Euclidean numbers and numerosities

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Abstract

Several different versions of the theory of numerosities have been introduced in the literature. Here, we unify these approaches in a consistent frame through the notion of set of labels, relating numerosities with the Kiesler field of Euclidean numbers. This approach allows to easily introduce, by means of numerosities, ordinals and their natural operations, as well as the Lebesgue measure as a counting measure on the reals.

Contents

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1 Introduction

The techniques of nonstandard analysis allow to construct several different hyperreal fields which, for many pratical purposes, are equivalent. However, there is a unique hyperreal field which is isomorphic to the closed real field having the cardinality of the first strongly inaccessible[1] uncountable cardinal number. Such a field has been introduced in [19] and we refer to it as to the Keisler field.

Given a ring of sets $\mathcal{R}$ (closed for cartesian product) and a non Archimedean field $\mathbb{K}$, the numerosity is a function

$$\mathfrak{num} : \mathcal{R} \to \mathbb{K} \tag{1}$$

which satisfies the following properties:

- **Finite sets principle**: if A is a finite set, then $\mathfrak{num}(A) = |A|$ ($|A|$ denotes the cardinality of A);
- **Euclid's principle**: if $A \subset B$, then $\mathfrak{num}(A) < \mathfrak{num}(B)$;
- **Sum principle**: if $A \cap B = \varnothing$, then $\mathfrak{num}(A \cup B) = \mathfrak{num}(A) + \mathfrak{num}(B)$;
- **Product principle**: $\mathfrak{num}(A \times B) = \mathfrak{num}(A) \cdot \mathfrak{num}(B)$.

The notion of numerosity has been introduced in [1],[5] and developed in several directions ([4],[6],[7],[8],[9],[14],[16],[17],[21])[2]. Since its beginning, numerosity theory has been strictly related to some hyperreal field, namely the field $\mathbb{K}$ in (1) must be hyperreal.

The aim of this paper is to relate the theory of numerosity to the Keisler field in such a way that most of the properties investigated in the previous papers are preserved and unified in a consistent frame.

In particular, we want at least the following three properties to be satisfied:

- (**Consistency with the theory of cardinal numbers**) if $A, B \subset \mathbb{A}$ then
$$|A| < |B| \Rightarrow \mathfrak{num}(A) < \mathfrak{num}(B).$$
where $|E|$ denotese the cardinality of E.

[1] κ is strongly inaccessible if it is uncountable, it is not a sum of fewer than κ cardinals smaller than κ and, for all $\alpha < \kappa$, $2^\alpha < \kappa$.

[2] See also [10] for a historical survey of the ideas related to the measure of the size of infinite sets.

- (**Consistency with the theory of ordinal numbers**) if **COrd** is the set of the Cantor ordinal numbers smaller than the first inaccessible uncountable cardinal number and **Num** is the set of numerosities, then there is a map
$$\Psi : \mathbf{COrd} \to \mathbf{Num}$$
such that

  1. $\Psi(\sigma) = \mathfrak{num}\left(\{\Psi(\tau) \mid \tau < \sigma\}\right);$
  2. $\Psi\left(\sigma \oplus \tau\right) = \Psi\left(\sigma\right) + \Psi\left(\tau\right);$
  3. $\Psi\left(\sigma \otimes \tau\right) = \Psi\left(\sigma\right) \cdot \Psi\left(\tau\right),$

where $\oplus$ and $\otimes$ denote the natural operations between ordinal numbers (see Section 4.2).

- (**Consistency with the Lebesgue measure**) if $E \subset \mathbb{R}$ is a Lebesgue measurable set, then
$$m_L\left(E\right) = st\left(\frac{\mathfrak{num}\left(E\right)}{\mathfrak{num}\left(\left[0,1\right)\right)}\right), \tag{2}$$
where $m_L\left(E\right)$ denotes the Lebesgue measure of E and $st(\xi)$ denotes the standard part of ξ.

To this aim we build a field which, following [9], we will call field of Euclidean numbers. This field is isomorphic to the Keisler field and its construction presents an extra structure that allows to build a numerosity theory which satisfies, among others, the above requests.

2 The Euclidean numbers

In this section we introduce the field of Euclidean numbers. As we are going to show, this is a hyperreal field constructed by means of a minor modification of the usual superstructure construction, so to implement a development of the theory of numerosity with certain useful peculiarities (see Remark 7).

2.1 Non Archimedean fields

Here, we recall the basic definitions and some facts regarding non Archimedean fields. In the following, $\mathbb{K}$ will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

Definition 1. *Let $\mathbb{K}$ be an ordered field. Let $\xi \in \mathbb{K}$. We say that:*

- *ξ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;*
- *ξ is finite if there exists $n \in \mathbb{N}$ such that $|\xi| < n$;*

- *ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).*

Definition 2. *An ordered field $\mathbb{K}$ is called non Archimedean if it contains an infinitesimal $\xi \neq 0$.*

Infinitesimal numbers can be used to formalize the notion of "infinitely close":

Definition 3. *We say that two numbers $\xi, \zeta \in \mathbb{K}$ are* ***infinitely close*** *if $\xi - \zeta$ is infinitesimal. In this case we write $\xi \sim \zeta$.*

Clearly, the relation "$\sim$" of infinite closeness is an equivalence relation.

Theorem 4. *If $\mathbb{K} \supset \mathbb{R}$ is an ordered field, then it is non Archimedean and every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the the* ***standard part*** *of ξ.*

The standard part can be regarded as a function:

$$st : \{x \in \mathbb{K} \mid x \text{ is finite}\} \to \mathbb{R}. \tag{3}$$

Moreover, with some abuse of notation, we can extend st to all $\mathbb{K}$ by setting

$$st(\xi) = \begin{cases} +\infty & \text{if } \xi \text{ is a positive infinite number;} \\ -\infty & \text{if } \xi \text{ is a negative infinite number.} \end{cases}$$

2.2 Construction of the Euclidean numbers

Given any set E we let $\mathbb{V}(E)$ be the superstructure on E, namely the family of sets which is inductively defined as follows:

$$\begin{aligned} \mathbb{V}_0(E) &= E; \\ \mathbb{V}_{n+1}(E) &= \mathbb{V}_n(E) \cup \wp(\mathbb{V}_n(E)); \\ \mathbb{V}(E) &= \bigcup_{n=0}^{\infty} \mathbb{V}_n(E). \end{aligned}$$

If an object $x \in \mathbb{V}_{n+1}(E) \setminus \mathbb{V}_n(E)$ we say that its rank is $n+1$, and we write $rank(x) = n+1$. With the usual identifications of pairs with Kuratowski pairs and functions and relations with their graphs, we have that $\mathbb{V}(E)$ contains all the usual mathematical objects that can be constructed from E. Moreover, notice that if E is finite then also each finite level $V_n(E)$ of the superstructure on E is finite.

Now we let $\mathbb{A}$ be a set of atoms whose cardinality κ is the first strongly uncountable inaccessible cardinal number, and we assume that $\mathbb{R} \subset \mathbb{A}$. The mathematical universe we will consider in this paper is

$$\Lambda = \{E \in \mathbb{V}(\mathbb{A}) \mid E \text{ is an atom or a set such that } |E| < \kappa\}$$

where $|E|$ denotes the cardinality of E.

We let $\mathfrak{L}$ be the family of finite subsets of Λ:

$$\mathfrak{L} = \wp_{fin}(\Lambda).$$

$\mathfrak{L}$, ordered by the inclusion relation $\subseteq$, is a directed set; if E is any set, we call **net** (with values in E) any function

$$\varphi : \mathfrak{L} \to E.$$

From now on, we will denote by $\sqsubseteq$ a partial order relation over Λ that extends the inclusion, namely such that $\forall \lambda, \mu \in \mathfrak{L}$,

$$\lambda \subseteq \mu \Rightarrow \lambda \sqsubseteq \mu.$$

We assume that also $(\mathfrak{L}, \sqsubseteq)$ is a directed set; for the moment we will not make any other assumption on E. One of the main task of this paper is to define $\sqsubseteq$ in such a way to get a numerosity theory which satisfies the requests described in the introduction.

Let

$$\mathfrak{F}(\mathfrak{L}, \mathbb{R}) = \left\{ \varphi \in \mathbb{R}^{\mathfrak{L}} \mid \exists A \in \Lambda,\ \varphi(\lambda \cap A) = \varphi(\lambda) \right\}$$ [3]

be endowed with the natural operations

$$\begin{aligned} (\varphi + \psi)(\lambda) &= \varphi(\lambda) + \psi(\lambda); \\ (\varphi \cdot \psi)(\lambda) &= \varphi(\lambda) \cdot \psi(\lambda) \end{aligned}$$

and the partial ordering

$$\varphi \geq \psi \Leftrightarrow \forall \lambda \in \mathfrak{L},\ \varphi(\lambda) \geq \psi(\lambda).$$

The field of Euclidean numbers is defined as follows[4]:

Definition 5. *The field of Euclidean numbers $\mathbb{E} \supset \mathbb{R}$ is a field so that there exists a surjective map*

$$J : \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \to \mathbb{E}$$

with the following properties:

(i) ***Ring homomorphism:*** *J is a ring homomorphism, namely for all $\varphi, \psi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R})$*

- $J(\varphi + \psi) = J(\varphi) + J(\psi);$
- $J(\varphi \cdot \psi) = J(\varphi) \cdot J(\psi).$

[3] The choice of this particular space is due to the fact that we want to end with the unique hyperreal field whose cardinality is the first inaccessible, see [19].

[4] This construction can be seen as an extension of α-theory, see e.g. [3].

(ii) ***Monotonicity:*** *for all* $\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R})$*, for all* $r \in \mathbb{R}$*, if eventually* $\varphi(\lambda) \geq r$ *(namely there exists* $\lambda_0 \in \mathfrak{L}$ *such that* $\forall \lambda \sqsupseteq \lambda_0, \ \varphi(\lambda) \geq r$*), then*

$$J(\varphi) \geq r.$$

Let us show that such a field exists[5].

Proof. Let $\mathcal{U}$ be a fine ultrafilter on $\mathfrak{L}$, namely a filter of sets such that

- **Maximality:** $Q \in \mathcal{U} \Leftrightarrow \mathfrak{L} \backslash Q \notin \mathcal{U}$;
- **Finess:** $\forall \lambda \in \mathfrak{L}, \ Q[\lambda] \in \mathcal{U}$, where

$$Q[\lambda] := \{\mu \in \mathfrak{L} \mid \mu \sqsupseteq \lambda\}. \tag{4}$$

The existence of $\mathcal{U}$ is a well known and easy consequence of Zorn's Lemma. We use $\mathcal{U}$ to introduce an equivalence relation on nets, by letting for all $\psi, \varphi \in \mathfrak{F}(\mathfrak{B}, \mathbb{R})$

$$\varphi \approx_{\mathcal{U}} \psi \Longleftrightarrow \exists Q \in \mathcal{U} \ \forall \lambda \in Q, \ \varphi(\lambda) = \psi(\lambda).$$

We set

$$\widetilde{\mathbb{E}} := \mathfrak{F}(\mathfrak{L}, \mathbb{R}) / \approx_{\mathcal{U}}$$

and we denote by $[\varphi]_{\mathcal{U}}$ the equivalence classes. Now we take a injective map

$$\Phi : \widetilde{\mathbb{E}} \to \mathbb{A}$$

such that $\forall r \in \mathbb{R}$,

$$\Phi([c_r]_{\mathcal{U}}) = r$$

were c_r is the net identically equal to r. Finally we set

$$\mathbb{E} = \Phi\left(\widetilde{\mathbb{E}}\right).$$

The operations on $\mathbb{E}$ can be easily defined by letting

$$\Phi([\varphi]_{\mathcal{U}}) + \Phi([\psi]_{\mathcal{U}}) = \Phi([\varphi + \psi]_{\mathcal{U}}); \quad \Phi([\varphi]_{\mathcal{U}}) \cdot \Phi([\psi]_{\mathcal{U}}) = \Phi([\varphi \cdot \psi]_{\mathcal{U}}).$$

It is very well known (see e.g. [19]) and simple to show that, thanks to $\mathcal{U}$ being an ultrafilter, $\mathbb{E}$ endowed with the above operations is a field; moreover, it can be made an ordered field by endowing it with the following ordering:

$$\forall \varphi, \psi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}), \ \Phi([\varphi]_{\mathcal{U}}) \geq \Phi([\psi]_{\mathcal{U}}) :\Longleftrightarrow \exists Q \in \mathcal{U}, \ \forall \lambda \in Q \ \ \varphi(\lambda) \geq \psi(\lambda). \ \square$$

Remark 6. $\mathbb{E}$ *is an hyperreal field whose cardinality is* κ*; such a field is unique up to isomorphisms (see [19]); namely, changing "*$\sqsubseteq$*" we get an isomorphic hyperreal field. However, we will choose "*$\sqsubseteq$*" in such a way to get interesting interactions with other mathematical structures.*

[5] Readers with a basic knowledge of nonstandard analysis will recognize immediately that our construction is a minor modification of the usual limit ultrapower construction.

The number $J(\varphi)$ is called the Λ-limit of the net φ and will be denoted by

$$J(\varphi) = \lim_{\lambda\uparrow\Lambda} \varphi(\lambda).$$

The reason of this name and notation is that the operation

$$\varphi \mapsto \lim_{\lambda\uparrow\Lambda} \varphi(\lambda)$$

satisfies many of the properties of the usual Cauchy limit, but with the stronger property of existing for every net. More exactly, it satisfies the following properties:

- **Existence:** Every net $\varphi : \mathfrak{L} \to \mathbb{R}$ has a unique limit $L \in \mathbb{E}$.
- **Monotonicity:** For all $r \in \mathbb{R}$ if eventually $\varphi(\lambda) \geq r$, then
$$\lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \geq r;$$
- **Sum and product:** For all $\varphi, \psi : \mathfrak{L} \to \mathbb{R}$
$$\begin{aligned} \lim_{\lambda\uparrow\Lambda} \varphi(\lambda) + \lim_{\lambda\uparrow\Lambda} \psi(\lambda) &= \lim_{\lambda\uparrow\Lambda} (\varphi(\lambda) + \psi(\lambda)), \\ \lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \cdot \lim_{\lambda\uparrow\Lambda} \psi(\lambda) &= \lim_{\lambda\uparrow\Lambda} (\varphi(\lambda) \cdot \psi(\lambda)). \end{aligned}$$

Notice that, if $\lim_{\lambda\to\Lambda} \varphi(\lambda)$ denotes the usual Cauchy limit of φ, the relationship between the Cauchy limit and the Λ-limit is

$$\lim_{\lambda\to\Lambda} \varphi(\lambda) = st\left(\lim_{\lambda\uparrow\Lambda} \varphi(\lambda)\right).$$

Remark 7. *The notion of Eucliean field defined by Definition 5 has been used in several papers with "$\subseteq$" instead of "$\sqsubseteq$" (e.g. [2], [11], [12]). Now, we will explain the main technical reason for using an Euclidean field rather than a "generic" hyperreal field. A set $F \subset \Lambda^*$ is called hyperfinite if*

$$F = \lim_{\lambda\uparrow\Lambda} F_\lambda = \left\{ \lim_{\lambda\uparrow\Lambda} \ \varphi(\lambda) \mid \varphi(\lambda) \in F_\lambda \right\}$$

where the sets $F_\lambda \in \Lambda$ are finite. Hyperfinite sets play a crucial role in many applications of nonstandard analysis. If we use a Euclidean field, we can associate to every set $E \in \Lambda$ a ***unique*** *hyperfinite set $E^{\circledast}$ defined as follows*

$$E^{\circledast} = \lim_{\lambda\uparrow\Lambda} E \cap \lambda = \left\{ \lim_{\lambda\uparrow\Lambda} \ \varphi(\lambda) \mid \varphi(\lambda) \in E \cap \lambda \right\}.$$

The set $E^{\circledast}$ satisfies the property[6]

$$E^\sigma \subset E^{\circledast} \subset E^*$$

[6] Here, as usual, we have set

$$E^\sigma = \{x^* \mid x \in E\}.$$

which is very useful in the applications. Moreover, using an Euclidean field we can easily define the numerosity function over any set $E \in \Lambda$ by setting (see Section 3)

$$\mathfrak{num}(E) := \lim_{\lambda \uparrow \Lambda} |E \cap \lambda| . \tag{5}$$

In this paper, replacing with "$\subseteq$" with a suitable "$\sqsubseteq$", the numerosity theory given by Equation (5) is consistent with the main features of the numerosity theories present in the litterature (e.g. [1], [4], [5], [6], [7], [8], [9], [14], [16], [17], [21]).

2.3 Labelled sets

The notion of labelled set has been introduced in [1] and [5] to construct a numerosity theory for countable sets. Here we extend this notion to adapt it to the study of numerosity theories for larger sets.

Definition 8. *We call* ***label set*** *a family of sets $\mathfrak{B} \subset \mathfrak{L}$ such that*

(i) $\forall \mathfrak{s}, \mathfrak{t} \in \mathfrak{B},\ \mathfrak{s} \cap \mathfrak{t},\ \mathfrak{s} \cup \mathfrak{t} \in \mathfrak{B}$;

(ii) $\forall \mathfrak{s} \in \mathfrak{B},\ \mathfrak{s} \cap \mathfrak{L} = \varnothing$;

(iii) $\bigcup_{\mathfrak{s} \in \mathfrak{B}} \mathbb{V}(\mathfrak{s}) = \Lambda$.

Requirement (i) gives to $\mathfrak{B}$ a lattice structure, whilst requirement (ii) entails that the elements of a label are either atoms or infinite sets.

Having fixed the notion of "labels", we can now introduce the notion of "labelling":

Definition 9. *Let $\mathfrak{B}$ be a set of labels. We call $\mathfrak{B}$-**labelling** the map*

$$\ell : \Lambda \to \mathfrak{B}$$

defined as follows:

$$\ell(a) = \bigcap_{\mu \in I_a} \mu,$$

where $I_a = \{\mu \in \mathfrak{B} \mid a \in \mathbb{V}(\mu)\}$. For every $a \in \mathfrak{L}$ we call $\ell(a)$ the label of a.

Roughly speaking, the label of an object $a \in \Lambda$ is a finite set whose elements allow to define a by the fundamental finitistic set operations.

There is plenty of sets of labels: just set

$$\mathfrak{B}_{\max} = \{\mathfrak{t} \in \mathfrak{L} \mid\ \mathfrak{t} \cap \mathfrak{L} = \emptyset\} . \tag{6}$$

Obviously, $\mathfrak{B}_{\max}$ is a label set; more importantly, every label set $\mathfrak{B}$ is a subset of $\mathfrak{B}_{\max}$.

Example 10. *Let $a = \mathbb{N}$. Then, using the $\mathfrak{B}_{\max}$-labelling,*

$$\ell(\mathbb{N}) = \{\mathbb{N}\}.$$

Now we will describe some properties of a set of labels $\mathfrak{B}$ and the corresponding $\mathfrak{B}$-labelling:

Proposition 11. *Let $\mathfrak{B}$ be a set of labels, and let $\mathfrak{s} \in \mathfrak{B}$. Then*

(i) $\mathbb{V}(\mathfrak{s})$ *is countable;*

(ii) $\mathbb{V}(\mathfrak{s}) \setminus \mathfrak{s}$ *consists only of finite sets;*

(iii) for all $\mathfrak{s}, \mathfrak{t} \in \mathfrak{B}$, *for all* $m \in \mathbb{N}$ *we have that*

$$\mathbb{V}_m(\mathfrak{s}) \subseteq \mathbb{V}_m(\mathfrak{t}) \Leftrightarrow \mathfrak{s} \subseteq \mathfrak{t};$$

(iv) for all $\mathfrak{s}, \mathfrak{t} \in \mathfrak{B}$, $\mathbb{V}(\mathfrak{s} \cap \mathfrak{t}) = \mathbb{V}(\mathfrak{s}) \cap \mathbb{V}(\mathfrak{t})$*;*

(v) If $a \in \mathfrak{L}$ *and* $\mathfrak{t} \in \mathfrak{B}$, *then* $\{a\} \in \mathbb{V}(\mathfrak{t}) \Leftrightarrow a \in \mathbb{V}(\mathfrak{t})$.

Proof. (i) $\mathfrak{s}$ is finite, hence by induction it trivially holds that $\mathbb{V}_n(\mathfrak{s})$ is finite for every $n \in \mathbb{N}$. Therefore $\mathbb{V}(\mathfrak{s})$ is countable.

(ii) As $\mathfrak{s}$ is finite, by induction it is immediate to prove that $\mathbb{V}_{n+1}(\mathfrak{s}) \setminus \mathfrak{s}$ consists only of finite sets, hence the thesis follows straightforwardly.

(iii) The implication $\Leftarrow$ is trivial. Let us prove the other implication. If $\mathbb{V}_m(\mathfrak{s}) \subseteq \mathbb{V}_m(\mathfrak{t})$ then, in particular, $\mathfrak{s} \in \mathbb{V}_m(\mathfrak{t})$. Now, if $\mathfrak{s} = \{a_1, \ldots, a_n\}$, all $a_1, \ldots, a_n$ are either atoms or infinite sets, hence by (i), $a_1, \ldots, a_n \in \mathfrak{t}$.

(iv) The inclusion $\subseteq$ is trivial. For the reverse inclusion, let $\eta \in \mathbb{V}(\mathfrak{s}) \cap \mathbb{V}(\mathfrak{t})$. In particular, $\eta \in \mathbb{V}_n(\mathfrak{s}) \cap \mathbb{V}_m(\mathfrak{t})$ and so, if $l = \max\{n, m\}$, we have that $\eta \in \mathbb{V}_l(\mathfrak{s}) \cap \mathbb{V}_l(\mathfrak{t})$. We proceed by induction on l to show that $\mathbb{V}_l(\mathfrak{s}) \cap \mathbb{V}_l(\mathfrak{t}) \subseteq \mathbb{V}(\mathfrak{s} \cap \mathfrak{t})$.

If $l = 0$, then $\eta \in \mathbb{V}_0(\mathfrak{s}) \cap \mathbb{V}_0(\mathfrak{t})$ if and only if $\eta = \mathfrak{s} = \mathfrak{t}$, and the desired inclusion trivially holds.

Now let us suppose the inclusion to hold for $l \in \mathbb{N}$, and let $\eta \in \mathbb{V}_{l+1}(\mathfrak{s}) \cap \mathbb{V}_{l+1}(\mathfrak{t})$. If $\eta \in \mathbb{V}_l(\mathfrak{s}) \cap \mathbb{V}_l(\mathfrak{t})$ we are done by inductive hypothesis; if not, there are $A \in \mathbb{V}_l(\mathfrak{s}), B \in \mathbb{V}_l(\mathfrak{t})$ such that $\eta \in \wp(A) \cap \wp(B)$. In particular, $\eta \in \wp(A \cap B)$. But $A \cap B \in \mathbb{V}_l(\mathfrak{s}) \cap \mathbb{V}_l(\mathfrak{t})$, so by induction $A \cap B \in \mathbb{V}(\mathfrak{s} \cap \mathfrak{t})$, hence $\eta \in \mathbb{V}(\mathfrak{s} \cap \mathfrak{t})$ as desired.

(v) The implication $\Leftarrow$ is trivial. For the reverse implication, let

$$l = \min\{n \in \mathbb{N} \mid a \in \mathbb{V}_n(\mathfrak{t})\}.$$

In particular, we have that $l \geq 1$. In fact, if $l = 0$ then $\{a\} = \mathfrak{t}$, and this cannot happen as $\mathfrak{t} \cap \mathfrak{L} = \emptyset$. Hence $a \in \mathbb{V}_{l-1}(\mathfrak{t})$ and we are done. $\square$

As we will see in Section 5, the freedom of choosing a particular set of labels allows to impose certain additional arithmetical properties on numerosities.

Proposition 12. *Let $\mathfrak{B}$ be a set of labels and let ℓ be a $\mathfrak{B}$-labelling. The following properties hold:*

(i) $\forall a, b \in \Lambda$, $a \subseteq b \Rightarrow \ell(a) \subseteq \ell(b)$;

(ii) $\forall a \in \Lambda$, $\ell(a) \supseteq \ell(\{a\})$, *and equality holds if* $a \in \mathfrak{L}$*;*

(*iii*) $\forall a, b \in \Lambda,\ a \in b \Rightarrow \ell(\{a\}) \subseteq \ell(b)$;

(*iv*) $\forall \lambda \in \mathfrak{B},\ \lambda \in \mathbb{V}\left(\ell(\lambda)\right)$;

(*v*) $\forall \mathfrak{s} \in \mathfrak{B},\ \ell\left(\mathfrak{s}\right) = \mathfrak{s}$;

(*vi*) $\forall \mathfrak{s} \in \mathfrak{B}, \forall m \in \mathbb{N},\ \ell\left(\mathbb{V}_m(\mathfrak{s})\right) = \mathfrak{s}$;

(*vii*) $\forall a \in \Lambda, \forall \mathfrak{s} \in \mathfrak{B},\ \ell\left(a\right) \subseteq \mathfrak{s} \Leftrightarrow a \in \mathbb{V}(\mathfrak{s})$*;*

(*viii*) $\forall a, b \in \Lambda,\ \ell\left(\{a, b\}\right) = \ell\left(a\right) \cup \ell(b)$*;*

(*ix*) $\forall a, b \in \Lambda,\ \ell\left((a, b)\right) = \ell(a) \cup \ell(b)$.

(*x*) $\forall E \in \Lambda,\ \forall \lambda \in \mathfrak{L}$ *if we set*

$$E_\lambda := \{x \in E \mid \ell\left(x\right) \subseteq \ell\left(\lambda\right)\}$$

then

$$E_\lambda = E \cap \mathbb{V}(\ell\left(\lambda\right));$$

(*xi*) $\forall E \in \Lambda$, *the set* E_λ *is finite.*

Proof. (i) If $a \subseteq b$, then trivially $a \in \mathbb{V}(\mathfrak{t})$ whenever $b \in \mathbb{V}(\mathfrak{t})$ and hence

$$\ell(a) = \bigcap\{\mathbb{V}(\mathfrak{t}) \mid \mathfrak{t} \in \mathfrak{B},\ a \in \mathbb{V}(\mathfrak{t})\} \subseteq \bigcap\{\mathbb{V}(\mathfrak{t}) \mid \mathfrak{t} \in \mathfrak{B},\ b \in \mathbb{V}(\mathfrak{t})\} = \ell(b).$$

(ii) Trivially, for all $\mathfrak{t} \in \mathfrak{B}$, if $a \in \mathbb{V}(\mathfrak{t})$ then also $\{a\} \in \mathbb{V}(\mathfrak{t})$, hence $\ell(\{a\}) \subseteq \ell(\{a\}$. The second claim follows from the fact that, by Proposition 11.(v), $a \in \mathbb{V}(\mathfrak{t}) \Leftrightarrow \{a\} \in \mathbb{V}(\mathfrak{t})$.

(iii) If $a \in b$, then $\{a\} \subseteq b$ and by (i) and (ii), we have that

$$\ell(a) = \ell(\{a\}) \subseteq \ell(b).$$

(iv) By definition, $\forall \mu \in I_a,\ a \in \mathbb{V}(\mu)$; hence

$$a \in \bigcap_{\mu \in I_a} \mathbb{V}(\mu) = \mathbb{V}\left(\bigcap_{\mu \in I_a} \mu\right) = \mathbb{V}\left(\ell(a)\right).$$

(v) We have that $\mathfrak{s} = \mathbb{V}_0(\mathfrak{s}) \in \mathbb{V}(\mathfrak{s})$; hence $\mathfrak{s} \in I_{\mathfrak{s}}$ and so $\ell(\mathfrak{s}) \subseteq \mathfrak{s}$. Moreover, if $\mathfrak{t} \in I_{\mathfrak{s}}$, $\mathfrak{s} \in \mathbb{V}(\mathfrak{t})$ and since $\mathfrak{s}$ is a label, $\mathfrak{s} \subset \mathfrak{t}$ and so

$$\mathfrak{s} \subseteq \bigcap_{\mathfrak{t} \in I_{\mathfrak{s}}} \mathfrak{t} = \ell(\mathfrak{s}).$$

(vi) If $\mathfrak{s} \in \mathfrak{B}$, then $\forall \mathfrak{t} \in \mathfrak{B}$, by Proposition 11.(iv) we have that

$$\mathbb{V}_m(\mathfrak{s}) \in \mathbb{V}(\mathfrak{t}) \Leftrightarrow \mathfrak{s} \subseteq \mathfrak{t} \Leftrightarrow \mathfrak{s} \in \mathbb{V}(\mathfrak{t}).$$

Then

$$I_{\mathbb{V}_m(\mathfrak{s})} = \{\mathfrak{t} \in \mathfrak{B} \mid \mathbb{V}_m(\mathfrak{s}) \in \mathbb{V}(\mathfrak{t})\} = \{\mathfrak{t} \in \mathfrak{B} \mid \mathfrak{s} \subseteq \mathfrak{t}\} \qquad (7)$$
$$= \{\mathfrak{t} \in \mathfrak{B} \mid \mathfrak{s} \in \mathbb{V}(\mathfrak{t})\} = I_{\mathfrak{s}},$$

hence the thesis follows by the definition of $\mathfrak{B}$-labeling.

(vii) ($\Rightarrow$) If $\ell(a) \subseteq \mathfrak{s}$, then, by (i), $\mathbb{V}(\ell(a)) \subseteq \mathbb{V}(\mathfrak{s})$; therefore, by (iv), $a \in \mathbb{V}(\mathfrak{s})$.

($\Leftarrow$) If $a \in \mathbb{V}(\mathfrak{s})$, then $\mathfrak{s} \in I_a$ and so $\ell(a) \subseteq \mathfrak{s}$.

(viii) For $\mathfrak{t} \in I_{\{a,b\}}$, since $\{a,b\} \in \mathbb{V}(\mathfrak{t})$, but $\{a,b\} \notin \mathfrak{t}$, we have that $a \in \mathbb{V}(\mathfrak{t})$ and $b \in \mathbb{V}(\mathfrak{t})$; then

$$I_{\{a,b\}} = \{\mathfrak{t} \in \mathfrak{B} \mid a \in \mathbb{V}(\mathfrak{t}) \text{ and } b \in \mathbb{V}(\mathfrak{t})\}$$
$$= \{\mathfrak{t} \in \mathfrak{B} \mid a \in \mathbb{V}(\mathfrak{t})\} \cap \{\mathfrak{t} \in \mathfrak{B} \mid b \in \mathbb{V}(\mathfrak{t})\} = I_a \cap I_b.$$

Hence

$$\ell(\{a,b\}) = \bigcap_{\mathfrak{t} \in I_{\{a,b\}}} \mathfrak{t} = \bigcap_{\mathfrak{t} \in I_a \cap I_b} \mathfrak{t} = \{x \in \Lambda \mid x \in \mathfrak{t} \text{ and } \mathfrak{t} \in I_a \cap I_b\}$$
$$= \{x \in \Lambda \mid (x \in \mathfrak{t} \text{ and } \mathfrak{t} \in I_a) \text{ or } (x \in \mathfrak{t} \text{ and } \mathfrak{t} \in I_b)\}$$
$$= \{x \in \Lambda \mid (x \in \mathfrak{t} \text{ and } \mathfrak{t} \in I_a)\} \cup \{x \in \Lambda \mid (x \in \mathfrak{t} \text{ and } \mathfrak{t} \in I_b)\}$$
$$= \left(\bigcap_{\mathfrak{t} \in I_a} \mathfrak{t}\right) \cup \left(\bigcap_{\mathfrak{t} \in I_b} \mathfrak{t}\right) = \ell(a) \cup \ell(b).$$

(ix) We have that

$$\ell((a,b)) = \ell(\{a,\{a,b\}\}) = \ell(a) \cup \ell(\{a,b\}) = \ell(b)\,\ell(a) \cup \ell(a) \cup \ell(b) = \ell(a) \cup \ell(b).$$

(x) First set $\mathfrak{s} = \ell(\lambda)$. Let us first prove the inclusion $\subseteq$. Let $x \in E$ be such that $\ell(x) \subseteq \mathfrak{s}$. By the definition of labelling then $x \in \mathbb{V}(\mathfrak{s})$, and the inclusion is proven. For the reverse inclusion, let $x \in E \cap \mathbb{V}(\mathfrak{s})$. In particular, it must be $\ell(x) \subseteq \mathfrak{s}$, and we are done.

(xi) If $E \in \Lambda$ then E has a finite rank n, which means that $E \cap \mathbb{V}(\mathfrak{s}) = E \cap \mathbb{V}_n(\mathfrak{s})$, and the conclusion follows by (x) as, by construction, $\mathbb{V}_n(\mathfrak{s})$ is finite. □

The notion of $\mathfrak{B}$-labelling allows to equip $\mathfrak{L}$ with a partial order structure $\sqsubseteq$:

Definition 13. *We set*

$$\mathfrak{L}_0(\mathfrak{B}) := \{\mathbb{V}_m(\mathfrak{t}) \mid \ m \in \mathbb{N}_0,\ \mathfrak{t} \in \mathfrak{B}\}$$

and for every $\lambda, \mu \in \mathfrak{L}$, we set

$$\lambda \sqsubseteq \mu \Leftrightarrow \lambda \subseteq \bigcap \{\tau \in \mathfrak{L}_0 \mid \mu \subseteq \tau\}.$$

Notice that, by definition

$$\lambda \sqsubseteq \mu \Leftrightarrow I_\lambda \subseteq I_\mu,$$

where I_λ has been introduced in Definition 9, and that

$$\mathfrak{L}_0\left(\mathfrak{L}_\mathfrak{o}\left(\mathfrak{B}\right)\right) = \mathfrak{L}_0\left(\mathfrak{B}\right).$$

Clearly $\sqsubseteq$ induces a lattice structure on $\mathfrak{L}_0\left(\mathfrak{B}\right)$, since

$$\lambda \vee \mu := \bigcap \left\{\tau \in \mathfrak{L}_0 \mid \lambda \cup \mu \subseteq \tau\right\};$$

$$\lambda \wedge \mu := \bigcup \left\{\tau \in \mathfrak{L}_0 \mid \tau \subseteq \lambda \cap \mu\right\}.$$

Since $\lambda \subseteq \mu \Rightarrow \lambda \sqsubseteq \mu$, we can use the directed set $\left(\mathfrak{L}_0\left(\mathfrak{B}\right), \sqsubseteq\right)$ to define a field of Euclidean numbers as in Definition 5. From now on, $\mathbb{E}$ will denote such a field.

Remark 14. *Now the idea is to construct a suitable set of labels in such a way that the relation $\sqsubseteq$ carry all the informations needed for a "good" numerosity theory. All these informations depend on $\sqsubseteq$ and not on the choice of the ultrafilter used in the construction of $\mathbb{E}$.*

3 The general theory of numerosities

Different versions of the notion of numerosity have already been studied in several previous papers [1, 5, 6, 7, 8, 14, 16, 17]; we refer also to the book [4] for a complete overview of the countable case. In this paper, we want to show how the new definition of labels and of the Euclidean field allows to easily provide the most interesting features of the theory of numerosities. In particular, we show how numerosities can be used to simultaneously unify and generalize objects and results coming from different areas, like (a version of) Lagrange's Theorem for groups, the Peano-Jordan measure and the Lebesgue measure.

3.1 Definition and first properties

Definition 15. *Let E be a set in Λ. We call **numerosity** of E the Euclidean number*

$$\mathfrak{num}\left(E\right) := \lim_{\lambda \uparrow \Lambda} \left|E \cap \lambda\right|.$$

The set of numerosities will be denoted by **Num**.

The notion of numerosity allows to "give a name" to some hyperreal number. We set

$$\alpha = \mathfrak{num}\left(\mathbb{N}\right); \ \beta = \mathfrak{num}\left(\left[0, 1\right)\right). \tag{8}$$

The numerosity of a set depends on the choice of the set of labels $\mathfrak{B}$, as well as on the ultrafilter $\mathcal{U}$ on $\mathfrak{B}$ chosen to construct $\mathbb{E}$. However the properties which will be listed below are independent of any choice.

Theorem 16. *Let E, F be sets in Λ. Numerosities satisfy the following properties:*

(i) ***Finite sets principle:*** *if E is a finite set, then $\mathfrak{num}(E) = |E|$;*

(ii) ***Euclid's principle:*** *if $E \subset F$ then $\mathfrak{num}(E) < \mathfrak{num}(F)$;*

(iii) ***Labels principle****: if*

$$E_\lambda = \{x \in E \mid \ell(x) \subseteq \ell(\lambda)\}.$$

then, if $\lambda \in \mathfrak{L}_0(\mathfrak{B})$, $E_\lambda = E \cap \lambda$ and hence

$$\mathfrak{num}(E) = \lim_{\lambda \uparrow \Lambda} |E_\lambda|;$$

(iv) ***Comparison principle:*** *if $\Phi : E \to F$ is a bijection that preserves labels, namely such that for all $x \in E$*

$$\ell(\Phi(x)) = \ell(x),$$

then $\mathfrak{num}(E) = \mathfrak{num}(F)$;

(v) ***Sum principle:*** *if $E \cap F = \varnothing$ then $\mathfrak{num}(E \cup F) = \mathfrak{num}(E) + \mathfrak{num}(F)$;*

(vi) ***Product principle:*** *$\mathfrak{num}(E \times F) = \mathfrak{num}(E) \cdot \mathfrak{num}(F)$;*

(vii) ***Finite parts principle:*** *$\mathfrak{num}(\wp_{fin}(E)) = 2^{\mathfrak{num}(E)}$;*

(viii) ***Finite functions principle:*** *let E be nonempty, and*

$$\mathfrak{F}_{fin}(X, E) := \{f : D \to E \mid D \in \wp_{fin}(X)\}.$$

Then, if $a \in E$, we have

$$\mathfrak{num}(\mathfrak{F}_{fin}(X, E \backslash \{a\})) = \mathfrak{num}(E)^{\mathfrak{num}(X)}.$$

Proof. (i) If $|E| = n < \infty$, then for every $\lambda \in \mathfrak{L}$, we have $|E \cap \lambda| = n$, and the thesis then follows by taking the λ-limit.

(ii) If $E \subset F$, eventually $|E \cap \lambda| < |F \cap \lambda|$, so $\lim_{\lambda \uparrow \Lambda} |E \cap \lambda| < \lim_{\lambda \uparrow \Lambda} |E \cap \lambda|$.

(iii) Take $\lambda = \mathbb{V}_m(\mathfrak{s})$ with $m \geq rank(E)$ and $\mathfrak{s} \in \mathfrak{B}$. Then, by Proposition 12.12

$$E \cap \lambda = E \cap \mathbb{V}_m(\mathfrak{s}) = E \cap \mathbb{V}(\mathfrak{s}) = E_\lambda.$$

(iv) By hypothesis we have that for all $\lambda \in \mathfrak{L}$ $|E_\lambda| = |F_\lambda|$, and so by the labels principle

$$\mathfrak{num}(E) = \lim_{\lambda \uparrow \Lambda} |E \cap \lambda| = \lim_{\lambda \uparrow \Lambda} |F \cap \lambda| = \mathfrak{num}(F). \tag{9}$$

(v) Just notice that $|E \cup F|_\lambda = |E_\lambda| + |F_\lambda|$ for every $\lambda \in \mathfrak{L}$, hence the thesis follows by Definition 5.(2) and by the labels principle.

(vi) Let $\lambda \in \mathfrak{L}$. By property (ix) in Proposition 12, we have that $(E \times F)_\lambda = E_\lambda \times F_\lambda$, hence $|\,(E \times F)_\lambda\,| = |E_\lambda \times F_\lambda| = |E_\lambda| \cdot |F_\lambda|$, and the thesis then follows immediately, again by the labels principle.

(vii) Let $\lambda = \mathbb{V}_m(\mathfrak{s}) \in \mathfrak{L}_0$ $(m > rank(E))$, and let $a \in \wp_{fin}(E) \cap \mathbb{V}(\mathfrak{s})$. Then by Proposition 12.(x) we have that it must be $a \in \wp_{fin}(E_\lambda)$. Conversely, if $a \in \wp_{fin}(E_\lambda)$ it is immediate to see that $a \in \wp_{fin}(E) \cap \mathbb{V}(\mathfrak{s})$. Hence, by Proposition 12 we have

$$\left|[\wp_{fin}(E)]_\lambda\right| = |\wp_{fin}(E) \cap \mathbb{V}(\mathfrak{s})| = |\wp_{fin}(E_\lambda)| = 2^{|E_\lambda|},$$

and so by the labels principle

$$\mathfrak{num}(\wp_{fin}(E)) = \lim_{\lambda \uparrow \Lambda} 2^{|E_\lambda|} = 2^{\mathfrak{num}(E)}.$$

(viii) We set $\lambda = \mathbb{V}_m(\mathfrak{s}) \in \mathfrak{L}_0$, $m > rank(f)$. Let $f \in \mathfrak{F}_{fin}(X, E \backslash \{a\}) \cap \mathbb{V}(\mathfrak{s})$, and let D be the domain of f. By identifying functions with Kuratowski pairs, and by our definition of labellings on pairs, it is immediate to see that $f \in \mathfrak{F}_{fin}(X, E \backslash \{a\}) \cap \mathbb{V}(\mathfrak{s})$ if and only if $D(f) \subset X \cap \mathbb{V}(\mathfrak{s}) = X_\lambda$ and $Im(f) \subset (E \backslash \{a\}) \cap \mathbb{V}(\mathfrak{s}) = E_\lambda \backslash \{a\}$. Therefore

$$\mathfrak{F}_{fin}(X, E \backslash \{a\}) \cap \mathbb{V}(\mathfrak{s}) = \mathfrak{F}_{fin}(X_\lambda, E_\lambda \backslash \{a\}).$$

Notice that

$$|\mathfrak{F}_{fin}(X_\lambda, E_\lambda \backslash \{a\})| = |\mathfrak{F}(X_\lambda, E_\lambda)|.$$

In fact, the association $g \in \mathfrak{F}_{fin}(X_\lambda, E_\lambda \backslash \{a\}) \to \widetilde{g} \in \mathfrak{F}(X_\lambda, E_\lambda)$, with

$$\widetilde{g}(x) = \begin{cases} g(x), & \text{if } x \in X_\lambda; \\ a, & \text{otherwise} \end{cases}$$

is a bijection. Hence, again by the labels principle,

$$\begin{aligned} \mathfrak{num}(\mathfrak{F}_{fin}(X, E \backslash \{a\})) &= \lim_{\lambda \uparrow \Lambda} |\mathfrak{F}_{fin}(X, E \backslash \{a\}) \cap \mathbb{V}(\lambda)| = \lim_{\lambda \uparrow \Lambda} |\mathfrak{F}(X_\lambda, E_\lambda)| \\ &= \lim_{\lambda \uparrow \Lambda} |E_\lambda|^{|X_\lambda|} = \mathfrak{num}(E)^{\mathfrak{num}(X)}. \end{aligned}$$

□

4 Ordinal numbers and numerosities

In this section we will select a subset of the numerosities which we will call ordinal numerosities (or simply ordinals). This set, equipped with its natural order relation $<$, is isomorphic to the set of ordinal numbers. In Section 4.2 we will show that this correspondence is deeper than expected since it preserves also the natural operations between ordinals.

4.1 The ordinal numerosities

Let **Num** be the set of numerosities.

Definition 17. *The set* **Ord** $\subset$ **Num** *of ordinal numerosities is defined as follows:* $\tau \in$ **Ord** *if and only if*

$$\tau = \mathfrak{num}\left(\Omega_\tau\right),$$

where

$$\Omega_\tau = \{x \in \mathbf{Ord} \mid x < \tau\}.$$

It is easy to see by transfinite induction that this is a good definition. In fact, it is immediate to check that

- $0 \in$ **Ord**;
- if $\tau \in$ **Ord**, then $\tau + 1 = \mathfrak{num}\left(\Omega_\tau \cup \{\tau\}\right) \in$ **Ord** (and hence $\mathbb{N} \subset$ **Ord**).

Moreover, if $\tau_k = \mathfrak{num}\left(\Omega_k\right)$, $k \in K$, ($|K| < \kappa$) are ordinal numerosities, then

$$\tau := \mathfrak{num}\left(\bigcup_{k\in K} \Omega_k\right) \in \mathbf{Ord}.$$

In fact, this holds as $\bigcup_{k\in K} \Omega_k = \{x \in \mathbf{Ord} \mid x < \tau\}$: the inclusion $\bigcup_{k\in K} \Omega_k \subseteq \{x \in \mathbf{Ord} \mid x < \tau\}$ holds trivially, as if $x \in \bigcup_{k\in K} \Omega_k$ then $x \in$ **Ord** and $x \in \Omega_k$ for some k, and so $x < \tau_k < \tau$; conversely, if $x \in$ **Ord** is such that $x < \tau$, if $x \notin \bigcup_{k\in K} \Omega_k$ we would have that $\Omega_x \supseteq \bigcup_{k\in K} \Omega_k$, and so by taking numerosities we would get $x \geq \tau$, which is absurd.

Definition 18. *If* τ_k, $k \in K$, *(*$|K| < \kappa$*) are ordinals, we set*

$$\sup_{k\in K} \tau_k = \mathfrak{num}\left(\bigcup_{k\in K} \Omega_{\tau_k}\right),$$

where $\tau_k = \mathfrak{num}\left(\Omega_{\tau_k}\right)$.

Then $\tau = \sup_{k\in K} \tau_k$ is the least element in **Ord** equal or greater than every τ_k, namely $\tau \in$ **Ord** and

$$\forall k \in K,\ \tau \geq \tau_k; \tag{10}$$

$$\forall k \in K,\ \forall \xi \geq \tau_k \Rightarrow \xi \geq \tau. \tag{11}$$

However τ is not the least element in **Num** greater or equal to every τ_k. In fact, as we have seen, if $\sup_{k\in K} \tau_k$ is not a maximum, there are numerosities $\xi \in \mathbb{E}$, greater that every τ_k and smaller than τ, e.g. $(\sup_{k\in K} \tau_k) - 1$.

Our construction of the ordinal numbers is similar to the construction of Von Neumann. However, whilst a Von Neumann ordinal τ is the set of all the Von Neumann ordinals contained in τ, in our construction an ordinal τ is the numerosity of the set of ordinals smaller than τ. Hence, here, an ordinal number, as any other numerosity, is an atom.

Obviously, not all numerosities are ordinals: for example, $\mathfrak{num}(\mathbb{N})$ is not an ordinal. In fact, if $\alpha = \mathfrak{num}(\mathbb{N})$ were an ordinal then:

$$\begin{aligned} \alpha &= \mathfrak{num}(\{x \in \mathbf{Ord} \mid x < \mathfrak{num}(\mathbb{N})\}) = \mathfrak{num}(\mathbb{N}_0) \\ &= \mathfrak{num}(\mathbb{N} \cup \{0\}) = \alpha + 1. \end{aligned}$$

In a similar way, one can prove that no infinite numerosity smaller than $\mathfrak{num}(\mathbb{N})$ is an ordinal. However, $\alpha + 1$ is an ordinal:

$$\alpha + 1 = \mathfrak{num}(\mathbb{N}_0) = \mathfrak{num}(\{x \in \mathbf{Ord} \mid x < \alpha\}).$$

Actually $\alpha + 1$ is the smallest infinite ordinal. From now on, we will call it ω.

As we expect, $\mathbf{Ord}$ is a well ordered set; in fact is $E \subset \mathbf{Ord}$, the minimum is given by

$$\min\ E = \sup\{x \in \mathbf{Ord} \mid \forall a \in E,\ x \leq a\}.$$

4.2 Sums and products of ordinals

In this section we will show that the set of ordinal numerosities is closed under sums and products, and we will show that there is relationship between sums and products of ordinal numerosities and the natural operations between Cantor ordinals.

First, we start by showing that the operations between numerosities are consistent with the order structure over the ordinals.

Theorem 19. *For all ordinal numbers $\sigma, \tau \in \mathbf{Ord}$ we have that*

$$\begin{aligned} \mathfrak{num}(\Omega_\sigma) + \mathfrak{num}(\Omega_\tau) &= \mathfrak{num}(\Omega_{\sigma+\tau}); \\ \mathfrak{num}(\Omega_\sigma) \cdot \mathfrak{num}(\Omega_\tau) &= \mathfrak{num}(\Omega_{\sigma\tau}). \end{aligned}$$

In particular, $\sigma + \tau \in \mathbf{Ord}$ and $\sigma\tau \in \mathbf{Ord}$.

Proof. First let us prove that

$$\mathfrak{num}(\Omega_{\sigma+\tau}) = \mathfrak{num}(\Omega_\sigma) + \mathfrak{num}(\Omega_\tau)$$

acting by induction on τ. If $\tau = 0$, then this relation is obvious. If $\tau = \gamma + 1$, then

$$\begin{aligned} \mathfrak{num}(\Omega_{\sigma+\tau}) &= \mathfrak{num}(\Omega_{\sigma+\gamma+1}) = \mathfrak{num}(\Omega_{\sigma+\gamma} \cup \{\sigma + \gamma + 1\}) \\ &= \mathfrak{num}(\Omega_{\sigma+\gamma}) + \mathfrak{num}(\{\sigma + \gamma + 1\}) = \mathfrak{num}(\Omega_\sigma) + \mathfrak{num}(\Omega_\gamma) + 1 \\ &= \mathfrak{num}(\Omega_\sigma) + \mathfrak{num}(\Omega_\gamma \cup \{\gamma + 1\}) = \mathfrak{num}(\Omega_\sigma) + \mathfrak{num}(\Omega_\tau). \end{aligned}$$

If $\tau = \sup_{k\in K} \tau_k$, (where $\tau_k = \mathfrak{num}(\Omega_k)$), is a limit ordinal, then

$$\mathfrak{num}(\Omega_{\sigma+\tau}) = \sup_{k\in K} \mathfrak{num}(\Omega_{\sigma+\tau_k}) = \sup_{k\in K} [\mathfrak{num}(\Omega_\sigma) + \mathfrak{num}(\Omega_{\tau_k})] .$$

Since $\sigma + \tau_k = \mathfrak{num}(\Omega_\sigma) + \mathfrak{num}(\Omega_{\tau_k})$ is an ordinal number, τ satisfies (10) and (11) and hence,

$$\forall k \in K,\ \sigma + \tau \geq \sigma + \tau_k;$$

$$\forall k \in K,\ \forall \xi \in \mathbf{Ord}\ \sigma + \xi \geq \sigma + \tau_k \Rightarrow \sigma + \xi \geq \sigma + \tau.$$

Then,

$$\sup_{k\in K} (\sigma + \tau_k) = \sigma + \sup_{k\in K} \tau_k$$

and so

$$\mathfrak{num}(\Omega_{\sigma+\tau}) = \sigma + \sup_{k\in K} \tau_k = \mathfrak{num}(\Omega_\sigma) + \sup_{k\in K} [\mathfrak{num}(\Omega_{\tau_k}] = \mathfrak{num}(\Omega_\sigma) + \mathfrak{num}(\Omega_\tau).$$

Similarly we act with the product. If $\tau = 0$, then this relation is obvious. If $\tau = \gamma + 1$, then

$$\begin{aligned}
\mathfrak{num}(\Omega_{\sigma\tau}) &= \mathfrak{num}(\Omega_{\sigma(\gamma+1)}) = \mathfrak{num}(\Omega_{\sigma\gamma+\sigma}) = \mathfrak{num}(\Omega_{\sigma\gamma}) + \mathfrak{num}(\Omega_\sigma) \\
&= \mathfrak{num}(\Omega_\sigma) \cdot \mathfrak{num}(\Omega_\gamma) + \mathfrak{num}(\Omega_\sigma) = \mathfrak{num}(\Omega_\sigma) [\mathfrak{num}(\Omega_\gamma) + 1] \\
&= \mathfrak{num}(\Omega_\sigma) \cdot \mathfrak{num}(\Omega_\tau).
\end{aligned}$$

If $\tau = \sup_{k\in K} \tau_k$ (where $\tau_k = \mathfrak{num}(\Omega_k)$), is a limit ordinal, then

$$\mathfrak{num}(\Omega_{\sigma\tau}) = \sup_{k\in K} \mathfrak{num}(\Omega_{\sigma\tau_k}) = \sup_{k\in K} [\mathfrak{num}(\Omega_\sigma) \cdot \mathfrak{num}(\Omega_{\tau_k})] .$$

Since τ satisfies (10) and (11),

$$\forall k \in K,\ \sigma\tau \geq \sigma\tau_k;$$

$$\forall k \in K,\ \forall \xi \in \mathbf{Ord}\ \sigma\xi \geq \sigma\tau_k \Rightarrow \sigma\xi \geq \sigma\tau.$$

Then

$$\sup_{k\in K} (\sigma\tau_k) = \sigma \cdot \sup_{k\in K} \tau_k,$$

hence

$$\mathfrak{num}(\Omega_{\sigma\tau}) = \sigma \cdot \sup_{k\in K} \tau_k = \mathfrak{num}(\Omega_\sigma) \cdot \mathfrak{num}(\Omega_\tau).$$ □

4.3 Numerosities and Cantor ordinals

The relation with the Cantor definition of ordinal is the following: if $\tau \in \mathbf{Ord}$, Ω_τ is a well ordered set and hence $ot(\Omega_\tau)$ (the *order type* of Ω_τ) is a Cantor ordinal. From now on, to avoid confusion, we will denote the Cantor ordinals by $\bar{\tau}$ and their set by **COrd**. Whilst a Cantor ordinal is an equivalence class of well-ordered sets, in our definition an ordinal is the numerosity of a suitable well ordered set; in particular, if we let ω be the smallest infinite ordinal, then $\omega = \mathfrak{num}(\mathbb{N}_0)$ and $\bar{\omega} = ot(\mathbb{N}_0)$.

Now, let us consider the map

$$\Phi : \mathbf{Ord} \to \mathbf{COrd} : \ \Phi(\tau) = ot(\Omega_\tau) := \bar{\tau} \tag{12}$$

which identifies the "numerosity ordinals" with the "Cantor ordinals". So, by construction Φ is an isomorphism between the ordered sets $(\mathbf{Ord}, <)$ and $(\mathbf{COrd}, <)$.

In general, the map does not preserve the operations $+, \cdot$, as $+$ and $\cdot$ are commutative on $\mathbf{Num} \subset \mathbb{N}^*$ but not on $\mathbf{COrd}$. However, the situation is more interesting if we consider the natural operations $\oplus, \otimes$ between ordinals. We recall that each ordinal $\bar{\sigma}$ has a unique *normal form*

$$\bar{\sigma} = \sum_{n=0}^{m} \bar{\omega}^{j_n} a_n$$

where $a_n \in \mathbb{N}$ and $n_1 < n_2 \Rightarrow j_{n_1} > j_{n_2}$.

By using the normal form, the *natural ordinal operations* can be defined as follows: given

$$\bar{\sigma} = \sum_{n=0}^{m} \bar{\omega}^{j_n} a_n \ \text{ and } \ \bar{\tau} = \sum_{n=0}^{m} \bar{\omega}^{j_n} b_n \tag{13}$$

we let

$$\bar{\sigma} \oplus \bar{\tau} = \sum_{n=0}^{m} \bar{\omega}^{j_n} (a_n + b_n) \ \text{ and } \ \bar{\sigma} \otimes \bar{\tau} = \bigoplus_{n,h=0}^{m} a_n b_h \bar{\omega}^{j_n \oplus j_h}, \tag{14}$$

where $a_n + b_n$ and $a_l b_m$ are the usual operations on natural numbers.

In order to compare the operations between numerosities and the natural ordinal operations, we extend a notion used for the Cantor ordinals to the numerosities .

Definition 20. *An ordinal $\theta > 0$ is called irreducible if*

$$\sigma, \tau, \gamma < \theta \Rightarrow \sigma\tau + \gamma < \theta$$

If θ is irreducible then

$$\sigma, \tau \in \Omega_\theta \Rightarrow \sigma + \gamma < \theta \text{ and } \sigma\tau < \theta;$$

we need to prove that $\sigma + \gamma$ and $\sigma\tau \in \Omega_\theta$.

We denote by θ_j, $j \in \mathbf{Ord}$ the sequence of irreducible ordinals, namely

- $\theta_0 = \omega$,
- $\theta_j = \min \{x \in \mathbf{Ord} \mid \forall m \in \mathbb{N}_0, \ \forall k < j, \ x > \theta_k^m\}$.

Proposition 21. *If $\tau \in \mathbf{Ord}$, we have that*

$$\tau < \theta_{j+1} \Leftrightarrow \tau = \sum_{k=0}^{m} b_k \theta_j^k$$

with $b_k \in \Omega_{\theta_j}$.

Proof. This proof is based only on the order structure of **Ord** and hence it could be considered well known. However we will report it for completeness and for the sake of the reader.

($\Leftarrow$) trivial.

($\Rightarrow$) If $\tau < \theta_{j+1}$, we take

$$n = \max\left\{m \in \mathbb{N}_0 \mid \theta_j^m \leq \tau\right\}$$

Such an m exists by the definition of θ_{j+1}. Then we set

$$b_m = \sup\left\{x \in \Omega_{\theta_j} \mid x\theta_j^m \leq \tau\right\}$$

and

$$y_{j,m} = \tau - b_m \theta_j^m$$

Then,

$$\forall z \in \Omega_{\theta_j},\ y_{j,m} \leq z. \tag{15}$$

Now, by inducion over $k = m-1, ..., 0$, we set

$$b_k = \sup\left\{x \in \Omega_{\theta_j} \mid \sum_{l=k+1}^{m} b_l \theta_j^l + x\theta_j^k \leq \tau\right\}$$

and

$$y_{j,k} = \tau - \sum_{l=k}^{m} b_l \theta_j^l$$

so we have that,

$$\forall z \in \Omega_{\theta_j},\ y_{j,m} \leq z \tag{16}$$

Now we claim that

$$\tau - \sum_{k=0}^{m} b_k \theta_j^k = 0 \tag{17}$$

In order to prove this we argue by iduction over $j \in \mathbf{Ord} \cup \{-1\}$ by proving that

$$y_{jk} = 0. \tag{18}$$

If $j = -1,\ \tau \in \Omega_{\theta_0} = \mathbb{N}_0$, then $\forall n \in \mathbb{N}_0,\ y_{00} \leq 0$ and hence $y_{00} = 0$. If (18) holds $\forall \tau \in \Omega_{\theta_j}$, then by (15) and (16), equality (18) holds also for $\tau \in \Omega_{\theta_{j+1}}$. $\square$

Corollary 22. *If $\sigma, \tau \in \mathbf{Ord}$, then $\sigma + \tau \in \mathbf{Ord}$ and $\sigma\tau \in \mathbf{Ord}$*

Proof. By Prop. 21,

$$\sigma = \sum_{k=0}^{n} a_k \theta_j^k, \ \tau = \sum_{k=0}^{n} b_k \theta_j^k$$

for some $j \in \mathbf{Ord}$ and hence

$$\sigma + \tau = \sum_{k=0}^{n} (a_k + b_k)\, \theta_j^k; \ \sigma\tau = \sum_{h,k=0}^{n} (a_h b_k)\, \theta_j^{h+k}. \qquad \square$$

Now we describe the sequence of the irreducible ordinal numerosities: we set

- $\theta_0 = \bar{\omega}$
- $\theta_{\bar{j}} = \sup \left\{ \theta_{\bar{k}}^n \mid n \in \mathbb{N}, \ \bar{k} < \bar{j} \right\}$

So we have that

$$\begin{array}{rcl}
\theta_0 & = & \omega, \\
\theta_1 & = & \omega^{\omega}, \\
\theta_2 & = & \omega^{\omega^{\omega}} \\
 & & \ldots \\
\theta_{j+1} & = & \theta_j^{\omega} \\
 & & \ldots \\
\theta_{\omega} & = & \varepsilon_0 \\
 & & \ldots
\end{array}$$

and so on. Since the definiton of θ_j depends only on the order structure of $(\mathbf{Ord}, <)$, then

$$\Phi(\theta_j) = \bar{\theta}_j.$$

It is well known and easy to check that any ordinal number $\bar{\tau} \in \mathbf{COrd}, \bar{\tau} < \bar{\theta}_{j+1}$, can be written as follows:

$$\bar{\tau} = \bigoplus_{n=0}^{m} \bar{a}_n \otimes \bar{\theta}_j^n; \quad \bar{a}_n < \bar{\theta}_j$$

and the natural operations $\oplus, \otimes$ take the following form:

$$\left(\bigoplus_{n=0}^{m} \bar{a}_n \otimes \bar{\theta}_j^n \right) \oplus \left(\bigoplus_{n=0}^{m} \bar{b}_n \otimes \bar{\theta}_j^n \right) = \bigoplus_{n=0}^{m} \left(\bar{a}_n \oplus \bar{b}_n \right) \otimes \bar{\theta}_j^n$$

$$\bar{\sigma} \otimes \bar{\tau} = \bigoplus_{n,h=0}^{m} \left(\bar{a}_n \otimes \bar{b}_h \right) \otimes \bar{\theta}_j^{n+h}$$

Theorem 23. *The map (12) is an isomorphism between the semirings* $(\mathbf{Ord},+,\cdot)$ *and* $(\mathbf{COrd},\oplus,\otimes)$*, namely*

$$\begin{aligned}\Phi(\sigma+\tau) &= \bar{\sigma}\oplus\bar{\tau}\\ \Phi(\sigma\tau) &= \bar{\sigma}\otimes\bar{\tau}\end{aligned}$$

Proof. Let $\tau=\sum_{k=0}^{m}b_k\theta_j^k$ be an ordinal numerosity. Then

$$\begin{aligned}\sum_{n=0}^{m}\bar{b}_n\otimes\bar{\theta}_j^n &= ot\left(\left\{\bigoplus_{n=0}^{m}\bar{a}_n\otimes\bar{\theta}_j^n\in\mathbf{COrd}\mid\bigoplus_{n=0}^{m}\bar{a}_n\otimes\bar{\theta}_j^n<\bigoplus_{n=0}^{m}\bar{b}_n\otimes\bar{\theta}_j^n\right\}\right)\\ &= ot\left(\left\{\sum_{n=0}^{m}a_n\theta_j^n\in\mathbf{Ord}\mid\sum_{n=0}^{m}a_n\theta_j^n<\sum_{n=0}^{m}b_n\theta_j^n\right\}\right)=ot\left(\Omega_\tau\right)=\bar{\tau},\end{aligned}$$

namely

$$\Phi(\tau)=\Phi\left(\sum_{n=0}^{m}b_n\theta_j^n\right)=\bigoplus_{n=0}^{m}\bar{b}_n\otimes\bar{\theta}_j^n=\bar{\tau}.$$

Hence Φ is an isomorphism. □

Remark 24. *Theorems 23 and 19 provide a new interpretation for the natural operations* $\oplus$ *and* $\otimes$ *namely*

$$\bar{\sigma}\oplus\tau=ot(\Omega_{\sigma+\tau})\ \text{ and }\ \bar{\sigma}\otimes\bar{\tau}=ot(\Omega_{\sigma\tau})$$

This fact is somewhat surprising since the operation $+$ *and* $\cdot$ *between numerosities have been introduced in a natural way for the numerosity theory and,* a priori*, they should not have any realation with the natural operations between ordinal numbers.*

Notice, however, that not all operations are the same between numerosity ordinals and Cantor ordinals: for example, let $\bar{\varepsilon}_0=\bar{\theta}_{\bar{\omega}}$ be the Cantor ordinal that corresponds to the numerosity ordinal θ_ω. If we use the ordinal exponentiation, we have that

$$\bar{\omega}^{\bar{\varepsilon}_0}=\bar{\varepsilon}_0,$$

whilst on the contrary, if we use the Euclidean exponentiation, we get that

$$\omega^{\varepsilon_0}>\theta_\omega.$$

In particular the equation

$$\bar{\omega}^x=\bar{\varepsilon}_0$$

in the world of Cantor ordinals has the solution $x=\bar{\varepsilon}_0$ while the equation

$$\omega^x=\varepsilon_0$$

in the world of Euclidean numbers, has the solution $\xi=\log_\omega\varepsilon_0$. ξ is a well defined Euclidean number, but it is not an ordinal number since

$$\xi<\varepsilon_0=\mathfrak{num}\left(\bigcup_{k<\omega}\Omega_{\theta_k}\right).$$

However it is easy to prove that the ordinal exponentiation agrees with the Euclidean exponentiation for numbers in Ω_{θ_ω}.

5 Numerosities of some denumerable sets

There are many different ways of definining a label set according to Definition 8. Different label sets might give different algebraical properties to the numerosity; moreover, in some cases particular choices of the label sets may lead to other concepts (e.g. Lebesgue measure for the reals). In this and the next Sections, we want to show several examples of these facts.

5.1 The general strategy

Theorem 16 describes the fundamental properties of numerosities, which are satisfied for all choices of the label set $\mathfrak{B}$ (and of the ultrafilter $\mathcal{U}$). However, certain additional properties are satisfied only for some choices of $\mathfrak{B}$: in fact, they depend on the ultrafilter $\mathcal{U}$ over $\wp_{fin}(\mathfrak{B})$, whose existence depends on Zorn's lemma which cannot be explicit and hence it is impossible to prove or disprove some of them. However, if we choose a suitable label set $\mathfrak{B}$ (and, consequently we restrict the choice of $\mathcal{U}$), it is possible to show that some properties, as the ones mentioned in the Introduction, are satisfied independently of $\mathcal{U}$. The goal of Section 5 is to show how a suitable choice of $\mathfrak{B}$ allows the numerosity function to satisfy interesting properties in many specific cases.

The smaller the set $\mathfrak{B}$ is, the more properties are satisfied by the numerosity function. So the idea is to begin with a set $\mathfrak{B}_{\max} = \{\mathfrak{t} \in \mathfrak{L} \mid \mathfrak{t} \cap \mathfrak{L} = \varnothing\}$ and to construct smaller label sets $\mathfrak{B}_{\max} \supset \mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \ldots$ which provide a richer and richer structure to the theory. In this paper we are interested in the numerosity of some specific subsets of $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and so we will construct set of labels $\mathfrak{B}_{\max} \supset \mathfrak{B}(\mathbb{N}_0) \supset \cdots \supset \mathfrak{B}(\mathbb{R})$. Each set of labels allows to enrich the theory with new theorems; all these theorems are independent of the ultrafilter employed in the sense that every ultrafilter which satisfies the finess property[7] does the job[8].

The construction which we will present in the next sections is based on the following definition:

Definition 25. *If $\mathfrak{D} \subset \mathfrak{B}_{\max}$ is a directed set (wih respect to $\subseteq$), we define*

$$\overline{\mathfrak{D}} = \mathfrak{G}\left(\{\mathfrak{s} \in \mathfrak{B}_{\max} \mid \exists \mathfrak{t} \in \mathfrak{D},\ \mathfrak{s} \sqsupseteq \mathfrak{t}\}\right),$$

where $\mathfrak{G}F$ denotes the smallest lattice containing F.

Notice that, by Definition, if $\mathfrak{D} \subset \mathfrak{B}_{\max}$ is a directed set then

$$\overline{\overline{\mathfrak{D}}} = \overline{\mathfrak{D}}.$$

[7]The finess property has been introduced in the proof of the existence of the field of Euclidean numbers.

[8]Of course a smaller set of labels reduces the choice of the ultrafilter. More precisely if $\mathfrak{B}_1 \supset \mathfrak{B}_2$, an ultrafilter constructed over $\mathfrak{B}_2$ makes $\mathfrak{B}_1$ to be a qualified set.

Lemma 26. *For every* $\mathfrak{D} \subset \mathfrak{B}_{\max}$, $\overline{\mathfrak{D}}$ *is a label set.*

Proof. Let us check that $\overline{\mathfrak{D}}$ satisfies the properties of Definition 8.

Property 8.(i) holds as $\overline{\mathfrak{D}}$ is a lattice by definition.

Property 8.(ii) holds as $\overline{\mathfrak{D}} \subseteq \mathfrak{B}_{\max}$.

Property 8.(iii) holds as $\forall a \in \Lambda, \exists \mathfrak{s} \in \mathfrak{B}_{\max},\ a \in \mathbb{V}(\mathfrak{s})$ and hence, if you take any $\mathfrak{t} \in \mathfrak{D}$, $a \in \mathbb{V}(\mathfrak{s} \cup \mathfrak{t})$; on the other hand $\mathfrak{s} \cup \mathfrak{t} \in \overline{\mathfrak{D}}$ and so $\bigcup_{\mathfrak{s} \in \overline{\mathfrak{D}}} \mathbb{V}(\mathfrak{s}) = \Lambda$. □

The numerosity of a set depends on the set of labels $\mathfrak{B}$ and an ultrafilter $\mathcal{U}$ consistent with $\mathfrak{B}$. As in this section we will discuss also coherence properties between different label sets, we will use the notation $\mathfrak{num}_{\mathfrak{B}}^{\mathcal{U}}$ to denote the numerosity function obtained using labels in $\mathfrak{B}$ and the ultrafilter $\mathcal{U}$ and similarly we denote by $\ell_{\mathfrak{B}}(x)$ the label relative to $\mathfrak{B}$ (see Definition 9). This notation will be used only when there is danger of confusion, as multiple sets of labels are used at once. We will keep to use to the simpler notation $\mathfrak{num}$ whenever there is no danger of such confusion.

By definition, the $\overline{\mathfrak{D}}$-labelling of $a \in \Lambda$ is given by

$$\ell_{\overline{\mathfrak{D}}}(a) = \bigcap \{\mathfrak{s} \in \overline{\mathfrak{D}} \mid a \in \mathbb{V}(\mathfrak{s})\} = \bigcap \{\mathfrak{s} \in \mathfrak{B}_{\max} \mid a \in \mathbb{V}(\mathfrak{s}) \text{ and } \exists \mathfrak{t} \in \mathfrak{D},\ \mathfrak{s} \supseteq \mathfrak{t}\};$$

in particular, we have that

$$\mathfrak{s} \in \mathfrak{D} \Rightarrow \ell_{\overline{\mathfrak{D}}}(\mathfrak{s}) = \mathfrak{s}. \tag{19}$$

Proposition 27. *If* $\mathfrak{D}_1 \subset \mathfrak{D}_2$ *then* $\overline{\mathfrak{D}}_1 \subset \overline{\mathfrak{D}}_2$, *hence for all ultrafilter* $\mathcal{U}$ *consistent with* $\overline{\mathfrak{D}}_1$, *for every set* A *in* Λ

$$\mathfrak{num}_{\overline{\mathfrak{D}}_1}^{\mathcal{U}}(A) = \mathfrak{num}_{\overline{\mathfrak{D}}_2}^{\mathcal{U}}(A).$$

Proof. The inclusion $\overline{\mathfrak{D}}_1 \subset \overline{\mathfrak{D}}_2$ holds trivially from Definition 25. The consistency is immediate as if $\mathcal{U}$ contains $\overline{\mathfrak{D}}_1$ and $\overline{\mathfrak{D}}_1 \subset \overline{\mathfrak{D}}_2$ then necessarily $\mathcal{U}$ contains $\overline{\mathfrak{D}}_2$. □

Lemma 28. *If* $\lambda \in \overline{\mathfrak{D}}$, *then* λ *can be split as follows*

$$\lambda = \mathfrak{s} \cup \mathfrak{t}$$

where $\mathfrak{s} \in \mathfrak{D}$ *and* $\mathfrak{t}$ *is such that*

$$\forall \sigma \in \mathfrak{D},\ \mathfrak{t} \cap \sigma = \varnothing.$$

Proof. Given $\lambda \in \overline{\mathfrak{D}}$, we set

$$\mathfrak{s} := \bigcup \{\mathfrak{u} \in \mathfrak{D} \mid \mathfrak{u} \subset \lambda\}$$

and

$$\mathfrak{t} := \lambda \backslash \mathfrak{s}.$$

Then $\mathfrak{s} \in \overline{\mathfrak{D}}$, as $\overline{\mathfrak{D}}$ is a lattice and the union defining $\mathfrak{s}$ is finite, and $\forall \sigma \in \mathfrak{D}$, $\mathfrak{t} \cap \sigma = \varnothing$. In fact, if we set $\mathfrak{u} = \mathfrak{t} \cap \sigma$ then, as $\mathfrak{s} \supseteq \sigma$, we have

$$\varnothing = \mathfrak{s} \cap \mathfrak{t} \supseteq \sigma \cap \mathfrak{t} = \mathfrak{u},$$

hence $\mathfrak{u} = \varnothing$. □

Remark 29. *We can look at the splitting given by Lemma 28 thinking of* $\mathfrak{B}$ *as a vector space over* $\mathbb{Z}_2$*; in this case we can write*

$$\mathfrak{B} = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$$

and the splitting $\lambda = \mathfrak{s} \cup \mathfrak{t}$ *implies that*

$$\mathfrak{s} \in \mathfrak{D} \text{ and } \mathfrak{t} \in \mathfrak{D}^{\perp}.$$

5.2 Numerosity of the natural numbers

In what follows, we set $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and we let α denote the numerosity of $\mathbb{N}$. We will consider numbers in $\mathbb{N}_0$, as well as more generally in $\mathbb{R}$, as atoms. Our goal is to find a label set $\mathfrak{B}(\mathbb{N}_0) \subset \mathfrak{B}_{\max}$ so that we can prove some properties of α and to describe the numerosity of some subsets of $\mathbb{N}_0$ by functions of α.

We define $\mathfrak{D}(\mathbb{N}_0)$ as follows:

$$\lambda \in \mathfrak{D}(\mathbb{N}_0) \Leftrightarrow \exists m \in \mathbb{N} \text{ such that } \lambda = \left\{0, \dots, m!^{m!}\right\},$$

and we set

$$\mathfrak{B}(\mathbb{N}_0) := \overline{\mathfrak{D}(\mathbb{N}_0)}.$$

which, by Lemma 26, is a label set. By Definition, we have that for every $n \in \mathbb{N}_0$,

$$\ell(n) = \{0, \dots, f(n)\},$$

where

$$f(n) := \min\left\{m!^{m!} \mid m \in \mathbb{N},\ m!^{m!} \geq n\right\}.$$

The main reason for such a peculiar labelling is to ensure the following algebraical properties for α:

Proposition 30. *Let* $n \in \mathbb{N}$*. Then*

(*i*) $\mathfrak{num}(\{nm \mid m \in \mathbb{N}\}) = \frac{\alpha}{n}$*;*

(*ii*) $\mathfrak{num}(\{m^n \mid m \in \mathbb{N}\}) = \alpha^{\frac{1}{n}}$*.*

Proof. (i) For $i = 0, \dots, n-1$ let

$$A_i = \{m \in \mathbb{N}_0 \mid m \equiv i \mod n\}.$$

Then for every $\lambda \sqsupseteq \{0,1,\ldots,n!^{n!}\}$, with $\lambda \in \mathfrak{B}(\mathbb{N}_0)$, for every $0 \le i,j < n$ we have

$$|A_i \cap \lambda| = |A_j \cap \lambda|,$$

as $\lambda \cap \mathbb{N} = \{0,1,\ldots,f(m)\}$ for some $m \ge n$, and n divides $f(m)$ for every such m. In particular, this shows that $\mathfrak{num}(A_i) = \mathfrak{num}(A_j)$ for every $0 \le i,j < n$, hence

$$\alpha = \mathfrak{num}(\mathbb{N}_0) = \sum_{i=0}^{n-1} \mathfrak{num}(A_i) = n \cdot \mathfrak{num}(A_0).$$

(ii) Let $\lambda \sqsupseteq \{1,\ldots,n!^{n!}\}$, with $\lambda \in \mathfrak{B}(\mathbb{N}_0)$. As noticed in (i) above, it must be $\lambda \cap \mathbb{N} = \{1,\ldots,m!^{m!}\}$ for some $m \ge n$. If $a = m!^{\frac{m!}{n}}$, we can rewrite $\{1,\ldots,m!^{m!}\}$ as $\{1,\ldots,a^n\}$. Hence $|\{m^n \mid m \in \mathbb{N}\} \cap \lambda| = a = |\mathbb{N} \cap \lambda|^{\frac{1}{n}}$. The thesis is reached by taking the Λ-limit on the above equality. □

Remark 31. *Of course, the choice of $\mathfrak{D}(\mathbb{N}_0)$ is not intrinsic, and has been done so to make it possible to have the properties listed in Proposition 30. Some additional motivations for this choice of $\mathfrak{D}(\mathbb{N}_0)$ can be found in [4]; different motivations have lead the authors of [9] to make the following different choice:*

$$\lambda \in \mathfrak{D}_1(\mathbb{N}_0) \Leftrightarrow \exists m \in \mathbb{N} \text{ such that } \lambda = \{0,\ldots,2^m-1\}.$$

This can be seen as a feature of this approach: different algebraical properties of the numerosity can be rather easily obtained by changing the label set.

5.3 Numerosity of the integers

We proceed as in the case of the natural numbers. We define $\mathfrak{D}(\mathbb{Z})$ as follows:

$$\lambda \in \mathfrak{D}(\mathbb{Z}) \Leftrightarrow \exists m \in \mathbb{N} \text{ such that } \lambda = \left\{-m!^{m!},\ldots,m!^{m!}\right\}.$$

Clearly $\mathfrak{D}(\mathbb{Z}) \subset \mathfrak{B}(\mathbb{N}_0)$ and hence, by Lemma 26,

$$\mathfrak{B}(\mathbb{Z}) := \overline{\mathfrak{B}(\mathbb{N}_0) \cap \mathfrak{D}(\mathbb{Z})}$$

is a label set. Using this label basis for every $z \in \mathbb{Z}$,

$$\ell(z) \cap \mathbb{Z} = \{-n(z),\ldots,n(z)\},$$

where

$$n(z) := \min\left\{m!^{m!} \mid m \in \mathbb{N},\ m!^{m!} \ge |z|\right\}.$$

Moreover, as $\mathfrak{B}(\mathbb{Z}) \subseteq \mathfrak{B}(\mathbb{N})$, by Proposition 27 the numerosities constructed with $\mathfrak{B}(\mathbb{Z})$ are coherent with those constructed with $\mathfrak{B}(\mathbb{N})$.

With this choice of $\mathfrak{D}(\mathbb{Z})$, $\mathfrak{num}(\mathbb{Z}) = 2\alpha + 1$ and we have that

$$\mathfrak{num}(\mathbb{Z}_{<0}) = \mathfrak{num}(\mathbb{Z}_{>0}) = \alpha; \tag{20}$$

this equality agrees with the intuition that the positive numbers are as many as the negative numbers.

Just as an example of a possible application, let us prove the following result for subgroups of $\mathbb{Z}$, which reminds Lagrange's Theorem for finite groups:

Theorem 32. *Let $S := m\mathbb{Z}$ be a subgroup of $(\mathbb{Z}, +)$. Then*

$$\frac{\mathfrak{num}(\mathbb{Z})}{\mathfrak{num}(S)} \sim m = \mathfrak{num}(\mathbb{Z}_m). \tag{21}$$

Proof. By definition, $S = \{mn \mid n \in \mathbb{Z}\}$. We write $S = S_+ \cup S_- \cup \{0\}$, where

$$S_+ = \{a \in S \mid a > 0\}, S_- = \{a \in S \mid a < 0\}.$$

By Proposition (30) we know that $\mathfrak{num}(S_+) = \frac{\alpha}{m}$, and it is trivial to show that $\mathfrak{num}(S_-) = \mathfrak{num}(S_+)$. Hence $\mathfrak{num}(S) = \mathfrak{num}(S_+) + \mathfrak{num}(S_-) + 1 = 2\frac{\alpha}{m} + 1$. As $\mathfrak{num}(\mathbb{Z}) = 2\alpha + 1$, we have

$$\frac{\mathfrak{num}(\mathbb{Z})}{\mathfrak{num}(S)} = \frac{2\alpha + 1}{\frac{2\alpha}{m} + 1} = \frac{2\alpha + 1}{\frac{1}{m}(2\alpha + m)} \sim m,$$

as α is infinite. □

Remark 33. *Let us notice that, with our labelling, in the above Proposition we do not have the equality*

$$\frac{\mathfrak{num}(\mathbb{Z})}{\mathfrak{num}(S)} = m \tag{22}$$

because not all lateral classes $[k]$ in the quotient have the same numerosity:

$$\mathfrak{num}([k]) = \frac{2\alpha}{m} \quad \textit{if} \quad k \neq 0; \quad \mathfrak{num}([0]) = \frac{2\alpha}{m} + 1.$$

If we want the equality in (21), then we can replace $\mathfrak{D}(\mathbb{Z})$ with $\mathfrak{D}_1(\mathbb{Z})$ defined as follows:

$$\lambda \in \mathfrak{D}_1(\mathbb{Z}) \Leftrightarrow \exists m \in \mathbb{N} \text{ such that } \lambda = \left\{-m!^{m!} + 1, \ldots, m!^{m!}\right\}.$$

In this case, we get (22), but $\mathfrak{num}(\mathbb{Z}) = 2\alpha$ and the equality (20) is violated.

5.4 Numerosity of the rationals

The labelling of $\mathbb{Z}$ given in Section 5.3 can be extended in several ways to the rationals. A natural one is obtained by setting

$$\mathfrak{D}(\mathbb{Q}) := \left\{ \mathbb{H}_n \mid \exists m \in \mathbb{N},\ n = m!^{m!} \right\},$$

where

$$\mathbb{H}_n := \left\{ \frac{a}{n} \mid a \in \mathbb{Z},\ -n^2 < a < n^2 \right\}.$$

By Lemma 26,

$$\mathfrak{B}(\mathbb{Q}) := \overline{\mathfrak{D}(\mathbb{Q})}$$

is a label set.

As $\mathfrak{D}(\mathbb{Q}) \subset \mathfrak{B}(\mathbb{Z})$, by Proposition 27 the numerosities constructed with $\mathfrak{B}(\mathbb{Q})$ are coherent with those constructed with $\mathfrak{B}(\mathbb{Z})$. Using the label basis $\mathfrak{B}(\mathbb{Q})$ we have that, for every $q \in \mathbb{Q}$,

$$\ell(q) \cap \mathbb{Q} = \mathbb{H}_{n(q)},$$

where

$$n(q) := \min\left\{m!^{m!} \mid m \in \mathbb{N},\ m!^{m!} \geq |q|\right\}.$$

This labelling has been chosen in order to have the following results:

Proposition 34. *Using the labelling $\mathfrak{B}(\mathbb{Q})$, the following properties hold*

(i) *for all $n \in \mathbb{N}_0$, $\mathfrak{num}(\mathbb{Q} \cap [n, n+1)) = \alpha$;*

(ii) *for all $p, q \in \mathbb{R}$ with $p < q$, $\frac{\mathfrak{num}(\mathbb{Q}\cap[p,q))}{\mathfrak{num}(\mathbb{Q}\cap[0,1))} \sim (p-q)$;*

(iii) $\mathfrak{num}(\mathbb{Q}) = 2\alpha^2 + 1$.

Proof. (i) Take $H_m \in \mathfrak{B}_{\mathbb{Q}}$ with m larger than $n+1$. Then $|(\mathbb{Q} \cap [n, n+1)) \cap \mathbb{H}_m| = m$, hence eventually $|(\mathbb{Q} \cap [n, n+1)) \cap \mathbb{H}_m| = |\mathbb{N} \cap \mathbb{H}_m|$, and the thesis follows by taking the Λ-limit.

(ii) Take $\mathbb{H}_m \in \mathfrak{B}_{\mathbb{Q}}$ with m larger than $|p|, |q|$. Then $(\mathbb{Q} \cap [p, q)) = (p-q)m$ if $p \in \mathbb{H}_m$, $(\mathbb{Q} \cap [p, q)) = (p-q)m - 1$ if $p \notin \mathbb{H}_m$. By taking the Λ-limit we have that either $\mathfrak{num}(\mathbb{Q} \cap [p, q)) = (p-q)\alpha - 1$ or $\mathfrak{num}(\mathbb{Q} \cap [p, q)) = (p-q)\alpha$, and the thesis follows as, by (i), $\mathfrak{num}(\mathbb{Q} \cap [0, 1)) = \alpha$.

(iii) Let us first compute $\mathfrak{num}(\mathbb{Q}_{>0})$. Let $\lambda = \mathbb{H}_n \in \mathfrak{B}_{\mathbb{Q}}$. Then $|\mathbb{H}_n \cap \mathbb{Q}_{>0}| = n^2$, hence if φ is the enumeration of $\mathbb{Q}_{>0}$ and ψ is the enumeration of $\mathbb{N}$, we have that $\varphi(\lambda) = \psi(\lambda)^2$, so

$$\mathfrak{num}(\mathbb{Q}_{>0}) = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = \lim_{\lambda \uparrow \Lambda} \psi^2(\lambda) = \left(\lim_{\lambda \uparrow \Lambda} \psi(\lambda)\right)^2 = \alpha^2.$$

Therefore, as each $\mathbb{H}_n$ is symmetrical with respect to 0, we also have that $\mathfrak{num}(\mathbb{Q}_{<0}) = \alpha^2$, and so

$$\mathfrak{num}(\mathbb{Q}) = \mathfrak{num}(\mathbb{Q}_{<0}) + \mathfrak{num}(\mathbb{Q}_{>0}) + 1 = 2\alpha^2 + 1. \qquad \square$$

An example of a possible application, let us prove the following result:

Theorem 35. *Let m_{PJ} denote the Peano-Jordan measure of a m_{PJ}-measurable set E. Then*

$$m_{PJ}(E) = st\left(\frac{\mathfrak{num}(E \cap \mathbb{Q})}{\mathfrak{num}([0,1) \cap \mathbb{Q})}\right) = st\left(\frac{1}{\alpha} \cdot \mathfrak{num}(E \cap \mathbb{Q})\right). \tag{23}$$

Proof. If E is an interval then the result follows from Proposition 34. We can extend this result to a plurinterval $E = \bigcup E_i$ by the Sum Principle (see

Theorem 16.(v)). In general, if E is m_{PJ}-measurable, $\forall\varepsilon \in \mathbb{R}_{>0}$ there are two plurintervals A and B such that

$$A \subseteq E \subseteq B$$

$$|m_{PJ}(B) - m_{PJ}(E)| < \varepsilon \text{ and } |m_{PJ}(E) - m_{PJ}(A)| < \varepsilon \tag{24}$$

By the Euclid Principle (see Theorem 16.(ii)), we have that

$$\mathfrak{num}\,(A \cap \mathbb{Q}) \subseteq \mathfrak{num}\,(E \cap \mathbb{Q}) \subseteq \mathfrak{num}\,(E \cap \mathbb{Q}),$$

then

$$m_{PJ}(A) \leq st\left(\frac{1}{\alpha} \cdot \mathfrak{num}\,(E \cap \mathbb{Q})\right) \leq m_{PJ}(B).$$

The conclusion follows by the inequality above, Equation (24) and the arbitrariness of ε. □

6 Numerosities of non-denumerable sets

6.1 A suitable labelling

Let $\hat{\mathbb{R}}^N$, $N \in \mathbb{N}$, $\hat{\mathbb{R}}^N \subset \mathbb{A}$, be a family of sets such that

$$\begin{array}{rcl} \hat{\mathbb{R}}^0 & = & \mathbb{R}, \\ \hat{\mathbb{R}}^N & \subset & \hat{\mathbb{R}}^{N+1} \end{array}$$

and each $\hat{\mathbb{R}}^{N+1}$ is isomorphic to $\mathbb{R}^N$. This akward distinction between $\hat{\mathbb{R}}^N$ and $\mathbb{R}^N$ is useful since, in this contest, it is easier to deal with atoms and the points of $\mathbb{R}^N$ are N-ples. Moreover, we need to assume that the isomorphism

$$\Psi : \mathbb{R}^N \to \hat{\mathbb{R}}^N$$

preserves also the labels, namely, if $(x_1, \ldots, x_N) \in \mathbb{R}^N$, then

$$\ell\left[\Psi\left(x_1, \ldots, x_N\right)\right] = \max\left\{\ell\left(x_1\right), \ldots, \ell\left(x_N\right)\right\}. \tag{25}$$

If $A \in \wp\left(\hat{\mathbb{R}}^N\right)$ for some N, we denote by $\mathcal{H}_d(A)$ the normalized d-dimensional Hausdorff measure[9] in $\hat{\mathbb{R}}^N$. We introduce on $\wp(\mathbb{A})$ the following order relation: given $A, B \in \wp(\mathbb{A})$, if $|A| \neq \mathfrak{c}$, we let $A \sqsubseteq B \Leftrightarrow |A| \leq |B|$; if $|A| = |B| = \mathfrak{c}$, we let

$$A \sqsubseteq B \Leftrightarrow \mathcal{H}_d\left(A \cap \hat{\mathbb{R}}^d\right) \leq \mathcal{H}_d\left(B \cap \hat{\mathbb{R}}^d\right). \tag{26}$$

[9]The normalized d-dimensional Hausdorff measure is given by

$$\mathcal{H}_d(A) = N_d H_d(A),$$

where H_d is the usual Hausdorff measure and the normalization factor is such that $\mathcal{H}_d(A)$ coincides with the usual Lebesgue m_d measure for $d \in \mathbb{N}$.

If $A \sqsubseteq B$ and $B \sqsubseteq A$, we will write $A \equiv B$.

We define $\mathfrak{D}(\mathbb{A})$ as follows: $\lambda \in \mathfrak{D}(\mathbb{A})$ if and only if

$$\lambda = \Xi \cup \mathfrak{A},$$

where

- $\Xi \in \mathfrak{B}(\mathbb{Q})$;
- $\mathfrak{A} \in \wp_{fin}(\wp(\mathbb{A}))$;
- for all $A, B \in \mathfrak{A}$, the following property holds:

$$A \sqsubseteq B \Rightarrow |A \cap \Xi| \leq |B \cap \Xi| ; \tag{27}$$

$$|A| > \aleph_0 \Rightarrow |\Xi \cap A| > |\mathbb{Q} \cap \Xi|^2 . \tag{28}$$

Lemma 36. *If* $A_1, \ldots, A_l \subset \mathbb{A}$ *and* $F \subset \mathbb{A}$ *is a finite set, there exists* $\Xi \in \wp_{fin}(\mathbb{A})$ *such that* $F \subseteq \Xi$ *and*

$$\Xi \cup \{A_1, \ldots, A_l\} \in \mathfrak{D}(\mathbb{A}) .$$

Proof. Let $A_1, \ldots, A_l \subset \mathbb{A}$ and $F \subset \mathbb{A}$ be given; first we prove the Lemma in the case in which

$$A_j \cap A_k = \varnothing \quad \text{for} \quad j \neq k. \tag{29}$$

We order the A_j's so that

$$j < k \Rightarrow A_j \sqsubseteq A_k,$$

and we construct a sequence of labels

$$\lambda_k = \Xi_k \cup \{A_1, \ldots, A_k\} , \ k \leq l$$

such that $\lambda_k \in \mathfrak{D}(\mathbb{A})$, $\lambda_k \subset \lambda_{k+1}$ and $\Xi_k \supseteq F$. We do it by induction: for $k = 1$, we set

$$\lambda_1 = \Xi_1 \cup \{A_1\} ; \quad \Xi_1 = F.$$

Trivially $\lambda_1 \in \mathfrak{D}(\mathbb{A})$, since there is nothing to verify.

Now, if $k < l$, in order to define Ξ_{k+1}, we consider four cases:

(i) $|A_{k+1}| \leq \aleph_0$;

(ii) $|A_{k+1}| > \aleph_0$, and $\mathcal{H}_d\left(A_{k+1} \cap \hat{\mathbb{R}}^N\right) = 0$;

(iii) $|A_{k+1}| > \aleph_0$, and $\mathcal{H}_d\left(A_{k+1} \cap \hat{\mathbb{R}}^N\right) > 0$;

(iv) $|A_{k+1}| > \mathfrak{c}$.

(i) We take a finite set $F_{k+1} \subset A_{k+1}$ such that $|F_{k+1}| > |A_k \cap \Xi_k|$ and we set

$$\lambda_{k+1} = \Xi_{k+1} \cup \{A_1, \ldots, A_k, A_{k+1}\}, \quad \Xi_{k+1} = \Xi_k \cup F_{k+1}.$$

Then $\lambda_k \in \mathfrak{D}(\mathbb{A})$ since (27) holds.

(ii) We take a finite set $F_{k+1} \subset A_{k+1} \backslash \mathbb{Q}$ such that

$$|F_{k+1}| > \max \left\{ |A_k \cap \Xi_k|, \ |\mathbb{Q} \cap \Xi_k|^2 \right\}$$

and we set

$$\lambda_{k+1} = \Xi_{k+1} \cup \{A_1, \ldots, A_{k+1}\}, \quad \Xi_{k+1} = \Xi_k \cup F_{k+1}.$$

Then (27) trivially holds; moreover

$$|A_{k+1} \cap \Xi_{k+1}| = |F_{k+1}| > \ |\mathbb{Q} \cap \Xi_k|^2.$$

Then also (28) is satisfied.

(iii) if $\mathcal{H}_d \left(A_{k+1} \cap \hat{\mathbb{R}}^N \right) > 0$, then $|A_{k+1}| \geq \mathfrak{c}$; if $|A_{k+1}| > \mathfrak{c}$ we are in case (iv); if $|A_{k+1}| = \mathfrak{c}$, we take $F_{k+1} \subset \left(A_{k+1} \cap \hat{\mathbb{R}}^N \right) \setminus (\mathbb{Q} \cup \Xi_k)$ such that $|F_{k+1}| > \max \left\{ |A_k \cap \Xi_k|, \ |\mathbb{Q} \cap \Xi_k|^2 \right\}$ and we argue as in point (ii).

(iv) $|A_{k+1}| > \mathfrak{c}$, we argue as in point (ii).

Now let us consider the case in which $A_1, \ldots, A_l$ does not satisfy (29). In this case we take a finite family of sets $\{B_1, \ldots, B_m\}$ which satisfies (29) and such that every A_k is the union of some B_j's. Then, given $\{B_1, \ldots, B_m\}$ and Ξ, we have proved that there exists

$$\bar{\lambda} = \bar{\Xi} \cup \{B_1, \ldots, B_m\} \in \mathfrak{D}(\mathbb{A}), \text{ with } F \subseteq \bar{\Xi}.$$

Now, it is easy to chek that

$$\lambda = \bar{\Xi} \cup \{A_1, \ldots, A_l, B_1, \ldots, B_m\} \in \mathfrak{D}(\mathbb{A})$$ □

Lemma 37. *We have that $\mathfrak{D}(\mathbb{Q}) \subset \mathfrak{D}(\mathbb{A})$ and $(\mathfrak{D}(\mathbb{A}), \subseteq)$ is a directed set.*

Proof. Given $\mathbb{H}_n \in \mathfrak{D}(\mathbb{Q})$, eventually, there is a set $A \in \mathfrak{A}$ such that $\mathbb{H}_n \in \mathfrak{D}(\mathbb{A})$ and so $\mathfrak{D}(\mathbb{Q}) \subset \mathfrak{D}(\mathbb{A})$.

For $i = 1, 2$ let

$$\lambda_i = \Xi^i \cup \left\{ A_1^i, \ldots, A_{l_i}^i \right\} \in \mathfrak{D}_0(\mathbb{A}).$$

We set

$$F = \Xi^1 \cup \Xi^2;$$

then, by Lemma (36), we can add points to F and get a set $\Xi \supset \Xi^1 \cup \Xi^2$ so that (27) and (28) are satisfied. □

Using Lemma 26, we define the label set

$$\mathfrak{B}(\mathbb{A}) := \overline{\mathfrak{D}(\mathbb{A})}. \tag{30}$$

6.2 Cardinal numbers and numerosities

A property that is natural to expect, when one has a numerosity theory for all sets in Λ, is that it must be coherent with cardinalities, namely it must satisfy the following property:

Cantor property: if $A, B \subset \Lambda \backslash \mathbb{A}$ then

$$|A| < |B| \Rightarrow \mathfrak{num}(A) < \mathfrak{num}(B). \tag{31}$$

Using the labelling $\mathfrak{B}(\mathbb{A})$ defined by (30), the following result holds:

Theorem 38. *If $A, B \subset \mathbb{A}$ then*

$$|A| < |B| \Rightarrow \mathfrak{num}(A) < \mathfrak{num}(B).$$

Proof. Given two sets $A, B \subset \mathbb{A}$ with $|A| < |B|$, we take a label $\lambda \supseteq \lambda_0 := \Xi \cup \{A, B\} \in \mathfrak{B}(\mathbb{A})$. Then, by (27)

$$|A \cap \lambda| = |A \cap \Xi| < |B \cap \Xi| = |B \cap \lambda|.$$

The conclusion follows taking the Λ-limit $\square$

By Theorem 16, it follows that the numerosity function is well defined for every set belonging to the family

$$K = \{E \in \mathbb{V}(F) \mid F \in \wp_{fin}(\mathbb{A})\},$$

since $\mathfrak{D}(\mathbb{A})$ provides a label to the elements of K. In particular, by the Comparison Principle (Theorem 16.(iv)) and (25), we have that for every set $E \subset \mathbb{R}^N$,

$$\mathfrak{num}(E) = \mathfrak{num}[\Psi(E)].$$

Now, we want to extend the notion of numerosity to any set A in Λ in such a way that the Cantor property (31) be satisfied. The simplest way to realize this task is to consider the family of infinite sets

$$\mathbb{S} := \Lambda \backslash (\mathfrak{L} \cup \mathbb{A})$$

and to assign a label to each of them. We can take an injective map

$$\Phi : \mathbb{S} \to \mathbb{A}$$

and set

$$\ell(A) = \ell(\Phi(A)).$$

Then, every set in $\Lambda \backslash \mathbb{A}$ has a label in $\mathfrak{B}(\mathbb{A})$. By the Comparison Principle (Theorem 16.(iv)), we get our desired final result:

Theorem 39. *If $A, B \in \Lambda \backslash \mathbb{A}$, then*

$$|A| < |B| \Rightarrow \mathfrak{num}(A) < \mathfrak{num}(B).$$

7 Numerosity and measures

7.1 The general theory

Given a numerosity theory and a set $E \in \Lambda$, we put

$$\mu_\gamma(E) = st\left(\frac{\mathfrak{num}(E)}{\gamma}\right),$$

where $\gamma \in \mathbb{N}^*$. μ_γ is called numerosity measure. As we will see, an interesting case occurs if you take $\gamma = \mathfrak{num}([0,1))^d$ with $d \in \mathbb{R}_{\geq 0}$. In this case we will say that μ_γ is the canonical d-dimensional numerosity measure.

Theorem 40. *The numerosity measure μ_γ satisfies the following properties:*

(i) it is finitely additive: for all sets A, B

$$\mu_\gamma(A \cup B) = \mu_\gamma(A) + \mu_\gamma(B) - \mu_\gamma(A \cap B);$$

(ii) it is superadditive, namely given a denumerable partition $\{A_n\}_{n\in\mathbb{N}}$ of a set $A \subset \mathbb{R}$, then

$$\mu_\gamma(A) \geq \sum_{n=0}^{\infty} \mu_\gamma(A_n).$$

Proof. (i) This is a trivial consequence of the additivity of the numerosity.

(ii) By Theorem 16, we have that for all $N \in \mathbb{N}$,

$$\mathfrak{num}(A) \geq \mathfrak{num}\left(\bigcup_{n=0}^{N} A_n\right) = \sum_{n=0}^{N} \mathfrak{num}(A_n),$$

hence

$$st\left(\frac{\mathfrak{num}(A)}{\gamma}\right) \geq st\left(\sum_{n=0}^{N} \frac{\mathfrak{num}(A_n)}{\gamma}\right) = \sum_{n=0}^{N} st\left(\frac{\mathfrak{num}(A_n)}{\gamma}\right);$$

therefore,

$$\mu_\gamma(A) \geq \sum_{n=0}^{N} \mu_\gamma(A_n).$$

The conclusion follows taking the Cauchy limit in the above inequality for $N \to \infty$. □

7.2 Numerosity of the subsets of $\mathbb{R}^N$

In this section, we will show that μ_β agrees with the Lebesgue measure, namely, if E is a Lebesgue measurable set, then

$$m_L(E) = \mu_\beta(E) = st\left(\frac{\mathfrak{num}(E)}{\beta}\right), \tag{32}$$

where

$$\beta := \mathfrak{num}\left([0,1)\right). \tag{33}$$

First, let we show that this holds for intervals:

Theorem 41. *The numerosity measure μ_γ is translation invariant for any $\gamma \in \mathbb{N}^*$. In particular, if $\gamma = \beta$ then for any $\varepsilon = \frac{a}{b} \in [0,1)$ we have that $\mu_\beta\left([0,\frac{a}{b})\right) = \frac{a}{b}$.*

Proof. Let $r \in \mathbb{R}, E \subseteq \mathbb{R}$. By Property 27, as $E \equiv r + E$ (in the sense of the ordering $\sqsubseteq$), we have that for every $\lambda \in \mathfrak{B}(\mathbb{A}), \lambda = \Xi \cup \mathfrak{A}$ necessarily $|E \cap \Xi| = |(E + r) \cap \Xi|$. By taking the Λ-limit, we get our first claim.

As for the second, we just have to observe that $[0,1) = [0,\frac{1}{b}) \cup [\frac{1}{b},\frac{2}{b}) \cdots \cup [\frac{b-1}{b},1)$, so by finite additivity and translation invariance we get $\mu_\beta\left([0,\frac{1}{b})\right) = \frac{1}{b}$, and the thesis follows as, similarly, $[0,\frac{a}{b}) = [0,\frac{1}{b}) \cup [\frac{1}{b},\frac{2}{b}) \cdots \cup [\frac{a-1}{b},1)$. □

Moreover, we have the following property:

Proposition 42. *The numerosity measure μ_β is subadditive on the σ-algebra of Lebesgue measurable sets.*

Proof. Let $E \in \wp(\mathbb{R})$; wlog, we assume $E \in \wp(\mathbb{R}_{\geq 0})$, as the result for a generic E will then follow easily by splitting $E = E^+ \cup E^-$. Let

$$E = \bigcup_{j\in\mathbb{N}} E_j$$

be a partition of E, with all E_j's Lebesgue measurable. Let $\varepsilon = \frac{a}{b} \in [0,1)$; for N large enough we have

$$m_L(E) \leq m_L\left(\bigcup_{j=1}^{N} E_j\right) + \varepsilon.$$

Now $E \cap [-1,-1+\varepsilon] = \varnothing$, so

$$m_L(E) \leq m_L\left(\bigcup_{j=1}^{N} E_j \cup [-1,-1+\varepsilon]\right).$$

By Property 36 of our labelling, where $d = 1$, we have

$$\mathfrak{num}(E) \leq \mathfrak{num}\left(\bigcup_{j=1}^{N} E_j \cup [-1,-1+\varepsilon]\right).$$

By Theorem 41

$$\mathfrak{num}\left(\bigcup_{j=1}^{N} E_j \cup [-1,-1+\varepsilon]\right) = \sum_{j=1}^{N} \mathfrak{num}(E_j) + \mathfrak{num}([-1,-1+\varepsilon]) \sim \sum_{j=1}^{N} \mathfrak{num}(E_j) + \varepsilon\beta,$$

hence

$$\mu_\beta(E) \leq \sum_{j=1}^{N} \mu_\beta(E_j) + \varepsilon \leq \sum_{j=1}^{\infty} \mu_\beta(E_j) + \varepsilon.$$

The arbitrariness of ε gives the desired inequality

$$\mu_\beta(E) \leq \sum_{j=1}^{\infty} \mu_\beta(E_j). \qquad \square$$

We can now prove our desired final result:

Theorem 43. *$\mu_\beta(E) = \mu_L(E)$ for all Lebesgue measurable sets $E \subseteq \mathbb{R}$.*

Proof. By Theorems 40,41 and by Proposition 42 we have that μ_β, restricted to Lebesgue measurable sets, has the empty set property, it is countably additive (as it is both subadditive and superadditive), it is invariant under translation and it is normalized. Hence it must coincide with the Lebesgue measure. $\square$

The arguments above could be generalized to prove that for any measurable set $A \subset \mathbb{R}^N$ we have that

$$m_N(A) = st\left(\frac{\mathfrak{num}(A)}{\beta^N}\right).$$

Similarly, we can define the "fractal measure" any fractal set $A \subset \mathbb{R}^N$ as follows:

$$\mathfrak{m}_d(A) = st\left(\frac{\mathfrak{num}(A)}{\beta^d}\right), \quad d \in [0, N].$$

We are not gonna study this fractal measure in detail here; however, it is not difficult to chek that $\mathfrak{m}_d(A)$ concides with the normalized Hausdorff measure H_d.

7.3 Numerosity and nonstandard measures

It is well known that the Lebesgue measure can be realized using a counting procedure based on hyperfinite sets: this is, e.g., at the core of the construction of Loeb measures, which is the most known and used of such constructions. Loeb measures were introduced in mid-70's, see [20]; see also [22] for an overview of Loeb methods and applications, and [23] for an overview of other applications of nonstandard analysis in measure theory. To confront Loeb construction with our approach, here we shortly recall Loeb construction following Goldblatt's presentantion, see [18], Section 16.8.

Let N be an infinite hypernatural number, and let $S = \{\frac{k}{N} \mid -N^2 \leq k \leq N^2, k \text{ hyperinteger}\}$. Let $\wp_I(S)$ the set of internal subsets of S, and for every $A \in \wp_I(S)$ let

$$m(A) := st\left(\frac{|A|}{N}\right),$$

where $|A|$ denotes the internal cardinality of A. Then $m : \wp_I(S) \to [0, +\infty]$ is a finitely additive measure on $\wp_I(S)$. The Loeb measure is obtained by means of the usual Carathéodory extension procedure applied to m (we will denote also the Loeb measure by m). What Loeb proved is that the Lebesgue measure can be seen as a restriction of m, in the sense that for every Lebesgue measurable set X the Lebesgue measure $m_L(X)$ is equal to the Loeb measure of the so-called pre-shadow $st^{-1}(X)$ of X, namely

$$m_L(X) = st\left(m\left(st^{-1}(X)\right)\right),$$

where $st^{-1}(X) = \{\xi \in S \mid st(\xi) \in X\}$.

The similarity between our approach is that we have that, actually, μ_β is obtained as the standard part of a quotient similar to Loeb's one. In fact, $\mu_\beta(A) = st\left(\frac{|A^* \cap \Gamma|}{|[0,1)^* \cap \Gamma|}\right)$, where:

1. $|\cdot|$ denotes the internal cardinality of a set;
2. Γ is the hyperfinite set obtained by taking $\lim_{\lambda\uparrow\Lambda} \lambda \cap \mathbb{R}$.

However, in our approach the use of Carathéodory extension procedure, as well as of pre-shadows, is substituted with the choice of a particular labelling set, which can be equivalently seen as a particular choice of the hyperfinite set used in the quotient. A similar result in a general nonstandard setting was first obtained by Bernstein and Wattenberg (see [13]; see also [15], Section 2, for a comparison of Bernstein-Wattenberg's result and Loeb measures), who in fact proved that there exists hyperfinite subsets $S \subseteq [0, 1]^*$ such that for all Lebesgue measurable $A \subseteq [0, 1]$

$$m_L(A) = st\left(\frac{|A^* \cap S|}{|S|}\right).$$

As we said before, Theorem 43 provides a new proof of the above result by taking

$$S = \lim_{\lambda\uparrow\Lambda} (\lambda \cap \mathbb{R}).$$

Finally, the problem of the relationship between numerosities and Lebesgue measure in general has been addressed in [7, 8]. In these papers, the authors introduced the notion of "elementary numerosity" (see [7], Definition 1.1), that we recall:

Definition. *An elementary numerosity on a set Ω is a function $n : \wp(\Omega) \to [0, +\infty)$ defined on all subsets of Ω, taking values in the non-negative part of an ordered field $\mathbb{F} \supseteq \mathbb{R}$, and such that the following two conditions are satisfied:*

1. *$n(x) = 1$ for every point $x \in \Omega$;*
2. *$n(A \cup B) = n(A) + n(B)$ whenever A and B are disjoint.*

The main connection between the "elementary numerosity" and Lebesgue measure is given by the following result, which is one of the instances of Theorem 3.1 in [8]:

Theorem. *There exists an elementary numerosity* $\mathfrak{n} : \wp(\mathbb{R}) \to [0, +\infty)_{\mathbb{F}}$ *such that* $m_L(X) = st\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0,1))}\right)$ *for every Lebesgue measurable set* X*.*

Once again, Theorem 43 provides another proof of the above result, as $\mathfrak{num}$, when restricted to $\wp(\mathbb{R})$ is, in fact, an elementary numerosity on $\mathbb{R}$.

The interest of Theorem 43 lies on the fact that it is based on a numerosity theory which satisfies many other additional properties.

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