which is based on the belief in something outside us: a general order of the universe. The two share the fact that they proceed from the particular to the universal and, therefore, mathematics is a science, too.

Poincaré added that the two types of mathematicians are not such on the basis of the subject they study: one is not an analyst because one studies analysis, but because one has an approach that privileges pure number intuition.

> It is the very nature of their mind which makes them logicians or intuitionalists, and they can not lay it aside when they approach a new subject. Nor is it education which has developed in them one of the two tendencies and stifled the other. The mathematician is born, not made, and it seems he is born a geometer or an analyst. (Poincaré, 2014, p. 210).

He specified further: "The two sorts of minds are equally necessary for the progress of science; both the logicians and the intuitionalists have achieved great things that others could not have done. Who would venture to say whether he preferred that Weierstrass had never written or that there had never been a Riemann? (Poincaré, 2014, p. 212).

Poincaré, who classified himself a 'geometer', expressed admiration for analysts who work without the aid of the imagination: "The majority of us, if we wished to see afar by pure intuition alone, would soon feel ourselves seized with vertigo". (Poincaré, 2014, p. 221).

He named two Germans with different mentalities and two Frenchmen with equally different mentalities. The Frenchmen were Bertrand and Hermite:

> They were scholars of the same school at the same time; they had the same education, were under the same influences; and yet what a difference! […] M. Bertrand is always in motion; now he seems in combat with some outside enemy, now he outlines with a gesture of the hand the figures he studies. Plainly he sees and he is eager to paint, this is why he calls gesture to his aid. With M. Hermite,[19] it is just the opposite; his eyes seem to shun contact with the world; it is not without, it is within he seeks the vision of truth. (Poincaré, 2014, p. 211).

For the Germans, he offered us the analytical Weierstrass ("you may turn through all his books without finding a figure" (Poincaré, 2014, p. 212)) and the geometer Riemann "each of his conceptions is an image that no one can forget, once he has caught its meaning" (ibid.).

At school, intuition and concrete examples are always the necessary starting point before moving on to the rigour of logic, otherwise students will not understand the content of what is being discussed, feel motivated, understand the genesis of the theories nor their applicability. Logic teaches us that certain specific paths

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[19] Still, at the end of the paper, Poincaré reconsidered Hermite and stated that Hermite cannot be entirely classified as one of the geometers using primarily sensible intuition, nor can he be considered a logician due to his repulsion for deductive arguments.

are viable, i.e. free from contradictions. Poincaré appreciated this aspect, which is a guarantee of the existence of mathematical entities, and shared J. Stuart Mill's view: every definition implicitly contains the axiom that establishes the existence of the objects described. Poincaré noted that this can happen because their definition is non-contradictory, but he also pointed out that logic does not tell us which of the paths allows us to reach the goal. For this reason, intuition is needed: "Without it the geometer would be like a writer who should be versed in grammar but had no ideas. Now how could this faculty develop if, as soon as it showed itself, we chase it away and proscribe it, if we learn to set it at naught before knowing the good of it." (Poincaré, 2014, p. 438).

Some similarities between Klein's and Poincaré's reflections on intuition are visible. Both insist on the importance of intuition to mathematical invention and are sensitive to the psychological aspects present in education (and thus insist on presenting mathematical entities first intuitively and only then formally). However, for Klein intuition is strictly visual-concrete, while in Poincaré intuition's most essential characteristic is that it is a unitary view of a demonstration, and for this reason, somehow present even in analysts, who proceed after formalization, because their reasoning by recurrence (i.e. mathematical induction), 'contains, as it was, condensed into a single formula, an infinity of hypothetical syllogisms' and therefore is a kind of unitary view. In addition, Klein ventured a racial classification, while Poincaré believed that there are various skills involved in doing mathematics which vary from individual to individual. Poincaré's reference to the word 'race' is in the context of 'coarse space', the space of superior animals, the space relative to one's own body (designed to control one's territory, defend oneself from enemies which Poincaré calls 'distribution table'), with 'race' meaning our ancestors[20] common to all of today's human race:

> We have selected the most convenient space, but experience has guided our choice; as this choice has been unconscious, we think it has been imposed upon us […] In this progressive education whose outcome has been the construction of space, it is very difficult to determine what is the terms of use, part of the individual, what the part of the race. How far could one of us, transported from birth to an entirely different world, where were dominant, for instance, bodies moving in conformity to the laws of motion of non-Euclidean solids, renounce the ancestral space to build a space completely new? (Poincaré, 2014, p. 429)

2.3 Brouwer

Brouwer's first particularly significant comments regarding intuition can be identified in his 1907 doctoral thesis *Grondslagen der wiskunde (Foundations of Mathematics):* 'to exist in mathematics means: to be constructed by intuition' (Brouwer, 1975: from now on CW I, p. 96) and 'Mathematics is created by a free action independent of experience: it develops from a single aprioristic basic intuition, which

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[20] He distinguished this space from the geometric space, rigorous, born from the first, but fertilized 'by the faculty that we have of constructing mathematical concepts, such as that of group; it was necessary to search among the pure concepts for the one that best suited this coarse space.' (CW I, p. 81).

may be called invariance in change as well as unity in multitude' (CW I, p. 97). This, then, is a first understanding of intuition as a means of construction whose action is described (in the case of a theorem asserting a property of some mathematical objects) in the following terms:

> Usually mathematics is expressed […] by means of a chain of syllogisms. But the conceptions which are evoked by the words used in such an explanation, consist in the following: Where mathematical objects are given by their relations to the basic or complex parts of a mathematical structure [this means that the object in question is built in connection with the components to which it is said to be related], we transform these given relations by a sequences of tautologies [i.e. by fixing one's attention to different substructures of the mathematical system] and thus gradually proceed to the relations of the object to other component of the structure. (CW I, p. 72)

In the case of an 'affirmative' theorem which seems to start from a structure defined via certain relations embedded within another structure whose construction is not immediately clear (i.e. it seems to start from mere hypotheses) it happens that:

> One starts by setting up a structure which fulfills part of the required relations, thereupon one tries to deduce from these relations, by means of tautologies, other relations, in such a way that the new relations, combined with those that have not yet been used, yield a system of conditions, suitable as a starting-point for the construction of the required structure. (ibidem)

In the case of a theorem denying that a property belongs to a mathematical entity, the construction comes to an end: "I simply perceive that the construction no longer *goes*, that the required structure cannot be imbedded in the given basic structure."(CW I, p. 73).[21]

It should be noted here that for Brouwer mathematics is made up of constructions, that is, it is alinguistic, and based on attempts at constructions, which may succeed or fail. They are 'creative' attempts, i.e. they do not follow any fixed rule. So mathematics does not follow logic, it does not use logic. Logic records the regularities present in expressions of mathematical constructions, which are carried out to support memory and communicate one's results, with an awareness that there are no guarantees of success in another person's same mathematical construction, and that the emotions accompanying the mathematical experience are inevitably linked to the subject (and hence not repeatable).

The second meaning of intuition is the one that originates basic mathematical entities, first of all natural numbers. Brouwer describes this meaning of intuition as the basic phenomenon, the 'simple intuition of time, in which repetition is possible in the form: 'thing in time and thing again', as a consequence of which moments of life break up into sequences of things which differ qualitatively.' (CW I, p. 53)

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[21] Here Brouwer criticises Poincaré's definition of mathematical existence as non-contradictory, on the grounds that this latter is a linguistic characteristic which does not guarantee the constructability of the object.

In a note he calls this intuition 'intuition of two-ity'. (CW I, p. 97) It is a priori in that it is independent of experience, while it is not a necessary condition for experience (CW I, p. 70), because mathematics and experience exist independently of each other; but it is a necessary condition of the 'mathematical receptacle of experience'. Consequently, synthetic a priori judgments are "the very possibility of mathematical synthesis, of thinking many-one-ness, and the repetition thereof in a new many-one-ness; the possibility of intercalation; the possibility of infinite continuation (axiom of complete induction)". (CW I, p. 70).

In the inaugural address to his chair, "Intuitionisme en formalisme", besides defining his foundational school as 'neo-intuitionism' in order to link it and, at the same time, distinguish it from the intuitionism of Kant and Poincaré,[22] Brouwer attempted a more precise description of temporal intuition:

> This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the fundamental phenomenon of the human intellect. (CW I, p. 127)

Brouwer traced his idea of basing the natural number on the intuition of time back to Kant, and his *Doktorvater* Korteweg doubted this. In fact, for Kant the number was the scheme of temporal intuition i.e. it was the medium between concept and imagination. Brouwer, on the contrary, described the intuition of time without referring to the notion of schematism. For him, temporal intuition was abstraction from inner experience. It was the intuition of two and one together, and this also constituted a difference from Kant: "The first act of construction has two discrete things thought together […] F. Meyer says that one thing is sufficient, […] this is false, for exactly this adding (i.e. setting [one thing] while the former is retained) presupposes the intuition of two-ity; only afterwards this simplest mathematical

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[22] As Heinzmann-Nabonnand 2008 have stressed, the main difference between Brouwer's and Poincaré's arithmetical intuition is the role played by experience, which was meaningful only for Poincaré. Namely, according to the latter, experience provides the innate capacities of our minds with an occasion to express themselves both in arithmetic and in geometry, grounded on two different intuitions. If we are obliged to make use of such capacities, we obtain the recursion principle; if we have a choice regarding them, 'experience helps us choose and the result is a convention', as in the case of geometrical intuition. According to Brouwer, mathematics develops out of an *inner* intuition (that is *time* intuition, not mentioned by Poincaré) and must remain inward to avoid causing pain. The continuum is also grounded on inner intuition. For further details, see Heinzmann-Nabonnand 2008. Regarding pre-intuitionism, see Michel 2007. Already in his dissertation, Brouwer (CW I, p. 96) noted that, in 1905, Poincaré had stated: 'En mathématiques le mot exister ne peut avoir qu'un sens, il signifie *exempt de contradiction*' (Poincaré 1905, p. 819), and remarked that, in this sense, Poincaré's views were very similar to those of Russell criticised. If we look at Poincaré in 1905 and read a few lines after the above quotation, Poincaré considers the logicistic search for consistency in axiomatic systems to point out that, in the real world, where a model for these is not at hand, the search presupposes the principle of induction, that is a mathematical (and not a purely logical) principle (Poincaré 1905, pp. 819–820). Heinzmann-Nabonnand 2008 stress that, in any case, Poincaré's appeal to the non-contradictoriness of axiom systems must be seen in a non-Hilbertian light: 'Poincaré rejects the idea of ensuring mathematical reliability with a Hilbert-style consistency proof'.

system is projected on the first thing and the ego which thinks the thin". (CW I, p. 97) Kant, as we have seen above, spoke of the successive addition of equal things.

In his doctoral thesis, in a summary table comparing his mathematical system with Kant's *Transcendental Aesthetics* and Russell's *Foundations of Geometry*, Brouwer stressed the fact that only his mathematics was independent of experience outside the intellect. In fact, for both Kant and Russell three-dimensional Euclidean space is "inseparably linked up with external experience", whereas for himself "nothing is inseparably linked up with external experience". (CW I, p. 71) In 1912, Brouwer placed Kant among the precursors of his intuitionism, since he had theorised temporal intuition, but described his own intuitionism as "neo-intuitionism" (CW I, p. 127), underlining his abandonment of spatial intuition as apriori. Thereafter, all reference to Kant disappeared.

In 1918, Brouwer expanded the mathematical content of this intuition, highlighting the way it is foundational to the concept of species ('Unter einer Spezies erster Ordnung verstehen wir eine Eigenschaft welche nur eine mathematische Entitaet besitzen kann […] Unter einer Spezies zweiter Ordnung verstehen wir eine Eigenschaft welche nur eine mathematische Entitaet oder Spezies erster Ordnung besitzen kann […] In analoger Weise definieren wir Spezies n- ter Ordnung'—CW I, p. 151) and that of 'spread' (spreiding). This is initially described with reference to the universal tree (thought of as a growing structure), i.e. a tree with all possible branches: at each branching point—called 'node'—one assigns either sterilization or a term or nothing. The possibility of assigning a sterilization, which causes the sterilization of the entire branch, is introduced to model the tree, by cutting out the branch; the possibility of assigning nothing, generating finite successions, was used to homologate their construction to that of infinite successions. Later (after Griss's criticism of his definition of negation[23]), from his Cambridge lectures on, Brouwer replaced the sterilization procedure with a direct indication to prosecute only certain nodes, i.e. describing the construction of the tree without passing through the universal tree by saying that it has:

1) for initial nodes (of order 1) either all natural numbers or only those not exceeding a certain given m;
2) for nodes of order $n+1$ (for each n) or all immediate descendants of the node p of order n or only those whose $(n+1)$-th constituent joined to the constituents of p does not exceed a certain number m_p.

To achieve a spread, to each node either objects or nothing are attached.

Within the production of the spread, during the construction of the tree, Brouwer contemplated freedom of choice in the continuation (and intended each branch as a succession of free choices). Still, 'freedom' encompasses everything and, therefore, can also allow for its progressive restriction and even restriction of restriction. Brouwer had numerous second thoughts on the subject, but from 1946 onwards he maintained a definitive opinion, saying: "In some former publications of the author restrictions of freedom of future restrictions of freedom, restrictions of freedom of

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[23] See Franchella 1994a, 1994b and 1995.

future restrictions of freedom of future restrictions of freedom, and so on were also admitted. But at present the author is inclined to think this admission superfluous and perhaps leading to unnecessary complications." (Brouwer, 1981, p. 13)

Finally, Brouwer rethought his definition of temporal intuition, contextualising it with the original *Weltanschauung* context that he had not been allowed to make explicit in his 1907 thesis[24]: man can find serenity only in the interiority of his own consciousness. He is compelled by karma to go out, but it is appropriate for him to do so only minimally. In particular, scientific work must avoid being applicative and take place inward form: for mathematics, the perfect starting point is the intuition of time. It was at the 1948 conference ("Consciousness, Philosophy, and Mathematics") that Brouwer set out the steps from the inner self to the sciences in greater depth. On that occasion, Brouwer explained temporal intuition within the description of the path of man's consciousness (an unquestionable starting point for him) towards externality: consciousness oscillates between sensation and tranquillity, followed by another sensation and therefore distinguishes between present and past; then it distinguishes itself from both (becoming 'mind'); it identifies complexes of sensations that repeat themselves (if the order never changes, they are called 'things', among which there are human bodies) driven by 'causal attention', i.e. the desire to know and obtain objects.

> Mathematics comes into being, when the two-ity created by a move of time is divested of all quality by the subject, and when the remaining empty form of the common substratum of all two-ities, as basing intuition of mathematics, is left to the unlimited unfolding, creating new mathematical entities in the shape of predeterminately or more or less freely proceeding infinite sequences of mathematical entities previously acquired, and in the shape of mathematical species. (CW I, p. 482)

Thus, his intuitionist proposal achieved a gnoseological framework. The reference to intuition in his later writings was always according to his first meaning; he did not explicitly take up again the second meaning of intuition that he had proposed in his thesis, i.e. as a control on the mental path leading to a construction, but in 1937 he mentioned an analogy with the accountant whose accounts are correct in order to describe what inwardly guarantees something to be true: "For the bookkeeper this connection is then dissolved into a general feeling that he has conscientiously performed his duty, a feeling that is positively akin to the sentiment that constitutes for the mathematicians the notion of truth". (CW I, pp. 551–552).

Brouwer (far from setting up questions of race) tried to impose this type of foundation on the European, and even world, mathematical scene (leading to periods of suspension from research activity due to serious disputes with colleagues), for general well-being, despite the fact that, at the conference of 1948, he came to support the impossibility of a plurality of minds (while a plurality of bodies can be observed):

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[24] Regarding the reasons for this fact, see van Stigt 1990a, pp. 35–44 and 405–415 (where he includes the rejected parts of Brouwer's 1907 thesis).

It is not unreasonable to derive this behaviour [the behaviour of individuals in general] from 'reason'. But unreasonable to derive it from 'mind'. For by the choice of this term the subject in its scientific thinking is induced to place in each individual a mind with free-will dependent on this individual, thus elevating itself to a mind of second order experiencing incognizable alien consciousness as sensations. Quod non est. And which moreover would have the consequence that the mind of second order would causally think about the pluralified mind of first order, then cooperatively study the science of the pluralified mind, and in consequence of this study assign a mind of second order with sensation of alien consciousness to other individuals, thus once more elevating itself, this time to a mind of third order. And so on. Usque ad infinitum. […] In default of a plurality of mind, there is no exchange of thought either. Thoughts are inseparably bound up with the subject. […] By so-called exchange of thought with another being the subject only touches the outer wall of an automaton. This can hardly be called mutual understanding. […] Only through the sensation of the other's soul sometimes a deeper approach is experienced. (CW I, p. 485)

The problem remains open as to how it is possible to be certain that in every human being there is a conscience in which she/he can rest peacefully, but Brouwer mentioned neither the problem nor a possible answer.

2.4 Bieberbach

As a full professor in Basel, in 1914 Ludwig Bieberbach gave an 'inaugural address entitled *Über die Grundlagen der Modernen Mathematik*, in which he expressed doubts about the feasibility of Hilbert's program, while defining mathematical existence (in the manner of Cantor and Hilbert) as non-contradictoriness, and lamented that intuitionism refused to recognize promising areas of mathematical research, while praising Brouwer for his article "Intuitionism and Formalism" (Segal, 2003, p. 348). In 1926, after he had moved to Berlin, Bieberbach shifted sharply in the direction of Brouwer's intuitionism,[25] as is clear in the (never published) lecture entitled "Concerning the Scientific Ideal of the Mathematics", which 'ended with the total rejection of Weierstrass/Hilbert-style formalism as a transitory period between Klein's view of mathematics and the coming (in Bieberbach's view) ascendancy of Brouwer's intuitionism.' (Segal, 2003, p. 346). In it, Bieberbach criticised Hilbert's formalism, arguing that 'Hilbert's scientific deal is directly inimical to the needs of applications. Under the aegis of Formalism, applied mathematics have, so to say, died out, and this just a short time after Klein's initiative had inaugurated a new

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[25] Various scholars have been puzzled by Biebarbach's motivations (Mehrtens 1987, 217–218; Biermann 1988, p. 198; Segal 2003, p. 356). Segal suggests that Bieberbach was 'a person of no truly fixed or well-thought-out philosophical or political ideas […] that seems to have sought the main chance for himself.' (Segal 2003, pp. 356–357) He may have become an intuitionist because, for him, being a Berliner meant opposition to Goettingen, and hence to Formalism. 'When Hitler came to power, Bieberbach […] may be aspired to no less than becoming a czar (or a *Führer*) of mathematics.' (Segal 2003, pp. 356–357).

blooming of applied mathematics, a short time after Klein had succeeded in rescuing it from the assaults of the Weierstrass school.' Bieberbach's discourse appears confused, however, because the Klein intuition praised is (correctly) geometric intuition, in which 'every conclusion that appears has an immediately visible, concretely serviceable meaning,' but it is difficult to see Brouwerian pure intuition in continuity with it. Moreover, Bieberbach condemns the 'French encyclopedists' as 'opportunists,' while Brouwer interprets his own intuitionism as relating to French intuitionism. In 1926, however, there is still no racial connotation in Bieberbach's identification of types of mathematicians.

In 1928 Bieberbach and Brouwer held the same position[26] on the occasion of the International Congress of Mathematics organized in Bologna in 1928 under the aegis of the Union Mathématique Internationale (which had expressly excluded Germans and their allies from belonging to the Union and which, during the previous conferences, had forbidden even the participation of German mathematicians). The Society of German Mathematicians (*Deutsche Mathematiker Vereinigung*), represented by Erhard Schmidt, invited members to attend only on an individual basis, only in cases of scientific necessity and avoiding large numbers. According to Brouwer and Bieberbach, German mathematicians should have boycotted the conference altogether, in response to their exclusion from the two previous ones and from the Union. For Brouwer it was (according to Dirk van Dalen, 2005, p. 598) a matter of obtaining a true union of all the mathematicians of the world, without politically motivated preclusions while Bieberbach's stance, on the other hand, was part of his pro-Nazi nationalism. For his part, Hilbert, pragmatically urged everyone to take the opportunity to reactivate the international exchange in a progressive way. At the same time he saw Brouwer as having inspired Bieberbach's attitude both towards the Bologna congress and from a foundational point of view. Brouwer's aim would have been to divide German mathematicians, in order to set himself up as a point of reference. This fear was supported by the fact that Berlin, where Bieberbach taught, Göttingen's only mathematical excellence rival, was still overwhelmed by enthusiasm for the lectures given by Brouwer in the 1926/1927 academic year and Hilbert's student Hermann Weyl had also approached intuitionism.

Ultimately 76 German mathematicians attended the conference (22% of the non-Italian participants), a good number,[27] but Hilbert was irritated and also exhausted by the pernicious anaemia that had caused him a heart attack shortly before the congress. In October 1928 he tried to parry Brouwer's (hypothetical) blows by immediately expelling him from the prestigious *Mathematische Annalen* journal.[28] Brouwer tried to recover from the blow, in 1933, by founding a new journal, *Compositio matematica*. On this occasion he included Bieberbach in the editorial board, but the

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[26] Van Dalen points out, quoting a letter from Bieberbach to Courant, that 'Brouwer's disposition had not been of decisive influence on him. It had not exercised any influence at all'. (Van Dalen 2005, p. 595) For Brouwer 'it was a fight between good and evil, between a closed shop of *Conseil*-connected scholars and institutions, and the free world of science' (p. 598). Furthermore, Van Dalen 2005 specifies that 'Brouwer's actions in Italy were his own; Bieberbach only learned about them from von Mises' (p. 593), but Hilbert was convinced that there was a 'huge conspiracy' from Brouwer, von Mises and Bieberbach.

[27] See Segal 2003, pp. 354.

[28] See van Dalen 2005, pp. 599–636.

latter later asked him to remove his name given the inclusion of Jews and immigrants on it, at a time in which the international Jewish community was plotting war against Germany: 'I feel obliged to make the disappearance of the Jews from the editorial board a condition for my presence in the editorial board.' (van Dalen, 2005, p. 707) He also stressed that such a composition of the board would cause difficulties for the distribution of the journal in Germany. Brouwer created a protected atmosphere for Bieberbach inside the board by sending a circular to all editors to advise them that 'any editor's public participation in manifestations which could harm the mutual esteem of people and nations was incompatible with his function' (van Dalen, 2005, p. 708) Still, Bieberbach resigned and pressed the other German editors to follow him. In March 1936 all the German editors resigned from the editorial board. In any case, 'things ran smoothly for the journal. The relationship between Brouwer and Bieberbach ended and, at the same time, this was, for Brouwer, a period of only sporadic scientific interventions.' (Ibid.)

In 1934 Bieberbach gave a racial orientation to his mathematician classifications, writing two articles on the subject: "Persoenlichkeitsstruktur und mathematisches Schaffen" ["The Structure of Personality and Mathematical Creation"], and "Stilarten mathematischen Schaffens" ["Styles of Mathematical Creation"]. He was inspired by the racist work of Erich Jaensch, in particular by what the psychologist from Marburg (a convinced Nazi) had written in 1931 in *Grundlagen der menschlichen Erkenntnis* [*Foundations of Human Knowledge*]. Jaensch did not simply compare[29] 'Germans' and 'not Germans /Jews', using the two capital letters I and S, with I standing for Integrationstypus and S for Strahltypus, but he also analysed a number of possible nuances within them (types I1, I2, I3, I2/I3 etc.) to enable him to reconcile his scheme with historical reality, returning on various occasions in his writings to his classifications to change them. Bieberbach introduced his distinction between S-types and I-types by recalling Poincaré's remark about the difficulties, as a Frenchman, of reading James Maxwell (Segal, 2003, p.362), He then defined the Germanic 'I-types' (Integrationstypus), which 'let the influence of experience stream into them' (Segal, 2003, pp. 362–363) and the S-types (Strahltypus—radiating type), which 'only value those things in Reality which their intellect infers in it'. In the latter group he included (like Jaensch) the French and the Jews—in particular Jacobi, Poincaré, Minkowski and

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[29] Jaensch went into further detail in his analysis together with his student Althoff, with whom he wrote *Mathematisches Denken und Seelenform* [Mathematical Thought and the Shape of the Soul] in 1939. Jaensch considered himself a failed mathematician. In fact, he turned to the humanities following his participation in the team that designed the philosophical and pedagogical volume of the *Enzyklopaedie* in 1909–1910. However mathematics remained one of his interests. He also added the following remark: "Mathematical thinking and knowing is dependent on its material far less than any other kind of thought. Its various forms are therefore many fewer than is the case in other branches of knowing which are determined through the varied forms of the material to which the epistemological process applies itself. In mathematics the forms of knowledge receive much more the impress of the psychic organism itself " (Jaensch and Althoff 1939, p. 71). It justified his and Althoff's opinion that mathematical thought lent itself better than any other to being analysed in relation to the personality that generated it: 'We obtain in this way, though we look at thinking from the viewpoint of mathematics, an insight into the varied forms of any sort of thought, the way such thought is embedded in the whole person and in the forms of personality and their corresponding forms of thought.' (Ibid.).

Lejeune-Dirichlet; in the former group he placed Klein *with* Weierstrass,[30] Gauss, Euler and even Dedekind and Hilbert (who 'do show a certain preference for thinking over intuition, but this is distinct from the S-type, who denies the connection to an outer reality that is not mentally constructed'). (Segal, 2003 p. 365). Ultimately he proclaimed: 'I am of the opinion that the whole dispute over the foundations of mathematics is a dispute of contrary psychological types, therefore in the first place a dispute between races. The rise of intuitionism seems to me only a corroboration of this interpretation.' (Segal, 2003, p. 365) Bieberbach realised that putting Brouwer and Hilbert on the same side of the boat was very strange, but he justified his choice by stating that the difference between the two 'is quite compatible with the fact that both Hilbert and Brouwer should be classified under the psychological type I3/I2. The fact that two men approach their science with an ideal norm, does not necessarily imply that it has to be the same norm in both case' (Mehrtens, 1987, pp. 228 and van Dalen, 2005 p. 702).

The distinction between abstract mathematics, which is typically Jewish, and concrete mathematics, which is typically Aryan, had positive consequences: it advocated saving the teaching of mathematics in a Germany where the Nazi call for irrationality certainly did not facilitate its retention in schools, on condition that it be taught and practiced with a strong applicative attitude.[31]

In 1940, in *Die voelkische Verwurzelung der Wissenschaft* (*The Rootedness of Science in the People*) Bieberbach resumes his previous classification,[32] but expresses himself in a milder way: 'In the face of such different types of mathematical thought, one notices that the content of mathematics, despite that, largely seems to be independent of the thought-type. In fact, it would be hard to give a correct mathematical theorem that not every mathematician recognized as correct. As soon, however, as the question arises whether the theorem concerned might be important […] opinions about it depends largely upon the [Jaenschian] type of the judge.' (Quote in Segal, 2003, p. 384) There are manifold potential reasons for this tempering of his argument: there were no more Jewish mathematicians to purge, Brouwer had withdrawn from the foundational scene, a more moderate tone could favour the acceptance of racial types abroad too and with them the *Deutsche Mathematik* journal newly founded by Bieberbach in 1936.[33]

Bieberbach was dismissed from his teaching post war in 1945 and the Aryan 'intuition' disappeared. His racial mathematical approach connotation must be

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[30] He justified his inclusion of Weierstrass for 'the unity, which, however at least still existed in his [Weierstrass] inner person'. (Segal 2003, 363),

[31] It is useful to stress that 'Although mathematicians like Bieberbach, Tornier, and Vahlen, and perhaps Blaschke, cared deeply about promoting Nazi ideology, there were others, like Knopp, Behnke, and Hecke, who thought differently and the latter were always in the majority in the mathematical community. ' (Segal 1986, p. 132).

[32] 'In short, I-types are inwardly conceptual, S-types outwardly computational. The key descriptive word for I-types is *anschaulich*, indicating the intuitive understanding of true Germanic types' (Segal 2003, p. 363).

[33] He was helped by Theodor Vahlen who was, at the time, head of the Prussian Scientific Office. Having graduated in mathematics, become full professor, served in the First World War and had been rector of the University at Greifswald in 1923 when Hitler's failed Putsch of that year convinced him to become a National Socialist. He visited Hitler in Landsburg prison and, in 1924, became a Nazi member of the Reichstag. He took down the Weimar flag from his university and was thus dismissed. Shortly after Hitler's accession to power, he rose rapidly through the ranks as an educational bureaucrat in the Prussian ministry of education.

remembered in order to keep the memory of what happened alive. From a conceptual point of view it had, and has, no theoretical credibility, as is made clear by the many obvious falsifications that Bieberbach (and, before him, Jaensch and Althoff) were obliged to enact in his attempt to reconcile them with a historical reality that contradicted their classification. Brouwer will suffice as an example. He was classified as an 'intuitionist' (according to Bieberbach's definition), but intuition was not geometric but temporal for Brouwer and he advocated *pure* mathematics.

2.5 Heyting

In the thirties, Brouwer's student Arend Heyting came onto the mathematical scene. His first approach was one he would never abandon: building bridges for the sake of understanding and cooperation among mathematicians. He wrote a series of articles on the formal presentation of arithmetic and intuitionistic logic, despite sharing with Brouwer[34] the idea that mathematics is a mental construction and that in it language serves solely expressive, not demonstrative, purposes. Moreover, he took part at the mathematicians meeting in Koenigsberg, the round table on foundational currents, where John von Neumann represented formalism and Rudolf Carnap represented logicism. The atmosphere was one of co-operation (to the extent that individual speakers expressly sought meeting points with the thought of the others) and Heyting entered the stage effortlessly, as a representative of intuitionism, avoiding propaganda.

In his volume *Beweistheorie. Intuitionismus* of 1934, in which he also contextualised intuitionism and formalism from a historical point of view, Heyting took care to separate intuitionist mathematics from Brouwer's mystical *Weltanschauung*, because he feared that it might also keep people away from their mathematics. He asserted that, according to Brouwer, mathematics is the exact part of our thinking and, therefore, the basis for all other sciences (including logic) and has no presuppositions, with its only source being 'an intuition which sets before our eyes its concepts and conclusions as immediately clear.' (Heyting, 1934, p. 14) It is, Heyting points out, no more than the faculty of considering concepts and conclusions that habitually occur in our thinking separately. It is a faculty that one must train oneself to exercise, 'a peculiar mental aptitude' that allows mathematics to develop in full autonomy from any philosophical presupposition. By which he meant that, an existence 'seen' intellectually requires no other ontological reference, while existence according to Hilbert's formalism, that is existence within an axiomatic system, must be granted from outside, because internally there is only logical consequence.

Heyting described intuition in his 1956 *Intuitionism: an Introduction* as follows: 'A mathematical construction ought to be so immediate to the mind and its result

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[34] Heyting prepared the original version of his formalisation for a Mathematical Society competition organized by Gerrit Mannoury. When Brouwer received this manuscript he asked his pupil to prepare a German version of it for publication in *Mathematische Annalen*. Due to the quarrel between Brouwer and Hilbert the paper was published in the *Sitzungsberichte der preuszischen Akademie von Wissenschaften*. In any case, Brouwer praised the work and accepted it as an adequate representation of 'his own' intuitionism (see van Stigt 1990, pp. 288–294).

so clear that it needs no foundation whatsoever. One may very well know whether a reasoning is sound without using any logic; *a clear scientific conscience* suffices.' (Heyting, 1956, p. 6).

He also supported the belief in the commonality of mathematical thoughts as follows:

'My mathematical thoughts belong to my individual intellectual life and are confined to my personal mind, as in the case for other thoughts as well. We are generally convinced that other people have thoughts analogous to our own and that they can understand us when we express our thoughts in words, but we also know that we are never quite sure of being faultlessly understood'. (Heyting, 1956, p. 8).

Here Heyting expressed his consciousness of the fact that neither Brouwer nor himself had provided proof for the belief in a common mathematical though but such belief is part of our common beliefs.

Then he specified, in the course of various writings, the entities that intuition can attest to. Regarding two-oneness, he wrote:

'We know how to build up the sequence of natural numbers in such a way that we begin to think in terms of a unity, in the same spiritually constructive way that had to be done in forming the observation "a pencil". Then we think "another unit", and finally we think that this last step is repeated again and again. The three concepts "one", "another one" and "again and again one" are sufficient to explain the theory of natural numbers.'[35] (Heyting, 1980, from now on CP, pp. 278-279)

Referring to the formation of two-oneness, Heyting also wrote (Heyting, 1956, p. 13):

Class: Are these considerations not metaphysical in nature?
Int: They become so if one tries to build up a theory about them, e.g. to answer the question whether we form the notion of an entity by abstraction from actual perceptions of objects, or if, on the contrary, the notion of entity must be present in our mind in order to enable us to perceive an object apart from the rest of the world. But such questions have nothing to do with mathematics. We simply state the fact that the concepts of an abstract entity and of a sequence of such entities are clear to every normal human being, even to young children.

Heyting constructed the same mathematical entities as his teacher but he felt the need to specify, with respect to alternative label "choice sequences" for them, that he preferred 'infinitely proceeding sequences' because 'to arrive at the notion of infinitely proceeding sequences, we need not introduce new ideas, in particular the notion of choice' (Heyting, 1956, p. 33), which seemed to him overly linked to the

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[35] "Wij kunnen de rij der naturlijke getallen op deze wijzen opbouwen, dat wij beginnen, een eenheid te denken, dit is dezelfde geestelijken bouwdaad, die ook bij de vorming van de waarneming "een potlood" verricht moest worden. Daarna denken wij ons "nog een eenheid", en ten slotte denken wij ons, dat deze laatste stap telkens weer herhaald wordt. De drie begrippen "een", "nog een" en "telkens weer een" zijn voldoende, om de theorie van de natuurlijke."

psychology of the subject. In particular, when Brouwer died, he stated that he had glimpsed in the latter's 1948 writing a solipsistic turning point (with a pronounced role accorded individual psychology) that he could not share. That is, in 1948 Brouwer had introduced the expression 'creative subject' in a very short article written in response to the criticism levelled by Dutch mathematician George François Cornelis Griss[36] (as an intuitionist) against his definition of negation.

Griss had arrived to intuitionism from his own *Weltanschauung* that he had outlined in his 1946 book *Idealistische filosofie*. There, he had based his *Weltanschauung* on the original datum that consciousness grasps by attaining its own fullness: the subject distinguishes himself from the object, but the one has no meaning without the other. Mathematics is the specific way to analyse the original datum that focuses on the subject-object link. For this reason, mathematical objects cannot be thought of independently of a mathematician that produces them: a platonic existence for them is excluded. Griss' *Weltanschauung* had led him to intuitionism. This did not imply a total acceptance of Brouwer's system. In particular, he criticized Brouwer's definition of negation as a reasoning that ends in a contradiction, i.e. that cannot be carried out, by explaining that: "Faire la supposition qu'un preuve soit donnée, tandis que cette preuve parait être impossible, est incompatible avec le point de départ constructive et evidente, car l'existence d'une preuve est identique au fait qu'elle a été donnée". (Griss, 1948, 71) Griss criticised the Brouwerian definition of negation, because an intuitionist demonstration must start with something evident and end with something evident. Brouwerian negation had been described as arriving at proof of the impossibility of a construction. The point at which the proof stops can be considered as evident, because one *sees* that, metaphorically, one hits a wall, but no status of evidence can be attached to the starting point of the proof, otherwise there would be an evidence that is then disproved: which would remove all foundation for intuitionist mathematics. Hence, Brouwer's definition of negation cannot be considered acceptable within mathematics: it can only be kept at a pre-mathematical stage. A new definition of negation within mathematics is needed. Griss suggested a comparison between two already constructed entities and the realisation that one has more properties than the other.[37]

Brouwer responded by constructing a real number which was certainly not zero (i.e. regarding which a negative property was known) but which could not be said to be greater than or less than zero (positive properties), to show that it is not always possible to find a positive substitute for a property defined through disputed negation: he responded to Griss's criticism by arguing that it would be a loss for intuitionist mathematics if the properties defined through the disputed negation were to be eliminated, because some properties would be irretrievably lost.

Heyting did not follow or comment on Brouwer's continuous second thoughts regarding limiting freedom of choice, while he took seriously the doubts that, even from the intuitionist side, one could ever arrive at some of the constructions

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[36] Brouwer had supported the publication of Griss's papers by presenting them to the Academy of Sciences (see van Atten's entry "The Development of Intuitionistic Logic" in the Sitography). Still, in his 1948 paper, Brouwer did not expressly mention Griss's name.

[37] Griss could give only a few details of both mathematics and logic built according to his new definition of negation because he died in 1953. Still, for further details about his thought see Franchella 1994a.

presented by Brouwer. In particular, reflecting on Griss's critique, he realized that its core was directed against hypothetical constructions which fail, and he grouped together the various types of 'bedingte Konstruktionen' present in intuitionist concepts. He began drawing up a list of these in 1949 (CP, pp. 459–460), refining it in 1958 (CP, pp. 560–564, pp. 103–104), and providing a detailed and final version of it in 1962 (CP, p. 641) within a scale of degrees of evidence:

> the highest grade is that of such assertions as $2 + 2 = 4$. $1002 + 2 = 1004$ belongs to a lower grade; we show this not by actual counting, but by a reasoning which shows that in general $(n+2) + 2 = n + 4$. Such general statements about natural numbers belong to a next grade. They have already the character of an implication [...] This level is formalized in the free variable calculus. I shall not try to arrange the other levels in a linear order; it will suffice to mention some notions which by their introduction lower the grade of evidence.

1) The notion of the order type ω, as it occurs in the definition of constructible ordinals.
2) The notion of negation, which involves a hypothetical construction which is shown afterwards to be impossible.
3) The theory of quantification. The interpretation of the quantifiers themselves is not problematical, but the use of quantified expressions in logical formulas is.
4) The introduction of infinitely proceeding sequences.
5) The notion of a species

reiterating that individual intuitionists' willingness to accept hypothetical constructions varies. The starting point is strictly finite mathematics and then one decides how far the arc of mathematical entities acceptable as evident can be stretched. Accepting the existence of entities of which we only know the impossibility of non-existence would be very different: that would not be stretching the arc, but going in a completely different direction from the others on the scale. It would be a leap into metaphysical darkness.

However, this does not mean that Heyting showed a peaceful and benevolent attitude only within intuitionism, accepting the various shades of constructability, because Heyting emphasises – in both mathematical (as early as his 1956 handbook) and philosophical contexts (traceable in his unpublished manuscripts[38]) – that views of what is accepted as existing vary. Along the ascending scale of abstraction, ranging from one's consciousness to real numbers and beyond to God, some stop early on and do not accept even very large natural numbers (truly unconstructable by the human mind), and there are those who believe that there are also Platonic ideas of number or notion-limits for human reason, such as God.

One of his considerations thus contains, as an aphorism, the possibility of a whole range of meanings of the word 'exist' in different situations:

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[38] For a survey of these, see Franchella 1994b.

1) the immediate environment, 2) representations of the environment; 3) memories of the environment; 4) communication with other people; 5) space-relations among represented environments; 6) systematizations of these relations by means of maps and globes, 7) the fitting of all structures into a spatial generalisation stretching out to infinity; 8) astronomy; 9) microscopically small objects; 10) theoretical physics. (F11.5)

We also find a question mark at the end of the following quote[39]:

Each of our abstract concepts starts with something simple. So does "existence". First it is the objects from my immediate surroundings that exist; after all, they are stars and mesons. How many stairs lie between them and how does the concept change during its journey along them? God, real numbers and cardinal numbers are at the top of such stairs. On which stairs do the logical rules for the existential quantifier apply? (F5 20-21)

Heyting did not feel up to taking the last step to the top but he understood that others might, on conscience grounds, and he let them do so, without feeling the need to convince them forcibly and, at the same time, declaring that he could not be convinced.

In 1956 he had already shown great openness towards other approaches, which he saw as constantly intertwined (often unconsciously) throughout the history of mathematics. He proposed to maintain pluralism within mathematics, and, in parallel, within logic, with the idea that different fields of exploration demanded different mathematics. In particular, intuitionism explored the limits of what can be constructed through the human mind. One might approximate this approach to Carnap's principle of tolerance: '[…] das Toleranzprinzip: wir wollen nicht Verbote aufstellen, sondern Festsetzungen treffen. […] In der Logik gibt es keine Moral. Jeder mag seine Logik, d.h. seine Sprachform aufbauen wie er will' (Carnap, 1934, pp. 44–45). Carnap referred to logic, but we know that logic, according to intuitionism, is the expression of mathematics: therefore, tolerance in mathematics implies tolerance in logic. However, it should be noted, from a historical point of view, that Heyting mentioned the principle of tolerance in the fictitious debate at the beginning of this 1956 volume, putting it into the mouth of the representative of the 'formalists' (Heyting, 1956, 2), but did not cite it as the source of his own pluralism. In addition, we must remember a warning that comes to us from Elio Franzini's reflections in the context of the Enlightenment legacy[40]: 'The Enlightenment taught, with all its limitations, tolerance (a necessary value, and certainly not sufficient, which is nevertheless the basis for its dialogical evolution) […]' (Franzini, 2009, p.38) '[…] with all the limits of tolerance, while still convinced of possessing a superior point of view' (Franzini, 2009, p. 41). Tolerance

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[39] „Ieder van onze abstracte begrippen begint met iets eenvoudigs aanschouwelijks. Zo ook "bestaan". Eerst zijn het de voorwerpen uit mijn direkte omgeving, die bestaan; tenslotte zijn het sterren en mesons. Hoeveel trappen liggen daartussen en hoe verandert het begrip bij zijn tocht daarlangs? God, reele getallen en hoge kardinaalgetallen staan bovenaan zulke trappen. Op welke trap gelden de logische regels voor de existentiekwantor? "

[40] He was inspired by Todorov 1992 *The Conquest of America. The problem of the Other*, in particular from the Epilogue. (pp. 297–303)

simply means marking out one's own territory and those of others in order that each can cultivate their own garden separately and in isolation: of course, it is a recognition of the legitimacy of one's own existence and that of the other(s), and it implies a peaceful attitude, which can still be arrogant and which is not yet interested in others, a desire to know others: it is the premise for, but not yet the definitive step towards, dialogue. Therefore, the word 'tolerance' does not fully express Heyting's attitude: he did not draw furrows in the mathematical ground in order to barricade himself inside his own territory and carry out his work in blissful isolation but encouraged methodological self-awareness during mathematicians' research and suggested that each identify the most suitable ground for the growth of their seeds, showing an ever lively desire to make their own seeds known to those near and far. This comes across throughout his search for a symbolic expression of intuitionist logic and arithmetic carried out in order to make his theory more understandable to his formalist counterparts, as well as in all the efforts he made to divulge the intuitionist perspective in clear, straightforward language, an effort rewarded by his happy admission in his 1962 paper "After Thirty Years":

> Let me compare the situation of 1930 with that of today. The spirit of peaceful cooperation has gained the victory over the ruthless contest. No direction of research has any longer the pretension to represent the only true mathematics […] we know at every moment whether we work on an intuitive basis or not, which part of the work is purely formal, and which platonistic assumption we make (CP, p. 640).

2.6 Final considerations

We have seen that it was only in the Evanston lectures that Klein distinguished between 'naïve ' and 'refined' intuition, of which the first, visual-spatial, seemed to him more Teutonic, while the more logical latter appeared more linked to the Latin and Jewish lineage. He did not, however, resume this racial classification in later writings.

Poincaré cited two types of approach to mathematics: intuitive and analytical, but did not relate these to nationality or ethnicity. He saw them as on a par with hair colour: his opinion might be summed up by stating that your mathematical approach is in your DNA. These are flip sides of the same coin, however: the overall view of proof. There are people who see this abstractly and those who see it graphically, but it is, in any case, an overall view. Poincaré considers this overview unteachable, and thus condemns those without it to understanding mathematics but not creating it – if they have sufficient memory – or even to an inability to understand it at all in addition to not creating it if they do not have a powerful memory.

Bieberbach distinguished between Germanic 'I-types' (Integrationstypus), who 'let the influence of experience stream into them' (Segal, 2003, pp. 362–363) and 'S-types' (radiating typus), who 'only value those things in Reality which their intellect infers in it'. His teaching of mathematics was oriented towards the concreteness of its applications, in order to educate German youth with the type of mathematics suitable to the Aryan race. It was, therefore, a racial not DNA distinction (unlike

Poincaré's non-racial DNA concept) which did not, in any case, square well with historical reality, to the point of requiring continuous revisions and the addition of internal nuances in order to reconcile his descriptions of mathematicians with those who really existed.

The presentation Brouwer gave of intuitionism contained no reference to race. Nonetheless, in the Stanford Encyclopedia entry "Brouwer", we find the following statement about Brouwer's mathematical and philosophical nine notebooks towards his dissertation: "Some remarks betray a form of anti-semitism that, unfortunately, often appeared in language at the time, as codification of a common prejudice; an occasional remark goes further and contrasts Jewish people unfavourably to Germanic ones." Therefore, Brouwer's juvenile remarks on the Jews deserve here a further analysis to point out their link with his *Weltanschauung.* They have been transcribed and studied[41] by John Kuiper in his 2004 Ph.D. Thesis *Ideas and Explorations: Brouwer's Road to Intuitionism.* They are basically notes that appear as aphorisms *rari nantes in gurgite magno.* Over and over he blames the Jews for having constructed a mathematics aimed at controlling the external world, separated from the Self:

> "How the Jews, through mathematics, control the farmers and the farmers, through mathematics, control the cows. Mathematics is a part of cultural engineering, which has been commercialised."[42]

> "Life has no special needs; only the Jews can see the world as a mechanism [i.e., a part of the world as a partial aspect of a mechanism], and then also see you as a mechanism, and thus pull the soil under your feet, because they saw the bottom in you as necessarily acting in the mechanism (seeing you as a mechanism is only possible for them, because they see the world as something outside themselves, i.e. they are divided in the head); for you then appear in your world the `evil', which you have to suffer. […] A human being feels looser from the world than an animal, can live higher, but as a flip side is more susceptible to Jewishness."[43]

> "All these investigations thus lead to the result that mathematics in life is the first stage of sin. (You can only tampering with it as a `hindrance'); it can only be defended as the frivolous entertainment of building, but it is a trick of the Jews to place this in the full movement of life. Now a wise man does not participate in frivolities and even less in the tricks of the Jews. But then again,

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[41] Each notebook and the thesis are available at: https://www.cs.ru.nl/~freek/brouwer/index.html.

[42] Hoe de joden door wiskunde de boeren, en de boeren door wiskunde de koeien beheerschen. Wiskunde een deel der cultuurtechniek, die in den handel is gebracht. (I-22).

[43] Het leven heeft geen speciale behoeften; alleen de joden kunnen de wereld zien als een mechanisme [d.w.z. een deel van de wereld als deel-aspect van een mechanisme], en dan ook jou zien als een mechanisme, en zoo jou een bodem onder de voeten wegtrekken, omdat ze die bodem bij jou zagen als noodzakelijk optredend in het mechanisme (jou zien als mechanisme kunnen ze alleen, doordat ze de wereld aanschouwen als iets buiten zich, d.w.z. gepartieerd zijn in 't hoofd); voor jou verschijnt dan in jou(w?) wereld het `kwaad', dat je lijdelijk hebt te dulden; want doe je dat niet, dan zit er niets voor je op, dan met de joden mee te leven in de partieering, om je daar te kunnen verdedigen, d.w.z. ook de wereld te gaan aanschouwen. Een jood heeft alleen vat op je, als je je aan de wereld gebonden hebt. Een mensch voelt zich losser van de wereld dan een dier, kan hooger leven, maar als keerzijde is meer vatbaar voor jodigheid. (II-39).

he feels that as long as the gates to worlds are not opened for him, he has to learn to participate in all earthly misery, grateful for his weakness and incapacity in that business, and without in that company, and without bondage".[44]

Here appears the core of his *Weltanschauung*, according to which the kingdom of serenity lies for man in being enclosed within himself in contact with his soul, but destiny obliges him, during this journey that is our life, to leave his interiority, letting reason put a demarcation between him and the world, and then trying to dominate the latter through mathematics, reducing the nature to a mechanism. This core had also appeared in *Life, Art, and Mysticism* (Brouwer, 1905, p. 400) and it would re-appear at the 1948 conference "Consciousness, Philosophy, and Mathematics" (CW I, pp. 485–486): what was not and would not be present in these two works is the apostrophising of the demarcation between Self and world and of this mathematics in general as "Jewish". The fact that he was able to present his thought without these references makes it clear that they did not form the basis of his thought, that they did not constitute it.

Furthermore, we can stop and feel the tone in which he expressed the opposition between "We Germans"and the Jews:

"The reason made the world ready to be attacked by the Jews in the partition of the 3 dimensions, and thus to be forced to live in three dimensions. [And for ourselves - free Germans - the 3 dimensions make sense only as frivolity]".[45]
"A German does not do mathematics of his own accord, but is forced to go along with the fools and Jews who brought it into the world".[46]

It is the same haughty, over-the-top tone that had appeared in his letters of those years to his friend Carel Adama van Scheltema in 1904, in which he called himself and his friend 'kings' (van Dalen, 2011, p. 18), disavowed his socialist inclinations (van Dalen, 2011, p. 19) and recommended him: ''Take care that an overhasty ambition will not lead us, out of yearning for quick success, to assimilation and to consorting with low company" (van Dalen, 2011, p. 19).

The anti-semitical framework that van Atten in his entry defines as a common prejudice of the period was better suited than socialism to the ambitious impulses that the young Brouwer felt within himself, but it did not forge his mathematics, as

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[44] Al deze onderzoekingen voeren dus tot het resultaat, dat de wiskunde in het leven optreedt als eerste fase van zonde. (je kunt er de natuur alleen mee knoeien als `hindernis'); dat ze alleen nog te verdedigen is als het lichtzinnig vertier van bouwen, maar dat het een jodenstreek is, om dat in het volle beweeg van het leven te plaatsen. Aan lichtzinnigheden hen puzzlesi doet nu een wijs mensch niet mee, en aan joden-streeken nog minder. Maar dan weer voelt hij, dat zoolang de poorten naar betere werelden niet voor hem worden opengezet, hij te leeren heeft, aan al het aardsche ellendig bedrijf mee te doen, dankbaar voor zijn zwakte en onvermogen in dat bedrijf, en zonder gebondenheid. (VIII-5).

[45] De verstandhouding maakte de wereld rijp, om in de partieering der 3 afm. te worden aangevallen door de joden, en zoo te worden gedwongen, ook in 3 afmetingen te gaan leven. [En van onszelf—vrije Germanen—hebben de 3 afm. alleen zin als lichtzinnigheid]. (II-12).

[46] Wiskunde doet een Germaan niet uit zichzelf, maar uit gedwongen-zijn-tot meegaan met de knoeiers en joden, die het in de wereld hebben gebracht. (II-11).

evidenced by the fact that the stereotypical counterposition between Jew and German would never be re-proposed by Brouwer, not even in the period of the Nazi occupation of Holland.

Also van Atten's entry concludes: "No such utterances are known from other periods in his life […]. There is no meaningful connection between this fleeting anti-semitism and the philosophical views on mathematics, language, and society he set out in his dissertation and developed over the rest of his life."

Regarding Brouwer's personal stance on Nazi ideology, we know that at the end of World War II he was tried by the Committee of Restoration, which found him guilty of the following actions[47]:

1. the posting of the advice that students could sign the loyalty declaration 'without essential scruple' and re-posting it after the statement of the exiled government in London,
2. the opposition in the Senate to the resistance movements of the professors and of the students,
3. the financial assistance to the *Nederlandse Volksdienst*. (van Dalen, 2005, p. 791)

When the Minister received the statement, he sent a reprimand to Brouwer but left the university free to reinstate him. Eventually Brouwer was reinstated. Nevertheless, it is interesting to understand the motivations that led Brouwer to each of the above actions, in order to realize how involved he was in Nazi ideology.

The first two actions refer to the strong request put by the Secretary for Education on 10/03/1943: each student had to sign a declaration of loyalty to the German commands in force in the Netherlands if he/she wanted to follow courses or take parts in examinations. Two meetings of the Senate took place in March and April. A first draft of Senate statement concluded that if the ordinance were carried out in the given form, "the members of the Senate had no choice but to resign". (van Dalen, 2005, p. 762) Brouwer presented an alternative statement with the deletion of the final passage: this represented his action nr. 2. Furthermore, he encouraged the students to sign the loyalty declaration (required to attend the courses) even after the exiled government had expressed a negative opinion (action 1). Both action 1 and action 2 aimed to keep the university running in order to allow students to take their exams. Finally, as for action nr. 3, *Nederlandse Volksdienst* was a German association which had been set up by the occupiers in order to replace all existing charities in the Netherlands. Brouwer made his contribution, because his hope that the funds would actually go to the needy Dutch people, regardless of their ideologies, had been strengthened by seeing a fellow communist receive a grant from the association.

We have seen that Brouwer did not link intuitionism to race, but made it an obligation for all men in order to practice mathematics without excessively compromising their mystical inner serenity: the purpose of his foundational vision might be said to have been for the sake of good, taking for granted that all men have the same inner lives (while, at the same time, failing to demonstrate the possibility of other

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[47] About this subject see van Dalen 2005 ch. 17 and 18.

minds). Gradually, over the course of his inner considerations on the intuition of time, he discovered the faculty to construct natural numbers, species and free choice sequences, from which he then proceeded to construct all mathematics in a creative way. He followed no specific rules, but rather checked the evidence, the intuitiveness of each step, by experiencing the sense of correctness that also an accountant has when the results "come to him".

While to Brouwer all his notions generated by the intuition of time appeared equally evident, some intuitionists cast doubt on these. Of these, it was G.F.C Griss who challenged Brouwer's notion of negation as contradicting an initial sketch of a construction. Namely, intuitionist constructions should be evident at every step, but what evidence can there be of the initial step in a construction that is never built?

Brouwer responded to this by showing that the non-acceptance of this notion of negation would impoverish mathematics. However, such an answer would seem to be limited by the fact that it could backfire on the whole of intuitionism, subjecting classical mathematics to considerable mutilation.

Heyting, on the other hand, addressed this issue along with other criticisms of the intuitiveness of Brouwerian concepts (e.g., the notion of free-choice sequence), organized the various notions along a scale of degrees of evidence, and admitted (without specifically labelling it) a kind of pluralism within intuitionism, underlining, however, that the distinction between those who call themselves intuitionists (while disagreeing with each other on the admissible mathematical entities) and those who do not remains clearly visible, because intuitionists limit their acceptance of the existence of the mathematical entities proposed by 'classical' mathematicians. Unlike Brouwer, moreover, Heyting did not claim that intuitionism was the mathematics to convince others of, but stated that it was possible for some people to believe in the existence of entities (and not only mathematical ones) that were unacceptable for others. This was not a problem: logics would be generated (i.e. linguistic-symbolic transcriptions of the two mathematics), which would differ because they originate from different mathematics. This is the crux of Heyting's logical pluralism (derived from his "mathematical pluralism") and his adherence to the principle of tolerance. 'Tolerance', however, might not be the most appropriate expression, because it does not necessarily involve dialogue between the parties, whereas Heyting desired and sought dialogue. In particular, he wanted to be able to make the other side understand what 'his' mathematics consisted of as well. Hence, the most appropriate expression is 'dialogue'. Heyting proposed logical pluralism and tirelessly sought dialogue.

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