

Conditional Systemic Risk Measures

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Abstract

We investigate to which extent the relevant features of (static) Systemic Risk Measures can be extended to a conditional setting. After providing a general dual representation result, we analyze in greater detail Conditional Shortfall Systemic Risk Measures. In the particular case of exponential preferences, we provide explicit formulas that also allow us to show a time consistency property. Finally, we provide an interpretation of the allocations associated to Conditional Shortfall Systemic Risk Measures as suitably defined equilibria. Conceptually, the generalization from static to conditional Systemic Risk Measures can be achieved in a natural way, even though the proofs become more technical than in the unconditional framework.

Keywords: Conditional Risk; Systemic Risk; Conditional Equilibrium; Dynamic Risk Measures.

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1 Introduction

We provide a natural extension of static Systemic Risk Measures to a dynamic, conditional setting, and we study related concepts of time consistency and equilibrium.

To put the principal findings of this paper into perspective, we briefly review the literature pertaining to Systemic Risk Measures. We let $X = [X^1, \dots, X^N] \in (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$ be a vector of N \mathbb{P} -a.s. finite random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, representing a configuration of risky (financial) factors at a future time T associated to a system of N financial institutions/banks. A traditional approach to evaluate the risk of each institution $j \in \{1, \dots, N\}$ is to apply a *univariate monetary Risk Measure* η^j to the single financial position X^j , yielding $\eta^j(X^j)$. Let L be a subspace of $L^0(\Omega, \mathcal{F}, \mathbb{P})$. A monetary Risk Measure (see [38]) is a map $\eta : L \rightarrow \mathbb{R}$ that can be interpreted as the minimal capital needed to secure a financial position with payoff $Z \in L$, i.e., the minimal amount $m \in \mathbb{R}$ that must be added to Z in order to make the resulting (discounted) payoff at time T acceptable

$$\eta(Z) := \inf\{m \in \mathbb{R} \mid Z + m \in \mathbb{A}\}, \quad (1)$$

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where the acceptance set $\mathbb{A} \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be monotone, i.e., $Z \geq Y \in \mathbb{A}$ implies $Z \in \mathbb{A}$. Then η is monotone decreasing and satisfies the cash additivity property

$$\eta(Z + m) = \eta(Z) - m, \text{ for all } m \in \mathbb{R} \text{ and } Z \in L. \quad (2)$$

Under the assumption that the set \mathbb{A} is convex (resp. is a convex cone) the maps in (1) are convex (resp. convex and positively homogeneous) and are called *convex* (resp. coherent) *Risk Measures*, see Artzner et al. (1999) [7], Föllmer and Schied (2002) [37], Frittelli and Rosazza Gianin (2002) [42]. Once the risk $\eta^j(X^j)$ of each institution $j \in \{1, \dots, N\}$ has been determined, the quantity

$$\rho(X) := \sum_{j=1}^N \eta^j(X^j)$$

could be used as a very preliminary and naive assessment of the risk of the entire system.

1.1 Static Systemic Risk Measures

The approach sketched above does not clearly capture systemic risk of an interconnected system, and the design of more adequate Risk Measures for financial systems is the topic of a vast literature on systemic risk. Let $L_{\mathcal{F}}$ be a vector subspace of $(L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$. For example, we may take as $L_{\mathcal{F}}$ the space $(L^p(\Omega, \mathcal{F}, \mathbb{P}))^N$, $p \in [1, \infty]$, of (equivalence classes of) p -integrable (or essentially bounded if $p = \infty$) N -dimensional vectors of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. A Systemic Risk Measure is a map $\rho : L_{\mathcal{F}} \rightarrow \mathbb{R}$ that evaluates the risk $\rho(X)$ of the complete system $X \in L_{\mathcal{F}}$ and satisfies additionally financially reasonable properties. The subspace $L_{\mathcal{F}}$ of $(L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$ may represent possible additional integrability or boundedness requirements.

First aggregate then allocate. In Chen et al. (2013) [18] the authors investigated under which conditions a Systemic Risk Measure could be written in the form

$$\rho(X) = \eta(U(X)) = \inf\{m \in \mathbb{R} \mid U(X) + m \in \mathbb{A}\}, \quad (3)$$

for some univariate monetary Risk Measure η and some aggregation rule

$$U : \mathbb{R}^N \rightarrow \mathbb{R}$$

that aggregates the N -dimensional risk factors into a univariate risk factor. We also refer to Kromer et al. [48] (2013) for extension to general probability space.

Such systemic risk might again be interpreted as the minimal cash amount that secures the system when it is added to the total aggregated system loss $U(X)$, given that $U(X)$ allows for a monetary loss interpretation. Note, however, that in (3) systemic risk is the minimal capital added to secure the system *after aggregating individual risks*.

First allocate then aggregate. A second approach consisted in measuring systemic risk as the minimal cash that secures the aggregated system by adding the capital into the single institutions *before aggregating their individual risks*. This way of measuring systemic risk can be expressed by

$$\rho(X) := \inf \left\{ \sum_{j=1}^N m^j \mid m = [m^1, \dots, m^N] \in \mathbb{R}^N, U(X + m) \in \mathbb{A} \right\}. \quad (4)$$

Here, the amount m^j is added to the financial position X^j of institution $j \in \{1, \dots, N\}$ before the corresponding total loss $U(X+m)$ is computed. Such Systemic Risk Measures were introduced and analyzed by Biagini et al. (2019) [11]. Feinstein et al. (2017) [32] introduced a similar approach for set-valued Risk Measures. We refer to Armenti et al. (2018) [6] and Biagini et al. (2020) [12] for a detailed study of Shortfall Systemic Risk Measures - a relevant subclass of Risk Measures in the form (4) - and their dual representations. More recently, dual representations of Systemic Risk Measures based on acceptance sets have been studied in Arduca et al. (2019) [5] for the real-valued case, in Ararat and Rudloff (2020) [4] in the set-valued case.

Scenario dependent allocations. The “*first allocate and then aggregate*” approach was then extended in Biagini et al. (2019) [11] and (2020) [12] by adding to X not merely a vector $m = [m_1, \dots, m_N] \in \mathbb{R}^N$ of deterministic amounts but, more generally, a random vector $Y \in \mathcal{C}$, for some given class \mathcal{C} . In particular, one main example considered in [12] is given by the class \mathcal{C} such that

$$\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap L_{\mathcal{F}}, \text{ where } \mathcal{C}_{\mathbb{R}} := \left\{ Y \in (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N \mid \sum_{j=1}^N Y^j \in \mathbb{R} \right\}. \quad (5)$$

Here, the notation $\sum_{j=1}^N Y^j \in \mathbb{R}$ means that $\sum_{j=1}^N Y^j$ is equal to some deterministic constant in \mathbb{R} , even though each single Y^j , $j = 1, \dots, N$, is a random variable. It is possible to model additional constraints on the allocation $Y \in \mathcal{C}_{\mathbb{R}}$ by requiring $Y \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$. The set \mathcal{C} represents the class of feasible allocations and it is assumed that $\mathbb{R}^N \subseteq \mathcal{C}$.

Under these premises the Systemic Risk Measure considered in [12] takes the form

$$\rho(X) := \inf \left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{C}, U(X+Y) \in \mathbb{A} \right\} \quad (6)$$

and can still be interpreted, since $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$, as the minimal total cash amount $\sum_{j=1}^N Y^j \in \mathbb{R}$ needed today to secure the system by distributing the cash at the future time T among the components of the risk vector X . However, while the total capital requirement $\sum_{j=1}^N Y^j$ is determined today, contrary to (4) the individual allocation $Y^j(\omega)$ to institution j does not need to be decided today but in general depends on the scenario ω realized at time T . As explained in details in [12], this total cash amount $\rho(X)$ can be composed today through the formula

$$\sum_{j=1}^N a^j(X) = \rho(X), \quad (7)$$

where each cash amount $a^j(X) \in \mathbb{R}$ can be interpreted as a *risk allocation* of bank j . The exact formula for the risk allocation $a^j(X)$ will be introduced later in (10).

We remark that by selecting $\mathcal{C} = \mathbb{R}^N$ in (6), one recovers the deterministic case (4); while when $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ no further requirements are imposed on the set of feasible allocations.

Under minimal simple properties on the sets $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$, $\mathbb{A} \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ and on the aggregator $U : \mathbb{R}^N \rightarrow \mathbb{R}$ the Systemic Risk Measures in (6) satisfy the key properties of: (i) decreasing monotonicity, (ii) convexity, (iii) systemic cash additivity:

$$\rho(X+c) = \rho(X) - \sum_{j=1}^N c^j \quad \text{for all } c \in \mathbb{R}^N \text{ and } X \in L_{\mathcal{F}}.$$

Shortfall Systemic Risk Measures. A special, relevant case of Systemic Risk Measures of the form (6) “*first allocate and then aggregate, with scenario dependent allocation*” is given by the class of Shortfall Systemic Risk Measures, where the acceptance set has the form $\mathbb{A} = \{Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \mid \mathbb{E}_{\mathbb{P}}[Z] \geq B\}$ for a given constant $B \in \mathbb{R}$, namely:

$$\rho(X) := \inf \left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{C}, \mathbb{E}_{\mathbb{P}}[U(X + Y)] \geq B \right\}. \quad (8)$$

For the financial motivation behind these choices and for a detailed study of this class of measures, we refer to [11] and [12] when $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$, and to Armenti et al. (2018) [6] for the analysis of such Risk Measures in the special case $\mathcal{C} = \mathbb{R}^N$, i.e. when only deterministic allocations are allowed. The choice of the aggregation functions $U : \mathbb{R}^N \rightarrow \mathbb{R}$ is also a key ingredient in the construction of ρ and we refer to Acharia et al. (2017) [1], Adrian and Brunnermeier (2016) [2], Huang and Zhou (2009) [47], Lehar (2005) [50], Brunnermeier and Cheridito (2019) [15], Biagini et al. (2019) [11], and (2020) [12] for the many examples of aggregators adopted in literature. In order to obtain more specific and significant properties of ρ , [12] selected the aggregator

$$U(x) = \sum_{j=1}^N u_j(x^j), \quad x \in \mathbb{R}^N, \quad (9)$$

for strictly concave increasing utility functions $u_j : \mathbb{R} \rightarrow \mathbb{R}$, for each $j = 1, \dots, N$.

Systemic Risk Measures can be applied not only to determine the overall risk $\rho(X)$ of the system, but also to establish the riskiness of each individual financial institution. As explained in [12] it is possible to determine the risk allocations $a^j(X) \in \mathbb{R}$ of each bank j that satisfy (7) and additional meaningful properties. It was there shown that, with the choice (9), a *fair risk allocation* of bank j is given by:

$$a^j(X) := \mathbb{E}_{\mathbb{Q}^j(X)}[Y^j(X)], \quad j = 1, \dots, N, \quad (10)$$

where the vector $Y(X)$ is the optimizer in (8) and $\mathbb{Q}(X) = [\mathbb{Q}^1(X), \dots, \mathbb{Q}^N(X)]$ is the vector of probability measures that optimizes the dual problem associated to $\rho(X)$.

In this paper we will adopt the generalization of the aggregation function (9) defined by

$$U(x) = \sum_{j=1}^N u_j(x_j) + \Lambda(x), \quad x \in \mathbb{R}^N, \quad (11)$$

where the multivariate aggregator $\Lambda : \mathbb{R}^N \rightarrow \mathbb{R}$ is concave and increasing (not necessarily in a strict sense). Thus the selection $\Lambda = 0$ is possible and hence, in this case, (11) reduces to (9). The term Λ allows additionally for modeling interdependence among agents also from the point of view of the preferences.

Just to mention a few examples (see also the examples in [23]), any of the following multivariate utility functions satisfy our assumptions:

$$U(x) := \sum_{j=1}^N u_j(x^j) + u \left(\sum_{j=1}^N \beta_j x^j \right), \quad \text{with } \beta_j \geq 0, \text{ for all } j,$$

where $u : \mathbb{R} \rightarrow \mathbb{R}$, for some $p > 1$, is any one of the following functions:

$$u(x) := 1 - \exp(-px); \quad u(x) := \begin{cases} p \frac{x}{x+1} & x \geq 0 \\ 1 - |x-1|^p & x < 0 \end{cases}; \quad u(x) := \begin{cases} p \arctan(x) & x \geq 0 \\ 1 - |x-1|^p & x < 0 \end{cases}$$

and u_1, \dots, u_N are exponential utility functions ($u_j(x^j) = 1 - \exp(-\alpha_j x^j)$, $\alpha_j > 0$) for any choice of u as above. As shown in this paper, a fairness property for the risk allocation of each bank can be established also in a conditional setting and for the aggregator expressed by (11).

1.2 Conditional Systemic Risk Measures

The temporal setting in the approaches described above is static, meaning that the Risk Measures do not allow for possible dynamic elements, such as additional information, or the possibility of risk monitoring in continuous time, or the possibility of intermediate payoffs and valuation. In order to model the conditional setting we then assume that $\mathcal{G} \subseteq \mathcal{F}$ is a sub- σ -algebra of \mathcal{F} and we consider Risk Measures $\rho_{\mathcal{G}}$ with range in $L^0(\Omega, \mathcal{G}, \mathbb{P})$ and interpret $\rho_{\mathcal{G}}(X)$ as the risk of the whole system X given the information \mathcal{G} .

Conditional Risk Measures have mostly been studied in the framework of *univariate* dynamic Risk Measures, where one adjusts the risk measurement in response to the flow of information that is revealed when time elapses. The conditional coherent case was treated in Riedel (2004) [54]. Detlefsen and Scandolo (2006) [21] was one of the first contributions in the study of conditional convex Risk Measures and since then a vast literature appeared. Among the early works on the topic we refer to Barrieu and El Karoui (2005) [8], Tutsch (2008) [60], Weber (2006) [61]. Several results have been obtained for the case of quasi-convex conditional maps and Risk Measures, see Frittelli and Maggis (2011) [39], and Frittelli and Maggis (2014) [40], [41]. Conditional counterparts to classical static results (e.g. dual representation and separation properties) have been obtained exploiting the theory of L^0 -modules. Among the many contributions in this stream of research we mention Filipović et al. (2009) [33] and (2012) [34], Drapeau et al. (2016) [24], Drapeau et al. (2019) [25], Guo (2010) [44] and the references therein. Overall, the fact that the natural conditional counterparts hold for static results is not so surprising. The two are intrinsically related by a Boolean Logic principle. As seen in Carl and Jammeshan (2018) [17], traditional theorems carry over to the conditional setup assuming that suitable concatenation properties hold. Time consistency properties have been considered, in the univariate case, in Roorda and Schumacher (2007) [55], (2013) [56] and (2016) [57].

We refer the reader to [38] Chapter 11 for a good overview on univariate dynamic Risk Measures. We observe that such a conditional and dynamic framework generated a florilegium of interesting ramification in different fields, including the relationships with BSDEs (Barrieu and El Karoui (2005) [8], Rosazza Gianin (2006) [58], Bion-Nadal (2008) [14], Delbaen et al. (2011) [20]) and Non Linear Expectations (Peng (2004) [52]).

A conditional Systemic Risk Measure is a map $\rho_{\mathcal{G}} : L_{\mathcal{F}} \rightarrow L^0(\Omega, \mathcal{G}, \mathbb{P})$ that associates to a N -dimensional risk factor $X \in L_{\mathcal{F}} \subseteq (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$ a \mathcal{G} -measurable random variable. A conditional Systemic Risk Measure thus models the risk of a system as new information arises in the course of time. The study of conditional Systemic (multivariate) Risk Measures was initiated by Hoffmann et al. (2016) [45] and (2018) [46]. However, as pointed out in [46], in the context of multivariate Risk Measures, a second interesting and important dimension of conditioning arises, besides dynamic conditioning: a risk measurement, of the N -dimensional vector X , conditional on some specific substructure of the system, for example induced by mergers and acquisitions. In this paper we will

not elaborate on this topic and refer the reader to Follmer (2014) [35] or Follmer and Kluppelberg (2014) [36] for some details. However, in order to allow both interpretations, a general σ -algebra $\mathcal{G} \subseteq \mathcal{F}$ will be considered in the sequel.

The papers [45], [46], as well as Kromer et al. (2019) [49], consider only the conditional extension of (static) Systemic Risk Measures of the “first aggregate, then allocate” form expressed by (3) and study related consistency issues. Multivariate/Systemic and set-valued conditional Risk Measures, and related time consistency aspects, have also been analyzed in Feinstein and Rudloff (2013) [27], (2015) [29], (2017) [30] and (2021) [31], Tahar and Lepinette (2014) [59], Chen and Hu (2018) [19]. Although apparently similar, our approach in the present work is significantly different. Once we clarify our setup, we will elaborate more on this after Theorem 5.4 and in Remark 6.4.

Contribution and outline of the paper. Our aim in the present paper is the study of general conditional convex Systemic Risk Measures and the detailed analysis of Conditional Shortfall Systemic Risk Measures. Our findings show that most properties of Shortfall Systemic Risk Measures carry over to the conditional setting, even if the proofs become more technical, and that a new vector type consistency, with respect to sub σ -algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$, replaces the scalar recursiveness property of univariate Risk Measures.

More precisely, we define axiomatically a *Conditional Systemic Risk Measure* (CSRМ) on $L_{\mathcal{F}}$ as a map $\rho_{\mathcal{G}} : L_{\mathcal{F}} \rightarrow L^0(\Omega, \mathcal{G}, \mathbb{P})$ satisfying monotonicity, conditional convexity and the conditional monetary property (see Definition 3.5). Our first result (Theorem 3.9) shows, under fairly general assumptions, that: (i) $\rho_{\mathcal{G}}$ admits the conditional dual representation

$$\rho_{\mathcal{G}}(X) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \alpha(\mathbb{Q}) \right), \quad \mathbb{P}\text{-a.s.}, \text{ for } X \in L_{\mathcal{F}}, \quad (12)$$

where $\mathcal{Q}_{\mathcal{G}}$, defined in Equation (20), is a set of vectors of probability measures and the penalty $\alpha(\mathbb{Q}) \in L^0(\Omega, \mathcal{G}, \mathbb{P})$ is defined in Equation (21); (ii) the supremum in (12) is attained.

We then specialize our analysis by considering the *Conditional Shortfall Systemic Risk Measure, associated to multivariate utility functions* U of the form (11), defined by

$$\rho_{\mathcal{G}}(X) := \operatorname{ess\,inf} \left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}} [U(X + Y) | \mathcal{G}] \geq B \right\}, \quad (13)$$

where B is now a random variable in $L^{\infty}(\Omega, \mathcal{G}, \mathbb{P})$ and the set of \mathcal{G} -admissible allocations is

$$\mathcal{C}_{\mathcal{G}} \subseteq \left\{ Y \in (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N \text{ such that } \sum_{j=1}^N Y^j \in L^{\infty}(\Omega, \mathcal{G}, \mathbb{P}) \right\}.$$

Thus, with these definitions that mimic those in (5) and in (8), the same motivations, mutatis mutandis, explained in the unconditional setting remain true in the conditional one.

Observe that even for the trivial selection $\mathcal{G} = \{\emptyset, \Omega\}$, for which conditional Risk Measures reduce to static ones, this paper extends the results in [12] to the more general aggregator U in the form (11).

In Theorem 5.4 we prove the main properties of the Conditional Shortfall Systemic Risk Measure $\rho_{\mathcal{G}}$ and, in particular, we show that (i) $\rho_{\mathcal{G}}$ is continuous from above and from below; (ii) the essential infimum in (13) is attained by a vector $Y(\mathcal{G}, X) = [Y^1(\mathcal{G}, X), \dots, Y^N(\mathcal{G}, X)] \in \mathcal{C}_{\mathcal{G}}$; (iii) $\rho_{\mathcal{G}}$ admits the dual representation described in (33); (iv) the supremum in the dual formulation (33) of $\rho_{\mathcal{G}}$ is attained by a vector $\mathbb{Q}(\mathcal{G}, X) = [\mathbb{Q}^1(\mathcal{G}, X), \dots, \mathbb{Q}^N(\mathcal{G}, X)]$ of probability measures satisfying:

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j(\mathcal{G}, X)} [Y^j(\mathcal{G}, X) | \mathcal{G}] = \sum_{j=1}^N Y^j(\mathcal{G}, X) = \rho_{\mathcal{G}}(X) \quad \mathbb{P} - \text{a.s.}$$

In the same spirit of [12], we will then interpret the quantity

$$a^j(\mathcal{G}, X) := \mathbb{E}_{\mathbb{Q}^j(\mathcal{G}, X)} [Y^j(\mathcal{G}, X) | \mathcal{G}]$$

as a *fair risk allocation of institution j , given \mathcal{G}* .

Section 6 is then devoted to the particular case of exponential utility functions $u_j(x^j) := -e^{-\alpha_j x^j}$, $\alpha_j > 0$, $j = 1, \dots, N$, and with $\Lambda = 0$. As in the static case (see [12]), also in the conditional case it is possible to find the explicit formulas for: (i) the value of the Conditional Shortfall Systemic Risk Measure $\rho_{\mathcal{G}}(X)$; (ii) the optimizer $Y(\mathcal{G}, X)$ in (13) of $\rho_{\mathcal{G}}(X)$; (iii) the vector $\mathbb{Q}(\mathcal{G}, X)$ of probability measures that attains the supremum in the dual formulation. Such formulas provide a conditional counterpart to the results in [12].

Finally, for sub σ -algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ we prove a particular consistency property, which does not have a counterpart in the univariate case. Indeed, a recursive property of the type $\rho_{\mathcal{H}}(-\rho_{\mathcal{G}}(X)) = \rho_{\mathcal{H}}(X)$ is not even well defined in the systemic setting, as $\rho_{\mathcal{G}}(X)$ is a random variable but the argument of $\rho_{\mathcal{H}}$ is a vector of random variables. However, we explain that consistency properties may be well defined for: (i) the vector optimizers $Y(\mathcal{G}, X)$ of $\rho_{\mathcal{G}}(X)$ and $Y(\mathcal{H}, -Y(\mathcal{G}, X))$ of $\rho_{\mathcal{H}}(-Y(\mathcal{G}, X))$; (ii) the fair risk allocations vectors $[a(\mathcal{G}, X)]_j := [\mathbb{E}_{\mathbb{Q}^j(\mathcal{G}, X)} [Y^j(\mathcal{G}, X) | \mathcal{G}]]_j$ of $\rho_{\mathcal{G}}(X)$ and $a(\mathcal{H}, -a(\mathcal{G}, X))$ of $\rho_{\mathcal{H}}(-a(\mathcal{G}, X))$. The consistency properties are shown in (56) and (58) and proven in Theorem 6.3 for the entropic Conditional Systemic Risk Measure. In Remark 6.4 we compare this consistency properties with the ones for the set-valued case from [29] and [19].

In a final Section we elaborate on the concept of a Systemic Optimal Risk Transfer Equilibrium, a notion introduced in [10]. We defer the interested reader to [10] for the economic motivation and for the applications of this equilibrium. Based on the results on the Conditional Shortfall Systemic Risk Measure developed in Section 5, we are able to provide in Theorem 7.3 a direct extension of this equilibrium in the conditional setting. At the same time, we show that the optimal allocations for Shortfall Systemic Risk Measures, in both the static and dynamic cases, admit an interpretation in the sense of a suitably defined equilibrium. By the choice of the trivial $\mathcal{G} = \{\emptyset, \Omega\}$ our findings in this last section cover the static setup, and provide an explicit link between [10] and [12].

2 Static Systemic Risk Measures

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We take two vector subspaces $L_{\mathcal{F}}, L^*$ of $(L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$, for $N \geq 1$.

Definition 2.1. A functional $\rho_0 : L_{\mathcal{F}} \rightarrow \mathbb{R}$ will be called a **(Static) Convex Systemic Risk Measure** if it satisfies: **Monotonicity**, that is $X \leq Y$ componentwise $\Rightarrow \rho_0(X) \geq \rho_0(Y)$, **Convexity**, that is $0 \leq \lambda \leq 1 \Rightarrow \rho_0(\lambda X + (1-\lambda)Y) \leq \lambda \rho_0(X) + (1-\lambda)\rho_0(Y)$ and the **Monetary property** (or **Cash Additivity**), that is $X \in L_{\mathcal{F}}, c \in \mathbb{R}^N \Rightarrow \rho_0(X + c) = \rho_0(X) - \sum_{j=1}^N c^j$.

For $L_{\mathcal{F}} = (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N$, we also say that $\rho_0 : (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N \rightarrow \mathbb{R}$ is continuous from below (resp. from above) if for any sequence $(X_n)_n$ such that $X_n \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N$ and $X_n \uparrow_n X \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N$ (resp. $X_n \downarrow_n X \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N$) we have $\rho_0(X) = \lim_n \rho_0(X_n)$. If $\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N]$ is a vector of probability measures on (Ω, \mathcal{F}) , we write $\mathbb{Q} \ll \mathbb{P}$ for $\mathbb{Q}^j \ll \mathbb{P} \forall j = 1, \dots, N$ and use the notation $\frac{d\mathbb{Q}}{d\mathbb{P}} := \left[\frac{d\mathbb{Q}^1}{d\mathbb{P}}, \dots, \frac{d\mathbb{Q}^N}{d\mathbb{P}} \right]$. We set

$$\mathcal{Q} := \left\{ \mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N] \ll \mathbb{P} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^* \right\}.$$

Definition 2.2. We say that a (Static) Convex Systemic Risk Measure $\rho_0 : L_{\mathcal{F}} \rightarrow \mathbb{R}$ is **nicely representable** (with respect to the $\sigma(L_{\mathcal{F}}, L^*)$ topology) if

$$\rho_0(X) = \max_{\mathbb{Q} \in \mathcal{Q}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j] - \alpha_0(\mathbb{Q}) \right), \quad X \in L_{\mathcal{F}} \quad (14)$$

where

$$\alpha_0(\mathbb{Q}) := \rho_0^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right) = \sup_{X \in L_{\mathcal{F}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j] - \rho_0(X) \right), \quad \mathbb{Q} \in \mathcal{Q}. \quad (15)$$

and ρ_0^* is the convex conjugate of ρ_0 .

Remark 2.3. For univariate ($N = 1$) Convex Risk Measures, there are well known sufficient conditions for nice representability, which can be split in two categories: either continuity conditions (order upper semicontinuity or continuity from below); or structural properties of the vector spaces. In particular:

1. If $L_{\mathcal{F}} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $L^* = L^1(\Omega, \mathcal{F}, \mathbb{P})$ and if $\rho_0 : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ is continuous from below then it is nicely representable (see [13] Lemma 7 or [38] Corollary 4.35). This in turns implies $\sigma(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), L^1(\Omega, \mathcal{F}, \mathbb{P}))$ -lower semicontinuity and continuity from above.
2. If $L_{\mathcal{F}} = L^p(\Omega, \mathcal{F}, \mathbb{P})$, $L^* = L^q(\Omega, \mathcal{F}, \mathbb{P})$ with $p \in [1, +\infty)$ and q the conjugate exponent, or if $L_{\mathcal{F}} = M^\Phi(\Omega, \mathcal{F}, \mathbb{P}) \neq \emptyset$, $L^* = L^{\Phi^*}(\Omega, \mathcal{F}, \mathbb{P})$ (see Section 5.2.2 and Equation (45) for the definitions), then any univariate Convex Systemic Risk Measure $\rho_0 : L_{\mathcal{F}} \rightarrow \mathbb{R}$ is nicely representable, due to the Extended Namioka-Klee Theorem in [13].

We will now extend one dimensional classical results to our systemic setup. Only slight modifications are needed in the proofs, but we add them in the Appendix, Section A.2 for the sake of completeness.

Theorem 2.4. *i) If $\rho_0 : (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N \rightarrow \mathbb{R}$ is a (Static) Convex Systemic Risk Measure continuous from below then it is nicely representable with $L^* = (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$ and therefore it is $\sigma((L^\infty(\mathcal{F}))^N, (L^1(\mathcal{F}))^N)$ lower semicontinuous and continuous from above. ii) If $L_{\mathcal{F}} \subseteq (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$ is a Banach lattice with order continuous topology and if $L^* \subseteq (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$*

is its topological dual space then any (Static) Convex Systemic Risk Measure $\rho_0 : L_{\mathcal{F}} \rightarrow \mathbb{R}$ is nicely representable.

Obviously item ii) in the theorem covers the multidimensional versions of the cases described in item 2 in Remark 2.3.

3 Conditional Systemic Risk Measures

We now present the conditional framework which acts as a counterpart to the static one presented before. We introduce the conditional versions of usual properties of Systemic Risk Measures (convexity, additivity), provide the general definition of Conditional Systemic Risk Measure and related continuity concepts we will use in the following. We also present a general duality result in Section 3.2.

3.1 Setup and notation

We let $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ -algebra and recall that $L_{\mathcal{F}} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$. Throughout all the paper we will often need to change underlying σ -algebras. In order to avoid unnecessarily heavy notation, we will explicitly specify the one or the other only when some confusion might arise. For example, $L^\infty(\mathcal{F})$, $L^\infty(\mathcal{G})$ stand for $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ respectively. Unless differently stated, all inequalities between random variables hold \mathbb{P} -a.s.. If $A \in \mathcal{F}$ we write A^c for its complement.

Remark 3.1. In the following (MON) and (DOM) are references to Monotone and Dominated convergence Theorem respectively. (cMON) and (cDOM) refer to their conditional counterparts. We will use without explicit mention the properties of essential suprema (and essential infima) collected in Proposition A.5.

Definition 3.2. $L_{\mathcal{F}}$ is \mathcal{G} -decomposable if $(L^\infty(\mathcal{F}))^N \subseteq L_{\mathcal{F}}$ and if for any $Y \in (L^\infty(\mathcal{G}))^N$ and $X \in L_{\mathcal{F}}$ the random vector Z defined as $Z^j = X^j Y^j$, $j = 1, \dots, N$, belongs to $L_{\mathcal{F}}$.

Remark 3.3. Observe that by decomposability whenever $A \in \mathcal{G}$ and $X, Y \in L_{\mathcal{F}}$ we also have $X1_A + Y1_{A^c} \in L_{\mathcal{F}}$. We stress the fact that \mathcal{G} -decomposability is a very mild requirement, which is clearly satisfied for example if $L_{\mathcal{F}} = L^p$ for some $p \in [1, +\infty]$ or $L_{\mathcal{F}}$ is an Orlicz Space (see Section 5.2.2 and [23] Section 2.1).

Definition 3.4. A subset $\mathcal{C} \subseteq L_{\mathcal{F}}$ is:

- \mathcal{G} -conditionally convex if $\forall \lambda \in L^0(\mathcal{G})$, $0 \leq \lambda \leq 1$ and $\forall X, Y \in \mathcal{C}$, $\lambda X + (1 - \lambda)Y \in \mathcal{C}$.
- a \mathcal{G} -conditional cone if $\forall \lambda \in L^\infty(\mathcal{G})$ and $\forall X \in \mathcal{C}$, $\lambda X \in \mathcal{C}$.
- closed under \mathcal{G} -truncation if $\forall Y \in \mathcal{C}$ there exists $k_Y \in \mathbb{N}$ and a $Z_Y \in (L^\infty(\mathcal{F}))^N$ such that $\sum_{j=1}^N Z_Y^j = \sum_{j=1}^N Y^j$ and $\forall k \geq k_Y$, $k \in \mathbb{N}$

$$Y_{(k)} := Y1_{\bigcap_j \{|Y^j| < k\}} + Z_Y 1_{\bigcup_j \{|Y^j| \geq k\}} \in \mathcal{C}. \quad (16)$$

We will explicitly specify the σ -algebra (\mathcal{G} in the notation above) with respect to which the properties are required to hold only when some confusion might arise.

Definition 3.5. A map $\rho_{\mathcal{G}} : L_{\mathcal{F}} \rightarrow L^0(\mathcal{G})$ is a **Conditional Systemic Risk Measure (CSR)** if it satisfies for all $X, Y \in L_{\mathcal{F}}$

1. **Monotonicity**, that is

$$X \leq Y \text{ componentwise} \Rightarrow \rho_{\mathcal{G}}(X) \geq \rho_{\mathcal{G}}(Y) \quad (17)$$

2. **Conditional Convexity**, that is

$$\rho_{\mathcal{G}}(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_{\mathcal{G}}(X) + (1 - \lambda) \rho_{\mathcal{G}}(Y) \quad \text{for all } 0 \leq \lambda \leq 1, \lambda \in L^{\infty}(\mathcal{G}) \quad (18)$$

3. **Conditional \mathcal{G} -Additivity** (or the conditional monetary property), that is

$$\rho_{\mathcal{G}}(X + Y) = \rho_{\mathcal{G}}(X) - \sum_{j=1}^N Y^j \quad \text{if } Y \in (L^{\infty}(\mathcal{G}))^N \cap L_{\mathcal{F}} \quad (19)$$

One may easily show, as in the one dimensional case, that a map $\rho_{\mathcal{G}} : (L^{\infty}(\mathcal{F}))^N \rightarrow L^0(\mathcal{G})$ satisfying $\rho_{\mathcal{G}}(0) \in L^{\infty}(\mathcal{G})$, monotonicity and the conditional monetary property has range in $L^{\infty}(\mathcal{G})$ and $|\rho_{\mathcal{G}}(X) - \rho_{\mathcal{G}}(0)| \leq \sum_{j=1}^N \|X^j\|_{\infty}$ \mathbb{P} -a.s.. For the Conditional Shortfall Systemic Risk Measure in Section 5 we prove first that $\rho_{\mathcal{G}}(X) \in L^{\infty}(\mathcal{G})$ for all $X \in (L^{\infty}(\mathcal{F}))^N$ and then show all the properties of the Risk Measure. When $L_{\mathcal{F}} \neq (L^{\infty}(\mathcal{F}))^N$, in order to apply the scalarization procedure $\rho_0(\cdot) = \mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(\cdot)]$ we will need, in Theorem 3.9, the assumption that the range of $\rho_{\mathcal{G}}$ is contained in $L^1(\mathcal{G})$, and for this we will require that $\rho_{\mathcal{G}} : L_{\mathcal{F}} \rightarrow L^0(\mathcal{G}) \cap L_{\mathcal{F}}$.

Definition 3.6. For the particular choice $L_{\mathcal{F}} = (L^{\infty}(\mathcal{F}))^N$ we say that a CSR $\rho_{\mathcal{G}} : (L^{\infty}(\mathcal{F}))^N \rightarrow L^0(\mathcal{G})$ is

- **continuous from above** if for any sequence $(X_n)_n \subseteq (L^{\infty}(\mathcal{F}))^N$ and $X \in (L^{\infty}(\mathcal{F}))^N$ such that for each $j = 1, \dots, N$ $X_n^j \downarrow_n X^j$ we have $\rho_{\mathcal{G}}(X_n) \uparrow_n \rho_{\mathcal{G}}(X)$ \mathbb{P} -a.s.
- **continuous from below** if for any sequence $(X_n)_n \subseteq (L^{\infty}(\mathcal{F}))^N$ and $X \in (L^{\infty}(\mathcal{F}))^N$ such that for each $j = 1, \dots, N$ $X_n^j \uparrow_n X^j$ we have $\rho_{\mathcal{G}}(X_n) \downarrow_n \rho_{\mathcal{G}}(X)$ \mathbb{P} -a.s.
- **Lebesgue continuous** (or that $\rho_{\mathcal{G}}(\cdot)$ has the Lebesgue property) if for any sequence $(X_n)_n \subseteq (L^{\infty}(\mathcal{F}))^N$ and $X \in (L^{\infty}(\mathcal{F}))^N$ such that for each $j = 1, \dots, N$ $\sup_n \|X_n^j\|_{\infty} < +\infty$ and $X_n^j \rightarrow_n X^j$ \mathbb{P} -a.s. we have $\rho_{\mathcal{G}}(X_n) \rightarrow_n \rho_{\mathcal{G}}(X)$ \mathbb{P} -a.s.

Remark 3.7. Observe that continuity from above and continuity from below of a CSR $\rho_{\mathcal{G}}$ yield the Lebesgue property, by simple computations similar to the univariate case.

3.2 Dual representation of Conditional Systemic Risk Measures

This section follows the lines of the scalarization procedure in [21] and [51] and we defer the proofs to the Appendix, Section A.3.

Define the following subset of \mathcal{Q} :

$$\mathcal{Q}_{\mathcal{G}} := \left\{ \mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N] \ll \mathbb{P} \mid \frac{d\mathbb{Q}^j}{d\mathbb{P}} \in L^*, \mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}^j}{d\mathbb{P}} \mid \mathcal{G} \right] = 1 \quad \forall j = 1, \dots, N \right\}, \quad (20)$$

and set

$$\alpha(\mathbb{Q}) := \operatorname{ess\,sup}_{X \in L_{\mathcal{F}}, \rho_{\mathcal{G}}(X) \leq 0} \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}], \quad \mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}. \quad (21)$$

Remark 3.8. Observe that each component \mathbb{Q}^j of elements in $\mathcal{Q}_{\mathcal{G}}$ satisfies $\mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}^j}{d\mathbb{P}} \Big| \mathcal{G} \right] = 1$. In this case, $\mathbb{Q}^j = \mathbb{P}$ on \mathcal{G} and $\mathbb{E}_{\mathbb{Q}^j} [X^j | \mathcal{G}] = \mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}^j}{d\mathbb{P}} X^j \Big| \mathcal{G} \right]$ is defined not only \mathbb{Q}^j -a.s. but also \mathbb{P} -a.s.. Hence (21) and (22) are well defined \mathbb{P} -a.s..

Theorem 3.9. *Suppose that $L_{\mathcal{F}}$ is \mathcal{G} -decomposable and that for any $X \in L_{\mathcal{F}}, Z \in L^*$ we have $\sum_{j=1}^N X^j Z^j \in L^1(\mathcal{F})$. Let $\rho_{\mathcal{G}} : L_{\mathcal{F}} \rightarrow L^0(\mathcal{G}) \cap L_{\mathcal{F}}$ satisfy monotonicity, conditional convexity and conditional additivity (that is, let $\rho_{\mathcal{G}}$ be a CSRМ) and let $\rho_0(\cdot) := \mathbb{E}_{\mathbb{P}} [\rho_{\mathcal{G}}(\cdot)] : L_{\mathcal{F}} \rightarrow \mathbb{R}$ be nicely representable (with respect to the $\sigma(L_{\mathcal{F}}, L^*)$ topology). Then $\rho_{\mathcal{G}}$ admits the following dual representation:*

$$\rho_{\mathcal{G}}(X) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \alpha(\mathbb{Q}) \right), \quad X \in L_{\mathcal{F}}. \quad (22)$$

Furthermore, there exists $\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}$ such that $\rho_{\mathcal{G}}(X) = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [-X^j | \mathcal{G}] - \alpha(\widehat{\mathbb{Q}})$ and $\rho_{\mathcal{G}}$ is continuous from above.

4 Multivariate utility functions

We will now turn our attention to Conditional Systemic Risk Measures of shortfall type, which consider as eligible for securing the system those terminal time allocations which produce a utility (for the system) above a given threshold. Before formulating the precise definition of such a risk measurement regime, we need to specify a model for preferences of the agents in the system. To this end, we exploit multivariate utility functions. This allows for modeling the fact that a single agent's preferences might depend on the actions of the others.

Definition 4.1. *We say that $U : \mathbb{R}^N \rightarrow \mathbb{R}$ is a **multivariate utility function** if it is strictly concave and increasing with respect to the partial componentwise order. When $N = 1$ we will use the term univariate utility function instead.*

The following assumption, as well as Standing Assumption II below, holds true throughout the paper without further mention.

Standing Assumption I. *We will consider multivariate utility functions in the form*

$$U(x) := \sum_{j=1}^N u_j(x^j) + \Lambda(x) \quad (23)$$

where $u_1, \dots, u_j : \mathbb{R} \rightarrow \mathbb{R}$ are univariate utility functions and $\Lambda : \mathbb{R}^N \rightarrow \mathbb{R}$ is concave, increasing with respect to the partial componentwise order and bounded from above. Inspired by Asymptotic Satiability as defined in Definition 2.13 [16] we will furthermore assume that for every $\varepsilon > 0$ there exist a point $z_{\varepsilon} \in \mathbb{R}^N$ and a selection $\nu_{\varepsilon} \in \partial \Lambda(z_{\varepsilon})$, such that $\sum_{j=1}^N |\nu_{\varepsilon}^j| < \varepsilon$.

For each $j = 1, \dots, N$, we also assume the Inada conditions

$$\lim_{x \rightarrow +\infty} \frac{u_j(x)}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{u_j(x)}{x} = +\infty$$

and that, without loss of generality, $u_j(0) = 0$.

Observe that such a multivariate utility function is split in two components: the sum of single agent utility functions and a universal part Λ that could be either selected upon agreement by all the agents or could be imposed by a regulatory institution. As Λ is not necessarily strictly convex nor strictly increasing, we may choose $\Lambda = 0$, which corresponds to the case analyzed in [12] for the non conditional case.

Remark 4.2. U defined in (23) is a multivariate utility function since it inherits strict concavity and strict monotonicity from u_1, \dots, u_N . We may assume without loss of generality that $u_j(0) = 0 \quad \forall j = 1, \dots, N$, since we can always write

$$U(x) = \sum_{j=1}^N (u_j(x^j) - u_j(0)) + \left(\Lambda(x) + \sum_{j=1}^N u_j(0) \right)$$

Thus, we can always redefine the univariate utilities and the multivariate one, without affecting other assumptions, in such a way that univariate utilities are null in 0.

We will make use of the following properties, without explicit mention: for every $f : \mathbb{R} \rightarrow \mathbb{R}$ nondecreasing and such that $f(0) = 0$ it holds that

$$f(x) = f(x^+) + f(-x^-), \quad (f(x))^+ = f(x^+).$$

Moreover, if u_1, \dots, u_N are all null in 0 (w.l.o.g. by the argument above), for any $x^1, \dots, x^N \geq 0$

$$\sum_{j=1}^N u_j(x^j) \leq \max_{j=1, \dots, N} \left(\frac{du_j}{dx^j}(0) \right) \sum_{j=1}^N x^j \quad (24)$$

where $\frac{du_j}{dx^j}(0)$, by an abuse of notation, stands for any choice in $\partial u_j(0)$, i.e. in the subdifferential of u_j at the point 0, for each $j = 1, \dots, N$. Inequality (24) can be showed observing that $\partial u_j(0) \neq \emptyset$ by concavity and that $\partial u_j(0) \subseteq [0, +\infty)$ since u_j is nondecreasing.

5 Conditional Shortfall Systemic Risk Measures on $(L^\infty(\mathcal{F}))^N$

Once we fixed our model for the preferences in the system, we discuss the set of allocations we admit for the terminal time exchanges, of scenario dependent nature. This will allow us to formalize the problem which will be the protagonist of our analysis in the subsequent parts of the paper, and to state some of its main features in Theorem 5.4.

Given a sub σ -algebra $\mathcal{G} \subseteq \mathcal{F}$ we introduce the set

$$\mathcal{D}_{\mathcal{G}} := \left\{ Y \in (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N \mid \sum_{j=1}^N Y^j \in L^0(\Omega, \mathcal{G}, \mathbb{P}) \right\}. \quad (25)$$

We would like to consider as the set of admissible allocations a subset

$$\mathcal{B}_{\mathcal{G}} \subseteq \mathcal{D}_{\mathcal{G}}$$

satisfying appropriate conditions (see the Standing Assumption II).

At the same time, we observe that the constraints in (25) can be interpreted saying that the risk can be shared by all the agents in the single group $\mathbf{I} := \{1, \dots, N\}$. This can be generalized by introducing the set of constraints corresponding to a cluster of agents conditional on the information in \mathcal{G} , inspired by an example in [12] for the static case.

Definition 5.1. For $h \in \{1, \dots, N\}$, let $\mathbf{I} := (I_m)_{m=1, \dots, h}$ be some partition of $\{1, \dots, N\}$. Then we set

$$\mathcal{B}_{\mathcal{G}}^{(\mathbf{I})} := \left\{ \mathbf{Y} \in (L^0(\mathcal{F}))^N \mid \exists d = [d_1, \dots, d_h] \in (L^0(\mathcal{G}))^h \mid \sum_{i \in I_m} Y^i = d_m \text{ for } m = 1, \dots, h \right\}, \quad (26)$$

$$\mathcal{B}_{\mathcal{G}}^{(\mathbf{I}, \infty)} := \left\{ \mathbf{Y} \in (L^0(\mathcal{F}))^N \mid \exists d = [d_1, \dots, d_h] \in (L^\infty(\mathcal{G}))^h \mid \sum_{i \in I_m} Y^i = d_m \text{ for } m = 1, \dots, h \right\}. \quad (27)$$

We stress that the family $\mathcal{B}_{\mathcal{G}}^{(\mathbf{I})}$ admits two extreme cases:

- (i) when we have only one group $h = 1$ then $\mathcal{B}_{\mathcal{G}}^{(\mathbf{I})} = \mathcal{D}_{\mathcal{G}}$ is the largest possible class, corresponding to risk sharing among all agents in the system;
- (ii) on the opposite side, the strongest restriction occurs when $h = N$, i.e., we consider exactly N groups, and in this case $\mathcal{B}_{\mathcal{G}}^{(\mathbf{I})} = (L^0(\mathcal{G}))^N$ corresponds to no risk sharing. This generalizes to a dynamic setting the case of deterministic allocations, when no further information is available (i.e. \mathcal{G} is trivial). This case has been treated in the literature, especially in the set-valued approach we mentioned in the Introduction. See also the comments after Theorem 5.4 below for further details.

Suppose now a partition \mathbf{I} has been fixed. We will consider a subset

$$\mathcal{B}_{\mathcal{G}} \subseteq \mathcal{B}_{\mathcal{G}}^{(\mathbf{I})}$$

and note that each component of $Y \in \mathcal{B}_{\mathcal{G}}$ is required to be \mathcal{F} -measurable, while the sums $\sum_{i \in I_m} Y^i$ are \mathcal{G} -measurable, and so is consequently $\sum_{j=1}^N Y^j$. Thus $\mathcal{B}_{\mathcal{G}} \subseteq \mathcal{B}_{\mathcal{G}}^{(\mathbf{I})} \subseteq \mathcal{D}_{\mathcal{G}}$.

We define

$$\mathcal{C}_{\mathcal{G}} := \mathcal{B}_{\mathcal{G}} \cap \mathcal{B}_{\mathcal{G}}^{(\mathbf{I}, \infty)} \cap (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N. \quad (28)$$

Standing Assumption II. $\mathcal{B}_{\mathcal{G}}$ is closed in probability, it is conditionally convex and it is a conditional cone. Moreover $\mathcal{B}_{\mathcal{G}} + (L^0(\mathcal{G}))^N = \mathcal{B}_{\mathcal{G}}$ and the set $\mathcal{C}_{\mathcal{G}}$ is closed under \mathcal{G} -truncation. We finally consider a $B \in L^\infty(\mathcal{G})$ with $\text{ess sup}(B) < \sup_{z \in \mathbb{R}^N} U(z) \leq +\infty$.

Example 5.2. It is easily seen that taking $\mathcal{B}_{\mathcal{G}} = \mathcal{B}_{\mathcal{G}}^{(\mathbf{I})}$ and consequently $\mathcal{C}_{\mathcal{G}} = \mathcal{B}_{\mathcal{G}}^{(\mathbf{I}, \infty)} \cap (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$ Standing Assumption II is satisfied. Closedness under truncation in particular is verified as follows: for $Y \in \mathcal{C}_{\mathcal{G}}$, for $j \in I_m$ we can take $Z_Y^j = \frac{1}{|I_m|} \sum_{i \in I_m} Y^i$ where $|I_m|$ is the cardinality of I_m . Then it is easily verified that $Y_{(k)}$ defined as in (16) satisfies for every $m = 1, \dots, h$

$$\sum_{i \in I_m} Y_{(k)}^i = \left(\sum_{i \in I_m} Y^i \right) 1_{\cap_j \{|Y^j| < k\}} + \left(\sum_{i \in I_m} \left(\frac{1}{|I_m|} \sum_{i \in I_m} Y^i \right) \right) 1_{\cup_j \{|Y^j| \geq k\}} = \sum_{i \in I_m} Y^i \in L_{\mathcal{G}}^\infty$$

which proves that $Y_{(k)} \in \mathcal{B}_{\mathcal{G}}^{(\mathbf{I}, \infty)} \cap (L^\infty(\mathcal{F}))^N \subseteq \mathcal{C}_{\mathcal{G}}$ and that also

$$\sum_{j=1}^N Y_{(k)}^j = \sum_{m=1}^h \sum_{i \in I_m} Y_{(k)}^i = \sum_{m=1}^h \sum_{i \in I_m} Y^i = \sum_{j=1}^N Y^j.$$

Finally, we point out that we can cover the setup of [12] in our framework (clearly, here we work with bounded positions and not in an Orlicz setup). Indeed, we may take the trivial partition $\mathbf{I} = \{\{1, \dots, N\}\}$ and, to cover the static case, we may choose $\mathcal{G} = \{\emptyset, \Omega\}$. Then we select the set $\mathcal{B}_{\mathcal{G}}$ equal to the set \mathcal{C}_0 , defined in [12], which is assumed to be closed under truncation in the sense of [12] Definition 4.18. Then our assumptions here are satisfied as well.

Definition 5.3. For each $X \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N$ we set

$$\rho_{\mathcal{G}}^\infty(X) := \text{ess inf} \left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N, \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B \right\}, \quad (29)$$

$$\rho_{\mathcal{G}}(X) := \text{ess inf} \left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B \right\}. \quad (30)$$

and we call $\rho_{\mathcal{G}}(X)$ the **Conditional Shortfall Systemic Risk Measure** associated to the multivariate utility function U and the set of allocations $\mathcal{C}_{\mathcal{G}}$.

The difference between the two definitions only resides on the additional constraint $Y \in (L^\infty(\mathcal{F}))^N$ appearing in $\rho_{\mathcal{G}}^\infty(X)$. As stated in our next main result, the two Risk Measures coincide under our Standing Assumptions I and II. The proof, which is quite lengthy, is split in separate results in the following Section 5.1.

For every $\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N] \in \mathcal{Q}_{\mathcal{G}}$ defined in (20), we set

$$\alpha^1(\mathbb{Q}) := \text{ess sup} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[-Z^j|\mathcal{G}] \mid Z \in (L^\infty(\mathcal{F}))^N \text{ and } \mathbb{E}_{\mathbb{P}}[U(Z)|\mathcal{G}] \geq B \right\} \quad (31)$$

and we introduce the set

$$\mathcal{Q}_{\mathcal{G}}^1 := \left\{ \mathbb{Q} \in \mathcal{Q}_{\mathcal{G}} \mid \begin{array}{l} \alpha^1(\mathbb{Q}) \in L^1(\mathcal{G}) \quad \text{and} \\ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j|\mathcal{G}] \leq \sum_{j=1}^N Y^j, \forall Y \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N \end{array} \right\}. \quad (32)$$

As the set $\mathcal{Q}_{\mathcal{G}}^1$ is included in $\mathcal{Q}_{\mathcal{G}}$, the observation made in Remark 3.8 on the conditional expectation applies also here.

Theorem 5.4. Consider the maps $\rho_{\mathcal{G}}^\infty$ and $\rho_{\mathcal{G}}$ defined in (29) and (30).

1. $\rho_{\mathcal{G}}^\infty(X) \in L^\infty(\mathcal{G})$ for all $X \in (L^\infty(\mathcal{F}))^N$ and $\rho_{\mathcal{G}}^\infty$ is a Conditional Systemic Risk Measure as $\rho_{\mathcal{G}}^\infty$ is monotone (17), conditionally convex (18) and conditionally monetary (19). It is also continuous from above and from below in the sense of Definition 3.6.

2. For every $X \in (L^\infty(\mathcal{F}))^N$

$$\rho_{\mathcal{G}}^\infty(X) = \rho_{\mathcal{G}}(X)$$

and the essential infimum in (30) is attained.

3. The CSRM $\rho_{\mathcal{G}}^\infty$ admits the following dual representation:

$$\rho_{\mathcal{G}}^\infty(X) = \text{ess sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[-X^j|\mathcal{G}] - \alpha^1(\mathbb{Q}) \right), \quad \forall X \in (L^\infty(\mathcal{F}))^N. \quad (33)$$

Furthermore, for every $X \in (L^\infty(\mathcal{F}))^N$ there exists $\widehat{\mathbb{Q}} \in \mathcal{Q}_G^1$ such that

$$\rho_G^\infty(X) = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [-X^j | \mathcal{G}] - \alpha^1(\widehat{\mathbb{Q}}).$$

As anticipated in the Introduction, several works have focused on the set-valued theory for Systemic Risk Measures, in both the static and dynamic case. A key difference with our approach is marked by our use of *random allocations*. In [19], [27], [28], [29], [30], [31], [59] one associates, to each risky position X , “the set $R_t(X)$ of eligible portfolios at time t that cover the risk of the portfolio X ”, quoting from [29]. The risk of X is quantified in these works using a set of vectors which are measurable with respect \mathcal{F}_t (the information known at time t). Here, instead, we are primarily interested in random allocations which happen at terminal time. Taking $\mathcal{G} = \mathcal{F}_t$ in Definition 5.3 and Theorem 5.4 to uniform notation, we stress once again that the amount $\rho_{\mathcal{F}_t}(X)$ is known once the information of \mathcal{F}_t is known, but this is not the case for $Y \in \mathcal{C}_{\mathcal{F}_t}$ since the latter vectors are \mathcal{F} -measurable, hence known only at terminal time. The label \mathcal{F}_t in $\mathcal{C}_{\mathcal{F}_t}$ only points out that $\sum_{j=1}^N Y^j$ is \mathcal{F}_t -measurable. An evident consequence of this can be found in the dual representation result (33): the dual variables are taken in the set \mathcal{Q}_G^1 and satisfy the fairness condition $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y^j | \mathcal{F}_t] \leq \sum_{j=1}^N Y^j, \forall Y \in \mathcal{C}_{\mathcal{F}_t} \cap (L^\infty(\mathcal{F}))^N$. This would be a triviality taking vectors Y which are componentwise \mathcal{F}_t -measurable, but becomes an additional characteristic feature in our setup. Additionally, recall the scalarization procedure in [31] for weights $[w^1, \dots, w^N]$ which are \mathcal{F}_t -measurable, namely

$$\rho_{\mathcal{F}_t}^w(X) := \text{ess inf} \left\{ \sum_{j=1}^N w^j Y^j \mid Y \in R_t(X) \right\}.$$

This is meaningful whenever the eligible allocations $Y \in R_t(X)$ are \mathcal{F}_t -measurable. We point out that our eligible assets for $\rho_{\mathcal{F}_t}$ satisfy $Y \in \mathcal{C}_{\mathcal{F}_t}$, a condition purposely designed for the valuation $Y \rightarrow \sum_{j=1}^N Y^j$. Using any other type of weights $w \in L^\infty(\mathcal{F}_t)^N$ would produce an amount $\sum_{j=1}^N w^j Y^j$ which would be in general only \mathcal{F} -measurable. This would violate the basic idea that the (scalar) measurement of risk, given the information in \mathcal{F}_t , should only depend on the information in \mathcal{F}_t .

5.1 Proof of Theorem 5.4

In the notation (30), the expression $\mathbb{E}_{\mathbb{P}} [U(X + Y) | \mathcal{G}] \geq B$ stands for a shortened version of the following set of conditions: $U(X + Y) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}_{\mathbb{P}} [U(X + Y) | \mathcal{G}]$, which is well defined, is not \mathbb{P} -a.s smaller than the random variable B . Recall also that for any random variable W taking values in $[0, +\infty]$ $\mathbb{E}_{\mathbb{P}} [W | \mathcal{G}]$ is always well defined via the Radon-Nikodym Theorem (see [9], Theorems 17.10-11), and in this case the notation $\mathbb{E}_{\mathbb{P}} [W | \mathcal{G}]$ will be used with this meaning.

For technical reasons we first study the functional ρ_G defined in (30). We will first prove in Claim 5.5 that the range of ρ_G is $L^\infty(\mathcal{G})$ and then we will show all the properties in Theorem 5.4 Item 1, made exception for continuity from above and below, and existence of an allocation for ρ_G . We will then prove that $\rho_G \equiv \rho_G^\infty$ on $(L^\infty(\mathcal{F}))^N$ (Claim 5.6), which yields Theorem 5.4 Item 2, and move on proving continuity from below and from above (Claim 5.7). Finally, in Claim 5.8 we prove Theorem 5.4 Item 3.

Claim 5.5. *The functional $\rho_{\mathcal{G}}$ on $(L^\infty(\mathcal{F}))^N$ takes values in $L^\infty(\mathcal{G})$, the infimum is attained by a $\hat{Y} \in \mathcal{C}_{\mathcal{G}}$, $\rho_{\mathcal{G}}$ is monotone (17) conditionally convex (18) and conditionally monetary (19).*

Proof.

STEP 1: $\rho_{\mathcal{G}}$ takes values in $L^\infty(\mathcal{G})$.

First we see that the set over which we take the essential infimum defining $\rho_{\mathcal{G}}$ is nonempty. We have by monotonicity (for m an N -dimensional deterministic vector) $\mathbb{E}_{\mathbb{P}}[U(X+m)|\mathcal{G}] \geq U(-\|X\|_\infty + m)$ where $\|X\|_\infty$ stands for the vector $[\|X^1\|_\infty, \dots, \|X^N\|_\infty] \in \mathbb{R}^N$. Since by assumption

$$\sup_{m \in \mathbb{R}^N} U(-\|X\|_\infty + m) = \sup_{z \in \mathbb{R}^N} U(z) > \text{ess sup}(B),$$

we have consequently $\mathbb{E}_{\mathbb{P}}[U(X+m)|\mathcal{G}] \geq B$, for some $m \in \mathbb{R}^N$.

We claim that the set over which we take the essential infimum is downward directed. To show this, suppose that $Z, Y \in (L^1(\mathcal{F}))^N$ are such that $\sum_{j=1}^N Y^j, \sum_{j=1}^N Z^j \in L^\infty(\mathcal{G})$ and

$$\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B, \quad \mathbb{E}_{\mathbb{P}}[U(X+Z)|\mathcal{G}] \geq B$$

Define the set $A := \{\sum_{j=1}^N Y^j \leq \sum_{j=1}^N Z^j\} \in \mathcal{G}$ and the random variable $W := 1_A Y + 1_{A^c} Z \in (L^1(\mathcal{F}))^N \cap \mathcal{B}_{\mathcal{G}}$ (observe that it belongs to $\mathcal{B}_{\mathcal{G}}$ since $\mathcal{B}_{\mathcal{G}}$ is conditionally convex). It is easy to see that $\sum_{j=1}^N W^j = 1_A \sum_{j=1}^N Y^j + 1_{A^c} \sum_{j=1}^N Z^j = \min\left(\sum_{j=1}^N Y^j, \sum_{j=1}^N Z^j\right) \in L^\infty(\mathcal{G})$, so that the set is downward directed. Furthermore

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[U(X+W)|\mathcal{G}] &= \mathbb{E}_{\mathbb{P}}[U(X+W)|\mathcal{G}] 1_A + \mathbb{E}_{\mathbb{P}}[U(X+W)|\mathcal{G}] 1_{A^c} = \\ &= \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] 1_A + \mathbb{E}_{\mathbb{P}}[U(X+Z)|\mathcal{G}] 1_{A^c} \geq B 1_A + B 1_{A^c} = B \end{aligned}$$

which concludes the proof of our claim.

Since the set is downward directed, there exists a minimizing sequence $(Y_n)_n \subseteq \mathcal{C}_{\mathcal{G}}$ such that $\sum_{j=1}^N Y_n^j \downarrow \rho_{\mathcal{G}}(X)$ and, having $\rho_{\mathcal{G}}(X) \leq \sum_{j=1}^N Y_1^j \in L^\infty$, we conclude that $\|(\rho_{\mathcal{G}}(X))^+\|_\infty < +\infty$. Suppose now by contradiction that for a sequence $k_n \uparrow +\infty$ we had $\mathbb{P}(A_n) > 0$ for all n , where $A_n := \{\rho_{\mathcal{G}}(X) \leq -k_n\} \in \mathcal{G}$. Since for all $M \in \mathbb{N}$ we have $-\|B\|_\infty \leq B \leq \mathbb{E}_{\mathbb{P}}[U(X+Y_M)|\mathcal{G}]$ we deduce:

$$\begin{aligned} -\|B\|_\infty \mathbb{P}(A_n) &\leq \mathbb{E}_{\mathbb{P}}[B 1_{A_n}] \leq \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[U(X+Y_M)|\mathcal{G}] 1_{A_n}] = \mathbb{E}_{\mathbb{P}}[U(X+Y_M) 1_{A_n}] \\ &\stackrel{\text{Lemma A.6.(ii)}}{\leq} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left(a(X^j + Y_M^j) + b\right) 1_{A_n}\right] \\ &\leq \left(a \sum_{j=1}^N \|X^j\|_\infty + b\right) \mathbb{P}(A_n) + a \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y_M^j 1_{A_n}\right], \text{ with } a > 0. \end{aligned}$$

Consequently

$$\begin{aligned} -\|B\|_\infty \mathbb{P}(A_n) &\leq \left(a \sum_{j=1}^N \|X^j\|_\infty + b\right) \mathbb{P}(A_n) + a \lim_M \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y_M^j 1_{A_n}\right] \\ &\stackrel{(\text{MON})}{=} \left(a \sum_{j=1}^N \|X^j\|_\infty + b\right) \mathbb{P}(A_n) + a \mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(X) 1_{A_n}] \\ &\leq \left(a \sum_{j=1}^N \|X^j\|_\infty + b\right) \mathbb{P}(A_n) - k_n a \mathbb{P}(A_n). \end{aligned}$$

Dividing by $\mathbb{P}(A_n)$ and sending n to infinity we would get a contradiction. This proves that $\|(\rho_{\mathcal{G}}(X))^{-}\|_{\infty} < +\infty$. Recalling that we already proved $\|(\rho_{\mathcal{G}}(X))^{+}\|_{\infty} < +\infty$, we obtain $\rho_{\mathcal{G}}(X) \in L^{\infty}(\mathcal{G})$.

STEP 2: the infimum in the definition of $\overline{\rho}_{\mathcal{G}}$ is attained.

For the minimizing sequence $(Y_n)_n$, from the budget constraint $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$ and the fact that $\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[X^j + Y_n^j]$ is bounded in n because of what we just proved ($L^{\infty} \ni \rho_{\mathcal{G}}(X) \leq \sum_{j=1}^N Y_n^j \leq \sum_{j=1}^N Y_1^j \in L^{\infty}$), we obtain that the sequence $(Y_n)_n$ is bounded in $(L^1(\mathcal{F}))^N$ using Lemma A.7.

Applying Corollary A.2 we can find a subsequence and a $\widehat{Y} \in (L^1(\mathcal{F}))^N$ such that

$$W_K := \frac{1}{K} \sum_{k=1}^K Y_{n_k} \xrightarrow[H \rightarrow \infty]{\mathbb{P}\text{-a.s.}} \widehat{Y}.$$

Furthermore $\sum_{j=1}^N Y^j \in L^1(\mathcal{G})$, $W_K \in \mathcal{B}_{\mathcal{G}}$ by convexity of the set and $\widehat{Y} \in \mathcal{B}_{\mathcal{G}}$ since this set is closed in probability. Additionally we have that

$$\sum_{j=1}^N \widehat{Y}^j = \lim_K \frac{1}{K} \sum_{k=1}^K \sum_{j=1}^N Y_{n_k}^j \stackrel{\text{Rem. A.4}}{=} \lim_k \sum_{j=1}^N Y_{n_k}^j = \rho_{\mathcal{G}}(X) \in L^{\infty}(\mathcal{G}) \quad (34)$$

which yields that also $\sum_{j=1}^N \widehat{Y}^j \in L^{\infty}(\mathcal{G})$. To prove that $\widehat{Y} \in \mathcal{C}_{\mathcal{G}}$ we need to show that $\sum_{i \in I_m} \widehat{Y}^i \in L^{\infty}(\mathcal{G})$ for every $m = 1, \dots, h$. This will be a consequence of Proposition A.9, once we show that $\mathbb{E}_{\mathbb{P}}[U(X + \widehat{Y})|\mathcal{G}] \geq B$. Hence we now focus on the latter inequality. We observe now that setting $Z_K := X + \frac{1}{K} \sum_{k=1}^K Y_{n_k}$ and $Z = X + \widehat{Y}$ Items 2 and 3 in Lemma A.8 are satisfied. Moreover if we take

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[Z_K^j|\mathcal{G}] = \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[X^j|\mathcal{G}] + \frac{1}{K} \sum_{k=1}^K \sum_{j=1}^N Y_{n_k}^j$$

we see that the first term in the sum in RHS does not depend on K , while the Césaro means almost surely converge. Hence also Item 1 in Lemma A.8 is satisfied, and we get that $\mathbb{E}_{\mathbb{P}}[U(Z)|\mathcal{G}] = \mathbb{E}_{\mathbb{P}}[U(X + \widehat{Y})|\mathcal{G}] \geq B$. As mentioned above, we now get also $\widehat{Y} \in \mathcal{C}_{\mathcal{G}}$. We finally recall from (34) that $\sum_{j=1}^N \widehat{Y}^j = \rho_{\mathcal{G}}(X)$ so that the infimum is in fact attained at \widehat{Y} , which satisfies the constraints for $\rho_{\mathcal{G}}(X)$.

STEP 3: $\rho_{\mathcal{G}}$ satisfies equations (17), (18), (19).

These have to be checked directly using definition of $\rho_{\mathcal{G}}(\cdot)$. We start with (17): if $X \leq Z$ componentwise a.s., for all $Y \in (L^1(\mathbb{P}))^N$ such that $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$ we have automatically (by monotonicity of U) that $\mathbb{E}_{\mathbb{P}}[U(Z+Y)|\mathcal{G}] \geq \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$ so that

$$\left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B \right\} \subseteq \left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}}[U(Z+Y)|\mathcal{G}] \geq B \right\}$$

and taking essential infima equation (17) follows.

As to (18), fix $0 \leq \lambda \leq 1$, $\lambda \in L^{\infty}(\mathcal{G})$ and $X, Z \in (L^{\infty}(\mathcal{F}))^N$. For $Y, W \in \mathcal{C}_{\mathcal{G}}$ such that $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$, $\mathbb{E}_{\mathbb{P}}[U(Z+W)|\mathcal{G}] \geq B$ we then have by concavity of utilities and \mathcal{G} -measurability of λ

$$\mathbb{E}_{\mathbb{P}}[U(\lambda X + (1-\lambda)Z + \lambda Y + (1-\lambda)W)|\mathcal{G}] = \mathbb{E}_{\mathbb{P}}[U(\lambda(X+Y) + (1-\lambda)(Z+W))|\mathcal{G}]$$

$$\geq \lambda \mathbb{E}_{\mathbb{P}} [U(X+Y)|\mathcal{G}] + (1-\lambda) \mathbb{E}_{\mathbb{P}} [U(Z+W)|\mathcal{G}] \geq \lambda B + (1-\lambda)B = B.$$

Moreover obviously $\lambda Y + (1-\lambda)W \in \mathcal{C}_{\mathcal{G}}$, so that by definition

$$\rho_{\mathcal{G}}(\lambda X + (1-\lambda)Z) \leq \lambda \sum_{j=1}^N Y^j + (1-\lambda) \sum_{j=1}^N W^j.$$

Taking essential infima in RHS over Y and W yields equation (18).

Finally we come to (19). For $Y \in (L^\infty(\mathcal{G}))^N$ the assumption $\mathcal{B}_{\mathcal{G}} + (L^0(\mathcal{G}))^N = \mathcal{B}_{\mathcal{G}}$ implies that $W := Z + Y \in \mathcal{C}_{\mathcal{G}}$ for all $Z \in \mathcal{C}_{\mathcal{G}}$. Hence

$$\begin{aligned} \rho_{\mathcal{G}}(X+Y) &= \text{ess inf} \left\{ \sum_{j=1}^N Z^j \mid Z \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}} [U(X+Y+Z)|\mathcal{G}] \geq B \right\} \\ &= \text{ess inf} \left\{ \sum_{j=1}^N (W^j - Y^j) \mid W \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}} [U(X+W)|\mathcal{G}] \geq B \right\} = \rho_{\mathcal{G}}(X) - \sum_{j=1}^N Y^j. \end{aligned}$$

□

Claim 5.6. *We have that $\rho_{\mathcal{G}}^\infty(X) = \rho_{\mathcal{G}}(X)$ for every $X \in (L^\infty(\mathcal{F}))^N$.*

Proof. It is clear that

$$\rho_{\mathcal{G}}(X) \leq \text{ess inf} \left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N, \mathbb{E}_{\mathbb{P}} [U(X+Y)|\mathcal{G}] \geq B \right\}$$

since the infimum on RHS is taken over a smaller set.

We prove now the reverse inequality: by Claim 5.5 an allocation exists, call it $Y \in \mathcal{C}_{\mathcal{G}}$. Use closedness under truncation to see that for $k \geq k_Y$ $Y_{(k)} \in \mathcal{C}_{\mathcal{G}}$ where $Y_{(k)}$, defined as in (16), satisfies $Y_{(k)} \rightarrow_k Y$ a.s.. We want to show that the convergence $U(X+Y_{(k)}+\varepsilon\mathbf{1}) \rightarrow_k U(X+Y+\varepsilon\mathbf{1})$ is dominated, where $\mathbf{1}$ is the N -components vector with all components equal to 1. To see this observe that $|U(X+Y+\varepsilon\mathbf{1})|$ and $|U(X+Z_Y+\varepsilon\mathbf{1})|$ are integrable:

$$L^1(\mathcal{F}) \ni a \left(\sum_{j=1}^N (X^j + Y^j) \right) + aN\varepsilon + b \stackrel{\text{Lemma.A.6.(ii)}}{\geq} U(X+Y+\varepsilon\mathbf{1}) \geq U(X+Y) \in L^1(\mathcal{F})$$

while integrability of $|U(X+Z_Y+\varepsilon\mathbf{1})|$ is trivial by boundedness of the vectors X, Z_Y and continuity of U . Moreover

$$\begin{aligned} |U(X+Y_{(k)}+\varepsilon\mathbf{1})| &= \left| U(X+Y+\varepsilon\mathbf{1}) \mathbf{1}_{\cap_j \{|Y^j| < k\}} + U(X+Z_Y+\varepsilon\mathbf{1}) \mathbf{1}_{\cup_j \{|Y^j| \geq k\}} \right| \\ &\leq \max(|U(X+Y+\varepsilon\mathbf{1})|, |U(X+Z_Y+\varepsilon\mathbf{1})|) \\ &\leq |U(X+Y+\varepsilon\mathbf{1})| + |U(X+Z_Y+\varepsilon\mathbf{1})|. \end{aligned}$$

Applying (cDOM) we then get that for all $\varepsilon > 0$

$$\mathbb{E}_{\mathbb{P}} [U(X+Y_{(k)}+\varepsilon\mathbf{1})|\mathcal{G}] \rightarrow_k \mathbb{E}_{\mathbb{P}} [U(X+Y+\varepsilon\mathbf{1})|\mathcal{G}] > B.$$

From the last expression we infer that

$$\mathbb{P} \left(\Gamma_K := \bigcap_{k \geq K} \{ \mathbb{E}_{\mathbb{P}} [U(X+Y_{(k)}+\varepsilon\mathbf{1})|\mathcal{G}] \geq B \} \right) \uparrow_K 1. \quad (35)$$

Fix K and take $\alpha_K \in \mathbb{R}^N$ with

$$U(-\|X\|_\infty - \|Y_{(K)}\|_\infty + \varepsilon \mathbf{1} + \alpha_K) \geq \text{ess sup}(B)$$

where again $\|X\|_\infty$ denotes the vector $[\|X^1\|_\infty, \dots, \|X^N\|_\infty]$ and similar notation is used for $\|Y_{(K)}\|_\infty$. Notice that such an α_K exists since $\sup_{z \in \mathbb{R}^N} U(z) > \text{ess sup}(B)$. Define Z_K by $Z_K^j := Y_{(K)}^j + \varepsilon + \alpha_K^j \mathbf{1}_{\Gamma_K^c}$, $j = 1, \dots, N$ and observe that since $\Gamma_K \in \mathcal{G}$, $Z_K \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N$. Furthermore

$$\mathbb{E}_{\mathbb{P}}[U(X + Z_K)|\mathcal{G}] = \mathbb{E}_{\mathbb{P}}[U(X + Z_K)|\mathcal{G}] \mathbf{1}_{\Gamma_K} + \mathbb{E}_{\mathbb{P}}[U(X + Z_K)|\mathcal{G}] \mathbf{1}_{\Gamma_K^c}$$

and

$$\mathbb{E}_{\mathbb{P}}[U(X + Z_K)|\mathcal{G}] \mathbf{1}_{\Gamma_K} = \mathbb{E}_{\mathbb{P}}[U(X + Y_{(K)} + \varepsilon \mathbf{1})|\mathcal{G}] \mathbf{1}_{\Gamma_K} \geq B \mathbf{1}_{\Gamma_K}$$

by definition of Γ_K and the fact that $\mathbf{1}_{\Gamma_K}$ can be moved inside conditional expectation.

Moreover by definition of α_K

$$\mathbb{E}_{\mathbb{P}}[U(X + Z_K)|\mathcal{G}] \mathbf{1}_{\Gamma_K^c} = \mathbb{E}_{\mathbb{P}}[U(X + Y_{(K)} + \varepsilon \mathbf{1} + \alpha_K)|\mathcal{G}] \mathbf{1}_{\Gamma_K^c} \geq B \mathbf{1}_{\Gamma_K^c}.$$

Hence we have that $Z_K \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N$, $\mathbb{E}_{\mathbb{P}}[U(X + Z_K)|\mathcal{G}] \geq B$, and we conclude that

$$\rho_{\mathcal{G}}^\infty(X) \leq \sum_{j=1}^N Z_K^j. \quad (36)$$

Now, by (35), for almost all $\omega \in \Omega$ there exists a $K(\omega) \in \mathbb{N}$ such that $\omega \in \Gamma_K$ for all $K \geq K(\omega)$, which implies for all $j = 1, \dots, N$ $Z_K^j(\omega) = Y_{(K)}^j(\omega) + \varepsilon \forall K \geq K(\omega)$.

By definition $Y_{(K)} \rightarrow_K Y$ a.s., so that by (36) we can write for almost all $\omega \in \Omega$:

$$\begin{aligned} \rho_{\mathcal{G}}^\infty(X) &\leq \liminf_{K \rightarrow +\infty} \sum_{j=1}^N Z_K^j = \liminf_{K \rightarrow +\infty} \left(\sum_{j=1}^N (Y_{(K)}^j + \varepsilon) \right) \\ &= \lim_{K \rightarrow +\infty} \sum_{j=1}^N Y_{(K)}^j + N\varepsilon = \rho_{\mathcal{G}}(X) + N\varepsilon. \end{aligned}$$

Hence $\rho_{\mathcal{G}}^\infty(X) \leq \rho_{\mathcal{G}}(X)$ \mathbb{P} -a.s., which implies $\rho_{\mathcal{G}}^\infty(X) = \rho_{\mathcal{G}}(X)$ \mathbb{P} -a.s.. \square

Claim 5.7. *The CSRМ $\rho_{\mathcal{G}}$ on $(L^\infty(\mathcal{F}))^N$ is continuous from below and from above, in the sense of Definition 3.6.*

Proof. Consider a sequence $X_n \uparrow_n X$ and take any $Y \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty)^N$ such that $\mathbb{E}_{\mathbb{P}}[U(X + Y)|\mathcal{G}] \geq B$. Then for any $\varepsilon > 0$

$$B < \mathbb{E}_{\mathbb{P}}[U(X + Y + \varepsilon \mathbf{1})|\mathcal{G}] \stackrel{(\text{eMON})}{=} \lim_n \mathbb{E}_{\mathbb{P}}[U(X_n + Y + \varepsilon \mathbf{1})|\mathcal{G}]$$

Hence the sequence $(A_K)_K$, where

$$A_K := \{\mathbb{E}_{\mathbb{P}}[U(X_n + Y + \varepsilon \mathbf{1})|\mathcal{G}] \geq B, \forall n \geq K\}$$

satisfies $\mathbb{P}(A_K) \uparrow_K 1$. Take $\alpha_K \in \mathbb{R}^N$ such that

$$U(-\|X_n\|_\infty - \|Y\|_\infty + \varepsilon \mathbf{1} + \alpha_K) \geq \text{ess sup}(B) \forall n \geq K$$

where the notation for $\|X_n\|_\infty$ and $\|Y\|_\infty$ is the same as in the proof of Claim 5.6. Define $Z_K \in (L^\infty(\mathcal{F}))^N$ by $Z_K^j := Y^j + \varepsilon \mathbf{1} + \alpha_K \mathbf{1}_{A_K^c}$ for $j = 1, \dots, N$. Since $A_K \in \mathcal{G}$ we have $Z \in \mathcal{C}_G$. Furthermore for all $n \geq K$

$$\begin{aligned} \mathbb{E}_\mathbb{P}[U(X_n + Z_K)|\mathcal{G}] &= \mathbb{E}_\mathbb{P}[U(X_n + Z_K)|\mathcal{G}] \mathbf{1}_{A_K} + \mathbb{E}_\mathbb{P}[U(X_n + Z_K)|\mathcal{G}] \mathbf{1}_{A_K^c} \\ &\geq B \mathbf{1}_{A_K} + \text{ess sup}(B) \mathbf{1}_{A_K^c} \geq B. \end{aligned}$$

Hence by definition of $\rho_G(X_n)$

$$\begin{aligned} \rho_G(X_n) &\leq \sum_{j=1}^N Z_K^j = \sum_{j=1}^N Y^j + N\varepsilon + \sum_{j=1}^N \alpha_K^j \mathbf{1}_{A_K^c}, \\ \lim_n \rho_G(X_n) &\leq \liminf_K \left(\sum_{j=1}^N Y^j + N\varepsilon + \sum_{j=1}^N \alpha_K^j \mathbf{1}_{A_K^c} \right). \end{aligned}$$

Recall now that $\mathbb{P}(A_K) \rightarrow_K 1$ and $A_K \subseteq A_{K+1}$. Hence almost all $\omega \in \Omega$ are such that $\mathbf{1}_{A_K^c}(\omega) = 0$ definitely in K . As a consequence

$$\liminf_K \left(\sum_{j=1}^N Y^j + N\varepsilon + \sum_{j=1}^N \alpha_K^j \mathbf{1}_{A_K^c} \right) = \liminf_K \left(\sum_{j=1}^N Y^j + N\varepsilon \right) = \sum_{j=1}^N Y^j + N\varepsilon.$$

It follows that

$$\lim_n \rho_G(X_n) \leq \sum_{j=1}^N Y^j \quad \mathbb{P} - \text{a.s.}$$

and this holds for all $Y \in \mathcal{C}_G$ such that $\mathbb{E}_\mathbb{P}[U(X + Y)|\mathcal{G}] \geq B$. Taking essential infimum on RHS for $Y \in \mathcal{C}_G \cap (L^\infty(\mathcal{F}))^N$, $\mathbb{E}_\mathbb{P}[U(X + Y)|\mathcal{G}] \geq B$ by Claim 5.6 we obtain

$$\lim_n \rho_G(X_n) \leq \rho_G^\infty(X) = \rho_G(X) \stackrel{(17)}{\leq} \lim_n \rho_G(X_n),$$

which shows continuity from below. By monotone convergence, the continuity from below of ρ_G yields the continuity from below of $\rho_0(\cdot) := \mathbb{E}_\mathbb{P}[\rho_G(\cdot)] : (L^\infty(\mathcal{F}))^N \rightarrow \mathbb{R}$ so that Theorem 2.4 item i) shows that ρ_0 is nicely representable. The continuity from above then follows from Theorem 3.9. \square

We now study the dual representation of the CSRM ρ_G^∞ . Notice that we just showed that Theorem 3.9 applies and so it yields the dual representation (22) for ρ_G^∞ , using $L_\mathcal{F} := (L^\infty(\mathcal{F}))^N$ and $L^* := (L^1(\mathcal{F}))^N$. However, in view of Claim 5.6, we can apply an argument inspired by [43] Proposition 3.6 to get a more specific dual representation. Observe that the set \mathcal{Q}_G defined in (20) takes the form

$$\mathcal{Q}_G := \left\{ \mathbb{Q} \ll \mathbb{P} : \frac{d\mathbb{Q}}{d\mathbb{P}} \in (L^1(\mathcal{F}))^N, \mathbb{E}_\mathbb{P} \left[\frac{d\mathbb{Q}^j}{d\mathbb{P}} \middle| \mathcal{G} \right] = 1 \quad \forall j = 1, \dots, N \right\} \quad (37)$$

and let

$$\rho_G^*(Y) := \text{ess sup}_{X \in L_\mathcal{F}} \left\{ \sum_{j=1}^N \mathbb{E}_\mathbb{P}[X^j Y^j | \mathcal{G}] - \rho_G(X) \right\}, \quad Y \in L^*. \quad (38)$$

Claim 5.8. *Let $\rho_G : (L^\infty(\mathcal{F}))^N \rightarrow L^\infty(\mathcal{G})$ be defined by (29) and take $\alpha^1(\cdot)$ as in (31). Then the following are equivalent for fixed $p \in \{0, 1\}$ and $\mathbb{Q} \in \mathcal{Q}_G$:*

1. $\rho_{\mathcal{G}}^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \in L^p(\mathcal{G})$.
2. $\alpha(\mathbb{Q}) \in L^p(\mathcal{G})$, where α is defined in (21) for $L_{\mathcal{F}} = (L^\infty(\mathcal{F}))^N$ and $L^* = (L^1(\mathcal{F}))^N$.
3. $\alpha^1(\mathbb{Q}) \in L^p(\mathcal{G})$ and $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y^j | \mathcal{G}] \leq \sum_{j=1}^N Y^j$ for all $Y \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N$.

Moreover $\rho_{\mathcal{G}}$ admits the dual representation in (33) and, for every $X \in (L^\infty(\mathcal{F}))^N$, there exists $\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}^1$ such that $\rho_{\mathcal{G}}(X) = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [-X^j | \mathcal{G}] - \alpha^1(\widehat{\mathbb{Q}})$.

Proof. Recall that in this specific setup we have by Claim 5.5 that $\rho_{\mathcal{G}} \in L^\infty(\mathcal{G})$ and that $\rho_{\mathcal{G}} = \rho_{\mathcal{G}}^\infty$ on $(L^\infty(\mathcal{F}))^N$ by Claim 5.6. We argued before that Theorem 3.9 applies here. From its proof, more precisely from STEP 5, selecting the setup $L_{\mathcal{F}} = (L^\infty(\mathcal{F}))^N$ and $L^* = (L^1(\mathcal{F}))^N$ we have:

$$\alpha(\mathbb{Q}) = \rho_{\mathcal{G}}^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \quad (39)$$

for all $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}$. Moreover we have:

$$\begin{aligned} \rho_{\mathcal{G}}^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right) &= \operatorname{ess\,sup}_{X \in (L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \rho_{\mathcal{G}}(X) \right) \\ &\stackrel{(29)}{=} \operatorname{ess\,sup}_{X \in (L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \operatorname{ess\,inf}_{\substack{Y \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N \\ \mathbb{E}_{\mathbb{P}}[U(X+Y) | \mathcal{G}] \geq B}} \left(\sum_{j=1}^N Y^j \right) \right) \\ &= \operatorname{ess\,sup}_{\substack{X, Y \in (L^\infty(\mathcal{F}))^N \\ Y \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}}[U(X+Y) | \mathcal{G}] \geq B}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \sum_{j=1}^N Y^j \right) \\ &= \operatorname{ess\,sup}_{\substack{Z, Y \in (L^\infty(\mathcal{F}))^N \\ Y \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}}[U(Z) | \mathcal{G}] \geq B}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-(Z^j - Y^j) | \mathcal{G}] - \sum_{j=1}^N Y^j \right). \end{aligned}$$

We conclude that

$$\alpha(\mathbb{Q}) = \operatorname{ess\,sup}_{\substack{Z \in (L^\infty(\mathcal{F}))^N \\ \mathbb{E}_{\mathbb{P}}[U(Z) | \mathcal{G}] \geq B}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-Z^j | \mathcal{G}] \right) + \operatorname{ess\,sup}_{Y \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y^j | \mathcal{G}] - \sum_{j=1}^N Y^j \right). \quad (40)$$

The equivalence among Items 1-2-3 is now clear, once we observe that for every $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}$ such that $\alpha(\mathbb{Q}) \in L^0(\mathcal{G})$ we must have $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y^j | \mathcal{G}] - \sum_{j=1}^N Y^j \leq 0$ \mathbb{P} -a.s. since $\mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N$ is a conditional cone.

All the claims then follow from Theorem 3.9, observing that for the optimum $\widehat{\mathbb{Q}}$ provided there we must have $\alpha(\mathbb{Q}) \in L^1(\mathcal{G})$ (since $\rho_{\mathcal{G}}(X) \in L^\infty(\mathcal{G})$). \square

Remark 5.9. We stress the fact that by Claim 5.8 we have for every $Y \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N$

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y^j | \mathcal{G}] \leq \sum_{j=1}^N Y^j \quad \mathbb{P} - \text{a.s.} \quad \text{for all } \mathbb{Q} \in \mathcal{Q}_{\mathcal{G}} \text{ such that } \alpha(\mathbb{Q}) \in L^0(\mathcal{G}). \quad (41)$$

5.2 Uniqueness and integrability of optima of $\rho_{\mathcal{G}}$

In this Section, under suitable additional assumptions on U we prove uniqueness for primal optimal allocations of $\rho_{\mathcal{G}}$. We also provide a fairness condition in the form (41) for such optima \widehat{Y} and measures in $\mathcal{Q}_{\mathcal{G}}^1$ (notice that, a priori, at the moment we do not even know if \widehat{Y} is integrable under the various measures $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$).

5.2.1 Uniqueness

Assumption 5.10. *The function $U : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies:*

$$X \in (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N, (U(X))^- \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \Rightarrow \exists \delta > 0 \text{ s.t. } (U(X - \varepsilon \mathbf{1}))^- \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \forall 0 \leq \varepsilon < \delta. \quad (42)$$

Observe that for example taking $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N > 0$ the function

$$U(x) := \sum_{j=1}^N (1 - \exp(-\alpha_j x_j)) + \left(1 - \exp\left(-\sum_{j=1}^N \beta_j x_j\right) \right)$$

satisfies Assumption 5.10.

Proposition 5.11. *Under Assumption 5.10 $\rho_{\mathcal{G}}(X)$ defined in (30) admits a unique optimum in $\mathcal{C}_{\mathcal{G}}$ for every $X \in (L^\infty(\mathcal{F}))^N$.*

Proof. Suppose $\widehat{Y}_1 \neq \widehat{Y}_2$ were two optima. Then clearly so is $\widehat{Y}_\lambda = \lambda \widehat{Y}_1 + (1 - \lambda) \widehat{Y}_2$ for $\lambda \in \mathbb{R}, 0 < \lambda < 1$ by concavity of U . At the same time, we have that $\Gamma := \{\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_\lambda) | \mathcal{G}] > \lambda \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_1) | \mathcal{G}] + (1 - \lambda) \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_2) | \mathcal{G}]\} \in \mathcal{G}$ satisfies $\mathbb{P}(\Gamma) = 1$ by strict concavity: if this were not the case, from concavity and

$$\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_\lambda) \mathbf{1}_{\Gamma^c}] = \lambda \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_1) \mathbf{1}_{\Gamma^c}] + (1 - \lambda) \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_2) \mathbf{1}_{\Gamma^c}]$$

we would get that on Γ^c , which has positive probability, $U(X + \widehat{Y}_\lambda) = \lambda U(X + \widehat{Y}_1) + (1 - \lambda) U(X + \widehat{Y}_2)$ which contradicts strict concavity of U .

Fix now for some $\varepsilon > 0$. Recall that we showed

$$\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_\lambda) | \mathcal{G}] > \lambda \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_1) | \mathcal{G}] + (1 - \lambda) \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_2) | \mathcal{G}] \geq B \quad \mathbb{P} - \text{a.s.}$$

By monotonicity of U and Assumption 5.10 we have the convergence

$$\mathbb{E}_{\mathbb{P}} \left[U \left(X + \widehat{Y}_\lambda - \frac{1}{H} \mathbf{1} \right) \middle| \mathcal{G} \right] \uparrow_H \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_\lambda) | \mathcal{G}] > B \quad \mathbb{P} - \text{a.s.}$$

where $\mathbf{1} := [1, \dots, 1] \in \mathbb{R}^N$. By Egorov Theorem A.3, there exists a $\Xi \in \mathcal{G}$, with $\mathbb{P}(\Xi) > 0$, such that on Ξ both the following conditions hold: $\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_\lambda) | \mathcal{G}] \geq B + \varepsilon$ and the convergence

$$\mathbb{E}_{\mathbb{P}} \left[U \left(X + \widehat{Y}_\lambda - \frac{1}{H} \mathbf{1} \right) \middle| \mathcal{G} \right] \uparrow_H \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}_\lambda) | \mathcal{G}]$$

is uniform. Hence, definitely in $H \in \mathbb{N}$,

$$\mathbb{E}_{\mathbb{P}} \left[U \left(X + \widehat{Y}_\lambda - \frac{1}{H} \mathbf{1}_{\Xi} \right) \middle| \mathcal{G} \right] \geq B.$$

At the same time, $\widehat{Y}_\lambda - \frac{1}{H} \mathbf{1}_{\Xi} \in \mathcal{C}_{\mathcal{G}}$ and by definition $\rho_{\mathcal{G}}(X) \mathbf{1}_{\Xi} \leq \left(\sum_{j=1}^N \widehat{Y}_\lambda^j - N \frac{1}{H} \right) \mathbf{1}_{\Xi} < \left(\sum_{j=1}^N \widehat{Y}_\lambda^j \right) \mathbf{1}_{\Xi}$ which contradicts the optimality of \widehat{Y} , as $\mathbb{P}(\Xi) > 0$. \square

5.2.2 Integrability

We now wish to establish a conditional fairness property for any optimum \widehat{Y} from Theorem 5.4, namely we aim to prove (see Proposition 5.13): $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [\widehat{Y}^j | \mathcal{G}] \leq \sum_{j=1}^N \widehat{Y}^j$ for any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ (defined in (32)). Notice that this is not automatic from the fairness condition (41) coming from the definition of $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$, since we do not know in general if $\widehat{Y} \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N$ (we only know, from Theorem 5.4, that $\widehat{Y} \in \mathcal{C}_{\mathcal{G}}$). We point out that Proposition 5.13 will be also needed in the proof of Theorem 7.3. In order to show such a fairness property, we need to establish an integrability result for such a \widehat{Y} , and the theory of multivariate Orlicz spaces will come in handy for this purpose. To each univariate Young function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ we can associate its conjugate $\phi^*(y) := \sup_{x \in \mathbb{R}} (xy - \phi(x))$. As in [53], we can associate to both ϕ and ϕ^* the Orlicz spaces and Hearts $L^\phi, M^\phi, L^{\phi^*}, M^{\phi^*}$. Univariate Young functions naturally arise from univariate utility functions u , setting $\phi(x) := u(0) - u(-x)$, $x \geq 0$. We now recall from [23] how to produce multivariate Orlicz functions and spaces from multivariate utility functions, inspired by [6], Appendix B. Indeed, for a multivariate utility function U specified in (23), we define the function Φ on $(\mathbb{R}_+)^N$ by

$$\Phi(y) := U(0) - U(-y). \quad (43)$$

and

$$\phi_j(z) := u_j(0) - u_j(-z), \quad z \in \mathbb{R}_+ \quad (44)$$

as the (univariate) functions associated to the univariate utilities u_1, \dots, u_N . Note that Φ defined in (43) is a multivariate Orlicz function ([23] Lemma 3.5.(i) observing that U satisfying Standing Assumption I in this paper is well controlled according to [23] Definition 3.4, as shown in [23] Proposition 7.1). As such, Φ in (43) generates a multivariate Orlicz space and a multivariate Orlicz Heart:

$$\begin{aligned} L^\Phi &:= \{X \in L^0((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N) \mid \exists \lambda \in (0, +\infty) \text{ s.t. } \mathbb{E}_{\mathbb{P}}[\Phi(\lambda|X|)] < +\infty\}, \\ M^\Phi &:= \{X \in L^0((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N) \mid \forall \lambda \in (0, +\infty) \mathbb{E}_{\mathbb{P}}[\Phi(\lambda|X|)] < +\infty\}. \end{aligned} \quad (45)$$

Moreover, ϕ_1, \dots, ϕ_N are univariate Orlicz functions.

The Köthe dual K_Φ of the space L^Φ is defined as

$$K_\Phi := \left\{ Z \in L^0((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N) \mid \sum_{j=1}^N X^j Z^j \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \quad \forall X \in L^\Phi \right\}. \quad (46)$$

Section 2.1 in [23] collects some useful properties on multivariate Orlicz spaces, Orlicz Hearts and Köthe duals.

Assumption 5.12. $L^\Phi = L^{\phi_1} \times \dots \times L^{\phi_N}$.

Assumption 5.12 is a request on the utility functions we allow for. It can be rephrased as: if for $X \in (L^0((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]))^N$ there exist $\lambda_1, \dots, \lambda_N > 0$ such that $\mathbb{E}_{\mathbb{P}}[u_j(-\lambda_j |X^j|)] > -\infty$, then there exists $\alpha > 0$ such that $\mathbb{E}_{\mathbb{P}}[\Lambda(-\alpha |X|)] > -\infty$. This request is rather weak and there are many examples of choices of U and Λ that guarantee this condition is met, see [23] Proposition 7.3. Note however that this is not a request on the topological spaces, but just an integrability requirement, and it is automatically satisfied if $\Lambda = 0$.

Proposition 5.13. *Suppose Assumption 5.12 is fulfilled. Then for any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ (defined in (32)), any optimum \widehat{Y} from Theorem 5.4 satisfies $\widehat{Y} \in L^1((\Omega, \mathcal{F}, \mathbb{Q}^1) \times \cdots \times L^1((\Omega, \mathcal{F}, \mathbb{Q}^N))$ and*

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [\widehat{Y}^j | \mathcal{G}] \leq \sum_{j=1}^N \widehat{Y}^j.$$

Proof. Postponed to Section A.4. □

5.3 Optimization with a fixed measure $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$

We will now study a counterpart of $\rho_{\mathcal{G}}$ which can be obtained by fixing a vector of pricing measures $\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N]$. For $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ we define the following optimization problem:

$$\rho_{\mathcal{G}}^{\mathbb{Q}}(X) := \text{ess inf} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y^j | \mathcal{G}] \mid Y \in (L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}))^N \text{ and } \mathbb{E}_{\mathbb{P}} [U(X + Y) | \mathcal{G}] \geq B \right\}. \quad (47)$$

Notice that in the problem (47) the constraint $\sum_{j=1}^N Y^j \in L^{\infty}(\mathcal{G})$ does not appear anymore. The problem still makes sense, however, since a valuation of Y is now assigned by the pricing vector \mathbb{Q} . By the fairness condition (41), it is easy to verify that $\rho_{\mathcal{G}}(X) \leq \rho_{\mathcal{G}}^{\mathbb{Q}}(X)$ for all $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ and $X \in (L^{\infty}(\mathcal{F}))^N$. Such a bound is actually tight, as the following Theorem 5.14 shows. The result we are to present will also be useful when studying equilibrium properties for primal and dual optima of $\rho_{\mathcal{G}}$ in Section 7. In order to state the result more easily, and since we will need to change underlying probability measures, some additional notation is in place: from now on, given a vector of probability measures $\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N]$ and a number $p \in \{0, 1\}$, we set

$$L^p(\mathcal{F}, \mathbb{Q}) := L^p(\Omega, \mathcal{F}, \mathbb{Q}^1) \times \cdots \times L^p(\Omega, \mathcal{F}, \mathbb{Q}^N).$$

Similarly, when some confusion might arise, we will write explicitly also the measure \mathbb{P} , that is we will use $L^p(\mathcal{F}, \mathbb{P}) := L^p(\Omega, \mathcal{F}, \mathbb{P})$ in place of the shortened $L^p(\mathcal{F})$.

Theorem 5.14. *Suppose Assumption 5.12 is fulfilled and let $X \in (L^{\infty}(\mathcal{F}, \mathbb{P}))^N$. Then for any optimum $\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}^1$ of (33), the optimum $\widehat{Y} \in \mathcal{C}_{\mathcal{G}}$ of Theorem 5.4 is an optimum for $\rho_{\mathcal{G}}^{\widehat{\mathbb{Q}}}(X)$ in the following extended sense: $\widehat{Y} \in (L^1(\mathcal{F}, \mathbb{P}))^N \cap \bigcap_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1} L^1(\mathcal{F}, \mathbb{Q})$, $\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}) | \mathcal{G}] \geq B$ and*

$$\rho_{\mathcal{G}}^{\widehat{\mathbb{Q}}}(X) = \text{ess inf} \left\{ \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [Y^j | \mathcal{G}] \mid Y \in (L^1(\mathcal{F}, \mathbb{P}))^N \cap \bigcap_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1} L^1(\mathcal{F}, \mathbb{Q}), \mathbb{E}_{\mathbb{P}} [U(X + Y) | \mathcal{G}] \geq B \right\} \quad (48)$$

$$= \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [\widehat{Y}^j | \mathcal{G}] = \rho_{\mathcal{G}}(X).$$

Proof. See Section A.4. □

6 The exponential case

A specific, rather canonical choice for the multivariate utility U is the aggregation of univariate exponential utility functions for single agents. It allows for obtaining explicit formulas for $\rho_{\mathcal{G}}$, as well for the corresponding optima.

Throughout the whole Section 6 we take $u_j(x) = -e^{-\alpha_j x}$, $j = 1, \dots, N$ for real numbers $\alpha_1, \dots, \alpha_N > 0$ and $\Lambda = 0$. We set:

$$\bar{X} := \sum_{j=1}^N X^j; \quad \beta := \sum_{j=1}^N \frac{1}{\alpha_j}; \quad A_j := \frac{1}{\alpha_j} \log \left(\frac{1}{\alpha_j} \right); \quad A := \sum_{j=1}^N A_j. \quad (49)$$

We consider only the case $\mathcal{B}_{\mathcal{G}} = \mathcal{D}_{\mathcal{G}}$ (recall (25) and (28)), which corresponds to the case of full sharing among all agents in the system (i.e., the extreme case of one single group, as described in Item (i) on page 13).

Theorem 6.1. *Consider a general sub σ -algebra $\mathcal{G} \subseteq \mathcal{F}$, $X \in (L^\infty(\mathcal{F}))^N$, $B \in L^\infty(\mathcal{G})$, with $\text{ess sup}(B) < 0$ and $\rho_{\mathcal{G}}$ defined in (30). Then*

$$\rho_{\mathcal{G}}(X) = \beta \log \left(-\frac{\beta}{B} \mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\bar{X}}{\beta} \right) \middle| \mathcal{G} \right] \right) - A;$$

$\hat{Y} = [\hat{Y}^1, \dots, \hat{Y}^N] \in (L^\infty(\mathcal{F}))^N$ is an optimal allocation for $\rho_{\mathcal{G}}(X)$ and $\hat{\mathbb{Q}} = [\hat{\mathbb{Q}}^1, \dots, \hat{\mathbb{Q}}^N]$ is an optimum for the dual representation of $\rho_{\mathcal{G}}(X)$, where for $j = 1, \dots, N$

$$\hat{Y}^j := -X^j + \frac{1}{\beta \alpha_j} (\bar{X} + \rho_{\mathcal{G}}(X) + A) - A_j, \quad (50)$$

$$\frac{d\hat{\mathbb{Q}}^j}{d\mathbb{P}} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} := \frac{\exp \left(-\frac{\bar{X}}{\beta} \right)}{\mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\bar{X}}{\beta} \right) \middle| \mathcal{G} \right]}. \quad (51)$$

Proof. See Section A.5. □

Remark 6.2. In order to avoid more complex notations and lengthy proofs, we provided the explicit formulas only for the case $\mathcal{B}_{\mathcal{G}} = \mathcal{D}_{\mathcal{G}}$. The reader may obtain similar formulas for the cluster cases in Example 5.2, using the corresponding deterministic formulas in Biagini et al. (2020) [12], Section 6.

6.1 Time consistency

A rather natural issue we now wish to tackle is whether a consistency or concatenation property can be associated to the Conditional Shortfall Systemic Risk Measures, at least in the exponential case where explicit computations are feasible. More precisely, we consider now two sub σ -algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ and study the relations between primal and dual allocations corresponding to the various possible risk measurements (from \mathcal{F} to \mathcal{H} , from \mathcal{G} to \mathcal{H} , and combinations). In this subsection we will need to exploit explicitly the dependence of optimal allocations and minimax measures given by Theorem 6.1 on initial datum and sub σ -algebras. To fix the notation, given $X \in (L^\infty(\mathcal{F}))^N$ and $\mathcal{G} \subseteq \mathcal{F}$ we define for each $k = 1, \dots, N$:

$$\hat{Y}^k(\mathcal{G}, X) := -X^k + \frac{1}{\beta \alpha_k} (\bar{X} + \rho_{\mathcal{G}}(X) + A) - A_k, \quad (52)$$

$$\frac{d\hat{\mathbb{Q}}^k(\mathcal{G}, X)}{d\mathbb{P}} := \frac{\exp \left(-\frac{\bar{X}}{\beta} \right)}{\mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\bar{X}}{\beta} \right) \middle| \mathcal{G} \right]}, \quad (53)$$

$$\hat{a}^k(\mathcal{G}, X) := \mathbb{E}_{\hat{\mathbb{Q}}^k(\mathcal{G}, X)}[\hat{Y}^k(\mathcal{G}, X) | \mathcal{G}], \quad (54)$$

$$\rho_{\mathcal{G}}(X) = \beta \log \left(-\frac{\beta}{B} \mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\bar{X}}{\beta} \right) \middle| \mathcal{G} \right] \right) - A = \sum_{j=1}^N \hat{Y}^j(\mathcal{G}, X) = \sum_{j=1}^N \hat{a}^j(\mathcal{G}, X). \quad (55)$$

Theorem 6.3. *Let $X \in (L^\infty(\mathcal{F}))^N$. The following time consistency property holds whenever $B \in L^\infty(\mathcal{H})$ is given: for every $k = 1, \dots, N$*

$$\hat{Y}^k(\mathcal{H}, -\hat{Y}(\mathcal{G}, X)) = \hat{Y}^k(\mathcal{H}, X) + \hat{Y}^k(\mathcal{H}, 0), \quad (56)$$

$$\frac{d\hat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{G}, X) \frac{d\hat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\hat{Y}(\mathcal{G}, X)) = \frac{d\hat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{G}, X) \frac{d\hat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\hat{a}(\mathcal{G}, X)) = \frac{d\hat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, X), \quad (57)$$

$$\hat{a}^k(\mathcal{H}, -\hat{a}(\mathcal{G}, X)) = \hat{a}^k(\mathcal{H}, X) + \hat{a}^k(\mathcal{H}, 0). \quad (58)$$

Proof. See Section A.5. □

Remark 6.4. Various concepts of time consistency have already been explored in the literature for the dynamic and set-valued case. In particular, those of “time consistency” in [59] and “multiportfolio time consistency” in [19] and [29], were shown to be equivalent under mild assumptions in [28]. Let us point out that in these approaches consistency is required for the whole set of eligible portfolio that covers the risk of X . Instead, we only request consistency of particular allocations, enjoying some peculiar optimality property, as well as for the dual optimizers. Furthermore, as mentioned already after Theorem 5.4 and adopting the same notation introduced there for \mathcal{F}_t , we use terminal-time, random allocations for securing the system. This enlightens a further difference from the aforementioned works where the whole set of allocations is required to be \mathcal{F}_t -measurable, making the properties hardly comparable. Possible links might become clearer with a more detailed inspections of the properties of our allocations \hat{a} , and we wish to leave this topic for further research.

The proof of Theorem 6.3 is entirely based on the availability of explicit formulas, as well as on the nice combinations of logarithms and exponentials one sees also in the univariate case.

7 Conditional Shortfall Systemic Risk Measures and equilibrium: dynamic mSORTE

In the papers [10] and [23] the equilibrium concepts of Systemic Optimal Risk Transfer Equilibrium (SORTE) and of its multivariate extension Multivariate Systemic Optimal Risk Transfer Equilibrium (mSORTE) were introduced and analyzed in a static setup. We refer the reader to these papers for the economic motivation, applications and unexplained notation. Here we show that a generalization to the conditional setting is possible and prove the existence of a time consistent family of dynamic mSORTE in the exponential setup. Consider a multivariate utility function U . For each $j = 1, \dots, N$ consider a vector subspace $L_{\mathcal{F}}^j$ with $L^\infty(\mathcal{F}) \subseteq L_{\mathcal{F}}^j \subseteq L^0(\mathcal{F})$ and set

$$L_{\mathcal{F}} := L_{\mathcal{F}}^1 \times \dots \times L_{\mathcal{F}}^N \subseteq (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N.$$

With

$$\mathcal{M} \subseteq \mathcal{Q}_{\mathcal{G}}$$

we will denote a subset of probability vectors.

Remark 7.1. We impose the condition $\mathcal{M} \subseteq \mathcal{Q}_{\mathcal{G}}$, which implies that for every $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}$ and for every $j = 1, \dots, N$, \mathbb{Q}^j coincides with \mathbb{P} on \mathcal{G} and

$$(L^1(\mathcal{G}, \mathbb{P}))^N = L^1(\mathcal{G}, \mathbb{Q}) := L^1(\Omega, \mathcal{F}, \mathbb{Q}^1) \times \dots \times L^1(\Omega, \mathcal{F}, \mathbb{Q}^N).$$

For $(Y, \mathbb{Q}, \alpha, A) \in (L_{\mathcal{F}} \cap L^1(\mathcal{F}, \mathbb{Q})) \times \mathcal{M} \times (L^1(\mathcal{G}, \mathbb{P}))^N \times L^\infty(\mathcal{G})$ define for $j = 1, \dots, N$

$$\begin{aligned} Y^{[-j]} &:= [Y^1, \dots, Y^{j-1}, Y^{j+1}, \dots, Y^N] \in L^0(\mathcal{F}, \mathbb{P})^{N-1}, \\ [Y^{[-j]}, Z] &:= [Y^1, \dots, Y^{j-1}, Z, Y^{j+1}, \dots, Y^N], \quad Z \in L^0(\mathcal{F}, \mathbb{P}), \\ U_j^{Y^{[-j]}}(Z) &:= \mathbb{E}[u_j(X^j + Z)|\mathcal{G}] + \mathbb{E}\left[\Lambda(X + [Y^{[-j]}, Z])\middle|\mathcal{G}\right], \quad Z \in L^0(\mathcal{F}, \mathbb{P}), \end{aligned} \quad (59)$$

$$U_j^{\mathbb{Q}^j, Y^{[-j]}}(\alpha^j) := \text{ess sup} \left\{ U_j^{Y^{[-j]}}(Z) \mid Z \in L_{\mathcal{F}}^j \cap L^1(\Omega, \mathcal{F}, \mathbb{Q}^j), E_{\mathbb{Q}^j}[Z|\mathcal{G}] \leq \alpha^j \right\}, \quad (60)$$

and

$$T^{\mathbb{Q}}(\alpha) := \text{ess sup} \left\{ \mathbb{E}_{\mathbb{P}}[U(X + Y)|\mathcal{G}] \mid Y \in L_{\mathcal{F}} \cap L^1(\mathcal{F}, \mathbb{Q}), E_{\mathbb{Q}^j}[Y^j|\mathcal{G}] \leq \alpha^j, \forall j \right\}, \quad (61)$$

$$S^{\mathbb{Q}}(A) := \text{ess sup} \left\{ T^{\mathbb{Q}}(\alpha) \mid \alpha \in (L^1(\mathcal{G}, \mathbb{P}))^N, \sum_{j=1}^N \alpha^j \leq A \right\}. \quad (62)$$

Obviously, all such quantities depend also on $X \in (L^\infty(\mathcal{F}))^N$, but as X will be kept fixed throughout the analysis, we may avoid to explicitly specify this dependence in the notations. As $u_1, \dots, u_N, \Lambda, U$ are increasing we can replace, in the definitions (60), (61), (62), the inequalities in the budget constraints with equalities.

Definition 7.2. *The triple $(Y_X, \mathbb{Q}_X, \alpha_X) \in L_{\mathcal{F}} \times \mathcal{M} \times (L^1(\mathcal{G}, \mathbb{P}))^N$ with $Y \in L^1(\mathcal{F}, \mathbb{Q}_X)$ is a **Dynamic Multivariate Systemic Optimal Risk Transfer Equilibrium** (Dynamic mSORTE) with budget $A \in L^\infty(\mathcal{G})$ if*

1. (Y_X, α_X) is an optimum for

$$\text{ess sup}_{\substack{\alpha \in (L^1(\mathbb{P}, \mathcal{G}))^N \\ \sum_{j=1}^N \alpha_j = A}} \left\{ \text{ess sup} \left\{ \mathbb{E}_{\mathbb{P}}[U(X + Y)|\mathcal{G}] \mid Y \in L_{\mathcal{F}} \cap L^1(\mathcal{F}, \mathbb{Q}_X), \mathbb{E}_{\mathbb{Q}_X^j}[Y^j|\mathcal{G}] \leq \alpha^j, \forall j \right\} \right\}; \quad (63)$$

2. $Y_X \in \mathcal{C}_{\mathcal{G}}$ and $\sum_{j=1}^N Y_X^j = A$ \mathbb{P} -a.s..

Theorem 7.3. *Suppose Assumption 5.10 and Assumption 5.12 hold and let $X \in (L^\infty(\mathcal{F}))^N$. Let \widehat{Y} be the optimum of $\rho_{\mathcal{G}}$ in (30) and let $\widehat{\mathbb{Q}}$ be an optimum of (33). Define $\widehat{\alpha}^j := \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j|\mathcal{G}]$. Then $(\widehat{Y}, \widehat{\mathbb{Q}}, \widehat{\alpha})$ is a Dynamic mSORTE for $\mathcal{M} := \mathcal{Q}_{\mathcal{G}}^1$, $L_{\mathcal{F}} := (L^1(\mathcal{F}, \mathbb{P}))^N \cap \bigcap_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1} L^1(\mathcal{F}, \mathbb{Q})$, $A := \rho_{\mathcal{G}}(X)$.*

Proof.

STEP 1: Item 2 of Definition 7.2 is satisfied. We start observing that by Theorem 5.4, $\widehat{Y} \in \mathcal{C}_{\mathcal{G}}$ and trivially being an optimum it satisfies $\sum_{j=1}^N \widehat{Y}^j = \rho_{\mathcal{G}}(X) =: A$.

STEP 2: we prove that for any optimum $\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}^1$ of (33), the optimization problem

$$\pi_A^{\mathcal{G}, \widehat{\mathbb{Q}}}(X) := \text{ess sup} \left\{ \mathbb{E}_{\mathbb{P}} [U(X+Y)|\mathcal{G}] \left| \begin{array}{l} Y \in (L^1(\mathcal{F}, \mathbb{P}))^N \cap \bigcap_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1} L^1(\mathcal{F}, \mathbb{Q}) \text{ and} \\ \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [Y^j | \mathcal{G}] \leq A \end{array} \right. \right\} \quad (64)$$

satisfies $\pi_A^{\mathcal{G}, \widehat{\mathbb{Q}}}(X) = B$.

We start showing that the optimal allocation \widehat{Y} for $\rho_{\mathcal{G}}(X)$ provided by Theorem 5.4 satisfies $\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [Y^j | \mathcal{G}] = A$ (directly from Theorem 5.14) and $\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}) | \mathcal{G}] = B$. To see the latter equality, observe that we already know that $\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}) | \mathcal{G}] \geq B$. If on a set Ξ of positive measure we had that the inequality is strict, we would have for some $N > 0$ that $\Xi_N := \{\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}) | \mathcal{G}] > B + \frac{1}{N}\} \in \mathcal{G}$ has positive probability. By Assumption 5.10 and (cDOM) we have

$$\mathbb{E}_{\mathbb{P}} \left[U \left(X + \widehat{Y} - \frac{1}{H} \mathbf{1} \right) \middle| \mathcal{G} \right] \uparrow_H \mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}) | \mathcal{G}] \text{ on } \Xi_N.$$

By Egorov Theorem A.3 there exists a $\Theta_N \in \mathcal{G}$, with both $\Theta_N \subseteq \Xi_N$ and $\mathbb{P}(\Theta_N) > 0$, on which the convergence above is uniform (in H).

Hence, definitely in H , $\mathbb{E}_{\mathbb{P}} \left[U \left(X + \widehat{Y} - \frac{1}{H} \mathbf{1}_{\Theta_N} \right) \middle| \mathcal{G} \right] \geq B$. Putting things together, we have then

$$\widehat{Y} - \frac{1}{H} \mathbf{1}_{\Theta_N} \mathbf{1} \in \mathcal{C}_{\mathcal{G}}, \quad \mathbb{E}_{\mathbb{P}} \left[U \left(X + \widehat{Y} - \frac{1}{H} \mathbf{1}_{\Theta_N} \right) \middle| \mathcal{G} \right] \geq B.$$

Clearly then $\rho_{\mathcal{G}}(X) \leq \sum_{j=1}^N \widehat{Y}^j - \frac{N}{H} \mathbf{1}_{\Theta_N}$. This in turns gives a contradiction, since \widehat{Y} is supposed to be an optimum for $\rho_{\mathcal{G}}(X)$.

Now we prove that $\pi_A^{\mathcal{G}, \widehat{\mathbb{Q}}}(X) = B$. Take \widehat{Y} as before. We stress that it satisfies $\widehat{Y} \in (L^1(\mathcal{F}, \mathbb{P}))^N$ by Theorem 5.4 and $\widehat{Y} \in L^1(\mathcal{F}, \mathbb{Q})$ for every $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ by Proposition 5.13. We showed above that $\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [\widehat{Y}^j | \mathcal{G}] = A$ and $\mathbb{E}_{\mathbb{P}} [U(X + \widehat{Y}) | \mathcal{G}] = B$. Hence by (64) we have $\pi_A^{\mathcal{G}, \widehat{\mathbb{Q}}}(X) \geq B$.

Since the set over which we take the essential supremum to define $\pi_A^{\mathcal{G}, \widehat{\mathbb{Q}}}(X)$ is upward directed, we can take a maximizing sequence for $\pi_A^{\mathcal{G}, \widehat{\mathbb{Q}}}(X)$, call it $(Y_n)_n$. W.l.o.g. we may suppose that $\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [Y_n^j | \mathcal{G}] = A$. Suppose for $\Delta := \{\pi_A^{\mathcal{G}, \widehat{\mathbb{Q}}}(X) > B\}$ we had $\mathbb{P}(\Delta) > 0$. Then setting $\Delta_N := \{\pi_A^{\mathcal{G}, \widehat{\mathbb{Q}}}(X) > B + \frac{1}{N}\} \in \mathcal{G}$ we have $\mathbb{P}(\Delta_N) > 0$ for some N big enough. By Egorov Theorem A.3, we have that on a subset $\widetilde{\Delta}_N$ of Δ_N , having positive probability, the pointwise convergence of $\mathbb{E}_{\mathbb{P}} [U(X + Y_n) | \mathcal{G}]$ to the essential supremum is uniform. Hence given $\varepsilon > 0$ small enough, for n big enough and for $\widetilde{Y}_n = Y_n - \varepsilon \mathbf{1}_{\widetilde{\Delta}_N} \mathbf{1} \in \mathcal{C}_{\mathcal{G}} \cap (L^\infty(\mathbb{P}, \mathcal{F}))^N$ we have $\mathbb{E}_{\mathbb{P}} [U(X + \widetilde{Y}_n) | \mathcal{G}] \geq B$. Clearly $\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [\widetilde{Y}_n^j | \mathcal{G}] < A$ with positive probability, by definition of \widetilde{Y}_n . By the definition of $\rho_{\mathcal{G}}^{\widehat{\mathbb{Q}}}$ in (47), we obtain that with positive probability (i.e. on $\widetilde{\Delta}_N$)

$$\rho_{\mathcal{G}}^{\widehat{\mathbb{Q}}}(X) \leq \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [\widetilde{Y}_n^j | \mathcal{G}] < A.$$

We then get a contradiction to $A := \rho_{\mathcal{G}}(X)$, since by Theorem 5.14 we have $\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}^{\widehat{\mathbb{Q}}}(X)$.

STEP 3: the optimal allocation \widehat{Y} of $\rho_{\mathcal{G}}(X)$ given in Theorem 5.4 (which is an optimum by Theorem 5.14) is an optimum for the RHS of (64). This follows trivially from the arguments in the previous steps, as $\widehat{Y} \in (L^1(\mathcal{F}, \mathbb{P}))^N$ by Theorem 5.4, $\widehat{Y} \in L^1(\mathcal{F}, \mathbb{Q})$ for every $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ by

Proposition 5.13, and $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j|\mathcal{G}] = A$. Thus \widehat{Y} satisfies the constraints of RHS of (64). Moreover we proved in STEP 1 that $\mathbb{E}_{\mathbb{P}} \left[U \left(X + \widehat{Y} \right) \middle| \mathcal{G} \right] = B = \pi_A^{\mathcal{G}, \mathbb{Q}}(X)$.

STEP 4: conclusion. We easily see that \widehat{Y} is an optimum for

$$\text{ess sup} \left\{ \mathbb{E}_{\mathbb{P}} [U(X+Y)|\mathcal{G}] \mid Y \in L_{\mathcal{F}}, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j|\mathcal{G}] \leq A \right\} =$$

$$\text{ess sup}_{\substack{\alpha \in (L^1(\mathcal{G}, \mathbb{P}))^N \\ \sum_{j=1}^N \alpha_j = A}} \left\{ \text{ess sup} \left\{ \mathbb{E}_{\mathbb{P}} [U(X+Y)|\mathcal{G}] \mid Y \in L_{\mathcal{F}}, \mathbb{E}_{\mathbb{Q}^j}[Y^j|\mathcal{G}] \leq \alpha^j \forall j = 1, \dots, N \right\} \right\}.$$

Hence $(\widehat{Y}, \widehat{\alpha})$ are optimum for (63), and also Item 1 of Definition 7.2 is satisfied. This completes the proof. \square

Corollary 7.4. *For the exponential utilities there exists a time consistent family of Dynamic mSORTES.*

Proof. Follows from Theorem 6.3 and Theorem 7.3. \square

A Appendix

A.1 Miscellaneous results

We recall the original Komlós Theorem:

Theorem A.1 (Komlós). *Let $(f_n)_n \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ be a sequence with bounded L^1 norms. Then there exists a subsequence $(f_{n_k})_k$ and a random variable g in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that for any further subsequence the Cesaro means satisfy:*

$$\frac{1}{N} \sum_{i \leq N} f_{n_{k_i}} \rightarrow_N g \text{ } \mathbb{P}\text{- a.s..}$$

Corollary A.2. *Consider probability measures $\mathbb{P}_1, \dots, \mathbb{P}_N \ll \mathbb{P}$. Let a sequence $(X_n)_n$ be given in $L^1(\Omega, \mathcal{F}, \mathbb{P}_1) \times \dots \times L^1(\Omega, \mathcal{F}, \mathbb{P}_N)$ such that*

$$\sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} \left[|X_n^j| \frac{d\mathbb{P}_j}{d\mathbb{P}} \right] < \infty.$$

Then there exists a subsequence $(X_{n_h})_h$ and an $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}_1) \times \dots \times L^1(\Omega, \mathcal{F}, \mathbb{P}_N)$ such that every further subsequence $(X_{n_{h_k}})_k$ satisfies

$$\frac{1}{K} \sum_{k=1}^K X_{n_{h_k}}^j \xrightarrow[K \rightarrow +\infty]{\mathbb{P}_j\text{-a.s.}} Y^j \quad \forall j = 1, \dots, N.$$

Proof. See [23] Corollary A.12. \square

Theorem A.3 (Egorov). *Let $(X_n)_n$ be a sequence in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ almost surely converging to $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$. For every $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{F}$ with $\mathbb{P}(A_\varepsilon) < \varepsilon$ satisfying*

$$\| (X_n - X) 1_{(A_\varepsilon)^c} \|_\infty \rightarrow_n 0.$$

Proof. See [3] Theorem 10.38. □

Remark A.4. Observe that for any sequence of real numbers $(a_n)_n$ converging to an $a \in \mathbb{R}$ and for any sequence $(N_h)_h \uparrow +\infty$ we have $\frac{1}{N_h} \sum_{j \leq N_h} a_j \rightarrow_h a$. This can be seen as follows: for $\varepsilon > 0$ fixed, take K s.t. $|a_j - a| \leq \varepsilon$ for all $j \geq K$. Take h big enough to have $N_h > K$. Then

$$\left| \frac{1}{N_h} \sum_{j \leq N_h} a_j - a \right| \leq \frac{1}{N_h} \sum_{j \leq N_h} |a_j - a| \leq \frac{K}{N_h} \sup_{j \leq K} |a_j - a| + \frac{N_h - K}{N_h} \varepsilon$$

and we can send h to infinity.

A.1.1 Essential suprema and infima

In this Section A.1.1 we write $L^0(\Omega, \mathcal{F}, \mathbb{P}; [-\infty, +\infty])$ for the set of (equivalence classes of) $[-\infty, +\infty]$ -valued random variables. $L^0(\Omega, \mathcal{F}, \mathbb{P}; [0, +\infty])$ is defined analogously.

Proposition A.5. *Let $\mathcal{A}, \mathcal{B} \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}; [-\infty, +\infty])$ be nonempty, $\lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P}; [0, +\infty])$, $f : \mathcal{A} \times \mathcal{B} \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P}; [-\infty, +\infty])$, $g : \mathcal{A} \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P}; [-\infty, +\infty])$, a sequence $(\alpha_n)_n \subseteq \mathcal{A}$ be given. Then:*

$$\begin{aligned} \operatorname{ess\,sup}_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} (\alpha + \beta) &= \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \alpha + \operatorname{ess\,sup}_{\beta \in \mathcal{B}} \beta = \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \left(\alpha + \operatorname{ess\,sup}_{\beta \in \mathcal{B}} \beta \right); \\ \operatorname{ess\,sup}_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} f(\alpha, \beta) &= \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \operatorname{ess\,sup}_{\beta \in \mathcal{B}} f(\alpha, \beta) = \operatorname{ess\,sup}_{\beta \in \mathcal{B}} \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} f(\alpha, \beta); \\ \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \lambda g(\alpha) &= \lambda \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} g(\alpha); \quad \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \alpha \geq \limsup_n \alpha_n; \quad \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} -g(\alpha) = -\operatorname{ess\,inf}_{\alpha \in \mathcal{A}} g(\alpha). \end{aligned}$$

A.1.2 Additional properties of multivariate utility functions

Recall that we are working under Standing Assumption I.

Lemma A.6.

(i) *There exist $a > 0$, $b \in \mathbb{R}$ such that*

$$U(x) \leq a \sum_{j=1}^N x^j + a \sum_{j=1}^N -(x^j)^- + b \quad \forall x \in \mathbb{R}^N.$$

(ii) *There exist $a > 0$, $b \in \mathbb{R}$ such that*

$$U(x) \leq a \sum_{j=1}^N x^j + b \quad \forall x \in \mathbb{R}^N.$$

(iii) *For every $\varepsilon > 0$ there exist a constant b_ε such that*

$$U(x) \leq \varepsilon \sum_{j=1}^N (x^j)^+ + b_\varepsilon \quad \forall x \in \mathbb{R}^N.$$

Proof. (i) See [23] Lemma 3.5 (as pointed out before, a function U satisfying Standing Assumption I in this paper is well controlled according to [23] Definition 3.4, as shown in [23] Proposition 7.1).

(ii) Use Item (i) and observe that $a \sum_{j=1}^N x^j + a \sum_{j=1}^N -(x^j)^- + b \leq a \sum_{j=1}^N x^j + b$.

(iii) See [23] Lemma 3.5. □

Lemma A.7. Let $(Z_n) \in (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$ satisfy $\mathbb{E}_{\mathbb{P}}[U(Z_n)] \geq B$ for all $n \in \mathbb{N}$ and for some constant $B \in \mathbb{R}$. If $\sup_n \left| \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[Z_n^j] \right| < +\infty$ then $\sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[|Z_n^j|] < \infty$.

Proof. See [23] Lemma A.1. □

Lemma A.8. Suppose $(Z_n)_n$ is a sequence in $(L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$. Suppose furthermore that the following conditions are met for some $B \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$:

1. $\sup_n \left| \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[Z_n^j | \mathcal{G}] \right| < +\infty$ \mathbb{P} -a.s.;
2. $\inf_n \mathbb{E}_{\mathbb{P}}[U(Z_n) | \mathcal{G}] \geq B$ \mathbb{P} -a.s.;
3. $Z_n \rightarrow_n Z$ \mathbb{P} -a.s..

Then $\mathbb{E}_{\mathbb{P}}[U(Z) | \mathcal{G}] \geq B$ a.s..

Proof.

STEP 1: $\sup_n \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^+ | \mathcal{G}] \right) < +\infty$ \mathbb{P} -a.s..

Define the sets

$$A^+ := \left\{ \sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^+ | \mathcal{G}] = +\infty \right\} \quad A^- := \left\{ \sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^- | \mathcal{G}] = +\infty \right\}.$$

We prove that $\mathbb{P}(A^-) = 0$: suppose by contradiction that $\mathbb{P}(A^-) > 0$. Apply Item 2 together with the fact that A^- is \mathcal{G} measurable to see that for some $a > 0, b \in \mathbb{R}$

$$B1_{A^-} \leq \mathbb{E}_{\mathbb{P}}[U(Z_n) | \mathcal{G}] 1_{A^-} \stackrel{\text{Lemma A.6(i)}}{\leq} \left(a \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[Z_n^j | \mathcal{G}] + a \sum_{j=1}^N (Z_n^j)^- + b \right) 1_{A^-}$$

which is a contradiction, by definition of A^- and boundedness of B . Hence $\mathbb{P}(A^-) = 0$. By Item 1, together with

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[Z_n^j | \mathcal{G}] = \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^+ | \mathcal{G}] - \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^- | \mathcal{G}]$$

we have that the symmetric difference $A^+ \Delta A^-$ is \mathbb{P} -null (equivalently $1_{A^+} = 1_{A^-}$), so that from $\mathbb{P}(A^-) = 0$ we get $\mathbb{P}(A^+) = 0$, and the claim follows.

STEP 2: Fatou Lemma and conclusion.

By Lemma A.6 (iii) for every $\varepsilon > 0$ there exist $b_\varepsilon > 0$ such that $\Gamma_\varepsilon(x) := -U(x) + \varepsilon \sum_{j=1}^N (x^j)^+ + b_\varepsilon \geq 0$ for all $x \in \mathbb{R}^N$. By Fatou Lemma (Γ_ε is continuous) we have that

$$\begin{aligned} -\mathbb{E}_{\mathbb{P}}[U(Z) | \mathcal{G}] + \varepsilon \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z^j)^+ | \mathcal{G}] + b_\varepsilon &= \mathbb{E}_{\mathbb{P}}[\Gamma_\varepsilon(Z) | \mathcal{G}] = \mathbb{E}_{\mathbb{P}} \left[\liminf_n \Gamma_\varepsilon(Z_n) | \mathcal{G} \right] \\ &\leq \liminf_n \mathbb{E}_{\mathbb{P}}[\Gamma_\varepsilon(Z_n) | \mathcal{G}] = \liminf_n \left(-\mathbb{E}_{\mathbb{P}}[U(Z_n) | \mathcal{G}] + \varepsilon \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^+ | \mathcal{G}] + b_\varepsilon \right). \end{aligned}$$

This chain of inequalities yields, since b_ε disappears on both sides:

$$-\mathbb{E}_{\mathbb{P}}[U(Z) | \mathcal{G}] + \varepsilon \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z^j)^+ | \mathcal{G}] \leq \liminf_n \left(-\mathbb{E}_{\mathbb{P}}[U(Z_n) | \mathcal{G}] + \varepsilon \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^+ | \mathcal{G}] \right).$$

We can thus exploit Item 2 in RHS to get

$$-\mathbb{E}_{\mathbb{P}} [U(Z)|\mathcal{G}] + \varepsilon \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [(Z^j)^+|\mathcal{G}] \leq \left(-B + \varepsilon \sup_n \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [(Z_n^j)^+|\mathcal{G}] \right) \right).$$

From the latter inequality we deduce

$$-\mathbb{E}_{\mathbb{P}} [U(Z)|\mathcal{G}] \leq -B + \varepsilon \sup_n \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [(Z_n^j)^+|\mathcal{G}] \right)$$

which holds \mathbb{P} -a.s. for all $\varepsilon > 0$. By STEP 1, $\sup_n \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [(Z_n^j)^+|\mathcal{G}] \right) < \infty$ and therefore $-\mathbb{E}_{\mathbb{P}} [U(Z^j)|\mathcal{G}] \leq -B$. \square

The following result is of technical nature and is used in the proof of the existence of an optimal allocation \hat{Y} in Claim 5.5, STEP 2.

Proposition A.9. *Suppose the vectors $X \in (L^\infty(\mathcal{F}))^N$ and $Y \in (L^1(\mathcal{F}))^N$ satisfy $\sum_{j=1}^N Y^j \in L^\infty(\mathcal{G})$ and*

$$\mathbb{E}_{\mathbb{P}} [U(X+Y) | \mathcal{G}] \geq B.$$

Suppose $\sum_{i \in I} Y^i \in L^0(\mathcal{G})$ for some family of indexes $I \subseteq \{1, \dots, N\}$. Then $\sum_{i \in I} Y^i \in L^\infty(\mathcal{G})$.

Proof. Set $A_H := \{\sum_{i \in I} Y^i < -H\} \in \mathcal{G}$ for $H > 0$ and suppose $\mathbb{P}(A_H) > 0$ for all $H > 0$. Then we have by Lemma A.6 (i) and $\mathbb{E}_{\mathbb{P}} [U(X+Y)|\mathcal{G}] \geq B$ that

$$B1_{A_H} \leq a \left(\sum_{j=1}^N X^j + \sum_{j=1}^N Y^j \right) 1_{A_H} - a \left(\sum_{j=1}^N (X^j + Y^j)^- \right) 1_{A_H} + b1_{A_H}. \quad (65)$$

At the same time from $\sum_{j=1}^N Y^j \in L^\infty(\mathcal{G})$ we must have for some index $k \in I$ (depending on H) that $A_H^k := \{Y^k < -\frac{1}{N+1}H\} \cap A_H \subseteq A_H$ satisfies $\mathbb{P}(A_H^k) > 0$ (otherwise we would get that $\sum_{i \in I} Y^i \geq -\frac{N}{N+1}H$ on A_H , which is a contradiction). From (65) and H big enough we also have

$$\begin{aligned} B1_{A_H^k} &\leq a \left(\sum_{j=1}^N X^j + \sum_{j=1}^N Y^j \right) 1_{A_H^k} + a(-(X^k + Y^k)^-) 1_{A_H^k} + b1_{A_H^k} \\ &\leq a \left(\sum_{j=1}^N X^j + \sum_{j=1}^N Y^j \right) 1_{A_H^k} + a(-(\|X^k\|_\infty + Y^k)^-) 1_{A_H^k} + b1_{A_H^k} \\ &\leq a \left(\left\| \sum_{j=1}^N X^j + \sum_{j=1}^N Y^j \right\|_\infty \right) 1_{A_H^k} + a \left(\|X^k\|_\infty - \frac{H}{N+1} \right) 1_{A_H^k} + b1_{A_H^k}. \end{aligned}$$

As B is bounded, for an even bigger H in this inequality, we get a contradiction.

Now set $B_H := \{\sum_{i \in I} Y^i > H\}$. Assume that $\mathbb{P}(B_H) > 0$ for all $H > 0$. Then from $\sum_{j=1}^N Y^j \in L^\infty(\mathcal{G})$ we get that

$$\mathbb{P} \left(\left\{ \sum_{i \notin I} Y^i < \sum_{j=1}^N Y^j - H \right\} \right) > 0.$$

The argument in the first part of the proof can then be replicated, since $\sum_{i \notin I} Y^i \in L^0(\mathcal{G})$, yielding a contradiction. \square

A.2 Proofs for Section 2

Proof of Theorem 2.4. In Item i) we consider $L_{\mathcal{F}} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $L^* = (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$. We denote by ba_1 the set of N -dimensional vectors of finitely additive functionals on \mathcal{F} taking values in $[0, 1]$ and taking value 1 on Ω . Applying the Namioka-Klee Theorem in [13] together with a standard argument regarding the monetary property, we see that

$$\rho_0(X) = \max_{\mu \in \text{ba}_1} \left(\sum_{j=1}^N \mu^j(-X^j) - \rho_0^*(-\mu) \right). \quad (66)$$

We now follow the lines of [38], Theorem 4.22 and Lemma 4.23.

Take any optimum $\hat{\mu} = [\hat{\mu}^1, \dots, \hat{\mu}^N]$ in the dual representation (66), so that $\rho_0^*(-\hat{\mu}) < +\infty$, and select a real number $c > -\rho_0(0)$ big enough so that $c \geq \rho_0^*(-\hat{\mu})$. Take any sequence of sets $(A_n)_n$ in \mathcal{F} increasing to Ω . We claim that $\hat{\mu}^k(A_n) \rightarrow_n 1$ for all $k \in \{1, \dots, N\}$, which allows us to conclude that each $\hat{\mu}^k$ is σ -additive. Hence any optimum of (66) belongs to \mathcal{Q} , which as a consequence can replace ba_1 in (66), obtaining thus the thesis. To prove the claim fix any $k \in \{1, \dots, N\}$. By definition of ρ_0^*

$$c \geq \rho_0^*(-\hat{\mu}) \geq \hat{\mu}(-\lambda 1_{A_n} e^k) - \rho_0(\lambda 1_{A_n} e^k),$$

which implies

$$\hat{\mu}(1_{A_n} e^k) = \hat{\mu}^k(A_n) \geq \frac{1}{\lambda} (-c - \rho_0(\lambda 1_{A_n} e^k)).$$

Now using continuity from below of ρ_0 we deduce that for each $\lambda > 0$:

$$\liminf_n (\hat{\mu}^k(A_n)) \geq \lim_n \frac{1}{\lambda} (-c - \rho_0(\lambda 1_{A_n} e^k)) = \frac{1}{\lambda} (-c - \rho_0(\lambda e^k)) \stackrel{\text{Monetary prop.}}{=} 1 - \frac{c + \rho_0(0)}{\lambda}.$$

Letting $\lambda \rightarrow +\infty$ we see that

$$\hat{\mu}^k(A_n) \rightarrow_n 1.$$

The $\sigma((L^\infty(\mathcal{F}))^N, (L^1(\mathcal{F}))^N)$ -lower semicontinuity of ρ_0 follows directly from (14) and continuity from above is a consequence of the $\sigma((L^\infty(\mathcal{F}))^N, (L^1(\mathcal{F}))^N)$ -lower semicontinuity (see [13]).

The proof of Item ii) is a simple consequence of the Namioka-Klee Theorem in [13] applied to the dual system $(L_{\mathcal{F}}, L^*)$. In particular, the thesis follows from Lemma 7 [13] and the application of the monetary property. \square

A.3 Proofs for Section 3

Proof of Theorem 3.9. Set $\rho_0(X) := \mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(X)]$ and let

$$\gamma(\mathbb{Q}) := \text{ess sup}_{X \in L_{\mathcal{F}}} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[-X^j | \mathcal{G}] - \rho_{\mathcal{G}}(X) \right\} := \rho_{\mathcal{G}}^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right), \quad \mathbb{Q} \in \mathcal{Q}_{\mathcal{G}},$$

where $\rho_{\mathcal{G}}^*$ was introduced in (38). From the definition of $\gamma(\mathbb{Q})$ we immediately deduce

$$\rho_{\mathcal{G}}(X) \geq \text{ess sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[-X^j | \mathcal{G}] - \gamma(\mathbb{Q}) \right\} \geq \left\{ \sum_{j=1}^N \mathbb{E}_{\hat{\mathbb{Q}}^j}[-X^j | \mathcal{G}] - \gamma(\hat{\mathbb{Q}}) \right\} \quad (67)$$

for any $\hat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}$.

STEP 1: $\rho_{\mathcal{G}}$ has the local property, i.e. for any $A \in \mathcal{G}$ and $X \in L_{\mathcal{F}}$ $\rho_{\mathcal{G}}(X) 1_A = \rho_{\mathcal{G}}(1_A X) 1_A$.

Observe that

$$\begin{aligned} \rho_{\mathcal{G}}(1_A X) &\stackrel{(18)}{\leq} \rho_{\mathcal{G}}(X) 1_A + \rho_{\mathcal{G}}(0) 1_{A^c} \stackrel{(18)}{\leq} 1_A (\rho_{\mathcal{G}}(1_A X) 1_A + \rho_{\mathcal{G}}(X 1_{A^c}) 1_{A^c}) + \rho_{\mathcal{G}}(0) 1_{A^c} \\ &= \rho_{\mathcal{G}}(1_A X) 1_A + \rho_{\mathcal{G}}(0) 1_{A^c}, \end{aligned}$$

then multiply by 1_A . The local property of $\rho_{\mathcal{G}}$ is equivalent to:

$$\rho_{\mathcal{G}}(X 1_A + Z 1_{A^c}) = \rho_{\mathcal{G}}(X) 1_A + \rho_{\mathcal{G}}(Z) 1_{A^c} \quad (68)$$

if $A \in \mathcal{G}$ and $X, Z \in L_{\mathcal{F}}$.

STEP 2: for every $\mathbb{Q} \in \mathcal{Q}$ the set $\{\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \rho_{\mathcal{G}}(X), X \in L_{\mathcal{F}}\}$ is upward directed.

For $X, Z \in L_{\mathcal{F}}$ we set

$$\begin{aligned} \xi_X &:= \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \rho_{\mathcal{G}}(X), & \xi_Z &:= \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-Z^j | \mathcal{G}] - \rho_{\mathcal{G}}(Z), \\ A &:= \{\xi_X \geq \xi_Z\} \in \mathcal{G}, & W &:= X 1_A + Z 1_{A^c}. \end{aligned}$$

As explained in Remark 3.3, by the decomposability of $L_{\mathcal{F}}$ we get $W \in L_{\mathcal{F}}$. Then one can check, using (68), that

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-W^j | \mathcal{G}] - \rho_{\mathcal{G}}(W) = \xi_X 1_A + \xi_Z 1_{A^c} = \max(\xi_X, \xi_Z)$$

proving the claim.

STEP 3: For $\mathbb{Q} \in \mathcal{Q}$, $\gamma_0(\mathbb{Q}) = \mathbb{E}_{\mathbb{P}}[\gamma(\mathbb{Q})]$ where we set

$$\gamma_0(\mathbb{Q}) := \sup_{X \in L_{\mathcal{F}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j] - \rho_0(X) \right) = \rho_0^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right), \quad \mathbb{Q} \in \mathcal{Q}.$$

Recall that $\mathbb{Q}^j = \mathbb{P}$ on \mathcal{G} for all $\mathbb{Q} \in \mathcal{Q}$. By Step 2 we deduce:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[\gamma(\mathbb{Q})] &= \mathbb{E}_{\mathbb{P}} \left[\operatorname{ess\,sup}_{X \in L_{\mathcal{F}}} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \rho_{\mathcal{G}}(X) \right\} \right] \\ &= \sup_{X \in L_{\mathcal{F}}} \mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \rho_{\mathcal{G}}(X) \right] = \sup_{X \in L_{\mathcal{F}}} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [\mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}]] - \mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(X)] \right\} \\ &= \sup_{X \in L_{\mathcal{F}}} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j] - \rho_0(X) \right\} = \gamma_0(\mathbb{Q}). \end{aligned}$$

Notice that we have also shown: $\rho_0^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right) = \mathbb{E}_{\mathbb{P}} \left[\rho_{\mathcal{G}}^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right]$.

STEP 4 Dual Representation and attainment of the supremum.

As in the univariate case, applying the monetary property one may show that $\gamma_0(\mathbb{Q}) = +\infty$ if

$\mathbb{Q} \in \mathcal{Q} \setminus \mathcal{Q}_{\mathcal{G}}$. Then by nice representability of ρ_0

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(X)] &= \rho_0(X) = \max_{\mathbb{Q} \in \mathcal{Q}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j] - \gamma_0(\mathbb{Q}) \right) = \max_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j] - \gamma_0(\mathbb{Q}) \right) \\ &= \sup_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [\mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}]] - \mathbb{E}_{\mathbb{P}}[\gamma(\mathbb{Q})] \right) \\ &= \sup_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \gamma(\mathbb{Q}) \right] \leq \mathbb{E}_{\mathbb{P}} \left[\operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \gamma(\mathbb{Q}) \right) \right]. \end{aligned}$$

From this inequality and (67) we then deduce the dual representation:

$$\rho_{\mathcal{G}}(X) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \gamma(\mathbb{Q}) \right). \quad (69)$$

Similarly, for the vector $\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}$ attaining the maximum in $\rho_0(X)$ we have:

$$\mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(X)] = \mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [-X^j | \mathcal{G}] - \gamma(\widehat{\mathbb{Q}}) \right]$$

and then again from (67) we deduce

$$\rho_{\mathcal{G}}(X) = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [-X^j | \mathcal{G}] - \gamma(\widehat{\mathbb{Q}}).$$

STEP 5 We prove that $\alpha(\mathbb{Q}) = \gamma(\mathbb{Q})$ for all $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}$, where α is defined in (21). Then (22) follows from (69). Applying the monetary property of $\rho_{\mathcal{G}}$, for each $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}$

$$\begin{aligned} \gamma(\mathbb{Q}) &= \operatorname{ess\,sup}_{X \in L_{\mathcal{F}}} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \rho_{\mathcal{G}}(X) \right\} = \operatorname{ess\,sup}_{X \in L_{\mathcal{F}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[\left(-X^j - \frac{1}{N} \rho_{\mathcal{G}}(X) \right) | \mathcal{G} \right] \right) \\ &\leq \operatorname{ess\,sup}_{Z \in L_{\mathcal{F}}, \rho_{\mathcal{G}}(Z) \leq 0} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-Z^j | \mathcal{G}] \right) = \alpha(\mathbb{Q}) \\ &\leq \operatorname{ess\,sup}_{Z \in L_{\mathcal{F}}, \rho_{\mathcal{G}}(Z) \leq 0} \sum_{j=1}^N \left(\mathbb{E}_{\mathbb{Q}^j} [-Z^j | \mathcal{G}] - \frac{1}{N} \rho_{\mathcal{G}}(Z) \right) \leq \operatorname{ess\,sup}_{Z \in L_{\mathcal{F}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-Z^j | \mathcal{G}] - \rho_{\mathcal{G}}(Z) \right) = \gamma(\mathbb{Q}). \end{aligned}$$

STEP 6 Continuity from above of $\rho_{\mathcal{G}}$

Continuity from above is an easy consequence of the dual representation (22) just proved. Take $X_n \downarrow X \Leftrightarrow -X_n \uparrow -X$ and observe that:

$$\begin{aligned} \rho_{\mathcal{G}}(X) &= \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \alpha(\mathbb{Q}) \right) \\ &\stackrel{\text{(cMON)}}{=} \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\operatorname{ess\,sup}_n \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X_n^j | \mathcal{G}] - \alpha(\mathbb{Q}) \right) \end{aligned}$$

$$\begin{aligned}
& \stackrel{\text{Prop. A.5}}{=} \operatorname{ess\,sup}_n \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X_n^j | \mathcal{G}] - \alpha(\mathbb{Q}) \right) \\
& = \sup_n \rho_{\mathcal{G}}(X_n) = \lim_n \rho_{\mathcal{G}}(X_n) \leq \rho_{\mathcal{G}}(X).
\end{aligned}$$

□

A.4 Proofs for Section 5

The proof of Proposition 5.13 needs some preparation. Recall the definition of $\rho_{\mathcal{G}}^*$ in (38).

Proposition A.10. *There exists an extension, ρ_0^{Φ} , of $\rho_0(\cdot) := \mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(\cdot)]$ to M^{Φ} which is convex, nondecreasing and $\|\cdot\|_{\Phi}$ -continuous.*

Proof. Observe that because of the downward directness proved in STEP 1 of Claim 5.5, together with (MON), we have

$$\rho_0(X) = \inf \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [Y^j] \mid Y \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}} [U(X+Y) | \mathcal{G}] \geq B \right\} \quad X \in (L^{\infty}(\mathcal{F}))^N.$$

Define now

$$\rho_0^{\Phi}(X) := \inf \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [Y^j] \mid Y \in \mathcal{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}} [U(X+Y) | \mathcal{G}] \geq B \right\} \quad X \in M^{\Phi}. \quad (70)$$

We easily see that $\rho_0^{\Phi}(X) < +\infty$ for every $X \in M^{\Phi}$ (since the set over which we take infima in (70) is nonempty). Moreover $\rho_0^{\Phi}(X) > -\infty$ since if this were the case, for a minimizing sequence $(Y_n)_n$ we would have $\inf_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [Y_n^j] = -\infty$. Now, by Lemma A.6.(ii)

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}} [B] & \leq \mathbb{E}_{\mathbb{P}} [U(X+Y_n)] \\
& \leq a \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [X^j] + a \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [Y_n^j] + b \leq \text{const} + a \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [Y_n^j],
\end{aligned}$$

which gives a contradiction. Clearly, mimicking what we did in the proof of Claim 5.5 Step 3, we can check that $\rho_0^{\Phi}(\cdot)$ is also convex and nondecreasing. Now by the Extended Namioka-Klee Theorem in [13]) it is also norm continuous. □

Lemma A.11. *For any $Z \in (L^1(\mathcal{F}))^N$ we have that $\rho_{\mathcal{G}}^*(Z) \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ if and only if $\rho_0^*(Z) < +\infty$ and, if any of the two conditions is met, we have $Z \in K_{\Phi}$.*

Proof. Notice first that, for any $Z \in (L^1(\mathcal{F}))^N$, $\rho_{\mathcal{G}}^*(Z) \geq -\rho_{\mathcal{G}}(0) \in L^{\infty}(\Omega, \mathcal{G}, \mathbb{P})$ and $\rho_0^*(Z) \geq -\rho_0(0) > -\infty$. As in STEP 3 of Theorem 3.9, one can show that $\rho_0^*(Z) = \mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}^*(Z)]$ for any $Z \in (L^1(\mathcal{F}))^N$. The first claim then follows. Suppose now $\rho_{\mathcal{G}}^*(Z) \in L^1(\mathcal{G})$. By [3] Theorem 5.43 Item 3 ρ_0^{Φ} is bounded on a ball B_{ε} (defined using the norm $\|\cdot\|_{\Phi}$) of M^{Φ} centered at 0. We have as a consequence

$$+\infty > \sup_{X \in B_{\varepsilon}} (\rho_0^{\Phi}(X) + \rho_0^*(Z)).$$

Now we use the fact that ρ_0^Φ , when restricted to $(L^\infty(\mathcal{F}))^N$, coincides with ρ_0 , and continuity of ρ_0^Φ (by Proposition A.10) to see that

$$\begin{aligned} +\infty &> \sup_{X \in B_\varepsilon} (\rho_0^\Phi(X) + \rho_0^*(Z)) \geq \sup_{X \in B_\varepsilon \cap (L^\infty(\mathcal{F}))^N} (\rho_0^\Phi(X) + \rho_0^*(Z)) \\ &= \sup_{X \in B_\varepsilon \cap (L^\infty(\mathcal{F}))^N} (\rho_0(X) + \rho_0^*(Z)) \geq \sup_{X \in B_\varepsilon \cap (L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [X^j Z^j] \right) \end{aligned}$$

where we used Fenchel inequality to obtain the last inequality. Furthermore, using the fact that given Z , for any $X \in B_\varepsilon \cap (L^\infty(\mathcal{F}))^N$ the vector \widehat{X} defined by $\widehat{X}^j = \text{sgn}(Z^j) |X^j|$ still belongs to $B_\varepsilon \cap (L^\infty(\mathcal{F}))^N$, we have

$$\sup_{X \in B_\varepsilon \cap (L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [X^j Z^j] \right) = \sup_{X \in B_\varepsilon \cap (L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [|X^j Z^j|] \right).$$

To conclude, we observe that an approximation with simple functions yields:

$$\sup_{X \in B_\varepsilon \cap (L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [|X^j Z^j|] \right) = \sup_{X \in B_\varepsilon} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}} [|X^j Z^j|] \right).$$

This completes the proof using [23] Proposition 2.5 Item 1. \square

Lemma A.12. *Suppose Assumption 5.12 holds. Let $Z \in (L^1(\mathcal{F}))^N$ be given and suppose that $\mathbb{E}_{\mathbb{P}} [U(Z)|\mathcal{G}] \geq B$. Then for any $W \in K_\Phi$ we have $\sum_{j=1}^N (Z^j)^- W^j \in L^1(\mathcal{F})$.*

Proof. Observe that $\mathbb{E}_{\mathbb{P}} [U(Z)] = \mathbb{E}_{\mathbb{P}} [\mathbb{E}_{\mathbb{P}} [U(Z)|\mathcal{G}]] \geq \mathbb{E}_{\mathbb{P}} [B]$. Furthermore

$$U(Z) = \sum_{j=1}^N u_j(Z^j) + \Lambda(Z) = \sum_{j=1}^N u_j((Z^j)^+) + \sum_{j=1}^N u_j(-(Z^j)^-) + \Lambda(Z).$$

As $u_j(0) = 0$ and u_j is increasing, for each j , this implies

$$0 \leq - \sum_{j=1}^N u_j(-(Z^j)^-) \leq \max_{j=1, \dots, N} \left(\frac{du_j}{dx^j}(0) \right) \sum_{j=1}^N (Z^j)^+ + \sup_{z \in \mathbb{R}^N} \Lambda(z) - U(Z)$$

where in the last inequality we used (24). It then follows that $\sum_{j=1}^N (-u_j(-(Z^j)^-)) \in L^1(\mathcal{F})$, which in turns yields $(Z)^- \in L^{\phi_1} \times \dots \times L^{\phi_N}$, for ϕ_j defined in (44). Since by [23] Proposition 2.5 Item 3 we have $W \in L^{\phi_1^*} \times \dots \times L^{\phi_N^*}$, we get by [26] Proposition 2.2.7 that $(Z^j)^- W^j \in L^1(\mathcal{F})$ for every $j = 1, \dots, N$, and the last claim is proved. \square

Proof of Proposition 5.13. Observe that $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ implies $\alpha^1(\mathbb{Q}) \in L^1(\mathcal{G})$ by definition of $\mathcal{Q}_{\mathcal{G}}^1$, which in turns implies that $\rho_{\mathcal{G}}^* \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \in L^1(\mathcal{G})$ by Claim 5.8. By Lemma A.11, then, for any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ we have $\frac{d\mathbb{Q}}{d\mathbb{P}} \in K_\Phi$, so that, by Lemma A.12, for any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ we get $[(\widehat{Y}^1)^-, \dots, (\widehat{Y}^N)^-] \in L^1(\mathbb{Q}^1) \times \dots \times L^1(\mathbb{Q}^N)$. Given \widehat{Y} , we use the notation $\widehat{Y}_{(k)}$ and Z_Y from (16) for large values

$k \geq k_{\widehat{Y}}$. By Fatou Lemma we have for any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$

$$\begin{aligned} \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}^j)^+ \middle| \mathcal{G} \right] &\leq \liminf_k \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}_{(k)}^j)^+ \middle| \mathcal{G} \right] \\ &= \liminf_k \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[\widehat{Y}_{(k)}^j \middle| \mathcal{G} \right] + \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}_{(k)}^j)^- \middle| \mathcal{G} \right] \right) \\ &\stackrel{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}}{\leq} \liminf_k \left(\sum_{j=1}^N \widehat{Y}_{(k)}^j + \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}_{(k)}^j)^- \middle| \mathcal{G} \right] \right). \end{aligned}$$

We conclude that

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}^j)^+ \middle| \mathcal{G} \right] \leq \sum_{j=1}^N \widehat{Y}^j + \lim_k \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}_{(k)}^j)^- \middle| \mathcal{G} \right] \stackrel{\text{(cDOM)}}{=} \sum_{j=1}^N \widehat{Y}^j + \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}^j)^- \middle| \mathcal{G} \right]$$

where we used in the last step that $Y_{(k)} \rightarrow \widehat{Y} \mathbb{P}$ - a.s. and that $(\widehat{Y})^- \in L^1(\mathbb{Q}^1) \times \dots \times L^1(\mathbb{Q}^N)$ to apply (cDOM): $(\widehat{Y}_{(k)}^j)^- \leq \max \left((\widehat{Y}^j)^-, (Z_{Y^j}^j)^- \right) \in L^1(\mathbb{Q}^j)$, $j = 1, \dots, N$. This yields both integrability and the fact that, rearranging terms,

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[\widehat{Y}^j \middle| \mathcal{G} \right] \leq \sum_{j=1}^N \widehat{Y}^j.$$

□

To prove Theorem 5.14 we first state two preliminary Propositions.

Proposition A.13. *Let $X \in (L^\infty(\mathcal{F}, \mathbb{P}))^N$. For any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ we have:*

$$\rho_{\mathcal{G}}^{\mathbb{Q}}(X) = \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \alpha^1(\mathbb{Q}) \quad (71)$$

and

$$\rho_{\mathcal{G}}(X) = \max_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1} \rho_{\mathcal{G}}^{\mathbb{Q}}(X) = \rho_{\widehat{\mathbb{Q}}}^{\widehat{\mathbb{Q}}}(X) \text{ for any optimum } \widehat{\mathbb{Q}} \text{ of (33)}. \quad (72)$$

Proof. We first prove (71): observe that by (31) we have for any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$

$$\begin{aligned} \alpha^1(\mathbb{Q}) &= \text{ess sup} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-W^j | \mathcal{G}] \mid W \in (L^\infty(\mathcal{F}, \mathbb{P}))^N, \mathbb{E}_{\mathbb{P}} [U(W) | \mathcal{G}] \geq B \right\} \\ &= - \text{ess inf}_{\substack{W-X \in (L^\infty(\mathcal{F}, \mathbb{P}))^N \\ \mathbb{E}_{\mathbb{P}} [U(X+(W-X)) | \mathcal{G}] \geq B}} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [(W^j - X^j) | \mathcal{G}] + \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [X^j | \mathcal{G}] \right\} \\ &= \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \text{ess inf} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Z^j | \mathcal{G}] \mid Z \in (L^\infty(\mathcal{F}, \mathbb{P}))^N, \mathbb{E}_{\mathbb{P}} [U(X+Z) | \mathcal{G}] \geq B \right\} \\ &= \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \rho_{\mathcal{G}}^{\mathbb{Q}}(X). \end{aligned}$$

Observe now that by Theorem 5.4 Item 3 $\rho_{\mathcal{G}}(X) \geq \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [-X^j | \mathcal{G}] - \alpha^1(\mathbb{Q})$ for every $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$, and equality holds for any optimum $\widehat{\mathbb{Q}}$ of (33). Direct substitution yields then (72). □

Proposition A.14. *Suppose Assumption 5.12 is fulfilled and let $X \in (L^\infty(\mathcal{F}, \mathbb{P}))^N$. Then for any $\mathbb{Q} \in \mathcal{Q}_G^1$ we have:*

$$\rho_G^{\mathbb{Q}}(X) = \text{ess inf} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y^j | \mathcal{G}] \mid Y \in L^1(\mathcal{F}, \mathbb{Q}) \cap (L^1(\mathcal{F}, \mathbb{P}))^N, \mathbb{E}_{\mathbb{P}} [U(X+Y) | \mathcal{G}] \geq B \right\}. \quad (73)$$

Proof. Clearly the inequality (\geq) is trivial, since we are enlarging the set over which we take the essential infimum. As to the converse (\leq), observe that whenever $Y \in L^1(\mathcal{F}, \mathbb{Q}) \cap (L^1(\mathcal{F}, \mathbb{P}))^N$ is given with $\mathbb{E}_{\mathbb{P}} [U(X+Y) | \mathcal{G}] \geq B$, then, for any $\varepsilon > 0$, $\mathbb{E}_{\mathbb{P}} [U(X+Y+\varepsilon \mathbf{1}) | \mathcal{G}] > B$ \mathbb{P} -a.s. by strict monotonicity of U . Hence, given $Y_{(k)}$ as in (16), $k \geq k_Y$, defining

$$\Gamma_K := \bigcap_{k \geq K} \{ \mathbb{E}_{\mathbb{P}} [U(X+Y_{(k)}+\varepsilon \mathbf{1}) | \mathcal{G}] \geq B \} \in \mathcal{G}$$

we have that $\Gamma_K \subseteq \Gamma_{K+1}$ and $\mathbb{P}(\cup_K \Gamma_K) = 1$. The argument is now similar to the one in the proof of Claim 5.6. As a consequence, we have

$$1_{\Gamma_K^c} = 0 \text{ definitely in } K \text{ } \mathbb{P}\text{-a.s.} \quad (74)$$

For each K , take a vector $\alpha_K \in \mathbb{R}^N$ such that

$$U(-\|X\|_\infty - \|Y_{(K)}\|_\infty + \varepsilon \mathbf{1} + \alpha_K) \geq \text{ess sup}(B)$$

where the notation for the vectors $\|X\|_\infty, \|Y_{(K)}\|_\infty$ is the same used in the proof of Claim 5.6 and define

$$Z_K := Y_{(K)} + \varepsilon \mathbf{1} + \alpha_K 1_{\Gamma_K^c} \in (L^\infty(\mathcal{F}, \mathbb{P}))^N.$$

Then clearly

$$\mathbb{E}_{\mathbb{P}} [U(X+Y_{(K)}+\varepsilon \mathbf{1}) | \mathcal{G}] 1_{\Gamma_K} + \mathbb{E}_{\mathbb{P}} [U(Y_{(K)}+\varepsilon \mathbf{1} + \alpha_K 1_{\Gamma_K^c}) | \mathcal{G}] 1_{\Gamma_K^c} \geq B$$

which implies $\mathbb{E}_{\mathbb{P}} [U(X+Z_K) | \mathcal{G}] \geq B$. Hence

$$\begin{aligned} \rho_G^{\mathbb{Q}}(X) &\leq \liminf_K \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Z_K^j | \mathcal{G}] \\ &= \liminf_K \left(\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y_{(K)}^j | \mathcal{G}] \right) 1_{\Gamma_K} + \left(\sum_{j=1}^N \alpha_K^j \right) 1_{\Gamma_K^c} \right) + N\varepsilon \\ &= \lim_K \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y_{(K)}^j | \mathcal{G}] + \lim_K \left(\sum_{j=1}^N \alpha_K^j 1_{\Gamma_K^c} \right) + N\varepsilon \\ &\stackrel{(74)}{=} \stackrel{(cDOM)}{=} \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} [Y^j | \mathcal{G}] + N\varepsilon. \end{aligned}$$

□

Proof of Theorem 5.14. By Proposition 5.13 we have that $\widehat{Y} \in (L^1(\mathcal{F}, \mathbb{P}))^N \cap \bigcap_{\mathbb{Q} \in \mathcal{Q}_G^1} L^1(\mathcal{F}, \mathbb{Q})$.

We also know that $\mathbb{E}_{\mathbb{P}} [U(X+\widehat{Y}) | \mathcal{G}] \geq B$. Hence we have

$$\rho_G^{\widehat{\mathbb{Q}}}(X) \stackrel{(73)}{\leq} \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [\widehat{Y}^j | \mathcal{G}] \stackrel{\text{Prop. 5.13}}{\leq} \sum_{j=1}^N \widehat{Y}^j.$$

Moreover, by optimality of \widehat{Y} for $\rho_{\mathcal{G}}(X)$ and (72), we have $\sum_{j=1}^N \widehat{Y}^j = \rho_{\mathcal{G}}(X) = \rho_{\widehat{\mathcal{G}}}^{\widehat{\mathbb{Q}}}(X)$. We then conclude jointly optimality in the extended sense of \widehat{Y} for $\rho_{\widehat{\mathcal{G}}}^{\widehat{\mathbb{Q}}}(X)$ and (48). \square

A.5 Proofs for Section 6

A.5.1 Proof of Theorem 6.1

Let us rename

$$H_{\mathcal{G}}(X) := \beta \log \left(-\frac{\beta}{B} \mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\overline{X}}{\beta} \right) \middle| \mathcal{G} \right] \right) - A \in L^{\infty}(\mathcal{G}). \quad (75)$$

We consider \widehat{Y} and $\widehat{\mathbb{Q}}$ assigned in (50) and (51). One immediately checks that $\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}$. As all the components $\widehat{\mathbb{Q}}^j$ of $\widehat{\mathbb{Q}}$ are all equal, to prove that $\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}^1$ it is sufficient to show that $\alpha^1(\widehat{\mathbb{Q}}) \in L^1(\mathcal{G})$.

Let

$$V(y) := \sup_{x \in \mathbb{R}^N} \left\{ U(x) - \sum_{j=1}^N x^j y^j \right\} = \sum_{j=1}^N \left(\frac{y^j}{\alpha_j} \log \left(\frac{y^j}{\alpha_j} \right) - \frac{y^j}{\alpha_j} \right), \quad y \in (0, +\infty)^N,$$

be the the convex conjugate of U and let $\lambda \in L^{\infty}(\mathcal{G})$ with $\lambda \geq c$ \mathbb{P} -a.s. for some constant $c \in (0, +\infty)$. Then the Fenchel inequality $U(W) - V\left(\frac{1}{\lambda} \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right) \leq \sum_{j=1}^N W^j \frac{1}{\lambda} \frac{d\widehat{\mathbb{Q}}^j}{d\mathbb{P}}$ holds \mathbb{P} -a.s. for any $W \in (L^{\infty}(\mathcal{F}))^N$. Take any $j \in \{1, \dots, N\}$ and set $I_{\mathcal{G}}(\widehat{\mathbb{Q}}, \mathbb{P}) := \mathbb{E}_{\mathbb{P}} \left[\frac{d\widehat{\mathbb{Q}}^j}{d\mathbb{P}} \log \left(\frac{d\widehat{\mathbb{Q}}^j}{d\mathbb{P}} \right) \middle| \mathcal{G} \right]$. Then

$$\begin{aligned} \alpha^1(\widehat{\mathbb{Q}}) &= \operatorname{ess\,sup}_{\substack{W \in (L^{\infty}(\mathcal{F}))^N \\ \mathbb{E}_{\mathbb{P}}[U(W)|\mathcal{G}] \geq B}} \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [-W^j | \mathcal{G}] \stackrel{\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}}{=} \operatorname{ess\,sup}_{\substack{W \in (L^{\infty}(\mathcal{F}))^N \\ \mathbb{E}_{\mathbb{P}}[U(W)|\mathcal{G}] \geq B}} \sum_{j=1}^N \lambda \mathbb{E}_{\mathbb{P}} \left[-W^j \left(\frac{1}{\lambda} \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \right) \middle| \mathcal{G} \right] \\ &\leq \operatorname{ess\,sup}_{\substack{W \in (L^{\infty}(\mathcal{F}))^N \\ \mathbb{E}_{\mathbb{P}}[U(W)|\mathcal{G}] \geq B}} \left(\lambda \mathbb{E}_{\mathbb{P}} \left[V \left(\frac{1}{\lambda} \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \right) \middle| \mathcal{G} \right] - \lambda \mathbb{E}_{\mathbb{P}} [U(W)|\mathcal{G}] \right) \leq \lambda \mathbb{E}_{\mathbb{P}} \left[V \left(\frac{1}{\lambda} \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \right) \middle| \mathcal{G} \right] - \lambda B \\ &= \sum_{j=1}^N \frac{1}{\alpha_j} \log \left(\frac{1}{\alpha_j} \right) - \sum_{j=1}^N \frac{1}{\alpha_j} + \sum_{j=1}^N \frac{1}{\alpha_j} I_{\mathcal{G}}(\widehat{\mathbb{Q}}, \mathbb{P}) + \sum_{j=1}^N \frac{1}{\alpha_j} \log \left(\frac{1}{\lambda} \right) - \lambda B \\ &= A - \beta + \beta I_{\mathcal{G}}(\widehat{\mathbb{Q}}, \mathbb{P}) + \beta \log \left(\frac{1}{\lambda} \right) - \lambda B \\ &= A + \beta I_{\mathcal{G}}(\widehat{\mathbb{Q}}, \mathbb{P}) + \beta \log \left(-\frac{B}{\beta} \right) \quad \text{if } \lambda := -\frac{\beta}{B} \\ &= \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [-X^j | \mathcal{G}] - H_{\mathcal{G}}(X), \end{aligned}$$

where the last equality is obtained by direct computation using (51) and (75). Hence $\alpha^1(\widehat{\mathbb{Q}}) \in L^1(\mathcal{G})$, $\widehat{\mathbb{Q}} \in \mathcal{Q}_{\mathcal{G}}^1$ and

$$H_{\mathcal{G}}(X) \leq \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j} [-X^j | \mathcal{G}] - \alpha^1(\widehat{\mathbb{Q}}). \quad (76)$$

Notice that when $\mathcal{B}_G = \mathcal{D}_G$ then $\mathcal{C}_G \cap (L^\infty(\mathcal{F}))^N = \mathcal{D}_G \cap (L^\infty(\mathcal{F}))^N$. The Theorem is proved once we show the following chain of inequalities:

$$\rho_G(X) = \rho_G^\infty(X) := \text{ess inf} \left\{ \sum_{j=1}^N Y^j \mid Y \in \mathcal{D}_G \cap (L^\infty(\mathcal{F}))^N, \mathbb{E}_\mathbb{P}[U(X+Y)|\mathcal{G}] \geq B \right\} \quad (77)$$

$$\leq \sum_{j=1}^N \widehat{Y}^j = H_G(X) \quad (78)$$

$$\leq \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[-X^j|\mathcal{G}] - \alpha^1(\widehat{\mathbb{Q}}) \leq \text{ess sup}_{\mathbb{Q} \in \mathcal{Q}_G^1} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[-X^j|\mathcal{G}] - \alpha^1(\mathbb{Q}) \right) = \rho_G(X). \quad (79)$$

The equalities in (77) and the last equality in (79) follow from Theorem 5.4. By direct computation, \widehat{Y} satisfies: $\widehat{Y} \in (L^\infty(\mathcal{F}))^N$, $\sum_{j=1}^N \mathbb{E}_\mathbb{P}[-\exp(-\alpha_j(X^j + \widehat{Y}^j))|\mathcal{G}] = B$, $\sum_{j=1}^N \widehat{Y}^j = H_G(X) \in L^\infty(\mathcal{G})$. Hence \widehat{Y} satisfies the constraints of $\rho_G^\infty(X)$, which proves the inequality (and the equality) in (78). The first inequality in (79) is shown in (76), while the second one is a direct consequence of $\widehat{\mathbb{Q}} \in \mathcal{Q}_G^1$.

A.5.2 Proof of Theorem 6.3

Equation (56): we start observing that a straightforward computation yields

$$\widehat{Y}^k(\mathcal{G}, X) = \widehat{Y}^k(\mathcal{H}, X) + \frac{1}{\beta\alpha_k} (\rho_G(X) - \rho_{\mathcal{H}}(X)) \quad \forall k = 1, \dots, N. \quad (80)$$

We also have, recalling $\sum_{j=1}^N \widehat{Y}^j(\mathcal{G}, X) = \rho_G(X)$ and fixing k , that

$$\begin{aligned} \widehat{Y}^k(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)) &= \widehat{Y}^k(\mathcal{G}, X) + \frac{1}{\beta\alpha_k} \left(-\rho_G(X) + \rho_{\mathcal{H}}(-\widehat{Y}(\mathcal{G}, X)) \right) + \frac{1}{\beta\alpha_k} A - A_k \\ &\stackrel{\text{Eq.(80)}}{=} \widehat{Y}^k(\mathcal{H}, X) + \frac{1}{\beta\alpha_k} (-\rho_{\mathcal{H}}(X)) + \\ &\quad \frac{1}{\beta\alpha_k} \left(\rho_{\mathcal{H}}(-\widehat{Y}(\mathcal{G}, X)) - \rho_{\mathcal{H}}(0) \right) + \frac{1}{\beta\alpha_k} (\rho_{\mathcal{H}}(0) + A) - A_k. \end{aligned}$$

It is then enough to show that $\rho_{\mathcal{H}}(-\widehat{Y}(\mathcal{G}, X)) = \rho_{\mathcal{H}}(X) + \rho_{\mathcal{H}}(0)$, since $\widehat{Y}^k(\mathcal{H}, 0) = \frac{1}{\beta\alpha_k} (\rho_{\mathcal{H}}(0) + A) - A_k$. A direct computation yields

$$\begin{aligned} \rho_{\mathcal{H}}(-\widehat{Y}(\mathcal{G}, X)) &= \beta \log \left(-\frac{\beta}{B} \right) - A + \beta \log \left(\mathbb{E}_\mathbb{P} \left[\exp \left(-\frac{1}{\beta} \left(-\sum_{j=1}^N \widehat{Y}^j(\mathcal{G}, X) \right) \right) \middle| \mathcal{H} \right] \right) \\ &\stackrel{\text{Eq.(55)}}{=} \beta \log \left(-\frac{\beta}{B} \right) - A + \beta \log \left(-\frac{A}{\beta} \right) + \\ &\quad + \beta \log \left(\mathbb{E}_\mathbb{P} \left[\exp \left(\frac{\beta}{\beta} \log \left(-\frac{\beta}{B} \mathbb{E}_\mathbb{P} \left[\exp \left(-\frac{1}{\beta} \bar{X} \right) \middle| \mathcal{G} \right] \right) \right) \middle| \mathcal{H} \right] \right) \\ &= \rho_{\mathcal{H}}(0) - A + \beta \log \left(-\frac{\beta}{B} \mathbb{E}_\mathbb{P} \left[\mathbb{E}_\mathbb{P} \left[\exp \left(-\frac{1}{\beta} \bar{X} \right) \middle| \mathcal{G} \right] \middle| \mathcal{H} \right] \right). \end{aligned}$$

Hence we have

$$\rho_{\mathcal{H}}(-\widehat{Y}(\mathcal{G}, X)) = \rho_{\mathcal{H}}(0) + \rho_{\mathcal{H}}(X). \quad (81)$$

Equation (57): we have by (53) and using (55) that

$$\frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{G}, X) \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)) = \frac{\exp\left(-\frac{\bar{X}}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\bar{X}}{\beta}\right)\middle|\mathcal{G}\right]} \frac{\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right)\middle|\mathcal{H}\right]}. \quad (82)$$

We now see, just using (55), that

$$\begin{aligned} \exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right) &= \left(-\frac{\beta}{B}\right) \exp\left(-\frac{A}{\beta}\right) \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\bar{X}}{\beta}\right)\middle|\mathcal{G}\right], \\ \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right)\middle|\mathcal{H}\right] &= \left(-\frac{\beta}{B}\right) \exp\left(-\frac{A}{\beta}\right) \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\bar{X}}{\beta}\right)\middle|\mathcal{H}\right]. \end{aligned}$$

Direct substitution in (82) yields

$$\frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{G}, X) \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)) = \frac{\exp\left(-\frac{\bar{X}}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\bar{X}}{\beta}\right)\middle|\mathcal{H}\right]} \stackrel{\text{Eq. (53)}}{=} \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, X).$$

Equation (58): by definition (54) and using the fact that

$$\mathbb{E}_{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X))\middle|\mathcal{H}\right] = 1 \quad \forall k = 1, \dots, N$$

we have

$$\widehat{a}^k(\mathcal{H}, -\widehat{a}(\mathcal{G}, X)) = \mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^k(\mathcal{H}, -\widehat{a}(\mathcal{G}, X)) \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\middle|\mathcal{H}\right] = E + F + G + H$$

where

$$\begin{aligned} E &:= \mathbb{E}_{\mathbb{P}}\left[-(-\widehat{a}^k(\mathcal{G}, X)) \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\middle|\mathcal{H}\right], \\ F &:= \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\beta\alpha_k} \sum_{j=1}^N (-\widehat{a}^j(\mathcal{G}, X)) \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\middle|\mathcal{H}\right], \\ G &:= \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\beta\alpha_k} \rho_{\mathcal{H}}(-\widehat{a}(\mathcal{G}, X)) \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\middle|\mathcal{H}\right], \\ H &:= \mathbb{E}_{\mathbb{P}}\left[\left(\frac{1}{\beta\alpha_k} A - A_k\right) \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\middle|\mathcal{H}\right] = \frac{1}{\beta\alpha_k} A - A_k. \end{aligned}$$

We now work separately on each of the above random variables:

- considering E , by (54), observing that $\frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X)) \in L^\infty(\mathcal{G})$ and using the fact that $\rho_{\mathcal{H}}(X) \in L^\infty(\mathcal{H})$ we get

$$E = \widehat{a}^k(\mathcal{H}, X) + \frac{1}{\beta\alpha_k} \mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}(X) \frac{d\widehat{\mathbb{Q}}^k}{d\mathbb{P}}(\mathcal{H}, X)\middle|\mathcal{H}\right] - \frac{1}{\beta\alpha_k} \rho_{\mathcal{H}}(X). \quad (83)$$

- We now move to F . First, computing $\frac{d\widehat{Q}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))$ we get

$$\frac{d\widehat{Q}^k}{d\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X)) = \frac{\mathbb{E}_{\mathbb{P}} \left[\exp\left(-\frac{\overline{X}}{\beta}\right) \middle| \mathcal{G} \right]}{\mathbb{E}_{\mathbb{P}} \left[\exp\left(-\frac{\overline{X}}{\beta}\right) \middle| \mathcal{H} \right]} . \quad (84)$$

After some additional tedious computation we obtain

$$F = -\frac{1}{\beta\alpha_k} \mathbb{E}_{\mathbb{P}} \left[\rho_{\mathcal{G}}(X) \frac{d\widehat{Q}^k}{d\mathbb{P}}(\mathcal{H}, X) \middle| \mathcal{H} \right] . \quad (85)$$

- To compute G , we first see that $\rho_{\mathcal{H}}(-\widehat{a}(\mathcal{G}, X)) = \rho_{\mathcal{H}}(-\widehat{Y}(\mathcal{G}, X))$. We can thus exploit (81) to see that $\rho_{\mathcal{H}}(-\widehat{a}(\mathcal{G}, X)) = \rho_{\mathcal{G}}(0) + \rho_{\mathcal{H}}(X)$. Using also (84) we get

$$G = \frac{1}{\beta\alpha_k} \left(\rho_{\mathcal{H}}(0) + \mathbb{E}_{\mathbb{P}} \left[\rho_{\mathcal{H}}(X) \frac{\mathbb{E}_{\mathbb{P}} \left[\exp\left(-\frac{\overline{X}}{\beta}\right) \middle| \mathcal{G} \right]}{\mathbb{E}_{\mathbb{P}} \left[\exp\left(-\frac{\overline{X}}{\beta}\right) \middle| \mathcal{H} \right]} \middle| \mathcal{H} \right] \right) \quad (86)$$

$$\stackrel{\rho_{\mathcal{H}}(X) \in L^\infty(\mathcal{H})}{=} \frac{1}{\beta\alpha_k} (\rho_{\mathcal{H}}(0) + \rho_{\mathcal{H}}(X)) .$$

- Recalling (54) we have $\widehat{a}^k(\mathcal{H}, 0) = H + \frac{1}{\beta\alpha_k} \rho_{\mathcal{H}}(0)$ hence

$$H = \widehat{a}^k(\mathcal{H}, 0) - \frac{1}{\beta\alpha_k} \rho_{\mathcal{H}}(0) . \quad (87)$$

More detailed computations can be found in [22], proof of Theorem 3.5.8. Summing (83), (85), (86), (87) most terms simplify and we get

$$\widehat{a}^k(\mathcal{H}, -\widehat{a}(\mathcal{G}, X)) = E + F + G + H = \widehat{a}^k(\mathcal{H}, X) + \widehat{a}^k(\mathcal{H}, 0) \quad k = 1, \dots, N .$$

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