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A two-periodic generalization of the $Q_{\rm V}$ equation

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In this article, we introduce a two-periodic generalization of the $Q_{\rm V}$ equation introduced by Viallet. All the equations of Boll's classification appear in it for special choices of the parameters. Using the algebraic entropy test, we infer that the equation should be integrable and with the aid of a formula introduced by Xenitidis we find its three point generalized symmetries.

Keywords: integrable partial difference equation; quad graph equation; symmetries; algebraic entropy.

1. Introduction

In the book *What is Integrability?* edited by V.E. Zakharov [32] in 1991 we find a series of contributions on the various definitions of integrability or exact solvability of nonlinear partial differential equations and maps. The theory of integrability of partial difference equations is more recent. The first classification of integrable partial difference equations on a quad graph, obtained by the *Compatibility around a cube*, is due to Adler Bobenko and Suris [1] in 2003, the so-called ABS equations. Different integrability detectors for partial difference equations have been introduced starting from the work of Grammaticos, Ramani and Papageorgiou [8] which extended to the discrete world the Painlevé test [7]. Here we will consider as integrability criteria the algebraic entropy [3, 14, 24–26] and existence of generalized symmetries [18].

In two previous papers [12, 13], we calculated the three point generalized symmetries of equations belonging to Boll classification [2, 4–6] of quad graph equations compatible around the cube, i.e. the trapezoidal H^4 equations and the H^6 equations. The symmetries of the rhombic H^4 equations have been considered in [28]. In [12], we also noticed that all the flows generated by such symmetries

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = Q,$$

where Q is the characteristic of the symmetry and t is the group parameter, are related to some particular cases of the two-periodic Yamilov discretization of the Krichever-Novikov equation (YdKN) equation [17][1]:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \frac{A_n(u_n)u_{n+1}u_{n-1} + B_n(u_n)(u_{n+1}+u_{n-1}) + C_n(u_n)}{u_{n+1}-u_{n-1}}. \tag{1}$$

[1] For the rhombic H^4 equations this was already known from [28].

Here the n-dependent coefficients are given by:

$$A_n(u_n) = au_n^2 + 2b_n u_n + c_n, \tag{2a}$$

$$B_n(u_n) = b_{n+1}u_n^2 + du_n + e_{n+1}, \tag{2b}$$

$$C_n(u_n) = c_{n+1}u_n^2 + 2e_n u_n + f, \tag{2c}$$

where b_n, c_n and e_n are two-periodic functions, i.e.

$$b_n = b_0 + b_1(-1)^n,\ c_n = c_0 + c_1(-1)^n,\ e_n = e_0 + e_1(-1)^n. \tag{3}$$

Ref. [12] naturally extended the results contained in [15] where the three points generalized symmetries of the ABS class of lattice equations [1], found in [26], which are at the base of Boll's classification, are shown to be sub-cases of the autonomous YdKN equation [29–31]. We recall that the autonomous YdKN equation is obtained from (1) just taking b_k, c_k and e_k as pure constants, i.e.:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \frac{A(u_n)u_{n+1}u_{n-1} + B(u_n)(u_{n+1} + u_{n-1}) + C(u_n)}{u_{n+1} - u_{n-1}}, \tag{4}$$

where:

$$A(u_n) = au_n^2 + 2bu_n + c, \tag{5a}$$

$$B(u_n) = bu_n^2 + du_n + e, \tag{5b}$$

$$C(u_n) = cu_n^2 + 2eu_n + f. \tag{5c}$$

In particular in [12] we noted that the symmetries of all equations in the Boll classification are sub-cases of the general two-periodic YdKN (1) corresponding to $a = b_k = 0$.

In [27], it was shown that the Q_{V} equation, introduced in [26]:

$$\begin{aligned} Q_{\mathrm{V}} = {} & a_1 u_{n,m}u_{n+1,m}u_{n,m+1}u_{n+1,m+1} \\ & + a_{2,0}\left(u_{n,m}u_{n,m+1}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1}u_{n+1,m+1}\right. \\ & \left. + u_{n,m}u_{n+1,m}u_{n+1,m+1} + u_{n,m}u_{n+1,m}u_{n,m+1}\right) \\ & + a_{3,0}\left(u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1}\right) \\ & + a_{4,0}\left(u_{n,m}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1}\right) \\ & + a_{5,0}\left(u_{n+1,m}u_{n+1,m+1} + u_{n,m}u_{n,m+1}\right) \\ & + a_{6,0}\left(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}\right) \\ & + a_7 = 0 \end{aligned} \tag{6}$$

admits a symmetry in the direction n

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}t} = \frac{h_n}{u_{n+1,m} - u_{n-1,m}} - \frac{1}{2}\partial_{u_{n+1,m}} h_n, \tag{7}$$

where:

$$h_n(u_{n,m}, u_{n+1,m}) = Q_V \partial_{u_{n,m+1}} \partial_{u_{n+1,m+1}} Q_V - \left(\partial_{u_{n,m+1}} Q_V\right)\left(\partial_{u_{n+1,m+1}} Q_V\right) \tag{8}$$

and a symmetry in the direction m

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}t} = \frac{h_m}{u_{n,m+1} - u_{n,m-1}} - \frac{1}{2}\partial_{u_{n,m+1}} h_m, \tag{9}$$

where:

$$h_m(u_{n,m}, u_{n,m+1}) = Q_V \partial_{u_{n+1,m}} \partial_{u_{n+1,m+1}} Q_V - \left(\partial_{u_{n+1,m}} Q_V\right)\left(\partial_{u_{n+1,m+1}} Q_V\right) \tag{10}$$

of the same form as the YdKN (4).

In [19–21], its recursion operator is constructed and thus an infinite number of generalized symmetries can be constructed showing that the equation is integrable by a spectral transform or linearizable.

The connection formulae between the coefficients of Q_V and the n direction YdKN (4) are:

$$\begin{aligned} &a = a_{3,0}a_1 - a_{2,0}^2, \qquad b = \frac{1}{2}[a_{2,0}(a_{3,0} - a_{5,0} - a_{4,0}) + a_{6,0}a_1], \\ &c = a_{2,0}a_{6,0} - a_{4,0}a_{5,0}, \qquad d = \frac{1}{2}[a_{3,0}^2 - a_{4,0}^2 - a_{5,0}^2 + a_1 a_7], \\ &e = \frac{1}{2}[a_{6,0}(a_{3,0} - a_{4,0} - a_{5,0}) + a_{2,0}a_7], \qquad f = a_{3,0}a_7 - a_{6,0}^2 \end{aligned} \tag{11}$$

and these show that (a, b) may be different from $(0, 0)$. The connection formulae between the coefficients of Q_V and the m direction YdKN (4) are:

$$\begin{aligned} &a = a_{5,0}a_1 - a_{2,0}^2, \qquad b = \frac{1}{2}[a_{2,0}(a_{5,0} - a_{3,0} - a_{4,0}) + a_{6,0}a_1], \\ &c = a_{2,0}a_{6,0} - a_{4,0}a_{3,0}, \qquad d = \frac{1}{2}[a_{5,0}^2 - a_{4,0}^2 - a_{3,0}^2 + a_1 a_7], \\ &e = \frac{1}{2}[a_{6,0}(a_{5,0} - a_{4,0} - a_{3,0}) + a_{2,0}a_7], \qquad f = a_{5,0}a_7 - a_{6,0}^2. \end{aligned} \tag{12}$$

So the three point generalized symmetries of the Q_{V} equation belong to the class of the general YdKN equation (4).

From the results obtained in [12], we were led to conjecture the existence of a two-periodic generalization of the Q_{V} equation and suggested some possible ways to obtain such a generalization. In this article, we follow them to generalize the Q_{V} equation. It is known [26] that Q_{V} is the most general multi-linear equation on a quad graph possessing Klein discrete symmetries, i.e. such that:

$$\begin{aligned} Q\left(u_{n+1,m}, u_{n,m}, u_{n+1,m+1}, u_{n,m+1}\right) &= \tau Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right), \\ Q\left(u_{n,m+1}, u_{n+1,m+1}, u_{n,m}, u_{n,m+1}\right) &= \tau' Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right), \end{aligned} \tag{13}$$

where $\tau, \tau' = \pm 1$. A way to extend the Klein discrete symmetry is to consider a multi-linear function Q with two-periodic coefficients in n and m such that the following equations holds:

$$\begin{aligned} &Q\left(u_{n+1,m}, u_{n,m}, u_{n+1,m+1}, u_{n,m+1}; (-1)^n, (-1)^m\right) = \\ &\quad \tau Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; -(-1)^n, (-1)^m\right), \\ &Q\left(u_{n,m+1}, u_{n+1,m+1}, u_{n,m}, u_{n+1,m}; (-1)^n, (-1)^m\right) = \\ &\quad \tau' Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; (-1)^n, -(-1)^m\right). \end{aligned} \tag{14}$$

If a two-periodic system Q satisfies the discrete symmetries (14) we will say that Q admits a *two-periodic discrete Klein symmetry*. The name follows from the fact that if Q is autonomous then the discrete symmetry (14) reduces to the Klein one (13). Furthermore all the equations belonging to the Boll's classification satisfy these symmetry conditions (14) when $\tau = \tau' = 1$.

In Section 2, we will show that up to multiplication by a function and redefinition of the parameters there is only one quad graph equation possessing the two-periodic Klein symmetry. We will prove that this equation, which we will call the *two-periodic* Q_{V} *equation* has as sub-cases all the equations of Boll classification and it satisfies the algebraic entropy integrability test. Furthermore, we will show that (7, 8) will provide an n directional and (9, 10) an m directional symmetry for the two-periodic Q_{V} equation and that such symmetries belong to the class of the two-periodic YdKN equation (1, 2). We will provide then the appropriate connection formulae. In Section 3, we will present some concluding remarks.

2. The two-periodic Q_{V} equation

Let us consider the most general multi-linear equation in the lattice variables with two-periodic coefficients:

$$\begin{aligned} &p_1 u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \\ &+ p_2 u_{n,m} u_{n,m+1} u_{n+1,m+1} + p_3 u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \\ &+ p_4 u_{n,m} u_{n+1,m} u_{n+1,m+1} + p_5 u_{n,m} u_{n+1,m} u_{n,m+1} \\ &+ p_6 u_{n,m} u_{n+1,m} + p_7 u_{n,m+1} u_{n+1,m+1} + p_8 u_{n,m} u_{n+1,m+1} \\ &+ p_9 u_{n+1,m} u_{n,m+1} + p_{10} u_{n+1,m} u_{n+1,m+1} + p_{11} u_{n,m} u_{n,m+1} \\ &+ p_{12} u_{n,m} + p_{13} u_{n+1,m} + p_{14} u_{n,m+1} + p_{15} u_{n+1,m+1} + p_{16} = 0 \end{aligned} \tag{15}$$

i.e. the p_i coefficients have the following expression:

$$p_i = p_{i,0} + p_{i,1}(-1)^n + p_{i,2}(-1)^m + p_{i,3}(-1)^{n+m}, \quad i = 1, \dots, 16. \tag{16}$$

If we impose the two-periodic Klein symmetry condition (14) with $\tau = \tau' = 1$ the 64 coefficients of (15) turn out to be related among themselves and we can choose among them 16 independent coefficients. In term of the 16 independent coefficients (15) reads:

$$\begin{aligned} &a_1 u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \\ &+ \left[a_{2,0} - (-1)^n a_{2,1} - (-1)^m a_{2,2} + (-1)^{n+m} a_{2,3}\right] u_{n,m} u_{n,m+1} u_{n+1,m+1} \end{aligned}$$

$$
\begin{aligned}
&+\left[a_{2,0}+(-1)^n a_{2,1}-(-1)^m a_{2,2}-(-1)^{n+m} a_{2,3}\right] u_{n+1,m}u_{n,m+1}u_{n+1,m+1}\\
&+\left[a_{2,0}+(-1)^n a_{2,1}+(-1)^m a_{2,2}+(-1)^{n+m} a_{2,3}\right] u_{n,m}u_{n+1,m}u_{n+1,m+1}\\
&+\left[a_{2,0}-(-1)^n a_{2,1}+(-1)^m a_{2,2}-(-1)^{n+m} a_{2,3}\right] u_{n,m}u_{n+1,m}u_{n,m+1}\\
&+\left[a_{3,0}-(-1)^m a_{3,2}\right] u_{n,m}u_{n+1,m}\\
&+\left[a_{3,0}+(-1)^m a_{3,2}\right] u_{n,m+1}u_{n+1,m+1}\\
&+\left[a_{4,0}-(-1)^{n+m} a_{4,3}\right] u_{n,m}u_{n+1,m+1}\\
&+\left[a_{4,0}+(-1)^{n+m} a_{4,3}\right] u_{n+1,m}u_{n,m+1}\\
&+\left[a_{5,0}-(-1)^n a_{5,1}\right] u_{n+1,m}u_{n+1,m+1}\\
&+\left[a_{5,0}+(-1)^n a_{5,1}\right] u_{n,m}u_{n,m+1}\\
&+\left[a_{6,0}+(-1)^n a_{6,1}-(-1)^m a_{6,2}-(-1)^{n+m} a_{6,3}\right] u_{n,m}\\
&+\left[a_{6,0}-(-1)^n a_{6,1}-(-1)^m a_{6,2}+(-1)^{n+m} a_{6,3}\right] u_{n+1,m}\\
&+\left[a_{6,0}+(-1)^n a_{6,1}+(-1)^m a_{6,2}+(-1)^{n+m} a_{6,3}\right] u_{n,m+1}\\
&+\left[a_{6,0}-(-1)^n a_{6,1}+(-1)^m a_{6,2}-(-1)^{n+m} a_{6,3}\right] u_{n+1,m+1}\\
&+a_7=0.
\end{aligned}
\tag{17}
$$

Upon the substitution $a_{2,1}=a_{2,2}=a_{2,3}=a_{3,2}=a_{4,3}=a_{5,1}=a_{6,1}=a_{6,2}=a_{6,3}=0$ (17) reduces to the Q_{V} equation (6). Therefore, we will call (17) *the two-periodic Q_{V} equation*.

If we impose the two-periodic Klein symmetry condition (14) with the choice $\tau=1$ and $\tau'=-1$ we will get an expression which can be reduced to (17) by multiplying by $(-1)^n$ and redefining the coefficients. In an analogous manner, the two remaining cases $\tau=-1$, $\tau'=1$ and $\tau=\tau'=-1$ can be identified with the case $\tau=\tau'=1$ multiplying by $(-1)^n$ and $(-1)^{n+m}$, respectively and redefining the coefficients. Therefore the only equation belonging to the class of the lattice equation possessing the two-periodic Klein symmetries is just the two-periodic Q_{V} equation (17).

We note that the two-periodic Q_{V} equation contains as particular cases the rhombic H^4 equations, the trapezoidal H^4 equations and the H^6 equations. The explicit identification of the coefficients of such equations is given in Table 1. The reader can refer to Appendix A for the explicit expressions of these equations or to [11] for a complete derivation following the prescription of [6].

In order to establish if (17) is integrable we use the algebraic entropy integrability test [3, 14, 24–26], using the program `ae2d.py` [9, 10]. Applying it to the two-periodic Q_{V} equation, we find the following degree of growth in all directions:

$$1,3,7,13,21,31,43,57,73,91,111,133\ldots, \tag{18}$$

which is the same as for the autonomous Q_{V} equation [26]. The generating function for the sequence (18) is:

$$g(z)=\frac{1+z^2}{(1-z)^3}, \tag{19}$$

TABLE 1. Identification of the coefficients of the two-periodic Q_{V} equation with those of the Boll's equations (A.2,A.3, A.4). Since $a_1 = a_{2,i} = 0$ for every equation these coefficients absent in the table.

Equation	$a_{3,0}$	$a_{3,2}$	$a_{4,0}$	$a_{4,3}$	$a_{5,0}$	$a_{5,1}$	$a_{6,0}$	$a_{6,1}$	$a_{6,2}$	$a_{6,3}$	a_7
${}_rH_1^{\varepsilon}$	1	0	$\frac{1}{2}\varepsilon(\alpha-\beta)$	$\frac{1}{2}\varepsilon(\alpha-\beta)$	-1	0	0	0	0	0	$\beta-\alpha$
${}_rH_2^{\varepsilon}$	1	0	$2\varepsilon(\beta-\alpha)$	$2\varepsilon(\beta-\alpha)$	-1	0	$-(\alpha-\beta)(\varepsilon\alpha+1+\varepsilon\beta)$	0	0	$\varepsilon\left(\beta^2-\alpha^2\right)$	$-(\alpha-\beta)\left(2\varepsilon\alpha^2+\alpha+2\varepsilon\beta^2+\beta\right)$
${}_rH_3^{\varepsilon}$	α	0	$\frac{1}{2}\frac{\varepsilon(\beta^2-\alpha^2)}{\alpha\beta}$	$\frac{1}{2}\frac{\varepsilon(\beta^2-\alpha^2)}{\alpha\beta}$	$-\beta$	0	0	0	0	0	$\delta\left(\alpha^2-\beta^2\right)$
${}_tH_1^{\varepsilon}$	$-\frac{1}{2}\alpha_2\varepsilon^2$	$-\frac{1}{2}\alpha_2\varepsilon^2$	-1	0	1	0	0	0	0	0	$-\alpha_2$
${}_tH_2^{\varepsilon}$	$\varepsilon\alpha_2$	$\varepsilon\alpha_2$	-1	0	1	0	$\frac{1}{2}\alpha_2\left[2+\varepsilon(2\alpha_2+\alpha_3)\right]$	0	$\varepsilon\alpha_2\alpha_3+\frac{1}{2}\varepsilon{\alpha_2}^2$	0	$\alpha_2\left[\alpha_2+2\alpha_3+\varepsilon\left(\alpha_2+\alpha_3\right)^2\right]$
${}_tH_3^{\varepsilon}$	$\frac{1}{2}\frac{\varepsilon^2(1-{\alpha_2}^2)}{\alpha_3\alpha_2}$	$\frac{1}{2}\frac{\varepsilon^2(1-{\alpha_2}^2)}{\alpha_3\alpha_2}$	α_2	0	-1	0	0	0	0	0	$\delta^2\alpha_3\left(1-\alpha_2^2\right)$
${}_1D_2$	$\frac{1}{2}\delta_1$	$\frac{1}{2}\delta_1$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_2)$	$\frac{1}{4}(\delta_2-\delta_1)$	$-\frac{1}{4}(\delta_1+\delta_2)$	$\frac{1}{2}-\frac{1}{4}(\delta_1+\delta_2)$	0
${}_2D_2$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\delta_1$	$\frac{1}{2}\delta_1$	0	0	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_2+\delta_1\lambda)$	$\frac{1}{4}(\delta_1\lambda-\delta_1+\delta_2)$	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_1\lambda+\delta_2)$	$-\frac{1}{4}(\delta_1+\delta_1\lambda+\delta_2)$	$-\delta_1\delta_2\lambda$
${}_3D_2$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}\delta_1$	$-\frac{1}{2}\delta_1$	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_2+\delta_1\lambda)$	$\frac{1}{4}(\delta_1+\delta_1\lambda+\delta_2)$	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_1\lambda+\delta_2)$	$\frac{1}{4}(\delta_1-\delta_1\lambda-\delta_2)$	$-\delta_1\delta_2\lambda$
D_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0
${}_1D_4$	$\frac{1}{2}\delta_2$	$\frac{1}{2}\delta_2$	1	0	$\frac{1}{2}\delta_1$	$-\frac{1}{2}\delta_1$	0	0	0	0	δ_3
${}_2D_4$	1	0	$\frac{1}{2}\delta_2$	$\frac{1}{2}\delta_2$	$\frac{1}{2}\delta_1$	$-\frac{1}{2}\delta_1$	0	0	0	0	δ_3

which implies that we have the following quadratic fit for the growth:

$$d_k = k(k+1) + 1 \tag{20}$$

and thus the algebraic entropy is zero. This is a strong indication of the integrability of the two-periodic $Q_{\rm V}$ equation (17).

Using (7, 8) or (9, 10) with $Q_{\rm V}$ substituted by its two-periodic version, we get a version of the two-periodic YdKN (1), however the proof that this is effectively a symmetry of the two-periodic $Q_{\rm V}$ encounters serious computational difficulties. We can prove by a direct computation its validity for the following sub-cases:

- When $Q_{\rm V}$ equation is two-periodic with respect to one direction only, either n or m. All the trapezoidal H^4 equations belong to these two sub-classes;
- For all the H^6 equations, which are two-periodic in both directions.

Its validity for the autonomous $Q_{\rm V}$ and for all the rhombic H^4 equations was already shown, respectively, in [27] and [28]. However, we cannot prove its validity for the general case (17) due to our computer power limitation.

Here follows the connection formulae for the general two-periodic case (17). For the n directional symmetry, we have:

$$\begin{aligned}
a &= a_1a_{3,0} - a_{2,0}^2 + \\
&\quad + a_{2,1}^2 - a_{2,2}^2 + a_{2,3}^2 - (-1)^m(2a_{2,0}a_{2,2} - 2a_{2,1}a_{2,3} + a_1a_{3,2}),\\
b_0 &= \frac{1}{2}\{a_{2,0}(a_{3,0} - a_{5,0} - a_{4,0}) + a_1a_{6,0} + a_{2,2}a_{3,2} - a_{2,3}a_{4,3} - a_{2,1}a_{5,1} - \\
&\quad - (-1)^m[a_{2,2}(a_{5,0} + a_{3,0} + a_{4,0}) + a_{2,3}a_{5,1} + a_1a_{6,2} + a_{2,0}a_{3,2} + \\
&\quad + a_{2,1}a_{4,3}]\},\\
b_1 &= \frac{1}{2}\{a_{2,1}(a_{3,0} - a_{4,0} + a_{5,0}) + a_{2,3}a_{3,2} - a_{2,2}a_{4,3} + a_{2,0}a_{5,1} - a_1a_{6,1} + \\
&\quad + (-1)^m[a_1a_{6,3} - a_{2,3}(a_{3,0} + a_{4,0} - a_{5,0}) - a_{2,1}a_{3,2} - a_{2,0}a_{4,3} + \\
&\quad + a_{2,2}a_{5,1}]\},\\
c_0 &= a_{2,0}a_{6,0} - a_{4,0}a_{5,0} - a_{2,1}a_{6,1} - a_{2,3}a_{6,3} + a_{2,2}a_{6,2} + \\
&\quad - (-1)^m[a_{2,2}a_{6,0} - a_{4,3}a_{5,1} - a_{2,3}a_{6,1} + a_{2,0}a_{6,2} - a_{2,1}a_{6,3}],\\
c_1 &= a_{4,0}a_{5,1} + a_{2,1}a_{6,0} - a_{2,0}a_{6,1} + a_{2,3}a_{6,2} - a_{2,2}a_{6,3} + \\
&\quad + (-1)^m[a_{2,2}a_{6,1} - a_{4,3}a_{5,0} - a_{2,3}a_{6,0} - a_{2,1}a_{6,2} + a_{2,0}a_{6,3}],\\
d &= \frac{1}{2}[a_{3,0}^2 - a_{4,0}^2 - a_{5,0}^2 + a_1a_7 - a_{3,2}^2 + a_{4,3}^2 + a_{5,1}^2 - \\
&\quad - 4(-1)^m(a_{2,2}a_{6,0} + a_{2,3}a_{6,1} + a_{2,0}a_{6,2} + a_{2,1}a_{6,3}],\\
e_0 &= \frac{1}{2}\{a_{6,0}[a_{3,0} - a_{4,0} - a_{5,0}] + a_{2,0}a_7 + a_{5,1}a_{6,1} - a_{3,2}a_{6,2} + a_{4,3}a_{6,3} + \\
&\quad + (-1)^m[a_{3,2}a_{6,0} + a_{4,3}a_{6,1} + a_{5,1}a_{6,3} - a_{6,2}(a_{3,0} + a_{4,0} + a_{5,0}) - a_{2,2}a_7]\},
\end{aligned} \tag{21}$$

$$
\begin{aligned}
e_1 &= \frac{1}{2}\{a_{6,1}[a_{3,0}-a_{4,0}+a_{5,0}]-a_{5,1}a_{6,0}+a_{4,3}a_{6,2}-a_{3,2}a_{6,3}-a_{2,1}a_7+\\
&\quad+(-1)^m[a_{4,3}a_{6,0}+a_{3,2}a_{6,1}-a_{5,1}a_{6,2}+a_{6,3}(a_{5,0}-a_{3,0}-a_{4,0})+a_{2,3}a_7\},\\
f &= a_{3,0}a_7-a_{6,0}^2-a_{6,2}^2+a_{6,3}^2+a_{6,1}^2-(-1)^m(2a_{6,0}a_{6,2}-2a_{6,1}a_{6,3}-a_{3,2}a_7).
\end{aligned}
$$

The two-periodic Q_V is not symmetric in the exchange of n and m, so its symmetries in the m direction are different in their dependence on the coefficients and so are the connection formulae, however, for the sake of the reader, as they are not essentially different from those presented above, we do not write them here but present in Appendix B, cfr. (B.4).

3. Conclusions

In this article, we propose a two-periodic extension of the Q_V equation (17) which satisfies an extended Klein symmetry (14) in such a way that in the autonomous sub-case reduces to the Q_V equation (6).

The so obtained two-periodic Q_V equation includes all equations of the Boll classification as its sub-cases and results integrable by the algebraic entropy test. Using the construction proposed by Xenitidis [27], one builds up a symmetry for the two-periodic Q_V with respect to one direction only; this symmetry turns out to belong to the class of the two-periodic YdKN equation proposed by Levi and Yamilov [16], an equation satisfying all the integrability conditions obtained by the formal generalized symmetry method. One is confident that the results obtained using the formulae (7, 9) by Xenitidis [27] are correct also in general setting, as for all the sub-cases of the two-periodic Q_V belonging to the Boll classification the three point generalized symmetries, calculated using both the definition or through the formula by Xenitidis, coincide and belong to the class of the two-periodic YdKN.

However, we have not been able to have a complete direct proof in the general two-periodic case due to computational complexities. We leave to a following work the calculation of the recurrence formula for the two-periodic extension of the Q_V and thus the proof that we have an infinite set of generalized symmetries.

In all generality, we have two possible connection formulae (21, B.4) between the 16 coefficients of the general two-periodic Q_V and the 9 coefficients of the two-periodic YdKN, one corresponding to the Xenitidis formula (7) along the n direction and another one corresponding to (9) along m.

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Appendix A. Explicit form of the H^4 and H^6 equations

Throughout this appendix, we will use the notation:

$$
F_k^{\pm} = \frac{1 \pm (-1)^k}{2}, \quad k \in \mathbb{Z}. \tag{A.1}
$$

Rhombic H^4 equations:

$$
\begin{aligned}
{}_rH_1^{\varepsilon}\colon \quad & (u_{n,m} - u_{n+1,m+1})\,(u_{n+1,m} - u_{n,m+1}) \,-\, (\alpha \,-\, \beta) && \text{(A.2a)}\\
& + \varepsilon(\alpha - \beta)\left(F_{n+m}^{(+)}\, u_{n+1,m}u_{n,m+1} + F_{n+m}^{(-)}\, u_{n,m}u_{n+1,m+1}\right) = 0, \\
{}_rH_2^{\varepsilon}\colon \quad & (u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) + && \text{(A.2b)}\\
& + (\beta - \alpha)(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) - \alpha^2 + \beta^2 \\
& - \varepsilon\,(\beta - \alpha)^3 - \varepsilon\,(\beta - \alpha)\,\left(2F_{n+m}^{(-)}u_{n,m} + 2F_{n+m}^{(+)}u_{n+1,m} + \alpha + \beta\right)\cdot \\
& \cdot\left(2F_{n+m}^{(-)}u_{n+1,m+1} + 2F_{n+m}^{(+)}u_{n,m+1} + \alpha + \beta\right) \,=\, 0, \\
{}_rH_3^{\varepsilon}\colon \quad & \alpha\,(u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1}) && \text{(A.2c)}\\
& - \beta(u_{n,m}u_{n,m+1} + u_{n+1,m}u_{n+1,m+1}) + (\alpha^2 - \beta^2)\delta \\
& - \frac{\varepsilon(\alpha^2 - \beta^2)}{\alpha\beta}\left(F_{n+m}^{(+)}\, u_{n+1,m}u_{n,m+1} + F_{n+m}^{(-)}\, u_{n,m}u_{n+1,m+1}\right) = 0.
\end{aligned}
$$

Trapezoidal H^4 equations:

$$
\begin{aligned}
{}_tH_1\colon\; & \left(u_{n,m} - u_{n+1,m}\right)\left(u_{n,m+1} - u_{n+1,m+1}\right) - && \text{(A.3a)}\\
& - \alpha_2\varepsilon^2\left(F_m^{(+)}u_{n,m+1}u_{n+1,m+1} + F_m^{(-)}u_{n,m}u_{n+1,m}\right) - \alpha_2 = 0, \\
{}_tH_2\colon\; & \left(u_{n,m} - u_{n+1,m}\right)\left(u_{n,m+1} - u_{n+1,m+1}\right) && \text{(A.3b)}\\
& + \alpha_2\left(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}\right) \\
& + \frac{\varepsilon\alpha_2}{2}\left(2F_m^{(+)}u_{n,m+1} + 2\alpha_3 + \alpha_2\right)\left(2F_m^{(+)}u_{n+1,m+1} + 2\alpha_3 + \alpha_2\right) \\
& + \frac{\varepsilon\alpha_2}{2}\left(2F_m^{(-)}u_{n,m} + 2\alpha_3 + \alpha_2\right)\left(2F_m^{(-)}u_{n+1,m} + 2\alpha_3 + \alpha_2\right) \\
& + (\alpha_3 + \alpha_2)^2 - \alpha_3^2 - 2\varepsilon\alpha_2\alpha_3\,(\alpha_3 + \alpha_2) = 0, \\
{}_tH_3\colon\; & \alpha_2\left(u_{n,m}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1}\right) && \text{(A.3c)}\\
& - \left(u_{n,m}u_{n,m+1} + u_{n+1,m}u_{n+1,m+1}\right) - \alpha_3\left(\alpha_2^2 - 1\right)\delta^2 + \\
& - \frac{\varepsilon^2(\alpha_2^2 - 1)}{\alpha_3\alpha_2}\left(F_m^{(+)}u_{n,m+1}u_{n+1,m+1} + F_m^{(-)}u_{n,m}u_{n+1,m}\right) = 0.
\end{aligned}
$$

H^6 equations:

$$
\begin{aligned}
{}_1D_2:\quad & \left(F^{(-)}_{n+m}-\delta_1 F^{(+)}_n F^{(-)}_m+\delta_2 F^{(+)}_n F^{(+)}_m\right)u_{n,m} \\
&+\left(F^{(+)}_{n+m}-\delta_1 F^{(-)}_n F^{(-)}_m+\delta_2 F^{(-)}_n F^{(+)}_m\right)u_{n+1,m}+ \\
&+\left(F^{(+)}_{n+m}-\delta_1 F^{(+)}_n F^{(+)}_m+\delta_2 F^{(+)}_n F^{(-)}_m\right)u_{n,m+1} \\
&+\left(F^{(-)}_{n+m}-\delta_1 F^{(-)}_n F^{(+)}_m+\delta_2 F^{(-)}_n F^{(-)}_m\right)u_{n+1,m+1}+ \\
&+\delta_1\left(F^{(-)}_m u_{n,m}u_{n+1,m}+F^{(+)}_m u_{n,m+1}u_{n+1,m+1}\right) \\
&+F^{(+)}_{n+m}u_{n,m}u_{n+1,m+1}+F^{(-)}_{n+m}u_{n+1,m}u_{n,m+1}=0,
\end{aligned}
\tag{A.4a}
$$

$$
\begin{aligned}
{}_2D_2:\quad & \left(F^{(-)}_m-\delta_1 F^{(+)}_n F^{(-)}_m+\delta_2 F^{(+)}_n F^{(+)}_m-\delta_1\lambda F^{(-)}_n F^{(+)}_m\right)u_{n,m} \\
&+\left(F^{(-)}_m-\delta_1 F^{(-)}_n F^{(-)}_m+\delta_2 F^{(-)}_n F^{(+)}_m-\delta_1\lambda F^{(+)}_n F^{(+)}_m\right)u_{n+1,m} \\
&+\left(F^{(+)}_m-\delta_1 F^{(+)}_n F^{(+)}_m+\delta_2 F^{(+)}_n F^{(-)}_m-\delta_1\lambda F^{(-)}_n F^{(-)}_m\right)u_{n,m+1} \\
&+\left(F^{(+)}_m-\delta_1 F^{(-)}_n F^{(+)}_m+\delta_2 F^{(-)}_n F^{(-)}_m-\delta_1\lambda F^{(+)}_n F^{(-)}_m\right)u_{n+1,m+1} \\
&+\delta_1\left(F^{(-)}_{n+m}u_{n,m}u_{n+1,m+1}+F^{(+)}_{n+m}u_{n+1,m}u_{n,m+1}\right) \\
&+F^{(+)}_m u_{n,m}u_{n+1,m}+F^{(-)}_m u_{n,m+1}u_{n+1,m+1}-\delta_1\delta_2\lambda=0,
\end{aligned}
\tag{A.4b}
$$

$$
\begin{aligned}
{}_3D_2:\quad & \left(F^{(-)}_m-\delta_1 F^{(-)}_n F^{(-)}_m+\delta_2 F^{(+)}_n F^{(+)}_m-\delta_1\lambda F^{(-)}_n F^{(+)}_m\right)u_{n,m} \\
&+\left(F^{(-)}_m-\delta_1 F^{(+)}_n F^{(-)}_m+\delta_2 F^{(-)}_n F^{(+)}_m-\delta_1\lambda F^{(+)}_n F^{(+)}_m\right)u_{n+1,m} \\
&+\left(F^{(+)}_m-\delta_1 F^{(-)}_n F^{(+)}_m+\delta_2 F^{(+)}_n F^{(-)}_m-\delta_1\lambda F^{(-)}_n F^{(-)}_m\right)u_{n,m+1} \\
&+\left(F^{(+)}_m-\delta_1 F^{(+)}_n F^{(+)}_m+\delta_2 F^{(-)}_n F^{(-)}_m-\delta_1\lambda F^{(+)}_n F^{(-)}_m\right)u_{n+1,m+1} \\
&+\delta_1\left(F^{(-)}_n u_{n,m}u_{n,m+1}+F^{(+)}_n u_{n+1,m}u_{n+1,m+1}\right) \\
&+F^{(-)}_m u_{n,m+1}u_{n+1,m+1}+F^{(+)}_m u_{n,m}u_{n+1,m}-\delta_1\delta_2\lambda=0,
\end{aligned}
\tag{A.4c}
$$

$$
\begin{aligned}
D_3:\quad & F^{(+)}_n F^{(+)}_m u_{n,m}+F^{(-)}_n F^{(+)}_m u_{n+1,m}+F^{(+)}_n F^{(-)}_m u_{n,m+1} \\
&+F^{(-)}_n F^{(-)}_m u_{n+1,m+1}+F^{(-)}_m u_{n,m}u_{n+1,m} \\
&+F^{(-)}_n u_{n,m}u_{n,m+1}+F^{(-)}_{n+m}u_{n,m}u_{n+1,m+1}+ \\
&+F^{(+)}_{n+m}u_{n+1,m}u_{n,m+1}+F^{(+)}_n u_{n+1,m}u_{n+1,m+1} \\
&+F^{(+)}_m u_{n,m+1}u_{n+1,m+1}=0,
\end{aligned}
\tag{A.4d}
$$

$$
\begin{aligned}
{}_1D_4:\quad & \delta_1\left(F^{(-)}_n u_{n,m}u_{n,m+1}+F^{(+)}_n u_{n+1,m}u_{n+1,m+1}\right)+ \\
&+\delta_2\left(F^{(-)}_m u_{n,m}u_{n+1,m}+F^{(+)}_m u_{n,m+1}u_{n+1,m+1}\right)+ \\
&+u_{n,m}u_{n+1,m+1}+u_{n+1,m}u_{n,m+1}+\delta_3=0,
\end{aligned}
\tag{A.4e}
$$

$$
\begin{aligned}
{}_2D_4:\quad & \delta_1\left(F^{(-)}_n u_{n,m}u_{n,m+1}+F^{(+)}_n u_{n+1,m}u_{n+1,m+1}\right)+ \\
&+\delta_2\left(F^{(-)}_{n+m}u_{n,m}u_{n+1,m+1}+F^{(+)}_{n+m}u_{n+1,m}u_{n,m+1}\right)+ \\
&+u_{n,m}u_{n+1,m}+u_{n,m+1}u_{n+1,m+1}+\delta_3=0.
\end{aligned}
\tag{A.4f}
$$

Appendix B. Correlation formulae between the two-periodic Q_{V} and the two-periodic YdKN in the *m* direction

For the sake of completeness, let us write down the two-periodic YdKN in the m direction:

$$\frac{\mathrm{d}u_m}{\mathrm{d}t} = \frac{A_m(u_m)u_{m+1}u_{m-1} + B_m(u_m)(u_{m+1}+u_{m-1}) + C_m(u_m)}{u_{m+1}-u_{m-1}}. \tag{B.1}$$

Here the m-dependent coefficients are given by:

$$A_m(u_m) = au_m^2 + 2b_m u_m + c_m, \tag{B.2a}$$

$$B_m(u_m) = b_{m+1}u_m^2 + du_m + e_{m+1}, \tag{B.2b}$$

$$C_m(u_m) = c_{m+1}u_m^2 + 2e_m u_m + f, \tag{B.2c}$$

where b_m, c_m and e_m are two-periodic functions, i.e.

$$b_m = b_0 + b_1(-1)^m,\ c_m = c_0 + c_1(-1)^m,\ e_m = e_0 + e_1(-1)^m. \tag{B.3}$$

The correlation formulae read:

$$\begin{aligned}
a &= a_1a_{5,0} - a_{2,0}^2 + \\
&\quad - a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2 + (-1)^n(2a_{2,0}a_{2,1} - 2a_{2,2}a_{2,3} + a_1a_{5,1}),\\
b_0 &= \frac{1}{2}\{a_{2,0}(a_{5,0} - a_{3,0} - a_{4,0}) + a_1a_{6,0} - a_{2,2}a_{3,2} - a_{2,3}a_{4,3} + a_{2,1}a_{5,1} + \\
&\quad + (-1)^n[a_{2,1}(a_{5,0} + a_{3,0} + a_{4,0}) + a_{2,3}a_{3,2} + a_1a_{6,1} + a_{2,0}a_{5,1} + \\
&\quad + a_{2,2}a_{4,3}]\},\\
b_1 &= \frac{1}{2}\{a_{2,2}(a_{4,0} - a_{3,0} - a_{5,0}) - a_{2,3}a_{5,1} + a_{2,1}a_{4,3} - a_{2,0}a_{3,2} + a_1a_{6,2} + \\
&\quad + (-1)^n[a_1a_{6,3} + a_{2,3}(a_{3,0} - a_{4,0} - a_{5,0}) + a_{2,1}a_{3,2} - a_{2,0}a_{4,3} - \\
&\quad - a_{2,2}a_{5,1}]\},\\
c_0 &= a_{4,0}a_{3,0} - a_{2,0}a_{6,0} - a_{2,1}a_{6,1} + a_{2,3}a_{6,3} + a_{2,2}a_{6,2} + \\
&\quad - (-1)^n[a_{2,2}a_{6,3} + a_{4,3}a_{3,2} + a_{2,3}a_{6,2} - a_{2,0}a_{6,1} - a_{2,1}a_{6,0}],\\
c_1 &= a_{2,1}a_{6,3} - a_{4,0}a_{3,2} + a_{2,0}a_{6,2} - a_{2,3}a_{6,1} - a_{2,2}a_{6,0} - \\
&\quad - (-1)^n[a_{2,2}a_{6,1} + a_{4,3}a_{3,0} + a_{2,3}a_{6,0} - a_{2,1}a_{6,2} - a_{2,0}a_{6,3}],\\
d &= \frac{1}{2}[a_{5,0}^2 - a_{4,0}^2 - a_{3,0}^2 + a_1a_7 + a_{3,2}^2 + a_{4,3}^2 - a_{5,1}^2 + \\
&\quad + 4(-1)^n(a_{2,1}a_{6,0} + a_{2,0}a_{6,1} + a_{2,3}a_{6,2} + a_{2,2}a_{6,3}],\\
e_0 &= \frac{1}{2}\{a_{6,0}[a_{5,0} - a_{4,0} - a_{3,0}] + a_{2,0}a_7 - a_{5,1}a_{6,1} + a_{3,2}a_{6,2} + a_{4,3}a_{6,3} + \\
&\quad + (-1)^n[a_{3,2}a_{6,3} + a_{4,3}a_{6,2} + a_{5,1}a_{6,0} - a_{6,1}(a_{3,0} + a_{4,0} + a_{5,0}) - a_{2,1}a_7]\},
\end{aligned} \tag{B.4}$$

$$e_1 = \frac{1}{2}\{a_{6,2}[a_{3,0} - a_{4,0} + a_{5,0}] - a_{3,2}a_{6,0} + a_{4,3}a_{6,1} - a_{5,1}a_{6,3} - a_{2,2}a_7 + \\ + (-1)^n[a_{4,3}a_{6,0} - a_{3,2}a_{6,1} + a_{5,1}a_{6,2} + a_{6,3}(a_{3,0} - a_{5,0} - a_{4,0}) + a_{2,3}a_7\},$$
$$f = a_{5,0}a_7 - a_{6,0}^2 + a_{6,2}^2 + a_{6,3}^2 - a_{6,1}^2 - (-1)^n(2a_{6,2}a_{6,3} - 2a_{6,0}a_{6,1} + a_{5,1}a_7).$$

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