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Algebraic Entropy of a Class of Five-Point Differential-Difference Equations

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Abstract: We compute the algebraic entropy of a class of integrable Volterra-like five-point differential-difference equations recently classified using the generalised symmetry method. We show that, when applicable, the results of the algebraic entropy agrees with the result of the generalised symmetry method, as all the equations in this class have vanishing entropy.

Keywords: generalised symmetries; algebraic entropy; integrability

1. Introduction

One of the most important topics in modern mathematical physics is the study of the so-called *integrable systems*. Roughly speaking integrable systems are important both from a theoretical and a practical point of view as they can be regarded as *universal models* for physics going beyond the linear regime [1]. The birth of the modern theory of integrable systems is usually recognized in the seminal works of Zabusky and Kruskal [2], Gardner, Greene, Kruskal and Miura [3] and Lax [4] on the Korteweg-deVries (KdV) equation [5].

The concept of integrability come from classical mechanics and means the existence of a sufficiently high number of *first integrals*. To be more specific a Hamiltonian with a Hamiltonian $H = H(p, q)$ system with N degrees of freedom is said to be integrable if there exist N well defined functionally independent and Poisson-commuting first integrals [6,7]. We recall that a function is well-defined on the phase space if it is *analytic* and *single-valued*. In the case of systems with infinitely many degrees of freedom, e.g., partial differential equations like the KdV equation, the existence of *infinitely many conservation laws* is then required. One of the most efficient ways to find these infinitely many is the existence a so-called *Lax pair* [4]. A Lax pair is an associated *overdetermined* linear problem whose compatibility condition is guaranteed if and only if the desired non-linear equation is satisfied.

Currently, a purely algorithmic method to prove or disprove the existence of a Lax pair is not available, and so many *integrability detectors* have been developed. Integrability detectors are algorithmic procedure which are sufficient conditions for integrability, or alternative definitions of integrability. This means that integrability detectors can be used to prove the integrability of a given equation without the need of a Lax pair.

One of the fundamental integrability detectors, which works both at continuous and discrete level, is the *generalised symmetry approach*. The generalised symmetry approach was mainly developed by the scientific school of A. B. Shabat in Ufa during the 1980s and has obtained many important results in the classification of partial differential equations [8–14], differential-difference equations [15–19] and partial difference equations [20–23].

Another integrability detector is the algebraic entropy test. The algebraic entropy test is specific to systems with discrete degrees of freedom which can be written as bi-rational maps. A bi-rational map is a rational map of algebraic varieties $\varphi\colon V \to W$ such that there exists a rational map $\psi\colon W \to V$ which is the inverse of φ were both are defined. Given a bi-rational map, which can represent an ordinary difference equation, a differential-difference equation or even a partial difference equation, the basic idea of algebraic entropy is that, is to examine the growth of the degree of its iterates, and extract a canonical quantity, which is an index of complexity of the map. This canonical quantity is what is called the algebraic entropy. The idea of algebraic entropy as measure of the complexity of the growth of bi-rational maps comes from the notion of complexity introduced by Arnold in [24] and was discussed for the first time in relation of discrete systems by Veselov [25]. For bi-rational maps algebraic entropy is often *used as a definition* of integrability. This statement is usually called the *algebraic entropy conjecture* [26].

In this paper we will compute the algebraic entropy of some first order, five-point differential-difference equation of the form:

$$\begin{aligned}\frac{\mathrm{d}u_n}{\mathrm{d}t} &= A\left(u_{n+1},u_n,u_{n-1}\right)u_{n+2} + B\left(u_{n+1},u_n,u_{n-1}\right)u_{n-2} \\ &\quad + C\left(u_{n+1},u_n,u_{n-1}\right),\end{aligned} \qquad u_n = u_n\left(t\right),\, n\in\mathbb{Z}, t\in\mathbb{R}, \tag{1}$$

integrable according to the algebraic entropy criterion. To be more specific we will consider the algebraic entropy of the *only* differential-difference equations integrable according to the generalised symmetry criterion as classified in [18,19]. We will prove that all the bi-rational equations in the classification given in [18,19] are integrable according to the algebraic entropy criterion. That is we are going to prove that the algebraic entropy conjecture holds true for differential-difference equations of the form (1). The plan of the paper is the following: In Section 2 we introduce the explicit form of the equations we are going to study. Moreover, we will present a new *rational* form of a new equation obtained in [18] and we will compute its continuum limit. In Section 3 we will give some details on how algebraic entropy is computed, then in Section 4 we will show the results for the equations of Lists 1–6 except the discrete Kaup-Kupershmidt Equation (35). Finally, in Section 5 we will discuss the results obtained in Sections 4 in the framework of the existing literature and we will give an outlook on future research in the field.

2. Integrable Volterra-Like Five-Point Differential-Difference Equations

Differential-difference equations of the form (1) are called Volterra-like five-point differential-difference equations due to the similarity with the well-known three-point Volterra equation

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = u_n\left(u_{n+1}-u_{n-1}\right). \tag{2}$$

Throughout this paper we are going to consider only autonomous equations of the form (1). Therefore, we will make use of the short-hand notation $u_{n+k} = u_k$ to simplify the formulæ.

In [18,19] integrable Volterra-like five-point differential-difference equations have been classified using the existence of a nine-point generalised symmetry as classifying tool. The results of this classification are two classes of equations. Equations belonging to the first class satisfy the following conditions:

$$A \neq \alpha\left(u_1,u_0\right)\alpha\left(u_0,u_{-1}\right), \quad B \neq \beta\left(u_1,u_0\right)\beta\left(u_0,u_{-1}\right). \tag{3}$$

Equations such that condition (3) does not hold are elements of the second class. Equations of these two classes are then divided into six smaller lists, which we are going to present. Equations within each list are related to each other by autonomous non-invertible non-point transformations or by simple non-autonomous point transformations.

Explicitly, the equations we are going to consider are the following:

List 1. Equations related to the double Volterra equation:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0(u_2 - u_{-2}), \tag{4}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0^2(u_2 - u_{-2}), \tag{5}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_0^2 + u_0)(u_2 - u_{-2}), \tag{6}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_2 + u_1)(u_0 + u_{-1}) - (u_1 + u_0)(u_{-1} + u_{-2}), \tag{7}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_2 - u_1 + a)(u_0 - u_{-1} + a) + (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) + b, \tag{8}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_2 u_1 u_0(u_0 u_{-1} + 1) - (u_1 u_0 + 1)u_0 u_{-1} u_{-2} + u_0^2(u_{-1} - u_1), \tag{9}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0\left[u_1(u_2 - u_0) + u_{-1}(u_0 - u_{-2})\right], \tag{10}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_1 {u_0}^2 u_{-1}\left(u_2 - u_{-2}\right). \tag{11}$$

Transformations $\tilde{u}_k = u_{2k}$ or $\tilde{u}_k = u_{2k+1}$ turn Equations (4)–(6) into the well-known Volterra equation and its modifications in their standard form. The other equations are related to the *double Volterra Equation* (4) through some autonomous non-invertible non-point transformations. We note that Equation (11) was presented in [27].

List 2. Linearizable equations:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (T - a)\left[\frac{(u_1 + au_0 + b)(u_{-1} + au_{-2} + b)}{u_0 + au_{-1} + b} + u_0 + au_{-1} + b\right] + cu_0 + d, \tag{12}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = \frac{u_2 u_0}{u_1} + u_1 - a^2\left(u_{-1} + \frac{u_0 u_{-2}}{u_{-1}}\right) + cu_0. \tag{13}$$

In both equations $a \neq 0$, in (12) $(a+1)d = bc$, and T is the translation operator $Tf_n = f_{n+1}$.

Both equations of List 2 are related to the linear equation:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_2 - a^2 u_{-2} + \frac{c}{2}u_0 \tag{14}$$

through an autonomous non-invertible non-point transformations. We note that (12) is linked to (14) with a transformation which is implicit in both directions, see [18] for more details.

List 3. Equations related to a generalised symmetry of the Volterra equation:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0\left[u_1(u_2 + u_1 + u_0) - u_{-1}(u_0 + u_{-1} + u_{-2})\right] + cu_0\left(u_1 - u_{-1}\right), \tag{15}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_0^2 - a^2)\left[(u_1^2 - a^2)(u_2 + u_0) - (u_{-1}^2 - a^2)(u_0 + u_{-2})\right] + c(u_0^2 - a^2)\left(u_1 - u_{-1}\right), \tag{16}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_1 - u_0 + a)(u_0 - u_{-1} + a)(u_2 - u_{-2} + 4a + c) + b, \tag{17}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0[u_1(u_2 - u_1 + u_0) - u_{-1}(u_0 - u_{-1} + u_{-2})], \tag{18}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_0^2 - a^2)\left[(u_1^2 - a^2)(u_2 - u_0) + (u_{-1}^2 - a^2)(u_0 - u_{-2})\right], \tag{19}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_1 + u_0)(u_0 + u_{-1})(u_2 - u_{-2}). \tag{20}$$

These equations are related between themselves by some transformations, for more details see [19]. Moreover Equations (15)–(17) are the generalised symmetries of some known three-point autonomous differential-difference equations [28].

List 4. Equations of the relativistic Toda type:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_0 - 1)\left(\frac{u_2(u_1-1)u_0}{u_1} - \frac{u_0(u_{-1}-1)u_{-2}}{u_{-1}} - u_1 + u_{-1}\right), \tag{21}$$

$$\begin{aligned}\frac{\mathrm{d}u_0}{\mathrm{d}t} &= \frac{u_2 u_1^2 u_0^2 (u_0 u_{-1} + 1)}{u_1 u_0 + 1} - \frac{(u_1 u_0 + 1) u_0^2 u_{-1}^2 u_{-2}}{u_0 u_{-1} + 1} \\ &\quad - \frac{(u_1 - u_{-1})(2u_1 u_0 u_{-1} + u_1 + u_{-1}) u_0^3}{(u_1 u_0 + 1)(u_0 u_{-1} + 1)},\end{aligned} \tag{22}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_1 u_0 - 1)(u_0 u_{-1} - 1)(u_2 - u_{-2}). \tag{23}$$

Equation (23) was known [29,30] to be is a relativistic Toda type equation. Since in [18] it was shown that the equations of List 4 are related through autonomous non-invertible non-point transformations, it was suggested that (21) and (22) should be of the same type. Finally, we note that Equation (21) appeared in [31] earlier than in [18].

List 5. Equations related to the Itoh-Narita-Bogoyavlensky (INB) equation:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0(u_2 + u_1 - u_{-1} - u_{-2}), \tag{24}$$

$$\begin{aligned}\frac{\mathrm{d}u_0}{\mathrm{d}t} &= (u_2 - u_1 + a)(u_0 - u_{-1} + a) \\ &\quad + (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) \\ &\quad + (u_1 - u_0 + a)(u_0 - u_{-1} + a) + b,\end{aligned} \tag{25}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_0^2 + a u_0)(u_2 u_1 - u_{-1} u_{-2}), \tag{26}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_1 - u_0)(u_0 - u_{-1})\left(\frac{u_2}{u_1} - \frac{u_{-2}}{u_{-1}}\right), \tag{27}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0(u_2 u_1 - u_{-1} u_{-2}), \tag{28}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_1 - u_0 + a)(u_0 - u_{-1} + a)(u_2 - u_1 + u_{-1} - u_{-2} + 2a) + b, \tag{29}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0(u_1 u_0 - a)(u_0 u_{-1} - a)(u_2 u_1 - u_{-1} u_{-2}), \tag{30}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_1 + u_0)(u_0 + u_{-1})(u_2 + u_1 - u_{-1} - u_{-2}). \tag{31}$$

Equation (24) is the well-known INB equation [32–34]. Equations (25) with $a = 0$ and (26) with $a = 0$ are simple modifications of the INB and were presented in [35,36], respectively. Equation (26) with $a = 1$ has been found in [37]. Up to an obvious linear transformation, it is equation (17.6.24) with $m = 2$ in [37]. Equation (28) is a well-known modification of INB Equation (24), found by Bogoyalavlesky himself [32]. Finally, Equation (30) with $a = 0$ was considered in [27]. All the equations in this list can be reduced to the INB equation using autonomous non-invertible non-point transformations. Moreover, Equations (25), (27) and (28) are related through non-invertible transformations to the equation:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_2 - u_0)(u_1 - u_{-1})(u_0 - u_{-2}). \tag{32}$$

For this reason, as it was done in [23], we will consider Equation (32), as independent. We note that Equation (32) and its relationship with Equations (25), (27) and (28) were first discussed in [38].

List 6. Other equations:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_0^2(u_2u_1 - u_{-1}u_{-2}) - u_0(u_1 - u_{-1}), \tag{33}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_0+1)\times \left[\frac{u_2u_0(u_1+1)^2}{u_1} - \frac{u_{-2}u_0(u_{-1}+1)^2}{u_{-1}} + (1+2u_0)(u_1 - u_{-1})\right], \tag{34}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = (u_0^2+1)\left(u_2\sqrt{u_1^2+1} - u_{-2}\sqrt{u_{-1}^2+1}\right), \tag{35}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = u_1u_0^3u_{-1}(u_2u_1 - u_{-1}u_{-2}) - u_0^2(u_1 - u_{-1}). \tag{36}$$

Equation (33) has been found in [39] and it is called the discrete Sawada-Kotera equation [39,40]. Equation (36) is a simple modification of the discrete Sawada-Kotera Equation (33). Equation (34) has been found in [31] and is related to (33). On the other hand, Equation (35) has been found as a result of the classification in [18] and seems to be a new equation. It was shown in [41] that Equation (35) is a discrete analogue of the Kaup-Kupershmidt equation [42]. Then we will refer to Equation (35) as the discrete Kaup-Kupershmidt equation. No transformation into known equations of Equation (35) is known.

Before going on we would like to present a new *rational* form of the discrete Kaup-Kupershmidt Equation (35). That is we have the following proposition:

Proposition 1. *There exists a point transformation which brings the discrete Kaup-Kupershmidt Equation* (35) *into the following rational form:*

$$\frac{\mathrm{d}v_0}{\mathrm{d}t} = \left(1+v_0^2\right)\left[\frac{1+v_1^2}{1-v_1^2}\frac{v_2}{1-v_2^2} - \frac{1+v_{-1}^2}{1-v_{-1}^2}\frac{v_{-2}}{1-v_{-2}^2}\right]. \tag{37}$$

Proof. We start with the substitution:

$$u_n = \sinh(\varphi_n), \tag{38}$$

which brings the discrete Kaup-Kupershmidt equation in hyperbolic form (D. Levi, private communication):

$$\dot{\varphi}_0 = \cosh(\varphi_n)\left[\cosh(\varphi_{n+1})\sinh(\varphi_{n+2}) - \cosh(\varphi_{n-1})\sinh(\varphi_{n-2})\right]. \tag{39}$$

Using the hyperbolic identities:

$$\sinh\alpha = \frac{2\tanh(\alpha/2)}{1-\tanh^2(\alpha/2)}, \quad \cosh\alpha = \frac{1+\tanh^2(\alpha/2)}{1-\tanh^2(\alpha/2)}, \tag{40}$$

and putting

$$\tanh\left(\frac{\varphi_n}{2}\right) = v_n \tag{41}$$

Equation (37) follows. □

Remark 1. *We note that under the scaling:*

$$u_n(t) = \imath\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{8}\epsilon^2 U\left(\tau - \frac{2}{135}\epsilon^5 t, x + \frac{4}{9}\epsilon t\right)\right],\ x = n\varepsilon, \tag{42}$$

Equation (37) admits the Kaup-Kaupershmit equation as continuum limit:

$$U_\tau = U_{xxxxx} + 10UU_{xxx} + 25U_xU_{xx} + 20U^2U_x, \tag{43}$$

just as the original (35) equation.

3. Algebraic Entropy

In this section we introduce the basic theoretic and computational notions needed to compute the algebraic entropy of Volterra-like differential-difference equations. Our introduction to algebraic entropy is based on the original paper on the algebraic entropy of differential-difference equations [43], and on the review contained in [44,45]. The computational rules given in this section follow the ones given in [45,46].

Heuristically integrability deals with the regularity of the solutions of a given system. In this sense a simple characterisation of chaotic behaviour is when two arbitrarily near initial values give rise to solutions diverging at infinity. For recurrence relations, i.e., equations where the solution is given by iteration of a formula, we could just try to compute the iteration to extract information about integrability, even if we cannot solve the equation explicitly. However it is usually impossible to calculate explicitly these iterates by hand or even with any state-of-the-art formal calculus software, simply because the expressions one should manipulate are rational fractions of increasing degree of the various initial conditions. The complexity and size of the calculation make it impossible to calculate the iterates. An example to give an idea of how this kind of computation can become unmanageable is shown in [45].

It was nevertheless observed that "integrable" maps are not as complex as generic ones. This was done primarily experimentally, by an accumulation of examples, and later by the elaboration of the concept of *algebraic entropy* for difference equations [25,47–50]. In [51,52] the method was developed in the case of quad equations and then used as a classifying tool [26]. Finally in [43] the same concept was introduced for differential-difference equation and later [53] to the very similar case of differential-delay equations. For a more complete discussion of the method in the context of the so-called integrability indicators we refer to [44,45].

As we stated in the introduction algebraic entropy is a measure of the growth of bi-rational maps with respect to general initial data. This kind of growth is not to be confused with other asymptotic properties of maps, like asymptotic stability, limit cycles and attractors. The most natural space for considering bi-rational maps is the projective space over a closed field rather than in the affine space one. We then transform a recurrence relation into a polynomial map in the homogeneous coordinates of the proper projective space over some closed field:

$$\varphi\colon x_i \mapsto \varphi_i\left(x_k\right), \tag{44}$$

with $x_i, x_k \in \mathcal{IN}$ where $\mathcal{IN}$ is the space of the initial conditions. The recurrence is then obtained by iterating the polynomial map φ. The map φ has to be *bi-rational* in the sense that it has to possesses an inverse map which is again a rational map.

The space of the initial condition depends on which type of recurrence relation we are considering. In the case of differential-difference equation of the discrete $k-k'$-th order and of the p-th continuous order:

$$u_{n+k} = f_n\left(\left\{\frac{\mathrm{d}^i u_{n+k-1}}{\mathrm{d}t^i},\dots,\frac{\mathrm{d}^i u_{n+k'+1}}{\mathrm{d}t^i}\right\}_{i=0}^{p};u_{n+k'}\right), \quad k',k,n\in\mathbb{Z},\, k'<k \tag{45}$$

the space of initial conditions is infinite dimensional. Indeed, in the case the order of the equation is $k-k'$, we need the initial value of $k-k'$-tuple as a function of the parameter t, but also the value of *all its derivatives*:

$$\mathcal{IN} = \left\{\frac{\mathrm{d}^i u_{k-1}}{\mathrm{d}t^i},\frac{\mathrm{d}^i u_{k-2}}{\mathrm{d}t^i},\dots,\frac{\mathrm{d}^i u_{k'}}{\mathrm{d}t^i},\right\}_{i\in\mathbb{N}_0}. \tag{46}$$

We need all the derivatives of $u_i(t)$ and not just the first p because at every iteration the order of the equation is raised by p. Therefore, to describe infinitely many iterations we need infinitely many derivatives. To obtain the map one only needs to pass to homogeneous coordinates in the equation and in (46).

Remark 2. *We remark that if we restrict to computing a finite number of iterates of a differential-difference equation of discrete $k-k'$-th order and of the p-th continuous order* (45) *then the space of initial conditions is finite dimensional. Indeed, let us assume that we wish to compute the Nth iterate of a differential-difference Equation* (45)*, then at most we will need the derivatives of order $N(p+1)$. That is, we need to consider the following restricted space of initial conditions:*

$$\mathcal{IN}^{(N)} = \left\{\frac{\mathrm{d}^i u_{k-1}}{\mathrm{d}t^i},\frac{\mathrm{d}^i u_{k-2}}{\mathrm{d}t^i},\dots,\frac{\mathrm{d}^i u_{k'}}{\mathrm{d}t^i},\right\}_{i=0}^{N(p+1)}. \tag{47}$$

If we factor out any common polynomial factors we can say that the degree with respect to the initial conditions is well defined. We can therefore form the sequence of degrees of the iterates of the map φ and call it $d_N = \deg\varphi^N$:

$$1, d_1, d_2, d_3, d_4, d_5, \dots, d_N, \dots. \tag{48}$$

The degree of the bi-rational projective map φ has to be understood as the *maximum of the total polynomial degree in the initial conditions $\mathcal{IN}$* of the entries of φ. The same definition in the affine case just translates to the *maximum of the degree of the numerator and of the denominator* of the Nth iterate in terms of the affine initial conditions. Degrees in the projective and in the affine setting can be different, but the global behaviour will be the same due to the properties of homogenization and de-homogenization.

The sequence of degree (48) is fixed in a given system of coordinates, but it is not invariant with respect to changes of coordinates. Therefore we need to introduce a canonical measure of the growth. It turn out that a good definition is the following one: Consider the following number

$$\eta_\varphi = \lim_{N\to\infty}\frac{1}{N}\log d_N, \tag{49}$$

called the *algebraic entropy* of the map φ. When no confusion is possible about the map φ we will usually omit the subscript φ in (49).

Algebraic entropy for bi-rational maps has the following properties [44,45,47]:

1. The algebraic entropy as given by (49) always exists.

2. The algebraic entropy has the following upper bound:

$$\eta_{\varphi} \leq \log \deg \varphi. \tag{50}$$

3. If $\eta_{\varphi} = 0$, i.e., the algebraic entropy is zero, then

$$d_N \sim N^{\nu}, \quad \text{with } \nu \in \mathbb{N}_0, \text{ as } N \to \infty. \tag{51}$$

4. The algebraic entropy is a *bi-rational invariant of bi-rational maps*. That is, if two bi-rational maps φ and ψ are conjugated by a bi-rational map χ,

$$\varphi = \chi \circ \psi \circ \chi^{-1} \tag{52}$$

then:

$$\eta_{\varphi} = \eta_{\psi}. \tag{53}$$

Properties 1 and 2 tell us that the definition of algebraic entropy is well posed, as it always exists and that its value cannot exceed the logarithm of the degree of the map itself. Property 3 gives us the characterisation of the maps with zero algebraic entropy, that is it states that maps with zero algebraic entropy have polynomial growth. Property 4 tells us that the algebraic entropy is a canonical measure of growth for bi-rational maps.

We will then have the following classification of equations according to their Algebraic Entropy [26]:

Linear growth: The equation is linearizable.
Polynomial growth: The equation is integrable.
Exponential growth: The equation is chaotic.

In our the following sections we will be dealing with differential-difference equations of first continuous order and fourth discrete order of the particular form:

$$u_{n+2} = f\left(u_{n+1}, u_n, u_{n-1}, u_{n-2}, \frac{\mathrm{d}u_n}{\mathrm{d}t}\right). \quad n \in \mathbb{Z}, \tag{54}$$

To practically compute the algebraic entropy we introduce some technical methods to reduce the computational complexity [45,46]. First, we fix the desired number of iterations to be some fixed $N \in \mathbb{N}$. Following Remark 2 this means that we need only finitely many initial conditions given by (47). Then we assume that the space of initial conditions is *linearly parametrised* in the appropriate projective space, i.e., in inhomogenous coordinates it has the following form:

$$u_i = \frac{\alpha_i t + \beta_i}{\alpha_0 t + \beta_0}, \quad u_i \in \mathcal{IN}^{(N)}. \tag{55}$$

We will assume that the parameter t is the same which describes the "time" evolution of the problem. To simplify the problem we choose all the parameters involved in the equations to be integers. Moreover, to avoid accidental factorisations which may alter the results we choose these integers to be *prime numbers*. A final simplification to speed up the computations is given by considering the factorisation of the iterates in some finite field $\mathbb{K}_r$, with r prime number.

Remark 3. *Several equations in Lists 1–6, e.g., (8) or (17), depend on some parameters. Depending on the value of the parameters their integrability properties can be, in principle, different. As was done in [23], in order* *to avoid ambiguities, we use some simple autonomous transformations to fix the values of some parameters. The remaining free parameters are then treated as free coefficients and then fixed to integers following the above discussion. We will describe these subcases when needed in the next section.*

Using the rules above we are able to avoid accidental cancellations and produce a finite sequence of degrees:

$$1, d_1, d_2, d_3, d_4, d_5, \ldots, d_N. \tag{56}$$

A standard way to extract the asymptotic behaviour from a finite sequence like (56) is to compute its *generating function*. A generating function is a function $g = g(s)$ such that the coefficients of its Taylor series

$$g(z) = \sum_{l=0}^{\infty} d_l z^l \tag{57}$$

up to order N coincides with the finite sequence (56). Generating functions are ubiquitous objects in mathematical sciences which have application is statistics [54], combinatorics [55], orthogonal polynomial theory [56] and networks [57,58].

If a generating function is rational, as is safe to assume, it can be computed exactly using a finite number of iterates using the method of Padé approximants [59,60].

Once obtained a generating function is *predictive* tool. Indeed one can readily compute the successive terms in the Taylor expansion for (57) and confront them with the degrees calculated with the iterations. This means that the assumption that the value of the algebraic entropy given by the approximate method is in fact very strong and very unlikely the real value will differ from it.

Having a rational generating function will also yield the value of the Algebraic Entropy from the modulus of the smallest pole of the generating function:

$$\eta_\varphi = \log \min \left\{ |z| \in \mathbb{R}^+ \;\middle|\; \frac{1}{g(z)} = 0 \right\}. \tag{58}$$

From the generating function one can also find an asymptotic fit for the degrees (56). This can be done by using the inverse $\mathcal{Z}$-transform [61,62]. Assume we are given a function $f = f(\zeta)$ of a complex variable $\zeta \in \mathbb{C}$ analytic in a region $|\zeta| > r$ for some $r \in \mathbb{R}^+$. We define its inverse $\mathcal{Z}$-transform of such function f to be the sequence:

$$\mathcal{Z}^{-1}[f(\zeta)]_l \equiv \frac{1}{2\pi\imath} \oint_C f(\zeta)\, \zeta^{l-1} \mathrm{d}\zeta, \quad l \in \mathbb{N}. \tag{59}$$

In Equation (59) the contour $C \subset \mathbb{C}$ is a counterclockwise closed path enclosing the origin and entirely in the region of convergence of f. From the definition of inverse $\mathcal{Z}$-transform (59) it can be readily proved that the sequence $\{d_l\}_{l\in\mathbb{N}}$ corresponding to the generating function (57) is given by:

$$d_l = \mathcal{Z}^{-1}\left[g\left(\frac{1}{\zeta}\right) \right]_l. \tag{60}$$

We note that the general asymptotic behaviour of the sequence $\{d_l\}_{l\in\mathbb{N}_0}$ can be obtained even without computing the inverse $\mathcal{Z}$-transform. This is the content of the following proposition:

Proposition 2 ([63])**.** *Assume that a sequence $\{d_l\}_{l\in\mathbb{N}_0}$ possesses a generating function of radius of convergence $\rho > 0$ and of the following form:*

$$g = A(z) + B(z) \left(1 - \frac{z}{\rho}\right)^{-\beta}, \quad \beta \in \mathbb{R} \setminus \{-n\}_{n\in\mathbb{N}}. \tag{61}$$

where A and B are analytic functions for $|z| < r$ such that $B(\rho) \neq 0$. Then the asymptotic behaviour of the sequence $\{d_l\}_{l\in\mathbb{N}_0}$ as $l \to \infty$ is given by:

$$d_l \sim \frac{B(\rho)}{\Gamma(\beta)} l^{\beta-1}\rho^{-l}, \quad l \to \infty, \tag{62}$$

where $\Gamma(z)$ is the Euler Gamma function. If additionally $\rho \equiv 1$, then

$$d_l \sim l^{\beta-1}, \quad l \to \infty, \tag{63}$$

i.e., the growth is asymptotically polynomial of degree $\beta - 1$.

The interested reader may found a proof of Proposition 2 in the case of rational generating functions, i.e., $\beta \in \mathbb{N}$ in Appendix A.

4. Results

In this section we describe the results of the procedure outlined in Section 3 for the differential-difference equations of Lists 1–6. Specifically, as described in Remark 3, we will underline the particular cases in which the parametric equations can be divided. We notice that certain equations are *symmetric* under the involution

$$u_n \to \tilde{u}_n = u_{-n}. \tag{64}$$

This implies that the recurrence defined by solving the equation with respect to u_2 and u_{-2} is the same. For equations satisfying this property the growth of the degree of the iterates can be computed just in one direction, as the growth in the other direction will be the same. Computations are performed using the `python` program for differential-difference equations presented in [46]. We remark that this program was already employed to discuss the integrability of some three-point differential-difference equations in [64].

4.1. List 1

4.1.1. Equation (4)

Equation (4) is symmetric and has the following growth of degrees:

$$1,2,2,4,4,7,7,11,11,16,16,22,22\ldots. \tag{65}$$

The generating function corresponding to the growth (65) is:

$$g(z) = -\frac{z^4 - 2z^2 + z + 1}{(z-1)^3(z+1)^2}. \tag{66}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (4) has quadratic growth.

4.1.2. Equation (5)

Equation (5) is symmetric and has the following growth of degrees:

$$1,3,3,7,7,13,13,21,21,31,31,43,43\ldots. \tag{67}$$

The generating function corresponding to the growth (67) is:

$$g(z) = -\frac{z^4 - 2z^2 + 2z + 1}{(z-1)^3(z+1)^2}. \tag{68}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (5) has quadratic growth.

4.1.3. Equation (6)

Equation (6) is symmetric and has the same growth of degrees as Equation (5). Therefore we have that Equation (6) has zero entropy and quadratic growth.

4.1.4. Equation (7)

Equation (7) is symmetric and has the following growth of degrees:

$$1, 2, 3, 5, 6, 8, 10, 14, 16, 20, 23, 29, 32, 38, 42, 50, 54 \dots . \tag{69}$$

The generating function corresponding to the growth (69) is:

$$g(z) = -\frac{z^7 + z^6 - z^5 - z^4 + z^3 + z + 1}{(z-1)^3(z+1)^2(z^2+1)}. \tag{70}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (7) has quadratic growth.

4.1.5. Equation (8)

Equation (8) depends on the parameter a. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. If $a = 1$ Equation (8) is asymmetric, but it has the following growth of degrees in both directions:

$$\begin{aligned} &1, 2, 3, 5, 6, 8, 10, 14, 16, 20, 23, 29, 32, 38, \\ &\quad 42, 50, 54, 62, 67, 77, 82, 92, 98, 110, 116 \dots . \end{aligned} \tag{71}$$

The generating function corresponding to the growth (71) is:

$$g(z) = -\frac{z^7 + z^6 - z^5 - z^4 + z^3 + z + 1}{(z-1)^3(z+1)^2(z^2+1)}. \tag{72}$$

If $a = 0$ Equation (8) is symmetric, but its growth of degrees is still given by the sequence (71) and fitted by the generating function (72). Therefore in both cases the entropy is zero since all the poles of g lie on the unit circle. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (8) has quadratic growth for all values of a.

4.1.6. Equation (9)

Equation (9) is symmetric and has the following growth of degrees:

$$1, 5, 8, 14, 19, 28, 35, 47, 56, 71, 82, 100, 113 \dots . \tag{73}$$

The generating function corresponding to the growth (73) is:

$$g(z) = -\frac{z^5 - 2z^3 + z^2 + 4z + 1}{(z-1)^3(z+1)^2}. \tag{74}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (9) has quadratic growth.

4.1.7. Equation (10)

Equation (10) is symmetric and has the following growth of degrees:

$$1, 3, 4, 6, 8, 12, 15, 19, 24, 29, 34, 40, 47, 54, 61, 69, 78, 87, 96, 106, 117 \dots . \tag{75}$$

The generating function corresponding to the growth (75) is:

$$g(z) = -\frac{z^9 - 3z^8 + 4z^7 - 4z^6 + 4z^5 - 3z^4 + 2z^3 - z^2 + 1}{(z-1)^3(z^2+1)}. \tag{76}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (10) has quadratic growth.

4.1.8. Equation (11)

Equation (11) is symmetric and has the following growth of degrees:

$$\begin{gathered} 1, 5, 7, 13, 18, 27, 34, 45, 54, 69, 80, 97, 110, \\ 131, 146, 169, 186, 213, 232, 261, 282 \dots . \end{gathered} \tag{77}$$

The generating function corresponding to the growth (77) is:

$$g(z) = -\frac{z^9 - z^8 + z^6 - z^5 + 2z^4 + 2z^3 + z^2 + 4z + 1}{(z-1)^3(z+1)^2(z^2+1)}. \tag{78}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (11) has quadratic growth.

4.2. List 2

4.2.1. Equation (12)

Equation (12) depends on four parameters a, b, c and d linked among themselves by the condition $(a+1)\,d = bc$. Using a linear transformation $u_{n,m} \to \alpha u_{n,m} + \beta$ we need to consider only three different cases:

1. $a \neq 0, a \neq -1, b = 0, d = 0,$
2. $a = -1, b = 1, c = 0,$
3. $a = -1, b = 0.$

Recall that $a \neq 0$ in all cases. See [23] for more details. In all the three cases Equation (12) is asymmetric. However, it has the same growth of degrees in both directions and in all the three cases:

$$1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37 \dots . \tag{79}$$

The generating function corresponding to the growth (79) is:

$$g(z) = -\frac{z^7 + z^6 - z^5 - z^4 + z^3 + z + 1}{(z-1)^3(z+1)^2(z^2+1)}. \tag{80}$$

In all cases the entropy is zero since all the poles of g lie on the unit circle. Moreover, due to the presence of the factor $(z-1)^2$ following Proposition 2 we have that Equation (12) has linear growth for all values of the parameters.

4.2.2. Equation (13)

Equation (13) is not symmetric, but in both directions has the following growth of degrees:

$$1, 4, 6, 9, 11, 14, 16, 19, 21, 24, 26, 29, 31 \ldots. \tag{81}$$

The generating function corresponding to the growth (81) is:

$$g(z) = \frac{z^2 + 3z + 1}{(z-1)^2(z+1)}. \tag{82}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^2$ following Proposition 2 we have that Equation (13) has linear growth.

4.3. List 3

4.3.1. Equation (15)

Equation (15) is symmetric and has the following growth of degrees:

$$1, 3, 6, 10, 16, 22, 29, 37, 46, 56, 67, 79, 92 \ldots. \tag{83}$$

The generating function corresponding to the growth (83) is:

$$g(z) = -\frac{z^6 - 2z^5 + z^4 + 1}{(z-1)^3}. \tag{84}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (15) has quadratic growth.

4.3.2. Equation (16)

Equation (16) depends on the parameter a. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. Equation (16) is symmetric for both $a = 1$ and $a = 0$. Moreover, in both cases it has the following growth of degrees:

$$1, 5, 11, 21, 31, 43, 57, 73, 91, 111, 133, 157, 183 \ldots. \tag{85}$$

The generating function corresponding to the growth (85) is:

$$g(z) = -\frac{2z^5 - 4z^4 + 2z^3 - z^2 + 2z + 1}{z-1)^3}. \tag{86}$$

Therefore in both cases the entropy is zero since all the poles of g lie on the unit circle. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (16) has quadratic growth for all values of a.

4.3.3. Equation (17)

Equation (17) is not symmetric, but it has the same growth of degrees in both directions:

$$1, 3, 4, 7, 10, 15, 19, 25, 31, 39, 46, 55, 64 \ldots. \tag{87}$$

The generating function corresponding to the growth (87) is:

$$g(z) = -\frac{z^5 - z^4 + 2z^3 - z^2 + z + 1}{(z-1)^3(z+1)(z^2+1)}. \tag{88}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (17) has quadratic growth.

4.3.4. Equation (18)

Equation (18) is symmetric and has the same growth of degrees as Equation (15). Therefore we have that Equation (18) has zero entropy and quadratic growth.

4.3.5. Equation (19)

Equation (19) depends on the parameter a. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. Equation (19) is symmetric for both $a = 1$ and $a = 0$. However, in both cases Equation (19) has the same growth of degrees as Equation (16). Therefore we have that Equation (19) has zero entropy and quadratic growth for all values of a.

4.3.6. Equation (20)

Equation (20) is symmetric and has the following growth of degrees:

$$1, 3, 4, 7, 10, 15, 19, 25, 31, 39, 46, 55, 64 \dots . \tag{89}$$

The generating function corresponding to the growth (89) is:

$$g(z) = -\frac{z^5 - z^4 + 2z^3 - z^2 + z + 1}{(z-1)^3(z+1)(z^2+1)}. \tag{90}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (20) has quadratic growth.

4.4. List 4

4.4.1. Equation (21)

Equation (21) is symmetric and has the following growth of degrees:

$$1, 5, 10, 16, 26, 38, 51, 65, 82, 102, 123, 145, 170, 198, 227, 257 \dots . \tag{91}$$

The generating function corresponding to the growth (91) is:

$$g(z) = -\frac{z^8 - 2z^7 + 2z^6 - 2z^5 + z^4 + 2z^3 - z^2 + 2z + 1}{(z-1)^3(z^2+1)}. \tag{92}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (21) has quadratic growth.

4.4.2. Equation (22)

Equation (22) is symmetric and has the following growth of degrees:

$$\begin{gathered} 1, 9, 19, 37, 55, 75, 101, 129, 163, 199, 237, \\ 281, 327, 379, 433, 489, 551, 615, 685, 757 \dots . \end{gathered} \tag{93}$$

The generating function corresponding to the growth (93) is:

$$g(z) = -\frac{2z^9 - 2z^8 - z^6 + z^5 + 8z^3 + 2z^2 + 7z + 1}{(z-1)^3(z^4 + z^3 + z^2 + z + 1)}. \tag{94}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (22) has quadratic growth.

4.4.3. Equation (23)

Equation (23) is symmetric and has the following growth of degrees:

$$1, 5, 7, 13, 19, 29, 37, 49, 61, 77, 91, 109, 127 \ldots . \tag{95}$$

The generating function corresponding to the growth (95) is:

$$g(z) = -\frac{z^5 - z^4 + 4z^3 - 2z^2 + 3z + 1}{(z-1)^3(z+1)(z^2+1)}. \tag{96}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (23) has quadratic growth.

4.5. List 5

4.5.1. Equation (24)

Equation (24) is symmetric and has the following growth of degrees:

$$1, 2, 3, 4, 6, 8, 10, 13, 16, 19, 23, 27, 31 \ldots . \tag{97}$$

The generating function corresponding to the growth (97) is:

$$g(z) = -\frac{z^4 - z^3 + 1}{(z-1)^3(z^2+z+1)}. \tag{98}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (24) has quadratic growth.

4.5.2. Equation (25)

Equation (25) depends on the parameter a. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. If $a = 1$ Equation (25) is asymmetric, but it has the following growth of degrees in both directions:

$$\begin{gathered} 1, 2, 2, 4, 5, 7, 8, 11, 12, 16, 18, 22, 24, 30, 31, \\ 38, 41, 47, 50, 59, 60, 70, 74, 82, 86, 98, 99 \ldots . \end{gathered} \tag{99}$$

The generating function corresponding to the growth (99) is:

$$g(z) = -\frac{z^{13} + z^{10} + z^9 - z^7 + 2z^5 + z^4 + z^3 + z^2 + 2z + 1}{(z-1)^3(z+1)^2(z^2-z+1)(z^2+z+1)^2}. \tag{100}$$

If $a = 0$ Equation (25) is symmetric, but its growth of degrees is still given by the sequence (99) and fitted by the generating function (100). Therefore in both cases the entropy is zero since all the poles of g lie on the unit circle. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (25) has quadratic growth for all values of a.

4.5.3. Equation (26)

Equation (26) depends on the parameter a. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. Equation (26) is symmetric for both $a = 1$ and $a = 0$. Moreover, in both cases it has the following growth of degrees:

$$1, 4, 6, 10, 16, 22, 29, 37, 46, 56, 67, 79, 92 \ldots. \tag{101}$$

The generating function corresponding to the growth (101) is:

$$g(z) = -\frac{(z^2 - z + 1)(z^4 - z^3 - 2z^2 + 2z + 1)}{(z-1)^3}. \tag{102}$$

If $a = 0$ Equation (26) is symmetric, but its growth of degrees is still given by the sequence (101) and fitted by the generating function (102). Therefore in both cases the entropy is zero since all the poles of g lie on the unit circle. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (26) has quadratic growth for all values of a.

4.5.4. Equation (27)

Equation (27) is symmetric and has the following growth of degrees:

$$1, 4, 7, 10, 15, 21, 27, 36, 45, 54, 65, 77, 89, 104, 119, 134, 151, 169, 187, 208 \ldots. \tag{103}$$

The generating function corresponding to the growth (103) is:

$$g(z) = -\frac{(z^4 + z + 1)(z^3 - z^2 + z + 1)}{(z-1)^3(z+1)(z^2 - z + 1)(z^2 + z + 1)}. \tag{104}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (27) has quadratic growth.

4.5.5. Equation (32)

Equation (32) is symmetric and has the following growth of degrees:

$$\begin{aligned} &1, 3, 1, 4, 5, 5, 6, 13, 7, 15, 17, 17, 19, 31, \\ &\quad 21, 34, 37, 37, 40, 57, 43, 61, 65, 65, 69 \ldots. \end{aligned} \tag{105}$$

The generating function corresponding to the growth (105) is:

$$g(z) = -\frac{z^{10} + z^9 + z^7 - z^6 + z^5 + z^4 + 3z + 1}{(z-1)^3(z+1)^2(z^2 - z + 1)(z^2 + z + 1)^2}. \tag{106}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Using the $\mathcal{Z}$-transform we obtain the following expression for the degrees:

$$\begin{aligned} d_n = {} & \frac{n^2}{9} + \frac{5n}{9} + \frac{191}{108} + \frac{5(-1)^n}{12} + \frac{(-1)^n n}{6} \\ & + \frac{\sqrt{3}}{36} \sin\left(\frac{n\pi}{3}\right) - \sqrt{3}\left(\frac{5n}{54} + \frac{7}{36}\right) \sin\left(\frac{2n\pi}{3}\right) \\ & + \frac{1}{12} \cos\left(\frac{n\pi}{3}\right) + \left(\frac{5n}{18} + \frac{79}{108}\right) \cos\left(\frac{2n\pi}{3}\right). \end{aligned} \tag{107}$$

Therefore the growth (107) is quadratic as $n \to \infty$, but we notice also the unusual presence of oscillating term proportional to $(-1)^n n$ which explains the high oscillations of the sequence (105). A similar occurrence was found in [65] on the degree pattern of some linearisable quad-equations.

4.5.6. Equation (28)

Equation (28) is symmetric and has the following growth of degrees:

$$1,3,4,7,11,15,20,25,31,38,45,53,62,71,81,92,103,115,128,141,155\ldots. \tag{108}$$

The generating function corresponding to the growth (108) is:

$$g(z) = -\frac{z^8 - z^7 - z^6 + z^5 + z^3 - z^2 + z + 1}{(z-1)^3(z^2+z+1)}. \tag{109}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (28) has quadratic growth.

4.5.7. Equation (29)

Equation (29) depends on the parameter a. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. Equation (29) is not symmetric for both $a = 1$ and $a = 0$. However, in both cases it has the following growth of degrees:

$$1,3,6,9,13,19,24,31,40,48,57,69,79,91,106,119,133,151,166,183,204\ldots. \tag{110}$$

The generating function corresponding to the growth (110) is:

$$g(z) = -\frac{z^9 + z^7 + z^6 + 3z^5 + 2z^4 + 2z^3 + 3z^2 + 2z + 1}{(z-1)^3(z+1)(z^2-z+1)(z^2+z+1)^2)}. \tag{111}$$

Therefore in both cases the entropy is zero since all the poles of g lie on the unit circle. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (29) has quadratic growth for all values of a.

4.5.8. Equation (30)

Equation (30) depends on the parameter a. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. Equation (30) is symmetric for both $a = 1$ and $a = 0$. When $a = 1$ Equation (30) has the following growth of degrees:

$$\begin{gathered}1,7,15,24,35,51,66,85,109,132,157,189,218,\\ 251,291,328,367,415,458,505,561,612,665\ldots.\end{gathered} \tag{112}$$

The generating function corresponding to the growth (112) is:

$$g_{a=1}(z) = -\frac{z^{10} + 2z^7 + 5z^6 + 8z^5 + 5z^4 + 8z^3 + 8z^2 + 6z + 1}{(z-1)^3(z+1)(z^2-z+1)(z^2+z+1)^2}. \tag{113}$$

When $a = 0$ Equation (30) has the following growth of degrees:

$$\begin{gathered}1,7,15,23,33,48,63,84,107,130,155,182,211,\\ 248,287,324,363,404,447,500,555,606,659,\\ 714,771,840,911,976,1043,1112,1183,1268,1355\ldots.\end{gathered} \tag{114}$$

The generating function corresponding to the growth (112) is:

$$g_{a=0}(z) = -\frac{\begin{pmatrix} 2z^{16} - 3z^{15} + 2z^{14} - z^{13} + 2z^{12} - 2z^{11} - z^{10} + 6z^{9} \\ +z^{8} + 6z^{7} + 3z^{6} + 10z^{5} + 5z^{4} + 5z^{3} + 3z^{2} + 5z + 1 \end{pmatrix}}{(z-1)^{3}(z+1)(z^{2}-z+1)^{2}(z^{2}+z+1)^{2}}. \tag{115}$$

All the poles of $g_{a=1}$ and $g_{a=0}$ lie on the unit circle, so that the entropy is zero in both cases. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (30) has quadratic growth in both cases.

4.5.9. Equation (31)

Equation (31) is symmetric and has the same growth of degrees as Equation (29). Therefore we have that Equation (31) has zero entropy and quadratic growth.

4.6. List 6

4.6.1. Equation (33)

Equation (33) is symmetric and has the following growth of degrees:

$$1, 4, 6, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92 \ldots. \tag{116}$$

The generating function corresponding to the growth (116) is:

$$g(z) = -\frac{z^5 - 3z^4 + 4z^3 - 3z^2 + z + 1}{(z-1)^3}. \tag{117}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (33) has quadratic growth.

4.6.2. Equation (34)

Equation (34) is symmetric and has the following growth of degrees:

$$1, 6, 13, 25, 42, 61, 85, 111, 139, 171, 207, 245, 287, 333, 381, 433, 489, 547 \ldots. \tag{118}$$

The generating function corresponding to the growth (118) is:

$$g(z) = -\frac{2z^{10} - z^9 - 3z^7 + z^4 + 4z^3 + 2z^2 + 4z + 1}{(z-1)^3(z^2+z+1)}. \tag{119}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (34) has quadratic growth.

4.6.3. Equation (35)

We proved that Equation (35) can be brought in in rational form (38), but this form is not bi-rational. So we cannot apply the algebraic entropy method to this equation.

4.6.4. Equation (36)

Equation (36) is symmetric and has the following growth of degrees:

$$1, 7, 15, 24, 35, 49, 67, 86, 107, 132, 159, 188, 219, 254, 291, 330, 371, 416, 463 \ldots. \tag{120}$$

The generating function corresponding to the growth (120) is:

$$g(z) = -\frac{z^{11} - 2z^{10} + z^9 + 2z^6 - 2z^5 + z^4 + z^3 + 2z^2 + 5z + 1}{(z-1)^3(z+1)(z^2+1)}. \tag{121}$$

All the poles of g lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z-1)^3$ following Proposition 2 we have that Equation (36) has quadratic growth.

5. Discussion

In the previous section we computed the algebraic entropy of all the integrable Volterra-like five-point differential-difference equations recently classified in [18,19]. When possible, we showed that the method of algebraic entropy and the method of generalised symmetries agree. That is, we showed that all the equations that are integrable according to the generalised symmetry test are also integrable according to the algebraic entropy method, i.e., the algebraic entropy is zero. The algebraic entropy method is unfortunately unable to treat the semi-discrete Kaup-Kaupershmidt Equation (35). This is because the generalised symmetry approach, differently from the algebraic entropy, makes no assumption on the nature of the recurrence and algebraic or even transcendental terms are allowed. That is, we proved that the for integrable Volterra-like five-point differential-difference equations the following version of the algebraic entropy conjecture holds true:

Conjecture 1. *The condition that algebraic entropy is zero is equivalent to the definition of integrability for bi-rational maps.*

Except for the two known non-trivially linearisable Equations (12) and (13) the growth is always quadratic. Equation (32) possesses an interesting non-standard highly oscillating growth, observed for the first time in differential-difference equations. Nevertheless the asymptotic growth is still quadratic.

In the case of two-dimensional difference equation it is known that the only possible polynomial, i.e., integrable, growth is quadratic [49]. Integrable higher order maps can exhibit higher rate of growth, see e.g., [66–68]. Despite being infinite-dimensional all the integrable Volterra-like five-point differential-difference equations possess this "minimal" integrable growth.

In the case of difference equations it has been observed that degree growth greater than quadratic is related to a procedure called *deflation* [66]. That is, a five-point equation is reduced to a four-point one using a non-point potential-like transformations of the form:

$$v_n = \frac{a_1 u_n u_{n+1} + a_2 u_{n+1} + a_3 u_n + a_4}{b_1 u_n u_{n+1} + b_2 u_{n+1} + b_3 u_n + b_4}. \tag{122}$$

The deflation transformation (122) is *non-point* as it depends on u_n and its shift u_{n+1}. Moreover, it is *potential* as it is a generalisation of the usual potential transformation:

$$v_n = u_{n+1} - u_n, \tag{123}$$

which is obtained for $a1_1 = a_4 = b_1 = b_2 = b_3 = 0$ and $a_2 = -a_3 = b_4 = 1$. The inflation transformation includes also the so-called discrete Cole-Hopf transformation [28,69]:

$$v_n = \frac{u_{n+1}}{u_n}. \tag{124}$$

Let us notice, that the inflated version of a differential-difference equation is not always a differential-difference equation of the same kind. For instance let us consider the linear semi-discrete heat equation:

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} = v_{n+1}. \tag{125}$$

It is well-known [28,69] that the discrete Cole-Hopf transformation (124) brings the semi-discrete heat Equation (125) into the semi-discrete Burgers equation:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = u_n\left(u_{n+1} - u_n\right). \tag{126}$$

First note that, even though Equation (125) is bi-rational, Equation (126) is not. This already means that there is no guarantee to preserve bi-rationality using the inflation transformation (122) even in simple cases like the discrete Cole-Hopf transformation (124). If we iterate the discrete Cole-Hopf transformation:

$$u_n = \frac{w_{n+1}}{w_n}, \tag{127}$$

we obtain:

$$w_n\frac{\mathrm{d}w_{n+1}}{\mathrm{d}t} - w_{n+1}\frac{\mathrm{d}w_n}{\mathrm{d}t} = w_{n+2}w_n - w_{n+1}^2. \tag{128}$$

Therefore the resulting equation is not bi-rational and not even in the form:

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = f\left(w_{n+k}, \ldots, w_{n+k'}\right). \quad k < k' \in \mathbb{Z}, \tag{129}$$

So Equation (128) belongs to a completely different kind of differential-difference equations. These facts makes very difficult to make predictions on the integrability properties of the inflated forms of differential-difference equations.

Finally, we notice that another interesting problem is to study the integrability properties of the stationary reductions of the integrable Volterra-like five-point differential-difference equations. The stationary reduction of a bi-rational five-point differential-difference equation is a fourth-order difference equation, i.e., a four-dimensional map of the projective space into itself. It will be important to understand how integrability arises inside these families of equations and if it fits with known cases of integrable families of fourth-order differential difference equations [67,68,70].

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Appendix A. Proof of Proposition 2

We carry out the proof in the case when $\beta \in \mathbb{N}$ as we are interested in rational generating functions. A similar argument, with the proper care in choosing an suitable path can be carried out in the general case $\beta \in \mathbb{R} \setminus \{-n\}_{n\in\mathbb{N}}$. To this end the interested reader can consult the paper [63].

From Equation (60) and the definition of inverse $\mathcal{Z}$-transform (59) applied to Equation (61) we have:

$$d_l = \frac{1}{2\pi\imath}\oint_C \left[A\left(\frac{1}{\zeta}\right) + B\left(\frac{1}{\zeta}\right)\left(1 - \frac{1}{\rho\zeta}\right)^{-\beta}\right]\zeta^{l-1}\mathrm{d}\zeta, \tag{A1}$$

where we can choose C to be the circle of radius $1/\rho + \varepsilon$ for a given $\varepsilon > 0$. Then $\zeta = 1/\rho$ is the greatest pole of $g\left(1/\zeta\right)$. This implies that the dominant behaviour as $l \to \infty$ of d_l is given from the residue of the pole in $\zeta = 1/\rho$:

$$d_l \sim \operatorname{Res}_{\zeta=1/\rho}\left\{\left[A\left(\frac{1}{\zeta}\right) + B\left(\frac{1}{\zeta}\right)\left(1 - \frac{1}{\rho\zeta}\right)^{-\beta}\right]\zeta^{l-1}\right\}, \quad l \to \infty. \tag{A2}$$

From the analyticity of $A(z)$ we obtain:

$$d_l \sim \mathrm{Res}_{\zeta=1/\rho}\left\{B\left(\frac{1}{\zeta}\right)\left(1-\frac{1}{\rho\zeta}\right)^{-\beta}\zeta^{l-1}\right\}, \quad l \to \infty. \tag{A3}$$

We introduce the change of variables $\chi = \zeta - 1/\rho$, so that (A3) becomes:

$$d_l \sim \mathrm{Res}_{\chi=0}\left\{\tilde{B}(\chi)\frac{(1+\rho\chi)^{l+\beta-1}}{\rho^{l+\beta-1}\chi^{\beta}}\right\}, \quad l \to \infty, \tag{A4}$$

where we defined:

$$\tilde{B}(\chi) = B\left(\frac{\rho}{1+\rho\chi}\right). \tag{A5}$$

Since B is an analytic function for $|z| < \rho$ we obtain that $\tilde{B}$ is analytic function in a neighbourhood of $\chi = 0$, that is:

$$\tilde{B}(\chi) = \sum_{k=0}^{\infty}\frac{\mathrm{d}^k\tilde{B}}{\mathrm{d}\chi^k}(0)\,\chi^k. \tag{A6}$$

Moreover, using the binomial expansion we get:

$$(1+\rho\chi)^{l+\beta-1} = \sum_{k=0}^{l+\beta-1}\binom{l+\beta-1}{k}\rho^k\chi^k. \tag{A7}$$

Plugging Equations (A6) and (A7) inside Equation (A4) we obtain using the definition of residue:

$$d_l \sim \sum_{k=0}^{\beta-1}\frac{\mathrm{d}^k\tilde{B}}{\mathrm{d}\chi^k}(0)\binom{l+\beta-1}{\beta-1-k}\rho^{-l-k}, \quad l \to \infty. \tag{A8}$$

In the sum in the right hand side of (A8) the dominant term is obtained for $k = 0$, therefore we can write:

$$d_l \sim B(\rho)\binom{\beta+l-1}{\beta-1}\rho^{-l}, \quad l \to \infty, \tag{A9}$$

where we used that $\tilde{B}(0) = B(\rho)$. From the definition of binomial coefficient we have:

$$\binom{\beta+l-1}{\beta-1} = \frac{(l+\beta-1)!}{(\beta-1)!l!}. \tag{A10}$$

Using the Stirling's expansion as $l \to \infty$ we obtain:

$$l! \sim \sqrt{2\pi l}\left(\frac{l}{e}\right)^{l}, \tag{A11a}$$

$$(l+\beta-1)! \sim \sqrt{2\pi(l+\beta-1)}\left(\frac{l+\beta-1}{e}\right)^{l+\beta-1} \sim \sqrt{2\pi l}\left(\frac{l}{e}\right)^{l+\beta-1}. \tag{A11b}$$

Inserting in (A10):

$$\binom{\beta+l-1}{\beta-1} \sim \frac{l^{\beta-1}}{(\beta-1)!}, \quad l \to \infty. \tag{A12}$$

Finally using this estimate in (A9) we obtain:

$$d_l \sim \frac{B(\rho)\, l^{\beta-1}}{(\beta-1)!}\rho^{-l}. \quad l \to \infty, \tag{A13}$$

Formula (A13) is just (62) in the case $\beta \in \mathbb{N}$ and the proof for this case is complete.

References

1. Calogero, F. Why are certain nonlinear PDEs both widely applicable and integrable? In *What Is Integrability?* Zakharov, V.E., Ed.; Springer: Berlin/Heidelberg, Germany, 1991.
2. Zabusky, N.J.; Kruskal, M.D. Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States. *Phys. Rev. Lett.* **1965**, *15*, 240–243.
3. Gardner, C.S.; Greene, J.M.; Kruskal, M.D.; Miura, R.M. Method for Solving the Korteweg-deVries Equation. *Phys. Rev. Lett.* **1967**, *19*, 1095–1097.
4. Lax, P.D. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* **1968**, *21*, 467–490.
5. Korteweg, D.J.; de Vries, G. XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **1895**, *39*, 422–443.
6. Liouville, J. Note sur l'intégration des équations différentielles de la Dynamique, présentée au Bureau des Longitudes le 29 juin 1853. *J. Math. Pures Appl.* **1855**, *20*, 137–138.
7. Whittaker, E.T. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*; Cambridge University Press: Cambridge, UK, 1999.
8. Adler, V.E.; Shabat, A.B.; Yamilov, R.I. Symmetry approach to the integrability problem. *Theor. Math. Phys.* **2000**, *125*, 1603–1661.
9. Habibullin, I.T.; V., S.V.; Yamilov, R.I. Multi-component integrable systems and nonassociative structures. In the proceedings of the 1st International Workshop on Nonlinear Physics: Theory and Experiment, Gallipoli, Turkey, 29 June–7 July 1995.
10. Mikhailov, A.V.; Shabat, A.B. Symmetries—Test of integrability. In *Important Developments in Soliton Theory*; Fokas, A.S., Zakharov, V.E., Eds.; Springer: Berlin, Germany, 1993; pp. 355–372.
11. Mikhailov, A.V.; Shabat, A.B.; Sokolov, V.V. The symmetry approach to classification of integrable equations. In *What is Integrability?* Zakharov, V.E., Ed.; Springer: Berlin, Germany, 1991; pp. 115–184.
12. Mikhailov, A.V.; Shabat, A.B.; Yamilov, R.I. The symmetry approach to the classification of nonlinear equations. Complete lists of integrable systems. *Russ. Math. Surv.* **1987**, *42*, 1–63.
13. Sokolov, V.V. On the symmetries of evolution equations. *Russ. Math. Surv.* **1988**, *43*, 165–204.
14. Sokolov, V.V.; Shabat, A.B. Classification of integrable evolution equations. In *Soviet Scientific Reviews, Section C, Mathematical Physics Reviews*; Harwood Academic Publishers: New York, NY, USA, 1984; Volume 4, pp. 221–280.
15. Yamilov, R.I. Classification of discrete evolution equations. *Uspekhi Mat. Nauk* **1983**, *38*, 155–156. (In Russian)
16. Yamilov, R.I. Classification of Discrete Integrable Equations. Ph.D. Thesis, Leningrad Branch of Steklov Mathematical Institute, Soviet Academy of Sciences, Leningrad, Russia, 1984.
17. Levi, D.; Yamilov, R.I. Conditions for the existence of higer symmetries of the evolutionary equations on the lattice. *J. Math. Phys.* **1997**, *38*, 6648–6674.
18. Garifullin, R.N.; Yamilov, R.I.; Levi, D. Classification of five-point differential-difference equations. *J. Phys. A Math. Theor.* **2017**, *50*, 125201.
19. Garifullin, R.N.; Yamilov, R.I.; Levi, D. Classification of five-point differential-difference equations II. *J. Phys. A Math. Theor.* **2018**, *51*, 065204.
20. Levi, D.; Yamilov, R.I. The generalized symmetry method for discrete equations. *J. Phys. A Math. Theor.* **2009**, *42*, 454012.
21. Levi, D.; Yamilov, R.I. Generalized symmetry integrability test for discrete equations on the square lattice. *J. Phys. A Math. Theor.* **2011**, *44*, 145207.

22. Garifullin, R.N.; Yamilov, R.I. Integrable discrete nonautonomous quad-equations as Bäcklund auto-transformations for known Volterra and Toda type semidiscrete equations. *J. Phys. Conf. Ser.* **2015**, *621*, 012005.
23. Garifullin, R.N.; Gubbiotti, G.; Yamilov, R.I. Integrable discrete autonomous quad-equations admitting, as generalized symmetries, known five-point differential-difference equations. *arXiv* **2018**, arXiv:1810.11184.
24. Arnold, V.I. Dynamics of complexity of intersections. *Bol. Soc. Bras. Mat.* **1990**, *21*, 1–10.
25. Veselov, A.P. Growth and integrability in the dynamics of mappings. *Comm. Math. Phys.* **1992**, *145*, 181–193.
26. Hietarinta, J.; Viallet, C.M. Searching for integrable lattice maps using factorization. *J. Phys. A Math. Theor.* **2007**, *40*, 12629–12643.
27. Adler, V.E.; Postnikov, V.V. On vector analogs of the modified Volterra lattice. *J. Phys. A Math. Theor.* **2008**, *41*, 455203.
28. Yamilov, R.I. Symmetries as integrability criteria for differential difference equations. *J. Phys. A* **2006**, *39*, R541–R623.
29. Garifullin, R.N.; Yamilov, R.I. Generalized symmetry classification of discrete equations of a class depending on twelve parameters. *J. Phys. A Math. Theor.* **2012**, *45*, 345205.
30. Garifullin, R.N.; Mikhailov, A.V.; Yamilov, R.I. Discrete equation on a square lattice with a nonstandard structure of generalized symmetries. *Theor. Math. Phys.* **2014**, *180*, 765–780.
31. Adler, V.E. Integrable Möbius invariant evolutionary lattices of second order. *arXiv* **2016**, arXiv:1605.00018.
32. Bogoyavlensky, O.I. Integrable discretizations of the KdV equation. *Phys. Lett.* **1988**, *134*, 34–38.
33. Itoh, Y. An H-theorem for a system of competing species. *Proc. Jpn. Acad.* **1975**, *51*, 374–379.
34. Narita, K. Soliton solution to extended Volterra equation. *J. Phys. Soc. Jpn.* **1982**, *51*, 1682–1685.
35. Mikhailov, A.V.; Xenitidis, P. Second order integrability conditions for difference equations: An integrable equation. *Lett. Math. Phys.* **2013**, doi:10.1007/s11005-013-0668-8.
36. Bogoyavlensky, O.I. Algebraic constructions of integrable dynamical systems-extensions of the Volterra system. *Russ. Math. Surv.* **1991**, *46*, 1–64.
37. Suris, Y.B. *The Problem of Integrable Discretization: Hamiltonian Approach*; Birkhäuser: Basel, Switzerland, 2003.
38. Garifullin, R.N.; Yamilov, R.I.; Levi, D. Non-invertible transformations of differential-difference equations. *J. Phys. A Math. Theor.* **2016**, *49*, 37LT01.
39. Tsujimoto, S.; Hirota, R. Pfaffian Representation of Solutions to the Discrete BKP Hierarchy in Bilinear Form. *J. Phys. Soc. Jpn.* **1996**, *65*, 2797–2806.
40. Adler, V.E. On a discrete analog of the Tzitzeica equation. *arXiv* **2011**, arXiv:1103.5139.
41. Garifullin, R.N.; Yamilov, R.I. On integrability of a discrete analogue of Kaup-Kupershmidt equation. *Ufa Math. J.* **2017**, *9*, 158–164.
42. Kaup, D.J. On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi x + 6R\psi = \lambda\psi$. *SIAM J. Appl. Math.* **1980**, *62*, 189–216.
43. Demskoy, D.K.; Viallet, C.M. Algebraic entropy for semi-discrete equations. *J. Phys. A Math. Theor.* **2012**, *45*, 352001.
44. Grammaticos, B.; Halburd, R.G.; Ramani, A.; Viallet, C.M. How to detect the integrability of discrete systems. *J. Phys A Math. Theor.* **2009**, *42*, 454002.
45. Gubbiotti, G. Integrability of difference equations through Algebraic Entropy and Generalized Symmetries. In *Symmetries and Integrability of Difference Equations: Lecture Notes of the Abecederian School of SIDE 12, Montreal 2016*; Levi, D., Verge-Rebelo, R., Winternitz, P., Eds.; CRM Series in Mathematical Physics; Springer International Publishing: Berlin, Germany, 2017; Chapter 3, pp. 75–152.
46. Gubbiotti, G. Integrability Properties of Quad Equations Consistent on the Cube. Ph.D. Thesis, Università degli Studi Roma Tre, Rome, Italy, 2017.
47. Bellon, M.; Viallet, C.M. Algebraic entropy. *Comm. Math. Phys.* **1999**, *204*, 425–437.
48. Falqui, G.; Viallet, C.M. Singularity, complexity, and quasi-integrability of rational mappings. *Comm. Math. Phys.* **1993**, *154*, 111–125.
49. Diller, J. Dynamics of birational maps of P^2. *Indiana Univ. Math. J.* **1996**, *45*, 721–772.
50. Russakovskii, A.; Shiffman, B. Value distribution of sequences of rational mappings and complex dynamics. *Indiana Univ. Math. J.* **1997**, *46*, 897–932.
51. Tremblay, S.; Grammaticos, B.; Ramani, A. Integrable lattice equations and their growth properties. *Phys. Lett.* **2001**, *278*, 319–324.

52. Viallet, C.M. Algebraic Entropy for lattice equations. *arXiv* **2006**, arXiv:math-ph/0609043.
53. Viallet, C.M. Algebraic entropy for differential-delay equations. *arXiv* **2014**, arXiv:1408.6161.
54. Curtiss, J.H. A Note on the Theory of Moment Generating Functions. *Ann. Math. Stat.* **1942**, *13*, 430–433.
55. Banderier, C.; Bousquet-Mélou, M.; Denise, A.; Flajolet, P.; Gardy, D.; Gouyou-Beauchamps, D. Generating functions for generating trees. *Discr. Math.* **2002**, *246*, 29–55.
56. Gautschi, W. On Generating Orthogonal Polynomials. *SIAM J. Sci. Stat. Comp.* **1982**, *3*, 289–317.
57. Shang, Y. Unveiling robustness and heterogeneity through percolation triggered by random-link breakdown. *Phys. Rev. E* **2014**, *90*, 032820.
58. Shang, Y. Effect of link oriented self-healing on resilience of networks. *J. Stat. Mech. Theory Exp.* **2016**, *2016*, 083403.
59. Padé, H. Sur la répresentation approchée d'une fonction par des fractions rationelles. *Annales Scientifiques de l'École Normale Supérieure* **1892**, *3*, 3–93.
60. Baker, G.A.; Graves-Morris, P.R. *Padé Approximants*; Cambridge University Press: Cambridge, UK, 1996.
61. Elaydi, S. *An Introduction to Difference Equations*, 3rd ed.; Springer: New York, NY, USA, 2005.
62. Jury, E. *Theory and Applications of the Z-Transform Method*; Wiley: New York, NY, USA, 1964.
63. Flajolet, P.; Odlyzko, A. Singularity Analysis of Generating Functions. *SIAM J. Discr. Math.* **1990**, *3*, 216–240.
64. Gubbiotti, G.; Scimiterna, C.; Levi, D. The non autonomous YdKN equation and generalized symmetries of Boll equations. *J. Math. Phys.* **2017**, *58*, 053507.
65. Gubbiotti, G.; Scimiterna, C.; Levi, D. Algebraic entropy, symmetries and linearization of quad equations consistent on the cube. *J. Nonlinear Math. Phys.* **2016**, *23*, 507–543.
66. Joshi, N.; Viallet, C.M. Rational Maps with Invariant Surfaces. *J. Int. Sys.* **2018**, *3*, xyy017.
67. Gubbiotti, G.; Joshi, N.; Tran, D.T.; Viallet, C.M. Integrability properties of a class of 4D bi-rational maps with two invariants. **2018**. In Preparation.
68. Gubbiotti, G.; Joshi, N.; Tran, D.T.; Viallet, C.M. Complexity and integrability in 4D bi-rational maps with two invariants. *arXiv* **2018**, arXiv:1808.04942.
69. Levi, D.; Ragnisco, O.; Bruschi, M. Continuous and discrete matrix Burgers' hierarchies. *Il Nuovo Cimento B (1971–1996)* **1983**, *74*, 33–51.
70. Capel, H.W.; Sahadevan, R. A new family of four-dimensional symplectic and integrable mappings. *Phys. A Stat. Mech. Appl.* **2001**, *289*, 80–106.