SPDEs with linear multiplicative fractional noise: continuity in law with respect to the Hurst index

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Abstract

In this article, we consider the one-dimensional stochastic wave and heat equations driven by a linear multiplicative Gaussian noise which is white in time and behaves in space like a fractional Brownian motion with Hurst index $H \in (\frac{1}{4}, 1)$. We prove that the solution of each of the above equations is continuous in terms of the index H, with respect to the convergence in law in the space of continuous functions. The proof is based on a tightness criterion on the plane and Malliavin calculus techniques in order to identify the limit law.

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1 Introduction

In this article, we consider the Hyperbolic Anderson Model

$$\begin{cases} \dfrac{\partial^2 u^H}{\partial t^2}(t,x) = \dfrac{\partial^2 u^H}{\partial x^2}(t,x) + u^H(t,x)\dot{W}^H(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u^H(0,x) = \eta, \ x \in \mathbb{R}, \\ \dfrac{\partial u^H}{\partial t}(0,x) = 0, \ x \in \mathbb{R}, \end{cases} \tag{SWE}$$

and the Parabolic Anderson Model

$$\begin{cases} \dfrac{\partial u^H}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u^H}{\partial x^2}(t,x) + u^H(t,x)\dot{W}^H(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u^H(0,x) = \eta, \ x \in \mathbb{R}. \end{cases} \tag{SHE}$$

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The initial condition $\eta \in \mathbb{R}$ is assumed to be constant. The random perturbation $\dot{W}^H$ is a Gaussian noise which is white in time and behaves in space like a fractional Brownian motion with Hurst index $H \in (\frac{1}{4}, 1)$. More precisely, it is given by a family of centered Gaussian random variables $W^H = \{W^H(\varphi),\, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R})\}$, indexed in the space of $\mathcal{C}^\infty$ functions with compact support on $\mathbb{R}_+ \times \mathbb{R}$, with the following covariance structure:

$$\mathbb{E}\left[W^H(\varphi)W^H(\psi)\right] = \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\mu_H(d\xi)dt,$$

for any $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, where the measure μ_H is given by $\mu_H(d\xi) = c_H\,|\xi|^{1-2H}d\xi$, with

$$c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}. \tag{1}$$

We denote by $\mathcal{F}$ the Fourier transform in the space variable, which is defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx, \quad f \in L^1(\mathbb{R}).$$

The solutions of (SWE) and (SHE) are understood in the mild Itô sense, as follows. We fix a time horizon $T > 0$ and we denote by $\{\mathcal{F}_t^H,\, t \geq 0\}$ the filtration generated by the noise W^H (conveniently completed). Then, we say that an adapted and jointly measurable random field $u^H = \{u^H(t,x),\, (t,x) \in [0,T] \times \mathbb{R}\}$ solves (SWE) (resp. (SHE)) if it holds, for all $(t,x) \in [0,T] \times \mathbb{R}$:

$$u^H(t,x) = \eta + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)u^H(s,y)W^H(ds,dy), \quad \mathbb{P}\text{-a.s.} \tag{2}$$

where G is the fundamental solution of the wave (resp. heat) equation in $\mathbb{R}$. We recall that

$$G_t(x) = \begin{cases} \frac{1}{2}1_{\{|x|<t\}}(x), & \text{wave equation,} \\[2ex] \dfrac{1}{(2\pi t)^{\frac{1}{2}}}\exp\Big(-\dfrac{|x|^2}{2t}\Big), & \text{heat equation.} \end{cases} \tag{3}$$

The stochastic integral appearing in (2) is understood in the Itô sense and will be described in detail in Section 2.3.

In this paper, we are interested in studying the continuity in law, in the space of continuous functions, of the solutions to (SWE) and (SHE) with respect to the Hurst index H. More precisely, the main result of the paper is the following:

Theorem 1.1. *Let $T > 0$. Let $H_0 \in (\frac{1}{4}, 1)$ and $\{H_n,\, n \geq 1\} \subset (\frac{1}{4}, 1)$ be any sequence converging to H_0. Then, u^{H_n} converges to u^{H_0}, as $n \to \infty$, in law in the space $\mathcal{C}([0,T] \times \mathbb{R}))$ of continuous functions, endowed with the metric of uniform convergence on compact sets.*

We point out that we restrict to Hurst indices greater than $\frac{1}{4}$. This is due to the fact that, as proved in [4, Prop. 3.7], $H > \frac{1}{4}$ is also a necessary condition in order to have a solution to (SWE) and (SHE).

The above theorem can be considered a continuation of the results obtained by the authors in [12] (see Theorem 4.2 therein), where the same kind of problem has been addressed for one-dimensional quasi-linear stochastic wave and heat equations with an additive fractional noise as the one described above. The proof of the latter result, which is indeed valid for any $H_0 \in (0,1)$, is based on the fact that the solution of the underlying SPDE can be represented as the image of the stochastic convolution through a continuous functional on the space $\mathcal{C}([0,T]\times\mathbb{R})$). In the present paper, this technique cannot be applied anymore because of the structure of the linear multiplicative noise. Instead, we consider the following strategy.

First, we prove that the sequence of probability measures induced by $\{u^{H_n},\, n\geq 1\}$ is tight in the space $\mathcal{C}([0,T]\times\mathbb{R})$ (see Section 3). Here, we split the proof taking into account that the sequence of Hurst indices is contained in $(\frac{1}{4},\frac{1}{2}]$ or $[\frac{1}{2},1)$, for the definition and properties of the stochastic integral in (2) differ significantly between those two cases. Indeed, the main difficulty here is concentrated in the *rough* case, where we carefully extend some moment estimates appearing in [4] in order to make them uniform with respect to $H\in(\frac{1}{4},\frac{1}{2})$.

Secondly, in order to identify the limit law, we prove the convergence of the corresponding finite dimensional distributions (see Section 4). The main problem here comes from the fact that the solution u^H is not a Gaussian process, and so identifying its covariance structure is not enough to characterize its law. However, thanks to a spectral representation in law of our noise W^H in terms of a complex-valued Gaussian measure (extending of some classical results in [21]), we are able to define the whole family of noises $\{W^H,\, H\in(0,1)\}$ in a single probability space and then check that, for any fixed $(t,x)\in[0,T]\times\mathbb{R}$, $u^{H_n}(t,x)$ converges to $u^{H_0}(t,x)$ in $L^2(\Omega)$. For this, we will use techniques of the Malliavin calculus, precisely the Wiener chaos expansion of the mild Skorohod solutions of (SWE) and (SHE). In the process of applying this methodology, we provide three preliminary results which have their own interest and turn out to be crucial in our main result's proof:

(i) For any $H\in(0,1)$, we prove that any multiple Wiener integral with respect to W^H admits a representation as a multiple Wiener integral with respect to the above-mentioned complex-valued Gaussian measure (see Theorem 2.7).

(ii) For any $H\in(\frac{1}{4},1)$, we prove an equivalence result between Itô and Skorohod stochastic integrals with respect to W^H (see Theorem 2.11). This result has already been proved in [4, Thm. 4.2] for the case $H<\frac{1}{2}$, and we extend it to $H\geq\frac{1}{2}$. We point out that the latter case, in which the noise is more regular, entails some extra difficulties due to the fact that the underlying Hilbert space associated to the noise's covariance contains distributions. An important consequence of Theorem 2.11 is that mild Itô and Skorohod solutions to (SWE) (resp. (SHE)) coincide, and the corresponding Picard iteration scheme admits a (finite) Wiener chaos decomposition.

(iii) In the setting $H\in(\frac{1}{4},\frac{1}{2})$, we prove a Sobolev embedding-type result for the norms of the Banach space on which we define our solutions (see Lemma 2.17). This result is similar to classical embedding results, e.g. the ones appearing in [11], but takes into account the different nature of the Sobolev norm in our setting.

The above strategy will be made clearer in Section 4 below, but let us remark at this point that the main strategy in this part of the paper does not require a separate analysis for the cases $H<\frac{1}{2}$ and $H\geq\frac{1}{2}$. Furthermore, the methodology used in both results on tightness and the limit identification cover equations (SWE) and (SHE) at the same time.

In the case of the stochastic heat equation (SHE) with $H>\frac{1}{2}$, the result in Theorem 1.1 is a particular case of [6, Thm. 1], where the author considers a general non-linear coefficient $\sigma(u^H(t,x))$ in front of the noise. We believe that such diffusion coefficient could be also considered in the case of the wave equation with $H>\frac{1}{2}$, but we have chosen to stick to the linear multiplicative noise in order to find a unified result that covers also the case $H<\frac{1}{2}$, which is more mathematically demanding.

Concerning other related results, we point out the recent article [19], in which the authors prove strong regularity properties in $H \in (0,1)$ of the Mandelbrot-van Ness representation of the fractional Brownian motion. As a consequence, it is proved that the solution of a scalar stochastic differential equation driven by the fractional Brownian motion is differentiable with respect to the Hurst parameter.

Finally, we also mention that continuity in law with respect to the Hurst index has been focused in other type of contexts beyond stochastic equations. For instance, in the series of papers [16, 17, 18], the authors study weak continuity with respect to H for different types of integrals with respect to fractional Brownian motion. In [15, 26], the same kind of continuity property has been tackled for the local time of the fractional Brownian motion and other Gaussian fields. Eventually, in the recent paper [1], the continuity property has been shown for additive functionals of the sub-fractional Brownian motion.

The paper is organized as follows. In Section 2, we give some preliminary tools that will be needed throughout the paper. Namely, we introduce the basic elements of the Malliavin calculus, we provide a new integral representation for the multiple Wiener integral with respect to W^H, we recall the construction of the stochastic integral with respect to W^H and, finally, we report about the existing well-posedness results for equations (SWE) and (SHE). Section 3 is devoted to prove the tightness property of the family of laws induced by the solution u^H, $H \in (\frac{1}{4}, 1)$. In Section 4, we deal with the limit identification, which allows us to conclude the proof of Theorem 1.1. In the Appendix, we collect some technical results and a tightness criterion that are used in the paper.

2 Preliminaries

2.1 Malliavin calculus

In this section, we recall some elements of Malliavin calculus and a useful result of [24]. We refer the reader to [20] for more details. We will work in the Gaussian space determined by the noise W^H, which is defined as follows.

Let $\langle \varphi, \psi \rangle_H := \mathbb{E}\left[W^H(\varphi) W^H(\psi)\right]$ and define $\mathcal{H}_H$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ with respect to the inner product $\langle \cdot, \cdot \rangle_H$. Then $\mathcal{H}_H$ defines a Hilbert space and it is well-known that, if $H \leq \frac{1}{2}$, it is a space of functions, while for $H > \frac{1}{2}$ it contains distributions (see [5, Thm. 4.3] and [14, Prop. 4.2]). Then, $\{W^H(\varphi),\ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R})\}$ can be extended to a family of Gaussian random variables indexed on the space $\mathcal{H}_H$, which we denote again by $W^H = \{W^H(\varphi),\ \varphi \in \mathcal{H}_H\}$. This family defines an isonormal Gaussian process on the Hilbert space $\mathcal{H}_H$: for any $\varphi \in \mathcal{H}_H$, $W^H(\varphi)$ is a centered Gaussian variable and

$$\mathbb{E}\left[W^H(\varphi), W^H(\psi)\right] = \langle \varphi, \psi \rangle_H, \quad \varphi, \psi \in \mathcal{H}_H.$$

Let $\mathcal{G}^H$ be the σ-algebra generated by $\{W^H(\varphi),\ \varphi \in \mathcal{H}_H\}$. Then, any $\mathcal{G}^H$-measurable random variable $F \in L^2(\Omega)$ admits the representation

$$F = \sum_{n \geq 0} J_n^H F, \tag{4}$$

where $J_n^H F$ is the projection of F on the n-th Wiener chaos space $\mathbb{H}_{H,n}$, for $n \geq 1$, and $J_0^H F = \mathbb{E}[F]$.

We denote by I_n^H the multiple Wiener integral of order n with respect to W^H, which defines a linear and continuous operator from $\mathcal{H}_H^{\otimes n}$ onto $\mathbb{H}_{H,n}$. We briefly recall the construction of I_n^H, since we will use some of its steps in the sequel. Let $\{e_k,\, k\geq 1\}$ be an orthonormal basis of $\mathcal{H}_H$ and consider an *elementary element* of $\mathcal{H}_H^{\otimes n}$ of the form

$$\varphi = e_{i_1}\hat{\otimes}\cdots\hat{\otimes}\, e_{i_n}, \tag{5}$$

where $\hat{\otimes}$ denotes the symmetrized tensor product, for some $i_1,\dots,i_n\geq 1$. Recall that the set of finite linear combinations of elementary elements is dense in $\mathcal{H}_H^{\otimes n}$. An elementary element of the form (5) can be more conveniently written as

$$\varphi = e_{j_1}^{\otimes k_1}\hat{\otimes}\dots\hat{\otimes}\, e_{j_m}^{\otimes k_m}, \tag{6}$$

where all $j_1,\dots j_m\geq 1$ are different and $k_1+\cdots+k_m=n$. The n-th order multiple Wiener integral of φ is defined as follows:

$$I_n^H(\varphi) = P_{k_1}\big(W^H(e_{j_1})\big)\cdots P_{k_m}\big(W^H(e_{j_m})\big), \tag{7}$$

where we denote by P_k the normalized k-th Hermite polynomial. The multiple Wiener integral is then extended by linearity to all finite linear combinations of elementary elements, and finally extended to the whole space $\mathcal{H}_H$ by density.

We also remind that any element in the n-th chaos $\mathbb{H}_{H,n}$ can be represented as $I_n^H(f)$, for some $f\in\mathcal{H}_H^{\otimes n}$. Hence, representation (4) can be written as follows:

$$F=\mathbb{E}[F]+\sum_{n\geq 1} I_n^H(f_n),$$

where $f_n\in\mathcal{H}_H^{\otimes n}$, for all $n\geq 1$. We recall that, for any $f\in\mathcal{H}_H^{\otimes n}$,

$$\mathbb{E}\left[|I_n^H(f)|^2\right]=\mathbb{E}\left[|I_n^H(\tilde{f})|^2\right]=n!\,\|\tilde{f}\|^2_{\mathcal{H}_H^{\otimes n}},$$

where $\tilde{f}$ stands for the symmetrization of f. We also remind that, for a general element f of $\mathcal{H}_H^{\otimes n}$, the norm $\|f\|_{\mathcal{H}_H^{\otimes n}}$ is given by

$$\|f\|^2_{\mathcal{H}_H^{\otimes n}}=\int_{\mathbb{R}_+^n}\int_{\mathbb{R}^n}|\mathcal{F}f(t_1,\cdot,t_2,\cdot,\dots,t_n,\cdot)(\xi_1,\dots,\xi_n)|^2\mu(d\xi_1)\cdots\mu(d\xi_n)dt_1\cdots dt_n.$$

Here, we still denoted by $\mathcal{F}$ the Fourier transform on the space of tempered distributions in $\mathbb{R}^n$.

Let $A\in\mathcal{B}([0,\infty))$. We define, for every $f\in\mathcal{H}_H^{\otimes n}$, the element $f1_A^{\otimes n}\in\mathcal{H}_H^{\otimes n}$ in the following way: if f is a function, we define it obviously as the function $f1_A^{\otimes n}$. If f is a general element of $\mathcal{H}_H^{\otimes n}$, we take any sequence $\{f_k,\, k\geq 1\}$ of functions in $\mathcal{H}_H^{\otimes n}$ such that $f_k\to f$ in $\mathcal{H}_H^{\otimes n}$, as $k\to\infty$, and we set

$$f1_A^{\otimes n}:=\lim_{k\to\infty} f_k 1_A^{\otimes n}.$$

This limit exists; indeed, we have that $\{f_k,\, k\geq 1\}$ is Cauchy in $\mathcal{H}_H^{\otimes n}$ and

$$\|f_k1_A^{\otimes n}-f_\ell 1_A^{\otimes n}\|_{\mathcal{H}_H^{\otimes n}}\leq\|f_k-f_\ell\|_{\mathcal{H}_H^{\otimes n}},$$

which implies that $\{f_k1_A^{\otimes n},\, k\geq 1\}$ is also a Cauchy sequence in $\mathcal{H}_H^{\otimes n}$. The limit clearly does not depend on the chosen approximating sequence. On the other hand, we define the σ-field

$$\mathcal{F}_A^H=\sigma\{W^H(1_D\varphi),\, D\in\mathcal{B}_0(\mathbb{R}_+),\, D\subset A,\, \varphi\in\mathcal{C}_0^\infty(\mathbb{R})\}\vee\mathcal{N},$$

where $\mathcal{N}$ are the null sets of $\mathcal{F}$ and $\mathcal{B}_0(\mathbb{R}_+)$ are the bounded Borel sets of $\mathbb{R}_+$.

We have the following result:

Lemma 2.1. *Let $F \in L^2(\Omega)$ with Wiener chaos expansion given by $F = \mathbb{E}[F] + \sum_{n\geq 1} I_n^H(f_n)$, where $f_n \in \mathcal{H}_H^{\otimes n}$ are symmetric, and let $A \in \mathcal{B}([0,\infty))$. Then, it holds*

$$\mathbb{E}\left[F|\mathcal{F}_A^H\right] = \sum_{n\geq 0} I_n^H\left(f_n 1_A^{\otimes n}\right).$$

Proof. The proof follows exactly as that of [4, Lem. A.1]. We only need to observe that, if $h \in \mathcal{H}_H^{\otimes n}$ is symmetric, it can be written as the limit of a sequence of symmetric functions, which in turn can be written as the limit of finite linear combinations of functions of the type $f^{\otimes n}$, where $f \in \mathcal{H}_H$ and $\|f\|_{\mathcal{H}_H} = 1$. □

Let us now introduce the Malliavin derivative operator and the Skorohod integral. Let $\mathcal{S}$ be the class of random variables F of the form

$$F = f(W^H(\varphi_1), \dots, W^H(\varphi_n)),$$

where $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and $\varphi_j \in \mathcal{H}_H$, for every $j = 1, \dots, n$. For any $F \in \mathcal{S}$, we define the *Malliavin derivative* of F as the $\mathcal{H}_H$-valued random variable DF given by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W^H(\varphi_1), \dots, W^H(\varphi_n))\varphi_j.$$

If we endow $\mathcal{S}$ with the norm $\|F\|_{\mathbb{D}^{1,2}} := \mathbb{E}\left[|F|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\|DF\|^2_{\mathcal{H}_H}\right]^{\frac{1}{2}}$, it turns out that the operator D can be extended to the completion of $\mathcal{S}$ with respect to $\|\cdot\|_{\mathbb{D}^{1,2}}$, which we will denote by $\mathbb{D}^{1,2}$. We define now the *divergence operator* δ, which is the adjoint of D. The divergence operator is defined on its domain $\mathrm{Dom}(\delta)$, which is the space of $\mathcal{H}_H$-valued random variables such that $u \in L^2(\Omega; \mathcal{H}_H)$ and

$$|\mathbb{E}\left[\langle DF, u\rangle_H\right]| \leq c\mathbb{E}\left[|F|^2\right]^{\frac{1}{2}}, \qquad \text{for all } F \in \mathbb{D}^{1,2},$$

where the constant c depends on u. Being the adjoint of D, the divergence operator $\delta(u)$ is defined for any $u \in \mathrm{Dom}(\delta)$ by the duality relation, holding for every $F \in \mathbb{D}^{1,2}$:

$$\mathbb{E}\Big[\langle DF, u\rangle_H\Big] = \mathbb{E}[F\delta(u)].$$

From the duality relation one can deduce that $\mathbb{E}[\delta(u)] = 0$, for every $u \in \mathrm{Dom}(\delta)$. For any $u \in \mathrm{Dom}(\delta)$, $\delta(u)$ is called the *Skorohod integral* of u and is denoted by

$$\int_0^\infty \int_{\mathbb{R}} u(t,x) W^H(\delta t, \delta x) := \delta(u).$$

We will need the following two results involving the Skorohod integral (cf. Propositions 1.3.3 and 1.3.6 in [20]).

Lemma 2.2. *Let $F \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}(\delta)$ such that $Fu \in L^2(\Omega; \mathcal{H}_H)$. Then, $Fu \in \mathrm{Dom}(\delta)$ and it holds*

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H.$$

Lemma 2.3. *Let $u \in L^2(\Omega;\mathcal{H}_H)$ and $\{u_n,\, n \geq 1\} \subset \mathrm{Dom}(\delta)$ such that*

$$\lim_{n\to\infty} \mathbb{E}\left[||u_n - u||^2_{\mathcal{H}_H}\right] = 0.$$

Suppose that there exists a random variable $G \in L^2(\Omega)$ such that, for all $F \in \mathcal{S}$,

$$\mathbb{E}\left[\delta(u_n)F\right] \to \mathbb{E}[GF].$$

Then $u \in \mathrm{Dom}(\delta)$ and $\delta(u) = G$.

We now define the contraction $\otimes_1$. For $h \in \mathcal{H}_H^{\otimes n}$ and any element $e_1 \otimes \cdots \otimes e_n$ of the canonical basis of $\mathcal{H}_H^{\otimes n}$, we define

$$(e_1 \otimes \cdots \otimes e_n) \otimes_1 h := (e_1 \otimes \cdots \otimes e_{n-1})\langle e_n, h\rangle_H,$$

and we extend it to a generic $f \in \mathcal{H}_H^{\otimes n}$ by linearity and density. The following lemma can be found in [24, Thm. 4.3.8]:

Lemma 2.4. *Let $F \in L^2(\Omega)$ with Wiener chaos expansion $F = \mathbb{E}[F] + \sum_{n\geq 1} I_n^H(f_n)$, where $f_n \in \mathcal{H}_H^{\otimes n}$ is symmetric, for all $n \geq 1$. Then $F \in \mathbb{D}^{1,2}$ if and only if*

$$\sum_{n\geq 1} n\, n! \|f_n\|^2_{\mathcal{H}_H^{\otimes n}} < \infty.$$

In this case, for every $h \in \mathcal{H}_H$, we have

$$\langle DF, h\rangle_H = \sum_{n\geq 1} n I_{n-1}^H(f_n \otimes_1 h).$$

2.2 Spectral representation of W^H

This section is devoted to prove that any multiple Wiener integral with respect to the noise W^H admits a representation as a multiple Wiener integral with respect to a complex-valued Gaussian measure. For this, we will provide a suitable spectral representation of W^H in terms of such a complex-valued Gaussian measure. We point out that the results in the present section will only be used in Section 4 in order to identify the underlying limit law.

Recall that $\{W^H(\varphi),\, \varphi \in \mathcal{H}_H\}$ denotes the underlying isonormal Gaussian process associated to our noise W^H. Using an approximation argument, one proves that, for any $t > 0$ and $x \in \mathbb{R}$, $1_{[0,t]\times[0,x]} \in \mathcal{H}_H$. Then, we can define the random field (making an abuse of notation)

$$W^H(t,x) := W^H\left(1_{[0,t]\times[0,x]}\right), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{8}$$

which is Gaussian, centered and satisfies, for all $s,t > 0$ and $x,y \in \mathbb{R}$:

$$\mathbb{E}\left[W^H(t,x)W^H(s,y)\right] = \frac{1}{2}(s \wedge t)\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right).$$

The latter equality is a consequence of the representation in law of the fractional Brownian motion as a Wiener type integral with respect to a complex Brownian motion (see, for instance, [21, p. 257]).

Let $\tilde{W} : \mathcal{B}_0(\mathbb{R}_+ \times \mathbb{R}) \to \mathbb{C}$ be a complex-valued Gaussian measure which can be written as $\tilde{W} = \tilde{W}_1 + i\tilde{W}_2$, where $\tilde{W}_1$ and $\tilde{W}_2$ are independent real-valued centered Gaussian measures such that, for any $A, B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$,

$$\mathbb{E}\big[\tilde{W}_j(A)\tilde{W}_j(B)\big] = \frac{|A \cap B|}{2}, \quad j = 1, 2,$$

where $|A \cap B|$ is the Lebesgue measure of $A \cap B$. In particular, $\mathbb{E}\big[|\tilde{W}(A)|^2\big] = |A|$, for all $A \in \mathcal{B}_0(\mathbb{R}_+ \times \mathbb{R})$. Note that $\tilde{W}_1$ and $\tilde{W}_2$ are essentially *white noises* in the sense of [9, Page 6, Example 3.13]. One can define the integral of any deterministic function $f \in L^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{C})$ with respect to $\tilde{W}$, as follows:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\tilde{W}(dt,dx) := \int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\tilde{W}_1(dt,dx) + i\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\tilde{W}_2(dt,dx),$$

and, for $j = 1, 2$,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\tilde{W}_j(dt,dx) := \int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathrm{Re}[f](t,x)\tilde{W}_j(dt,dx) + i\int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathrm{Im}[f](t,x)\tilde{W}_j(dt,dx).$$

The latter integrals can be interpreted, e.g., as integrals with respect to a martingale measure (see [25]). It holds that, for any $f, g \in L^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{C})$,

$$\mathbb{E}\left[\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\tilde{W}(dt,dx)\overline{\int_{\mathbb{R}_+}\int_{\mathbb{R}} g(t,x)\tilde{W}(dt,dx)}\right] = \int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\overline{g(t,x)}\,dxdt.$$

This yields, for all $f \in L^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{C})$, the isometry property

$$\mathbb{E}\left[\left|\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\tilde{W}(dt,dx)\right|^2\right] = \int_{\mathbb{R}_+}\int_{\mathbb{R}} |f(t,x)|^2\,dxdt.$$

We have the following result, whose proof follows immediately.

Proposition 2.5. *Set, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,*

$$\tilde{W}^H(t,x) := \sqrt{c_H}\int_0^t \int_{\mathbb{R}} \mathcal{F}[1_{[0,x]}](\xi)|\xi|^{\frac{1}{2}-H}\tilde{W}(ds,d\xi). \tag{9}$$

Then, $\tilde{W}^H$ is a Gaussian process which has the same distribution as the random field W^H defined in (8)*.*

At this point, we aim to extend the random field $\tilde{W}^H$ defined in (9) to an isonormal Gaussian process in $\mathcal{H}_H$. We need the following corollary of [5, Thm. 4.3]:

Proposition 2.6. *The space of finite linear combinations of functions of the form*

$$f(r,z) = 1_{(s,t]\times(x,y]}(r,z),$$

with $0 \le s < t$ and $x < y$, is dense in the Hilbert space $\mathcal{H}_H$.

Proof. The result is a direct consequence of [5, Thm. 4.3]. Indeed, in the latter paper it is proved that any predictable process $\{X(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ belonging to $L^2(\Omega; \mathcal{H}_H)$ can be approximated by finite linear combinations of processes of the form $(r,z,\omega) \mapsto 1_G(\omega)1_{(s,t]}(r)1_{(x,y]}(z)$, for some $G \in \mathcal{F}$. To prove our result, it suffices to observe that, if we choose a deterministic element φ in their proof, also its approximating sequence φ_n is deterministic, and the norm in the space $L^2(\Omega; \mathcal{H}_H)$ coincides with the norm in $\mathcal{H}_H$ for deterministic elements. $\square$

Let us now define, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\tilde{W}^H \left(1_{[0,t]\times[0,x]}\right) := \tilde{W}^H(t,x)$$

(again making an abuse of notation). This definition can be extended by linearity to any simple function on $\mathbb{R}_+ \times \mathbb{R}$. Then, thanks to Proposition 2.6 and using an approximation argument, one constructs an isonormal Gaussian process $\{\tilde{W}^H(\varphi),\ \varphi \in \mathcal{H}_H\}$ which has exactly the same law as $\{W^H(\varphi),\ \varphi \in \mathcal{H}_H\}$.

For the remainder of the paper, we will assume, without any loss of generality, that our Gaussian setting is the one determined by the isonormal Gaussian process $\tilde{W}^H = \{\tilde{W}^H(\varphi),\ \varphi \in \mathcal{H}_H\}$. For the sake of simplicity, we will use again the notation W^H instead of $\tilde{W}^H$. So, the main implications of this setting are that, first, we have the representation

$$W^H \left(1_{[0,t]\times[0,x]}\right) = \sqrt{c_H} \int_0^t \int_{\mathbb{R}} \mathcal{F}[1_{[0,x]}](\xi)|\xi|^{\frac{1}{2}-H} \tilde{W}(ds, d\xi) \tag{10}$$

and, secondly, the whole family of processes $\{W^H,\ H \in (0,1)\}$ are defined in a single probability space, which is the one where the Gaussian measure $\tilde{W}$ is defined. This last fact will be crucial in Section 4.

The main result of the section is the following:

Theorem 2.7. *Let $n \geq 1$, $f \in \mathcal{H}_H^{\otimes n}$ and $I_n^H(f)$ be the multiple Wiener integral of f with respect to W^H. Let $\hat{f}$ be the function defined by*

$$\hat{f}(t_1,x_1,t_2,x_2,\dots,t_n,x_n) = (c_H)^{\frac{n}{2}} \mathcal{F}[f(t_1,\cdot,t_2,\cdot,\dots,t_n,\cdot)](x_1,\dots,x_n)\, |x_1|^{\frac{1}{2}-H} \cdots |x_n|^{\frac{1}{2}-H},$$

where we recall that the constant c_H is given in (1) *Then, it holds that*

$$I_n^H(f) = \tilde{I}_n(\hat{f}),\ \mathbb{P}\text{-a.s.}, \tag{11}$$

where $\tilde{I}_n$ is the n-th order Wiener integral with respect to the complex Gaussian measure $\tilde{W}$.

Proof. We first check that the result is true for the first-order Wiener integral I_1^H. We aim to prove that, for any $\varphi \in \mathcal{H}_H$,

$$I_1^H(\varphi) = (c_H)^{\frac{1}{2}}\, \tilde{I}_1 \left(\mathcal{F}\varphi(t,\cdot)(x)\, |x|^{\frac{1}{2}-H}\right), \tag{12}$$

which means that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \varphi(t,x) W^H(dt,dx) = (c_H)^{\frac{1}{2}} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(x)\, |x|^{\frac{1}{2}-H} \tilde{W}(dt,dx).$$

By (10) and the linearity of the Wiener integral, the latter equality clearly holds in the case where $\varphi(t,x) = 1_{(r,s]\times(y,z]}(t,x)$, for $0 \leq r < s$ and $y < z$. Moreover, owing to Proposition 2.6, it can also be extended to the whole space $\mathcal{H}_H$, hence proving the theorem's statement for first-order Wiener integrals.

Let us now prove (11) for $n > 1$. We first consider the case where $f \in \mathcal{H}_H^{\otimes n}$ is an elementary element of the form (6). In this case, we use the definition of the multiple Wiener integral (see (7)) and the validity of the case $n = 1$ (see (12)), as follows:

$$\begin{aligned} I_n^H(f) &= P_{k_1}\left(W^H(e_{j_1})\right) \cdots P_{k_m}\left(W^H(e_{j_m})\right) \\ &= P_{k_1}\left(\tilde{I}_1(\hat{e}_{j_1})\right) \cdots P_{k_m}\left(\tilde{I}_1(\hat{e}_{j_m})\right) \\ &= \tilde{I}_n\left(\hat{e}_{j_1}^{\otimes k_1} \otimes \cdots \otimes \hat{e}_{j_m}^{\otimes k_m}\right) \\ &= \tilde{I}_n(\hat{f}). \end{aligned}$$

The extension to any element of $\mathcal{H}_H^{\otimes n}$ can be proved by recalling that the set of finite linear combinations of elementary elements of the form (6) is dense in $\mathcal{H}_H^{\otimes n}$. $\square$

2.3 Itô and Skorohod stochastic integrals

This section is devoted to recall the definition of stochastic integrals with respect to W^H, both in the case $H < \frac{1}{2}$ and $H \geq \frac{1}{2}$, and to prove that the Skorohod integral with respect to W^H of an adapted process coincides with the corresponding Itô integral (see Theorem 2.11 below). This result will allow us to express any Picard iteration associated to our underlying SDPEs as a finite sum of multiple Wiener integrals, and this fact will be used in the proof of Theorem 4.1 in Section 4.

Recall that we have a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in which we have our complex-valued Gaussian measure $\tilde{W}$ (see Section 2.2). Recall that our isonormal Gaussian process $W^H = \{W^H(\varphi),\, \varphi \in \mathcal{H}_H\}$ has been defined in such a way that we may assume that W^H is defined in $(\Omega, \mathcal{F}, \mathbb{P})$, for all $H \in (0,1)$. Regarding adaptability, we consider the natural filtration generated by $\tilde{W}$, which we denote by $\{\mathcal{F}_t,\, t \geq 0\}$ and can be defined as $\mathcal{F}_t = \sigma(\tilde{W}(s,x),\, (s,x) \in [0,t] \times \mathbb{R})$, where

$$\tilde{W}(s,x) := \int_{\mathbb{R}_+} \int_{\mathbb{R}} 1_{[0,s]\times[0,x]}(r,z) \tilde{W}(dr,dz).$$

Fix a time horizon $T > 0$. We denote by $\mathcal{E}$ the space of *simple processes* on $[0,T] \times \mathbb{R}$, that is the space of finite linear combinations of processes of the form

$$g(t,x,\omega) := Y(\omega) 1_{(r,s]\times(y,z]}(t,x), \tag{13}$$

for some $0 \leq r \leq s \leq T$ and $y \leq z$, and for some $\mathcal{F}_r$-measurable random variable Y. The (Itô) stochastic integral of g with respect to W^H is defined as follows: for any $t \in [0,T]$, set

$$\int_0^t \int_{\mathbb{R}} g(\tau,x) W^H(d\tau,dx) := Y\left(W^H(t \wedge s, z) - W^H(t \wedge s, y) - W^H(t \wedge r, z) + W^H(t \wedge r, y)\right).$$

This definition can be extended to all elements of $\mathcal{E}$ by linearity. Following [8] and [2], we endow $\mathcal{E}$ with the norm

$$\|g\|_0 := \left(\mathbb{E}\left[c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}g(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt\right]\right)^{\frac{1}{2}},$$

and we define $\mathcal{P}_0^T$ as the completion of $\mathcal{E}$ with respect to the norm $\|\cdot\|_0$. It turns out that $\mathcal{P}_0^T$ is the space of predictable processes g for which $\|g\|_0 < \infty$. The stochastic integral can be extended to the whole space $\mathcal{P}_0^T$.

The following result is a particular case of [10, Prop. 2.9]:

Theorem 2.8. *Suppose that $H \in [\frac{1}{2}, 1)$. Let $\Gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be such that, for all $t \in (0,T]$, the function $\Gamma(t,\cdot)$ defines non-negative distribution with rapid decrease and*

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}\Gamma(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt < \infty.$$

Moreover, we assume that, for all $t \in [0,T]$, $\Gamma(t,dx) := \Gamma(t,x)dx$ defines a non-negative measure on $\mathbb{R}$ such that

$$\sup_{t\in[0,T]} \Gamma(t,\mathbb{R}) < \infty.$$

Let $Z = \{Z(t,x),\, (t,x) \in [0,T] \times \mathbb{R}\}$ be a predictable stochastic process satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|Z(t,x)|^2\right] < \infty.$$

Then, the process $\{S(t,x) := Z(t,x)\Gamma(t,x),\, (t,x) \in [0,T] \times \mathbb{R}\}$ belongs to $\mathcal{P}_0^T$. Furthermore, if Z satisfies, for some $p \geq 2$, that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|Z(t,x)|^p\right] < \infty,$$

then we have the following Burkholder-Davis-Gundy's inequality:

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}} S(s,x) W^H(ds,dx)\right|^p\right] \\ &\qquad \leq z_p (\nu_{T,H})^{\frac{p}{2}-1} \int_0^T \sup_{x\in\mathbb{R}} \mathbb{E}\left[|Z(s,x)|^p\right] \int_{\mathbb{R}} c_H\, |\mathcal{F}\Gamma(s,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi ds, \end{aligned} \tag{14}$$

where the constant z_p is the one in the classical Burkholder-Davis-Gundy inequality for continuous martingales, and $\nu_{T,H}$ is given by

$$\nu_{T,H} := c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}\Gamma(s,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi ds.$$

As far as the case $H < \frac{1}{2}$ is concerned, we have the following result (see [2, Thm. 2.9]).

Theorem 2.9. *Suppose that $H \in (0, \frac{1}{2})$. Let $\{S(t,x),\, (t,x) \in [0,T] \times \mathbb{R}\}$ be a predictable process such that, for every (ω, t), $S(\omega,t,\cdot)$ defines a tempered function whose Fourier transform $\mathcal{F}S(\omega,t,\cdot)$ is a locally integrable function satisfying*

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\mathcal{F}S(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt\right] < \infty.$$

Then, $S \in \mathcal{P}_0^T$ and we have the isometry

$$\mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}} S(t,x) W^H(dt,dx)\right|^2\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\mathcal{F}S(t,\cdot)(\xi)|^2 c_H\, |\xi|^{1-2H} d\xi dt\right].$$

Moreover, we have the Burkholder-Davis-Gundy inequality: for any $p \geq 2$,

$$\mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}} S(t,x) W^H(dt,dx)\right|^p\right] \leq z_p\, c_H^{\frac{p}{2}}\, \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\mathcal{F}S(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt\right]^{\frac{p}{2}}, \tag{15}$$

where the constant z_p is the constant appearing in the classical Burkholder-Davis-Gundy inequality for continuous martingales.

Remark 2.10. *Owing to [2, Prop. 2.8], the isometry property in the above Theorem 2.9 can be equivalently written as*

$$\mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}} S(t,x) W^H(dy,dx)\right|^2\right] = \mathbb{E}\left[\tilde{c}_H \int_0^T \int_{\mathbb{R}^2} |S(t,x) - S(t,y)|^2 |x-y|^{2H-2} dy dx dt\right],$$

where $\tilde{c}_H = \frac{H(1-2H)}{2}$. Hence, (15) becomes

$$\mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}} S(t,x) W^H(dy,dx)\right|^p\right] \leq z_p\, \tilde{c}_H^{\frac{p}{2}}\, \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |S(t,x) - S(t,y)|^2 |x-y|^{2H-2} dy dx dt\right]^{\frac{p}{2}}.$$

The following result is an extension of [4, Thm. 4.2] to the case $H > \frac{1}{2}$. Note that, in this latter case, though the noise is more regular in space than a white noise, the corresponding Hilbert space $\mathcal{H}_H$ may be rather big, and indeed contains genuine distributions. This makes our proof different compared to the one of [4, Thm. 4.2], in which $\mathcal{H}_H$ is a space of functions (because $H < \frac{1}{2}$).

Theorem 2.11. *Let $H \in [\frac{1}{4}, 1)$ and $u = \{u(t,x), (t,x) \in [0,\infty) \times \mathbb{R}\}$ be a stochastic process such that, restricted to $t \in [0,T]$, belongs to $\mathcal{P}_0^T$. Then, for any $t > 0$, $u1_{[0,t]} \in \mathrm{Dom}(\delta)$ and its Skorohod integral coincides with the Itô integral, that is*

$$\int_0^\infty \int_{\mathbb{R}} u(s,x) 1_{[0,t]}(s) W^H(\delta s, \delta x) = \int_0^t \int_{\mathbb{R}} u(s,x) W^H(ds,dx), \quad \mathbb{P}\text{-}a.s.$$

Proof. The proof is an adaptation of that of [4, Thm. 4.2]. The only difference is that, here, a general element of $\mathcal{H}_H$ is not necessarily a function. It is enough to prove the statement in the case where u is an elementary process of the form (13). The extension to any arbitrary element of $\mathcal{P}_0^T$ can be done exactly as in Case 2 of the proof of [4, Thm. 4.2].

Let g be an elementary process of the form $g(\tau,x,\omega) = Y(\omega)1_{(r,s]}(\tau)1_{(y,z]}(x)$, with $0 \le r < s \le T$ and $y < z$, where we assume that Y is $\mathcal{F}_r$-measurable, bounded and belongs to $\mathbb{D}^{1,2}$. We have to check that $g1_{[0,t]} \in \mathrm{Dom}(\delta)$ and it holds

$$\delta\left(g1_{[0,t]}\right) = \int_0^t \int_{\mathbb{R}} g(\tau,x) W^H(d\tau,dx).$$

First, we note that $g1_{[0,t]} = Y1_{[r\wedge t, s\wedge t]\times[y,z]}$. Since $Y \in \mathbb{D}^{1,2}$ and $1_{[r\wedge t,s\wedge t]\times[y,z]} \in \mathrm{Dom}(\delta)$, we can apply Lemma 2.2 to conclude that $g1_{[0,t]} \in \mathrm{Dom}(\delta)$ and

$$\delta(g1_{[0,t]}) = Y\delta(1_{[r\wedge t,s\wedge t]\times[y,z]}) - \langle DY, 1_{[r\wedge t,s\wedge t]\times[y,z]}\rangle_H,$$

if the right-hand side above belongs to $L^2(\Omega)$. We clearly have that $Y\delta(1_{[r\wedge t,s\wedge t]\times[y,z]}) \in L^2(\Omega)$, and we will show that $\langle DY, 1_{[r\wedge t,s\wedge t]\times[y,z]}\rangle_H = 0$, which will allow us to conclude the proof.

Let $h := 1_{[r\wedge t,s\wedge t]\times[y,z]}$. Since Y is $\mathcal{F}_r$-measurable, we have, by Lemma 2.1,

$$Y = \mathbb{E}[Y|\mathcal{F}_r] = \sum_{n\ge 0} I_n^H\left(g_n 1_{[0,r]}^{\otimes n}\right),$$

for some symmetric $g_n \in \mathcal{H}_H^{\otimes n}$, $n \ge 1$. By Lemma 2.4 we have that

$$\langle DY, h\rangle_H = \sum_{n\ge 1} n I_{n-1}^H\left(g_n 1_{[0,r]}^{\otimes n} \otimes_1 h\right).$$

We claim that $g1_{[0,r]}^{\otimes n} \otimes_1 h = 0$, for all $g \in \mathcal{H}_H^{\otimes n}$. Indeed, if $g = e^{\otimes n}$ for some function $e \in \mathcal{H}_H$, we have

$$e^{\otimes n} 1_{[0,r]}^{\otimes n} \otimes_1 h = e^{\otimes(n-1)} 1_{[0,r]}^{\otimes(n-1)} \langle e1_{[0,r]}, h\rangle_H,$$

and we observe that

$$\langle e1_{[0,r]}, h\rangle_H = \int_0^\infty \int_{\mathbb{R}} \mathcal{F}e(s,\cdot)(\xi) 1_{[0,r]}(s) \overline{\mathcal{F}1_{[y,z]}(\xi)} 1_{[r\wedge t,s\wedge t]}(s) d\xi ds = 0.$$

This can be extended to a generic element in $\mathcal{H}_H^{\otimes n}$ by linearity and density (using Lemma 2.3). □

2.4 Existence and uniqueness of solution

This section is devoted to recall the well-posedness results for equations (SWE) and (SHE) and prove that the corresponding Picard iterations admit a suitable finite Wiener chaos expansion.

First, we recall that the solution to our equations is understood in the mild sense. Namely, an adapted and jointly measurable random field $u^H = \{u^H(t,x),\ (t,x) \in [0,T] \times \mathbb{R}\}$ solves (SWE) (resp. (SHE)) if it holds, for all $(t,x) \in [0,T] \times \mathbb{R}$:

$$u^H(t,x) = \eta + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) u^H(s,y) W^H(ds,dy), \tag{16}$$

where G is the fundamental solution of the wave (resp. heat) equation in $\mathbb{R}$ (see (3)).

The following result is a particular case of [10, Thm. 4.3], which covers the case $H \geq \frac{1}{2}$.

Theorem 2.12. *Let $H \in [\frac{1}{2}, 1)$. There exists a unique mild solution u^H to equation (16). Moreover, the solution u^H is $L^2(\Omega)$-continuous and satisfies, for every $p \geq 1$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u^H(t,x)|^p\right] < \infty.$$

Remark 2.13. *The case $H = \frac{1}{2}$ corresponds to the space-time white noise, while in the case $H \in (\frac{1}{2}, 1)$ the noise's spatial correlation is given by a Riesz kernel of order $2-2H$.*

The case $H \in (\frac{1}{4}, \frac{1}{2})$ has been considered in [2, Thm. 1.1]. In the latter reference, the authors proved that condition $H > \frac{1}{4}$ is necessary and sufficient in order to have a solution (see [2, Prop. 3.7]).

Theorem 2.14. *Let $H \in (\frac{1}{4}, \frac{1}{2})$. There exists a unique mild solution u^H to (16). Moreover, the solution u^H is $L^2(\Omega)$-continuous and satisfies, for every $p \geq 2$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u^H(t,x)|^p\right] < \infty \tag{17}$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^T \int_{\mathbb{R}^2} G^2_{t-s}(x-y) \frac{\mathbb{E}\left[|u^H(s,y) - u^H(s,z)|^p\right]^{\frac{2}{p}}}{|y-z|^{2-2H}} dydzds < \infty. \tag{18}$$

Remark 2.15. *In the case $H \in (\frac{1}{4}, \frac{1}{2})$, the solution u^H satisfies, in addition to (17), the further constraint (18). This comes from the fact that, in [2], the solution of (16) was proved to exist in the space of $L^2(\Omega)$-continuous, adapted and jointly measurable processes endowed with a Sobolev's type norm which included a term of the form (18).*

In the case $H \in (\frac{1}{4}, \frac{1}{2})$, the solution u^H of (SWE) (and (SHE)) has been found in [2] as a limit of the Picard iteration scheme, which is defined by

$$u_0^H(t,x) := \eta$$
$$u_{m+1}^H(t,x) := \eta + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) u_m^H(s,y) W^H(ds,dy), \quad m \geq 0,$$

where $(t,x) \in [0,T] \times \mathbb{R}$. The limit is found in the Banach space χ_H^p, for $p \geq 2$, which is defined as the space of $L^2(\Omega)$-continuous, adapted and jointly measurable processes $Y = \{Y(t,x),\ (t,x) \in [0,T] \times \mathbb{R}\}$ such that

$$\|Y\|_{\chi_H^p} := \|Y\|_{\chi_1^p} + \|Y\|_{\chi_{H,2}^p} < \infty,$$

where,

$$\|Y\|_{\chi_1^p} := \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|Y(t,x)|^p\right]^{\frac{1}{p}}$$

and

$$\|Y\|_{\chi_{H,2}^p} := \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(\tilde{c}_H \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y)-Y(s,z)|^p\right]\right)^{\frac{2}{p}}}{|y-z|^{2-2H}} dydzds\right)^{\frac{1}{2}}.$$

We recall that the constant $\tilde{c}_H$ has been defined in Remark 2.10. Notice that the L^p-part $\|\cdot\|_{\chi_1^p}$ of the norm $\|\cdot\|_{\chi_H^p}$ does not depend on H, as it is also pointed out by the notation itself, while the *Gagliardo-type* part $\|\cdot\|_{\chi_{H,2}^p}$ does depend on H.

Remark 2.16. *In [2], the norm $\|\cdot\|_{\chi_{H,2}^p}$ is defined without the constant $\tilde{c}_H = \frac{H(1-2H)}{2}$. Since the two definitions give rise to equivalent norms, the results about existence and uniqueness of solution for equation (16) when $H \in (\frac{1}{4}, \frac{1}{2})$ still hold true. On the other hand, we will see how adding this normalizing constant helps us proving some uniform (in H) results that will be needed in the sequel.*

Before stating the main result of the section, we consider the following Sobolev-type embedding for the space χ_H^p, which could be of independent interest.

Lemma 2.17. *Let $p \geq 2$ and $\frac{1}{4} < \alpha \leq \beta < \frac{1}{2}$. Then, it holds:*

$$\chi_\alpha^p \hookrightarrow \chi_\beta^p.$$

This means that there exists a constant C such that, for every adapted, jointly measurable and $L^2(\Omega)$-continuous process Y, we have

$$\|Y\|_{\chi_\beta^p} \leq C\|Y\|_{\chi_\alpha^p}. \tag{19}$$

Moreover, it holds the following stronger property for the Gagliardo-type seminorm $\|\cdot\|_{\chi_{\beta,2}^p}$:

$$\sup_{\beta\in[\alpha,\frac{1}{2})} \|Y\|_{\chi_{\beta,2}^p} \leq \tilde{C}\|Y\|_{\chi_\alpha^p}$$

where the constant $\tilde{C}$ only depends on p and T.

Proof. We follow the same lines as in the proof of [11, Prop. 2.1]. It suffices to prove (19) for the $\|\cdot\|_{\chi_{H,2}^p}$-part of the norm. It holds:

$$\begin{aligned}
&\left(\tilde{c}_\beta \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y)-Y(s,z)|^p\right]\right)^{\frac{2}{p}}}{|y-z|^{2-2\beta}} dydzds\right)^{\frac{1}{2}} \\
&= \left(\tilde{c}_\beta \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y)-Y(s,y-\overline{z})|^p\right]\right)^{\frac{2}{p}}}{|\overline{z}|^{2-2\beta}} dyd\overline{z}ds\right)^{\frac{1}{2}} \\
&\leq C(I_1+I_2),
\end{aligned} \tag{20}$$

where we label I_1 the term where we integrate in the variable $\overline{z}$ in the region $|\overline{z}| \geq 1$, and I_2 the term where we integrate in the region $|\overline{z}| < 1$. First, we have

$$\begin{aligned}I_1 &= \left(\tilde{c}_\beta \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|\geq 1} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y)-Y(s,y-\overline{z})|^p\right]\right)^{\frac{2}{p}}}{|\overline{z}|^{2-2\beta}}\, d\overline{z} dy ds\right)^{\frac{1}{2}} \\ &\leq C_p \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(\mathbb{E}\left[|Y(t,x)|^p\right]\right)^{\frac{1}{p}} \left(\tilde{c}_\beta \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|\geq 1} G_{t-s}^2(x-y) \frac{1}{|\overline{z}|^{2-2\beta}}\, d\overline{z} dy ds\right)^{\frac{1}{2}}\end{aligned}$$

Note that $\int_{|\overline{z}|\geq 1} \frac{1}{|\overline{z}|^{2-2\beta}} d\overline{z} = \frac{2}{1-2\beta}$. Hence,

$$\begin{aligned}&\tilde{c}_\beta \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|\geq 1} G_{t-s}^2(x-y) \frac{1}{|\overline{z}|^{2-2\beta}}\, d\overline{z} dy ds \\ &\qquad \leq \beta \int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y)\, dy ds \leq \beta C_T \leq \frac{C_T}{2}.\end{aligned}$$

Thus, we can conclude that

$$I_1 \leq C_{p,T} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(\mathbb{E}\left[|Y(t,x)|^p\right]\right)^{\frac{1}{p}}.$$

Regarding I_2, we observe that

$$\begin{aligned}I_2 &= \left(\tilde{c}_\beta \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|< 1} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y)-Y(s,y-\overline{z})|^p\right]\right)^{\frac{2}{p}}}{|\overline{z}|^{2-2\beta}}\, d\overline{z} dy ds\right)^{\frac{1}{2}} \\ &\leq \left(\tilde{c}_\alpha \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|< 1} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y)-Y(s,y-\overline{z})|^p\right]\right)^{\frac{2}{p}}}{|\overline{z}|^{2-2\alpha}}\, d\overline{z} dy ds\right)^{\frac{1}{2}} \\ &\leq \left(\tilde{c}_\alpha \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y)-Y(s,y-\overline{z})|^p\right]\right)^{\frac{2}{p}}}{|\overline{z}|^{2-2\alpha}}\, d\overline{z} dy ds\right)^{\frac{1}{2}} \\ &\leq \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(\tilde{c}_\alpha \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y)-Y(s,y-\overline{z})|^p\right]\right)^{\frac{2}{p}}}{|\overline{z}|^{2-2\alpha}}\, d\overline{z} dy ds\right)^{\frac{1}{2}} \\ &= \|Y\|_{\chi_{\alpha,2}^p}.\end{aligned}$$

Notice that both the estimate for I_1 and I_2 are independent of $(t,x) \in [0,T]\times\mathbb{R}$ and $\beta \in [\alpha, \frac{1}{2})$. Therefore, we can take the supremum with respect to $(t,x) \in [0,T]\times\mathbb{R}$ and $\beta \in [\alpha, \frac{1}{2})$ in the left-hand side of (20) and we conclude

$$\sup_{\beta\in[\alpha,\frac{1}{2})} \|Y\|_{\chi_{\beta,2}^p} \leq C_{p,T}\|Y\|_{\chi_1^p} + \|Y\|_{\chi_{\alpha,2}^p} \leq \tilde{C}\|Y\|_{\chi_\alpha^p},$$

which obviously implies

$$\|Y\|_{\chi_\beta^p} \leq (C_{p,T}+1)\|Y\|_{\chi_1^p} + \|Y\|_{\chi_{\alpha,2}^p} \leq C\|Y\|_{\chi_\alpha^p},$$

for some constant C. □

The path Hölder-continuity of the solution to (16) has been proved in [3] in the case $H \in (\frac{1}{4}, \frac{1}{2})$, while the case $H \in [\frac{1}{2}, 1)$ follows from the results in [25, 22, 23]. For the sake of completeness, we state a result which unifies both cases, and whose proof follows, indeed, as an immediate consequence of the stronger results Proposition 3.1 and Proposition 3.8 proven in Section 3.

Theorem 2.18. *Let $H \in (\frac{1}{4}, 1)$. Then, the solution u^H to (16) satisfies the following: for any $p \geq 2$, there exists a constant $C_p > 0$ (which indeed does not depend on H) such that, for all $t, t' \in [0, T]$ and $x, x' \in \mathbb{R}$, it holds*

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left[|u^H(t', x) - u^H(t, x)|^p\right] \leq C_p |t' - t|^{\gamma p}$$

and

$$\sup_{t\in[0,T]} \mathbb{E}\left[|u^H(t, x') - u^H(t, x)|^p\right] \leq C_p |x' - x|^{Hp},$$

where $\gamma = H$ for the wave equation and $\gamma = \frac{H}{2}$ for the heat equation. Thus, the process u^H has a modification whose trajectories are almost surely γ'-Hölder continuous in time, for all $\gamma' < \gamma$, and H'-Hölder continuous in space for all $H' < H$.

Proof. As already mentioned, the result follows from Propositions 3.1 and 3.8 in Section 3, where the same kind of estimates have obtained uniformly with respect to H. □

The above Theorems 2.14 and 2.12, together with Theorem 2.11 on the equivalence between Itô and Skorohod integrals, allow us to prove that equations (SWE) and (SHE) admit a unique *Skorohod mild solution.* By definition, it is a square integrable random field $\{u^H(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ such that, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$u^H(t, x) = \eta + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) u^H(s, y) W^H(\delta s, \delta y), \quad \mathbb{P}\text{-a.s.}, \tag{21}$$

that is, the process $v^{(t,x)} := \{1_{[0,t]}(s) G_{t-s}(x - y) u^H(s, y), (s, y) \in \mathbb{R}_+ \times \mathbb{R}\}$ belongs to $\mathrm{Dom}(\delta)$ and $u^H(t, x) = \eta + \delta\left(v^{(t,x)}\right)$.

Theorem 2.19. *Let $H \in (\frac{1}{4}, 1)$ and $T > 0$. Equation (21) admits a unique adapted solution in $[0, T] \times \mathbb{R}$.*

Proof. This result has already been proved in [4, Thm. 4.3] for the wave equation in the case $H \in (\frac{1}{4}, \frac{1}{2})$. In [13, p. 49], the authors notice that it is also true for the heat equation, still under the constraint $H \in (\frac{1}{4}, \frac{1}{2})$. The statement's validity in the case $H \in [\frac{1}{2}, 1)$ follows combining Theorems 2.11 and 2.12. □

Finally, the following result will be crucial in order to identify the limit law in Theorem 4.1.

Theorem 2.20. *Let $H \in (\frac{1}{4}, 1)$ and u^H be the solution to (16). Recall that the corresponding Picard iteration scheme is defined as follows: for any $m \geq 0$, set*

$$\begin{aligned} u_0^H(t, x) &:= \eta, \\ u_{m+1}^H(t, x) &:= \eta + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) u_m^H(s, y) W^H(ds, dy), \end{aligned}$$

where $(t,x) \in [0,T] \times \mathbb{R}$. *Then, for any* $m \geq 0$, *it holds*

$$u_m^H(t,x) = \sum_{n=0}^{m} I_n^H(g_n(\cdot,t,x)),$$

where I_n^H *is the* n*-th multiple Wiener integral with respect to* W^H *and the kernel* $g_n(\cdot,t,x)$ *is given by*

$$g_n(t_1,x_1,t_2,x_2,\dots,t_n,x_n,t,x) := G_{t-t_n}(x-x_n)\cdots G_{t_2-t_1}(x_2-x_1)\,\eta 1_{\{0<t_1<\cdots<t_n<t\}}. \quad (22)$$

Proof. The case of the wave equation with $H < \frac{1}{2}$ has already been proved in [4, Thm. 4.3]. Owing to Theorem 2.11, the arguments in the proof of the former theorem can be carried out to easily extend the result to the case $H \geq \frac{1}{2}$ as well as to the heat equation. □

3 Tightness

Recall that our main result (see Theorem 1.1) states that, if $H_0 \in (\frac{1}{4},1)$ and $\{H_n,\, n \in \mathbb{N}\} \subset (\frac{1}{4},1)$ is any sequence converging to H_0, then $u^{H_n} \to u^{H_0}$ in law in the space $\mathcal{C}([0,T]\times\mathbb{R})$ of continuous functions. The first step in order to prove the above result consists in checking that the laws of $\{u^{H_n},\, n \in \mathbb{N}\}$ define a tight family of probability measures on $\mathcal{C}([0,T]\times\mathbb{R})$.

We split the computations in the case $H_0 \in (\frac{1}{4},\frac{1}{2}]$, which has more involved calculations, and the case $H_0 \in [\frac{1}{2},1)$, in which the calculations are more straightforward. We explain briefly why: in the *rough* case, the Burkholder-Davis-Gundy inequality (15) forces us to consider the Fourier transform of the whole integrand process, while in the case $H \in [\frac{1}{2},1)$, when we use the Burkholder-Davis-Gundy inequality (14), we only have to compute the Fourier transform of the deterministic part of the integrand process, which will be explicit in our case.

3.1 Tightness in the case $(\frac{1}{4},\frac{1}{2})$

We suppose that the limiting Hurst exponent $H_0 \in (\frac{1}{4},\frac{1}{2}]$. If $H_0 \in (\frac{1}{4},\frac{1}{2})$, we can assume without loss of generality that the whole sequence $\{H_n,\, n \in \mathbb{N}\} \subset [\eta_1,\eta_2] \subset (\frac{1}{4},\frac{1}{2})$. If $H_0 = \frac{1}{2}$, we can assume at most that $\{H_n,\, n \in \mathbb{N}\} \subset [\eta_1,\frac{1}{2}) \subset (\frac{1}{4},\frac{1}{2})$. From now on we will denote both type of sets as K, meaning that $K = [\eta_1,\eta_2]$ if $H_0 \in (\frac{1}{4},\frac{1}{2})$ and $K = [\eta_1,\frac{1}{2})$ if $H_0 = \frac{1}{2}$. Clearly, if the limiting exponent $H_0 = \frac{1}{2}$, we cannot suppose that $H_n \to H_0$ always from below. In Section 3.2, we will also handle families of Hurst exponents with $K = (\frac{1}{2},\eta_2]$, so that our result will be complete (because the union of a finite number of tight families is a tight family itself).

We are ready to state the main result of the present section.

Proposition 3.1. *Let* $\mathcal{U}_K := \{u^H,\, H \in K\}$ *be the family of solutions of* (16), *where* K *is either of the form* $[\eta_1,\eta_2]$, *with* $\eta_1,\eta_2 \in (\frac{1}{4},\frac{1}{2})$ *and* $\eta_1 < \eta_2$, *or* $K = [\eta_1,\frac{1}{2})$, *where* $\eta_1 \in (\frac{1}{4},\frac{1}{2})$. *Then, the family* $\mathcal{U}_K$ *is tight in* $\mathcal{C}([0,T]\times\mathbb{R})$, *endowed with the metric of uniform convergence on compact sets.*

We postpone the proof of this result, since we need some preliminar results. We aim to apply the tightness criterion Theorem A.5. Indeed, we will check that conditions (i) and (ii) in the latter result are satisfied by the Picard iterations u_m^H, uniformly with respect to H, and then we will pass to the limit as $m \to \infty$.

First of all, we show that the the Picard iterations $\{u_m^H,\, m \geq 0\}$ are well-defined and satisfy some estimates uniformly with respect to H. The proof is very similar to that of [2, Thm. 3.7].

In fact, we will follow the same steps in its proof and take care of the fact that we need all estimates uniformly in H. Only the most significant parts of the proof will be written explicitly.

Proposition 3.2. *Let $p \geq 2$ and $H \in (\frac{1}{4}, \frac{1}{2})$. For any $m \geq 0$, we have that*

(i) $u_m^H(t,x)$ *is well-defined, for any $H \in K$ and $(t,x) \in [0,T] \times \mathbb{R}$.*

(ii) It holds

$$\sup_{H \in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u_m^H(t,x)|^p\right] < \infty.$$

(iii) It holds

$$\sup_{H \in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \tilde{c}_H \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|u_m^H(s,y) - u_m^H(s,z)|^p\right]\right)^{\frac{2}{p}}}{|y-z|^{2-2H}} dydzds < \infty.$$

Proof. Condition (i) is a direct consequence of [2, Thm. 3.7]. In order to prove (ii) and (iii), we use an induction argument. First, note that these two conditions clearly hold for $m = 0$.

Assume that conditions (ii) and (iii) are satisfied by u_m^H. We prove that they are also fulfilled by u_{m+1}^H. Precisely, arguing as in Step 2 in the proof of [2, Thm. 3.7] (see p. 18 therein), we have

$$\mathbb{E}\left[|u_{m+1}^H(t,x)|^p\right] \leq C\left\{\eta^p + \mathbb{E}\left[\left|\tilde{c}_H \int_0^T \int_{\mathbb{R}^2} \frac{|S_m^H(s,y) - S_m^H(s,z)|^2}{|y-z|^{2H-2}} dydzds\right|^{\frac{p}{2}}\right]\right\},$$

where we have used the notation $S_m^H(s,y) := G_{t-s}(x-y)u_m^H(s,y)$ and C is some positive constant. The expectation on the right hand-side above can be bounded, up to some constant independent of H, by $I_1^H + I_2^H$, where

$$I_1^H = \left(\tilde{c}_H \int_0^T \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|u_m^H(s,y) - u_m^H(s,z)|^p\right]\right)^{\frac{2}{p}}}{|y-z|^{2-2H}} dydzds\right)^{\frac{p}{2}}$$

and

$$I_2^H = \left(\tilde{c}_H \int_0^T \int_{\mathbb{R}^2} \left(\mathbb{E}\left[|u_m^H(s,z)|^p\right]\right)^{\frac{2}{p}} \frac{|G_{t-s}(x-y) - G_{t-s}(x-z)|^2}{|y-z|^{2-2H}} dydzds\right)^{\frac{p}{2}}$$

By the induction hypothesis, the term I_1^H is uniformly bounded in H and (t,x). Regarding I_2^H, using again the induction hypothesis and applying [2, Prop. 2.8], we get

$$\begin{aligned}
I_2^H &\leq \sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u_m^H(t,x)|^p\right] \left(\tilde{c}_H \int_0^T \int_{\mathbb{R}^2} \frac{|G_{t-s}(x-y) - G_{t-s}(x-z)|^2}{|y-z|^{2-2H}} dydzds\right)^{\frac{p}{2}} \\
&\leq C\left(c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi ds\right)^{\frac{p}{2}},
\end{aligned}$$

where we recall that the constant c_H is given by

$$c_H = \frac{\Gamma(1+2H)\sin(\pi H)}{2\pi}.$$

Notice that $c_H \leq \frac{1}{2\pi}$, for any $H \in (\frac{1}{4}, \frac{1}{2})$. Moreover, by Lemma A.1, it holds that

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi ds \leq \begin{cases} 2^{2H} C_{1-2H} \frac{1}{1+2H} T^{1+2H} & \text{wave equation,} \\ \\ \frac{1}{H}\Gamma(1-H) T^H & \text{heat equation.} \end{cases} \tag{23}$$

As explained in Step 1 of the proof of [12, Thm. 2.8], all constants appearing in (23) can be bounded uniformly in $H \in K$. This let us conclude that u^H_{m+1} satisfies condition (ii).

It remains to prove that u^H_{m+1} verifies (iii). The computations follow exactly as in Step 3 of the proof of [2, Thm. 3.7], in such a way that we apply the induction hypothesis, [2, Prop. 2.8] and Lemmas A.1 and A.4. We omit the details. Nevertheless, we point out why the presence of the constant $\tilde{c}_H$ in condition (iii) is crucial in order to get uniform estimates with respect to H. Precisely, one of the terms appearing in the treatment of the expression in (iii) for u^H_{m+1} can be bounded by

$$A := \tilde{c}_H\, C \int_0^t \int_{\mathbb{R}^2} \frac{G^2_{t-s}(x-y)}{|z|^{2-2H}}\, dyds \int_0^s \int_{\mathbb{R}} |1-e^{-i\xi z}|^2 |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{1-2H}\, d\xi dr.$$

By Lemma A.4, we have

$$\int_{\mathbb{R}} \frac{|1-e^{-i\xi z}|^2}{|z|^{2-2H}} dz = \frac{2\Gamma(2H+1)\sin(\pi H)}{H(1-2H)} |\xi|^{1-2H}.$$

Hence,

$$A \leq \tilde{c}_H \frac{2\Gamma(2H+1)\sin(\pi H)}{H(1-2H)}\, C \int_0^t \int_{\mathbb{R}} G^2_{t-s}(x-y)\, dyds \int_0^s \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)}\, d\xi dr. \tag{24}$$

Note that, by definition of $\tilde{c}_H$ (see Remark 2.10), it holds

$$\tilde{c}_H \frac{2\Gamma(2H+1)\sin(\pi H)}{H(1-2H)} = \Gamma(2H+1)\sin(\pi H),$$

and the latter is uniformly bounded for $H \in K$, since it is a continuous function of H. Regarding the integrals in (24), they can be estimated using the explicit expressions of the fundamental solutions of the wave and heat equations and applying Lemma A.1. □

We need to extend condition (ii) in the above proposition to a uniform estimate with respect to $m \geq 1$. For this, we follow the arguments of [2, Section 3.3], so we first need the following result, whose proof follows the same steps of [2, Thm. 3.8] and uses analogous arguments as those in Proposition 3.2.

Proposition 3.3. *Define, for any $m \geq 0$ and $t \in [0,T]$,*

$$V_m(t) := \sup_{H\in K} \sup_{x\in\mathbb{R}} \left(\mathbb{E}\left[|u^H_m(t,x) - u^H_{m-1}(t,x)|^p\right]\right)^{\frac{2}{p}}$$

and

$$W_m(t) := \sup_{H\in K} \sup_{x\in\mathbb{R}} C_H \int_0^t \int_{\mathbb{R}^2} G^2_{t-s}(x-y)|y-z|^{2H-2} \times \left(\mathbb{E}\left[|u^H_m(s,y) - u^H_{m-1}(s,y) - u^H_m(s,z) + u^H_{m-1}(s,z)|^p\right]\right)^{\frac{2}{p}} dydzds.$$

Then,

$$V_{m+1}(t) \le \int_0^t V_m(s) J_1(t-s) ds + C W_m(t)$$

and

$$W_{m+1}(t) \le \int_0^t V_m(s) J_2(t-s) ds + \int_0^t W_m(s) J_1(t-s) ds,$$

where J_1 *and* J_2 *are non-negative integrable functions on* $[0,T]$.

Next, we have the following result on the convergence of the underlying Picard iteration scheme, which extends [2, Thm. 3.9]:

Theorem 3.4. *Let* $H \in (\frac{1}{4}, \frac{1}{2})$ *and* $p \ge 2$. *The sequence* $\{u_m^H, m \ge 0\}$ *of Picard iterations converges in the space* χ_H^p *to a process* u^H *which is the unique mild solution of* (16). *Moreover, it holds:*

$$\lim_{m\to\infty} \sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u_m^H(t,x) - u^H(t,x)|^p\right] = 0. \tag{25}$$

Proof. As in the proof of [2, Thm. 3.9], we have to check that the modified definitions of V_m and W_m still work to show that the Picard iterations converge to the solution u^H, uniformly with respect to $H \in K$. There is no need to check that the solution is the same as the one found in [2], since for any fixed value of H the norm $\|\cdot\|_{\chi^H}$ is equivalent to the one defined in [2, Def. 3.6], as we noticed in Remark 2.16.

Set

$$M_m(t) := V_m(t) + W_m(t)$$

and

$$J(t) := C(J_1(t) + J_2(t)).$$

Then, by Proposition 3.3, we have

$$M_{m+1}(t) \le \int_0^t (M_m(s) + M_{m-1}(s)) J(t-s) ds.$$

The Grönwall type lemma [2, Lem. 3.10]) yields

$$\sum_{m\ge1} \sup_{H\in K} \|u_m^H - u_{m-1}^H\|_{\chi_H^p} < \infty.$$

This implies that $\{u_m^H\}_{m\ge0}$ is a Cauchy sequence in χ_H^p, uniformly with respect to $H \in K$, and so it converges, uniformly in H, to the limit u^H, which we already know that exists and is unique. ∎

Corollary 3.5. *Let* $H \in (\frac{1}{4}, \frac{1}{2})$ *and* $p \ge 2$. *Let* K *be of the form described in Proposition 3.1. Then, it holds that*

$$\sup_{H\in K} \sup_{m\ge0} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u_m^H(t,x)|^p\right] < \infty.$$

This corollary, together with the lemmas in the Appendix, allow us to prove the following result, which is an adaptation of [3, Prop. 2.2]. Indeed, as in the preceding result, one just needs to keep track on the constants depending on H.

Proposition 3.6. *Let $h_0 \in (0,1)$ and $p \geq 2$. Then, for all $|h| \leq h_0$,*

$$\sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u_m^H(t,x+h) - u_m^H(t,x)|^p\right] \leq C_m |h|^{\eta_1 p}$$

and

$$\sup_{H\in K} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \mathbb{E}\left[|u_m^H(t+h,x) - u_m^H(t,x)|^p\right] \leq C_m |h|^{\tilde{\eta}_1 p},$$

where $\tilde{\eta}_1 = \eta_1$ for the wave equation $\tilde{\eta}_1 = \frac{\eta_1}{2}$ for the heat equation. The constant C_m satisfies

$$C_m \leq C(c(h_0) + \overline{c}(h_0) C_{m-1}),$$

where the functions $c, \overline{c} : \mathbb{R} \to \mathbb{R}$ are non-negative and $\lim_{h_0\to 0} \overline{c}(h_0) = 0$. We define $C_{-1} = 0$.

Putting together (25) and Proposition 3.6, and taking into account that the sequence $\{C_m, m \geq 0\}$ in the latter result is bounded (see [3, Thm 1.1]), we finally have the following:

Proposition 3.7. *Let $p \geq 2$. There exists $h_0 > 0$ such that, for every $|h| \leq h_0$, it holds:*

$$\sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u^H(t,x+h) - u^H(t,x)|^p\right] \leq C |h|^{\eta_1 p}$$

and

$$\sup_{H\in K} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \mathbb{E}\left[|u^H(t+h,x) - u^H(t,x)|^p\right] \leq C |h|^{\tilde{\eta}_1 p},$$

where C is a constant depending only on p, $\tilde{\eta}_1 = \eta_1$ for the wave equation and $\tilde{\eta}_1 = \frac{\eta_1}{2}$ for the heat equation.

Now, we have all needed ingredients to prove our tightness result Proposition 3.1.

Proof of Proposition 3.1. We will apply Theorem A.5. First, we notice that condition (i) in this criterion is clearly satisfied, since $u^H(0,0)$ is deterministic and independent of H.

In order to check (ii) in Theorem A.5, we apply Proposition 3.7 and we deduce that, for any $t, t' \in [0,T]$ and $x, x' \in \mathbb{R}$ such that $|t'-t| < h_0$ and $|x'-x| < h_0$, it holds:

$$\mathbb{E}\left[|u^H(t',x') - u^H(t,x)|^p\right] \leq C(|t'-t|^{p\tilde{\eta}_1} + |x'-x|^{p\eta_1}). \tag{26}$$

One can easily deduce that estimate (26) holds for any $t, t' \in [0,T]$ and any x, x' in a compact set. □

3.2 Tightness in the case $[\frac{1}{2}, 1)$

We aim to prove an analogous tightness result as Proposition 3.1 for the case $H \geq \frac{1}{2}$. We state it in Proposition 3.8 below.

Now, we suppose that the limiting exponent $H_0 \in [\frac{1}{2}, 1)$, so whenever $H_n \to H_0$ we can suppose without loss of generality that $H_n \in K$, where K is of the form $[\eta_1, \eta_2]$, with $\eta_1, \eta_2 \in [\frac{1}{2}, 1)$ and $\eta_1 \leq \eta_2$. As we already observed at the beginning of Section 3.1, if we prove the tightness of the family of laws of $\{u^H, H \in K\}$ also for K of the form considered here, this will include also the case in which $H_0 = \frac{1}{2}$ and $H_n \to H_0$ either from above or from below.

The following tightness result will be proved directly, i.e. without going through the corresponding Picard iteration scheme. This is because the Burkholder-Davies-Gundy type inequality (14) is more practical than its *rough* counterpart (15).

Proposition 3.8. *Let $\mathcal{U}_K := \{u^H, H \in K\}$ be the family of solutions of* (16)*, where K is of the form $[\eta_1, \eta_2]$, with $\eta_1, \eta_2 \in [\frac{1}{2}, 1)$ and $\eta_1 \leq \eta_2$. Then, the family $\mathcal{U}_K$ is tight in $\mathcal{C}([0,T] \times \mathbb{R})$, endowed with the metric of uniform convergence on compact sets.*

Proof. We will apply again Theorem A.5. We split the proof in three steps.

Step 1: We show the uniform estimate

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u^H(t,x)|^p\right] < \infty. \tag{27}$$

We have

$$\mathbb{E}\left[|u^H(t,x)|^p\right] \leq C\left(1 + \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)u^H(s,y)W^H(ds,dy)\right|^p\right]\right).$$

By Theorem 2.8, we obtain that the expectation in the right hand-side above can be bounded, up to some positive constant, by

$$c_H(\nu_{t,H})^{\frac{p}{2}-1} \int_0^t \sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathbb{E}\left[|u^H(s,x)|^p\right] \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H}\, d\xi ds, \tag{28}$$

where $\nu_{t,H}$ is defined by

$$\nu_{t,H} = c_H \int_0^t \int_{\mathbb{R}} |\mathcal{F}G_s(\xi)|^2 |\xi|^{1-2H} d\xi ds.$$

We recall that $c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}$, which is bounded by $\frac{1}{\pi}$, for all H. Moreover, by Lemma A.1, it holds that

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{t\in[0,T]} \nu_{t,H} < \infty.$$

Note that this holds for both wave and heat equations. On the other hand, regarding the integral in $d\xi$ in (28), we can argue as follows. In the case of the wave equation, we have

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi &= 2\int_0^\infty \frac{\sin^2((t-s)\xi)}{\xi^{1+2H}} d\xi \\ &= 2(t-s)^{2H} 2^{2H-1} C_{1-2H} \\ &\leq T^{2H} 2^{2H} C_{1-2H},\end{aligned}$$

where the constant C_{1-2H} is the same one appearing in Lemma A.1. As showed in the proof of [12, Thm. 2.8], C_{1-2H} defines a continuous function with respect to $H \in (0,1)$, so it can be bounded by a constant when $H \in [\eta_1, \eta_2]$. Thus, for the wave equation we can conclude that

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathbb{E}\left[|u^H(t,x)|^p\right] \leq C\left(1 + \int_0^t \sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathbb{E}\left[|u^H(s,x)|^p\right] ds\right).$$

Hence, Grönwall lemma implies (27).

In the case of the heat equation, we have

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi &= \int_{\mathbb{R}} e^{-(t-s)|\xi|^2} |\xi|^{1-2H} d\xi \\ &= \frac{1}{2}(t-s)^{H-1} \int_0^\infty e^{-y} y^{-H} dy \\ &= \Gamma(1-H)(t-s)^{H-1}.\end{aligned}$$

Observe that, for all $H \in [\eta_1, \eta_2]$, it holds $\Gamma(1-H)(t-s)^{H-1} \le g(t-s)$, where

$$g(r) := \Gamma(1-\eta_2) \begin{cases} r^{\eta_1 - 1}, & r < 1 \\ 1, & r > 1. \end{cases}$$

Therefore,

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathbb{E}\left[|u^H(t,x)|^p\right] \le C\left(1 + \int_0^t \sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathbb{E}\left[|u^H(s,x)|^p\right] g(t-s)ds\right).$$

The Grönwall type lemma proved in [8, Lem. 15] let us conclude that (27) is also fulfilled in the case of the heat equation.

Step 2: In this part of the proof, we deal with the moments of the space increments of the solution u^H. Precisely, owing to Theorem 2.8, the estimate (27) and Lemma A.2, we can infer that, for all $p \ge 2$ and $|h| \le 1$,

$$\begin{aligned} &\mathbb{E}\left[|u^H(t,x+h) - u^H(t,x)|^p\right] \\ &\le C\, c_H^{\frac{p}{2}} \left(\int_0^t \int_{\mathbb{R}} |\mathcal{F}\left(G_{t-s}(x-\cdot) + G_{t-s}(x+h-\cdot)\right)(\xi)|^2\, |\xi|^{1-2H}\, d\xi ds\right)^{\frac{p}{2}} \\ &= C\, c_H^{\frac{p}{2}} \left(\int_0^t \int_{\mathbb{R}} (1-\cos(h\xi))|\mathcal{F}G_s(\xi)|^2 |\xi|^{1-2H}\, d\xi ds\right)^{\frac{p}{2}} \\ &\le C\, \tilde{C}_H^{\frac{p}{2}} |h|^{Hp}. \end{aligned}$$

The constant $\tilde{C}_H$ is the same appearing in [2, Lem. 3.4], and it is given by

$$\tilde{C}_H := \int_{\mathbb{R}} (1-\cos(\theta))|\theta|^{-1-2H} d\theta < \frac{1}{H} + \frac{1}{1-H} \le C,$$

provided that $H \in [\eta_1, \eta_2]$. Thus, we have proved that

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u^H(t,x+h) - u^H(t,x)|^p\right] \le C|h|^{\eta_1 p}.$$

Step 3: Here, we aim to prove that, for any $p \ge 2$ and $|h| < 1$,

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0\vee(-h), T\wedge(T-h)]\times\mathbb{R}} \mathbb{E}\left[|u^H(t+h,x) - u^H(t,x)|^p\right] \le \begin{cases} C|h|^{\eta_1 p} & \text{wave equation,} \\ C|h|^{\frac{\eta_1}{2} p} & \text{heat equation.} \end{cases} \tag{29}$$

Assume that $h > 0$ (the case $h < 0$ is completely analogous). Then,

$$\mathbb{E}\Big[|u^H(t+h,x) - u^H(t,x)|^p\Big] \le C(B_1 + B_2),$$

where

$$B_1 := \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} [G_{t+h-s}(x-y) - G_{t-s}(x-y)] u^H(s,y) W^H(ds,dy)\right|^p\right],$$

$$B_2 := \mathbb{E}\left[\left|\int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y) u^H(s,y) W^H(ds,dy)\right|^p\right].$$

Theorem 2.8, (27) and Lemma A.3 yield

$$\begin{aligned}
B_1 &\leq C\, c_H^{\frac{p}{2}} \left(\int_0^t \int_{\mathbb{R}} \left| \mathcal{F}\left(G_{t+h-s}(x-\cdot) - G_{t-s}(x-\cdot)\right)(\xi)\right|^2 |\xi|^{1-2H} d\xi ds \right)^{\frac{p}{2}} \\
&\leq C\, c_H^{\frac{p}{2}} \left(\int_0^T \int_{\mathbb{R}} \left| \mathcal{F}G_{s+h}(\xi) - \mathcal{F}G_s(\xi)\right|^2 |\xi|^{1-2H} d\xi ds \right)^{\frac{p}{2}} \\
&\leq C \begin{cases} |h|^{Hp}, & \text{wave equation,} \\ |h|^{\frac{H}{2}p}, & \text{heat equation.} \end{cases}
\end{aligned} \tag{30}$$

Regarding the term B_2, we can argue as before but we apply Lemma A.1. Indeed, we have that

$$\begin{aligned}
B_2 &\leq C\, c_H^{\frac{p}{2}} \left(\int_t^{t+h} \int_{\mathbb{R}} |\mathcal{F}G_{t+h-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi ds \right)^{\frac{p}{2}} \\
&= C\, c_H^{\frac{p}{2}} \int_0^h \int_{\mathbb{R}} |\mathcal{F}G_s(\xi)|^2 |\xi|^{1-2H} d\xi ds \\
&\leq C \begin{cases} |h|^{\frac{1+2H}{2}p}, & \text{wave equation,} \\ |h|^{\frac{H}{2}p}, & \text{heat equation.} \end{cases}
\end{aligned} \tag{31}$$

Putting together (30) and (31), and taking into account that $H \in [\eta_1, \eta_2]$, we end up with (29).

Finally, the results in Steps 2 and 3 let us conclude that, for any $t, t' \in [0, T]$ and x, x' in a compact of $\mathbb{R}$, we have

$$\mathbb{E}\left[|u^H(t', x') - u^H(t, x)|^p \right] \leq C \begin{cases} |t'-t|^{\eta_1 p} + |x'-x|^{\eta_1 p}, & \text{wave equation,} \\ |t'-t|^{\frac{\eta_1}{2}p} + |x'-x|^{\eta_1 p}, & \text{heat equation} \end{cases}.$$

Thus, it suffices to take $p > \frac{4}{\eta_1}$ for the heat equation and $p > \frac{2}{\eta_1}$ for the wave equation to be able to apply the tightness criterion Theorem A.5. □

The following result extends Corollary 3.5 to the case $H \geq \frac{1}{2}$. Its proof is very similar to that of [8, Thm. 13], and the terms that need to be estimated uniformly with respect to H are completely analogous as those appearing in Step 1 of the proof of the above Proposition 3.8.

Lemma 3.9. *Let $H \geq \frac{1}{2}$ and $\{u_m^H,\, m \geq 0\}$ be the sequence of Picard iterations corresponding to the mild formulation (16). Then, for any $p \geq 2$, u_m^H converges in $L^p(\Omega)$ to the solution u^H uniformly with respect to $H \in K$, i.e.*

$$\lim_{m\to\infty} \sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u_m^H(t,x) - u^H(t,x)|^p \right] = 0$$

4 Identification of the limit

Let $H_0 \in (\frac{1}{4}, 1)$ and $\{H_n,\, n \geq 1\}$ be any sequence such that $H_n \to H_0$, as $n \to \infty$. We may assume that there exists a compact set $K \subset (\frac{1}{4}, 1)$ such that $H_n \in K$, for all $n \geq 1$. The tightness results proved in Propositions 3.1 and 3.8 imply that there exists a subsequence $\{H_{n_k},\, k \geq 1\}$ such that $\{u^{H_{n_k}},\, k \geq 1\}$ converges in law in the space $\mathcal{C}([0,T]\times\mathbb{R})$ of contiuous functions. This section is devoted to prove that the limit law is the distribution of u^{H_0}.

Our strategy can be summarized as follows. We will verify that the finite dimensional distributions of u^{H_n} converge to those of u^{H_0} (see [7, Thm. 2.6]). For this, it suffices to prove that, for any fixed $(t,x) \in [0,T] \times \mathbb{R}$, $u^{H_n}(t,x)$ converges to $u^{H_0}(t,x)$ in $L^2(\Omega)$. This can be done thanks to the fact that the whole family of noises $\{W^H, H \in (0,1)\}$ can be defined on a single probability space (see Section 2.2). In order to prove the above $L^2(\Omega)$-convergence, we will check the same convergence for any of the corresponding Picard iterates, that is, for any $m \geq 1$, we show that $u_m^{H_n}(t,x) \to u_m^{H_0}(t,x)$ in $L^2(\Omega)$, as $n \to \infty$, and we will take into account that the Picard iteration scheme converges to the soution uniformly with respect to the Hurst index H. At this point, we recall (invoking Theorem 2.20) that any Picard iterate admits the following Wiener chaos expansion:

$$u_m^{H_n}(t,x) = \sum_{j=0}^{m} I_j^{H_n}(g_j(\cdot,t,x)),$$

where the latter is a finite sum of multiple Wiener integrals of order up to m and the kernels g_j are given by (22). Therefore, it will be sufficient to prove the $L^2(\Omega)$-convergence, as $n \to \infty$, of any of the above multiple Wiener integrals, for which we will make use of the representation result given in Theorem 2.7.

Here is the main result of the section:

Theorem 4.1. *Let $H_0 \in (\frac{1}{4}, 1)$ and $\{H_n, n \geq 1\}$ be any sequence such that $H_n \to H_0$, as $n \to \infty$. Let $u_n^{H_n}$ and u^{H_0} be the solutions of (16) corresponding the Hurst parameters H_n and H_0, respectively. Then, the finite dimensional distributions of u^{H_n} converge to those of u^{H_0}, as $n \to \infty$.*

Proof. We split the proof in three steps.

Step 1: To start with, we recall that, owing to Corollary 3.5 and Lemma 3.9 in the particular case $p = 2$, we have:

$$\lim_{m\to\infty} \sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u_m^H(t,x) - u^H(t,x)|^2\right] = 0, \tag{32}$$

where u_m^H denotes the associated mth Picard iterate.

As we already explained, in order to assure the statement's validity it is sufficient to show the following pointwise convergence in $L^2(\Omega)$: for any fixed $(t,x) \in [0,T] \times \mathbb{R}$, it holds

$$\lim_{n\to\infty} \mathbb{E}\left[|u^{H_n}(t,x) - u^{H_0}(t,x)|^2\right] = 0.$$

Note that

$$\begin{aligned}
&\mathbb{E}\left[|u^{H_n}(t,x) - u^{H_0}(t,x)|^2\right] \\
&\quad \leq C\left(\mathbb{E}\left[|u^{H_n}(t,x) - u_m^{H_n}(t,x)|^2\right] + \mathbb{E}\left[|u_m^{H_n}(t,x) - u_m^{H_0}(t,x)|^2\right] + \mathbb{E}\left[|u_m^{H_0}(t,x) - u^{H_0}(t,x)|^2\right]\right) \\
&\quad =: I_1(m,n) + I_2(m,n) + I_3(m).
\end{aligned}$$

By (32), we can infer that, for any $\varepsilon > 0$, we can choose m_0 big enough such that, for every $m \geq m_0$, we have

$$\sup_{n\geq 1}\{I_1(n,m) + I_3(m)\} < \varepsilon.$$

Thus, we are left to show that $I_2(m_0, n)$ tends to zero as $n \to \infty$. This means, in particular, that the m_0-th Picard iterate is continuous in $L^2(\Omega)$, with respect to H.

Theorem 2.20 implies that, for any $H \in (\frac{1}{4}, 1)$, $u^H_{m_0}$ has the Wiener chaos expansion

$$u^H_{m_0}(t,x) = \sum_{j=0}^{m_0} I^H_j(g_j(\cdot,t,x)),$$

where the functions g_j are defined by (22). Hence, in order to check that $I_2(m_0,n)$ tends to zero, it is enough to show that, for any $j = 1,\dots,m_0$, $I^{H_n}_j(g_j(\cdot,t,x))$ converges to $I^{H_0}_j(g_j(\cdot,t,x))$ in $L^2(\Omega)$, as $n \to \infty$. Indeed, by Theorem 2.7 we have that

$$\begin{aligned}
&I^{H_n}_j(g_j(\cdot,t,x)) - I^{H_0}_j(g_j(\cdot,t,x)) \\
&\quad = \int_{\{[0,T]\times\mathbb{R}\}^j} \left(c^j_{H_n}|\xi_1|^{\frac{1}{2}-H_n}\cdots|\xi_j|^{\frac{1}{2}-H_n} - c^j_{H_0}|\xi_1|^{\frac{1}{2}-H_0}\cdots|\xi_j|^{\frac{1}{2}-H_0}\right) \\
&\qquad\qquad \times \mathcal{F}g_j(t_1,\cdot,\dots,t_j,\cdot,t,x)(\xi_1,\dots,\xi_n)\tilde{W}(dt_1,d\xi_1)\cdots\tilde{W}(dt_j,d\xi_j).
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}\left[\left|I^{H_n}_j(g_j(\cdot,t,x)) - I^{H_0}_j(g_j(\cdot,t,x))\right|^2\right] \\
&\quad = \int_{\{[0,T]\times\mathbb{R}\}^j} \left|c^j_{H_n}|\xi_1|^{\frac{1}{2}-H_n}\cdots|\xi_j|^{\frac{1}{2}-H_n} - c^j_{H_0}|\xi_1|^{\frac{1}{2}-H_0}\cdots|\xi_j|^{\frac{1}{2}-H_0}\right|^2 \\
&\qquad\qquad \times |\mathcal{F}g_j(t_1,\cdot,\dots,t_j,\cdot,t,x)(\xi_1,\dots,\xi_n)|^2 d\xi_1\cdots d\xi_j\, dt_1\cdots dt_j.
\end{aligned}$$

We show that the last integral converges to 0 when $n \to \infty$. To do this, we have to compute explicitly the Fourier transform appearing in the above expression. Precisely, as detailed in [4, p. 10], we have

$$\begin{aligned}
&\mathcal{F}g_j(t_1,\cdot,\dots,t_j,\cdot,t,x)(\xi_1,\dots,\xi_j) \\
&\quad = \eta e^{-i(\xi_1+\cdots+\xi_j)x}\,\overline{\mathcal{F}G_{t_2-t_1}(\xi_1)}\,\overline{\mathcal{F}G_{t_3-t_2}(\xi_1+\xi_2)}\cdots\overline{\mathcal{F}G_{t-t_j}(\xi_1+\cdots+\xi_j)}\,1_{\{0<t_1<\cdots<t_j<t\}}
\end{aligned}$$

Therefore, making the change of variables $\eta_\ell := \xi_1 + \cdots + \xi_\ell$, for $\ell = 1,\dots,j$, we end up with

$$\begin{aligned}
&\mathbb{E}\left[|I^{H_n}_j(g_j(\cdot,t,x)) - I^{H_0}_j(g_j(\cdot,t,x))|^2\right] \\
&\quad \le \int_{T_j(t)}\int_{\mathbb{R}^j} \eta \prod_{\ell=1}^{j} |\mathcal{F}G_{t_{\ell+1}-t_\ell}(\eta_\ell)|^2 \left|c^j_{H_n}|\eta_1|^{\frac{1}{2}-H_n}|\eta_2-\eta_1|^{\frac{1}{2}-H_n}\cdots|\eta_j-\eta_{j-1}|^{\frac{1}{2}-H_n}\right. \\
&\qquad \left. - c^j_{H_0}|\eta_1|^{\frac{1}{2}-H_0}|\eta_2-\eta_1|^{\frac{1}{2}-H_0}\cdots|\eta_j-\eta_{j-1}|^{\frac{1}{2}-H_0}\right|^2 d\xi_1\cdots d\xi_j\, dt_1\cdots dt_j,
\end{aligned}$$

where $T_j(t) := \{(t_1,\dots,t_j),\ 0 < t_1 < \cdots < t_j < t\}$. We wish to prove that the latter integral converges to 0 as $n \to \infty$. For this, we will apply the Dominated convergence theorem. Note that the integrand clearly converges to 0 pointwise on $T_j(t) \times \mathbb{R}^j$. Indeed, the constant c_H (see (1)) defines a continuous function of $H \in (0,1)$. Now, we proceed to bound the integrand by an integrable function. First, we note that the integrand can be bounded, up to some positive constant, by

$$\begin{aligned}
&\prod_{\ell=1}^{j} |\mathcal{F}G_{t_{\ell+1}-t_\ell}(\eta_\ell)|^2 \left(c^{2j}_{H_n}|\eta_1|^{1-2H_n}|\eta_2-\eta_1|^{1-2H_n}\cdots|\eta_j-\eta_{j-1}|^{1-2H_n}\right. \\
&\qquad\qquad \left. + c^{2j}_{H_0}|\eta_1|^{1-2H_0}|\eta_2-\eta_1|^{1-2H_0}\cdots|\eta_j-\eta_{j-1}|^{1-2H_0}\right).
\end{aligned}$$

The two resulting terms in the above sum are of the same type, except the fact that the first one depends on n while the second does not, and they are equivalent to the integrands studied in [4, p. 11-13] (only in the case of wave equation with $H \in (\frac{1}{4}, \frac{1}{2})$). From now on, we will only consider the term of the integrand function that depends on n; the integrability of the other term will be an immediate consequence of the treatment of the first one.

Hence, we will find a suitable estimate for the term

$$|\eta_1|^{1-2H_n}|\eta_2 - \eta_1|^{1-2H_n} \cdots |\eta_j - \eta_{j-1}|^{1-2H_n} \prod_{\ell=1}^{j} |\mathcal{F}G_{t_{\ell+1}-t_\ell}(\eta_\ell)|^2. \tag{33}$$

Notice that we have bounded c_{H_n} by a constant, since we may assume that all H_n are included in a compact set of $(\frac{1}{4}, 1)$. We distinguish the cases $H_n < \frac{1}{2}$ and $H_n \geq \frac{1}{2}$.

Step 2: In the case $H_n < \frac{1}{2}$, we use the following fact: whenever $H \in (0, \frac{1}{2})$, we have

$$\prod_{\ell=2}^{j} |\eta_\ell - \eta_{\ell-1}|^{1-2H} \leq \sum_{\alpha \in D_j} \prod_{\ell=1}^{j} |\eta_\ell|^{\alpha_\ell},$$

where D_j is a set with cardinality 2^{j-1} and its elements are multi-indices $\alpha = (\alpha_1, \ldots, \alpha_j)$ whose component's sum equals to $(j-1)(1-2H)$ and satisfy

$$\alpha_1 \in \{0, 1-2H\}, \text{ and } \alpha_\ell \in \{0, 1-2H, 2(1-2H)\}, \text{ for } \ell = 2, \ldots, j.$$

When $H = H_n$, the corresponding α_ℓ will be denoted by $\alpha_{\ell,n}$. Thus, the integrand (33) may be bounded by

$$|\eta_1|^{1-2H_n} \left(\sum_{\alpha \in D_j} \prod_{\ell=1}^{j} |\eta_\ell|^{\alpha_{\ell,n}} \right) \left(\prod_{\ell=1}^{j} \left|\mathcal{F}G_{t_\ell - t_{\ell-1}}(\eta_\ell)\right|^2 \right) \tag{34}$$

Let $\beta := \min_{n \geq 1} H_n > 1/4$ and define the functions $f_0, f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ as follows: $f_0(r) = 1$ and

$$f_1(r) = \begin{cases} r^{1-2\beta}, & r \geq 1, \\ 1, & r < 1, \end{cases}$$

$$f_2(r) = \begin{cases} r^{2(1-2\beta)}, & r \geq 1, \\ 1, & r < 1. \end{cases}$$

We also set, for every $\alpha_{\ell,n}$,

$$N(\alpha_{\ell,n}) := \begin{cases} 0, & \alpha_{\ell,n} = 0, \\ 1, & \alpha_{\ell,n} = 1 - 2H_n, \\ 2, & \alpha_{\ell,n} = 2(1-2H_n). \end{cases}$$

Then, we have the following estimate for the term (34):

$$\begin{aligned} &|\eta_1|^{1-2H_n} \left(\sum_{\alpha \in D_j} \prod_{\ell=1}^{j} |\eta_\ell|^{\alpha_{\ell,n}} \right) \left(\prod_{\ell=1}^{j} \left|\mathcal{F}G_{t_\ell - t_{\ell-1}}(\eta_\ell)\right|^2 \right) \\ &\quad \leq f_1(|\eta_1|) \left(\sum_{\alpha \in D_j} \prod_{\ell=1}^{j} f_{N(\alpha_{\ell,n})}(|\eta_\ell|) \right) \left(\prod_{\ell=1}^{j} \left|\mathcal{F}G_{t_\ell - t_{\ell-1}}(\eta_\ell)\right|^2 \right). \end{aligned}$$

We have to prove that this function is integrable. To check this last fact, it is sufficient to show it for a single integrand of the form

$$\prod_{\ell=1}^{j} \left| \mathcal{F}G_{t_\ell - t_{\ell-1}}(\eta_\ell) \right|^2 |\eta_1|^\beta \prod_{\ell=1}^{j} |\eta_\ell|^{\alpha_\ell}$$

where, now, α_j does not take values in a discrete set, but they satisfy the weaker constraints:

$$\alpha_1 \in K_1 \subset [0, 1/2), \text{ and } \alpha_\ell \in K_2 \subset [0,1), \text{ for } \ell = 2, \dots, j,$$

where $K_1 = [0, 1 - 2\min_{n\geq 1} H_n]$ and $K_2 = [0, 2(1 - 2\min_{n\geq 1} H_n)]$ (we are assuming implicitly that $\min_{n\geq 1} H_n < \frac{1}{2}$; if this is not the case, then the entire sequence falls in the case $H_n \geq \frac{1}{2}$, which will be studied afterwards). It is important to notice that the sets K_1, K_2 do not depend on n. The fact that $1 - 2\min_{n\geq 1} H_n < \frac{1}{2}$ and $2(1 - 2\min_{n\geq 1} H_n) < 1$ turns out to be crucial for our estimates.

Thus, we want to prove that

$$\int_{T_j(t)} \left(\int_{\mathbb{R}} |\mathcal{F}G_{t_2 - t_1}(\eta_1)|^2 |\eta_1|^{\beta + \alpha_1} d\eta_1 \right) \prod_{\ell=2}^{j} \left(\int_{\mathbb{R}} |\mathcal{F}G_{t_{\ell+1} - t_\ell}(\eta_\ell)|^2 |\eta_\ell|^{\alpha_\ell} d\eta_\ell \right) dt_1 \cdots dt_j < \infty. \quad (35)$$

At this point, we have to consider separately the case of the wave equation case from that of the heat equation. It holds that, for any $\gamma \in (-1,1)$ (see the proof of Proposition 3.8):

$$\int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 |\xi|^\gamma d\xi \leq \begin{cases} C'_\gamma (2-\gamma) t^{1-\gamma}, & \text{wave equation,} \\ C''_\gamma \frac{1-\gamma}{2} t^{-\frac{(\gamma+1)}{2}}, & \text{heat equation.} \end{cases}$$

We recall that the constants C'_γ and C''_γ are continuous with respect to $\gamma \in (-1,1)$. We will apply the above estimate with $\gamma = 1 - 2H$ and $\gamma = 2(1-2H)$, and still we can bound them uniformly with respect to $H \in K \subset (\frac{1}{4}, \frac{1}{2}]$, with K compact. Hence, for the heat equation, the integral in (35) can be estimated by

$$\int_{T_j(t)} (t_2 - t_1)^{\frac{-\beta - \alpha_1}{2}} \prod_{\ell=2}^{j} (t_{\ell+1} - t_\ell)^{\frac{-\alpha_\ell - 1}{2}} dt_1 \cdots dt_j,$$

which is finite because all exponents are strictly greater than -1. For the wave equation, we end up with

$$\int_{T_j(t)} (t_2 - t_1)^{1 - \beta - \alpha_1} \prod_{\ell=2}^{j} (t_{\ell+1} - t_\ell)^{1 - \alpha_\ell} dt_1 \cdots dt_j,$$

which is also finite since all exponents are even greater than 0. This concludes the proof in the case $H \in (\frac{1}{4}, \frac{1}{2}]$.

Step 3: Let us now go back to expression (33), where we resettle the variables ξ_ℓ by means of the change of variables $\xi_\ell = \eta_\ell - \eta_{\ell-1}$. That is, we aim to bound the following term:

$$|\xi_1|^{1-2H_n} \cdots |\xi_j|^{1-2H_n} \prod_{\ell=1}^{j} \left| \mathcal{F}G_{t_{\ell+1} - t_\ell}(\xi_1 + \cdots + \xi_\ell) \right|^2, \quad (36)$$

where we assume that $H_n \in [\frac{1}{2}, 1)$. Here, the fact that $1 - 2H_n \leq 0$ helps us. Indeed, we can define the bounding function in a quite straightforward way:

$$g(r) := \begin{cases} 1, & r \geq 1, \\ r^{1-2(\max_{n\geq 1} H_n)}, & r < 1. \end{cases}$$

Clearly, the integrand function in (36) is bounded, for any $n \geq 1$, by

$$g(|\xi_1|) \cdots g(|\xi_j|) \prod_{\ell=1}^{j} \left|\mathcal{F}G_{t_{\ell+1}-t_\ell}(\xi_1 + \cdots + \xi_\ell)\right|^2.$$

We check that this upper bound function is integrable, namely

$$\int_{T_{j-1}(t_j)} \int_{\mathbb{R}^{j-1}} \prod_{\ell=1}^{j-1} \left|\mathcal{F}G_{t_{\ell+1}-t_\ell}(\xi_1 + \cdots + \xi_\ell)\right|^2 g(|\xi_\ell|) \\ \times \left(\int_{t_{j-1}}^{t} \int_{\mathbb{R}} \left|\mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j)\right|^2 g(|\xi_j|) d\xi_j dt_j \right) d\xi_1 \cdots d\xi_{j-1} dt_1 \cdots dt_{j-1} < \infty. \quad (37)$$

We have that

$$\int_{t_{j-1}}^{t} \int_{\mathbb{R}} \left|\mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j)\right|^2 g(|\xi_j|) d\xi_j dt_j \\ = \int_{t_{j-1}}^{t} \int_{|\xi_j|>1} \left|\mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j)\right|^2 d\xi_j dt_j \\ + \int_{t_{j-1}}^{t} \int_{|\xi_j|\leq 1} \left|\mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j)\right|^2 |\xi_j|^{1-2\min_{n\geq 1} H_n} d\xi_j dt_j.$$

We do the computations separately for the wave and heat equations. To start with, in the case of the wave equation, it clearly holds that

$$|\mathcal{F}G_t(\xi)| = \left|\frac{\sin(t|\xi|)}{|\xi|}\right| \leq t,$$

for all $(t, x) \in [0, T] \times \mathbb{R}$. Thus, we have

$$\int_{t_{j-1}}^{t} \int_{|\xi_j|\leq 1} \left|\mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j)\right|^2 |\xi_j|^{1-2\min_{n\geq 1} H_n} d\xi_j dt_j \\ \leq \int_{t_{j-1}}^{t} \int_{|\xi_j|\leq 1} |t - t_j|^2 |\xi_j|^{1-2\min_{n\geq 1} H_n} d\xi_j dt_j \\ \leq \frac{CT^3}{1 - \min_{n\geq 1} H_n} < \infty,$$

and

$$\int_{t_{j-1}}^{t} \int_{|\xi_j|>1} \left|\mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j)\right|^2 |\xi_j|^{1-2\min_{n\geq 1} H_n} \\ \leq \int_{t_{j-1}}^{t} \int_{\mathbb{R}} \frac{\sin^2\left[(t - t_j)|\xi_1 + \cdots + \xi_j|\right]}{|\xi_1 + \cdots + \xi_j|^2} d\xi_j dt_j \\ \leq C \int_{t_{j-1}}^{t} (t - t_j) dt_j < \infty,$$

since $\int_{\mathbb{R}} \frac{\sin^2(t|x|)}{|x|^2} dx = \pi t$. Therefore, we have got rid of the integral with respect to $d\xi_j dt_j$ in (37). Iterating this procedure one proves that the whole integral (37) is finite.

It remains to prove the analogous result for the heat equation. Here, we have

$$|\mathcal{F}G_t(\xi)| = e^{-\frac{t|\xi|^2}{2}} \leq 1,$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. Thus,

$$\begin{aligned}
&\int_{t_j}^{t} \int_{|\xi_j|\leq 1} \left|\mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j)\right|^2 |\xi_j|^{1-2\min_{n\geq 1} H_n} d\xi_j dt_j \\
&\quad \leq \int_{t_j}^{t} \int_{|\xi_j|\leq 1} |\xi_j|^{1-2\min_{n\geq 1} H_n} d\xi_j dt_j \\
&\quad \leq \frac{T}{1-\min_{n\geq 1} H_n} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_j}^{t} \int_{|\xi_j|> 1} \left|\mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j)\right|^2 |\xi_j|^{1-2\min_{n\geq 1} H_n} d\xi_j dt_j \\
&\quad \leq \int_{t_j}^{t} \int_{\mathbb{R}} \exp\left(-(t-t_j)|\xi_1 + \cdots + \xi_j|^2\right) d\xi_j dt_j \\
&\quad = C \int_{t_j}^{t} \sqrt{t-t_j} dt_j < \infty,
\end{aligned}$$

which, again by iterating this computation, shows that the integral in (37) is bounded also in the heat equation case. This completes the proof. □

A Auxiliary results

In this section, we state some results that have been applied throughout the paper. We start with four technical lemmas, proved in [2], which provide explicit estimates, depending on H, for the norm in the space $L^2(\mathbb{R}; \mu^H)$ of terms involving the Fourier transforms of the fundamental solutions of the wave and heat equations. Finally, we will also state a tightness criterion which will be applied in Section 3.

We recall that, for the wave and heat equations, we have, respectively:

$$\mathcal{F}G_t(\xi) = \frac{\sin(t|\xi|)}{|\xi|} \qquad \text{and} \qquad \mathcal{F}G_t(\xi) = \exp\left(\frac{-t\xi^2}{2}\right), \quad t>0,\, \xi \in \mathbb{R}.$$

In the following three lemmas, we will denote either one of these two functions by $\mathcal{F}G_t(\xi)$. We recall that the spatial spectral measure is given by $\mu^H(d\xi) = c_H |\xi|^{1-2H} d\xi$.

Lemma A.1 ([2], Lemma 3.1)**.** *Let $T>0$. Then, the integral*

$$A_T(\alpha) := \int_0^T \int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 |\xi|^{\alpha} \, d\xi \, dt$$

converges if and only if $\alpha \in (-1,1)$*. In this case, it holds:*

$$A_T(\alpha) = \begin{cases} 2^{1-\alpha} C_\alpha \dfrac{1}{2-\alpha} T^{2-\alpha} & \text{for the wave equation,} \\ \\ \dfrac{2}{1-\alpha} \Gamma\Big(\dfrac{\alpha+1}{2}\Big) T^{(1-\alpha)/2} & \text{for the heat equation,} \end{cases}$$

where the constant C_α *is given by*

$$C_\alpha = \begin{cases} \dfrac{\Gamma(\alpha)}{1-\alpha} \sin(\pi\alpha/2), & \alpha \in (-1,1) \setminus \{0\}, \\ \\ \dfrac{\pi}{2}, & \alpha = 0. \end{cases}$$

Lemma A.2 ([2], Lemma 3.4)**.** *Let* $T > 0$ *and* $\alpha \in (-1,1)$*. Then, for any* $h > 0$*, it holds:*

$$\int_0^T \int_{\mathbb{R}} (1 - \cos(\xi h)) \, |\mathcal{F} G_t(\xi)|^2 |\xi|^\alpha \, d\xi \, dt \leq \begin{cases} C|h|^{1-\alpha} & \text{for the heat equation,} \\ CT|h|^{1-\alpha} & \text{for the wave equation,} \end{cases}$$

where $C = \int_{\mathbb{R}} (1 - \cos\eta) |\eta|^{\alpha-2} d\eta$*.*

Lemma A.3 ([2], Lemma 3.5)**.** *Let* $T > 0$ *and* $\alpha \in (-1,1)$*. Then, for any* $h > 0$*, it holds:*

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F} G_{t+h}(\xi) - \mathcal{F} G_t(\xi)|^2 |\xi|^\alpha \, d\xi \, dt \leq \begin{cases} C_\alpha |h|^{(1-\alpha)/2} & \text{for the heat equation,} \\ C_\alpha T |h|^{1-\alpha} & \text{for the wave equation,} \end{cases}$$

where

$$C_\alpha = \int_{\mathbb{R}} \frac{(1 - e^{-\eta^2/2})^2}{|\eta|^{2-\alpha}} d\eta \quad \text{for the heat equation, and}$$

$$C_\alpha = 4 \int_{\mathbb{R}} \frac{\min(1, |\eta|^2)}{|\eta|^{2-\alpha}} d\eta \quad \text{for the wave equation.}$$

Lemma A.4 ([2], Lemma D.2)**.** *For any* $H \in (0, \frac{1}{2})$ *and for any* $\xi \in \mathbb{R}$*, we have:*

$$\int_{\mathbb{R}} \frac{|1 - e^{-i\xi x}|^2}{|x|^{2-2H}} dx = |\xi|^{1-2H} \frac{2\Gamma(2H+1)\sin(\pi H)}{H(1-2H)}$$

The following tightness criterion on the plane was proved in [27, Prop. 2.3].

Theorem A.5. *Let* $\{X_\lambda\}_{\lambda \in \Lambda}$ *be a family of random functions indexed on the set* Λ *and taking values in the space* $\mathcal{C}([0,T] \times \mathbb{R})$*, in which we consider the metric of uniform convergence over compact sets. Then, the family* $\{X_\lambda\}_{\lambda \in \Lambda}$ *is tight if, for any compact set* $J \subset \mathbb{R}$*, there exist* $p', p > 0$*,* $\delta > 2$*, and a constant* C *such that the following holds for any* $t', t \in [0,T]$ *and* $x', x \in J$*:*

(i) $\sup_{\lambda \in \Lambda} \mathbb{E}\left[|X_\lambda(0,0)|^{p'}\right] < \infty$*,*

(ii) $\sup_{\lambda \in \Lambda} \mathbb{E}\left[|X_\lambda(t',x') - X_\lambda(t,x)|^p\right] \leq C\left(|t'-t| + |x'-x|\right)^\delta$*.*

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