

Weight-Reducing Turing Machines*

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Abstract

It is well-known that one-tape Turing machines working in linear time are no more powerful than finite automata, namely they recognize exactly the class of regular languages. We prove that it is not decidable if a one-tape machine works in linear time, even if it is deterministic and restricted to use only the portion of the tape which initially contains the input. This motivates the introduction of a constructive variant of one-tape machines, called weight-reducing machine, and the investigation of its properties. We focus on the deterministic case. In particular, we show that, paying a polynomial size increase only, each weight-reducing machine can be turned into a halting one that works in linear time. Furthermore each weight-reducing machine can be converted into equivalent nondeterministic and deterministic finite automata by paying exponential and doubly-exponential increase in size, respectively. These costs cannot be reduced in general.

*This work contains, in an extended form, some material and results which were previously presented in a preliminary form in conference papers [10] and [2].

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1 Introduction

The characterization of classes of languages in terms of recognizing devices is a classical topic in formal languages and automata theory. The bottom level of the Chomsky hierarchy, *i.e.*, the class of type 3 or regular languages, is characterized in terms of deterministic and nondeterministic finite automata (DFAs and NFAs, respectively). The top level of the hierarchy, *i.e.*, type 0 languages, can be characterized by Turing machines (in both deterministic and nondeterministic versions), even in the one-tape restriction, namely with a unique infinite or semi-infinite tape containing, at the beginning of the computation, only the input, and whose contents can be rewritten to store information.

Considering machines that make a restricted use of space or time, it is possible to characterize other classes of the hierarchy. On the one hand, if the available space is restricted only to the portion of the tape which initially contains the input and nondeterministic transitions are allowed, the resulting model, known as *linear-bounded automaton*, characterizes type 1 or context-sensitive languages [7]. (The power does not increase when the space is *linear* in the input length.) On the other hand, when the length of the computations, *i.e.*, the time, is linear in the input length, one-tape Turing machines are no more powerful than finite automata, namely they recognize only regular languages, as proved by Hennie in 1965 [5].¹

The main purpose of this paper is the investigation of some fundamental properties of several variants of one-tape Turing machines working in linear time. We now give an outline of the motivation for this investigation and of the results we present. (Figure 1 summarizes the models we are going to discuss and their relationships.)

A natural question concerning models that share the same computational power is the comparison of the sizes of their descriptions. In this respect, one could be interested in comparing one-tape Turing machines working in linear time with equivalent finite automata.

Here, we prove that there exists no recursive function bounding the size blowup resulting from the conversion from one-tape Turing machines working in linear time into equivalent finite automata. Hence, one-tape linear-time Turing machines can be arbitrarily more succinct than equivalent finite automata. Furthermore, it cannot be decided whether or not a one-tape Turing machine works in linear time.² These results remain true in the re-

¹Actually, the model considered by Hennie was deterministic. Several extensions of this result, including that to the nondeterministic case and greater time lower bounds for nonregular language recognition, have been stated in the literature [14, 4, 8, 9, 13].

²For the sake of completeness, we mention that it is decidable whether or not a machine

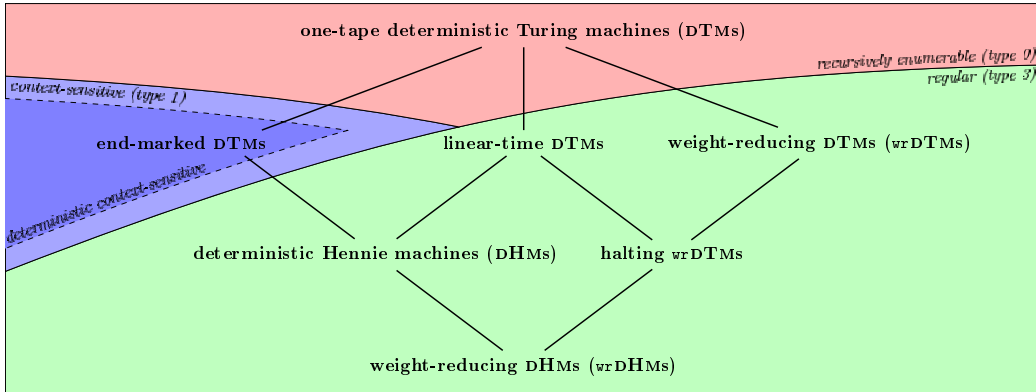


Figure 1: Variants of one-tape deterministic Turing machines and their expressive power confronted with the Chomsky hierarchy. In particular, end-marked DTMs are known as *deterministic linear bounded automata* in the literature, and recognize the so-called *deterministic context-sensitive languages*, a subclass of context-sensitive languages, see [16]. It is still unknown if such inclusion is strict.

stricted case of *end-marked machines*, namely one-tape deterministic Turing machines that do not have any extra space, besides the tape portion which initially contains the input. Deterministic end-marked machines working in linear time will be called *Hennie machines*.

To overcome the above-mentioned “negative” results, we consider a syntactical restriction on one-tape deterministic Turing machines, thus introducing *weight-reducing Turing machines*. This restriction aims to enforce the machine to work in linear time, by making any tape cell rewriting decreasing according to some fixed partial order on the working alphabet. However, due to the unrestricted amount of available tape space, these devices can have non-halting computations. Nevertheless, they work in linear time as soon as they are halting. Indeed, we prove that each computation either halts within a time which is linear in the input length, or is infinite. In the paper we show that it is possible to decide whether a weight-reducing Turing machine is halting. As a consequence, it is also possible to decide whether it works in linear time. Furthermore, with a polynomial size increase, any such machine can be made halting whence working in linear time. Our main result is that the tight size cost of converting a weight-reducing Turing machine into a DFA is double exponential. This cost reduces to a simple exponential when the target device is an NFA.

Considering end-marked Turing machines satisfying the weight-reducing makes at most $cm + d$ steps on input of length m , for any fixed $c, d > 0$ [1].

syntactical restriction, we obtain *weight-reducing Hennie machines*. These devices do not allow infinite computations whence always work in linear time. The above-stated double exponential blowup is easily extended to them.

The paper is organized as follows. Section 2 presents the fundamental notions and definitions, included those related to the computational models we are interested in. Section 3 is devoted to prove the above-mentioned undecidability and non-recursive trade-off results concerning Hennie machines. In Section 4, after proving that it can be decided if a deterministic Turing machine is weight-reducing, we describe a procedure that, given a linear-time machine together with the coefficient of its linear bound on time, turns it into an equivalent weight-reducing machine. Furthermore, we present a simulation of weight-reducing machines by finite automata, studying its size cost. In Section 5 we show how to decide if a weight-reducing machine halts on any input and if it works in linear time. We also prove that by a polynomial increase in size, each weight-reducing machine can be transformed into an equivalent one which always halts and which works in linear time.

2 Preliminaries

In this section we recall some basic definitions and notations. We also describe the main computational models considered in the paper.

We assume the reader familiar with notions from formal languages and automata theory (see, *e.g.*, [6]). Given a set S , $\#S$ denotes its cardinality and 2^S the family of all its subsets. Given an alphabet Σ , $|w|$ denotes the length of a string $w \in \Sigma^*$, w_i the i -th symbol of w , $i = 1, \dots, |w|$, and ε denotes the empty string.

The main computational model we consider is the *deterministic one-tape Turing machine* (DTM). Such a machine is a tuple $\langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ where Q is the *set of states*, Σ is the *input alphabet*, Γ is the *working alphabet* including symbols of Σ and the special *blank symbol*, denoted by \emptyset , that cannot be written by the machine, $q_0 \in Q$ is the *initial state*, $F \subseteq Q$ is the *set of final states*, and $\delta : Q \times \Gamma \rightarrow Q \times (\Gamma \setminus \{\emptyset\}) \times \{-1, +1\}$ is the partial *deterministic transition function*. In one step, depending on its current state p and on the symbol σ read by the head, a DTM changes its state to q , overwrites the corresponding tape cell with τ and moves the head one cell to the left or to the right according to $d = -1$ or $d = +1$, respectively, if $\delta(p, \sigma) = (q, \tau, d)$. Since δ is partial, it may happen that no transition can be applied. In this case, we say that the machine *halts*. At the beginning of computation the input string w resides on a segment of a bi-infinite tape, called *initial segment*, and the remaining infinitely many cells contain the blank symbol.

The computation over w starts in the initial state with the head scanning the leftmost symbol of w if $w \neq \varepsilon$ or a blank tape cell otherwise. The input is *accepted* if the machine eventually halts in a final state. The language accepted by a dTM \mathcal{A} is denoted by $L(\mathcal{A})$.

Let $\mathcal{A} = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ be a dTM. A *configuration* of \mathcal{A} is given by the current state q , the tape contents, and the position of the head. If the head is scanning a non-blank symbol, we describe it by zqu where $zu \in \Gamma^*$ is the finite non-blank contents of the tape, $u \neq \varepsilon$, and the head is scanning the first symbol of u . Otherwise, we describe it by $q\flat z$ or zq according to whether the head is scanning the first blank symbol to the left or to the right of the non-blank tape contents z , respectively. If the device may enter a configuration $z'q'u'$ from a configuration zqu in one step, we say that $z'q'u'$ is a *successor* of zqu , denoted $zqu \vdash z'q'u'$. A *halting configuration* is a configuration that has no successor. The reflexive and transitive closure of \vdash is denoted by \vdash^* . On an input string $w \in \Sigma^*$, the *initial configuration* is q_0w . An *accepting configuration* is a halting configuration zq_fu such that q_f is a final state of the machine. A *computation* is a (possibly infinite) sequence of successive configurations. It is *accepting* if it is finite, its first configuration is initial, and its last configuration is accepting. Therefore,

$$L(\mathcal{A}) = \{w \in \Sigma^* \mid q_0w \vdash^* zq_fu, \text{ where } q_f \in F \text{ and } zq_fu \text{ is halting}\}.$$

In the paper we consider the following restrictions of dTMs (see Figure 1).

End-marked machines. We say that a dTM is *end-marked*, if at the beginning of the computation the input string is surrounded by two special symbols belonging to Γ , \triangleright and \triangleleft respectively, called the *left* and the *right endmarkers*, which can never be overwritten, and that prevent the head to fall out the tape portion that initially contains the input. Formally, for each transition $\delta(p, \sigma) = (q, \tau, d)$, $\sigma = \triangleright$ (resp., $\sigma = \triangleleft$) implies $\tau = \sigma$ and $d = +1$ (resp., $d = -1$). This is the deterministic restriction of the well-known *linear-bounded automata* [7]. For end-marked machines, the initial configuration on input w is $q_0 \triangleright w \triangleleft$.

Weight-reducing Turing machines. A dTM is *weight-reducing* (*wr*dTM), if there exists a partial order $<$ on Γ such that each rewriting is decreasing, *i.e.*, $\delta(p, \sigma) = (q, \tau, d)$ implies $\tau < \sigma$. By this condition, in a *wr*dTM the number of visits to each tape cell is bounded by a constant. However, one *wr*dTM could have non-halting computations which, hence, necessarily visit infinitely many tape cells.

Linear-time Turing machines. A dTM is said to be *linear-time* if over each input w , its computation halts within $O(|w|)$ steps.

Hennie machines. A *Hennie machine* (DHM) is a linear-time DTM which is, furthermore, end-marked.

Weight-Reducing Hennie machines. By combining previous conditions, *weight-reducing Hennie machines* ($wrDHM$) are defined as particular DHM, for which there exists an order $<$ over $\Gamma \setminus \{\triangleright, \triangleleft\}$ such that $\delta(p, \sigma) = (q, \tau, d)$ implies $\tau < \sigma$ unless $\sigma \in \{\triangleright, \triangleleft\}$. Observe that each end-marked $wrDTM$ can execute a number of steps which is at most linear in the length of the input. Hence, end-marked $wrDTM$ are necessarily weight-reducing Hennie machines.

We also consider finite automata. We briefly recall their definition. A *non-deterministic finite automaton* (NFA) is a computational device equipped with a finite control and a finite read-only tape which is scanned by an input head in a one-way fashion. Formally, it is defined as a quintuple $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$, where Q is a finite *set of states*, Σ is a finite *input alphabet*, $q_0 \in Q$ is the *initial state*, $F \subseteq Q$ is a *set of final states*, and $\delta : Q \times \Sigma \rightarrow 2^Q$ is a *non-deterministic transition function*. At each step, according to its current state p and the symbol σ scanned by the head, \mathcal{A} enters one nondeterministically-chosen state from $\delta(p, \sigma)$ and moves the input head one position to the right. The machine *accepts* the input if there exists a computation starting from the initial state q_0 with the head on the leftmost input symbol, and ending in a final state $q \in F$ after having read the whole input with the head to the right of the rightmost input symbol. The language accepted by \mathcal{A} is denoted by $L(\mathcal{A})$. An NFA \mathcal{A} is said to be *deterministic* (DFA) whenever $\#\delta(q, \sigma) \leq 1$, for any $q \in Q$ and $\sigma \in \Sigma$.

The notions of configurations, successors, computations, and halting configurations, previously introduced in the context of DTMs, naturally transfer to NFAs.

The *size* of a machine is given by the total number of symbols used to write down its description. Therefore, the size of a one-tape Turing machine is bounded by a polynomial in the number of states and of working symbols. More precisely, the device is fully represented by its transition function which can be written in size $\Theta(\#Q \cdot \#\Gamma \cdot \log(\#Q \cdot \#\Gamma))$. In the case of NFAs (resp., DFAs), since no writings are allowed and hence the working alphabet is not provided, the size is linear in the number of instructions and states, which is bounded by a function quadratic (resp., subquadratic) in the number of states and linear in the number of input symbols. In this case, the description has size $\Theta(\#\Sigma \cdot \#Q^2)$ (resp., $\Theta(\#\Sigma \cdot \#Q \cdot \log \#Q)$).

3 Hennie Machines: Undecidability and Non-Recursive Trade-Offs

In this section we investigate some basic properties of DTMs. First of all, we prove that it cannot be decided whether an end-marked DTM works in linear time or not. As a consequence, it cannot be decided if a DTM is a Hennie machine. Since linear-time DTMs accept only regular languages [5], it is natural to investigate the size cost of their conversion into equivalent finite automata. Even in the restricted case of deterministic Hennie machines we obtain a “negative” result, by proving a non-recursive trade-off between the size of Hennie machines and that of the equivalent finite automata.

Let us start by proving the following undecidability result.

Theorem 1. *It is undecidable whether an end-marked DTM works in linear time.*

Proof. We show that the problem of deciding if a DTM halts on the empty word ε reduces to this problem. Let $\mathcal{T} = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ be a DTM. Without loss of generality, assume that \mathcal{T} has a tape infinite only to the right. Construct an end-marked Turing machine \mathcal{H} with the input alphabet $\{a\}$ as follows. Given an input $v \in a^*$, \mathcal{H} starts to simulate \mathcal{T} over ε . If, during the simulation, \mathcal{H} reaches the right endmarker, then it stops the simulation and performs additional $\Theta(|v|^2)$ computation steps.³ Otherwise, \mathcal{H} continues the simulation of \mathcal{T} and halts if \mathcal{T} halts. One can verify that the construction yields the following properties.

- If \mathcal{T} halts on ε in time t visiting s tape cells, then \mathcal{H} performs $O(t)$ computation steps on any input of length greater than s , while it performs $O(t^2)$ steps on shorter inputs. In both cases, the time is bounded by a constant in the input length.
- If \mathcal{T} does not halt on ε , then for any input v either the simulation reaches the right endmarker and then \mathcal{H} performs further $\Theta(|v|^2)$ computation steps, or it does not halt because \mathcal{T} enters an infinite loop,

³This can be achieved, for example, by a sequence of steps which overwrites every tape cell of the initial segment with a special marker $\# \notin \Gamma$, moving the head to the right endmarker after each rewriting. More precisely, from the cell containing the right endmarker, \mathcal{H} moves its head to the preceding cell and overwrites the contents with the special marker. After that, it moves its head to the right endmarker, then moves it backward to the rightmost cell not containing the special marker and writes $\#$, thus repeating the procedure until all tape cells but those containing the endmarkers have been overwritten with the special marker.

without reaching such a tape cell. In both cases \mathcal{H} is not a linear-time DTM.

This allows to conclude that \mathcal{H} is a linear-time DTM if and only if \mathcal{T} halts on input ε , which is known to be undecidable. \square

We now show that the size trade-off from linear-time DTM to finite automata is not recursive. More precisely, we obtain a non-recursive trade-off between the sizes of Hennie machines and finite automata.

Theorem 2. *There is no recursive function bounding the size blowup when transforming DHMs to finite automata.*

Proof. We recall that a busy beaver is an n -state deterministic Turing machine with a two-symbol working alphabet $\{\emptyset, 1\}$ that, starting its computation over a blank tape, halts after writing the maximum possible number $S(n)$ of 1's for its number of states. The function $S(n)$ of the space used by an n -state busy beaver is known to be non-recursive, *i.e.*, it grows asymptotically faster than any computable function [11].

Here we consider a modification of the busy beaver that operates on a semi-infinite tape (instead of bi-infinite) and starts the computation on the leftmost cell, according to [15]. This variant defines a different function $S(n)$ which is also non-recursive.

For each $n > 0$, let w_n be the string over $\{a\}$ of length $S(n)$ and let $L_n = \{w_n\}$. This language is accepted by an end-marked DTM \mathcal{H}_n with $O(n)$ states and $O(1)$ working tape symbols, which simulates a given n -state busy beaver (n -BB) and accepts an input $w \in a^*$ if and only if the space used by n -BB equals $|w|$. When n -BB uses more than $|w|$ space, at some point during the simulation the right endmarker is reached. At that point the simulation is aborted and the machine rejects. Furthermore, the simulation of n -BB does not depend on the input. Hence, it is made in constant time. This allows to conclude that, with respect to the input length, \mathcal{H}_n works in linear time, so it is a Hennie machine.

On the other hand, it is not difficult to see that the minimum DFA accepting L_n contains a path of $S(n) + 1$ states. This completes the proof. \square

4 Weight-Reducing Machines: Decidability, Expressiveness and Descriptive Complexity

In Section 3 we proved that it cannot be decided whether an end-marked DTM works in linear time. In this section we show that it is possible to

decide whether DTM's are weight reducing or not. Furthermore, every linear-time DTM \mathcal{T} with the length of each computation bounded by $Kn + C$, where K, C are constants and n denotes the input length, can be transformed into an equivalent weight-reducing machine whose size is bounded by a recursive function of K and the size of \mathcal{T} .

We also present a simulation of weight-reducing machines by finite automata, thus concluding that weight-reducing machines express exactly the class of regular languages. From such a simulation, we will obtain the size trade-off between weight-reducing machines and finite automata which, hence, is recursive. This contrasts with the non-recursive trade-off from Hennie machines to finite automata, proved in Section 3.

Proposition 1. *It is decidable whether a DTM is weight-reducing or not.*

Proof. Let $\mathcal{T} = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ be a DTM. To decide if there is any order $<$ on Γ proving that \mathcal{T} is weight-reducing, it suffices to check whether the directed graph $G = \langle \Gamma, E \rangle$, with

$$E = \{(\tau, \sigma) \mid \exists p, q \in Q \exists d \in \{-1, +1\} : \delta(p, \sigma) = (q, \tau, d)\},$$

is acyclic (each topological ordering of G acts as the required order $<$). \square

We now study how linear-time DTM's can be made weight-reducing. To this end, we use the fact that each DTM working in linear time makes a constant number of visits to each tape cell, hence linear time implies a constant number of visits per tape cell. This property is stated in the following lemma, which derives from [5, Proof of Theorem 3].

Lemma 1. *Let $\mathcal{T} = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ be a DTM. If there exist two constants K and C such that every computation of \mathcal{T} has length bounded by $Kn + C$, where n denotes the input length, then \mathcal{T} never visits a tape cell more than $2K \cdot (\#Q)^K + K$ times.*

The following lemma, which will be used in this section to study trade-offs between the computational models we are investigating and finite automata, presents a transformation from linear-time Turing and Hennie machines into equivalent weight-reducing ones.

Lemma 2. *Let $\mathcal{T} = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ be a DTM such that, for any input, \mathcal{T} performs at most k computation steps on each tape cell. Then there is a wrDTM \mathcal{A} accepting $L(\mathcal{T})$ with the same set of states Q as \mathcal{T} and working alphabet of size $O(k \cdot \#\Gamma)$. Furthermore, on each input \mathcal{A} uses the same space as \mathcal{H} . Hence, if \mathcal{T} is linear time or end-marked then so is \mathcal{A} .*

Proof. To obtain \mathcal{A} , we incorporate a counter into the working alphabet of \mathcal{T} . For each scanned cell, the counter says what is the maximum number of visits \mathcal{A} can perform during the remaining computation steps over the cell. More formally, denoting by $\Sigma_{\#}$ the alphabet $\Sigma \cup \{\#\}$, we define $\mathcal{A} = \langle Q, \Sigma, \Gamma', \delta', q_0, F \rangle$ with $\Gamma' = \Sigma_{\#} \cup ((\Gamma \setminus \Sigma_{\#}) \times \{0, \dots, k-1\})$ and, for all $q, q' \in Q$, $a, a' \in \Gamma$, $d \in \{-1, +1\}$ where $\delta(q, a) = (q', a', d)$, δ' fulfils

$$\begin{aligned} \delta'(q, a) &= (q', (a', k-1), d), & \text{if } a \in \Sigma_{\#}, \\ \delta'(q, (a, i)) &= (q', (a', i-1), d), & \text{otherwise, for } i = 1, \dots, k-1. \end{aligned}$$

Using an ordering $<$ on Γ' such that

$$\begin{aligned} (a, i) &< b & \text{for all } a \in (\Gamma \setminus \Sigma_{\#}), b \in \Sigma_{\#}, i = 0, \dots, k-1, \text{ and} \\ (a, i) &< (b, j) & \text{for all } a, b \in (\Gamma \setminus \Sigma_{\#}), i, j = 0, \dots, k-1, i < j, \end{aligned}$$

it is easy to see that \mathcal{A} is a wrDTM equivalent to \mathcal{T} . Furthermore, there is a natural bijection between computations of \mathcal{T} and those of \mathcal{A} , which preserves time (length of computations) and space (cells visited during the computation). Thus, if \mathcal{T} is linear time or end-marked, then so is \mathcal{A} . \square

By combining the above lemmas, we obtain a procedure to convert linear-time Turing machines into equivalent linear-time weight-reducing machines, as soon as a linear time bound of the input device is explicitly given.

Theorem 3. *Let $\mathcal{T} = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ be a dTM. If there exist two constants K and C such that every computation of \mathcal{T} has length bounded by $Kn + C$, where n denotes the input length, then there is an equivalent linear-time wrDTM with the same set of states Q as \mathcal{T} and working alphabet of size $O(k \cdot \#\Gamma)$, where $k = 2K \cdot (\#Q)^K + K$.*

Proof. Direct consequence of Lemmas 1 and 2. \square

We now investigate the transformation of weight-reducing machines into equivalent finite automata and its cost.

Theorem 4. *For every wrDTM $\mathcal{T} = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ there exist an NFA and a DFA accepting $L(\mathcal{T})$ with $2^{O(\#\Gamma \cdot \log(\#Q))}$ and $2^{2^{O(\#\Gamma \cdot \log(\#Q))}}$ states, respectively.*

Proof. Assume \mathcal{T} always ends each accepting computation with the head scanning a tape cell to the right of the initial segment. This can be obtained, at the cost of introducing one extra symbol in the working alphabet, by modifying the transition function in such a way that when \mathcal{T} enters a final state it starts to move its head to the right, ending when a blank cell is reached. Denote $n = \#Q$ and $m = \#\Gamma$.

We describe an NFA $\mathcal{A} = \langle Q', \Sigma, \delta', q_I, F' \rangle$ which accepts $L(\mathcal{T})$, working on the principle of guessing time-ordered sequences of states in which \mathcal{T} scans each of the tape cells storing the input, together with the input symbol of the cell. This is a variant of the classical crossing sequence argument. In this case, for a tape cell C , we consider the sequence of states in which the cell is scanned during a computation, while a crossing sequence is defined as the sequence of states of the machine when the border between two adjacent tape cells is crossed by the head.

Suppose that the time-ordered sequence of states in which a cell C is scanned in a computation ρ is (q_1, \dots, q_k) . Due to the weight-reducing property, there are k or $k + 1$ the different contents of C in ρ , depending on whether or not the computation stops in q_k . Since the working alphabet consists of m symbols, we can conclude that $k \leq m$.

The set of states Q' thus consists of a special initial state q_I , a special final state q_F , and all sequences of the form (a, q_1, \dots, q_k) where $a \in \Sigma \cup \{\emptyset\}$, $1 \leq k \leq m$, and $q_i \in Q$, for $i = 1, \dots, k$.

Let $w \in \Sigma^+$ be a non-empty input. Let τ_l , τ_{in} and τ_r denote the portion of \mathcal{T} 's tape which initially stores the blank symbols preceding w , the input w , and the blank symbols to the right of w , respectively. Let $\rho = (\mathcal{C}_0, \mathcal{C}_1, \dots)$ be the computation of \mathcal{T} over w . Let $q(j)$ denote the state of \mathcal{T} in configuration \mathcal{C}_j . Similarly, let $a(j)$ denote the symbol scanned by \mathcal{T} in \mathcal{C}_j . For a tape cell C in τ_{in} , let $\mathcal{C}_{j_1}, \dots, \mathcal{C}_{j_k}$, where $j_1 < \dots < j_k$, be the sequence of all configurations in which \mathcal{T} scans C . Observe that $a(j_1)$ and $q(j_1), \dots, q(j_k)$ determine $a(j_i)$ for all $i = 2, \dots, k$. For each configuration \mathcal{C}_{j_i} , it is also clear from which direction the head entered C and in which direction it moves out of it (\mathcal{C}_{j_1} is always entered from the left neighbouring cell, with the only exception of the initial configuration \mathcal{C}_1 , which is indicated by the initial state q_0 that is never re-entered; for $i > 1$, \mathcal{C}_{j_i} is entered from the opposite direction than $\mathcal{C}_{j_{i-1}+1}$ was entered). For this reason, we can determine for two neighbouring cells C_1 and C_2 of τ_{in} whether two sequences of states assigned to them are consistent with \mathcal{T} in the sense that the rightward head movements outgoing from C_1 have correspondent incoming leftward movements to C_2 and vice versa. Similarly, we can determine whether a sequence of states assigned to the first and to the last cell of τ_{in} is consistent with the computation of \mathcal{T} performed over the cell of τ_l and τ_r , respectively.

We are now going to formalize these ideas.

Given $(a, q_1, \dots, q_k), (b, p_1, \dots, p_\ell) \in Q'$, with $a, b \in \Sigma$, let:

- $a_0 = a$ and, for $i = 1, \dots, k$, $\delta(q_i, a_{i-1}) = (q'_i, a_i, d_i)$, $q'_i \in Q$, $a_i \in \Gamma$, $d_i \in \{-1, +1\}$;
- $b_0 = b$ and, for $i = 1, \dots, \ell$, $\delta(p_i, b_{i-1}) = (p'_i, b_i, e_i)$, $p'_i \in Q$, $b_i \in \Gamma$,

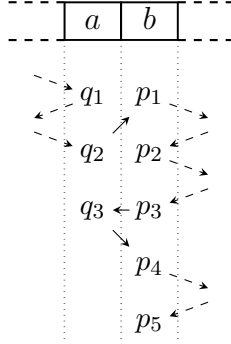


Figure 2: An example where (b, p_1, \dots, p_5) is consistent with (a, q_1, q_2, q_3) . Notice that $t = 3$, $i_1 = 2$, $i_2 = i_3 = 3$, $h_1 = 1$, $h_2 = 3$, $h_3 = 4$.

$$e_i \in \{-1, +1\}.$$

We say that (b, p_1, \dots, p_ℓ) is *consistent* with (a, q_1, \dots, q_k) when there are indices $i_1, i_2, \dots, i_t, h_1, h_2, \dots, h_t$, for some odd integer $t \geq 1$, with $1 \leq i_1 \leq i_2 \leq \dots \leq i_t = k$, $1 = h_1 \leq h_2 \leq \dots \leq h_t \leq \ell$, such that for $j = 1, \dots, t$ it holds that:

- if j is odd then $q'_{i_j} = p_{h_j}$, $d_{i_j} = +1$, and, when $j < t$, $i_j < i_{j+1}$,
- if j is even then $p'_{h_j} = q_{i_j}$, $e_{h_j} = -1$, and $h_j < h_{j+1}$,

while $d_i = -1$ for $i \notin \{i_1, \dots, i_t\}$, and $e_h = +1$ for $h \notin \{h_1, \dots, h_t\}$.

Notice that, for any odd j , in the transition from q_{i_j} to p_{h_j} the head crosses the border between the two adjacent cells by moving from left to right, while for any even j in the transition from p_{h_j} to q_{i_j} the head crosses the same border by moving in the opposite direction. Furthermore, the conditions t odd, $i_t = k$, $h_1 = 1$, derive from the fact that we are considering only sequences that could occur in accepting computations of \mathcal{T} . In such computations, each cell of the initial segment is entered from the left (with the exception of the leftmost one) and is finally left by moving the head to the right. (See Figure 2 for an example).

In a similar way, we are going to identify the sequences from Q' which can occur on the rightmost cell of the initial segment in an *accepting* computation (remember that we suppose that when τ reaches a final state it starts to move its head to the right, ending when a blank cell is reached). Using the above notations, we say that *the blank tape segment is consistent* with $(a, q_1, \dots, q_k) \in Q'$ when there are indices $1 \leq i_1 < i_2 < \dots < i_t = k$, and strings $\gamma_0 = \varepsilon, \gamma_1, \dots, \gamma_t \in \Gamma^*$, such that:

- $d_{i_j} = +1$, for $j = 1, \dots, t$, while $d_i = -1$ for $i \notin \{i_1, \dots, i_t\}$, namely, the indices i_j correspond to transitions moving the head to the right.
- For $j = 1, \dots, t - 1$, \mathcal{T} in the state q_{i_j} with the head scanning a tape cell C containing $a_{i_{j-1}}$ and the string γ_{j-1} written on the non-blank cells to the right of C , moves to the right and makes a finite sequence of moves, which ends when the cell C is re-entered. At this point the state $q_{i_{j+1}}$ and the string written on the non-blank cells to the right of C is γ_j .
- From $q_{i_t} = q_k$ the machine \mathcal{T} moves its head to the right and, at some point, reaches a blank cell in a final state, without re-entering the cell C in between.

Let us denote by Q'_R the set of states $(a, q_1, \dots, q_k) \in Q'$ such that the blank tape segment is consistent with (a, q_1, \dots, q_k) .

Finally, we are now going to identify the sequences from Q' which are consistent with τ_l , namely sequences that could occur, in accepting computations, on the tape cell which initially contains the leftmost input symbol.

Given $(a, q_1, \dots, q_k) \in Q'$, with $a \in \Sigma$ and, as before, $a_0 = a$ and, for $i = 1, \dots, k$, $\delta(q_i, a_{i-1}) = (q'_i, a_i, d_i)$, $q'_i \in Q$, $a_i \in \Gamma$, $d_i \in \{-1, +1\}$, let $1 \leq i_1 < i_2 < \dots < i_t < k$ be the indices corresponding to transitions moving the head to the left, *i.e.*, $\{i_1, i_2, \dots, i_t\} = \{i \mid d_i = -1\}$. We say that (a, q_1, \dots, q_k) is *consistent with the blank tape segment* when there are strings $\gamma_0 = \varepsilon, \gamma_1, \dots, \gamma_t \in \Gamma^*$, such that:

- q_1 is the initial state of \mathcal{T} ,
- for $j = 1, \dots, t$, \mathcal{T} in the state q_{i_j} with the head scanning a tape cell C containing $a_{i_{j-1}}$ and the string γ_{j-1} written on the non-blank cells to the left of C , moves to the left and makes a finite sequence of moves, up to re-enter C . At that point the state is $q_{i_{j+1}}$ and the string written on the non-blank cells to the left of C is γ_j .

Let Q'_L denote the set of states from Q' which are consistent with the blank tape segment.

At this point we developed all the tools to define the automaton $\mathcal{A} = \langle Q', \Sigma, \delta', q_I, F' \rangle$.

- The set of states is

$$Q' = \{q_I, q_F\} \cup \{(a, q_1, \dots, q_k) \mid a \in \Sigma \cup \{\emptyset\}, 1 \leq k \leq m+1, q_i \in Q, i = 1, \dots, k\}$$

as already mentioned.

- The initial state is q_I .
- The transition function is defined, for $(a, q_1, \dots, q_k) \in Q'$, $c \in \Sigma$, as

$$\delta'((a, q_1, \dots, q_k), c) = \begin{cases} \{(b, p_1, \dots, p_\ell) \mid (b, p_1, \dots, p_\ell) \text{ is} \\ \text{consistent with } (a, q_1, \dots, q_k)\}, & \text{if } c = a \text{ and } (a, q_1, \dots, q_k) \notin Q'_R \\ \{(b, p_1, \dots, p_\ell) \mid (b, p_1, \dots, p_\ell) \text{ is} \\ \text{consistent with } (a, q_1, \dots, q_k)\} \cup \{q_F\}, & \text{if } c = a \text{ and } (a, q_1, \dots, q_k) \in Q'_R \\ \emptyset, & \text{otherwise,} \end{cases}$$

The transitions from the initial state q_I are defined, for $a \in \Sigma$, as

$$\delta'(q_I, a) = \bigcup_{(a, q_1, \dots, q_k) \in Q'_L} \delta'((a, q_1, \dots, q_k), a),$$

while there are no transitions from q_F , namely $\delta'(q_F, a) = \emptyset$, for $a \in \Sigma$.

- The set of final states is:

$$F' = \begin{cases} \{q_F\} & \text{if } \varepsilon \notin L(\mathcal{T}) \\ \{q_I, q_F\}, & \text{otherwise.} \end{cases}$$

By summarizing, \mathcal{A} simulates \mathcal{T} as follows:

- In the initial state q_I , reading an input symbol a , \mathcal{A} implicitly guesses a sequence $(a, q_1, \dots, q_k) \in Q'_L$ and a sequence (b, p_1, \dots, p_ℓ) consistent with it, where b is supposed to be the symbol in the cell immediately to the right. The final state q_F can be also guessed, if $(a, q_1, \dots, q_k) \in Q'_R$,
- When scanning an input cell containing a symbol a , in a state $(a, q_1, \dots, q_k) \in Q'$, \mathcal{A} guesses a sequence (b, p_1, \dots, p_ℓ) consistent with it. If the next input symbol is b , then the simulation can continue in the same way, otherwise it stops because of an undefined transition.

Furthermore, when $(a, q_1, \dots, q_k) \in Q'_R$, \mathcal{A} can also guess to have reached the last input symbol, so entering the final state q_F . If the end of the input is effectively reached then \mathcal{A} accepts.

The number of states of \mathcal{A} is $2 + (\#\Sigma + 1) \sum_{i=1}^m n^i = 2^{O(m \log n)}$. If \mathcal{A} is in turn transformed to an equivalent DFA, using the classical powerset construction, the resulting automaton has $2^{2^{O(m \log n)}}$ states. \square

As a direct consequence of Theorem 4, we get that wrDTMs recognize exactly the class of regular languages.

Corollary 1. *A language is regular if and only if it is accepted by some wrDTM.*

Theorem 4 gives a double exponential upper bound for the size cost of the simulation of wrDTMs by DFAs. We now also prove a double exponential lower bound.

To this end, for each integer $n \geq 0$, we consider the language B_n over $\{0, 1, \$\}$ consisting of strings $v_1\$v_2\$ \cdots \v_k , where $k > 2$, $v_1, v_2, \dots, v_k \in \{0, 1\}^*$, $|v_k| \leq n$, $|v_i| \geq |v_k|$ for $i = 1, \dots, k-1$, and there exists $j < k$ such that $v_j = v_k$. Informally, every string in B_n is a sequence of binary blocks which are separated by the symbol $\$$, where the last block is of length at most n and it is a copy of one of the preceding blocks, which all are at least as long as the last one. For example,

$$v_1\$v_2\$v_3\$v_4\$v_5\$v_6 = 0011\$0101110\$011\$0011\$001\$011 \in B_4$$

since $|v_6| \leq 4$, $|v_i| \geq |v_6|$ for $i = 1, \dots, 5$, and $v_3 = v_6$.

Lemma 3. *For every integer $n \geq 0$, the language B_n is accepted by a wrDHM with $O(1)$ states and $O(n)$ working symbols.*

Proof. Let $\Sigma = \{0, 1, \$\}$. We first describe an end-marked dTM \mathcal{T} accepting the union of all B_i 's, for $i \geq 0$, that has a constant number of states, then we show how \mathcal{T} can be modified in order to recognize B_n , for a fixed integer n , by bounding the number of visits to each cell, thus obtaining a wrDHM with the desired properties. Let us define the working alphabet of \mathcal{T} as $\Gamma = \{0, 1, \$, x, f, \emptyset\}$.

Let $w \in \Sigma^*$ be an input string of the form $w = v_1\$v_2\$ \cdots \v_k , where $v_1, \dots, v_k \in \{0, 1\}^*$, and $v_k = a_1 \cdots a_\ell$, with $a_i \in \{0, 1\}$ for $i = 1, \dots, \ell$. The machine \mathcal{T} performs ℓ iterations. In each iteration it moves the head from the left end-marker to the right endmarker and back, thus visiting each input cell twice. It also rewrites some of the tape cells during this movement. The aim of the i -th iteration is comparing the i -th rightmost symbol of the last block with the i -th rightmost symbol of any other block. This is implemented as follows. Within the first iteration, \mathcal{T} memorizes a_ℓ in the states, rewrites it by the symbol x , and moves the head leftwards. Whenever it encounters the symbol $\$$ and enters the right end of a block v_j , it checks if its last symbol equals a_ℓ . If so, \mathcal{T} overwrites the cell contents with x , otherwise it writes f . During the i -th iteration, \mathcal{T} memorizes $a_{\ell+1-i}$ (which is in the rightmost input cell not containing the symbol x) in its finite control, overwrites the cell containing it by x and checks whether the i -th rightmost symbol of each v_j , with $j < k$, matches $a_{\ell+1-i}$ (if so, it overwrites the symbol with x , if not it writes f). Notice that, at the beginning of the i -th iteration, $i > 1$, the i -th

rightmost symbol of a block is located immediately to the left of a nonempty factor consisting only of symbols x and f . However, it could happen that for some $j < k$ there is no i -th rightmost symbol in the factor v_j , namely the block v_j is shorter than v_k . In this case the machine halts and rejects. The input w is accepted by \mathcal{T} if and only if, after some iteration, all symbols of v_k have been overwritten with x and there is some v_j with all symbols also rewritten to x (this ensures $v_j = v_k$). A constant number of states is sufficient to implement the procedure so far described.

It can be noticed that, for any fixed integer n , a word belongs to B_n if and only if it is accepted by \mathcal{T} within the first n iterations. Hence, as each iteration yields exactly two visits to each input cell, by bounding the number of visits to each cell by $2n$, we can restrict \mathcal{T} to accept words from B_n only. This can be obtained by using a construction similar as those used for proving Lemma 2. We thus obtain a halting wrDHM \mathcal{H} accepting B_n , which has $O(1)$ states and $O(n)$ working symbols. \square

Lemma 4. *Each DFA accepting B_n has at least 2^{2^n} states.*

Proof. Let \mathcal{S} be the family of all subsets of $\{0, 1\}^n$. Given a subset $S = \{w_1, \dots, w_k\} \in \mathcal{S}$, where $w_1 < \dots < w_k$ in the lexicographical order, consider the string $w(S) = w_1\$w_2\$ \dots \w_k . Let S_1 and S_2 be two different elements of \mathcal{S} and let $u \in \{0, 1\}^n$ be a string which is in S_1 but not in S_2 (or vice versa). Then, $w(S_1)\$u \in B_n$ and $w(S_2)\$u \notin B_n$ (or vice versa), hence $\$u$ is a distinguishing extension, and, by the Myhill-Nerode Theorem, each DFA accepting B_n has at least $\#\mathcal{S} = 2^{2^n}$ states. \square

From Theorem 4 and Lemmas 3 and 4, we obtain:

Corollary 2. *The size trade-offs from wrDTMs and wrDHMs to DFAs are double exponential.*

As shown in Theorem 2, by dropping the weight-reducing assumption for machines, the size trade-offs in Corollary 2 become not recursive. However, provided an explicit linear bound on computation lengths, we obtain the following result:

Corollary 3. *Let $\mathcal{T} = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ be a dTM. If there exist two constants K and C such that every computation of \mathcal{T} has length bounded by $Kn + C$, where n denotes the input length, then there exist an NFA and a DFA accepting $L(\mathcal{T})$ with $2^{O(k \cdot \#\Gamma \cdot \log(\#Q))}$ and $2^{2^{O(k \cdot \#\Gamma \cdot \log(\#Q))}}$ states, respectively, where $k = 2K \cdot (\#Q)^K + K$.*

Proof. Consequence of Theorems 3 and 4. \square

5 Weight-Reducing Machines: Space and Time Usage, Haltingness

Weight-reducing Turing machines generalize weight-reducing end-marked Turing machines (that are necessarily Hennie machines) by allowing to use additional tape cells that initially do not contain the input and to which we refer as *initially-blank cells*. This extension allows in particular infinite computations. For instance, a wrDTM can perform forward moves forever, rewriting each blank cell with some symbol. We now show that, however, due to the weight-reducing property, the amount of initially-blank cells that is really useful, *i.e.*, that is visited in some halting computation, is bounded by some constant which can be computed from the size of the wrDTM and does not depend on the input string. This allows us to transform any wrDTM into an equivalent halting one of polynomial size, which therefore operates in linear time. Notice that Theorem 4 already gave a simulation of wrDTMs by a halting and linear-time computational model.

Lemma 5. *Each computation of a wrDTM \mathcal{T} which visits in the same sequence of states two initially-blank cells of the tape, both located at the same side of the initial segment, is infinite.*

Proof. We give the proof in the case the two cells are located to the right of the initial segment. The proof for the other case can be obtained with a similar argument. For ease of exposition, we index the cell positions by integers, starting with the leftmost cell of the initial segment, whose index is 1, and we identify each cell with its position.

Let us consider a configuration \mathcal{C} of \mathcal{T} in which the head is scanning a tape cell c , located in the portion of the tape to the right of the initial segment, containing a symbol $a \in \Gamma$, the non-blank string written in the cells to the right of c , starting from the cell $c + 1$, is $\gamma \in (\Gamma \setminus \{\emptyset\})^*$,⁴ and the state is q . Thus $\mathcal{C} = zqa\gamma$ for some $z \in \Gamma^*$. Let us suppose that $\delta(q, a) = (q', a', d)$, with $q' \in Q$, $a' \in \Gamma$, and $d \in \{-1, +1\}$.

If $d = +1$, let us denote by $path_R(q, a\gamma)$ the longest computation path which starts in the configuration \mathcal{C} and, after \mathcal{C} , visits only cells to the right of c , possibly re-entering the cell c at the end. Notice that if the path does not re-enter the cell c then it could be infinite. Since \mathcal{T} is deterministic, we can observe the following facts:

- (1) The non-blank string which is written on the tape, starting from cell $c + 1$, after the execution of $path_R(q, a\gamma)$, if ending, only depends on q , a ,

⁴Notice that when the cell $c + 1$ contains \emptyset , each cell $c + h$ for $h > 0$ is not yet visited, hence $\gamma = \varepsilon$. For the same reason $a = \emptyset$ implies $\gamma = \varepsilon$.

and γ .

- (2) For any fixed integer $h > 0$, the finite sequence of states which are reached when the head visits the cell $c + h$ during $path_R(q, a\gamma)$, only depends on q , a , γ , and h . This is also true in the case $path_R(q, a\gamma)$ is infinite. Indeed, due to the fact that \mathcal{T} is weight reducing, each cell can be visited only a finite number of times.

These two facts will be now used in order to study a computation visiting c in configurations $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{k-1}$, $k \geq 1$. Let q_i, a_i, γ_i be the state, the symbol written in c and the non-blank contents of the cells to the right of c in configuration \mathcal{C}_i , respectively, $i = 0, \dots, k-1$. We are going to prove that, for each integer $h > 0$, the sequence P of states in which the cell $c + h$ is visited only depends on q_0, q_1, \dots, q_{k-1} and h .

Since the cell $c + h$ cannot be visited before the cell c , namely before configuration \mathcal{C}_0 , the sequence P can be decomposed as $P = P_1 P_2 \dots P_k$ where, for $i = 1, \dots, k$, P_i is the sequence of states which are reached when the head is visiting the cell $c + h$ after the configuration \mathcal{C}_{i-1} and, for $i < k$, before the configuration \mathcal{C}_i .

Let us start by proving the following claims, for $i = 1, \dots, k$:

(C1) P_i only depends on $q_{i-1}, a_{i-1}, \gamma_{i-1}$, and h ,

(C2) if $i < k$ then a_i and γ_i only depend on q_{i-1}, a_{i-1} and γ_{i-1} .

To this end, first we observe that when $i < k$, $\delta(q_{i-1}, a_{i-1}) = (q', a_i, d)$, for some state q' , $d \in \{-1, +1\}$. There are two possibilities:

- $d = +1$: in this case the cell c is re-entered in the state q_i , *i.e.*, q_i is the last state in $path_R(q_{i-1}, a_{i-1}\gamma_{i-1})$. Then γ_i is the string which is written in the cells to the right of c after executing $path_R(q_{i-1}, a_{i-1}\gamma_{i-1})$ and, by (1), it depends only on q_{i-1}, a_{i-1} , and γ_{i-1} . Furthermore, by (2), the (possibly empty) sequence P_i of states in which the cell $c + h$ is visited along this path depends only on $q_{i-1}, a_{i-1}, \gamma_{i-1}$, and h .
- $d = -1$: in this case, after a path which visits only cells to the left of c , the cell c is re-entered in state q_i . Since the contents of the cells to the right of c is not changed we have $\gamma_i = \gamma_{i-1}$. Furthermore, P_i is the empty sequence because the cell $c + h$ was not reached in this path.

Even for $i = k$, let $\delta(q_{k-1}, a) = (q', a', d)$. We also have two possibilities:

- $d = +1$: $path_R(q_{k-1}, a_{k-1}\gamma_{k-1})$ can be finite or infinite. By (2), the finite sequence P_k of states in which it visits $c + h$ only depends on $q_{k-1}, a_{k-1}, \gamma_{k-1}$, and h .
- $d = -1$: no more visits to the cell $c + h$ are performed. Hence P_k is the empty sequence.

This completes the proof of (C1) and (C2). Using these two statements and the fact that $a_0 = \emptyset$ and $\gamma_0 = \varepsilon$ are fixed, by proceeding in an inductive way, we obtain that P_i only depends on states q_0, q_1, \dots, q_{i-1} and on $h, i = 1, \dots, k$.

This allows us to conclude that the sequence $P = P_1 P_2 \dots P_k$ of states reached at the cell $c+h$ depends only on the sequence of states q_0, q_1, \dots, q_{k-1} reached at the cell c and on h , as we claimed.

Suppose now that P coincides with the sequence q_0, q_1, \dots, q_{k-1} of states visited at the cell c . By iterating the previous argument, the sequence of states which are reached in any cell $c+hj$ with $j > 0$ is q_0, q_1, \dots, q_{k-1} , thus implying the statement of the lemma. \square

Lemma 6. *Let \mathcal{T} be an n -state wrDTM which uses g working symbols. A computation of \mathcal{T} is infinite if and only if it visits $(n+1)^g$ consecutive initially-blank cells, i.e., tape cells to the left or to the right of the initial segment.*

Proof. Since \mathcal{T} is weight reducing, the number of visits to each tape cell is bounded by a constant which, in turn, is bounded by g . Thus, each infinite computation should visit infinitely many tape cells, hence at least $(n+1)^g$ consecutive initially-blank cells.

To prove the converse, let us consider a halting computation ρ of \mathcal{T} over an input word of length ℓ . By Lemma 5, ρ cannot visit two tape cells, laying at the same side of the initial segment, in the same sequence of states. Since there are less than $(n+1)^g$ nonempty distinct sequences of states of length at most g , we conclude that the number of consecutive initially-blank cells visited during the computation is less than $(n+1)^g$. \square

As a consequence of the above result, we obtain the following dichotomy of computations of wrDTMs.

Proposition 2. *Each computation of a wrDTM \mathcal{T} either is infinite and visits an infinite amount of tape cells, or is finite, has length linearly bounded in the input length, and visits at most C initially-blank cells, for some constant C which depends only on \mathcal{T} .*

Proof. Let \mathcal{T} be an n -state wrDTM having g working symbols, and let ρ be a computation of \mathcal{T} over some input w . If the amount of tape cells visited by ρ is k , for some finite k , then, as a cell cannot be visited more than g times by the weight-reducing property, ρ is finite and has length bounded by gk . Now, using Lemma 6 we have that ρ visits less than $(n+1)^g$ initially-blank cells to the left (resp., to the right) of the initial segment. Thus, $k < 2(n+1)^g - 1 + |w| \in O(|w|)$. Conversely, if ρ visits infinitely many tape cells then it is necessarily infinite. \square

Proposition 3. *By a polynomial size increase, each wrDTM can be transformed into an equivalent linear-time wrDTM.*

Proof. From an n -state wrDTM \mathcal{T} with a working alphabet of cardinality g , we can build an equivalent halting wrDTM \mathcal{T}' which works as follows. After an initial phase during which \mathcal{T}' marks $(n + 1)^g$ initially-blank cells to the left and to the right of the initial segment, it performs a direct simulation of \mathcal{T} while controlling that no further cells than those initially marked and those of the initial segment are used.

The initial phase is implemented using a counter in basis $(n + 1)$, stored on g consecutive tape cells, which is incremented up to $(n + 1)^g$ and shifted along the tape. We shall describe a procedure marking the $(n + 1)^g$ cells to the left of the initial segment. A similar procedure is repeated at the right of the initial segment.

At each step, the counter contains the number of marked cells minus one. Hence, at the beginning, it is initialized to value $g - 1$ (in basis $n + 1$) by writing the corresponding digits onto the g cells immediately to the left of the initial segment, the least significant digit being on the rightmost of these cells. Then the counter is incremented and shifted leftward by updating each digit, from right to left, namely starting from the least significant one. Let d be the digit scanned by the head. The value $d' = d + 1 \pmod{n + 1}$ is stored in the state control, along with a boolean variable c , for carry propagation. Then the head is moved one cell to the left, d' is written on the tape and its value updated according to the previous contents of the cell just overwritten and the value of c . After updating each of the g digits, the head is moved g positions to the right (on the least significant digit) and the counter is incremented again. This procedure stops when, moving rightward to reach the least significant digit, all the g scanned cells contain the symbol n . It is possible to notice that the number of states used to implement this procedure is $O(g + n)$. Furthermore, the number of visits to each initially-blank cell during this procedure is bounded by $2g$ because every cell is visited at most twice when containing the i -th digit of the current counter value, for $i = 1, \dots, g$. Thus, using Lemma 2, we can implement the procedure using a halting weight-reducing machine that uses $O(gn)$ symbols.

Once the space is marked, \mathcal{T}' simulates \mathcal{T} , stopping and rejecting if the simulation reaches a blank cell. Since \mathcal{T} is weight-reducing \mathcal{T}' is weight-reducing as well.

From Lemma 6, we can easily conclude that each halting computation of \mathcal{T} is simulated by an equivalent halting computation of \mathcal{T}' , while each infinite computation of \mathcal{T} is replaced in \mathcal{T}' by a computation which reaches a blank cell and then stops and rejects. Moreover, \mathcal{T}' uses $O(g + n)$ states

and $O(gn)$ working symbols. □

Using Lemma 6 and Proposition 3, we prove the following property.

Theorem 5. *It is decidable whether a wrDTM halts on each input string.*

Proof. From any given wrDTM \mathcal{T} , we construct a halting wrDTM \mathcal{T}' which, besides all the strings accepted by \mathcal{T} , accepts all the strings on which \mathcal{T} does not halt. To this end, we can slightly modify the construction used to prove Proposition 3, in such a way that when the head reaches a blank cell outside the initial segment and the initially marked space, the machine stops and accepts. Hence, the given wrDTM \mathcal{T} halts on each input string if and only if the finite automata which are obtained from \mathcal{T} and \mathcal{T}' according to Theorem 4 are equivalent. □

As a consequence:

Corollary 4. *It is decidable whether a wrDTM works in linear time.*

6 Conclusion

In this work, we investigated deterministic one-tape Turing machines working in linear time. Although these devices are known to be equivalent to finite automata [5], one cannot decide whether a given Turing machine is linear-time even in the case of end-marked devices, as we showed in Theorem 1. Furthermore, there is no recursive function bounding the size blowup of the conversion of DHMs into DFAs (Theorem 2). To avoid these negative results, we introduced the weight-reducing restriction, that forces one-tape Turing machines to work in linear time as long as they halt. Indeed, we proved that each computation of a wrDTM either is infinite or halts within a linear number of steps in the input length (Proposition 2). The weight-reducing restriction is syntactic and can be checked (Proposition 1). Furthermore, we proved in Theorem 4 that each wrDTM can be converted into an NFA (resp., DFA) whose size is exponential (resp., doubly-exponential) with respect to the size of the converted device. These costs are tight (Corollary 2).

Weight-reducing Turing machines are not restrictions of linear-time machines. Indeed, they allow infinite computations. However, the haltingness of wrDTMs on any input can be decided (Theorem 5). Furthermore, with a polynomial increase in size, each weight reducing machine can be made halting and linear-time (Proposition 3). Still, *halting* wrDTMs are not particular DHMs as they allow the use of extra space besides the initial segment contrary to DHMs which are end-marked. We do not know at the time of

writing whether this extra space usage is useful for concisely representing regular languages. In other words, we leave open the question of the size cost of turning wrDTMs into equivalent wrDHMs .

In a related paper [3], we continue the investigation of computational models considered here by focusing on the Sakoda and Sipser question about the size cost of the determinization of two-way finite automata [12]. We indeed propose a new approach to this famous open problem, which consists in converting two-way nondeterministic automata into equivalent deterministic extensions of two-way finite automata, paying a polynomial increase in size only. The considered extensions are variants of linear-time deterministic Turing machines, including wrDHMs and DHMs .

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